Unit 7 – Polygons and Circles

Diagonals of a Polygon

Overview: In this activity, participants investigate the number of diagonals from a given vertex of a polygon.

Objective: **TExES Mathematics Competencies**

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs.
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
IV.015.B. The beginning teacher organizes, displays, and interprets data in a variety of formats (e.g., tables, frequency distributions, scatter plots, stem-and-leaf plots, box-and-whisker plots, histograms, pie charts).
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.4. The student uses a variety of representations to describe geometric relationships and solve problems.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.

e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants should know that the sum of the measures of the angles in any given triangle is 180º. They should also be familiar with terms such as vertex, diagonal, and interior angle.

Materials: straightedge, graphing calculator

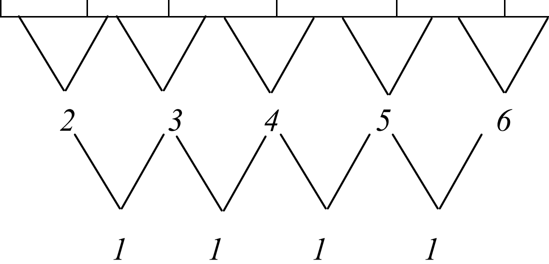
New Terms:

Procedures:

In this activity, participants will investigate the relationship between the number of sides of a polygon and the total number of diagonals.

1. Complete the following table relating the numbers of sides of a polygon to the total number of diagonals.

Number of sides of the polygon	**3**	**4**	**5**	**6**	**7**	**8**
Number of diagonals in the polygon	*0*	*2*	*5*	*9*	*14*	*20*



Participants may need to create the polygons and physically draw the diagonals to be able to complete the first few columns of the chart.

Create a scatter plot of your data on your graphing calculator. Place the number of sides of the polygon on the *x*-axis and the number of diagonals on the *y*-axis.

For an intensive tutorial on the use of the graphing calculator see Utilizing the graphing calculator in the secondary mathematics and science classroom at http://www.esc4.net/math.

Using the given window the scatter plot should look as follows:

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WINDOW
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=-3
 Ymax=25
 Yscl=1
 Xres=1
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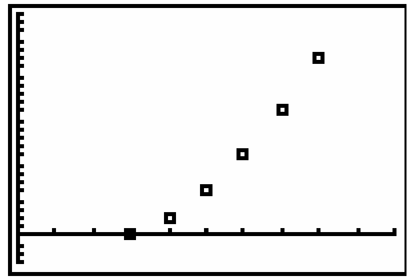


What patterns do you observe?
Graphically and numerically participants should see that the data is not linear. Finite differences (shown above) illustrate that the first finite differences are not constant; however the second finite differences have a constant value of one. This indicates that the data is quadratic, a fact that appears to be confirmed by the scatter plot.

2. How many diagonals are in a 24-gon? How many in an *n*-gon?
A 24-gon will have 252 diagonals. An n-gon will have $\frac{n(n-3)}{2}$ *diagonals.*

You may need to tell participants that a 24-gon is a polygon with 24 sides and an *n*-gon is a polygon with *n*-sides. Although participants may want to extend the pattern to find the number of diagonals in a 24-gon, they will soon realize that this is too tedious and that finding a rule to determine the number of diagonals would be more efficient. Participants may mistakenly believe from the scatter plot that the equation is a quadratic function with vertex at (3, 0). Others may attempt to perform a quadratic regression using the calculator to determine the function. If necessary, encourage participants to think about the underlying geometry in the situation.

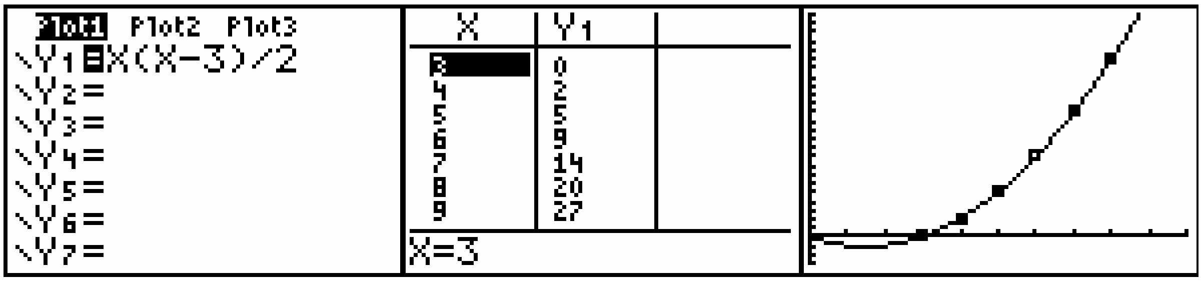
How many diagonals can be drawn from each vertex in an n-gon?
n-3 diagonals can be drawn from each vertex.

