SIZE MINIMIZATION AND APPROXIMATING PROBLEMS

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ABSTRACT. We consider Plateau type variational problems related to the size minimization of rectifiable currents. We realize the limit of a size minimizing sequence as a stationary varifold and a minimal set. Other examples of functionals to be minimized include the integral over the underlying carrying set of a power q of the multiplicity function, with $0 < q \leq 1$. Because minimizing sequences may have unbounded mass we make use of a more general object called a rectifiable scan for describing the limit. This concept is motivated by the possibility of recovering a flat chain from a sufficiently large collection of its slices. In case the given boundary is smooth and compact, the limiting scan has finite mass and corresponds to a rectifiable current.

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1. INTRODUCTION AND PRELIMINARIES

The general m dimensional Plateau problem is roughly the following: given an $m-1$ dimensional *boundary* B, find an m dimensional *surface* S *spanning* B of least m dimensional *area*. While this was classically studied for $m = 2$ using mappings of surfaces ([17]), geometric measure theory now provides for general m several precise formulations and definitions of the italicized terms ([18, 7, 10, 12, 1, 3]). The most popular involves the rectifiable currents of Federer and Fleming which we quickly review. With $\mathcal{H}^m$ denoting m dimensional Hausdorff measure on $\mathbb{R}^n$, a set $M \subset \mathbb{R}^n$ is called $(\mathcal{H}^m, m)$ *rectifiable* if $\mathcal{H}^m(M) < \infty$ and $\mathcal{H}^m(M \sim \cup_{i \in I} N_i) = 0$ for some

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finite or countable family $\{N_i : i \in I\}$ of $\mathcal{C}^1$ submanifolds of $\mathbb{R}^n$. It follows that for almost every $x \in M$, homothetic expansions by factors $r \uparrow \infty$ of the translated measure $\mathcal{H}^m \llcorner (M - x)$ converge weakly to a unique tangent measure $\mathcal{H}^m \llcorner T_x M$ where $T_x M$ is an m dimensional vector space. An m dimensional *rectifiable current* in $\mathbb{R}^n$ is given by three things:

(1) a bounded Borel $(\mathcal{H}^m, m)$ rectifiable set $M \subset \mathbb{R}^n$;
(2) an $\mathcal{H}^m$ measurable m vectorfield $\xi : M \longrightarrow \wedge_m \mathbb{R}^n$ such that for $\mathcal{H}^m$ a.e. $x \in M$, $\xi(x) = e_1 \wedge ... \wedge e_m$ for some orthonormal basis $\{e_1, ..., e_m\}$ of $T_x M$;
(3) an $\mathcal{H}^m \llcorner M$ summable multiplicity function $\theta : M \longrightarrow \{1, 2, 3, \dots\}$.

This is abbreviated $\mathcal{H}^m \llcorner M \wedge \theta\xi$ and its action on a differential m form $\phi \in \mathcal{D}^m(\mathbb{R}^n)$ is given by

$$(\mathcal{H}^m \llcorner M \wedge \theta\xi)(\phi) = \int_M \langle \xi, \phi \rangle \theta d\mathcal{H}^m ,$$

making it a current (in the sense of De Rham). For $m \geq 1$ the boundary of a general m dimensional current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is the current $\partial T \in \mathcal{D}_{m-1}(\mathbb{R}^n)$ defined by $(\partial T)(\psi) = T(d\psi)$ for $\psi \in \mathcal{D}^{m-1}(\mathbb{R}^n)$, thus generalizing, by the Stokes-Cartan Theorem, the special case T corresponds to a compact oriented (multiplicity-one) manifold with boundary. The mass norm, which is defined by

$$\mathbf{M}(T) = \sup\{T(\phi) : \phi \in \mathcal{D}^m(\mathbb{R}^n) \text{ with } \langle e_1 \wedge ... \wedge e_m, \phi(x) \rangle \leq 1 \\ \text{for each } x \in \mathbb{R}^n \text{ and } e_1, ..., e_m \in \mathbb{S}^{n-1}\}$$

has, for a rectifiable current, the simple form

$$\mathbf{M}(\mathcal{H}^m \llcorner M \wedge \theta\xi) = \int_M \theta d\mathcal{H}^m .$$

Let $\mathcal{R}_m(\mathbb{R}^n)$ denote the group of all m dimensional rectifiable currents in $\mathbb{R}^n$. Federer and Fleming proved the fundamental Compactness Theorem in [10] : *For $m \in \{1, ..., n\}$ and $0 < c < \infty$,*

$$\{T \in \mathcal{R}_m(\mathbb{R}^n) : \partial T \in \mathcal{R}_{m-1}(\mathbb{R}^n) \text{ and } \mathbf{M}(T) + \mathbf{M}(\partial T) \leq c\}$$

is weakly sequentially compact. With the lower semicontinuity of $\mathbf{M}$, this ensures, for a given $T_0 \in \mathcal{R}_m(\mathbb{R}^n)$, the existence of a minimizer for the basic Plateau problem:

$$(\mathcal{P}_{\mathbf{M},T_0}) \begin{cases} \text{minimize } \mathbf{M}(T) \text{ among} \\ T \in \mathcal{R}_m(\mathbb{R}^n),\ \partial T = \partial T_0 . \end{cases}$$

When $m = 2$, $n = 3$, a mass minimizing rectifiable current T provides a good model for some but not all soap films. Here $\operatorname{spt}(T) \sim \operatorname{spt}(\partial T)$ is necessarily ([11]) a smooth embedded surface in $\mathbb{R}^3$ whereas general soap films have interior singular curves that simultaneously border three surfaces. The model of Almgren using $(\mathbf{M}, 0, \infty)$ minimal sets is shown in the work of Taylor ([19]) to give variationally the exact observed geometric structure. However currents are more convenient than sets for a precise boundary condition. To use currents in a better model for soap films, Almgren ([4]) introduced the notion of *size* for a rectifiable current:

$$\mathbf{S}(\mathcal{H}^m \llcorner M \wedge \theta\xi) = \mathcal{H}^m(M) .$$

The use of size is illustrated by the case ∂T_0 is supported by two nearby coaxial circles in parallel planes. If the circles are oppositely oriented, then the mass minimizer for $(\mathcal{P}_{\mathbf{M},T_0})$ is an oriented catenoid. If they are similarly oriented, then the mass minimizer is two oriented planar disks. But in the latter case, there is another rectifiable current having the same boundary and smaller size (but larger mass); this is obtained by squeezing together the two disks onto a common middle disk (see Fig. 1.01 in [16]).

For general T_0, the problem

$$(\mathcal{P}_{\mathbf{S},T_0})\begin{cases}\text{minimize } \mathbf{S}(T) \text{ among}\\ T\in\mathcal{R}_m(\mathbb{R}^n)\,,\ \partial T=\partial T_0\end{cases}$$

seems quite difficult. It has been solved by Morgan ([16]) in the special case $m = n-1$ and $\operatorname{spt}(\partial T_0)$ is a smooth submanifold that lies on the boundary of its convex hull. The difficulty in the general case is that a size minimizing sequence may have masses approaching infinity. This is seen in Morgan's example ([16]). A two dimensional version of this is given by considering for two sequences $r_j \downarrow 0$, $h_j \downarrow 0$ corresponding oriented horizontal disks $D_j = \left[\mathcal{H}^2 \llcorner \mathbf{B}(0,r_j)\wedge e_1\wedge e_2\right] \times \boldsymbol{\delta}_{h_j}$ of radius r_j at height h_j. If $\sum_{j=0}^{\infty} r_j^2 < \infty$, then $T_0 = \sum_{j=0}^{\infty} D_j \in \mathcal{R}_2(\mathbb{R}^3)$ (with $\mathbf{M}(T_0) < \infty$). For suitable r_j, h_j, there is a mass minimizing catenoid C_j with $\partial C_j = \partial D_j - \partial D_{j+1}$ and, for each $k \geq 2$, the multiple catenoid $Q_k = \sum_{j=1}^{k} jC_j$ is size minimizing with

$$\partial Q_k = \partial\left(\sum_{j=1}^{k} D_j - kD_{k+1}\right) \longrightarrow \partial\left(\sum_{j=0}^{\infty} D_j\right)$$

$$\mathbf{S}(Q_k) = \sum_{j=1}^{k}\mathbf{M}(C_j) \longrightarrow \sum_{j=1}^{\infty}\mathbf{M}(C_j)$$

$$\mathbf{M}(Q_k) = \sum_{j=1}^{k} j\mathbf{M}(C_j) \longrightarrow \infty$$

as $k \to \infty$. For the fixed boundary $\partial\sum_{j=0}^{\infty} D_j$, the currents

$$T_k = Q_k + kD_{k+1} + \sum_{j=k+1}^{\infty} D_j$$

form a size minimizing sequence with $\mathbf{M}(T_k) \to \infty$. In fact, T_k is a minimizer for the modified size problem:

$$(\mathcal{P}_{\varepsilon_k,\mathbf{S},T_0})\begin{cases}\text{minimize } \mathbf{S}(T)+\varepsilon_k\mathbf{M}(T) \text{ among}\\ T\in\mathcal{R}_2(\mathbb{R}^3)\,,\ \partial T=\partial T_0\end{cases}$$

for some positive $\varepsilon_k \longrightarrow 0$ as $k \to \infty$.

This modified minimization is the general procedure we evoke in section 2 to obtain some positive results concerning size minimization. We renormalize each minimizer T_k of $(\mathcal{P}_{\varepsilon_k,\mathbf{S},T_0})$ to give a stationary weighted surface (varifold). A subsequence of these properly renormalized measures will then converge (as measures) to some weighted rectifiable set S. It is worth noting that we cannot achieve the convergence of the corresponding renormalized currents because the renormalization introduces new boundary. In codimension 1 we show that the corresponding set S obtained in the limit (the countable union of catenoids in our example) is $(\mathbf{M}, 0, \infty)$ minimal in the sense that $\mathcal{H}^m(S) \leq \mathcal{H}^m(f(S))$ for any Lipschitz map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $f|_{\operatorname{spt}(\partial T_0)} = \operatorname{id}|_{\operatorname{spt}(\partial T_0)}$. Next, in light of the examples, one looks for some conditions guaranteeing that the limit $(\mathbf{M}, 0, \infty)$ minimal set supports a finite mass rectifiable current $T \in \mathcal{R}_2(\mathbb{R}^3)$ with $\partial T = \partial T_0$. Such a T would then be a size minimizer. For $m = 2$, $n = 3$, the regularity theory of Taylor guarantees that S is (away from $\operatorname{spt}(\partial T_0)$) a Lipschitz neighborhood retract. This difficulty near $\operatorname{spt}(\partial T_0)$ is absent when Morgan [16] considers size minimization in a 2 dimensional homology class in a smooth compact 3 manifold. He obtains a rectifiable current by retracting an approximate minimizer to a local $(\mathbf{M}, 0, \infty)$ minimal set *before* passing to the limit. While the approximation idea of [16] has some problems, the procedure given here in section 2 works and so, along with his retraction method, completes the proof of this homology size minimization (see Remark 2.3.5). Treating the absolute size minimizing problem by this argument with $\operatorname{spt}(\partial T_0)$ being a smooth 1 dimensional manifold will require a suitable boundary regularity result for $(\mathbf{M}, 0, \infty)$ minimizing sets, which we are currently considering. With $\operatorname{spt}(\partial T_0)$ nonsmooth, the example suggests considering some infinite mass object which still carries the notions of boundary, rectifiability, local orientability and integer multiplicities. *Scans* were introduced in [14] to describe some such infinite mass objects that arose as limits of graphs in energy bubbling sequences.

In this paper an *m dimensional scan* in $\mathbb{R}^n$ is a measurable function which associates with almost every oriented $n - m$ plane P a 0 dimensional rectifiable current $\mathcal{T}(P) \in \mathcal{R}_0(P)$. Thus

$$\mathcal{T}(P) = \sum_{a \in A} \theta(a) \boldsymbol{\delta}_a$$

for some finite subset A of P and integers $\theta(a)$, $a \in A$. As in Proposition 3.1.5 one may represent any rectifiable (even flat) current in terms of some big enough collection of its slices. Thus the scans defined here generalize flat currents whereas the scans used in [14], which were related to graphs of smooth maps, generalized the *cartesian currents* of [13]. The measurability of the slice function corresponding to any flat chain is verified in Lemma 3.1.1. To solve variational problems with scans we need to understand when two scans have the same boundary, or equivalently when a scan has boundary 0. Our definition is motivated by the observation (using the Fourier transform) that *a flat current $T \in \mathbf{F}_m(\mathbb{R}^n)$ has $\partial T = 0$ if and only if the corresponding scan, evaluated at almost all $n - m$ planes P, has total multiplicity* 0; that is, the slice $\langle T, p, y\rangle(\mathbf{1}) = 0$ for almost every orthogonal projection $p : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and almost every $y \in \mathbb{R}^m$.

We work with *rectifiable scans* which are determined in Definition 3.1.7 by an $\mathcal{H}^m$ measurable $(\mathcal{H}^m, m)$ rectifiable set M, an orienting m vectorfield ξ for T_*M, and a positive integer multiplicity function θ. However we do *not* assume that $\int_M \theta d\mathcal{H}^m < \infty$. Besides size, other minimization problems may exhibit minimizing sequences with unbounded mass. These may fail to have rectifiable current minimizers but nevertheless admit rectifiable scan minimizers. For any increasing concave surjection $H : [0, \infty) \longrightarrow [0, \infty)$ with $H(0) = 0$ and $H(1) = 1$, we may define the H mass of a rectifiable current:

$$\mathbf{M}_H(\mathcal{H}^m \llcorner M \wedge \theta\xi) = \int_M H(\theta) d\mathcal{H}^m .$$

In particular for $0 < q \leq 1$, the q mass functionals $\int_M \theta^q d\mathcal{H}^m$ fill the gap between ordinary mass, with $q = 1$, and size. The concavity of H does guarantee the weak lower semicontinuity of $\mathbf{M}_H$ on the class of rectifiable currents (Lemma 3.2.14), but the failure of mass bounds indicates the need for a topology weaker than the weak topology of currents (for instance, $T_j = \partial\left(\mathcal{H}^2 \llcorner \mathbf{B}\left(0, j^{-1}\right) \wedge j^2 e_1 \wedge e_2\right) \in \mathcal{R}_1(\mathbb{R}^2)$ are such that $\mathbf{M}_H(T_j) \to 0$ as $j \to \infty$ for $H(\theta) = \theta^q$ with $0 < q < 2^{-1}$, yet $T_j \to \partial(\boldsymbol{\delta}_0 \wedge e_1 \wedge e_2) \neq 0$ weakly as $j \to \infty$). The H flat distance from T to $\tilde{T}$,

$$\inf\left\{\mathbf{M}_H(R) + \mathbf{M}_H(S) : T - \tilde{T} = R + \partial S\,,\ R \in \mathcal{R}_m(\mathbb{R}^n) \text{ and } S \in \mathcal{R}_{m+1}(\mathbb{R}^n)\right\},$$

which we use, was essentially introduced by Fleming ([12]) and occurs in the work of White ([21, 22]). It was first observed by Jerrard (see [15] or [6]) that the 0 dimensional slices of a normal current correspond to an MBV function. In Section 3.3, we find a corresponding estimate for integral currents involving the H mass and the H flat distance. This is precisely what is needed to apply our BV Compactness Theorem 3.4.1 which concerns maps from a Riemannian manifold to a weakly separable metric space.

In section 3.5 one finds the convergence of the scans corresponding to an $\mathbf{M}_H$ minimizing sequence of rectifiable currents. This limiting scan is also shown to be rectifiable. In the special case when the given boundary $\operatorname{spt}(\partial T_0)$ is a smooth compact $m-1$ dimensional submanifold, we observe that this scan corresponds to a rectifiable current, thus providing the existence of an $\mathbf{M}_H$ minimizer in the class of rectifiable currents.

By using H as an alternate norm on the group of integers, a rectifiable minimizer may also be found in a generalized class of flat chains following the works of Fleming [12] and White [21, 22]. The close relation between rectifiability and slicing explained in [6] and [22] was an important motivation for the definition of scans in [14] and for their use in the present paper. One dimensional flat $\mathbf{M}_H$ minimizers are also applied to describe transport paths in [23].

Use of scans accommodates as well treatment of the case when H is also allowed to depend continuously on the space variable x. Our underlying compactness argument relies on the fact that for a scan $\mathcal{T}$ one has $\mathbf{M}_H(\mathcal{T}(P)) \leq H(\mathbf{M}(\mathcal{T}(P)))$ thanks to the concavity of H and the fact that $\mathcal{T}(P)$ is rectifiable of dimension 0. As suggested in Proposition 3.1.5 one can also consider scans whose values are higher dimensional currents (they were 3 dimensional in [14]). We are currently considering the nature of scan minimizers of integral functionals involving $H(\mathbf{M}(\mathcal{T}(P)))$ instead of $\mathbf{M}_H(\mathcal{T}(P))$. In that case the rectifiability of the limit is not clear.

Most of our notation is consistent with that of Federer's book, which is summarized on pp 669-671 of [9]. In particular, for $T = \mathcal{H}^m \llcorner M \wedge \theta\xi$ as above, $\vec{T} = \xi$ and $\|T\| = \mathcal{H}^m \llcorner M \wedge \theta$. For the definition and notations concerning varifolds we refer to [1], whereas for minimal sets we refer to [3]. In addition we say that a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is *real rectifiable* if $T = \mathcal{H}^m \llcorner M \wedge \theta\xi$ for some M, ξ and θ as above but we drop the restriction that θ be integer valued. Following [4] we define the m dimensional *set* of a measure ϕ (usually $\phi = \|T\|$ for some real rectifiable current or $\phi = \|V\|$ for some rectifiable varifold) by

$$\operatorname{set}(\phi) := \mathbb{R}^n \cap \{x : \Theta^m(\phi, x) > 0\}\,.$$

We also define the *size* of a real rectifiable current T by

$$\mathbf{S}(T) := \mathcal{H}^m(\operatorname{set}(\|T\|))\,.$$

Finally we mention an elementary density property of stationary varifolds which we will refer to.

Proposition 1.0.1. *Let V be an m dimensional rectifiable varifold with compact support in $\mathbb{R}^n$, and suppose V is stationary in $\mathbb{R}^n \sim B$ where B is a $\mathcal{C}^{1,1}$ compact properly embedded $m-1$ dimensional submanifold of $\mathbb{R}^n$. Then the density function $\Theta^m(\|V\|,.)$ is bounded.*

The proof of this Proposition relies on [1, 5.1(2)] (monotonicity in the interior) and [2, 3.4] (monotonicity at the boundary). These two results can be used to show that (see [20, A.2]) the function

$$x \in \mathbb{R}^n \longmapsto \begin{cases} 2\,\Theta^m(\|V\|,x) \text{ if } x \in B \\ \Theta^m(\|V\|,x) \text{ if not} \end{cases}$$

is upper semicontinuous. Since $\operatorname{spt}(\|V\|)$ is compact, $\Theta^m(\|V\|,.)$ is clearly bounded.

