MATH 366: Assignment 12

Due Friday, April 16, 2010

Hyperbolic geometry

1. Do exercises K5, P6, P13 from chapter 7 in the textbook.

Complex numbers

2. Do exercises 45, 47 from chapter 9 in the textbook.

3. Compute the Möbius transformation representing the hyperbolic translation of hyperbolic distance $d$ to the right along the real axis in the Poincaré disk model. [Hint: either use the same approach as last week when we found the Möbius transformations representing rotation about $i$ in the upper half-plane model, or work directly with the hyperbolic distance formula.]

Hyperbolic trigonometry

4. We could define the usual trigonometric functions by their Taylor series:

\[
\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \quad \text{and}
\]

\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}.
\]

We define two new functions sinh and cosh, which will be important in hyperbolic trigonometry, by the following closely related series:

\[
\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \quad \text{and}
\]

\[
\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}.
\]

(a) Show that the series for cosh and sinh converge for all real $z$, and that $\frac{d}{dz} \sinh z = \cosh z$ and $\frac{d}{dz} \cosh z = \sinh z$.

(b) Show that for all real numbers $z$, we have $\cosh^2 z - \sinh^2 z = 1$.

(c) Show that $\cosh z = \frac{e^z + e^{-z}}{2}$ and $\sinh z = \frac{e^z - e^{-z}}{2}$.

(d) Show that for all real numbers $x$ and $y$, we have

\[
\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y, \quad \text{and}
\]

\[
\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.
\]
(e) Show that for all real numbers $z$ we have

$$\sinh 2z = 2 \sinh z \cosh z,$$
$$\cosh^2 z = \frac{1}{2}(\cosh 2z + 1), \text{ and}$$
$$\sinh^2 z = \frac{1}{2}(\cosh 2z - 1).$$

(f) Show that

$$\int_0^{\sinh a} \sqrt{1 + y^2} \, dy = \frac{a}{2} + \frac{\sinh a \cosh a}{2}.$$

(g) Show that $\cosh z = \cos \frac{z}{i}$ and $\sinh z = i \sin \frac{z}{i}$.

Extra Credit

5. Let $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ be distinct $w = (z_1, z_2; z_3, z_4)$. In class we claimed that $(z_3, z_2; z_1, z_4) = (z_1, z_4; z_3, z_2) = 1/w$.

(a) Show that $(z_1, z_3; z_2, z_4) = 1 - w$.

(b) Find $(z_1, z_3; z_4, z_2)$ and $(z_1, z_2; z_4, z_3)$ as functions of $w$.

(c) Among all the 24 permutations $z_{j_1}, z_{j_2}, z_{j_3}, z_{j_4}$ of the $z_1, z_2, z_3, z_4$, what are the possible values of $(z_{j_1}, z_{j_2}; z_{j_3}, z_{j_4})$?

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1. This shows that the hyperbolic trig functions could be defined in terms of the area of a “sector” of a hyperbola just as the usual trig functions can be defined in terms of the area of a sector of the circle:

![Hyperbolic sector](image)

Of course, the more usual geometric definition of cos and sin uses arc length rather than area, but the functions parametrizing the hyperbola by arc length are different (and less useful) functions than cosh and sinh. As such, the inverses of these hyperbolic trig functions are called the area hyperbolic cosine and the area hyperbolic sine and denoted arcosh and arsinh.

2. You need only check this “formally”; if you’re not comfortable with convergence of power series involving complex numbers, don’t worry about convergence.