Handout 1. A Fixed Point Theorem

Here we discuss the existence of a fixed point for a contraction mapping of a complete metric space. In Handouts 2 and 4, we apply this to the basic ODE Existence Theorem and the Inverse and Implicit Function Theorems.

Recall that, in a metric space (X, d), a sequence $x_1, x_2, \dots.$ is *Cauchy* if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$. Thus convergent sequences are Cauchy, and one says that the metric space (X, d) is *complete* if, conversely, every Cauchy sequence converges to some point of X. Then $\mathbf{R}$, or more generally any closed subset of $\mathbf{R}^n$, is complete with the usual distance $d(x, y) = |x - y|$.

We will also consider the following interesting infinite dimensional example. For any positive integers m, n, region $\Omega \subset \mathbf{R}^m$, point $y_0 \in \mathbf{R}^n$, and $0 < r < \infty$, let

$$\mathcal{C}\big(\Omega, \overline{\mathbf{B}_r(y_0)}\big) \ = \ \{\text{continuous } \ f : \Omega \to \overline{\mathbf{B}_r(y_0)}\} \ .$$

Here $\mathbf{B}_r(y_0)$ (respectively, $\overline{\mathbf{B}_r(y_0)}$) denotes the open (respectively, closed) ball of radius r in $\mathbf{R}^n$ centered at y_0.

Exercise. $\mathcal{C}\big(\Omega, \overline{\mathbf{B}_r(y_0)}\big)$ is a complete metric space with the (so called *uniform, sup*, or L^∞) distance

$$d(f, g) \ = \ \|f - g\|_{sup} \ = \ \sup_{x\in\Omega} |f(x) - g(x)| \ .$$

Definition. A mapping A from a metric space (X, d) to itself is called a *contraction* if there is a positive $\alpha < 1$ so that

$$d\big(A(x), A(y)\big) \ \le \alpha\, d(x, y) \ \text{ for all } \ x, y \in X \ .$$

An easy example is $A(x) = \frac{1}{2}x$ for $x \in X = \mathbf{R}^n$.

Fixed Point Theorem. *For any contraction A of a complete metric space (X, d) there exists a unique point $x_\infty \in X$ so that $A(x_\infty) = x_\infty$.*

Proof : Starting with any point $x_0 \in X$, we define inductively $x_{n+1} = A(x_n)$ for $n = 0, 1, \dots.$ We see inductively that

$$d(x_{k+1}, x_k) \ \le \ \alpha^k \ d(x_1, x_0)$$

for $k = 0, 1, 2, \dots$, So that, for positive integers $m < n$,

$$\begin{aligned} d(x_m, x_n) \ &\le \ \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \\ &\le \ (\alpha^{n-1} + \dots + \alpha^m) d(x_1, x_0) \ \le \ \frac{1}{1-\alpha} d(x_1, x_0)\alpha^m \ , \end{aligned}$$

which approaches 0 as $n \to \infty$. Thus, the sequence $x_0, x_1, x_2, \ldots$ is Cauchy and so, by the completeness of X, converges to a point $x_\infty \in X$. Being a contraction, A is continuous; hence,

$$A(x_\infty) \;=\; \lim_{n\to\infty} A(x_n) \;=\; \lim_{n\to\infty} x_{n+1} \;=\; x_\infty \;.$$

We conclude that x_∞ is a fixed point of A.

It is also the *unique* fixed point because for any fixed point y of A,

$$d(y, x_\infty) \;=\; d\big(A(y), A(x_\infty)\big) \;\leq\; \alpha\, d(y, x_\infty) \;,$$

which is only possible if $d(y, x_\infty) = 0$. ∎