2. Penalizing the Lack of Compactness

2.1. **Approximating Problems.** In this section we consider a functional $\mathfrak{F} : \mathcal{R}_m(\mathbb{R}^n) \longrightarrow \mathbb{R}$, a fixed rectifiable current $T_0 \in \mathcal{R}_m(\mathbb{R}^n)$, and a closed set C containing $\operatorname{spt}(T_0)$. We are interested in the following minimization problem:

$$(\mathcal{P}_{\mathfrak{F},T_0,C}) \begin{cases} \text{minimize } \mathfrak{F}(T) \\ \text{among } T \in \mathcal{R}_{m,K}(\mathbb{R}^n) \text{ such that } \partial T = \partial T_0 \text{ and } \operatorname{spt}(T) \subset C \end{cases}$$

We let $\Gamma(\mathcal{P}_{\mathfrak{F},T_0,C})$ be the infimum of that problem:

$$\Gamma(\mathcal{P}_{\mathfrak{F},T_0,C}) := \inf\{\mathfrak{F}(T) : T \in \mathcal{R}_m(\mathbb{R}^n) \text{ and } \partial T = \partial T_0 \text{ and } \operatorname{spt}(T) \subset C\}\,.$$

Of course the infimum of problem $(\mathcal{P}_{\mathfrak{F},T_0,C})$ is not necessarily achieved by any competitor. In order to obtain currents having a somewhat regular support and almost minimizing $\mathfrak{F}$, we introduce the following modified problems parametrized by $\varepsilon > 0$.

$$(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,C}) \begin{cases} \text{minimize } \mathfrak{F}_\varepsilon(T) := \mathfrak{F}(T) + \varepsilon^2\mathbf{M}(T) \text{ among} \\ T \in \mathcal{R}_m(\mathbb{R}^n) \text{ such that } \partial T = \partial T_0 \text{ and } \operatorname{spt}(T) \subset C \end{cases}$$

$$(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,C}) \begin{cases} \text{minimize } \mathfrak{F}(T) \text{ among} \\ T \in \mathcal{R}_m(\mathbb{R}^n) \text{ such that } \partial T = \partial T_0 \text{ and } \mathbf{M}(T) \le \varepsilon^{-1} \\ \qquad\qquad\qquad \text{as well as } \operatorname{spt}(T) \subset C \end{cases}$$

We denote by $\Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,C})$ and $\Gamma(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,C})$ the infima of these problems. The reasons for introducing these problems parametrized by ε are: (a) there exist minimizers for these approximating problems and (b) these minimizers have some regularity and variational properties that guarantee convergence, as $\varepsilon \downarrow 0$, to some limiting object (a rectifiable stationary varifold, an $(\mathbf{M},0,\infty)$ minimal set or a rectifiable scan but not necessarily a current). We now gather some basic observations.

Lemma 2.1.1. *Let $K \subset \mathbb{R}^n$ be a Lipschitz neighborhood retract, $T_0 \in \mathcal{R}_{m,K}(\mathbb{R}^n)$ and $\mathfrak{F} : \mathcal{R}_{m,K}(\mathbb{R}^n) \longrightarrow [0,\infty)$ a functional which is lower semicontinuous with respect to weak convergence. Let also $\mathfrak{F}_\varepsilon$, $\Gamma(\mathcal{P}_{\mathfrak{F},T_0,K})$, $\Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K})$ and $\Gamma(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K})$ be as above. Then the following holds true.*

1. *For each $\varepsilon > 0$ (resp. $0 < \varepsilon \le \mathbf{M}(T_0)^{-1}$) there exists at least one $T_\varepsilon \in \mathcal{R}_{m,K}(\mathbb{R}^n)$ (resp. $S_\varepsilon \in \mathcal{R}_{m,K}(\mathbb{R}^n)$) such that $\mathfrak{F}_\varepsilon(T_\varepsilon) = \Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K})$ (resp. $\mathfrak{F}(S_\varepsilon) = \Gamma(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K})$);*
2. *$\Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) = \lim_{\varepsilon\downarrow 0}\Gamma(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K})$;*

(3) $\Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) = \lim_{\varepsilon\downarrow 0}\Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K})$;
(4) $\lim_{\varepsilon\downarrow 0}\varepsilon^2\mathbf{M}(T_\varepsilon) = 0$ *for any* $T_\varepsilon \in \mathcal{R}_{m,K}(\mathbb{R}^n)$ *with* $\mathfrak{F}_\varepsilon(T_\varepsilon) = \Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K})$.

Proof. We will prove (1) for problem $(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K})$, the case $(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K})$ being analogous. Let $T_{\varepsilon,1}, T_{\varepsilon,2}, \ldots$ be a minimizing sequence. Then obviously

$$\begin{aligned}\mathbf{N}(T_{\varepsilon,j} - T_0) &= \mathbf{M}(T_{\varepsilon,j} - T_0)\\ &\le \mathbf{M}(T_0) + \varepsilon^{-2}\sup\{\mathfrak{F}_\varepsilon(T_{\varepsilon,k}) : k = 1,2,\ldots\}\end{aligned}$$

so that the Compactness Theorem of Federer and Fleming ([9, 4.2.17(2)]) implies that there are integers $\alpha(1),\alpha(2),\ldots$ and $R_\varepsilon \in \mathbf{I}_{m,K}(\mathbb{R}^n)$ with $\mathbf{F}_K(R_\varepsilon - T_{\varepsilon,\alpha(j)} + T_0) \to 0$ as $j \to \infty$. On letting $T_\varepsilon := R_\varepsilon + T_0 \in \mathcal{R}_{m,K}(\mathbb{R}^n)$ we see that $\partial T_\varepsilon = \partial T_0$ and $\mathfrak{F}_\varepsilon(T_\varepsilon) = \Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K})$ because $\mathfrak{F}_\varepsilon$ is weakly lower semicontinuous.

In order to prove (2) we notice first that $\Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) \le \Gamma(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K})$. Next we fix $\eta > 0$ and we choose $T \in \mathcal{R}_{m,K}(\mathbb{R}^n)$ such that $\mathfrak{F}(T) \le \Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) + \eta$. We then fix a positive $\varepsilon_0 \le \mathbf{M}(T)^{-1}$. Then for $0 < \varepsilon < \varepsilon_0$, T is a competitor for problem $(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K})$, whence $\Gamma(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K}) \le \mathfrak{F}(T) \le \Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) + \eta$.

For proving (3) we observe that $\Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) \le \Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K})$ because $\mathfrak{F} \le \mathfrak{F}_\varepsilon$. For fixed $\varepsilon > 0$, let S_ε be a minimizer for problem $(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K})$. Then

$$\begin{aligned}\Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K}) &\le \mathfrak{F}_\varepsilon(S_\varepsilon) = \mathfrak{F}(S_\varepsilon) + \varepsilon^2\mathbf{M}(S_\varepsilon)\\ &\le \mathfrak{F}(S_\varepsilon) + \varepsilon \le \Gamma(\mathcal{Q}_{\varepsilon,\mathfrak{F},T_0,K}) + \varepsilon\,,\end{aligned}$$

and we conclude (3) with the help of (2). Finally, to prove (4), let $\eta > 0$ and refer to (3) to find $\varepsilon_0 > 0$ such that $\Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K}) \le \Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) + \eta$ whenever $0 < \varepsilon < \varepsilon_0$. Then, for such $\varepsilon > 0$,

$$\Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) + \varepsilon^2\mathbf{M}(T_\varepsilon) \le \mathfrak{F}(T_\varepsilon) + \varepsilon^2\mathbf{M}(T_\varepsilon) = \Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,K}) \le \Gamma(\mathcal{P}_{\mathfrak{F},T_0,K}) + \eta\,,$$

whence $\varepsilon^2\mathbf{M}(T_\varepsilon) \le \eta$. □

2.2. **The Stationary Varifold Associated with a Modified Problem.** In this section we wish to evaluate $\mathfrak{F}(T)$ by calculating the mass of an associated real rectifiable current.

Definition 2.2.1. We say that a functional $\mathfrak{F} : \mathcal{R}_m(\mathbb{R}^n) \longrightarrow [0,\infty)$ is *mass-calculable* if there is associated to each $T \in \mathcal{R}_m(\mathbb{R}^n)$ a real compactly-supported rectifiable current $\Upsilon_{\mathfrak{F}}(T) \in \mathcal{D}_m(\mathbb{R}^n)$ having the following properties:

(1) $\mathbf{M}(\Upsilon_{\mathfrak{F}}(T)) = \mathfrak{F}(T)$.
(2) $f_\#\Upsilon_{\mathfrak{F}}(T) = \Upsilon_{\mathfrak{F}}(f_\# T)$ for every smooth diffeomorphism $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $\mathbb{R}^n \cap \{x : f(x) \ne x\}$ compact;

Example 2.2.2. In the next chapter we consider, for a *concave integrand* (see Definition 3.2.1) H, the H mass $\mathbf{M}_H(T)$ defined in 3.2.2 by integrating $H(\Theta^m(\|T\|,x))$. To get (1), we may simply take $\Upsilon_{\mathbf{M}_H}(T)$ to be the real rectifiable current obtained from T by replacing the density $\Theta^m(\|T\|,\cdot)$ by $H(\Theta^m(\|T\|,\cdot))$. Condition (2) follows readily from [9, 4.1.30] because f is one to one. Similarly replacing $\Theta^m(\|T\|,\cdot)$ by $\mathbf{1}_{\mathrm{set}(\|T\|)}$ shows that size $\mathbf{S}$ is mass-calculable.

We recall ([1, 3.5]) that specifying a rectifiable varifold $V \in \mathbf{RV}_m(U)$ is the same as specifying its weight $\|V\|$ which is an m rectifiable Radon measure on U.

Definition 2.2.3. For $T \in \mathcal{R}_m(\mathbb{R}^n)$, $\varepsilon > 0$ and $\mathfrak{F}$ a mass-calculable functional, we now associate an m rectifiable varifold $V_{\mathfrak{F},\varepsilon}(T) \in \mathbf{RV}_m(\mathbb{R}^n)$ defined by

$$\|V_{\mathfrak{F},\varepsilon}(T)\| := \|\Upsilon_{\mathfrak{F}}(T)\| + \varepsilon^2\|T\|\,.$$

Proposition 2.2.4. *Let $T_0 \in \mathcal{R}_m(\mathbb{R}^n)$, $\varepsilon > 0$, $\mathfrak{F}$ be a mass-calculable functional, and $T_\varepsilon \in \mathcal{R}_m(\mathbb{R}^n)$ be such that $\partial T_\varepsilon = \partial T_0$ and $\mathfrak{F}_\varepsilon(T_\varepsilon) = \Gamma(\mathcal{P}_{\varepsilon,\mathfrak{F},T_0,\mathbb{R}^n})$. Then $V_{\mathfrak{F},\varepsilon}(T_\varepsilon)$ is a stationary rectifiable varifold in $U := \mathbb{R}^n \sim \operatorname{spt}(\partial T_0)$.*

Proof. Let $f : U \longrightarrow U$ be a smooth diffeomorphism which differs from the identity only on a compact subset of U. Since f is one to one we have, by 2.2.1(2),

$$\begin{aligned} \|f_\# V_{\mathfrak{F},\varepsilon}(T_\varepsilon)\| &= \|f_\#(\Upsilon_{\mathfrak{F}}(T_\varepsilon))\| + \varepsilon^2 \|f_\# T_\varepsilon\| \\ &= \|\Upsilon_{\mathfrak{F}}(f_\# T_\varepsilon)\| + \varepsilon^2 \|f_\# T_\varepsilon\| = \|V_{\mathfrak{F},\varepsilon}(f_\# T_\varepsilon)\| \,. \end{aligned}$$

We also deduce from the fact that $f|_{\operatorname{spt}(\partial T_0)} = \operatorname{id}|_{\operatorname{spt}(\partial T_0)}$ and [9, 4.1.15] that $\partial f_\# T_\varepsilon = \partial T_\varepsilon = \partial T_0$. It then follows from the minimality of T_ε that

$$\begin{aligned} \|f_\# V_{\mathfrak{F},\varepsilon}(T_\varepsilon)\|(\mathbb{R}^n) &= \|V_{\mathfrak{F},\varepsilon}(f_\# T_\varepsilon)\|(\mathbb{R}^n) = \mathfrak{F}_\varepsilon(f_\# T_\varepsilon) \\ &\geq \mathfrak{F}_\varepsilon(T_\varepsilon) = \|V_{\mathfrak{F},\varepsilon}(T_\varepsilon)\|(\mathbb{R}^n) \,. \end{aligned}$$

From this clearly follows that $V_{\mathfrak{F},\varepsilon}(T_\varepsilon)$ is stationary in U. □

2.3. **Existence of $(\mathbf{M}, 0, \infty)$ Minimal Sets in Case $m = n-1$.** We particularize the setting of this section to the case when $\mathfrak{F} = \mathbf{S}$ is the size functional. We will study accumulation points of the collection of varifolds $V_{\mathbf{S},\varepsilon}(T_\varepsilon)$ (defined in the preceding section) as well as the sets associated with them.

Proposition 2.3.1. *Let $T_0 \in \mathcal{R}_m(\mathbb{R}^n)$. For every $\varepsilon > 0$ there exists $T_\varepsilon \in \mathcal{R}_m(\mathbb{R}^n)$ such that $\partial T_\varepsilon = \partial T_0$ and $\mathbf{S}_\varepsilon(T_\varepsilon) = \Gamma(\mathcal{P}_{\varepsilon,\mathbf{S},T_0,\mathbb{R}^n})$. Furthermore for every sequence $\varepsilon_j \downarrow 0$ there are integers $\alpha(1), \alpha(2), \ldots$ and a stationary rectifiable varifold V in $\mathbb{R}^n \sim \operatorname{spt}(\partial T_0)$ such that*

$$V_{\mathbf{S},\varepsilon_{\alpha(j)}}(T_{\varepsilon_{\alpha(j)}}) \longrightarrow V \;\; \textit{in} \;\; \mathbb{R}^n \sim \operatorname{spt}(\partial T_0) \;\; \textit{as} \;\; j \to \infty \,.$$

Finally, in case $m = n-1$, $\Theta^m(\|V\|, x) = 1$ for $\|V\|$ a.e. $x \in \mathbb{R}^n \sim \operatorname{spt}(\partial T_0)$.

Proof. We first observe that if $K \subset \mathbb{R}^n$ is a compact convex set containing $\operatorname{spt}(T_0)$, then $\Gamma(\mathcal{P}_{\varepsilon,\mathbf{S},T_0,K}) = \Gamma(\mathcal{P}_{\varepsilon,\mathbf{S},T_0,\mathbb{R}^n})$ because

$$\partial \pi_{K\,\#} T = \partial T_0 \;\; \text{and} \;\; \mathbf{S}_\varepsilon(\pi_{K\,\#} T) \leq \mathbf{S}_\varepsilon(T)$$

where $\pi_K : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denotes the nearest point projection onto K ([9, 4.1.15]).The existence of T_ε then follows from Lemma 2.1.1(1).

Again let $U := \mathbb{R}^n \sim \operatorname{spt}(\partial T_0)$. We first observe that

$$\Theta^m(\|V_{\mathbf{S},\varepsilon}(T_\varepsilon)\|, x) = \Theta^m(\mathcal{H}^m \llcorner \operatorname{set}(\|T_\varepsilon\|), x) + \varepsilon^2 \Theta^m(\|T_\varepsilon\|, x) \geq 1$$

for $\|T_\varepsilon\|$ almost every $x \in U$, whence also $\|V_{\mathbf{S},\varepsilon}(T_\varepsilon)\|$ almost every $x \in U$. According to Proposition 2.2.4 we have that $\delta V_{\mathbf{S},\varepsilon}(T_\varepsilon) = 0$ in U whereas Lemma 2.1.1(3) implies that $\sup\{\|V_{\mathbf{S},\varepsilon_j}(T_{\varepsilon_j})\|(U) : j = 1, 2, \ldots\} < \infty$. It then follows from [1, 5.6] (the compactness theorem for rectifiable varifolds) that there are integers $\alpha(1), \alpha(2), \ldots$ and a varifold V in U such that $\operatorname{set}(\|V\|)$ is $(\mathcal{H}^m, m)$ rectifiable and $V_{\mathbf{S},\varepsilon_{\alpha(j)}}(T_{\varepsilon_{\alpha(j)}}) \longrightarrow V$ as $j \to \infty$. It also follows from the same theorem that $\Theta^m(\|V\|, x) \geq 1$ for $\|V\|$ almost every $x \in U$. Clearly V is stationary in U. In order to keep the notation short for the remaining part of the proof we set $V_j := V_{\mathbf{S},\varepsilon_{\alpha(j)}}(T_{\varepsilon_{\alpha(j)}})$ and $T_j := T_{\varepsilon_{\alpha(j)}}$. We now assume that $m = n-1$ and we intend to show that $\Theta^m(\|V\|, x) \leq 1$ whenever

$$(\boldsymbol{\mu}_r \circ \boldsymbol{\tau}_{-x})_\# \mathcal{H}^m \llcorner \operatorname{set}(\|V\|) \longrightarrow \mathcal{H}^m \llcorner W \;\; \text{as} \;\; r \to \infty \tag{1}$$

for some $W \in \mathbf{G}(n,m)$. This will obviously finish the proof. Pick x and W as in (1) and assume for a contradiction that $\Theta^m(\|V\|,x) = 1+\lambda$ for some $\lambda > 0$. Choose $\eta > 0$ small enough for $2\eta(1-\eta)^{-1} \leq 2^{-1}$ and $4\sqrt{2}m\boldsymbol{\alpha}(m)\eta(1-\eta)^{-1} \leq 4^{-1}\lambda$. Next let $E \subset (0,\infty)$ be an $\mathcal{L}^1$ negligible set such that $\|V\|(\operatorname{Bdry}\mathbf{B}(x,\rho)) = 0$ and $\langle T_j, u, \rho\rangle \in \mathbf{I}_{m-1,K}(\mathbb{R}^n)$ whenever $\rho \in (0, \operatorname{dist}(x, \mathbb{R}^n \sim U)) \sim E$ and $j = 1,2,\ldots$, where $u(y) := |x-y|$. Referring to the monotonicity of $\|V\|$ in U (recall [1, 5.1(2)]) as well as the lower density bound we deduce (e.g. as in [8, 4.3]) from (1) that there exists $r_0 > 0$ such that for each $0 < r \leq r_0$:

$$\operatorname{spt}\big((\boldsymbol{\mu}_{r^{-1}} \circ \boldsymbol{\tau}_{-x})_\#\|V\|\big) \cap \mathbf{B}(0,1) \subset \mathbf{B}(W,\eta) \cap \mathbf{B}(0,1)$$

in other words,

$$\operatorname{spt}(\|V\|) \cap \mathbf{B}(x,r) \subset \mathbf{B}(x+W,\eta r) \cap \mathbf{B}(x,r)\ . \tag{2}$$