How many vertices are there in an *n*-gon?
An *n*-gon has *n* vertices.

Is the total number of diagonals *n*(*n*–3)?
No. Multiplying the number of vertices by the number of diagonals that can be drawn from each vertex "double counts" each diagonal. Therefore the total number of diagonals is $\frac{n(n-3)}{2}$.

Participants may have other approaches for determining this function. For example, some may see this as a combination problem. Determining the total number of segments that can be drawn among *n* vertices is equivalent to determining the number of groups of 2 that can be taken from a group of *n*. This is equivalent to $\frac{n(n-1)}{2}$. The number of segments that are not diagonals, the *n* sides, results in an expression of $\frac{n(n-1)}{2} - n$. If a participant uses this approach, demonstrate that this expression is algebraically equivalent to $\frac{n(n-3)}{2}$.

3. Use your graphing calculator to verify your function for determining the number of diagonals in an *n*-gon.



The graphing calculator may be used in several ways to verify the function. We used the variable n in this activity, because we were dealing with discrete data. To use the calculator, we will need to use the variable *x* which is more often associated with continuous functions. Participants may use a calculator table to see that their function reproduces their original table. They may also overlay their previously created scatter plot upon the graph of the function. The continuous function $f(x) = \frac{x(x-3)}{2}$ contains all the points of the discrete function for our geometric situation.

4. How many sides does a polygon with 860 diagonals have?
The polygon has 43 sides.

Participants have a variety of methods for finding this solution. They may use the table of values generated by the calculator to determine the number of sides. Or they may find the intersection point of the function $f(x) = \frac{x(x-3)}{2}$ and the line $g(x) = 860$. Or they may solve the quadratic equation . $\frac{x(x-3)}{2} = 860$.

In this activity participants are working at the van Hiele Descriptive Level as they determine properties of polygons. When they connect this with algebra, they approach the Relational Level.

Diagonals of a Polygon

1. Complete the following table relating the numbers of sides of a polygon to the total number of diagonals.

Number of sides of the polygon	**3**	**4**	**5**	**6**	**7**	**8**
Number of diagonals in the polygon						

Create a scatter plot of your data on your graphing calculator. Place the number of sides on the polygon on the *x*-axis and the number of diagonals on the *y*-axis.

What patterns do you observe?

2. How many diagonals are in a 24-gon? How many in an n-gon?

3. Use your graphing calculator to verify your function for determining the number of diagonals in an n-gon.

4. How many sides does a polygon with 860 diagonals have?

Interior and Exterior Angles of a Polygon

Overview: In this activity, participants investigate the sum of the measures of the interior and exterior angles of a polygon.

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Objective: **TExES Mathematics Competencies**

II.004.A. The beginning teacher recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs.
II.006.G. The beginning teacher models and solves problems involving linear and quadratic equations and inequalities using a variety of methods, including technology.
II.007.A. The beginning teacher recognizes and translates among various representations (e.g., written, tabular, graphical, algebraic) of polynomial, rational, radical, absolute value, and piecewise functions.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.C. The beginning teacher uses geometric patterns and properties (e.g., similarity, congruence) to make generalizations about two- and three-dimensional figures and shapes (e.g., relationships of sides, angles).
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
c.1.A. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need to have knowledge of the sum of the measures of a triangle's interior angles.

Materials: graphing calculator, straightedge, unlined 8.5 in. by 11 in. paper, scissors, tape, transparencies "Constructing a Polygon's Exterior Angles" and "Determining the Sum of a Polygon's Exterior Angles"

New Terms:

Procedures:

In this activity, participants will use information about the number of diagonals of a polygon from one vertex to determine the sum of the measures of the interior angles of a polygon as well as the measure of each interior angle of a regular polygon. Recall the definition of a regular polygon as a polygon with all sides congruent and all angles congruent.

Provide participants with unlined 8.5 in. by 11 in. paper.

1. In your group, draw polygons with 3 to 7 sides. Use a straightedge to draw all the diagonals from one vertex. Share your results to complete the table below.

