MATH 366: Assignment 4
Due Friday, February 5, 2010

Incidence geometry

1. Do exercise 15 and major exercise 6 from chapter 2 in the textbook.

Betweenness

2. Do exercises 1, 8, 9, 15, 16, from chapter 3 in the textbook.

Combinatorics of plane tilings

3. Consider the standard tiling of the Euclidean plane by regular hexagons. Given two hexagons $H$ and $H'$ of the tiling, we can define the (combinatorial) distance between them to be the smallest number $n$ so that there exist hexagons of the tiling $H = H_0, H_1, \ldots, H_n = H'$ so that for each $i = 1, \ldots, n$, the hexagons $H_{i-1}$ and $H_i$ share an edge.

Fix a hexagon $H$ of the tiling, and let $a_n$ be the number of hexagons of the tiling whose distance from $H$ is $n$. Find (with proof) a formula for $a_n$. \[\text{Hexagonal tiling of the Euclidean plane}\]

4. It is possible to tile the hyperbolic plane with regular heptagons, meeting three to a vertex (the heptagons in question are all congruent to one another, although they don’t look like it in the Poincaré disk model as depicted above). As in the previous problem, fix a heptagon, and let $b_n$ be the number of heptagons at distance $n$ from it. Compute $b_1, b_2, and b_3$ and show that in general, $b_{n+1} \geq 2b_n$.

You can think of this as a discrete hexagonal version of finding the “circumference of a circle.”

\[\text{Heptagonal tiling of the hyperbolic plane}\]
5. Imagine a tiling of a “plane” by regular pentagons, meeting three to a vertex. As in the previous problems, fix a pentagon, and let $c_n$ be the number of pentagons at distance $n$ from it. Compute $c_n$ for $n \leq 4$. Does your answer make sense? Could there really be a tiling of a “plane” by regular pentagons meeting three at a vertex?

**Extra credit**

6. (a) Find (with proof) an explicit formula for the $b_n$ of problem 4. [Hint: for $n \geq 2$, there are two different combinatorial “types” of heptagons at distance $n$ from the chosen one. Set $b_n = x_n + y_n$ based on the two types, find a recurrence for $x_n$ and $y_n$ that can be written in the form

$$
\begin{bmatrix}
    x_{n+1} \\
    y_{n+1}
\end{bmatrix} =
\begin{bmatrix}
    m_{11} & m_{12} \\
    m_{21} & m_{22}
\end{bmatrix}
\begin{bmatrix}
    x_n \\
    y_n
\end{bmatrix},
$$

and diagonalize the matrix.]

(b) More generally, for a tiling by regular $p$-gons, meeting three at a vertex, find a formula for the number of $p$-gons at distance $n$ from a fixed one. [What happens differently when $p = 6$?]

(c) What does your formula say in the cases $p = 3, 4, 5$? Explain.

7. Let $V$ be a 3-dimensional vector space over a field $F$. We define an interpretation of incidence geometry $\mathbf{P}(V)$ as follows: a point of $\mathbf{P}(V)$ is a 1-dimensional subspace of $V$, a line in $\mathbf{P}(V)$ is a 2-dimensional subspace of $V$, and if $P$ is a point and $\ell$ is a line, then $P \perp \ell \iff P \subset \ell$.

(a) Show that $\mathbf{P}(V)$ is a projective plane.

(b) Show that $\mathbf{P}(V) \cong \mathbf{P}^2(F)$. [Hint: choose a basis for $V$.]

(c) Show that $\mathbf{P}(V^*) \cong \mathbf{P}(V)^*$, where $V^* = \{\text{linear transformations } \phi: V \to F\}$ is the dual vector space of $V$ and $\mathbf{P}(V)^*$ is the dual projective plane of $\mathbf{P}(V)$ as defined on page 88 of the textbook.

Your isomorphism should be “natural” in the sense of not depending on the choice of a basis for $V$ (or the choice of an inner product, etc.); in particular, don’t use part (b).