Math 401 - Exam 1 (2 continuous hours, open textbook, notes)

Please show all your work, and turn in your completed exam at the beginning of class Friday, Oct.26, 3PM or before, to my office or the Mathematics Dept. office. There are 7 problems (of unequal values) so don't spend too much time on any one problem.

**1.** Suppose $\beta(t) = (1 + 3\cos t, 2 + 5\sin t, 3 + 4\cos t)$ for $t \in \mathbf{R}$.

(a) Compute the following functions of $t$: Velocity, speed, acceleration, tangent, normal, binormal, curvature, torsion.

velocity: $\beta'(t) = (-3\sin t, 5\cos t, -4\sin t)$

speed: $\frac{ds}{dt} = |\beta'(t)| = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t} = \sqrt{25} = 5$

acceleration: $\beta''(t) = (-3\cos t, -5\sin t, -4\cos t)$

tangent: $T(t) = \frac{\beta'(t)}{|\beta'(t)|} = \frac{1}{5}(-3\sin t, 5\cos t, -4\sin t)$

With $s(t) = 5t$, $t(s) = \frac{s}{5}$,

$$\kappa(s)N(s) = \frac{d}{ds}T\big(t(s)\big) = \frac{1}{25}(-3\cos\frac{s}{5}, -5\sin\frac{s}{5}, -4\cos\frac{s}{5})$$

so that the normal: $N(s) = \frac{1}{5}(-3\cos 5s, -5\sin 5s, -4\cos 5s)$

curvature: $\kappa(s) = \frac{1}{5}$

binormal: $B = T \wedge N = (\frac{1}{25}(-20, 0, 15) = \frac{1}{5}(-4, 0, 3)$

torsion: $\tau(s) = 0$ because $\tau(s)N(s) = \frac{d}{ds}B(s) = 0$

(b) Find an arc-length reparameterization of $\beta$.

$\beta(s) = \big(1 + 3\cos(\frac{s}{5}), 2 + 5\sin(\frac{s}{5}), 3 + 4\cos(\frac{s}{5})\big)$.

(c) Describe the geometric shape of the image of $\beta$.

This is a planar circle because the torsion is identically zero and the curvature is constant.

**2.** Suppose $\alpha(s)$ is a regular curve parameterized by arc-length. Find a formula for $\alpha'''(s)$ in terms of $\kappa$ (curvature), $\kappa'$, $\tau$ (torsion), $N$ (principal normal), $T$ (tangent), and $B$ (binormal).

The acceleration is $\alpha''(s) = T'(s) = \kappa(s)N(s)$ so

$$\alpha'''(s) = \kappa'(s)N(s) + \kappa(s)N'(s) = \kappa'(s)N(s) + \kappa(s)\big(-\kappa(s)T(s) - \tau(s)B(s)\big)$$

by the Frenet formulas.

**3.** (a) Find, if possible, a specific curve in $\mathbf{R}^3$ with curvature $\equiv 1$ everywhere and torsion $\equiv 0$ everywhere.

Any planar circle of radius 1. e.g. $\alpha(t) = (\cos t, \sin t, 0)$ for $t \in \mathbf{R}$.

(b) Find, if possible, a specific curve in $\mathbf{R}^3$ with torsion $\equiv 1$ everywhere and curvature $\equiv 0$ everywhere. There is no such curve because the curvature 0 condition implies that the curve is a straight line which automatically has torsion 0.

(c) Find, if possible, a specific curve in $\mathbf{R}^3$ with constant curvature and constant nonzero torsion. A helix has constant curvature and nonzero constant torsion, e.g. $\beta(t) = (\cos t, \sin t, t)$ for $t \in \mathbf{R}$.

(d) Does there exist a curve in $\mathbf{R}^3$ with curvature $\equiv 1$ everywhere and torsion not being constant? Yes, the Fundamental Theorem of Curves allows one to prescribe a smooth nonvanishing curvature function and a torsion functions arbitrarily.