We now choose some $\rho \in (0,(1-\eta)r_0) \sim E$ and we write $r := (1-\eta)^{-1}\rho$. The same monotonicity and lower density bound argument as above and (2) show that there exists an integer j_1 such that for each $j \geq j_1$:

$$\begin{aligned}\operatorname{spt}(\|V_j\|) \cap \mathbf{B}(x,\rho) &\subset \mathbf{B}(\operatorname{spt}(\|V\|),\eta r) \cap \mathbf{B}(x,\rho)\\ &\subset \mathbf{B}(x+W,2\eta r) \cap \mathbf{B}(x,\rho)\\ &= \mathbf{B}\big(x+W,2\eta(1-\eta)^{-1}\rho\big) \cap \mathbf{B}(x,\rho)\ .\end{aligned} \tag{3}$$

We let $\pi : \operatorname{Bdry}\mathbf{B}(x,\rho) \sim (x+W^\perp) \longrightarrow (x+W) \cap \operatorname{Bdry}\mathbf{B}(x,\rho)$ be on each hemisphere the central projection from the pole to the equator. For each $j \geq j_1$ let

$$\tilde{T}_j := T_j \llcorner (\mathbb{R}^n \sim \mathbf{B}(x,\rho)) + Q_j + \boldsymbol{\delta}_x \mathbin{⨳} \pi_\#\langle T_j,u,\rho\rangle$$

where $Q_j \in \mathbf{I}_m(\mathbb{R}^n)$ is such that

$$\partial Q_j = \langle T_j,u,\rho\rangle - \pi_\#\langle T_j,u,\rho\rangle$$

and $\operatorname{spt}(Q_j) \subset \big(\operatorname{Bdry}\mathbf{B}(x,\rho)\big) \cap \mathbf{B}\big(x+W,2\eta(1-\eta)^{-1}\rho\big)$. We compute that

$$\begin{aligned}\mathbf{S}(\tilde{T}_j) \ &\leq\ \mathcal{H}^m(\operatorname{set}(\|T_j\|) \sim \mathbf{B}(x,\rho))\\ &\quad + \mathcal{H}^m\big[(\operatorname{Bdry}\mathbf{B}(x,\rho)) \cap \mathbf{B}\big(x+W,2\eta(1-\eta)^{-1}\rho\big)\big] + \mathcal{H}^m[(x+W)\cap\mathbf{B}(x,\rho)]\\ &\leq \mathcal{H}^m(\operatorname{set}(\|T_j\|) \sim \mathbf{B}(x,\rho)) + 2\sqrt{2}m\boldsymbol{\alpha}(m)2\eta(1-\eta)^{-1}\rho^m + \boldsymbol{\alpha}(m)\rho^m\end{aligned}$$

which, according to the choice of η, is bounded by

$$\leq \mathcal{H}^m(\operatorname{set}(\|T_j\|) \sim \mathbf{B}(x,\rho)) + \boldsymbol{\alpha}(m)\rho^m\left(1+\frac{\lambda}{2}\right) - \frac{\lambda}{4}\boldsymbol{\alpha}(m)\rho^m\ . \tag{4}$$

On the other hand monotonicity implies that

$$(1+\lambda)\boldsymbol{\alpha}(m)\rho^m \leq \|V\|(\mathbf{B}(x,\rho))$$

and, since $\|V\|(\operatorname{Bdry}\mathbf{B}(x,\rho)) = 0$, we also have that

$$\|V\|(\mathbf{B}(x,\rho)) \leq \liminf_{j\to\infty}\|V_j\|(\mathbf{B}(x,\rho))$$

so that there is an integer j_2 such that

$$\begin{aligned}\left(1+\frac{\lambda}{2}\right)\boldsymbol{\alpha}(m)\rho^m &\leq \|V_j\|(\mathbf{B}(x,\rho))\\ &= \mathcal{H}^m\big(\operatorname{set}(\|T_j\|) \cap \mathbf{B}(x,\rho)\big) + \varepsilon^2_{\alpha(j)}\mathbf{M}(T_j \llcorner \mathbf{B}(x,\rho))\end{aligned} \tag{5}$$

whenever $j \geq j_1$. According to Lemma 2.1.1(3) we can also select an integer j_3 such that if $j \geq j_3$ then

$$\mathbf{S}(T_j) + \varepsilon^2_{\alpha(j)}\mathbf{M}(T_j) \leq \Gamma(\mathcal{P}_{\mathbf{S},T_0,K}) + \frac{\lambda}{8}\boldsymbol{\alpha}(m)\rho^m \,. \tag{6}$$

We observe from the definition of $\tilde{T}_j$ that $\partial\tilde{T}_j = \partial T_0$ and that $\operatorname{spt}(\tilde{T}_j) \subset K$. On letting $j := \max\{j_1, j_2, j_3\}$ and plugging (5) and (6) in (4) we obtain the following contradiction:

$$\begin{aligned}\mathbf{S}(\tilde{T}_j) &\leq -\frac{\lambda}{4}\boldsymbol{\alpha}(m)\rho^m + \mathcal{H}^m\left[\operatorname{set}(\|T_j\|) \sim \mathbf{B}(x,\rho)\right] + \boldsymbol{\alpha}(m)\rho^m\left(1+\frac{\lambda}{2}\right)\\ &\leq -\frac{\lambda}{4}\boldsymbol{\alpha}(m)\rho^m + \mathcal{H}^m\left[\operatorname{set}(\|T_j\|)\right] + \varepsilon^2_{\alpha(j)}\mathbf{M}(T_j)\\ &\leq -\frac{\lambda}{8}\boldsymbol{\alpha}(m)\rho^m + \Gamma(\mathcal{P}_{\mathbf{S},T_0,K})\end{aligned}$$

□

Before we go on with proving the main result of this section we need two elementary Lemmas.

Lemma 2.3.2. *Let $\mathcal{C}$ be a disjoint family of closed subsets of $\mathbb{R}^n$, $\Lambda > 0$, and assume that for each $C \in \mathcal{C}$ there is given a map $\psi_C : C \longrightarrow C$ with $\operatorname{Lip}(\psi_C) \leq \Lambda$ and $\psi_C(y) = y$ for each $y \in \operatorname{Bdry} C$. Then the map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\psi(y) := y$ if $y \notin \cup\mathcal{C}$ and $\psi(y) := \psi_C(y)$ whenever $y \in C \in \mathcal{C}$ is Lipschitzian with $\operatorname{Lip}(\psi) \leq \max\{1, \Lambda\}$.*

Proof. Let $y, z \in \mathbb{R}^n$: we need to show that $|\psi(y) - \psi(z)| \leq \max\{1,\Lambda\}|y-z|$. It is obvious in case $y, z \notin \cup\mathcal{C}$ as well as if $y, z \in C$ for some $C \in \mathcal{C}$. Assume that $y \in C \in \mathcal{C}$ and $z \notin \cup\mathcal{C}$: there exists $y' \in \operatorname{Bdry} C$ such that $|y-z| = |y-y'| + |y'-z|$ whence

$$\begin{aligned}|\psi(y) - \psi(z)| &\leq |\psi(y) - \psi(y')| + |\psi(y') - \psi(z)|\\ &\leq \Lambda|y-y'| + |y'-z| \leq \max\{1,\Lambda\}|y-z| \,.\end{aligned}$$

An analogous remark yields the same estimate in case $y \in C \in \mathcal{C}$ and $z \in D \in \mathcal{C}$. □

Lemma 2.3.3. *Let $x \in \mathbb{R}^n$, $r > 0$, $0 < \eta \leq h \leq 2^{-1}$, $\lambda > 0$, $W \in \mathbf{G}(n,m)$ and $u : (x+W) \cap \mathbf{B}(x,r) \longrightarrow x + W^\perp$ a map such that $|u - p_{W^\perp}(x)| \leq hr$ and $\operatorname{Lip}(u) \leq \lambda$. Let also $C := \mathbf{B}(x,r) \cap \mathbf{B}(x+W, 2hr)$. Then there exists a map $\psi_C : C \longrightarrow C$ satisfying the following conditions:*

1. *$\psi_C(y) = y$ whenever $y \in \operatorname{Bdry} C$;*
2. *$\operatorname{Lip}(\psi_C) \leq \max\left\{1, \sqrt{1+\lambda^2}, \sqrt{1+9h^2\eta^{-2}}\right\}$;*
3. *$\psi_C\left[\mathbf{U}(x,(1-\eta)r) \cap \mathbf{U}(x+W,hr)\right] \subset \operatorname{graph}(u) \cap \mathbf{B}\left(x, \sqrt{2}r\right)$.*

Proof. We can of course assume that $x = 0$. Let $p, p^\perp \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ be the orthogonal projections onto W and $W^\perp$ respectively. To keep the notation short let $\mathbf{U} := \mathbf{U}(0,(1-\eta)r) \cap \mathbf{U}(W, hr)$. Define

$$\varphi : \mathbf{U} \cup \operatorname{Bdry} C \longrightarrow \mathbb{R}^n$$

by $\varphi(y) := y$ if $y \in \operatorname{Bdry} C$ and $\varphi(y) := p(y) + u(p(y))$ if $y \in \mathbf{U}$. Obviously, the restriction of φ to $\operatorname{Bdry} C$ has Lipschitz constant 1 whereas its restriction to $\mathbf{U}$

has Lipschitz constant less than or equal to $\sqrt{1+\lambda^2}$. Moreover if $y \in \operatorname{Bdry} C$ and $z \in \mathbf{U}$ then

$$\begin{aligned} |\varphi(y)-\varphi(z)|^2 &= |y-p(z)-u(p(z))|^2 \\ &= |p(y)-p(z)|^2 + |p^\perp(y)-u(p(z))|^2 \\ &\le |y-z|^2 + |p^\perp(y)-u(p(z))|^2 . \end{aligned} \tag{7}$$

On the other hand it is easy to check that $|y-z| \ge r\min\{\eta,h\} = \eta r$ and hence

$$\begin{aligned} |p^\perp(y)-u(p(z))| &\le |p^\perp(y)| + |u(p(z))| \\ &\le 2hr + hr \le 3h\eta^{-1}|y-z| \end{aligned} \tag{8}$$

so that plugging (8) in (7) yields

$$|\varphi(y)-\varphi(z)|^2 \le \left(1+9h^2\eta^{-2}\right)|y-z|^2$$

and in turn

$$\operatorname{Lip}(\varphi) \le \max\left\{1, \sqrt{1+\lambda^2}, \sqrt{1+9h^2\eta^{-2}}\right\} .$$

Clearly $\varphi(\mathbf{U}) \subset \operatorname{graph}(u) \cap \mathbf{B}\left(0,\sqrt{2}r\right)$. Refering to [9, 2.10.43] (Kirszbraun's Theorem) we know φ extends to a map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $\operatorname{Lip}(\psi) = \operatorname{Lip}(\varphi)$. Finally we denote by $\pi_C : \mathbb{R}^n \longrightarrow C$ the nearest point projection (C is convex) and we define $\psi_C := (\pi_C \circ \psi)|_C$ and we see that ψ_C satisfies all the required conditions. □

L. Ambrosio, N. Fusco and J.E. Hutchinson recently proved in [5, 4.3] that the limit of a sequence of $(\mathbf{M},0,\infty)$ minimal sets of codimension 1 in $\mathbb{R}^n$ is itself $(\mathbf{M},0,\infty)$ minimal. In the next proof we will establish a similar result.

Theorem 2.3.4. *Let $T_0 \in \mathcal{R}_{n-1}(\mathbb{R}^n)$ and $V \in \mathbf{RV}_{n-1}(\mathbb{R}^n \sim \operatorname{spt}(\partial T_0))$ be as in Proposition 2.3.1 ($m = n-1$). Assume also that $\mathcal{H}^{n-1}(\operatorname{spt}(\partial T_0)) = 0$. Then* $\operatorname{set}(\|V\|)$ *is* $(\mathbf{M},0,\infty)$ *minimal with respect to* $\operatorname{spt}(\partial T_0)$.

Proof. We let $B := \operatorname{spt}(\partial T_0)$ and $U := \mathbb{R}^n \sim B$. First we notice that since V is stationary in U, monotonicity ([1, 5.1(2)]) and lower density bounds imply (for instance as in [8, 6.13]) that $\operatorname{set}(\|V\|) = \operatorname{spt}(\|V\|) \sim B$. In order to keep the notation short we let $S := \operatorname{set}(\|V\|)$ as well as $S_j := \operatorname{set}\left(\|V_{\mathbf{S},\varepsilon_{\alpha(j)}}(T_{\varepsilon_{\alpha(j)}})\|\right)$ and $T_j := T_{\varepsilon_{\alpha(j)}}$. It now suffices to show that

$$\mathcal{H}^{n-1}(S) \le \mathcal{H}^{n-1}(f(S)) \tag{9}$$

for any Lipschitz $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $f = \mathrm{id}_{\mathbb{R}^n}$ in a neighborhood of $\operatorname{spt}(\partial T_0)$.

Let $0 < \eta \le 2^{-1}$ and choose an open set $\tilde{U} \subset U$ such that $\|V\|(U \sim \tilde{U}) \le \eta$ and $f(y) = y$ for every $y \notin \tilde{U}$. We next set $G := S \cap \{x : \Theta^{n-1}(\|V\|,x) = 1\}$: since $\|V\| = \mathcal{H}^{n-1} \llcorner S$ (recall Proposition 2.3.1) and since S is $(\mathcal{H}^{n-1}, n-1)$ rectifiable we deduce that $\mathcal{H}^{n-1}(S \sim G) = 0$. The set G is an embedded $\mathcal{C}^1$ submanifold of dimension $n-1$ in U as follows from Allard's regularity Theorem [1, §8]. This means in particular that for each $x \in G$ there exist $r(x) > 0$, $W_x \in \mathbf{G}(n,m)$ and a class $\mathcal{C}^1$ map $u_x : (x+W_x) \cap \mathbf{B}(x,r(x)) \longrightarrow x + W_x^\perp$ satisfying the following:

(a) $u_x(x) = x$;
(b) $Du_x(x) = 0$;
(c) $\operatorname{graph}(u_x) \cap \mathbf{B}(x,r(x)) = G \cap \mathbf{B}(x,r(x))$.

In view of (a) and (b) above it is obvious that, by possibly decreasing $r(x)$, we can also assume that

(d) $\operatorname{Lip}(u_x|_{\mathbf{B}(x,r(x))}) \le \eta$;

(e) $|u_x - p_{W_x^\perp}(x)| \leq \eta r$.

By possibly decreasing $r(x)$ some more we may also require that $\mathbf{B}(x, r(x)) \subset \tilde{U}$ whenever $x \in \tilde{U}$ as well as

(f) $\mathcal{H}^{n-1}(G \cap \mathbf{B}(x,r)) = \|V\|(\mathbf{B}(x,r)) \leq (1+\eta)^{n-1}\boldsymbol{\alpha}(n-1)r^{n-1}$,

for each $0 < r \leq r(x)$ thanks to monotonicity. With the help of [9, 2.8.15] we secure countably many $x_1, x_2, \ldots \in G \cap \tilde{U}$ and $r_1, r_2, \ldots$ such that $r_i \leq 2^{-1/2} r(x_i)$, the balls $\mathbf{B}(x_i, r_i)$ are pairwise disjoint, $\|V\|(\operatorname{Bdry} \mathbf{B}(x_i, (1-\eta)r_i)) = 0$ and

$$\mathcal{H}^{n-1}\left[\tilde{U} \cap G \sim \cup_{i=1}^{\infty} \mathbf{B}(x_i, r_i)\right] = 0\,. \tag{10}$$

For each $i = 1, 2, \ldots$ we define C_i as in Lemma 2.3.3 applied with $x = x_i$, $r := r_i$, $\eta = h$, $\lambda := \eta$, $W := W_{x_i}$ and $u := u_{x_i}$. This Lemma ensures the existence, for each $i = 1, 2, \ldots$, of a map $\psi_i : C_i \longrightarrow C_i$ such that $\operatorname{Lip}(\psi_i) \leq \sqrt{10}$ and

$$\psi_i(A_i) \subset \operatorname{graph}(u_{x_i}) \cap \mathbf{B}\left(x_i, \sqrt{2} r_i\right) \subset G \subset S\,, \tag{11}$$

where we have set $A_i := \mathbf{U}(x_i, (1-\eta)r_i) \cap \mathbf{U}(x_i + W_{x_i}, \eta r_i)$. We now infer from monotonicity that

$$(1-\eta)^{n-1}\boldsymbol{\alpha}(n-1)r_i^{n-1} \leq \|V\|\left(\mathbf{B}(x_i, (1-\eta)r_i)\right) = \mathcal{H}^{n-1}\left(G \cap \mathbf{U}(x_i, (1-\eta)r_i)\right)$$

for each $i = 1, 2, \ldots$ so that, relying also on (f) above, we obtain

$$\mathcal{H}^{n-1}\left(G \cap \mathbf{B}(x_i, r_i)\right) - \mathcal{H}^{n-1}\left(G \cap \mathbf{U}(x_i, (1-\eta)r_i)\right) \leq 2^{n-1}\eta\boldsymbol{\alpha}(n-1)r_i^{n-1}\,. \tag{12}$$

Since $G \cap \mathbf{U}(x_i, (1-\eta)r_i) \subset \mathbf{U}(x_i + W_{x_i}, \eta r_i)$ (as follows from (a) and (d) above) we see that on letting $A := \cup_{i=1}^{\infty} A_i$, relations (10) and (12) above imply that

$$\mathcal{H}^{n-1}(\tilde{U} \cap G \sim A) \leq 2^{n-1}\eta \sum_{i=1}^{\infty} \boldsymbol{\alpha}(n-1)r_i^{n-1}$$

which, since $x_i \in G$ and $\|V\|$ is monotone, is bounded by

$$\leq 2^{n-1}\eta \sum_{i=1}^{\infty} \|V\|(\mathbf{B}(x_i, r_i)) \leq 2^{n-1}\eta\|V\|(U)\,. \tag{13}$$

We now deduce from the choice of $\tilde{U}$, inequality (13), the fact that A is open and Lemma 2.1.1(4) that

$$\begin{aligned} \mathcal{H}^{n-1}(S) - \eta - 2^{n-1}\eta\|V\|(U) &\leq \mathcal{H}^{n-1}(S \cap A) = \|V\|(A) \\ &\leq \|V_j\|(A) + \eta \leq \mathcal{H}^{n-1}(S_j \cap A) + 2\eta \end{aligned} \tag{14}$$

provided $j \geq j_1$ for some integer j_1. On the other hand, since $\mathcal{H}^{n-1}(\operatorname{spt}(\partial T_0)) = 0$, $\mathcal{H}^{n-1}(S_j) \longrightarrow \mathcal{H}^{n-1}(S)$ as $j \to \infty$ (recall Lemma 2.1.1(4)) so that

$$\mathcal{H}^{n-1}(S_j \sim A) + \mathcal{H}^{n-1}(S_j \cap A) = \mathcal{H}^{n-1}(S_j) \leq \mathcal{H}^{n-1}(S) + \eta\,. \tag{15}$$

provided $j \geq j_2$ for some integer j_2. From (14) and (15) follows that for $j \geq \max\{j_1, j_2\}$:

$$\mathcal{H}^{n-1}(S_j \sim A) \leq \eta\left(3 + 2^{n-1}\|V\|(U)\right)\,. \tag{16}$$

