THE FOURTH ASSIGNMENT.

Integration Theory 425.

This homework is due on Friday 29 September in class.
This homework will contribute to your grade.

ALGEBRAS OF SETS.

We start with a set \( X \). Recall that a family \( C \) of subsets of \( X \) is called a ring if for any \( A, B \in C \), we have \( A \cap B \in C \) and \( A \setminus B \in C \); an algebra if, additionally, \( X \in C \); and a sigma-algebra if, additionally, for any \( A_1, \ldots, A_n, \ldots \in C \), we have \( \bigcup_{n=1}^{\infty} A_n \in C \).

1. Let \( X, Y \) be two sets, let \( C \) be a sigma-algebra of subsets of \( Y \). Show that the family \( f^{-1}C = \{ f^{-1}A | A \in C \} \) is also a sigma-algebra.

2. Let \( X \) be a set, and let \( A \) be an algebra of sets. Show that \( A \) is a sigma-algebra if and only if one of the following equivalent conditions holds:

   1) for any \( A_n \in A \) such that \( A_1 \subset A_2 \subset \ldots \) we have \( \bigcup_{n=1}^{\infty} A_n \in A \).

   2) for any \( A_n \in A \) such that \( A_1 \supset A_2 \supset \ldots \) we have \( \bigcap_{n=1}^{\infty} A_n \in A \).

3. Let \( X \) be a set, let \( A \) be a sigma-algebra of sets, and let \( \mu \) be a finitely additive measure on \( A \) such that \( \mu(X) = 1 \). Show that \( \mu \) is countably additive if and only if one of the following equivalent conditions holds:

   1) for any \( A_n \in A \) such that \( A_1 \subset A_2 \subset \ldots \) we have
   \[ \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n). \]

   2) for any \( A_n \in A \) such that \( A_1 \supset A_2 \supset \ldots \) we have
   \[ \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n). \]

   **Remark.** Don’t forget to show that the limits at the right-hand side exist!

4. Give an example of a sigma-algebra containing exactly eight sets.

MEASURABLE FUNCTIONS.

Let \( X \) be a set, let \( A \) be a sigma-algebra of subsets of \( X \), and let \( f : X \to \mathbb{R} \) be a function. Unless otherwise specified, measurability of \( f \) will be understood with respect to the Borel sigma-algebra on \( \mathbb{R} \) and the sigma-algebra \( A \) on \( X \).

5. Let \( f \) be measurable. Show that so is \( |f| \). Is the converse true?

6. Let \( f_n \) be bounded measurable functions. Show that so are \( \sup f_n, \inf f_n \).
7. Prove that the following conditions (assumed to hold for any \( a \in \mathbb{R} \)) are equivalent to the measurability of \( f \).
   1) \( f^{-1}(-\infty, a) \in \mathcal{A} \);
   2) \( f^{-1}(-\infty, a] \in \mathcal{A} \);
   3) \( f^{-1}[a, \infty) \in \mathcal{A} \);
   4) \( f^{-1}(a, \infty) \in \mathcal{A} \).

8. Let \( f : (0, 1) \to \mathbb{R} \) be everywhere differentiable. Prove that its derivative is Borel measurable and, consequently, Lebesgue measurable.

9. Let \( f : [a, b] \to [c, d] \) be continuous. For \( t \in [c, d] \), let \( n(t) \) be the number of solutions to the equation \( f(x) = t \) (and we set \( n(t) = 0 \) if that number is infinite). Prove that the function \( n : [c, d] \to \mathbb{R} \) is measurable.

**Completeness in Metric Spaces**

We recall that a metric space is said to be complete if every Cauchy sequence converges in it.

10. Let \( X \) be a complete metric space and let \( B_1 \supseteq B_2 \supseteq \ldots B_n \supseteq \ldots \) be a sequence of closed balls whose radii converge to zero (in other words, \( B_n = B(x_n, \varepsilon_n) \) and \( \lim_{n \to \infty} \varepsilon_n = 0 \)). Prove that the intersection \( \bigcap_{n=1}^{\infty} B_n \) consists of exactly one point. Show, conversely, that every metric space having this property is complete.

11. Show that in a complete metric space there may exist balls \( B_1 \supset B_2 \supset \ldots B_n \supset \ldots \) such that the intersection \( \bigcap_{n=1}^{\infty} B_n \) is empty.

12. Introduce a metric \( d \) on \( \mathbb{R} \) by the formula \( d(x, y) = |\exp(x) - \exp(y)| \).
    Show that this formula does indeed give a metric. Is the resulting metric space complete?

13. Let \( X \) be the sets of all closed intervals on the real line and let the distance between two intervals \( \Delta_1, \Delta_2 \) be given by the formula
    \[
    d(\Delta_1, \Delta_2) = |\Delta_1| + |\Delta_2| - |\Delta_1 \cap \Delta_2|
    \]
    (here \( |\Delta| \) stands for the length of an interval \( \Delta \)). Show that this formula does indeed give a metric. Is this metric space complete?

14. Let \( X \) be an arbitrary set and let \( B(X) \) be the space of all bounded functions on \( X \), endowed with the metric \( d(f, g) = \sup_{x \in X} |f(x) - g(x)| \).
    Show that this formula does indeed give a metric. Is \( B(X) \) complete?

15. Let \( (X, d) \) be a metric space, let \( x \in X \). Introduce a function \( d_x : X \to \mathbb{R} \) by the formula \( d_x(y) = d(x, y) \). Prove that the function \( d_x \) is continuous.