**4.** Suppose $S$ is a smooth surface in $\mathbf{R}^3$, $f : \mathbf{R}^3 \to \mathbf{R}$ is smooth function, and $g : S \to \mathbf{R}$ is the restriction of $f$ to $S$, that is, $g(a) = f(a)$ for $a \in S$. Suppose that the maximum value of $g$ occurs at $p \in S$. Show that:

(a) $(\operatorname{grad} f)(p)$ is parallel to the normal line to $S$ at $p$. For any vector $v \in T_pS$ there is a smooth curve $\alpha$ in $S$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Since the function $f \circ \alpha(t)$ has a maximum at $t = 0$,

$$0 = (f \circ \alpha)'(0) =< (\operatorname{grad} f)(p), \alpha'(0) >=< (\operatorname{grad} f)(p), v > \ .$$

Being perpendicular to any tangent vector at p, $(\operatorname{grad} f)(p)$ must be parallel to the normal line at $p$.

(b) For any smooth curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$, the Hessian satisfies

$$(d^2 g)_p(\alpha'(0)) \ \leq \ - < (\operatorname{grad} f)(p), \alpha''(0) > \ .$$

Again since $f \circ \alpha(t)$ has a maximum at $t = 0$, so the second derivative test implies that

$$\begin{aligned} 0 \ \geq \ \frac{d^2}{dt^2}|_{t=0}\big(g \circ \alpha(t)\big) \ &= \ \frac{d}{dt}|_{t=0}\big[\big( < (\operatorname{grad} f) \circ \alpha(t)\big), \alpha'(t) > \big] \\ &= \ \sum_{i,j=1}^{3} \frac{\partial^2 g}{\partial x_i \partial x_j}(p)\alpha_i'(0)\alpha_j'(0) + \ < (\operatorname{grad} f)(p), \alpha''(0) > \end{aligned}$$

(c)

$$\max_{v \in T_pS, |v|=1} (d^2 g)_p(v) \ \leq \ |\operatorname{grad} f(p)| \max\{-|\kappa_1(p)|, -|\kappa_2(p)|\} \ .$$

Here, for $v \in T_pS, |v| = 1$, we can obtain $\alpha(t)$ with $\alpha(0) = p$ and $\alpha'(0) = v$ by choosing an arc-length parameterization of a normal section through $p$. Since $\alpha''(0)$ is in the normal directon which is parallel to $(\operatorname{grad} f)(p)$ and $|\alpha'(0)|$ is between $|\kappa_1(p)|$ and $|\kappa_2(p)|$, part (c) follows.

**5.** Suppose $X(u,v) = \big((3+\sin v)\sin u, (3+\sin v)\cos u, \cos v\big)$ for $-\pi \leq u < \pi$, $-\pi \leq v < 2\pi$, and $S$ is the image of $X$.

(a) Find a formula for the outward pointing unit normal $N(u,v)$ of $S$ at $X(u,v)$.

$X_u(u,v) = \big((3+\sin v)\cos u, -(3+\sin v)\sin u, 0\big)$

$X_v(u,v) = \big((\cos v)\sin u, (\cos v)\cos u, -\sin v\big)$

$X_u(u,v) \wedge X_v(u,v) = \big(-(3+\sin v)\sin u \sin v, -(3+\sin v)\cos u \sin v, (3+\sin v)\cos v\big)$

$N(u,v) = \big(\sin u \sin v, \cos u \sin v, \cos v\big)$

Checking at $X_u(u,v)(\frac{\pi}{2},\frac{\pi}{2}) = (4,0,0)$ that $N(\frac{\pi}{2},\frac{\pi}{2}) = (1,0,0)$, we see that $N$ is outward pointing.

(b) Find the area of $S$. (Hint: To find the formula for the area of such a parameterized surface $S$, recall from vector analysis that the area of the parallelogram $P = \{sv + tw : s,t \in [0,1]\}$ spanned by vectors $v,w$ is $|v \wedge w|$, and consider the approximate area of $X([u, u+\Delta u] \times [v, v_\Delta v])$ for $\Delta u, \Delta v$ small.)

The area of $S$ is

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |X_u(u,v) \wedge X_v(u,v)| du dv \ = \ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (3+\sin v)\, du\, dv = 12\pi^2 \ .$$

(c) What is the range of values of the curvatures of the *normal sections* of $S$ at the point $(4,0,0)$. (These are obtained by intersecting $S$ with planes containing the normal line through $(4,0,0)$.)

Similarly, what are they at the point $(2,0,0)$? (Hint: look at the picture.)

At both points one uses the intersections with the $X-Y$ and the $Y-Z$ planes.