Number of sides of the polygon	***3***	***4***	***5***	***6***	***7***	**…**	***n***
Number of diagonals from one vertex	*0*	*1*	*2*	*3*	*4*		$n-3$
Number of triangles created	*1*	*2*	*3*	*4*	*5*		$n-2$
Total measure of interior angles (degrees)	$1\cdot 180$ *180*	$2\cdot 180$ *360*	$3\cdot 180$ *540*	$4\cdot 180$ *720*	$5\cdot 180$ *900*		$(n-2)\cdot 180$ $180(n-2)$

2. Using the formula you derived in 1 for the sum of the measures of the interior angles of a polygon, complete the table below to determine the measure of each interior angle of a regular polygon.

Number of sides of the polygon	***3***	***4***	***5***	***6***	***7***	**…**	***n***
Sum of measures of interior angles (degrees)	*180*	*360*	*540*	*720*	*900*		$180(n-2)$

Measure of an interior angle (degrees)	$\frac{180}{3} =$ 60	$\frac{360}{4} =$ 90	$\frac{540}{5} =$ 108	$\frac{720}{6} =$ 120	$\frac{900}{7} =$ 128.6		$\frac{180(n-2)}{n}$

Stress with participants that the formula for the sum of the measures of interior angles is true for all polygons, but that this formula is applicable only to regular polygons, since by definition, all of the angles in a regular polygon must be congruent.

3. If $n = 50$, what is the measure of each interior angle of a regular polygon?

   The measure of the interior angle of a 50-gon is $\frac{180^o(50-2)}{50}$ *or* 172.8°.

4. If $n = 100$, what is the measure of each interior angle of a regular polygon?

   The measure of the interior angle of a 100-gon is $\frac{180^\circ(100-2)}{100}$ *or* 176.4°.

5. Is it possible for a regular polygon to have an interior angle measure of 175.5°? Explain.

   $175.5^\circ = \frac{180^\circ(n-2)}{n}$

   $175.5^\circ n = 180^\circ n - 360^\circ$

   $360^\circ = 4.5^\circ n$

   $n = 80$

   An 80-gon has an interior angle measure of 175.5°.

6. Is it possible for a regular polygon to have an interior angle measure of 169°? Explain.

   $169^\circ = \frac{180^\circ(n-2)}{n}$

   $169^\circ n = 180^\circ n - 360^\circ$

   $360^\circ = 11^\circ n$

   $n = 32.\overline{72}$

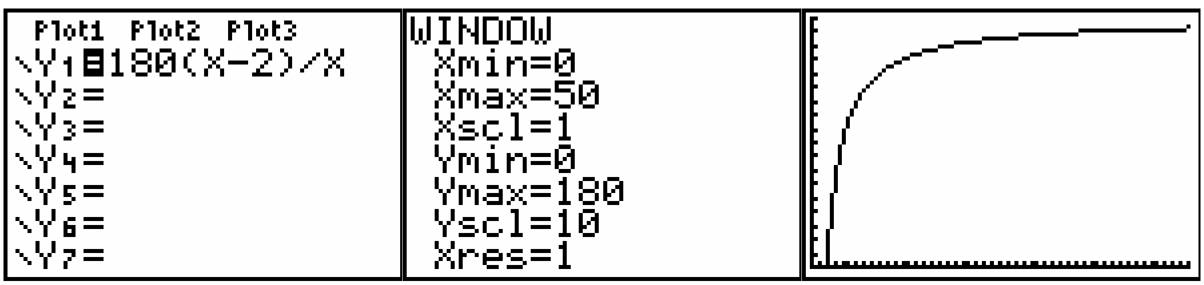
   Since n is not a natural number, there is no regular polygon with interior angle measure of 169°.

7. As n gets very large, ($n \to \infty$), what happens to the measure of an interior angle of a regular polygon? Illustrate this graphically.

   Measure of an interior angle of a regular polygon
   $$= \frac{180^\circ(n-2)}{n} = \frac{180^\circ n}{n} - \frac{360^\circ}{n} = 180^\circ - \frac{360^\circ}{n}$$

   As n gets very large, ($n \to \infty$), $\frac{360^\circ}{n}$ *approaches zero, and the measure of an interior angle approaches* 180°.

