Some Facts about L^p

Suppose $1 \leq p < \infty$ and $\|f\|_{L^p} = \left(\int |f|^p\,dx\right)^{1/p}$ for any measurable f on R^n. Then $L^p = \{f : \|f\|_{L^p} < \infty\}$ is a linear space which is topologized by $\operatorname{dist}(f,g) = \|f-g\|_{L^p}$ into a complete separable metric space. Under ordinary smoothing $f^\epsilon = \rho_\epsilon \cdot f$,

$$\|f^\epsilon - f\|_{L^p} \to 0 \quad \text{as} \quad \epsilon \to 0\ .$$

By multiplying by the characteristic function of a large ball before smoothing we see that the linear subspace $\mathcal{C}_0^\infty$ is dense in L^p.

Suppose that $\Upsilon : L^p \to \mathbf{R}$ is linear, that $\|\Upsilon\| \equiv \sup\{\Upsilon(f) : \|g\|_{L^q} \leq 1\}$, and that $q \in (1,\infty]$ is defined by $\frac{1}{q} = 1 - \frac{1}{p}$.

Riesz Representation Theorem.(see e.g. Royden §6.5.13) $\|\Upsilon\| < \infty \Leftrightarrow$ *there exists* $g \in L^q$ *with* $\|g\|_{L^q} = \|\Upsilon\|$ *so that* $\Upsilon(f) = \int fg$ *for all* $f \in L^p$.

For $f, f_i \in L^1(\mathbf{R}^n)$, one says that $f_i \rightharpoonup f$ *weakly* if $\int f_i g \to \int fg$ for all $g \in \mathcal{C}_0^\infty$. If, in addition, $\sup_i \|f_i\|_{L^p} < \infty$, then one says that $f_i \rightharpoonup f$ *weakly in* L^p. In this case one then has the relations

$$\|f\|_{L^p} \;\leq\; \liminf_{i\to\infty} \|f_i\|_{L^p}$$

and $\int f_i g \to \int fg$ for all $g \in L^q$.

Distribution Derivatives and H^1

For any $f \in L^1_{loc}(\mathbf{R}^n)$, the *distribution* (or *weak*) *derivative* is the linear function $D_f : \mathcal{C}_0^\infty(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}$, given by the formula

$$D_f[V] \;=\; -\int f \operatorname{div} V\,dx \quad \text{for} \quad V \in \mathcal{C}_0^\infty(\mathbf{R}^n, \mathbf{R}^n)\ .$$

In the classical case $f \in \mathcal{C}^1$, integration by parts shows that $D_f[V] = \int (Df)\cdot V\,dx$, which explains the definition and notation.

Exercise 1. In case $f = \chi_{\mathbf{B}_1}$, D_f is *not* represented by integration on $\mathbf{R}^n$ but rather by integration over the sphere $\partial\mathbf{B}_1$:

$$D_f[V] = -\int_{\partial\mathbf{B}_1} V(x)\cdot x\,d\omega_x \quad \big(\text{for } n=1,\ D_f[V] = V(-1) - V(+1)\big)\ .$$

Definition. *f belongs to the Sobolev space $H^1 = H^1(\mathbf{R}^n)$ (or alternately $D_f \in L^2$) if $D_f[V] = \int W\cdot V\,dx$ for some $W \in L^2(\mathbf{R}^n, \mathbf{R}^n)$.*

Exercise 2. $W = Df$ (the usual pointwise-defined gradient) a.e. Hint : To see that $\mathbf{e_i} \cdot W(a) = \frac{\partial}{\partial x_i} f(a)$ at each Lebesgue point a of W, one chooses, for a small positive h, a suitable sequence of smooth vectorfields V_j to approximate $h^{-n}\chi_{C_h}\mathbf{e}_i$ where C_h is the small cube $\Pi_{i=1}^n [a_i, a_i + h]$.

Theorem 1. *The following are equivalent for any $f \in L^1$:*
(I) $f \in H^1$
(II) $M \equiv \sup\{D_f[V] \ :\ \|V\|_{L^2(\mathbf{R}^n,\mathbf{R}^n)} \le 1\} \ < \ \infty$.
(III) $Df \in L^2$ and there exists a sequence $f_i \in \mathcal{C}^1$ such that $f_i \rightharpoonup f$ and $Df_i \rightharpoonup Df$.

