Handout 3. Fundamental Theorem of Curves

Theorem. *Given a smooth function* $\tau(s)$ *and a positive smooth function* $\kappa(s)$ *on an interval* I *containing* 0*, a point* $\alpha_0 \in \mathbf{R}^3$*, and two unit vectors* T_0 *and* N_0 *in* $\mathbf{R}^3$*, there exists a unique unit-speed curve* $\alpha(s)$ *on* I *with curvature* $\kappa(s)$*, torsion* $\tau(s)$*, initial position,* $\alpha(0) = \alpha_0$*, initial velocity* $\alpha'(s) = T_0$*, and initial acceleration* $\alpha''(s) = \kappa(0)N_0$*.*

Proof. For the vector-valued function $u \equiv (T, N, B) : I \to \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$, we may solve the O.D.E.

$$\begin{aligned} T' &= \kappa N \ , \\ N' &= -\kappa T \ + \ \tau B \ , \\ B' &= -\tau N \end{aligned}$$

with initial conditions $T(0) = T_0,\ N(0) = N_0,\ B(0) = T_0 \wedge N_0$.

We claim that (T, N, B) is then automatically an orthonormal frame. To see this, note that vector-valued function

$$v = (v_1, v_2, v_3, v_4, v_5, v_6) \equiv (T \cdot T, T \cdot N, T \cdot B, N \cdot N, N \cdot B, B \cdot B)$$

then satisfies the O.D.E.

$$\begin{aligned} v_1' &= -2\kappa v_2 \ , \\ v_2' &= -\kappa v_4 \ - \ \kappa v_1 \ + \ \tau v_3 \ , \\ v_3' &= \kappa v_5 \ - \ \tau v_3 \ , \\ v_4' &= -2\kappa v_2 \ + \ 2\tau v_6 \\ v_5' &= -\kappa v_3 \ + \ \tau v_6 \ - \tau v_4 \ , \\ v_6' &= -2\tau v_5 \end{aligned}$$

with $v(0) = (1, 0, 0, 1, 0, 1)$. But since the *constant* function $(1, 0, 0, 1, 0, 1)$ also satisfies the above O.D.E. with the same initial data, we conclude that $v \equiv (1, 0, 0, 1, 0, 1)$, so that (T, N, B) is indeed an orthonormal frame.

Also since the length of the vector function $u = (T, N, B)$ remains bounded, in fact identically $\sqrt{3}$, we see, by continuation, that the solution (T, N, B) exists not only near $s = 0$ but even over the whole interval I.

We conclude that

$$\alpha(s) \ = \ \alpha_0 \ + \ \int_0^s T(t)\, dt$$

has $\alpha(0) = \alpha_0$ and is unit-speed with tangent T because $\alpha'(s) = T(s)$. Also

$$\alpha''(s) \ = \ T'(s) \ = \ \kappa N(s)$$

so that α has curvature κ and principal normal N. Moreover,

$$(T \wedge N)'(s) \ = \ (T' \wedge N)(s) \ + \ (T \wedge N')(s) \ = \ \tau(s)N(s)$$

so that α also has torsion τ.

Finally, if $\overline{\alpha}$ is another unit-speed curve with curvature κ and torsion τ, $\overline{\alpha}(0) = \alpha_0$, $\overline{\alpha}'(0) = T_0$, and $\overline{\alpha}''(0) = \kappa(0)N_0$, then the function $f(s) = T \cdot \overline{T} + N \cdot \overline{N} + B \cdot \overline{B}$ satisfies $f(0) = 3$ and, by the Frenet formulas, $f' \equiv 0$. Thus f is the constant function 3, and each of the at most unit-sized terms $T \cdot \overline{T}$, $N \cdot \overline{N}$, $B \cdot \overline{B}$ must be the constant 1. Being unit vectors, $T \equiv \overline{T}$, $N \equiv \overline{N}$, $B \equiv \overline{B}$. In particular,

$$\overline{\alpha}(s) \;=\; \alpha_0 \;+\; \int_0^s \overline{T}(t)\,dt \;=\; \alpha_0 \;+\; \int_0^s T(t)\,dt \;=\; \alpha(s)\;.$$

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