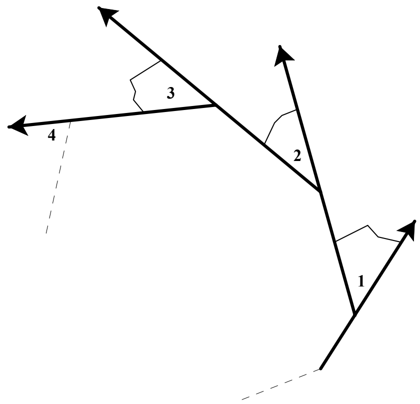
   On the graphing calculator, this appears as asymptotic behavior, with the line y = 180 serving as the horizontal asymptote to the curve $y = \frac{180(x-2)}{x}$.



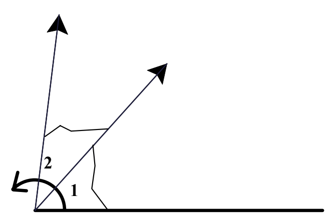
Participants will now investigate the exterior angles of a polygon. It is often difficult for students to understand intuitively that this sum does not depend on the number of sides of the polygon. Participants may use the polygons that they have already constructed to study exterior angles for the next part of the activity.

8. For the polygons that you created earlier, use a straightedge to extend each side of the polygon as a ray to construct the polygon's exterior angles as follows. Choose a vertex from which to begin, and extend the side to the right, thus making the side into a ray. The angle between the ray just drawn and the consecutive side to the right is $\angle 1$. Continue in a counter-clockwise direction until all exterior angles have been drawn. You will construct as many exterior angles as there are sides of the polygon.

   You will now carefully cut and tear the polygon so that each exterior angle keeps its distinct vertex. Cutting from each vertex slightly more than halfway along its corresponding side and then tearing away from the center will accomplish this best. When you have finished, you will have as many pieces as there are exterior angles of the polygon.



Align each exterior angle on the segment below, beginning with the vertex of $\angle 1$ at the arrow and above the line. Using a common vertex (align the vertex of $\angle 2$ with the vertex of $\angle 1$ and the edge of $\angle 2$ to the top edge of $\angle 1$), move counter-clockwise as shown by the arrow and align $\angle 1$ with $\angle 2$. Continue to align the exterior angles in this manner until all angles have been placed.



If necessary, use the transparencies provided to illustrate this process for participants.

How many complete revolutions do your combined exterior angles make?
The combined exterior angles make one complete revolution.

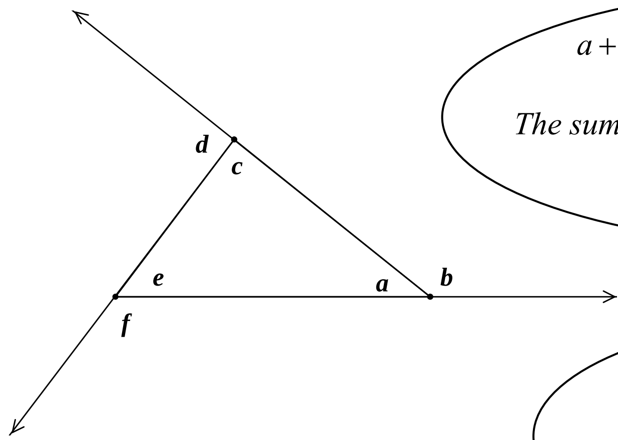
What is the sum of the measures of the exterior angles of your polygon? Is this true for the polygons of all members of your group?
The sum of the measures of the exterior angles of any polygon is 360°.

9. Complete the chart below, obtaining information from other groups. What conclusion can you draw from the chart?

Number of sides of the polygon	***4***	***5***	***6***	***7***	***8***	***9***	**…**	***n***
Sum of the measures of the exterior angles (degrees)	*360*	*360*	*360*	*360*	*360*	*360*		*360*

For any polygon, the sum of the measures of the exterior angles is 360°.

10. Complete the following flowchart proof for the sum of the measures of the exterior angles of a triangle.



$a + b = 180^\circ$; $c + d = 180^\circ$; $e + f = 180^\circ$

The sum of the measures of the angles that form a linear pair equals 180°.

↓

$a + b + c + d + e + f = 540^\circ$

Addition Property of Equality

↓

$a + c + e = 180^\circ$

The sum of the measures of the interior angles of a triangle equals 180°.