Proof : (I) $\Rightarrow$ (II): Here

$$M \ = \ \sup\{\int W \cdot V \, dx \ :\ \|V\|_{L^2(\mathbf{R}^n,\mathbf{R}^n)} \le 1\} \ = \ \|W\|_{L^2} \ < \ \infty \ .$$

(II) $\Rightarrow$ (I): Apply the Riesz Theorem to represent $\Upsilon_i(v) \equiv \int f \frac{\partial v}{\partial x_i}$ by an L^2 function W_i. Then

$$\int f \operatorname{div} V \ = \ \sum_{i=1}^n \Upsilon_i(V_i) \ = \ \sum_{i=1}^n \int V_i W_i \ = \ \int V \cdot W \ .$$

(I) $\Rightarrow$ (III): By Exercise 2, $Df = W \in L^2$. Ordinary smoothing then gives smooth functions f_{ϵ_i} converging strongly in L^1 , hence weakly, to f and smooth vectorfields $(Df)_{\epsilon_i}$ converging strongly in L^2, hence weakly, to Df. Thus, for (III) it is suffices to verify that $Df_\epsilon = (Df)_\epsilon$. We assume the mollifier is symmetric, i.e. $\rho_\epsilon(-x) = \rho_\epsilon(x)$, take any vectorfield $V = (V_1, \ldots, V_n) \in \mathcal{C}_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$, and use Fubini's Theorem and (I) to see that

$$\begin{aligned}
\int Df_\epsilon \cdot V \ &= \ -\int\int \rho_\epsilon(x-y)f(y)dy \operatorname{div} V(x)dx \\
&= \ \int\int \sum_{i=1}^n \frac{\partial}{\partial x_i}\rho_\epsilon(x-y)f(y)V_i(x)dxdy \\
&= \ -\int\int \sum_{i=1}^n \frac{\partial}{\partial y_i}\rho_\epsilon(x-y)f(y)V_i(x)dydx \\
&= \ \int\int \rho_\epsilon(x-y)\sum_{i=1}^n \frac{\partial}{\partial y_i}f(y)V_i(x)dydx \\
&= \ \int\int \rho_\epsilon(x-y)Df(y)\cdot V(x)dydx \ = \ \int (Df)_\epsilon \cdot V \ .
\end{aligned}$$

Thus $Df_\epsilon = (Df)_\epsilon$ almost everywhere, and by continuity, everywhere.

(III) $\Rightarrow$ (I) : For $V \in \mathcal{C}_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$, integration by parts implies that $-\int f_i \operatorname{div} V\, dx = \int (Df_i) \cdot V\, dx$. Noting, by Hölder's inequality, that

$$|\int (f - f_i) \operatorname{div} V| + |\int (Df - Df_i) \cdot V| \le \|f - f_i|_{L^2} \|\operatorname{div} V\|_{L^2} + \|Df - Df_i|_{L^2} \|V\|_{L^2}$$

approaches 0 as $i \to \infty$, we conclude that $D_f[V] = -\int f \operatorname{div} V\, dx = \int Df \cdot V\, dx$. ∎

Theorem 2. *If $f \in L^1 \cap H^1$ and $n \in \{3, 4, \ldots\}$, then $f \in L^p$ for all $p \in [1, \frac{2n}{n-2}]$.*
Proof : By the Sobolev inequality and the previous proof,

$$\begin{aligned} \|f\|_{L^{2n/(n-2)}} &\le \liminf_{\epsilon \to 0} \|f_\epsilon\|_{L^{2n/(n-2)}} \le \limsup_{\epsilon \to 0} c\|D(f_\epsilon)\|_{L^2} \\ &= \limsup_{\epsilon \to 0} c\|(Df)_\epsilon)\|_{L^2} \le c\|Df\|_{L^2} < \infty . \end{aligned}$$

Using Hölder's inequality with $\frac{1}{r} \equiv \frac{1}{p} - \frac{n-2}{2n}$ and the Chebychev inequality, we conclude

$$\begin{aligned} \int |f|^p\, dx &\le \int_{\{|f| \le 1\}} |f|^p\, dx + \int_{\{|f| > 1\}} |f|^p\, dx \\ &\le \int_{\{|f| \le 1\}} |f|\, dx + |\{|f| > 1\}|^{p/r} \|f\|^p_{L^{2n/(n-2)}} \\ &\le \|f\|_{L^1} + \|f\|^{p/r}_{L^1} \|f\|^p_{L^{2n/(n-2)}} < \infty . \end{aligned}$$

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Exercise 3. For $n = 1$, $f \in L^p$ for all $p \in [1, \infty]$. For $n = 2$, $f \in L^p$ for $p \in [1, \infty)$.

