Graph Calculations

Suppose $S = \{(x,y,f(x,y)) : (x,y) \in U\}$ for some smooth function $f(x,y)$ on an open subset of $\mathbf{R}^2$. Suppose also that $p = (0,0,0) \in S$ and that both the partial derivatives $f_x(0,0)$ and $f_y(0,0))$ vanish. Then T_pS equals the XY plane. For any vector $v = (v_1, v_2, 0) \in T_pS$, we can see that the curve $\alpha(t) = (tv_1, tv_2, f(tv_1, tv_2))$ has $\alpha(0) = p$ and $\alpha'(0) = v$. Let N denote the upward unit normal to S so that $N(p) = (0,0,1)$. Since $< N \circ \alpha, \alpha' > \equiv 0$, $0 =< N \circ \alpha, \alpha' >' =< (N \circ \alpha)', \alpha' > + < N \circ \alpha, \alpha'' >$, and we may compute the second fundamental form

$$\begin{aligned}\sigma_p(v,v) &= - < dN_p(v), v > = \; - < (N \circ \alpha)'(0), \alpha'(0) > = < N(p), \alpha''(0) > \\ &= < (0,0,1), (v_1, v_2, f_x(tv_1, tv_2)v_1 + f_y(tv_1, tv_2)v_2)'(0) > \\ &= \; f_{xx}(0,0)v_1^2 \; + \; 2f_{xy}(0,0)v_1v_2 \; + \; f_{yy}(0,0)v_2^2\end{aligned} .$$

If $h(x,y,z) = z$ for $(x,y,z) \in S$, then p is a critical point for h. Since $h \circ \alpha(t) = f(v_1t, v_2t)$, we see that the *Hessian* has the same value

$$d^2h_p(v) \; = \; \frac{d^2}{dt^2}|_{t=0}(h \circ \alpha)(t) \; = \; f_{xx}(0,0)v_1^2 \; + \; 2f_{xy}(0,0)v_1v_2 \; + \; f_{yy}(0,0)v_2^2 \; .$$

Here are some other formulas valid throughout U (not just at $p = (0,0,0)$) obtained by using the graph parameterization $X(u,v) = (u,v,f(u,v))$:

$$X_u = (1,0,f_u), \; X_v = (0,1,f_v), \; N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}} \; .$$

$$X_{uu} = (0,0,f_{uu}), \; X_{uv} = (0,0,f_{uv}), \; X_{vv} = (0,0,f_{vv}),$$

$$E =< X_u, X_u >= 1 + f_u^2, \; F =< X_u, X_v >= f_uf_v, \; G =< X_v, X_v >= 1 + f_v^2 \; .$$

$$e = \sigma(X_u, X_u) =< N, X_{uu} >= \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}},$$

$$\text{(the next coefficient)} \; \tilde{f} = \sigma(X_u, X_v) =< N, X_{uv} >= \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}},$$

$$g = \sigma(X_v, X_v) =< N, X_{vv} >= \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}},$$

$$\begin{aligned}A \; &= \; -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & \tilde{f} \\ \tilde{f} & g \end{pmatrix} \; = \; (EG - F^2)^{-1} \begin{pmatrix} -G & F \\ F & -E \end{pmatrix} \begin{pmatrix} e & \tilde{f} \\ \tilde{f} & g \end{pmatrix} \\ &= \; (1 + f_u^2 + f_v^2)^{-3/2} \begin{pmatrix} -1 - f_v^2 & f_uf_v \\ f_uf_v & -1 - f_u^2 \end{pmatrix}^{-1} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix}\end{aligned}$$