↓

$b + d + f = 360^\circ$

Subtraction Property of Equality

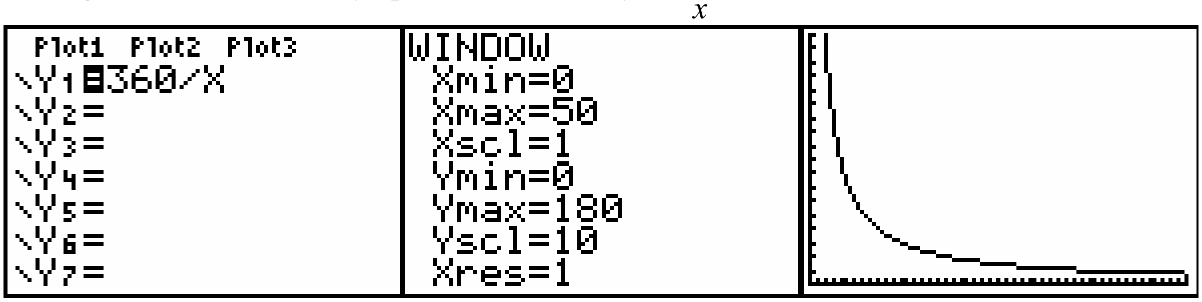
How may the proof above be extended to prove that the sum of the measures of the exterior angles of an *n*-gon is 360^o?

Since an interior angle and its related exterior angle form a linear pair at each vertex, there are n linear pairs of angles, totaling 180n degrees. As we discovered previously, the sum of the interior angles of a polygon is $180(n-2)$. *Therefore the exterior angles must have a sum of* $180n-[180(n-2)]$ *degrees. Simplifying this expression yields* $180n-(180n-360)$ *degrees or* 360^o.

11. Determine the measure of an exterior angle of a regular polygon. Complete the table below:

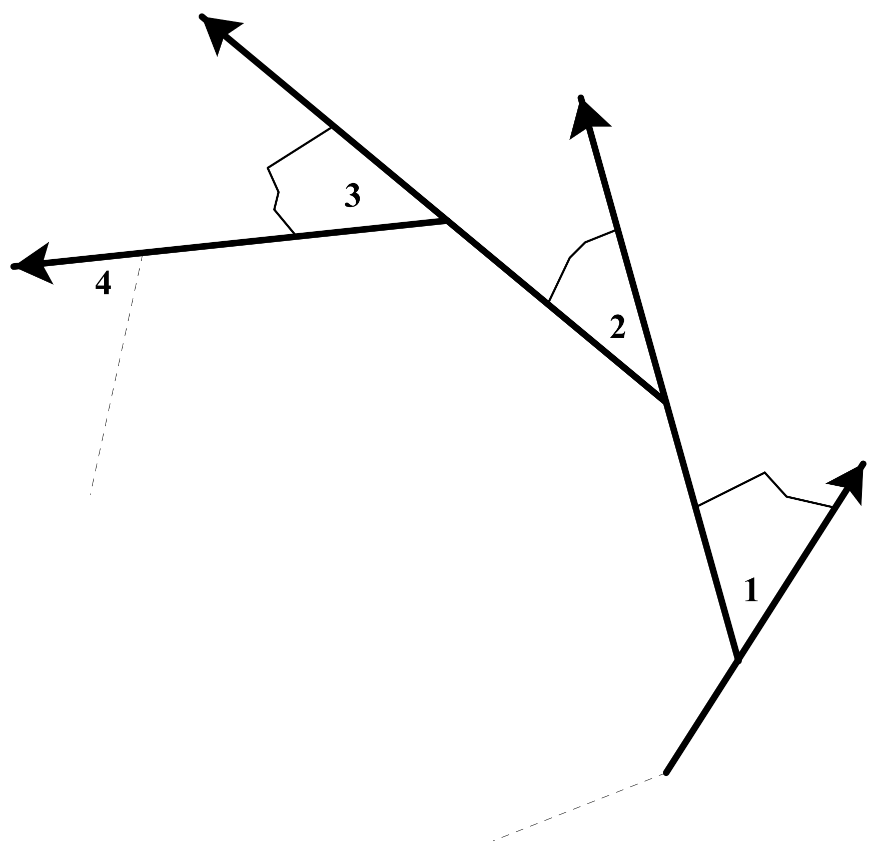
Number of sides of the regular polygon	***4***	***5***	***6***	***7***	***8***	***9***	**…**	***n***
Sum of the measures of the exterior angles (degrees)	*360*	*360*	*360*	*360*	*360*	*360*	..	*360*
Measure of an exterior angle (degrees)	$\frac{360}{4} =$ 90	$\frac{360}{5} =$ 72	$\frac{360}{6} =$ 60	$\frac{360}{7} =$ 51.7	$\frac{360}{8} =$ 45	$\frac{360}{9} =$ 40	..	$\frac{360}{n}$

12. As the number of sides of the polygon increases, what happens to the measures of each exterior angle of a regular polygon? Explain and illustrate graphically.
As the number of sides of the polygon increases, the measure of each exterior angle approaches zero. This results because the measure of the supplementary interior angle approaches 180°.
On the graphing calculator this appears as asymptotic behavior, with the x-axis serving as the horizontal asymptote to the curve $y = \frac{360}{x}$.