In order to finish the proof of inequality (9) we now associate with the C_i's and ψ_i's, $i = 1, 2, \ldots$, a map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as in Lemma 2.3.2 with $\operatorname{Lip}(\psi) \leq \sqrt{10}$ and

$\psi(y) = \psi_i(y)$ whenever $y \in C_i$ for some $i = 1, 2, \ldots$. We let $\tilde{f} := f \circ \psi$ and we calculate that:

$$\begin{aligned}
\mathcal{H}^{n-1}(S) &= \|V\|(\mathbb{R}^n \sim B) = \lim_{j\to\infty} \|V_j\|(\mathbb{R}^n \sim B) \\
&= \lim_{j\to\infty} \|V_j\|(\mathbb{R}^n) \text{ (because } \mathcal{H}^{n-1}(B) = 0) \\
&= \lim_{j\to\infty} \mathbf{S}(T_j) + \varepsilon^2_{\alpha(j)}\mathbf{M}(T_j)
\end{aligned}$$

which on noticing that $\tilde{f}(y) = y$ whenever y is in a neighborhood of $\operatorname{spt}(\partial T_0)$, whence $\partial \tilde{f}_\# T_j = \partial T_j$, can be bounded by

$$\begin{aligned}
&\le \lim_{j\to\infty} \mathbf{S}\left(\tilde{f}_\# T_j\right) + \varepsilon^2_{\alpha(j)}\mathbf{M}\left(\tilde{f}_\# T_j\right) \\
&\le \lim_{j\to\infty} \mathcal{H}^{n-1}\left(\tilde{f}(S_j)\right) + \left(\sqrt{10}\operatorname{Lip}(f)\right)^{n-1} \lim_{j\to\infty} \varepsilon^2_{\alpha(j)}\mathbf{M}(T_j)
\end{aligned}$$

which, according to Lemma 2.1.1(4), equals

$$= \lim_{j\to\infty} \left(\mathcal{H}^{n-1}\left(\tilde{f}(S_j \cap A)\right) + \mathcal{H}^{n-1}\left(\tilde{f}(S_j \sim A)\right)\right)$$

and to estimate this quantity we refer to the fact that $\psi(A) \subset S$ (recall (11)) as well as to inequality (16):

$$\le \mathcal{H}^{n-1}(f(S)) + \eta\left(\sqrt{10}\operatorname{Lip}(f)\right)^{n-1}\left(3 + 2^{n-1}\|V\|(U)\right).$$

Since $\eta > 0$ is arbitrary, this completes the proof. □

Remark 2.3.5. The results proved so far help to complete the argument of Theorem 2.11 in [16]: "*In a $\mathcal{C}^\infty$ compact 3 dimensional Riemannian manifold every 2 dimensional integral homology class supports a homologically size minimizing rectifiable current.*" In the proof of that Theorem one should replace minimizers (in a homology class) of $\mathbf{S}$ subject to the additional constraint $\mathbf{M}(T) \le k$ by minimizers of $\mathbf{S} + k^{-1}\mathbf{M}$ in order for the associated varifold to be stationary (as in Proposition 2.2.4 above). The $(\mathbf{M}, 0, \infty)$ minimality of the support of the varifold obtained in the limit can then be proved as in our Proposition 2.3.1 and Theorem 2.3.4.

3. Compactness and Existence for the H Mass Plateau Problem

3.1. **Measurability of Slicing and Rectifiable Scans.** In this section we supplement [9, 4.3] with some results about slicing and projections which motivate the definition of scans. The measurability of slicing was not addressed in [9, 4.3].

Lemma 3.1.1. *If $T \in \mathbf{F}_{m,K}(\mathbb{R}^n)$ with $K \subset \mathbb{R}^n$ compact, $k \in \{1, \ldots, m\}$, and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is locally Lipschitzian, then the slice $\langle T, f, \cdot\rangle$ is $\mathcal{L}^k$ measurable (with respect to the topology generated by $\mathbf{F}_K$).*

Proof. Following [9, 4.3.1], we let

$$\phi_{y,j} := \boldsymbol{\alpha}(k)^{-1} r_j^{-k}\,\Omega \wedge \mathbf{B}(y, r_j)$$

where $y \in \mathbb{R}^k$, $\Omega = DY_1 \wedge \ldots \wedge DY_k$ is the standard volume form in $\mathbb{R}^k$ and r_j, $j = 1, 2, \ldots$, is some *fixed* sequence of positive real numbers decreasing to 0. For $\mathcal{L}^k$ almost every $y \in \mathbb{R}^k$ the following conditions are met:

(a) the weak limit of $T \llcorner p^{\#}\phi_{y,j}$ exists (in which case this limit is denoted by $\langle T, p, y\rangle$ in accordance with [9, 4.3.1]);
(b) $\langle T, p, y\rangle \in \mathbf{F}_{m-k,K}(\mathbb{R}^n)$;
(c) $\mathbf{F}_K(T \llcorner p^{\#}\phi_{y,j} - \langle T, p, y\rangle) \to 0$ as $j \to \infty$;
(d) $\Theta^n(\mu, y) < \infty$ where μ is the Radon measure defined in [9, 4.3.1] in the middle of p. 437.

We observe that the conjuction of conditions (a), (b) and (c) above is equivalent to condition (c) which itself is equivalent to $T \llcorner p^{\#}\phi_{y,j}$, $j = 1, 2, ...$, being an $\mathbf{F}_K$ Cauchy sequence. It thus suffices to prove that each of the mappings

$$y \in \mathbb{R}^k \longmapsto T \llcorner p^{\#}\phi_{y,j} \in \mathbf{F}_{m-k,K}(\mathbb{R}^n)$$

is $\mathcal{L}^k$ measurable. For this, we observe that if $C \subset \mathcal{D}^{m-k}(\mathbb{R}^n)$ is a countable $\mathbf{F}_K$ dense subset of $\mathcal{D}^{m-k}(\mathbb{R}^n)$ and $Q \in \mathbf{F}_{m-k,K}(\mathbb{R}^n)$, then

$$\mathbf{F}_K(Q - T \llcorner p^{\#}\phi_{y,j}) = \sup\left\{Q(\psi) - (T \llcorner p^{\#}\phi_{y,j})(\psi) : \psi \in C \text{ and } \mathbf{F}_K(\psi) \leq 1\right\}.$$

Thus we will be done if we show that for each ψ the function $y \mapsto (T \llcorner p^{\#}\phi_{y,j})(\psi)$ is continuous. But this follows from the fact [9, 4.1.18] that there is an $\mathcal{L}^k$ summable k vectorfield $\xi_{p,\psi}$ on $\mathbb{R}^k$ so that

$$\begin{aligned}(T \llcorner p^{\#}\phi_{y,j})(\psi) &= (-1)^{k(m-k)}\left(p_{\#}(T \llcorner \psi)\right)(\phi_{y,j}) \\ &= (-1)^{k(m-k)}\left(\mathcal{L}^k \wedge \xi_{p,\psi}\right)(\phi_{y,j}) \\ &= (-1)^{k(m-k)}\boldsymbol{\alpha}(k)^{-1} r_j^{-k}\int_{\mathbf{B}(y,r_j)} \langle \xi_{p,\psi}, \Omega\rangle d\mathcal{L}^k .\end{aligned}$$

□

We will study slices $\langle T, p, y\rangle$ corresponding to an *orthogonal projection* $p : \mathbb{R}^m \longrightarrow \mathbb{R}^k$. Recall from [9, 1.7.4,2.7.16] the space $\mathbf{O}^*(n, k)$ of all orthogonal projections of $\mathbb{R}^n$ onto $\mathbb{R}^k$ with its $\mathbf{O}(n)$ invariant measure $\boldsymbol{\theta}^*_{n,k}$. It is a compact Riemannian manifold of dimension $N = \frac{n(n-1)-(n-k)(n-k-1)}{2}$ which admits a orienting unit N vectorfield $\vec{O}$ making it into a (multiplicity one) rectifiable current $[\![\mathbf{O}^*(n, k)]\!]$. To consider the variation in p of the slice $\langle T, p, y\rangle$, we first note how, by [9, 4.3.2(6)], the theory of slicing extends to maps to an oriented Riemannian manifold and then use the following handy formula:

Lemma 3.1.2. *If $T \in \mathbf{F}_{m,K}(\mathbb{R}^n)$ for some compact set $K \subset \mathbb{R}^n$ and $p \in \mathbf{O}^*(n, k)$, then, for $\mathcal{L}^k$ almost every $y \in \mathbb{R}^k$,*

$$\langle T, p, y\rangle = \Pi_{\#}\langle T \times [\![\mathbf{O}^*(n, k)]\!], \Psi, (y, p)\,\rangle$$

where

$$\Pi(x, q) = x \quad and \quad \Psi(x, q) = (q(x), q) \quad for \quad (x, q) \in \mathbb{R}^n \times \mathbf{O}^*(n, k)\,.$$

Proof. First we consider the special case when $T = (\mathcal{H}^m \llcorner R \wedge \theta\xi)$ with R being a convex region in an affine m plane transverse to p and with the orienting unit m vectorfield ξ and the density function θ being constant on R. Then $\dim(R \cap p^{-1}\{y\}) \leq m - k$ for all $y \in \mathbb{R}^k$, and Ψ is transverse to $R \times \mathbf{O}^*(n, k)$. The slice $\langle T, p, y\rangle$ is carried by the convex $m - k$ dimensional set $R \cap p^{-1}\{y\}$ which is the Π image of the set

$$(R \cap p^{-1}\{y\}) \times \{p\} \;=\; (R \times \mathbf{O}^*(n, k)) \cap \Psi^{-1}\{(y, p)\}\,,$$

the carrying set of the slice $\langle T \times [\![\mathbf{O}^*(n,k)]\!], \Psi, (y,p) \rangle$. The density of both slices is constantly θ, and the constant unit orienting $m-k$ vectors for the slices correspond under $\Lambda_{m-k} D\Pi$. See [9, 4.3.8]. Thus the formula holds for such a T.

By linearity it then holds for almost every m dimensional real polyhedral chain. By the density [9, 4.1.23] of such chains and [9, 4.3.1] it is finally true for any m dimensional real flat current T and $\mathcal{L}^k$ almost every $y \in \mathbb{R}^k$. □

Proposition 3.1.3. *For each $T \in \mathbf{F}_{m,K}(\mathbb{R}^n)$ and $k \in \{1, \dots, m\}$ the map*

$$\mathcal{S}(T) : \mathbf{O}^*(n,k) \times \mathbb{R}^k \longrightarrow \mathbf{F}_{m-k,K}(\mathbb{R}^n)\,, \quad \mathcal{S}(T)(p,y) = \langle T, p, y \rangle\,,$$

for all $p \in \mathbf{O}^*(n,k)$ and for $\mathcal{L}^k$ almost every $y \in \mathbb{R}^k$, is $\boldsymbol{\theta}^*_{n,k} \times \mathcal{L}^k$ measurable.

Proof. Combine Lemma 3.1.1, Lemma 3.1.2, and the flat continuity of $\Pi_{\#}$. □

We next observe how to recover a flat chain from its slices by coordinate projections [9, 1.7.4].

Lemma 3.1.4. *If $e_1, ..., e_n$ and $\omega_1, ..., \omega_n$ are dual bases of $\mathbb{R}^n$, then for every $m, k \in \{1, ..., n\}$ with $m \geq k$ and $\phi \in \wedge_m \mathbb{R}^n$ one has*

$$\phi = (-1)^{k(m-k)} \binom{m}{k}^{-1} \sum_{\lambda \in \Lambda(n,k)} \omega_\lambda \wedge (e_\lambda \,\lrcorner\, \phi)\,.$$

Proof. It is obviously sufficient to check it for each $\phi = \omega_\mu$, $\mu \in \Lambda(n,m)$. According to [9, 1.5.2], $e_\lambda \,\lrcorner\, \omega_\mu = 0$ if $\mathrm{im}(\lambda) \not\subset \mathrm{im}(\mu)$ and $e_\lambda \,\lrcorner\, \omega_\mu = (-1)^M \omega_\nu$ if $\mathrm{im}(\lambda) \subset \mathrm{im}(\mu)$, where $\nu \in \Lambda(n, m-k)$ is such that $\mathrm{im}(\lambda) \cup \mathrm{im}(\nu) = \mathrm{im}(\mu)$ and M is the number of pairs $(i,j) \in \mathrm{im}(\lambda) \times \mathrm{im}(\nu)$ with $i < j$. In turn, $\omega_\lambda \wedge (e_\lambda \,\lrcorner\, \omega_\mu) = 0$ if $\mathrm{im}(\lambda) \not\subset \mathrm{im}(\mu)$, whereas if $\mathrm{im}(\lambda) \subset \mathrm{im}(\mu)$ then, with the same notation as above,

$$\begin{aligned}
\omega_\lambda \wedge (e_\lambda \,\lrcorner\, \omega_\mu) &= (-1)^M \omega_\lambda \wedge \omega_\nu \\
&= (-1)^M (-1)^{k(m-k)} \omega_\nu \wedge \omega_\lambda \\
&= (-1)^{2M} (-1)^{k(m-k)} \omega_\mu\,.
\end{aligned}$$

Then the conclusion then follows because

$$\mathrm{card}\,(\Lambda(n,k) \cap \{\lambda : \mathrm{im}(\lambda) \subset \mathrm{im}(\mu)\}) = \binom{m}{k}.$$

□

Proposition 3.1.5. *Let $T \in \mathbf{F}_{m,K}(\mathbb{R}^n)$ for some compact set $K \subset \mathbb{R}^n$, and $\phi \in \mathcal{D}^m(\mathbb{R}^n)$, $k \in \{1, ..., m\}$. Then*

$$T(\phi) = (-1)^{k(m-k)} \binom{m}{k}^{-1} \sum_{\lambda \in \Lambda(n,k)} \int_{\mathbb{R}^k} \langle T, \mathbf{p}_\lambda, y \rangle (e_\lambda \,\lrcorner\, \phi) d\mathcal{L}^k(y)$$

where $\mathbf{p}_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, $\mathbf{p}_\lambda(x) := \sum_{i=1}^k \omega_{\lambda(i)}(x) \mathbf{e}_i$.

Proof. Note that $\omega_\lambda = \mathbf{p}_\lambda^\# \Omega$ where Ω is again the standard volume form on $\mathbb{R}^k$. By 3.1.4 and [9, 4.3.2(1)] with $\Phi \equiv 1$ and $\psi = e_\lambda \lrcorner \phi$,

$$\begin{aligned}(-1)^{k(m-k)}\binom{m}{k} T(\phi) &= \sum_{\lambda \in \Lambda(n,k)} T\left(\omega_\lambda \wedge (e_\lambda \lrcorner \phi)\right) \\ &= \sum_{\lambda \in \Lambda(n,k)} [T \llcorner \omega_\lambda](e_\lambda \lrcorner \phi) \\ &= \sum_{\lambda \in \Lambda(n,k)} [T \llcorner \mathbf{p}_\lambda^\# \Omega](e_\lambda \lrcorner \phi) \\ &= \sum_{\lambda \in \Lambda(n,k)} \int_{\mathbb{R}^k} \langle T, \mathbf{p}_\lambda, y\rangle (e_\lambda \lrcorner \phi) d\mathcal{L}^k(y)\,.\end{aligned}$$

□

We also observe that the condition that a flat current have boundary zero may also be recovered by a corresponding condition on its slices, even when the slices are only 0 dimensional.

Proposition 3.1.6. *If $m \in \{1, ..., n\}$, $T \in \mathbf{F}_m(\mathbb{R}^n)$, and $k \in \{1, ..., m-1\}$, then the following are equivalent:*

(1) $\partial T = 0$;
(2) $\partial \langle T, p, y\rangle = 0$ *for* $\boldsymbol{\theta}^*_{n,k} \times \mathcal{L}^k$ *almost every* $(p, y) \in \mathbf{O}^*(n,k) \times \mathbb{R}^k$;
(3) $\langle T, p, y\rangle(\mathbf{1}) = 0$ *for* $\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m$ *almost every* $(p, y) \in \mathbf{O}^*(n,m) \times \mathbb{R}^m$.

Proof. (1) ⇒ (2) follows from the formula [9, 4.3.1] $\partial\langle T, p, y\rangle = (-1)^k \langle \partial T, p, y\rangle$.

(2) ⇒ (1) because Fubini's Theorem shows that, for a.e. rotation T' of T and $\mathcal{L}^k$ a.e. $y \in \mathbb{R}^k$, $\partial\langle T', \mathbf{p}_\lambda, y\rangle = 0$ for all $\lambda \in \Lambda(n,k)$, and we may apply 3.1.6 to see that $\partial T'$, and hence ∂T vanishes.

The equivalence of (1) and (3) in case $m = n$ follows from the Constancy Theorem ([9, 4.1.10]) and the fact that top dimensional flat chains correspond to $\mathcal{L}^n$ summable n vectorfields.

Assume now that $m \le n-1$. Then (1) implies that $T = \partial S$ for some $S \in \mathbf{F}_{m+1}(\mathbb{R}^n)$ ([9, 4.4.6]) so that (3) follows from the relation

$$\langle T, p, y\rangle(\mathbf{1}) = (-1)^m \partial\langle S, p, y\rangle(\mathbf{1}) = 0\,,$$

valid for $\mathcal{L}^m$ almost every $y \in \mathbb{R}^m$.

