Solution of P.305(1)

The second and third equations show that the orthogonal projections of $V \circ X$ onto JX_u and JX_v are smooth functions. Since V is to be tangent to S, we see that $V \circ X$ and hence V are smooth functions. For the formula for div$V$, we may, as suggested, differentiate the second and third equations with respect to v and u respectively and subtract. The product gives 3 terms on each side of both equations. The 4 terms involving either $(N \circ X)_u$, $(N \circ X)_v$, $(N' \circ \phi \circ X)_u$, or $(N' \circ \circ X)_v$ all vanish because these latter vectors are tangent, and the determinant of a matrix with 3 tangent vectors must be zero. Also we have the cancelations

$$\det(V \circ X, N \circ X, X_{uv}) \; - \; \det(V \circ X, N \circ X, X_{vu}) \; = \; 0 \; ,$$

$$\det(W \circ \phi \circ X, N' \circ \phi \circ X, (\phi \circ X)_{uv}) \; - \; \det(W \circ \phi \circ X, N' \circ \phi \circ X, (\phi \circ X)_{uv}) \; = \; 0 \; .$$

To prove our formula for (div$V$)(p), we find it convenient to assume that $\{X_u(0), X_v(0)\}$ ae orthonormal where $X(0) = p$. Then we find that

$$\begin{aligned}
&\det((V \circ X)_v, N \circ X, X_u) - \det((V \circ X)_u, N \circ X, X_v) \\
&= \; - < (V \circ X)_v \wedge X_u, N \circ X > \; + \; < (V \circ X)_u \wedge X_v, N \circ X > \\
&= < (V \circ X)_v, X_v > \; + \; < (V \circ X)_u, X_u > \\
&= < (dV_p(X_v), X_v > \; + \; < (dV_p(X_u), X_u > = \; (\operatorname{div} V)(p) \; .
\end{aligned}$$

We treat the two remaining terms on the righthand side similarly. First we verify the following lemma:

For any linear maps of K, L of $\mathbf{R}^2 \times \{0\}$ to itself,

$$K(L(e_1)) \wedge L(e_2) \; - \; K(L(e_2)) \wedge L(e_1) \; = \; (\operatorname{trace} K)(\operatorname{Jac} L) e_3 \; .$$

where e_1, e_2 are the standard basis vectors of $\mathbf{R}^2$. We verify this formula by representing K and L as general 2×2 matrices and computing.We now essentailly apply this formula with $K = dW_{\phi(p)}, L = d\phi_p$ and e_1, e_2 replaced by $X_u(0), X_v(0)$. Thus

$$\begin{aligned}
&\det((W \circ \phi \circ X)_v, N' \circ \phi \circ X, (\phi \circ X)_u) - \det((W \circ \phi \circ X)_u, N' \circ \phi \circ X, (\phi \circ X)_v) \\
&=< -dW_{\phi(p)} df_p(X_v) \wedge d\phi_p(X_u) \; + \; dW_{\phi(p)} df_p(X_u) \wedge d\phi_p(X_v), \; N' \circ \phi \circ X > \\
&= \; [(\operatorname{div} W) \circ \phi](\operatorname{Jac} \phi) \; .
\end{aligned}$$