The second and third equations show that the orthogonal projections of \( V \circ X \) onto \( JX_u \) and \( JX_v \) are smooth functions. Since \( V \) is to be tangent to \( S \), we see that \( V \circ X \) and hence \( V \) are smooth functions. For the formula for \( \text{div} V \), we may, as suggested, differentiate the second and third equations with respect to \( v \) and \( u \) respectively and subtract. The product gives 3 terms on each side of both equations. The 4 terms involving either \((N \circ X)_u\), \((N' \circ \phi \circ X)_u\), or \((N' \circ X)_v\) all vanish because these latter vectors are tangent, and the determinant of a matrix with 3 tangent vectors must be zero. Also we have the cancelations

\[
\det (V \circ X, N \circ X, X_{uv}) - \det (V \circ X, N \circ X, X_{vu}) = 0,
\]
\[
\det (W \circ \phi \circ X, N' \circ \phi \circ X, (\phi \circ X)_{uv}) - \det (W \circ \phi \circ X, N' \circ \phi \circ X, (\phi \circ X)_{uv}) = 0.
\]

To prove our formula for \((\text{div} V)(p)\), we find it convenient to assume that \( \{X_u(0), X_v(0)\} \) are orthonormal where \( X(0) = p \). Then we find that

\[
\det ((V \circ X)_u, N \circ X, X_u) - \det ((V \circ X)_u, N \circ X, X_v)
\]
\[
= - <(V \circ X)_v \wedge X_u, N \circ X> + <(V \circ X)_u \wedge X_v, N \circ X>
\]
\[
= <(V \circ X)_v, X_u> + <(V \circ X)_u, X_v>
\]
\[
= <(dV_p(X_v)), X_u> + <(dV_p(X_u)), X_v> = (\text{div} V)(p).
\]

We treat the two remaining terms on the righthand side similarly. First we verify the following lemma:

For any linear maps of \( K, L \) of \( \mathbb{R}^2 \times \{0\} \) to itself,

\[
K(L(e_1)) \wedge L(e_2) - K(L(e_2)) \wedge L(e_1) = \text{(trace} K)(\text{Jac} L)e_3.
\]

where \( e_1, e_2 \) are the standard basis vectors of \( \mathbb{R}^2 \). We verify this formula by representing \( K \) and \( L \) as general \( 2 \times 2 \) matrices and computing. We now essentially apply this formula with \( K = dW_\phi(p), L = d\phi_p \) and \( e_1, e_2 \) replaced by \( X_u(0), X_v(0) \). Thus

\[
\det ((W \circ \phi \circ X)_v, N' \circ \phi \circ X, (\phi \circ X)_u) - \det ((W \circ \phi \circ X)_u, N' \circ \phi \circ X, (\phi \circ X)_v)
\]
\[
= < -dW_{\phi(p)}(df_p(X_v)) \wedge d\phi_p(X_u) + dW_{\phi(p)}(df_p(X_u)) \wedge d\phi_p(X_v), N' \circ \phi \circ X>
\]
\[
= [(\text{div} W) \circ \phi](\text{Jac} \phi).
\]