In this activity participants are working at the van Hiele Descriptive Level as they determine properties of polygons, specifically with respect to interior and exterior angles. They approach the Relational Level as they connect with algebra, and the Deductive Level if formal reasoning is used in the flowchart proof.

Transparency
Constructing a Polygon's Exterior Angles



Transparency
Sum of the Measures of the Exterior Angles of a Polygon
Determining the Sum of a Polygon's Exterior Angles

Interior and Exterior Angles of a Polygon

1. In your group, draw polygons with 3 to 7 sides. Use a straightedge to draw all the diagonals from one vertex. Share your results to complete the table below.

Number of sides of the polygon	**3**	**4**	**5**	**6**	**7**	**…**	***n***
Number of diagonals from one vertex							
Number of triangles created							
Total number of degrees in the polygon							

2. Using the formula you derived for the sum of the measures of the interior angles of the polygon, complete the table below to determine the measure of an interior angle of each regular polygon.

Number of sides of the polygon	**3**	**4**	**5**	**6**	**7**	**…**	***n***
Sum of measures of interior angles (degrees)							
Measure of an interior angle (degrees)							

3. If $n = 50$, what is the measure of each interior angle of a regular polygon?

4. If $n = 100$, what is the measure of each interior angle of a regular polygon?

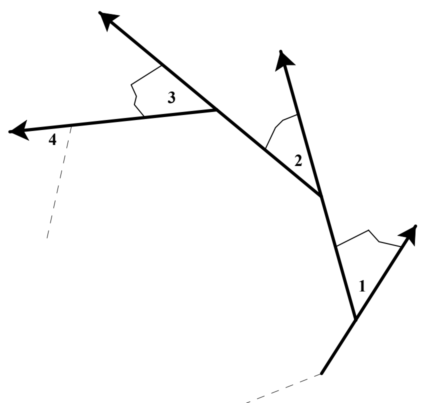
5. Is it possible for a regular polygon to have an interior angle measure of 175.5° ? Explain.

6. Is it possible for a regular polygon to have an interior angle measure of 169°? Explain.

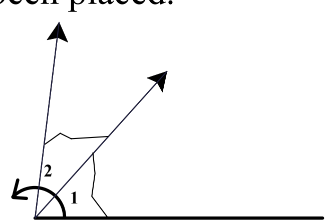
7. As n gets very large ($n \to \infty$) what happens to the measure of an interior angle of a regular polygon? Explain. Illustrate this graphically.

8. For the polygons that you created earlier, use a straightedge to extend each side of the polygon as a ray to construct the polygon's exterior angles as follows. Choose a vertex from which to begin, and extend the side to the right, thus making the side into a ray. The angle between the ray just drawn and the consecutive side to the right is $\angle 1$. Continue in a counter-clockwise direction until all exterior angles have been drawn. You will construct as many exterior angles as there are sides of the polygon.

   You will now carefully cut and tear the polygon so that each exterior angle keeps its distinct vertex. Cutting from each vertex slightly more than halfway along its corresponding side and then tearing away from the center will accomplish this best. When you have finished, you will have as many pieces as there are exterior angles of the polygon.



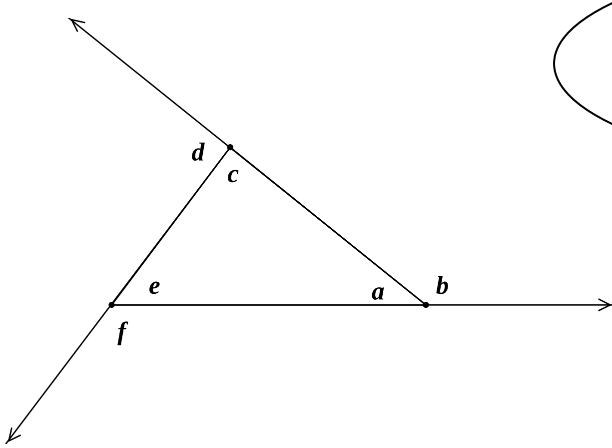
   Align each exterior angle on the segment below, beginning with the vertex of $\angle 1$ at the arrow and above the line. Using a common vertex (align the vertex of $\angle 2$ with the vertex of $\angle 1$ and the edge of $\angle 2$ to the top edge of $\angle 1$), move counter-clockwise as shown by the arrow and align $\angle 1$ with $\angle 2$. Continue to align the exterior angles in this manner until all angles have been placed.



