Handout 2. A Very Short Course in Local O.D.E. Theory

Suppose $0 < a < \infty$, $y_0 \in \mathbf{R}^n$, $0 < r < \infty$, and $f(x,y)$ is a continuous function defined on the cylinder $\Omega = [x_0 - a, x_0 + a] \times \mathbf{B}_r(y_0)$ in the $\mathbf{R} \times \mathbf{R}^n$. Consider the O.D.E. initial-value problem:

$$\frac{du}{dx} = f\big(x,\, u(x)\big) \quad \text{and} \quad u(x_0) = y_0 \ . \tag{$*$}$$

When $n = 1$, this has the following geometric interpretation. The function f defines a *line field* V on the rectangle where the line $V(x,y)$ through (x,y) has slope $f(x,y)$. Solving (*) is equivalent to finding the graph of a function u which passes through the fixed point (x_0, y_0) and which is everywhere tangent to the line field.

One cannot expect such a solution to exist for all x . For example, if $f(x,y) = 1+y^2$ and $(x_0,y_0) = (0,0)$, then the solution of (*) is found (by separation of variables) to be $u(x) = \tan(x)$ which blows up at $x = \pm\frac{\pi}{2}$. (It is useful to sketch the line field of this example.) The theorem below gives simple conditions under which the solution exists near x_0 .

Suppose

$$M \;=\; \sup_{(x,y)\in\Omega} |f(x,y)| \;<\; \infty \quad (\,f \text{ is } \textit{bounded}\,)\,,$$

$$L \;=\; \sup_{(x,y),(x,z)\in\Omega, y\neq z} \Big|\frac{|f(x,y)-f(x,z)|}{|y-z|}\Big| \;<\; \infty \quad (\,f \text{ is } \textit{Lipschitz} \text{ in } y)\ .$$

Both these conditions hold if f is *continuously differentiable* on the closed cylinder $\bar{\Omega}$.

Theorem. (Cauchy-Picard) *If* $c = \min\{a, \frac{r}{M}, \frac{1}{2L}\}$, *then there exists a unique solution* $u(x)$ *to* (*) *for* $|x - x_0| \le c$.

Lemma. u *is a solution of* (*) *if and only if*

$$u(x) \;=\; y_0 \;+\; \int_{x_0}^{x} f\big(t,\, u(t)\big)\, dt \;\; \text{for } |x - x_0| \le c\ . \tag{$**$}$$

Exercise. Prove this lemma using the fundamental theorems of calculus.

Proof of Theorem. For $u \in \mathcal{C}\big(\Omega, \overline{\mathbf{B}_r(y_0)}\big)$, let

$$A(u)(x) \;=\; y_0 \;+\; \int_{x_0}^{x} f\big(t,\, u(t)\big)\, dt\ ,$$

and note that $A(u)$ is continuous with

$$\|A(u) \; - \; y_0\|_{sup} \; \leq \; M|x - x_0| \; \leq \; Mc \; \leq \; r \; .$$

Thus $A(u) \in \mathcal{C}\big(\Omega, \overline{\mathbf{B}_r(y_0)}\big)$ and A is a contraction because

$$\|A(u) - A(v)\|_{sup} \; \leq \; \int_{x_0}^{x} |f\big(t,\, u(t)\big) - f\big(t,\, v(t)| \, dt \; \leq \; |x - x_0| L \|u - v\| \; \leq \; \frac{1}{2}\|u - v\| \; .$$

Thus A has a fixed point u which is, by the Lemma, the desired unique solution of (*). ■

Remark 1. Existence is still true for f being just continuous (we won't prove this here), but *uniqueness may fail without the Lipschitz condition.* For example, the equation

$$\frac{du}{dx} = |u|^{\frac{1}{2}}$$

has two solutions satisfying $u(0) = 0$; namely, 0 and $\frac{1}{2}|x|x$.

Exercise. Find 2 more distinct solutions.

Corollary. *Suppose* $f(t_1, \ldots, t_{m+1})$ *is bounded and Lipschitz near the point* $(x_0, y_0, y_1, \ldots, y_{m-1})$ *. Then the initial-value problem*

$$\begin{aligned}
\frac{d^m u}{dx^m} \; &= \; f\big(x, u, \frac{du}{dx}, \ldots, \frac{d^{m-1}u}{dx^{m-1}}\big) \; , \\
u(x_0) \; &= \; y_0 \; , \\
\frac{du}{dx}(x_0) \; &= \; y_1 \; , \\
\\
\frac{d^{m-1}u}{dx^{m-1}}(x_0) \; &= \; y_{m-1}
\end{aligned}$$

has a unique solution near x_0 *.*

Proof : Letting

$$\vec{u} = (u_1, \ldots, u_m) \;\; \vec{f} = (f_1, \ldots, f_m) \;\; \vec{y_0} = (y_0, \ldots, y_{m-1})$$

where

$$u_1 = u, \;\; u_2 = \frac{du}{dx}, \;\; \ldots \;\; , \; u_m = \frac{d^{m-1}u}{dx^{m-1}}$$

$$f_1(t_1, \ldots, t_{m+1}) = t_2, \;\; \ldots \;\; , \; f_{m-1}(t_1, \ldots, t_{m+1}) = t_m \; ,$$

$$f_m(t_1, \ldots, t_{m+1}) = f(t_1, \ldots, t_{m+1}) \; ,$$

our mth order problem reduces to the first order vector problem

$$\frac{d\vec{u}}{dx} = \vec{f}\big(x, \, \vec{u}(x)\big), \quad \vec{u}(x_0) = \vec{y_0} \; .$$

■