The *Sobolev inequality*, for an arbitrary H^1 function, now follows using ordinary smoothing from the $\mathcal{C}^1$ version of the Sobolev inequality. Similarly one defines, for an open $\Omega \subset \mathbf{R}^n$, the Sobolev space $H^1(\Omega)$ and proves by approximation the Poincaré inequality in case $\partial\Omega$ is smooth. One also has

Rellich's Theorem. *Any sequence $u_i \in H^1(\mathbf{B})$ with $\sup_i \big(\|u_i\|_{L^2} + \|Du_i\|_{L^2}\big) < \infty$ contains a subsequence $u_{i'}$ that converges strongly in L^2 to $u \in H^1(\mathbf{B})$. Moreover $Du_{i'}$ converges weakly in L^2 to Du and $\int_{\mathbf{B}} |Du|^2 \le \liminf_{i' \to \infty} \int_{\mathbf{B}} |Du|^2$.*

Next we discuss properties of a composition $\Phi \circ u$ where $u \in H^1$. In case Φ is Lipschitz, Φ is differentiable almost everywhere, but the exceptional set of non-differentiability may contain $u(A)$ for some set A with positive measure. Nevertheless, we still have:

Theorem 3. *If $u \in H^1$ and Φ is Lipschitz, then $\Phi \circ u \in H^1$ with*

$$\|D(\Phi \circ u)\|_{L^2} \ \leq \ \|\Phi'\|_{L^\infty} \|Du\|_{L^2} \ .$$

Proof : First note that

$$\int |\Phi \circ u - \Phi \circ u_\epsilon|^2 \ \leq \ \|\Phi'\|^2_{L^\infty} \int |u - u_\epsilon|^2 \ \to \ 0 \text{ as } \epsilon \to 0 \ .$$

In case $\Phi \in \mathcal{C}^\infty$, $V \in \mathcal{C}_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ and $\|V\|_{L^2} \leq 1$,

$$\begin{aligned} -\int (\Phi \circ u) \operatorname{div} V \ &= \ -\lim_{\epsilon \to 0} \int (\Phi \circ u_\epsilon) \operatorname{div} V \\ &= \ \lim_{\epsilon \to 0} \int D(\Phi \circ u_\epsilon) \cdot V \ = \ \lim_{\epsilon \to 0} \int \Phi'(u_\epsilon) Du_\epsilon \cdot V \\ &\leq \ \liminf_{\epsilon \to 0} \|\Phi'\|_{L^\infty} \|Du_\epsilon\|_{L^2} \leq \|\Phi'\|_{L^\infty} \|Du\|_{L^2} \ . \end{aligned}$$

Theorem 1 and its proof now imply that $\Phi \circ u \in H^1$ with $\|D(\Phi \circ u)\|_{L^2} \leq \|\Phi'\|_{L^\infty} \|Du\|_{L^2}$.

For a general Lipschitz Φ, $\Phi_\epsilon = \Phi * \rho_\epsilon \to \Phi$ uniformly as $\epsilon \to 0$. Thus, for V as above, the previous case and Hölder's inequality imply

$$\begin{aligned} -\int (\Phi \circ u) \operatorname{div} V \ &= \ \lim_{\epsilon \to 0} -\int (\Phi_\epsilon \circ u) \operatorname{div} V \\ &= \ \lim_{\epsilon \to 0} \int D(\Phi_\epsilon \circ u) \cdot V \\ &\leq \ \limsup_{\epsilon \to 0} \|\Phi'_\epsilon\|_{L^\infty} \|Du\|_{L^2} \ \leq \ \|\Phi'\|_{L^\infty} \|Du\|_{L^2} \ , \end{aligned}$$

and Theorem 3 follows as before from Theorem 1. ■

Corollary. *$\int D(\Phi_\epsilon \circ u) \cdot V \ \to \ \int D(\Phi \circ u) \cdot V$ as $\epsilon \to 0$ for all $V \in L^2(\mathbf{R}^n, \mathbf{R}^n)$.*
Proof : This follows from Theorem 3 in case $V \in \mathcal{C}_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$. For $V \in L^2(\mathbf{R}^n, \mathbf{R}^n)$ and $\delta > 0$, we first choose $\tilde{V} \in \mathcal{C}_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ with

$$\|V - \tilde{V}\|_{L^2} \ < \ \frac{\delta}{3\|\Phi'\|_{L^\infty} \|Du\|_{L^2}}$$

and then $\epsilon > 0$ so that $|\int D(\Phi_\epsilon \circ u) \cdot \tilde{V} \ - \int D(\Phi \circ u) \cdot \tilde{V}| < \frac{\delta}{3}$ to conclude, using Hölder's inequality, that

$$\begin{aligned} &|\int D(\Phi_\epsilon \circ u) \cdot V \ - \int D(\Phi \circ u) \cdot V| \\ &< \ |\int D(\Phi_\epsilon \circ u) \cdot (V - \tilde{V})| \ + \frac{\delta}{3} \ + \ |\int D(\Phi \circ u) \cdot (\tilde{V} - V)| \ < 3(\frac{\delta}{3}) = \delta \ . \end{aligned}$$

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