Finally we prove the implication (3)⇒(1). For each $\varepsilon > 0$ we define the mollified current T_ε as in [9, 4.1.2]. According to [9, 4.1.18] we see that $T_\varepsilon = \mathcal{L}^m \wedge \xi_\varepsilon$ for some $\xi_\varepsilon \in \mathcal{D}(\mathbb{R}^n, \wedge_m \mathbb{R}^n)$, $\partial T_\varepsilon = -\mathcal{L}^m \wedge \operatorname{div} \xi_\varepsilon$, and hence $T_\varepsilon \in \mathbf{N}_m(\mathbb{R}^n)$. We first claim that $\langle T_\varepsilon, p, y\rangle(\mathbf{1}) = 0$ for every $(p, y) \in \mathbf{O}^*(n,m) \times \mathbb{R}^m$. Indeed,

$$\begin{aligned}\langle T_\varepsilon, p, y\rangle(\mathbf{1}) &= \left\langle \int_{\mathbb{R}^n} \Phi_\varepsilon(-z)(\boldsymbol{\tau}_{x\,\#} T)(\mathbf{1}) d\mathcal{L}^n(z), p, y \right\rangle \\ &= \int_{\mathbb{R}^n} \Phi_\varepsilon(-z) \langle \boldsymbol{\tau}_{z\,\#} T, p, y\rangle(\mathbf{1}) d\mathcal{L}^n(z) \quad \text{([9, 4.3.1] and Fubini's)} \\ &= \int_{\mathbb{R}^n} \Phi_\varepsilon(-z) \langle T, p, y + p(z)\rangle(\mathbf{1}) d\mathcal{L}^n(z) \quad \text{([9, 4.3.2(7)])} \\ &= 0 \qquad \text{(by hypothesis)}\,.\end{aligned}$$

Next we observe that for each $(p, y) \in \mathbf{O}^*(n, m) \times \mathbb{R}^m$ the following follows from [9, 4.3.1]:

$$
\begin{aligned}
\langle T_\varepsilon, p, y\rangle(\mathbf{1}) &= \lim_{r\to 0^+} \boldsymbol{\alpha}(m)^{-1} r^{-m} (T_\varepsilon \llcorner p^{\#}[\mathbf{B}\,(y,r) \wedge \Omega])(\mathbf{1}) \\
&= \lim_{r\to 0^+} \boldsymbol{\alpha}(m)^{-1} r^{-m} \int_{p^{-1}\mathbf{B}(y,r)} \langle \xi_\varepsilon, p^{\#}\Omega\rangle d\mathcal{L}^n \\
&= \int_{p^{-1}\{y\}} \langle \xi_\varepsilon, p^{\#}\Omega\rangle d\mathcal{H}^{n-m}\,.
\end{aligned}
$$

We denote by $\widehat{\xi_\varepsilon}$ the Fourier transform of ξ_ε. Since

$$
\operatorname{div} \xi_\varepsilon = \sum_{j=1}^{n} D_j \xi_\varepsilon \llcorner DX_j
$$

we see that for each $u \in \mathbb{R}^n \sim \{0\}$ one has

$$
\begin{aligned}
\widehat{(\operatorname{div} \xi_\varepsilon)}(u) &= \sum_{j=1}^{n} \widehat{(D_j \xi_\varepsilon)}(u) \llcorner DX_j \\
&= \sum_{j=1}^{n} DX_j(u) \widehat{\xi_\varepsilon}(u) \llcorner DX_j \\
&= \|u\| \widehat{\xi_\varepsilon}(u) \llcorner DX_{u_1}
\end{aligned}
$$

where $u_1 := \|u\|^{-1}u$ and $X_u \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ is defined by $X_{u_1}(x) := x \bullet u_1$. We choose $u_2, ..., u_n$ so that $\{u_1, u_2, ..., u_n\}$ is an orthonormal family in $\mathbb{R}^n$. Fix $\lambda \in \Lambda(n, m-1)$ and observe that on letting $p := DX_{u_1} e_1 + \sum_{i=2}^{m} DX_{u_{\lambda(i)}} \mathbf{e}_i \in \mathbf{O}^*(n, m)$ the following holds true:

$$
\begin{aligned}
\langle \widehat{\xi_\varepsilon}(u) \llcorner DX_{u_1}, DX_{u_{\lambda(1)}} \wedge ... \wedge DX_{u_{\lambda(m-1)}}\rangle &= \langle \widehat{\xi_\varepsilon}(u), dp\rangle \\
&= \langle \widehat{\xi_\varepsilon}(u), p^{\#}\Omega\rangle \\
&= \int_{\mathbb{R}^n} \langle \xi_\varepsilon(x), p^{\#}\Omega\rangle \exp[-i\,(x \bullet u)] d\mathcal{L}^n(x) \\
&= \int_{\mathbb{R}^m} \exp[-i\, y_1] d\mathcal{L}^m(y) \int_{p^{-1}\{y\}} \langle \xi_\varepsilon, p^{\#}\Omega\rangle d\mathcal{H}^{n-m} \\
&= 0\,.
\end{aligned}
$$

We deduce that $\widehat{\operatorname{div} \xi_\varepsilon} = 0$ and, in turn, $\operatorname{div} \xi_\varepsilon = 0$ so that $\partial T_\varepsilon = 0$. Since $T_\varepsilon \to T$ weakly as $\varepsilon \to 0$ and the boundary operator ∂ is weakly continuous, $\partial T = 0$. ☐

For the remainder of the paper we will view currents in terms of their 0 *dimensional slices.* We now define the natural geometric object obtained as the limit of some minimizing sequences for the variational problems to be considered.

Definition 3.1.7. An m dimensional *rectifiable scan* is an $\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m$ measurable map

$$
\mathcal{T} : \mathbf{O}^*(n, m) \times \mathbb{R}^m \longrightarrow \mathbf{I}_{0,K}(\mathbb{R}^n)
$$

(where $K \subset \mathbb{R}^n$ is some compact set) such that there exist

(1) an $\mathcal{H}^m$ measurable $(\mathcal{H}^m, m)$ rectifiable set $R \subset K$,

(2) an $\mathcal{H}^m$ measurable function $\xi : R \longrightarrow \wedge_m \mathbb{R}^n$ such that for $\mathcal{H}^m$ almost every $x \in R$, $\xi(x)$ is one of the two simple, unit m vectors associated to the m dimensional vector space $\operatorname{Tan}^m(\mathcal{H}^m \llcorner R, x)$, and
(3) an $\mathcal{H}^m$ measurable function $\theta : R \longrightarrow \{1, 2, 3, \dots\}$,

giving, for $\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m$ almost every $(p, y) \in \mathbf{O}^*(n, m) \times \mathbb{R}^m$, the formula:

$$\mathcal{T}(p, y) = \sum_{x \in R \cap p^{-1}\{y\}} \operatorname{sign}(\xi(x) \llcorner p^{\#}\Omega)\theta(x)\boldsymbol{\delta}_x \,.$$

Moreover we say that $\mathcal{T}$ is a *scan cycle*, and we write $\boldsymbol{\partial}\mathcal{T} = 0$, provided

$$\mathcal{T}(p, y)(\mathbf{1}) = 0$$

for $\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m$ almost every $(p, y) \in \mathbf{O}^*(n, m) \times \mathbb{R}^m$.

Remark 3.1.8. The above Definition is motivated by the case when $\mathcal{T} = \mathcal{S}(T)$ for some $T \in \mathcal{R}_{m,K}(\mathbb{R}^n)$. In fact one may take $R := \operatorname{set}(\|T\|)$, $\xi = \vec{T}$ and $\theta = \Theta^m(\|T\|, x)$ because [9, 4.3.8] implies that, for each $p \in \mathbf{O}^*(n, m)$ and $\mathcal{L}^m$ almost every $y \in \mathbb{R}^m$,

$$\begin{aligned}\mathcal{S}(T)(p, y) = \langle T, p, y\rangle &= \sum_{x \in R \cap p^{-1}\{y\}} \operatorname{sign}\big[\vec{T}(x) \llcorner \langle \Omega, \wedge^m ap\, Dp_{|R}(x)\rangle\big]\, \theta(x)\boldsymbol{\delta}_x \\ &= \sum_{x \in R \cap p^{-1}\{y\}} \operatorname{sign}\big[\vec{T}(x) \llcorner p^{\#}\Omega\big]\, \theta(x)\boldsymbol{\delta}_x \,.\end{aligned}$$

Also $\boldsymbol{\partial}\mathcal{S}(T) = 0$ if and only if $\partial T = 0$ by Proposition 3.1.6.

A rectifiable scan $\mathcal{T}$ is, for $\boldsymbol{\theta}^*_{n,m}$ almost every fixed $p \in \mathbf{O}^*(n, m)$, actually determined completely by its values $\mathcal{T}(p, y)$ for $\mathcal{L}^m$ almost every $y \in \mathbb{R}^m$. In fact, we will see in the proof of Theorem 3.5.2 how $\mathcal{T}$ is determined by $\mathcal{T}(p, \cdot)$ for any regular projection of R. Here, for an $(\mathcal{H}^m, m)$ rectifiable set R, a projection $p \in \mathbf{O}^*(n, m)$ is called *regular* for R if $\mathcal{H}^m\,(\Sigma_p(R)) = 0$ where

$$\Sigma_p(R) := R \sim \{x \;:\; \operatorname{Tan}^m[(\mathcal{H}^m \llcorner R, x) \text{ is an } m \text{ dimensional subspace} \\ \text{and} \dim p\Big[\operatorname{Tan}^m(\mathcal{H}^m \llcorner R, x) = m\}\,.$$

One may check that $\boldsymbol{\theta}^*_{n,m}$ almost every $p \in \mathbf{O}^*(n, m)$ is a regular projections for R by applying [9, 3.2.22] to the map

$$(p, x) \in \mathbf{O}^*(n, m) \times R^* \longmapsto (p, p(x)) \in \mathbf{O}^*(n, m) \times \mathbb{R}^m \,.$$

Finally we notice that in special case $\mathcal{T} = \mathcal{S}(T)$ for some $T \in \mathcal{R}_m(\mathbb{R}^n)$, we have the integral-geometric identity

$$\boldsymbol{\beta}_1(m, n) \int_{\mathbf{O}^*(n,m) \times \mathbb{R}^m} \mathbf{M}(\mathcal{T}(p, y))\, d(\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m)(p, y) = \mathbf{M}(T) < \infty \,. \tag{17}$$

The finiteness of the above integral is however *not* required in the Definition of a rectifiable scan. In fact, minimizing sequences for the functional described in the next section may have unbounded mass and have limits being rectifiable scans that are not rectifiable currents.

3.2. The H Mass and the H Flat Distance.

Definition 3.2.1. A function $H : [0,\infty) \longrightarrow [0,\infty)$ is called a *concave integrand* whenever it satisfies the following conditions:

(1) $H(0) = 0$;
(2) $H(1) = 1$;
(3) $H(\theta_1) < H(\theta_2)$ for every $0 \le \theta_1 < \theta_2$;
(4) $H(\theta_1 + \theta_2) \le H(\theta_1) + H(\theta_2)$ for every $\theta_1 \ge 0$ and $\theta_2 \ge 0$;
(5) $\lim_{\theta\to\infty} H(\theta) = \infty$.

In fact for most of the first results in this section we would only need to assume a subset of these conditions. For instance condition (1) is critical in the proof of Lemma 3.2.12. Condition (2) is merely a normalization. Condition (4) is required for lower semicontinuity (Lemma 3.2.14) and Condition (5) will ensure suitable compactness. The functions $H(\theta) = \theta^q$, $0 < q \le 1$, provide examples of concave integrands.

Definition 3.2.2. For a concave integrand H and real rectifiable current $T \in \mathcal{D}_m(\mathbb{R}^n)$ with $\mathbf{S}(T) < \infty$, let

$$\mathbf{M}_H(T) := \int_{\mathbb{R}^n} H\left(\Theta^m(\|T\|, x)\right) d\mathcal{H}^m(x)\,.$$

Then, for the new rectifiable current

$$\Upsilon_H(T) := \mathcal{H}^m \llcorner H \circ \Theta^m(\|T\|, \cdot) \wedge \vec{T}\,,$$

one has $\mathbf{M}_H(T) = \mathbf{M}\left(\Upsilon_H(T)\right)$, and $\mathbf{M}_H$ is mass-calculable as in 2.2.1.

Obviously $H(\theta) \le \theta H(2)$ for each $\theta \ge 1$ in case H is a concave integrand. This implies that $\mathbf{M}_H(T) \le H(2)\mathbf{M}(T) + \mathbf{S}(T)$ and, in turn, that $\mathbf{M}_H(T) < \infty$ (hence $\Upsilon_H(T) \in \mathcal{D}_m(\mathbb{R}^n)$) whenever $\mathbf{S}(T) < \infty$. It also implies that $\mathbf{M}_H(T) \le H(2)\mathbf{M}(T)$ in case $T \in \mathcal{R}_m(\mathbb{R}^n)$.

Remark 3.2.3. Unfortunately the symbol Υ_H is clearly not linear and it does not commute with the boundary operator nor with pushing forward as elementary examples show. However there are several properties that will be useful.

Lemma 3.2.4. *Let $T \in \mathcal{D}_m(\mathbb{R}^n)$ be a real rectifiable current with $\mathbf{S}(T) < \infty$ and H be a concave integrand. The following hold true:*

(1) *For any real rectifiable $\tilde{T} \in \mathcal{D}_m(\mathbb{R}^n)$ with $\mathbf{S}(\tilde{T}) < \infty$,*
$$\|\Upsilon_H(T + \tilde{T})\| \le \|\Upsilon_H(T)\| + \|\Upsilon_H \tilde{T}\|\,;$$
(2) *For any Lipschitz map $F : \mathbb{R}^n \longrightarrow \mathbb{R}^\nu$,*
$$\|\Upsilon_H(F_\# T)\| \le \mathrm{Lip}(F)^m F_\# \|\Upsilon_H(T)\|\,;$$
(3) *For any Lipschitz $G : \mathbb{R}^n \longrightarrow \mathbb{R}^\nu$ which is one-to-one $\mathcal{H}^m$ a.e. on* $\mathrm{set}\|T\|$,
$$\Upsilon_H(G_\# T) = G_\# \Upsilon_H(T)\,;$$
(4) *For any $\tau > 0$,*
$$\Upsilon_H([\![0,\tau]\!] \times T) = [\![0,\tau]\!] \times \Upsilon_H(T)\,;$$
(5) *For any $p \in \mathbf{O}^*(n,k)$ with $k \in \{1, 2, \ldots, m\}$ and $\mathcal{L}^k$ a.e. $y \in \mathbb{R}^k$,*
$$\Upsilon_H\langle T, p, y\rangle \;=\; \langle \Upsilon_H T, p, y\rangle\,.$$

Proof. To verify (1) note that the inequality $\|T+\tilde{T}\| \leq \|T\| + \|\tilde{T}\|$ implies the pointwise inequality $\Theta^m(\|T+\tilde{T}\|,\cdot) \leq \Theta^m(\|T\|,\cdot) + \Theta^m(\|\tilde{T}\|,\cdot)$ is true $\|T\|+\|\tilde{T}\|$ almost everywhere, and by 3.2.1(4),

$$\begin{aligned}\|\Upsilon_H(T+\tilde{T})\| &= \mathcal{H}^m \wedge H \circ \Theta^m(\|T+\tilde{T}\|,\cdot)\\ &\leq \mathcal{H}^m \wedge H \circ \Theta^m(\|T\|,\cdot) + \mathcal{H}^m \wedge H \circ \Theta^m(\|\tilde{T}\|,\cdot)\\ &= \|\Upsilon_H(T)\| + \|\Upsilon_H(\tilde{T})\|\,.\end{aligned}$$

For (2) we recall from [9, 4.1.30] that

$$\Theta^m(\|F_\# T\|, y) \leq \sum_{x\in F^{-1}\{y\}} \Theta^m(\|T\|, x) \tag{18}$$

for $\mathcal{H}^m$ almost every $y \in \operatorname{set}(\|f_\# T\|)$. For a Borel $E \subset \mathbb{R}^\nu$, let $E^* := E \cap \operatorname{set}(\|F_\# T\|)$ so that $F^{-1}E^* \subset F^{-1}\operatorname{set}(\|F_\# T\|) \subset \operatorname{set}(\|T\|)$ is $(\mathcal{H}^m, m)$ rectifiable. Whence

$$\begin{aligned}\|\Upsilon_H(F_\# T)\|(E) &= \|\Upsilon_H(F_\# T)\|(E^*)\\ &= \int_{E^*} H\left[\Theta^m(\|F_\# T\|, y)\right] d\mathcal{H}^m(y)\\ &\leq \int_{E^*} \sum_{x\in F^{-1}\{y\}} H\left[\Theta^m(\|T\|, x)\right] d\mathcal{H}^m(y) \qquad \text{(thanks to (18))}\\ &= \int_{F^{-1}E^*} H\left[\Theta^m(\|T\|, x)\right] \operatorname{ap} J_m\left(F|\operatorname{set}\|T\|\right)(x) d\mathcal{H}^m(x) \;\; \text{([9, 3.2.22])}\\ &\leq \operatorname{Lip}(F)^m \|\Upsilon_H(T)\|(F^{-1}E)\end{aligned}$$

which proves (2). Identities (3), (4), and (5) follow from respectively, [9, 4.1.30], [9, 4.1.8], and [9, 4.3.8]. □

Corollary 3.2.5. *In the notation of 3.2.4,*

1. $\mathbf{M}_H(T+\tilde{T}) \leq \mathbf{M}_H(T) + \mathbf{M}_H(\tilde{T})$ *;*
2. $\mathbf{M}_H(F_\# T) \leq \operatorname{Lip}(F)^m \mathbf{M}_H(T)$ *;*
3. $\mathbf{M}_H(G_\# T) = \int H\left(\Theta^m(\|T\|, x)\right) \operatorname{ap} J_m\left(G|\operatorname{set}\|T\|\right)(x)\, d\mathcal{H}^m x$ *;*
4. $\mathbf{M}_H([\![0,\tau]\!] \times T) = \tau \dot{\mathbf{M}}_H(T)$ *;*
5. $\int_{\mathbb{R}^k} \mathbf{M}_H\langle T, p, y\rangle\, d\mathcal{L}^k y \;\leq\; \mathbf{M}_H(T)$.

Remark 3.2.6. It can happen that $\mathbf{M}_H(T_j) \to 0$ while $\mathbf{M}(T_j) \to \infty$ as $j \to \infty$. Consider for instance $T_j \in \mathbf{I}_1(\mathbb{R})$ defined by

$$T_j := \mathbf{E}^1 \llcorner \sum_{k=j}^{\nu_j} k[\![(k+1)^{-1}, k^{-1}]\!]$$

with ν_j sufficiently large to guarantee that $\lim_{j\to\infty} \mathbf{M}(T_j) = \infty$. Then obviously $\lim_{j\to\infty} \mathbf{M}_H(T_j) = 0$ if $H(\theta) = \theta^q$ for some small positive q. This behavior does not occur in the 0 dimensional case as is discussed in the next Remark.

Remark 3.2.7. *If H is a concave integrand, then*

$$H(\mathbf{M}(T)) \leq \mathbf{M}_H(T) \quad \text{for any } T \in \mathbf{I}_0(\mathbb{R}^n)\,.$$

Indeed $T = \sum_{a\in A} \nu_a \boldsymbol{\delta}_a$ for some finite subset A of $\mathbb{R}^n$ and some integers ν_a. Then $\mathbf{M}(T) = \sum_{a\in A} |\nu_a|$, whereas $\mathbf{M}_H(T) = \sum_{a\in A} H(|\nu_a|)$, hence $H(\mathbf{M}(T)) \leq \mathbf{M}_H(T)$.

