MATH 366: Assignment 10
Due Friday, March 26, 2010

Euclidean Geometry

1. Do exercises 5, 10, 13, 14, 16 from chapter 5 in the textbook.

Complex Numbers

2. (a) Let $T$ be a M"obius transformation. Show that $T$ sends the set
$$\hat{\mathbb{R}} := \{ z \in \mathbb{C} : \text{Im}(z) = 0 \} \cup \{ \infty \},$$
consisting of the real axis together with $\infty$, to itself if and only if $T$ can be written as
$$T(z) = \frac{az+b}{cz+d}$$
where $a, b, c, d$ are all real numbers. [Hint: to prove the “only if” part, pick three real numbers and use the problem from last week.]

(b) Let $a, b, c, d$ be real numbers with $ad - bc \neq 0$, and let $w = \frac{az+b}{cz+d}$. Show that
$$\text{Im}(w) = \frac{(ad-bc)}{|cz+d|^2} \text{Im}(z).$$

(c) Show that a M"obius transformation $T$ sends the upper half-plane
$$H := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$$
onto itself if and only if it can be written in the form $T(z) = \frac{az+b}{cz+d}$ where $a, b, c, d$ are real numbers with $ad - bc = 1$.

(d) Show that for any $z \in H$, there exists a M"obius transformation $T$ such that $T(H) = H$ and $T(z) = i$.

Extra Credit

3. In order to translate statements such as “$\mathcal{H}$ is semi-Euclidean if and only if there exists a triangle $\triangle ABC$ for which $(\angle A)^\circ + (\angle B)^\circ + (\angle C)^\circ = 180^\circ$” or “The sum of the angles in a convex $n$-gon in a semi-hyperbolic Hilbert plane is less than $(n-2)(180^\circ)$” into statements that do not rely on measuring angles by real numbers (which depends on Archimedes’ axiom), we use congruence classes of angles, just as we can use congruence classes of segments as a non-Archimedean replacement for real number lengths.

Given an angle $\angle A$ of a Hilbert plane $\mathcal{H}$, let $|\angle A| = \{ \angle B : \angle A \cong \angle B \}$ be its congruence class.

1Note that since any M"obius transformation $T$ is a one-to-one correspondence from $\hat{\mathbb{C}}$ to itself, $T$ sends $H$ onto itself if and only if $T(H) \subseteq H$ and $T^{-1}(H) \subseteq H$.

2You may assume the intuitively obvious fact that such a transformation must also send $\hat{\mathbb{R}}$ to itself.

3Major exercise 9 of chapter 4 describes how to do this for segments in more detail than we discussed in class.
(a) Show that $|\angle A| < |\angle B| \iff \angle A < \angle B$ defines a total order on the set of congruence classes of angles. (You should first check that it is well-defined, i.e. that if $|\angle A| = |\angle A'|$ and $|\angle B| = |\angle B'|$, then $\angle A < \angle B$ if and only if $\angle A' < \angle B'$.)

(b) Given acute angles $\angle A$ and $\angle B$, define a sum $|\angle A| + |\angle B|$ and show that it is well-defined.

(c) Given angles $\angle A$ and $\angle B$ with $\angle A < \angle B$, define a difference $|\angle B| - |\angle A|$ and show that it is well-defined.

Now, we fix a right angle $\angle R$, and consider the set

$$\mathfrak{A} = \{0\} \cup \{|\angle A| : |\angle A| < |\angle R|\}$$

of congruence classes of acute angles, together with 0. Our total order on the set of congruence classes induces a total order on $\mathfrak{A}$ where we say that 0 is less than $|\angle A|$ for every $\angle A$. We can likewise define $|\angle A| + 0 = 0 + |\angle A| = |\angle A|$.

Now, we take $G = \mathbb{Z} \times \mathfrak{A} = \{(n, \alpha) : n \text{ is an integer and } \alpha \in \mathfrak{A}\}$, on which we define an addition operation by

$$(n_1, \alpha_1) + (n_2, \alpha_2) = \begin{cases} (n_1 + n_2, \alpha_1 + \alpha_2) & \text{if } \alpha_1 + \alpha_2 < |\angle R| \\ (n_1 + n_2 + 1, (\alpha_1 + \alpha_2) - |\angle R|) & \text{if } \alpha_1 + \alpha_2 \geq |\angle R|. \end{cases}$$

We call $G$ the unwound circle group for the Hilbert plane $\mathcal{H}$. We define a total order on $G$ by

$$(n_1, \alpha_1) < (n_2, \alpha_2) \iff \text{either } n_1 < n_2, \text{ or } n_1 = n_2 \text{ and } \alpha_1 < \alpha_2.$$

(d) Prove the following properties of $(G, +, <)$:

i. $g_1 + g_2 = g_2 + g_1 \ \forall g_1, g_2 \in G$.

ii. $(g_1 + g_2) + g_3 = g_1 + (g_2 + g_3) \ \forall g_1, g_2, g_3 \in G$.

iii. There is a unique element of $0 \in G$ such that $g + 0 = g \ \forall g \in G$.

iv. For every $g \in G$, there is a unique $(-g) \in G$ so that $g + (-g) = 0$.

v. $g_1 < g_2 \implies g_1 + g_3 < g_2 + g_3 \ \forall g_1, g_2, g_3 \in G$.

(e) Prove that if Archimedes’ axiom is satisfied in $\mathcal{H}$, then there is a unique function $\phi : G \to \mathbb{R}$ with the following properties:

i. $\phi(0, |\angle A|) = (\angle A)^0$ for all acute angles $\angle A$,

ii. $\phi(g_1 + g_2) = \phi(g_1) + \phi(g_2)$ for all $g_1, g_2 \in G$, and

iii. $g_1 < g_2 \iff \phi(g_1) < \phi(g_2)$ for all $g_1, g_2 \in G$.

Prove also that if Dedekind’s axiom is satisfied in $\mathcal{H}$, then $\phi$ is a one-to-one correspondence.

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4 In addition to acting as angle measures for a non-Archimedean Hilbert plane, for non-Archimedean non-semi-Euclidean Hilbert planes, the defect of a triangle may be taken as a non-zero element of $G$, and hence we can measure areas by elements of $G$ as well as angles. This is part of why we take the “unwound” version of this construction, rather than one where $4|\angle R| = 0$; another reason is that there is not such a nice order relation on the “wound” version.

5 A set $G$ with a binary operation $+$ and total order $<$ satisfying these properties is called an ordered abelian group.