MATH 366: Assignment 11
Due Friday, April 9, 2010

Hyperbolic geometry

1. Do exercises K3, P7, P8, P9 from chapter 7 in the textbook.

Complex numbers

2. In class, we’ve mentioned that the Möbius transformation
   \[ T(z) = \frac{z - i}{z + i} \]
   and its inverse \( T^{-1}(z) = \frac{iz + i}{-z + 1} \)
induce an isomorphism between the Poincaré upper half-plane model of the hyperbolic plane to the Poincaré disk model.

   (a) Verify that \( T \) does in fact send the real axis to the unit circle and that \( T(i) = 0 \).

   (b) Show that a Möbius transformation \( S \) sends the unit disc onto itself if and only if the Möbius transformation \( T^{-1} \circ S \circ T \) sends the upper half-plane onto itself.

   (c) The Möbius transformation \( S(z) = e^{i\theta}z \) represents rotation about 0 with angle of \( \theta \) in the Poincaré disk model. Compute the Möbius transformation \( T^{-1} \circ S \circ T \) representing rotation about \( T^{-1}(0) = i \) with angle of \( \theta \) in the Poincaré upper half-plane model.

   (d) Show that for real numbers \( \theta \),
   \[ \frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i \tan \frac{\theta}{2}. \]
   [Hint: for a geometric proof, exercise 16 from chapter 5 may be helpful.]

   (e) Show that the Möbius transformation representing rotation by an angle of \( \theta \) around \( i \) in the upper half-plane model from part 2c can be rewritten as
   \[ z \mapsto \frac{(\cos \frac{\theta}{2}) z + \sin \frac{\theta}{2}}{(-\sin \frac{\theta}{2}) z + \cos \frac{\theta}{2}}. \]

Stereographic projection

3. Let \( S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \} \) be the unit sphere and \( N = (0, 0, 1) \), and let \( p \) be the stereographic projection function from \( S^2 \setminus \{N\} \) to the \( xy \)-plane.

   (a) Show that \( p(x, y, z) = \left( \frac{x}{1 - z}, \frac{y}{1 - z} \right) \).

\[^1\text{See page 306 in the textbook; we’re projecting to the equatorial plane while the text projects to the tangent plane at \( (0, 0, -1) \).}\]
(b) Show that \( p^{-1}(x, y) = \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right) \).

(c) Show that the plane \( ax + by + cz = d \) meets the unit sphere \( S^2 \) in a circle if and only if \( a^2 + b^2 + c^2 > d^2 \).

(d) Show that the stereographic projection \( p \) sends circles on the sphere to lines and circles in the plane.

**Extra Credit**

4. Show that a Möbius transformation sends the unit disk onto itself if and only if it can be written in the form

\[
T(z) = \frac{az + b}{bz + \bar{a}},
\]

where \( a \) and \( b \) are complex numbers and \( |a| > |b| \).