Definition 3.2.8. For a compact $K \subset \mathbb{R}^n$, and integral flat current $T \in \mathcal{F}_m(\mathbb{R}^n)$ and a concave integrand H, we define

$$\mathcal{F}_K^H(T) := \inf\Big\{\mathbf{M}_H(R) + \mathbf{M}_H(S) : T = R + \partial S \text{ with } R \in \mathcal{R}_{m,K}(\mathbb{R}^n) \text{ and } S \in \mathcal{R}_{m+1,K}(\mathbb{R}^n)\Big\}.$$

Flat distances of this type were considered for the first time by W.H. Fleming in [12]. It is worth comparing the following three notions of convergence of integral flat chains: $\mathcal{F}_K^H$ convergence, $\mathcal{F}_K$ convergence and weak convergence.

Remark 3.2.9. Obviously $\mathcal{F}_K^H(T) \leq H(2)\,\mathcal{F}_K(T)$ for every $T \in \mathcal{F}_{m,K}(\mathbb{R}^n)$.

Remark 3.2.10. *The fact that* $\lim_{j\to\infty} \mathcal{F}_K^H(T_j) = 0$ *does not necessarily imply that the sequence* $T_1, T_2, \ldots$ *converges weakly to* 0. Indeed, define $T_j \in \mathcal{R}_0(\mathbb{R})$, $j = 1, 2, \ldots$, by $T_j := j\boldsymbol{\delta}_{j^{-1}} - j\boldsymbol{\delta}_{-j^{-1}} = j\partial[\![-j^{-1}, j^{-1}]\!]$. It follows from the Definition that

$$\mathcal{F}_K^H(T_j) \leq \mathbf{M}_H\left(j\partial[\![-j^{-1}, j^{-1}]\!]\right) = \frac{2H(j)}{j},$$

which converges to 0 as j tends to ∞ for instance when $H(\theta) = \theta^q$ for some $0 < q < 1$. On the other hand, if $f \in \mathcal{D}^0(\mathbb{R})$ is such that $f(x) = x$ in a neighborhood of 0, then

$$\lim_{j\to\infty} T_j(f) = \lim_{j\to\infty} j\left(f(j^{-1}) - f(-j^{-1})\right) = 2.$$

Remark 3.2.11. It follows from 3.2.4(4) and the Definition of $\mathcal{F}_K^H$ that

$$\int_{\mathbb{R}^\nu} \mathcal{F}_K^H(\langle T, p, y\rangle)\, d\mathcal{L}^k(y) \;\leq\; \mathcal{F}_K^H(T)$$

whenever $T \in \mathcal{F}_{m,K}(\mathbb{R}^n)$ and $p \in \mathbf{O}^*(n,k)$.

Lemma 3.2.12. *Suppose that* $T \in \mathcal{F}_{m,K}(\mathbb{R}^n)$ *and* H *is a concave integrand. If* $\mathcal{F}_K^H(T) = 0$*, then* $T = 0$.

Proof. Observe that applying Remark 3.2.11 with $p = \mathbf{p}_\lambda$, $\lambda \in \Lambda(n,m)$, together with Proposition 3.1.5 reduces to proving the Lemma only in the particular case when $m = 0$. In this case, we may find, for each $j \in \{1, 2, \ldots\}$, representations $T = R_j + \partial S_j$ corresponding to currents $R_j \in \mathcal{R}_{0,K}(\mathbb{R}^n)$ and $S_j \in \mathcal{R}_{1,K}(\mathbb{R}^n)$ with $\mathbf{M}_H(R_j) + \mathbf{M}_H(S_j) \leq j^{-1}$. But then $R_j = 0$ for $j > H(1)^{-1}$. If $T \neq 0$, then

$$\infty > \operatorname{card}(\operatorname{spt}(T)) \geq 2$$

because $T = \partial S_j$ for each $j > H(1)^{-1}$. We deduce the contradiction

$$\mathbf{M}(S_j) \geq \min\{|a-b| \,:\, a, b \in \operatorname{spt}(T),\, a \neq b\} \;>\; 0.$$

□

For $T, \tilde{T} \in \mathcal{F}_{m,K}(\mathbb{R}^n)$ Lemma 3.2.5(1) clearly implies that $\mathcal{F}_K^H(T+\tilde{T}) \leq \mathcal{F}_K^H(T) + \mathcal{F}_K^H(\tilde{T})$. This together with Lemma 3.2.12 ensures that $(T, \tilde{T}) \longmapsto \mathcal{F}_K^H(T - \tilde{T})$ is indeed a distance on $\mathcal{F}_{m,K}(\mathbb{R}^n)$.

Lemma 3.2.13. *If* $T_1, T_2, \ldots \in \mathcal{F}_{m,K}(\mathbb{R}^n)$, $\lim_{j\to\infty} \mathcal{F}_K^H(T_j) = 0$ *and* $\sup_j \mathbf{N}(T_j) < \infty$*, then* $\lim_{j\to\infty} \mathcal{F}_K(T_j) = 0$.

Proof. The compactness theorem for integral currents [9, 4.2.17] implies that each subsequence $T_{\alpha(1)}, T_{\alpha(2)}, \ldots$ contains a further subsequence $T_{\alpha(\beta(1))}, T_{\alpha(\beta(2))}, \ldots$ converging in flat norm $\mathcal{F}_K$ to some $T_\beta \in \mathcal{F}_{m,K}(\mathbb{R}^n) \cap \mathbf{N}_{m,K}(\mathbb{R}^n) = \mathbf{I}_{m,K}(\mathbb{R}^n)$. It follows from Remark 3.2.9 that

$$\mathcal{F}_K^H(T_\beta) \le \lim_{j\to\infty} \mathcal{F}_K^H(T_\beta - T_{\alpha(\beta(j))}) + \lim_{j\to\infty} \mathcal{F}_K^H(T_{\alpha(\beta(j))}) = 0\,,$$

and hence that $T_\beta = 0$ (Lemma 3.2.12). The Lemma now follows from the arbitrariness of the subsequences. □

Lemma 3.2.14. *Suppose $K \subset \mathbb{R}^n$ is compact, H is a concave integrand, and $T, T_1, T_2, \dots \in \mathcal{R}_{m,K}(\mathbb{R}^n)$. If $\lim_{j\to\infty} \mathcal{F}_K^H(T_j - T) = 0$, then*

$$\mathbf{M}_H(T) \le \liminf_{j\to\infty} \mathbf{M}_H(T_j)\,.$$

Proof. First we consider the case $m = 0$. We may assume that $\liminf_{j\to\infty} \mathbf{M}_H(T_j)$ is finite, and, by passing to a subsequence and recalling 3.2.7, assume also that

$$\sup_j \mathbf{M}(T_j) \le \sup_j H^{-1}\left(\mathbf{M}_H(T_j)\right) < \infty\,. \tag{19}$$

By 3.2.13, $\lim_{j\to\infty} \mathcal{F}_K(T_j - T) = 0$; hence $T_j \to T$ weakly. For each $a \in \operatorname{spt}(T)$ let $r(a) := \frac{1}{2}\operatorname{dist}(a, \operatorname{spt}(T) \sim \{a\})$ and $U_a := \mathbf{U}(a, r(a))$ so that $\mathbf{M}(T \llcorner U_a) = \Theta^m(\|T\|, a)$ because $T = \sum_{a\in\operatorname{spt}(T)} \pm\Theta^0(\|T\|, a)\boldsymbol{\delta}_a$. The weak lower semicontinuity of $\mathbf{M}$ and the fact that all densities are integers imply that

$$\mathbf{M}(T \llcorner U_a) \le \liminf_{j\to\infty} \mathbf{M}(T_j \llcorner U_a)$$

and in turn

$$\begin{aligned}
\mathbf{M}_H(T) &= \sum_{a\in\operatorname{spt}(T)} H[\mathbf{M}(T \llcorner U_a)] \\
&\le \sum_{a\in\operatorname{spt}(T)} \liminf_{j\to\infty} H[\mathbf{M}(T_j \llcorner U_a)] \\
&\le \liminf_{j\to\infty} \sum_{a\in\operatorname{spt}(T)} \mathbf{M}_H(T_j \llcorner U_a) \qquad \text{(Remark 3.2.7)} \\
&\le \liminf_{j\to\infty} \mathbf{M}_H(T_j)\,.
\end{aligned}$$

Suppose now $m \ge 0$. By passing to a subsequence we may assume

$$\lim_{j\to\infty} \mathbf{M}_H(T_j) = \liminf_{j\to\infty} \mathbf{M}_H(T_j)\,.$$

Inasmuch as, by Remark 3.2.11,

$$\int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \mathcal{F}_K^H(\langle T_j - T, p, y\rangle)\, d(\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m)(p,y) \ \le \boldsymbol{\beta}_1(n,m)\mathcal{F}_K^H(T_j - T) \to 0$$

as $j \to \infty$, we may pass to another subsequence to guarantee that, for $\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m$ a.a. $(p,y) \in \mathbf{O}^*(n,m) \times \mathbb{R}^m$,

$$\lim_{j\to\infty} \mathcal{F}_K^H(\langle T_j, p, y\rangle - \langle T, p, y\rangle) = 0\,;$$

hence,

$$\mathbf{M}_H\langle T, p, y\rangle \le \liminf_{j\to\infty} \mathbf{M}_H\langle T_j, p, y\rangle \tag{20}$$

by the case $m = 0$. Also, for any $Q \in \mathcal{R}_{m,K}(\mathbb{R}^n)$ we deduce from [9, 3.2.26; 2.10.15; 4.3.8] the integral-geometric equality

$$\begin{aligned}
\mathbf{M}_H(Q) &= \int_{\mathrm{set}(\|Q\|)} H\left[\Theta^m(\|Q\|, x)\right] d\mathcal{H}^m(x) \\
&= \beta_1^{-1}(n,m) \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \sum_{x\in p^{-1}\{y\}\cap\mathrm{set}(\|Q\|)} H\left[\Theta^m(\|Q\|, x)\right]\, d(\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m)(p,y) \\
&= \beta_1^{-1}(n,m) \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \mathbf{M}_H\langle Q, p, y\rangle\, d(\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m)(p,y)\,.
\end{aligned} \tag{21}$$

Thus it suffices to integrate (20) and use Fatou's Lemma. □

Remark 3.2.15. We need one other elementary remark concerning 0 dimension. For $j = 1, 2, \ldots,$ the subsets

$$\mathbf{I}^j := \{T \in \mathbf{I}_{0,K}(\mathbb{R}^n) : \mathbf{M}(T) \le j\} = \{T \in \mathbf{I}_{0,K}(\mathbb{R}^n) : \mathbf{M}_H(T) \le H(j)\}$$

are $\mathbf{F}_K$, $\mathcal{F}_K$, and $\mathcal{F}_K^H$ closed. For $T = T_+ - T_-$ with $T_\pm \in \mathbf{I}^j$ and $0 < \varepsilon < 1$, the implications

$$\mathbf{F}_K(T) < \varepsilon \Leftrightarrow \mathcal{F}_K(T) < \varepsilon \Rightarrow \mathcal{F}_K^H(T) < \frac{\varepsilon}{H(2)} \Rightarrow \mathbf{F}_K(T) < jH^{-1}(\frac{\varepsilon}{H(2)})$$

show that $\mathbf{F}_K$, $\mathcal{F}_K$, and $\mathcal{F}_K^H$ all induce the same topology on each subspace $\mathbf{I}^j$. Thus $\mathbf{F}_K$ and $\mathcal{F}_K^H$ give $I_{0,K}(\mathbb{R}^n)$ the same Borel subsets (though different topologies). In particular, *a map from a measure space into $I_{0,K}(\mathbb{R}^n)$ will be measurable with respect to $\mathbf{F}_K$ if and only if it is measurable with respect to $\mathcal{F}_K^H$.*

3.3. An *H* Flat Variation Bound for Slicing.

Notation 3.3.1. In this section, we fix a closed ball $K \subset \mathbb{R}^n$, an m dimensional integral current $T \in \mathbf{I}_{m,K}(\mathbb{R}^n)$, and a locally Lipschitz map $f = (f_1, \ldots, f_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ where $k \in \{1, \ldots, m\}$. For $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$, we also let

$$q_i(y) = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k) \in \mathbb{R}^{k-1}\,.$$

For $\mathcal{L}^k$ almost every $y \in \mathbb{R}^k$, $\mathcal{L}^1$ almost every $t \in \mathbb{R}$, and each $i \in \{1, \ldots, k\}$, the formula

$$T \cap f^{-1}[y, y + t\mathbf{e}_i] := \langle T, q_i \circ f, q_i(y)\rangle \llcorner (f_i^{-1})\{[y_i, y_i + t]\}$$

defines, by [9, 4.3.1,4.3.4], an $m - k + 1$ dimensional integral current in $\mathbb{R}^n$. This current has two useful elementary properties. First, by applying the boundary operator and recalling [9, 4.3.1, 4.3.2(6), 4.3.5, 4.3.4, 4.2.1], we obtain, for $\mathcal{L}^k \times \mathcal{L}^1$ almost every $(y,t) \in \mathbb{R}^k \times \mathbb{R}$, the relation

$$\begin{aligned}
&\langle T, f, y + t\mathbf{e}_i\rangle - \langle T, f, y\rangle \\
&= (-1)^{i-1}\left(\langle\langle T, q_i \circ f, q_i(y)\rangle, f_i, y_i + t+\rangle - \langle\langle T, q_i \circ f, q_i(y)\rangle, f_i, y_i-\rangle\right) \\
&= (-1)^i (\partial T) \cap f^{-1}[y, y + t\mathbf{e}_i] - (-1)^i \partial(T \cap f^{-1}[y, y + t\mathbf{e}_i])\,.
\end{aligned} \tag{22}$$

Second, we may use 3.2.4 and [9, 4.3.8(2)] to estimate the H masses. Assuming

$$\theta_T = \Theta^m(\|T\|, \cdot)\,, \quad \theta_{\partial T} = \Theta^m(\|\partial T\|, \cdot)\,,$$

with corresponding rectifiable carrying sets

$$R_T = \{x : \theta_T(x) \neq 0\}\,, \quad R_{\partial T} = \{x : \theta_{\partial T}(x) \neq 0\}\,,$$

we deduce that

$$\mathbf{M}_H(T \cap f^{-1}[y, y + t\mathbf{e}_i]) \leq \int_{R_T \cap f^{-1}[y,y+t\mathbf{e}_i]} H(\theta_T)\, d\mathcal{H}^{m-k+1}$$

$$\mathbf{M}_H\left((\partial T) \cap f^{-1}[y, y + t\mathbf{e}_i]\right) \leq \int_{R_{\partial T} \cap f^{-1}[y,y+t\mathbf{e}_i]} H(\theta_{\partial T})\, d\mathcal{H}^{m-k}\,.$$

Combining these with (22), we obtain the basic distance estimate

$$\begin{aligned}&\mathcal{F}_K^H(\langle T, f, y + t\mathbf{e}_i\rangle - \langle T, f, y\rangle)\\ &\leq \int_{R_{\partial T} \cap f^{-1}[y,y+t\mathbf{e}_i]} H(\theta_{\partial T}) d\mathcal{H}^{m-k} + \int_{R_T \cap f^{-1}[y,y+t\mathbf{e}_i]} H(\theta_T) d\mathcal{H}^{m-k+1}\,.\end{aligned}$$

Applying this estimate to intervals coming from a locally finite partition of the line $L = q_i^{-1}\{q_i(y)\}$, then summing, and finally taking the supremum over almost every such partition, we find that the $\mathcal{F}_K^H$ essential variation [9, 4.5.10] of $\langle T, f, \cdot\rangle$ on L is bounded by

$$\int_{R_{\partial T} \cap f^{-1}L} H(\theta_{\partial T}) d\mathcal{H}^{m-k} + \int_{R_T \cap f^{-1}L} H(\theta_T) d\mathcal{H}^{m-k+1}\,. \tag{23}$$

We can now obtain an MBV bound (see [6, sec.7]) on the slicing function.

Theorem 3.3.2.

$$\int_{\mathbb{R}^k} |D(\phi \circ \langle T, f, \cdot\rangle)| \leq k(\operatorname{Lip} f)^{k-1}\left(\mathbf{M}_H(T) + \mathbf{M}_H(\partial T)\right)$$

for any Lipschitz map $\phi : \left(\mathbf{I}_{m-k,K}(\mathbb{R}^n), \mathcal{F}_K^H\right) \longrightarrow \mathbb{R}$ *with* $\operatorname{Lip}\phi \leq 1$.

Proof. With $\chi_{i,z}(t) = (z_1, \ldots, z_{i-1}, t, z_{i+1}, \ldots, z_k)$, we apply [9, 4.5.9(27)], (23), and the coarea formula [9, 3.2.22(3)] to see that

$$\begin{aligned}
&\int_{\mathbb{R}^k} |D\left(\phi \circ \langle T, f, \cdot\rangle\right)|\\
&\leq \sum_{i=1}^k \int_{\mathbb{R}^{k-1}} (\mathcal{F}_K^H \mathrm{EssVar})\left(\phi \circ \langle T, f, \chi_{i,z}(\cdot)\rangle\right)\, d\mathcal{L}^{k-1}z\\
&\leq \sum_{i=1}^k \int_{\mathbb{R}^{k-1}} (\mathcal{F}_K^H \mathrm{EssVar})\left(\langle T, f,\ \cdot\ \rangle \,|\, q_i^{-1}\{z\}\right)\, d\mathcal{L}^{k-1}z\\
&\leq \sum_{i=1}^k \int_{\mathbb{R}^{k-1}} \Big[\int_{R_{\partial T} \cap (q_i \circ f)^{-1}\{z\}} H(\theta_{\partial T}) d\mathcal{H}^{m-k}\\
&\qquad\qquad + \int_{R_T \cap (q_i \circ f)^{-1}\{z\}} H(\theta_T) d\mathcal{H}^{m-k+1}\Big]\, d\mathcal{L}^{k-1}z\\
&\leq \sum_{i=1}^k \left[\int_{R_{\partial T}} H(\theta_{\partial T}) \mathrm{ap}J_{k-1}(q_i \circ f)\, d\mathcal{H}^{m-1} + \int_{R_T} H(\theta_T) \mathrm{ap}J_{k-1}(q_i \circ f)\, d\mathcal{H}^m\right]\\
&\leq k(\operatorname{Lip} f)^{k-1}\left[\int_{R_{\partial T}} H(\theta_{\partial T})\, d\mathcal{H}^{m-1} + \int_{R_T} H(\theta_T)\, d\mathcal{H}^m\right]\\
&= k(\operatorname{Lip} f)^{k-1}\left(\mathbf{M}_H(\partial T) + \mathbf{M}_H(T)\right)
\end{aligned}$$

□

Remark 3.3.3. In case the target $\mathbb{R}^k$ is replaced by an oriented k dimensional Riemannian manifold X, one still has the MBV estimate

$$\int_X |D(\phi \circ \langle T, f, \cdot \rangle)| \ \leq \ \mathbf{c}(X, f, K)\,(\mathbf{M}_H(T) + \mathbf{M}_H(\partial T))\ .$$

One may take, for example, $\mathbf{c}(X, f, K) = k(\operatorname{Lip} f)^{k-1} \sum_{j=1}^J |\operatorname{Lip} \psi_j||\operatorname{Lip} \psi_j^{-1}|^{k-1}$ whenever $f(K) \subset \cup_{j=1}^J X_j$ and $\psi_j : X_j \longrightarrow \mathbb{R}^k$ are bilipschitz embeddings.

