Remarks about Degree and the Gauss-Bonnet,Poincaré-Hopf Formula

In class we proved two theorems not included in the text.

Theorem 1. *For any 2 smoothly homotopic maps f, g between compact surfaces S and S' and any regular value $y \in S'$ of both f and g,* $\deg(f, y) = \deg(g, y)$.
Theorem 2. *For any smooth map f between compact surfaces S and S' and any two regular values $y, z \in S'$ of f,* $\deg(f, y) = \deg(f, z)$.

As in Proposition 8.7 (p.281) the set of regular values of a given map f is open and dense in S' and the integer $\deg(f, y)$ does not change as one moves in a connected component of these values. So, by slightly changing y, we may assume that y is also a regular value for the map $h : [0, 1] \times S \to S'$ defining the homotopy from f to g. This implied that $h^{-1}\{y\}$ is finitely many oriented curves with induced orientations. Taking the boundaries of these oriented curves gave a connection between the finite sets $f^{-1}\{y\}$ and $g^{-1}\{y\}$ as well as $\pm$ signs which correspond to the signs of the Jacobians of g and minus the signs of the Jacobians of f. Since each nonclosed oriented curve contributes a total of $+1 - 1 = 0$, one then obtains the formula $\deg(g, y) - \deg(f, y) = 0$.

In the proof of Theorem 2, one modifies the map f by distorting the values that lie in a small tube about a path from y to z in S'. The resulting distorted map g is homotopic to f, has y as a regular value, has $g^{-1}\{y\} = f^{-1}\{z\}$, and $\deg(g, y) = \deg(f, z)$.
Corollary. $\deg f = (\text{Area}(S'))^{-1} \int_{S'} \deg(f, y)\, dy$ equals the integer $\deg(f, z)$ for any regular value z of f.

The **Gauss-Bonnet,Poincaré-Hopf Formula** is

$$\chi(S) \equiv (2\pi)^{-1} \int_S K(y)\, dy = \sum_{a \in V^{-1}\{0\}} i(V, a)$$

for any smooth tangent vectorfield V with at most finitely many zeros on a compact surface S. We followed the proof of this in §8.5 (p.295).

A Corollary is that $\chi(S)$ does not change under a diffeomorphism. The (not proved in class) **classification of surfaces** shows that any smooth compact surface is diffeomorphic to a vertically placed multiple-holed surface with g holes for some $g \in \{0, 1, 2, \cdots\}$. Choosing in particular $V(p)$ to be the gradient of the height function at p (i.e. $V(p)$ is, the projection of $(0, 0, 1)$ onto the tangent space T_pS, we obtain the $2g + 2$ indices $+1, -1, -1, \cdots, -1, +1$. Thus $\chi(S)$ is the even integer $2g - 2$.