9. Complete the chart below, obtaining information from other groups. What conclusion can you draw from the chart?

Number of sides of the polygon	**4**	**5**	**6**	**7**	**8**	**9**	**…**	***n***
Sum of the measures of the exterior angles (degrees)								

10. Complete the following flowchart proof for the sum of the measures of the exterior angles of a triangle.



$a+b=$ _____ ; $c+d=$ _____ ;
$e+f=$ _____

↓

$a+b+c+d+e+f=$ _____

↓

$a+c+e=$ _____

↓

$b+d+f=$ _____

The above is taken from *Discovering Geometry: An Inductive Approach:* 3rd Edition, ©2003, p. 262, with permission from Key Curriculum Press

How may the proof above be extended to prove that the sum of the measures of the exterior angles of an *n*-gon is 360°?

11. Determine the measure of an exterior angle of a regular polygon. Complete the table below:

Number of sides of the regular polygon	**4**	**5**	**6**	**7**	**8**	**9**	**…**	***n***
Sum of the measures of the exterior angles (degrees)								
Measure of an exterior angle (degrees)								

12. As the number of sides of the polygon increases, what happens to the measure of each exterior angle of a regular polygon? Explain and illustrate graphically.

Polygons in Circles

Overview: In this activity, participants determine and apply the area formula for regular polygons and relate properties of regular polygons and their circumscribed circles.

Objective: **TExES Mathematics Competencies**
II.004.A. The beginning teacher Recognizes and extends patterns and relationships in data presented in tables, sequences, or graphs.
III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.013.A. The beginning teacher analyzes the properties of polygons and their components.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes.
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.
V.019.D. The beginning teacher communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphical, pictorial, symbolic, concrete).

Geometry TEKS
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.3.E. The student uses deductive reasoning to prove a statement.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds area of regular polygons and composite figures.
e.2.B. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Background: Participants need to know the area and perimeter formulas for triangles and circles.