3.4. **A BV Compactness Theorem.**

Theorem 3.4.1. *Suppose X is a k dimensional Riemannian manifold, Y is a weakly separable* [6] *metric space, $M : Y \longrightarrow \mathbb{R}^+$ is lower semicontinuous, and $M^{-1}([0, R])$ is sequentially compact in Y for all $R > 0$. If $f_j : X \longrightarrow Y$ is measurable, and*

$$\int_X M(f_j(x))\, d\mathcal{H}^k x + \int_X |D(\phi \circ f_j)| \ \leq \Lambda \ < \infty\ ,$$

for all $j = 1, 2, \ldots$ and maps $\phi : Y \longrightarrow \mathbb{R}$ with $\operatorname{Lip} \phi \leq 1$, *then some subsequence f_{j^*} converges pointwise $\mathcal{H}^k$ a.e. to a function $f : X \longrightarrow Y$ with*

$$\int_X [M(f(x))]\, d\mathcal{H}^k x + \int_X |D(\phi \circ f)| \ \leq \Lambda$$

for all such ϕ. Moreover,

$$\lim_{j \to \infty} \int_K \operatorname{dist}_Y (f_{j^*}(x), f(x))^p \ d\mathcal{H}^k x \ = \ 0$$

for all compact $K \subset X$ and all $p \in [1, \frac{k}{k-1}]$.

Proof. Since X admits bi-Lipschitz coordinates locally we may assume, for notational simplicity, that X is an open cube in $\mathbb{R}^k$ with $\mathcal{H}^k(B) = 1$ and $\operatorname{dist}_X(x, w) = |x - w|$ for $x, w \in X$.

Also, by the isometric embedding into ℓ^∞ of a weakly separable metric space [6],1.1, we may also assume that $Y \subset \ell^\infty$ with

$$\operatorname{dist}_Y(y, z) \ = \ \|y - z\|_\infty \ = \ \sup_i |y^i - z^i|$$

for $y = (y^1, y^2, \ldots)$, $z = (z^1, z^2, \ldots) \in \ell^\infty$.

The hypothesis implies that, for each $i \in \{1, 2, \ldots\}$, the ith component f_j^i of f_j satisfies

$$\int_X |Df_j^i| \ \leq \ \Lambda\,.$$

To apply a standard BV compactness theorem, we also need some control on the functions f_j^i. To get this, we first use the precompactness hypothesis to verify that

$$N_R \ := \ \sup\{\|y\|_\infty \ : \ y \in Y\,,\ M(y) \leq R\} \ < \ \infty$$

for each positive $R < \infty$. Thus,

$$\mathcal{H}^k\{x \in X \ : \ \|f_j(x)\|_\infty > N_{2\Lambda}\} \ \leq \ \frac{1}{2}\ ,$$

and any median t_j^i [9, 4.5.9(18] of f_j^i on X has absolute value bounded by $N_{2\Lambda}$. Thus, by [9, 4.5.9(18)],

$$\int_X |f_j^i|\, d\mathcal{H}^k \ \leq \ |t_j^i| \ + \ \int_X |f_j^i - t_j^i|\, d\mathcal{H}^k \ \leq \ N_{2\Lambda} + C \int_X |Df_j^i| \ \leq \ N_{2\Lambda} + C\Lambda\,.$$

We can now apply BV compactness [13, I,p.336] to f_j^i or any of its subsequences. With the Cantor diagonal trick, we find a single subsequence j' and single $\mathcal{H}^k$ null subset Z' of X so that, for each i, the sequence $f_{j'}^i$ converges pointwise on $X \sim Z'$ (and strongly in L^p for $p \in [1, \frac{k}{k-1}))$ to a BV function f^i.

However, for the desired convergence at each $x \in X \sim Z'$ of the vectors $f_j(x) = \big(f_j^1(x), f_j^2(x), \dots\big)$, we still need to get uniform convergent rates for the sequences $f_j^i(x)$ *independent of* i and also to show that the ℓ^∞ limit $f(x) = (f^1(x), f^2(x), \dots)$ is actually a point in Y. To achieve this, we will find one more subsequence (j^*) of (j') along with another $\mathcal{H}^k$ null subset Z^* of X so that we have the additional *pointwise* bound

$$\sup_j M\left(f_{j^*}(x)\right) \;<\; \infty\,.$$

for every $x \in X \sim Z^*$. To obtain (j^*) and Z^*, we first choose, by the argument in the next paragraph, a Borel subset X_1 of X with $\mathcal{H}^k(X_1) \geq \frac{1}{2}\mathcal{H}^k(X)$ along with a subsequence (j'') of (j') so that

$$\tau_1 \;:=\; \sup_{x\in X_1}\sup_j M\Big(f_{j''}(x)\Big) \;<\; \infty\,.$$

Then we repeat with X replaced by $X \sim X_1$ to get a Borel subset X_2 of $X \sim X_1$ with $\mathcal{H}^k(X_2) \geq \frac{}{2}\mathcal{H}^k(X_1)$ along with a subsequence (j''') of (j'') giving another bound τ_2 for $M\Big(f_{j'''}\Big)$ on X_2. Continuing, we finally let $Z^* = X \sim \cup_{\ell=1}^\infty X_\ell$ and let (j^*) be the diagonal sequence so that, for each $x \in X \sim Z^*$, x belongs to some X_ℓ, and we have the bound

$$\sup_j M\left(f_{j^*}(x)\right) \;\leq\; \max\{M\left(f_{1^*}(x)\right), \dots, M\left(f_{\ell^*}(x)\right), \tau_\ell\} \;<\; \infty\,.$$

To find (j'') and X_1, we may use a dyadic-cube, Calderon-Zygmund construction with the uniformly integrable functions $M\left(f_{j^*}^i(x)\right)$. Referring to [13, p.188], we first choose the parameter $\tau \geq 2^{2m}\Lambda$ so that, at each stage, each function h_j has average τ over at most one of the 2^k subcubes. Starting with $h_{n_1(j)} = M\left(f_{j^*}^i(x)\right)$, one chooses consecutive subsequences $h_{n_1(j)}, h_{n_1(j)}, \dots$ and cubes $Q_1, Q_2, \dots$ so that, for each $\ell = 1, 2, \dots$, the averages,

$$\mathcal{H}^k(Q)^{-1}\int_Q h_{n_k\ell(j)}\, d\mathcal{H}^k \quad for:\; j = 1, 2, \dots\;,$$

either are all $< \tau$ *over each of the* 2^k *subcubes* Q (and we let $Q_\ell = \emptyset$)
or are all $\geq \tau$ *over precisely one subcube* Q_ℓ.

Taking the diagonal subsequence $(j'') = (n_j(j))$ and $X_1 = X \sim \cup_{\ell=1}^\infty Q_\ell$ then gives the uniform bound $M\Big(f_{j''}(x)\Big) \leq \tau$ on X_1 by differentiation theory.

With the final subsequence (j^*) and null set Z^* now in hand, we see that, for each $x \in X \sim (Z' \cup Z^*)$, the bound on $M\left(f_{j^*}(x)\right)$ implies, by hypothesis, that the sequence $f_{j*}(x)$ is ℓ^∞ sequentially compact in Y. Thus any of its subsequences contains a subsequence convergent in ℓ^∞ to some point of Y. Let

$$\lim_{j^{**}\to\infty} f_{j^{**}}(x) \;=\; z \;=\; (z_1, z_2, \dots) \;\in\; Y$$

be any such limit of such a convergent subsequence j^{**} of j^*. But this limit is uniquely determined by our earlier convergences,

$$z^i \;=\; \lim_{j\to\infty} f^i_{j^{**}}(x) \;=\; \lim_{j\to\infty} f^i_{j'}(x) \;=\; f^i(x) \quad for:\; i=1,2,\ldots .$$

We conclude that for our specific sequence (j^*),

$$\lim_{j>\infty} \|f_{j^*}(x) - f(x)\|_\infty \;=\; 0 \quad and: \quad f(x)\in Y$$

for all $x \in X \sim (Z' \cup Z^*)$.

The integral estimate for the limit function f next follows from the lower semicontinuity hypothesis, Fatou's Lemma, and BV lower semicontinuity.

Finally, to establish the L^p convergence, we observe that the scalar functions $g_{j^*}(x) := \|f_{j^*}(x) - f(x)\|_\infty$ satisfy

$$\int_X \sup_{w\in X 0<\operatorname{dist}_X(x,w)<1} \frac{|g_{j^*}(x) - g_{j^*}(w)|}{|x-w|}\, d\mathcal{H}^k x \;<\; 2\Lambda$$

by the triangular inequality. As before, this implies that $\sup_j \int_X |Dg_{j^*}| < \infty$. Also

$$\mathcal{H}^k\{x\in X \;:\; |g_{j^*}(x)| > N_{4\Lambda}\} \;\le\; \frac{1}{2}\;,$$

which allows again, by [9, 4.5.9(18)], use of BV compactness to get, for any subsequence of g_{j^*}, strong convergence in L^p for all $p \in [1, \frac{k}{k-1})$ of some subsequence. Since the limit of any such subsequence is necessarily 0 by the pointwise a.e. convergence, we conclude the L^p convergence of the original sequence g_{j^*} to 0, and the proof is complete. □

3.5. Existence of *H* Mass Minimizing Rectifiable Scans.

Lemma 3.5.1. *Let $m \in \{1,\ldots,n-1\}$, $K \subset \mathbb{R}^n$ compact, $T \in \mathcal{R}_{m,K}(\mathbb{R}^n)$, $W \in \mathbf{G}(n,m)$, $r>0$, $0<h<\frac{1}{2}$, $x\in\mathbb{R}^n$ and $p_0, p_1 \in \mathbf{O}^*(n,m)$. Assume the following conditions are met:*

1. $\operatorname{spt}(T) \cap \mathbf{B}(x,r) \subset \mathbf{B}(x+W, rh)$*;*
2. $\operatorname{spt}(\partial T)\cap \mathbf{B}(x,r) = \emptyset$*;*
3. $p_i^{-1}\{p_i(x)\} \cap \operatorname{Bdry}\mathbf{B}(x,r) \cap \mathbf{B}(x+W,2hr) = \emptyset$ *for* $i=0,1$*;*
4. $p_i(x)$ *is a Lebesgue point of the map*

$$y\in\mathbb{R}^m \longmapsto \langle T, p_i, y\rangle \in \mathbf{F}_{0,K}(\mathbb{R}^n) \;\; for \;\; i=0,1\,.$$

Then

$$\operatorname{sign}(\zeta \llcorner p_0^{\#}\Omega)\,\langle T, p_0, p_0(x)\rangle(\mathbf{1}_{\mathbf{B}(x,r)}) = \operatorname{sign}(\zeta\llcorner p_1^{\#}\Omega)\,\langle T,p_1,p_1(x)\rangle(\mathbf{1}_{\mathbf{B}(x,r)})$$

for any simple unit m vector $\zeta \in \wedge_m\mathbb{R}^n$ associated with W.

Proof. Let π be the nearest point projection of $\mathbb{R}^n$ onto the affine m plane $x+W$, $P = \pi_\#(T\llcorner\mathbf{B}(x,r))$, and $h : \mathbb{R}^\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the affine homotopy between the identity and π, that is,

$$h(t,z) = (1-t)z + t\pi(z) \;\; \text{for} \;\; (t,z)\in\mathbb{R}^\times\mathbb{R}^n\,.$$

The homotopy formula [9, 4.1.9] gives that

$$P - T = \partial h_\#\left([0,1]\times(T\llcorner\mathbf{B}(x,r))\right) + h_\#\left([0,1]\times\partial(T\llcorner\mathbf{B}(x,r))\right)\,. \qquad (24)$$

Also since

$$\begin{aligned}&\pi^{-1}\mathbf{B}(x,r/2)\cap \operatorname{spt}\partial(T\llcorner\mathbf{B}(x,r))\\ &\subset \pi^{-1}\mathbf{B}(x,r/2)\cap \operatorname{Bdry}\mathbf{B}(x,r)\cap\mathbf{B}(x+W,hr)=\emptyset\,,\end{aligned}$$

$\mathbf{B}(x,r/2)\cap\operatorname{spt}\partial P=\emptyset$, and the Constancy Theorem [9, 4.1.7] implies that

$$P\llcorner\mathbf{B}(x,r/2)=\mathcal{H}^m\llcorner(W\cap\mathbf{B}(x,r/2))\wedge j\zeta$$

for some integer j. Since $x+W$ meets the two $n-m$ planes $p_i^{-1}\{p_i(x)\}$ transversally, it is elementary to compute the 2 slices

$$\langle P,p_i,p_i(x)\rangle = j\operatorname{sign}(\zeta\llcorner p_i^{\#}\Omega)\boldsymbol{\delta}_x \quad \text{for } i=1,2\,. \tag{25}$$

Also we readily check that

$$\begin{aligned}&p_i^{-1}\{p_i(x)\}\cap h([0,1]\times\operatorname{spt}\partial(T\llcorner\mathbf{B}(x,r)))\\ &\subset p_i^{-1}\{p_i(x)\}\cap\mathbf{B}(x+W,hr)\sim p_i^{-1}\mathbf{B}(x,r/2)=\emptyset\,.\end{aligned} \tag{26}$$

So we can slice equation (24) and use (25), (26), and the boundary slice relation [9, 4.3.1] to obtain the formulas

$$\begin{aligned}\langle T,p_i,p_i(x)\rangle(\mathbf{1}_{\mathbf{B}(x,r)}) &= \langle P,p_i,p_i(x)\rangle(\mathbf{1}_{\mathbf{B}(x,r)})\\ &\quad -(-1)^m\partial\langle h_{\#}([0,1]\times(T\llcorner\mathbf{B}(x,r))),p_i,p_i(x)\rangle(\mathbf{1}_{\mathbf{B}(x,r)})+0\\ &= j\operatorname{sign}(\zeta\llcorner p_i^{\#}\Omega)\ +\ 0\ +\ 0 \quad \text{for } i=1,2\,,\end{aligned}$$

which give the desired conclusion. □

By the identity (21) it is natural to define the H mass of a rectifiable scan $\mathcal{T}$ by

$$\mathbf{M}_H(\mathcal{T})=\beta(m,n)\int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m}\mathbf{M}_H(\mathcal{T}(p,y))d(\boldsymbol{\theta}^*_{n,m}\times\mathcal{L}^m)(p,y)\,. \tag{27}$$

Theorem 3.5.2. *Let $T_0\in\mathcal{R}_m(\mathbb{R}^n)$ with $\mathcal{I}_1^m(\operatorname{spt}(\partial T_0))=0$ and let H be a concave integrand. Then there exists an m dimensional rectifiable scan $\mathcal{T}$ in $\mathbb{R}^n$ such that $\boldsymbol{\partial}(\mathcal{T}-\mathcal{S}(T_0))=0$ and $\mathbf{M}_H(\mathcal{T})=\Gamma(\mathcal{P}_{\mathbf{M}_H,T_0,\mathbb{R}^n})$. Moreover, if $\operatorname{spt}(\partial T_0)$ is an $m-1$ dimensional compact properly embedded $\mathcal{C}^{1,1}$ submanifold then there exists $T\in\mathbf{I}_m(\mathbb{R}^n)$ with $\partial T=\partial T_0$ and $\mathbf{M}_H(T)=\Gamma(\mathcal{P}_{\mathbf{M}_H,T_0,\mathbb{R}^n})$.*

Proof. We first notice (as in the beginning of the proof of Proposition 2.3.1) that $\Gamma(\mathcal{P}_{\mathbf{M}_H,T_0,\mathbb{R}^n})=\Gamma(\mathcal{P}_{\mathbf{M}_H,T_0,K})$ whenever $K\subset\mathbb{R}^n$ is a compact convex set containing $\operatorname{spt}(T_0)$, because $\mathbf{M}_H(\pi_{K\,\#}T)\le\mathbf{M}_H(T)$ for each $T\in\mathcal{R}_m(\mathbb{R}^n)$.

We may, by Lemma 2.1.1(1)), choose a sequence $\varepsilon_j\downarrow 0$ as well as currents $T_{\varepsilon_j}\in\mathcal{R}_m(\mathbb{R}^n)$ with $\partial T_{\varepsilon_j}=\partial T_0$ and

$$\mathbf{M}_H(T_{\varepsilon_j})+\varepsilon_j^2\mathbf{M}(T_{\varepsilon_j})=\Gamma(\mathcal{P}_{\varepsilon_j,\mathbf{M}_H,T_0,K})=\Gamma(\mathcal{P}_{\varepsilon_j,\mathbf{M}_H,T_0,\mathbb{R}^n})$$

Next we may argue as in the proof of Proposition 2.3.1 to secure a subsequence $\alpha(1),\alpha(2),\ldots$ and an m rectifiable stationary varifold V in $\mathbb{R}^n\sim\operatorname{spt}(\partial T_0)$ such that

$$\|\Upsilon_H(T_{\varepsilon_{\alpha(j)}})\|+\varepsilon^2_{\alpha(j)}\|T_{\varepsilon_{\alpha(j)}}\|\longrightarrow\|V\| \ \text{ in } \mathbb{R}^n\sim\operatorname{spt}(\partial T_0)\ \text{ as } j\to\infty\,. \tag{28}$$

Now we we are ready to apply the our Compactness Theorem 3.4.1 using

$$\begin{aligned}&X=\ \text{the Riemannian manifold } \mathbf{O}^*(n,m)\times\mathbb{R}^m\\ &Y=\mathbf{I}_{0,K}(\mathbb{R}^n)\ \text{with } \operatorname{dist}_Y(T,\tilde T)=\mathcal{F}_K^H(T-\tilde T),\\ &f_j=\mathcal{S}(T_{\varepsilon_{\alpha(j)}}-T_0)=\langle T_{\varepsilon_{\alpha(j)}}-T_0,\cdot,\cdot\rangle,\ \text{ and } \ M=\mathbf{M}_H\,.\end{aligned}$$

The space $Y = \mathbf{I}_{0,K}(\mathbb{R}^n)$ is clearly separable (hence, weakly separable), a dense subset being given by finite sums of atomic masses with rational coordinates and rational coefficients. The lower semicontinuity of $\mathbf{M}_H$ on Y was established in Lemma 3.2.14. A sequence $Q_j \in \mathbf{I}_{0,K}(\mathbb{R}^n$ with $\mathbf{M}_H(Q_j) \leq R$ has $\mathbf{N}(Q_j) = \mathbf{M}(Q_j) \leq H^{-1}(R)$ so that a subsequence converges in $\mathcal{F}_K$ to some $Q \in \mathbf{I}_{0,K}(\mathbb{R}^n)$. This convergence is also in the $\mathcal{F}_K^H$ metric by 3.2.9, and $\mathbf{M}_H(Q) \leq R$ again by the lower semicontinuity. Thus $M^{-1}([0,R])$ is sequentially compact in Y. The measurability of the f_j with respect to the $\mathcal{F}_K^H$ topology follows from Proposition 3.1.3 and Remark 3.2.15. The L^1 bound

$$\begin{aligned}\int_X M\left(f_j(x)\right) d\mathcal{H}^k x &= \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \mathbf{M}_H\langle T_{\varepsilon_{\alpha(j)}} - T_0, p, y\rangle,\, d(\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m)(p,y)\\ &= \boldsymbol{\beta}_1(m,n)^{-1}\mathbf{M}_H(T_{\varepsilon_{\alpha(j)}} - T_0)\\ &\leq \boldsymbol{\beta}_1(m,n)^{-1}[\mathbf{M}_H(T_{\varepsilon_{\alpha(j)}}) + \mathbf{M}_H(T_0)]\\ &\leq \boldsymbol{\beta}_1(m,n)^{-1}[2\mathbf{M}_H(T_0) + \mathbf{M}(T_0)] \quad < \quad \infty\end{aligned}$$

follows from (21), Corollary 3.2.5(1), and the minimizing property of $T_{\varepsilon_{\alpha(j)}}$.