At $(4,0,0)$ the resulting circles have radii 4 and 1 and the normal curvatures with respect to the inward (resp. outward) normal vary between $\frac{1}{4}$ and 1 (resp. $-\frac{1}{4}$ and $-1$). At $(2,0,0)$ the resulting circles have radii 2 and 1 and the normal curvatures with respect to the inward (resp. outward) normal vary vary between $-\frac{1}{2}$ and 1 (resp. $\frac{1}{2}$ and $-1$).

(d) What are all the possible values for $|k_\alpha(p)|$ where $p \in S$ and $k_\alpha$ is the curvature at $p$ of *any* smooth curve $\alpha$ in $S$ passing through $p$.? The smallest is $\frac{1}{4}$, and there is no upper bound on the size of curvature of a curve in $S$.q

**6.** Suppose that $S$ is a compact connected surface with $\kappa_1\kappa_2$ positive everywhere and with expression $\kappa_1^2\kappa_2 + \kappa_2^2\kappa_1$ being a constant function on $S$. Show that $S$ s a sphere.

Choose, by the compactness of $S$ a point $p \in S$ where $\kappa_1$ has a minimum. Then $p$ is automatically a maximum point for $\kappa_2$. In fact an inequality $\kappa_2(q) > \kappa_2(p)$ would imply that

$$(\kappa_1^2\kappa_2 + \kappa_2^2\kappa_1)(q) \ > \ (\kappa_1^2\kappa_2 + \kappa_2^2\kappa_1)(p) \ ,$$

contradicting the constancy assumption. So now we may apply the Hilbert Theorem to find that $p$ is an umbillic point of $S$. Thus for all $q \in S$,

$$\kappa_2(q) \ \le \ \kappa_2(p) \ = \ \kappa_1(p)\ \kappa_1(q) \ \le \ \kappa_1(q) \ \le \ \kappa_2(q) \ ,$$

and $S$ is totally umbillic. By compactness it can't be a plane, so it must be a sphere.

**7.** Suppose $g$ is a smooth real-valued function on the unit interval $[0,1]$ and $G = \{(x,y) \in \mathbf{R}^2 \ : 0 \le x \le 1, \ y = g(x)\}$.

(a) Find a formula for the total length of $G$. $L \ = \ \int_0^1 \sqrt{1 + |g'(x)|^2}\, dx$.

(b) Find a formula for the curvature $\kappa$ of $G$ at each point $\big(x, g(x)\big)$ with $0 < x < 1$. We use the formula

$$\kappa \ = \ \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

with the parameter $t = x$, $y = g(x)$ so that

$$\kappa(x) \ = \ \frac{1 \cdot g''(x) - g'(x) \cdot 0}{(1 + |g'(x)|^2)^{3/2}} \ = \ \frac{g''(x)}{(1 + |g'(x)|^2)^{3/2}} \ .$$

(c) For a fixed small positive number $\varepsilon$, find a formula for the total length of the set

$$G_\varepsilon = \{p \in \mathbf{R}^2 \ : \ \mathrm{dist}\,\big(p, G\big) = \varepsilon\} \ .$$

(Recall that $\mathrm{dist}\,(p, A) = \min\{|p - a| \ : \ a \in A\}$ for any closed set $A$. You may parameterize $G_\varepsilon$ by using, for each $p \in G_\varepsilon$ the *nearest* point in $G$.) The set $G_\varepsilon$ is a long loop consisting of the two curves $x \mapsto (x, g(x)) \pm \varepsilon N(x)$ and two semi-circles of radius $\varepsilon$ connecting the ends of the two curves. Since $N' = -\kappa(1, g')$, we see that these two curves have velocity vectors $(1, g') \mp \varepsilon\kappa(1, g') \ = \ (1 \mp \varepsilon\kappa)(1, g')$. For $\varepsilon$ small, $|\varepsilon\kappa| < 1$ and $|1 + \varepsilon\kappa| + |1 - \varepsilon\kappa| = 2$ . So the total length of $G_\varepsilon$ is

$$2\pi\varepsilon \ + \ \int_0^1 \big(|1 + \varepsilon\kappa(x)| + |1 - \varepsilon\kappa(x)|\big)\sqrt{1 + |g'(x)|^2}\, dx \ = \ 2\pi\varepsilon + 2L \ .$$