Finally, to verify the needed MBV bound, let $\phi : Y \longrightarrow \mathbb{R}$ have $\operatorname{Lip}\phi \leq 1$, recall Lemma 3.1.2, and use the general slicing bound 3.3.3 with

$$T = (T_{\varepsilon_{\alpha(j)}} - T_0) \times [\![\mathbf{O}^*(n,m)]\!] \quad \text{and} \quad f = \Psi\,,$$

hence $\partial T = 0$ and $\operatorname{Lip} f \leq \mathbf{c}_0 < \infty$, to estimate

$$\begin{aligned}&\int_X |D(\phi \circ f_j)|\\ &= \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \left|D\left(\phi \circ \mathcal{S}(T_{\varepsilon_{\alpha(j)}} - T_0)\right)\right|\\ &= \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} |D(\phi \circ \langle T_{\varepsilon_{\alpha(j)}} - T_0,\, \cdot\,,\, \cdot\,\rangle|\\ &= \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \left|D\left((\phi \circ \Pi_\#) \circ \langle (T_{\varepsilon_{\alpha(j)}} - T_0) \times [\![\mathbf{O}^*(n,m)]\!], \Psi,\, \cdot\,\rangle\right)\right|\\ &\leq \mathbf{c}_1\, \mathbf{M}_H\left((T_{\varepsilon_{\alpha(j)}} - T_0) \times [\![\mathbf{O}^*(n,m)]\!]\right)\\ &\leq \mathbf{c}_2[\mathbf{M}_H(T_{\varepsilon_{\alpha(j)}}) + \mathbf{M}_H(T_0)]\\ &\leq \mathbf{c}_2[2\mathbf{M}_H(T_0) + \mathbf{M}(T_0)] \quad < \quad \infty\,,\end{aligned}$$

where the constants $\mathbf{c}_0$, $\mathbf{c}_1$, $\mathbf{c}_2$ depend only on n and K.

From Theorem 3.4.1 and Remark 3.2.15 we now conclude the existence of a subsequence $\beta(1), \beta(2), \ldots$ of $\alpha(1), \alpha(2), \ldots$ and a measurable (for the $\mathbf{F}_K$ topology) map $\mathcal{R} : \mathbf{O}^*(n,m) \times \mathbb{R}^m \longrightarrow \mathbf{I}_{0,K}(\mathbb{R}^n)$ so that, for $\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m$ almost every (p,y),

$$\mathcal{F}_K^H\left[\mathcal{S}(T_{\varepsilon_{\beta(j)}} - T_0)(p,y) - \mathcal{R}(p,y)\right] \longrightarrow 0 \text{ as } j \to \infty\,.$$

We now change and simplify our notations by writing T_j in place of $T_{\varepsilon_{\beta(j)}}$ and V_j in place of $V_{\mathbf{M}_H, \varepsilon_{\beta(j)}}(T_j)$ for $j = 1, 2, \ldots$. Letting $\mathcal{T} = \mathcal{R} + \mathcal{S}(T_0)$, we deduce that,

$$\lim_{j\to\infty} \mathcal{F}_K^H\left[\langle T_j, p, y\rangle - \mathcal{T}(p,y)\right] = 0 \text{ for } \boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m \text{ almost every } (p,y)\,, \qquad (29)$$

which gives the desired scan boundary condition

$$\boldsymbol{\partial}(\mathcal{T} - \mathcal{S}(T_0)) = 0$$

because, by Proposition 3.1.6,

$$(\mathcal{T} - \mathcal{S}(T_0)(p,y))\,(1) = \lim_{j\to\infty} \langle T_j - T_0, p, y\rangle(1) = 0$$

for almost every (p,y). Also by (27), Fatou's Lemma, and (21),

$$\begin{aligned}\mathbf{M}_H(\mathcal{T}) &= \boldsymbol{\beta}_1^{-1}(n,m)\int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \mathbf{M}_H(\mathcal{T}(p,y))d(\boldsymbol{\theta}^*_{n,m}\times\mathcal{L}^m)(p,y)\\ &\le \boldsymbol{\beta}_1^{-1}(n,m)\int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \liminf_{j\to\infty}\mathbf{M}_H(\mathcal{S}(T_j)(p,y))d(\boldsymbol{\theta}^*_{n,m}\times\mathcal{L}^m)(p,y)\\ &\le \liminf_{j\to\infty}\mathbf{M}_H(T_j) \;=\; \Gamma(\mathcal{P}_{\mathbf{M}_H,T_0,\mathbb{R}^n}) < \infty\,,\end{aligned} \tag{30}$$

the desired minimizing property.

It only remains to verify that $\mathcal{T}$ is a rectifiable scan. We begin with the Borel $(\mathcal{H}^m, m)$ rectifiable set $R := \operatorname{set}(\|V\|)$. We first claim that

$$\operatorname{spt}(\mathcal{T}(p,y)) \sim \operatorname{spt}(\partial T_0) \subset R \tag{31}$$

for $\boldsymbol{\theta}^*_{n,m}\times\mathcal{L}^m$ almost every (p,y). To prove this, we deduce from the monotonicity of $\|V\|$ ([1, 5.1(2)]) and the lower density bound that R is relatively closed in $\mathbb{R}^n \sim \operatorname{spt}(\partial T_0)$ (see for instance [8, 6.13]) so that $R\cup\operatorname{spt}(\partial T_0)$ is closed in $\mathbb{R}^n$. If $x\notin R\cup\operatorname{spt}(\partial T_0)$, then also $\mathbf{U}(x,r)\cap(R\cup\operatorname{spt}(\partial T_0)) = \emptyset$ for some $r>0$ and the monotonicity of each $\|V_j\|$ implies that $\operatorname{spt}(T_j)\cap\mathbf{U}(x,r)=\emptyset$ if j is sufficiently large because $\operatorname{spt}(T_j)$, $j=1,2,\ldots$, converge in Hausdorff distance to R in compact subsets of $\mathbb{R}^n\sim\operatorname{spt}(\partial T_0)$ (see for instance [8, 4.2]). Now for each pair (p,y) such that (29) holds true, we deduce that $x\notin\operatorname{spt}(\mathcal{T}(p,y))$.

We will call a projection $p\in\mathbf{O}^*(n,m)$ *good* if it satisfies the following properties:

(1) $\mathcal{H}^m(\Sigma_p(R)) = 0$, (recall (3.1.8)),
(2) $\mathcal{L}^m(p(\partial T_0)) = 0$,
(3) $\lim_{j\to\infty}\mathcal{F}^H_K[\mathcal{S}(T_j)(p,y) - \mathcal{T}(p,y)] = 0$ for $\mathcal{L}^m$ almost every $y\in\mathbb{R}^m$.
(4) $\liminf_{j\to\infty}\mathbf{M}_H(\mathcal{S}(T_j)(p,y)) < \infty$ for $\mathcal{L}^m$ almost every $y\in\mathbb{R}^m$.

By Fubini's Theorem, (3.1.8), [9, 2.10.15], (29), Fatou's Lemma, and (30), we see that $\boldsymbol{\theta}^*_{n,m}$ almost every $p\in\mathbf{O}^*(n,m)$ is good. We now define the multiplicity θ and orienting m vectorfield ξ by using one *fixed* good projection p_0. For $x\in R_{p_0}$, let

$$\theta(x) := |\mathcal{T}(p_0,p_0(x))(\mathbf{1}_{\{x\}})|\,.$$

and $\qquad \xi(x) :=$ *the unique unit* m *vector orienting* $\operatorname{Tan}^m(\|V\|,x)$ *with*

$$\operatorname{sign}(\xi(x)\llcorner p^{\#}\Omega) = \operatorname{sign}[\mathcal{T}(p_0,p_0(x))(\mathbf{1}_{\{x\}})]\,.$$

To check that θ and ξ are $\mathcal{H}^m$ measurable, we first recall that Lusin's Theorem allows us to assume that $\mathcal{T}$ is fact Borel. Next we let $\mathcal{C}_k$, $k=1,2,\ldots$, be the partition of $\mathbb{R}^n$ into dyadic cubes of side length 2^{-k} and such that $\mathcal{C}_{k+1}$ is a refinement of $\mathcal{C}_k$. Then, since $\operatorname{spt}[\mathcal{T}(p_0,p_0(x))]$ is finite,

$$\theta(x) = \lim_{k\to\infty}\sum_{C\in\mathcal{C}_k}\mathbf{1}_C(x)|\mathcal{T}(p_0,p_0(x))|(\mathbf{1}_C)$$

and the fact that θ is Borel becomes clear. Similarly, one verifies that the function $\operatorname{sign}[\mathcal{T}(p_0,p_0(x))(\mathbf{1}_{\{x\}})]$ is Borel, and the measurability of ξ follows from the rectifiability of R.

To finish the proof of the fact that $\mathcal{T}$ is a rectifiable scan, we need only show that, for every good projection $p \in \mathbf{O}^*(n,m)$,

$$\mathcal{T}(p,y) = \sum_{x \in R \cap p^{-1}\{y\}} \operatorname{sign}(\xi(x) \llcorner p^{\#}\Omega)\theta(x)\boldsymbol{\delta}_x \tag{32}$$

is true for $\mathcal{L}^m$ almost every $y \in \mathbb{R}^m$. The definitions of θ and ξ give (32) when $p = p_0$ and $y \in \mathbb{R}^m \sim p_0(\Sigma_{p_0})$.

Let $p_1 \in \mathbf{O}^*(n,m)$ be another good projection, and for $i = 0,1$ we define

$$\begin{aligned} Y_i \; := \; & p_i(\Sigma_{p_i}) \cup p_i(\operatorname{spt}\partial T_0) \\ & \cup \{y \; : \; \limsup_{j\to\infty} \mathcal{F}_K^H\big[\mathcal{S}(T_j)(p,y) - \mathcal{T}(p,y)\big] > 0\} \\ & \cup \{y \; : \; \liminf_{j\to\infty} \mathbf{M}_H\left(\mathcal{S}(T_j)(p,y)\right) = \infty\} \\ & \cup \{y \; : \; y \text{ is not a Lebesgue point of } \langle T_j, p_i, \cdot\rangle \text{ for some } \; j = 1,2,\dots\} \end{aligned}$$

so that $\mathcal{L}^m(Y_i) = 0$ by the goodness of p_i. Applying [9, 3.2.22] to each $p_i|R \sim \Sigma_{p_i}$ we see that $Z_i := R \cap p_i^{-1}(Y_i)$ has $\mathcal{H}^m(Z_i) = 0$.

We now verify (32) for $p = p_1$ and $y \in \mathbb{R}^m \sim p_1(Z_1 \cup \mathbb{Z}_2)$. By (31) and the fact that $y \notin p_1(\operatorname{spt}\partial T_0)$,

$$\mathcal{T}(p_1,y) = \sum_{x \in R \cap p_1^{-1}\{y\}} \mathcal{T}(p_1,y)(\mathbf{1}_{\{x\}})\boldsymbol{\delta}_x\,. \tag{33}$$

It remains to compute the coefficient $\mathcal{T}(p_1,y)(\mathbf{1}_{\{x\}})$ for each fixed $x \in R \cap p_1^{-1}(y)$. The monotonicity of the measure $\|V\|$ as well as that of the measures $\|\Upsilon_H(T_j)\| + \varepsilon^2_{\hat{\alpha}(j)}\|T_j\|$ implies (as in the proof of Proposition 2.3.1) convergence of supports when letting $j \to \infty$ or when rescaling $\|V\|$ to its weak tangent plane at x. We conclude that, for some $r > 0$ sufficiently small, all the hypotheses of Lemma 3.5.1 are satisfied with $h = 1/3$, $W = \operatorname{Tan}^m(\|V\|,x)$, $\zeta = \xi(x)$, and $T = T_j$ for all j sufficiently large. Inasmuch as

$$\lim_{j\to\infty} \mathcal{F}_K^H\big[\mathcal{S}(T_j)(p_i,p_i(x)) - \mathcal{T}(p_i,p_i(x))\big] = 0 \text{ and } \liminf_{j\to\infty} \mathbf{M}_H\left(\mathcal{S}(T_j)(p_i,p_i(x))\right) < \infty\,,$$

we may also find, by Lemma 3.2.13, a single subsequence j' so that

$$\lim_{j'\to\infty} \mathbf{F}_K\big[\langle T_{j'} p_i, p_i(x)\rangle - \mathcal{T}(p_i,p_i(x))\big] = 0$$

for $i = 0,1$. We deduce from Lemma 3.5.1 that

$$\begin{aligned} \operatorname{sign}(\xi(x) \llcorner p_1^{\#}\Omega)\mathcal{T}(p_1,y)(\mathbf{1}_{\{x\}}) &= \operatorname{sign}(\xi(x) \llcorner p_1^{\#}\Omega)\mathcal{T}(p_1,y)(\mathbf{1}_{\mathbf{B}(x,r)}) \\ &= \lim_{j\to\infty} \operatorname{sign}(\xi(x) \llcorner p_1^{\#}\Omega)\langle T_{j'}, p_1, p_1(x)\rangle(\mathbf{1}_{\mathbf{B}(x,r)}) \\ &= \lim_{j\to\infty} \operatorname{sign}(\xi(x) \llcorner p_0^{\#}\Omega)\langle T_{j'}, p_0, p_0(x)\rangle(\mathbf{1}_{\mathbf{B}(x,r)}) \\ &= |\mathcal{T}(p_0,p_0(x))(\mathbf{1}_{\mathbf{B}(x,r)})| \\ &= \theta(x)\,. \end{aligned}$$

This calculation, combined with (33) gives (32).

We turn to the proof of the second part of the Theorem where we assume $\operatorname{spt}(\partial T_0)$ to have the stated extra regularity property. We define a measure ϕ on $\mathbb{R}^n$ by the formula

$$\phi(B) := \beta_1^{-1}(n,m) \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \|\Upsilon_H(\mathcal{T}(p,y))\|(B) d(\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m)(p,y)$$

for each Borel set $B \subset \mathbb{R}^n$ (recall Lemma 3.2.14) and we observe that

$$\phi(B) = \int_R H(|\theta|) d\mathcal{H}^m$$

according to the first part of this proof. On the other hand, for each open set $U \subset \mathbb{R}^n \sim \operatorname{spt}(\partial T_0)$ we refer to (28) and Fatou's Lemma to deduce that

$$\begin{aligned}\phi(U) &\leq \beta_1^{-1}(n,m) \int_{\mathbf{O}^*(n,m)\times\mathbb{R}^m} \liminf_{j\to\infty} \|\Upsilon_H \mathcal{S}(T_j)(p,y)\|(U) d(\boldsymbol{\theta}^*_{n,m} \times \mathcal{L}^m)(p,y) \\ &\leq \liminf_{j\to\infty} \left(\|\Upsilon_H(T_j)\|(U) + \varepsilon^2_{\hat{\alpha}(j)} \|T_j\|(U)\right)\end{aligned}$$

(because $\varepsilon^2_{\hat{\alpha}(j)} \mathbf{M}(T_j) \to 0$ according to Lemma 2.1.1(4))

$$= \liminf_{j\to\infty} \|V_j\|(U) \leq \|V\|(\operatorname{Clos} U)$$

and hence we infer that for each closed ball $B \subset \mathbb{R}^n$ we have $\phi(B) \leq \|V\|(B)$. In particular $\Theta^{*m}(\phi, x) \leq \Theta^{*m}(\|V\|, x)$ for every $x \in \mathbb{R}^n$. According to the differentiation theory of [9, 2.9] we know that $H(|\theta(x)|) = \Theta^m(\phi, x)$ for $\mathcal{H}^m$ almost every $x \in R$. On the other hand we know from Allard's boundary regularity theory (recall Proposition 1.0.1) that $c := \sup\{\Theta^m(\|V\|, x) : x \in \mathbb{R}^n\} < \infty$. Then $|\theta(x)| \leq H^{-1}(c)$ for $\mathcal{H}^m$ almost every $x \in R$ so that $\mathcal{H}^m \llcorner R \wedge \theta\xi \in \mathcal{R}_m(\mathbb{R}^n)$. Finally $\partial[\mathcal{H}^m \llcorner R \wedge \theta\xi] = \partial T_0$ according to Proposition 3.1.6. □

Remark 3.5.3. We will now give a different proof of the second part of the above Theorem, namely the existence result in the class of currents under the extra regularity condition on $\operatorname{spt}(\partial T_0)$. In fact Proposition 1.0.1 implies the following (seemingly stronger, but equivalent) result. *Let $B \subset \mathbb{R}^n$ be an $m-1$ dimensional, compact, properly embedded $\mathcal{C}^{1,1}$ submanifold and let $V_1, V_2, \ldots$ be m rectifiable stationary varifolds in $\mathbb{R}^n \sim B$, all supported in some common compact subset of $\mathbb{R}^n$. Assume also that $\Theta^m(\|V_j\|, x) \geq 1$ for $\|V_j\|$ almost every $x \in \mathbb{R}^n$, $j = 1, 2, \ldots$, and that $\sup\{\|V_j\|(\mathbb{R}^n) : j = 1, 2, \ldots\} < \infty$. Then there exist integers $\alpha(1), \alpha(2), \ldots$ such that*

$$\sup\{\Theta^m(\|V_{\alpha(j)}\|, x) : x \in \mathbb{R}^n \ , j = 1, 2, \ldots\} < \infty \, .$$

We apply this result to the varifolds $\|V_j\|$ from the proof of Theorem 3.5.2 and we observe it implies that $\sup\{\mathbf{M}(T_{\alpha(j)}) : j = 1, 2\ldots\} < \infty$ so that Federer and Fleming's compactness theorem applies. This shows in fact that there are $\mathbf{M}_H$ minimizing (sub)sequences converging in the *weak* topology of currents to an $\mathbf{M}_H$ minimizer, in contrast with the general behavior of $\mathbf{M}_H$ minimizing sequences.

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