Materials: calculator, centimeter ruler

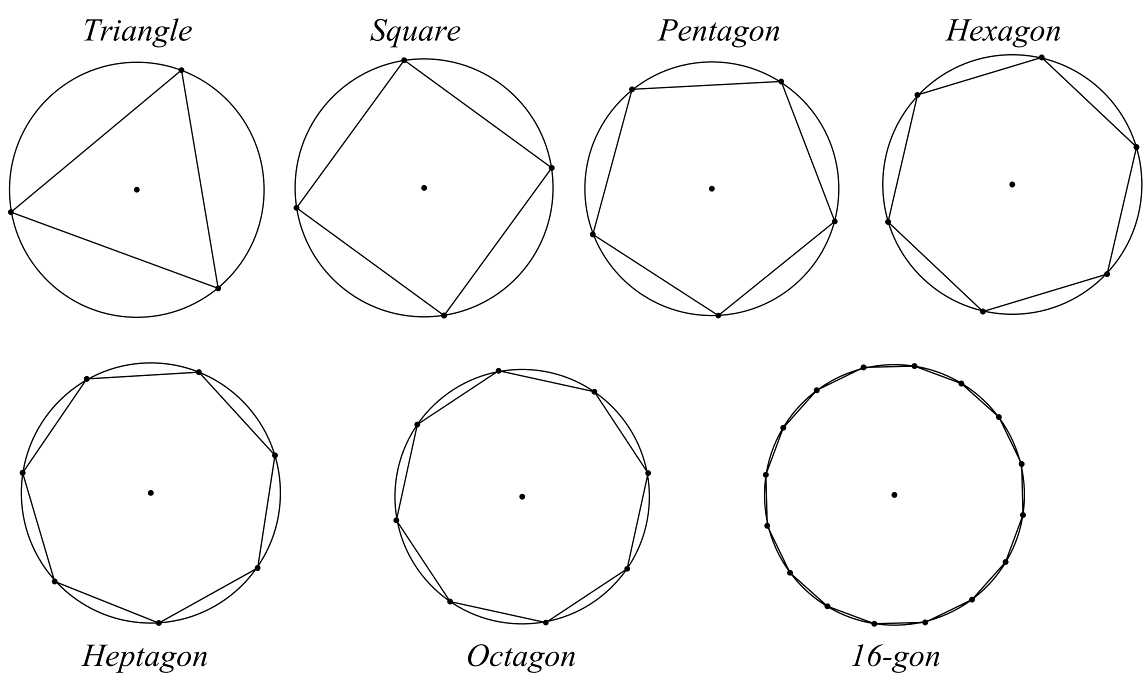
New Terms: apothem, radius of a polygon

Procedures:

Participants will derive the formula for the area of a regular polygon in relation to its circumscribed circle. They will also relate properties and parts of inscribed polygons to the corresponding parts and properties of circles.

Each group member should complete the activity for one or more different polygons so that all the polygons are considered. Share your findings with others in your group as you complete the activity.

1. To construct a regular polygon with *n* sides, connect *n* equally spaced points on the circumference of a circle. Connect the points on the following congruent circles with a straightedge to complete the regular polygons. Name each polygon.



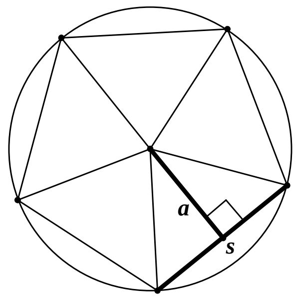
2. Draw segments connecting the center of each circle to the vertices of its inscribed polygon. The segments are called the *radii* of the polygons.

   The isosceles triangles formed by the radii and the sides of the polygon can be used to find the area of each polygon. In each polygon draw an altitude to the base of one of the isosceles triangles. This segment is called the *apothem* of the polygon.

Remind participants to add the new terms apothem and radius of a polygon to their glossaries.

Find the area of a regular polygon in terms of the number of sides, *n*, the length of a side, *s*, and the length of the apothem, *a*.

Example,



Area of polygon $= n(\text{Area of one triangle})$

Area of polygon $= n\left(\frac{1}{2}sa\right)$

Area of polygon $= \frac{1}{2} \cdot nsa$

3. Find the area of a regular polygon in terms of its perimeter, *P*, and the length of the apothem, *a*.
   $P = ns$
   From 2, the area of polygon $= \frac{1}{2}nsa = \frac{1}{2}Pa$.

4. Use a ruler to measure, to the nearest millimeter the length of a side and the length of an apothem for each of the regular polygons. Calculate the area of your polygon(s). Complete the following table by sharing your measurements and calculations with members of your group.

Number of sides n	Side Length s (mm)	Apothem a (mm)	Area (mm^2)
3	*38*	*11*	*627*
4	*31*	*16*	*992*
5	*26*	*18*	*1170*
6	*22*	*19*	*1254*
7	*19*	*20*	*1330*
8	*17*	*21*	*1428*
16	*8*	*22*	*1408*

The above measurements are for guidance only. Your actual measurements may vary due to variation in the module reproduction processes and measurement error.

5. Based on observation, complete the following statements for polygons and circles with congruent radii.
   - As *n* increases, the length of side *s* *decreases*.

- As n increases, the apothem length a *increases*.
- As n increases, the area of the polygon *increases*.
- As the chord length s decreases, the distance, a, of the chord from the center of the circle *increases*.
- As the chord length s decreases, the length of its intercepted arc *decreases*.

As n approaches infinity ($n \to \infty$), determine the limits of P, the perimeter, and a, the apothem. Then determine the limit of the area of the polygon in terms of the limits of P and a.

As $n \to \infty$, the length of the perimeter, P, approaches the length of the circumference, C, of the circle, and the length of the apothem, a, approaches the length of the radius of the circle, r.

$P \to C = 2\pi r$.

$$\text{Area of polygon} = \frac{1}{2}Pa$$

$$\text{Area of polygon} \to \frac{1}{2}Cr = \frac{1}{2}(2\pi r)r = \pi r^2.$$

The area of a regular polygon approaches the area of its circumscribed circle as the number of sides of the polygon approaches infinity.

Measure the length of the radius in millimeters and determine the area of the circle.
The radius measures 22 mm. The area of the circle is approximately 1520 mm^2.

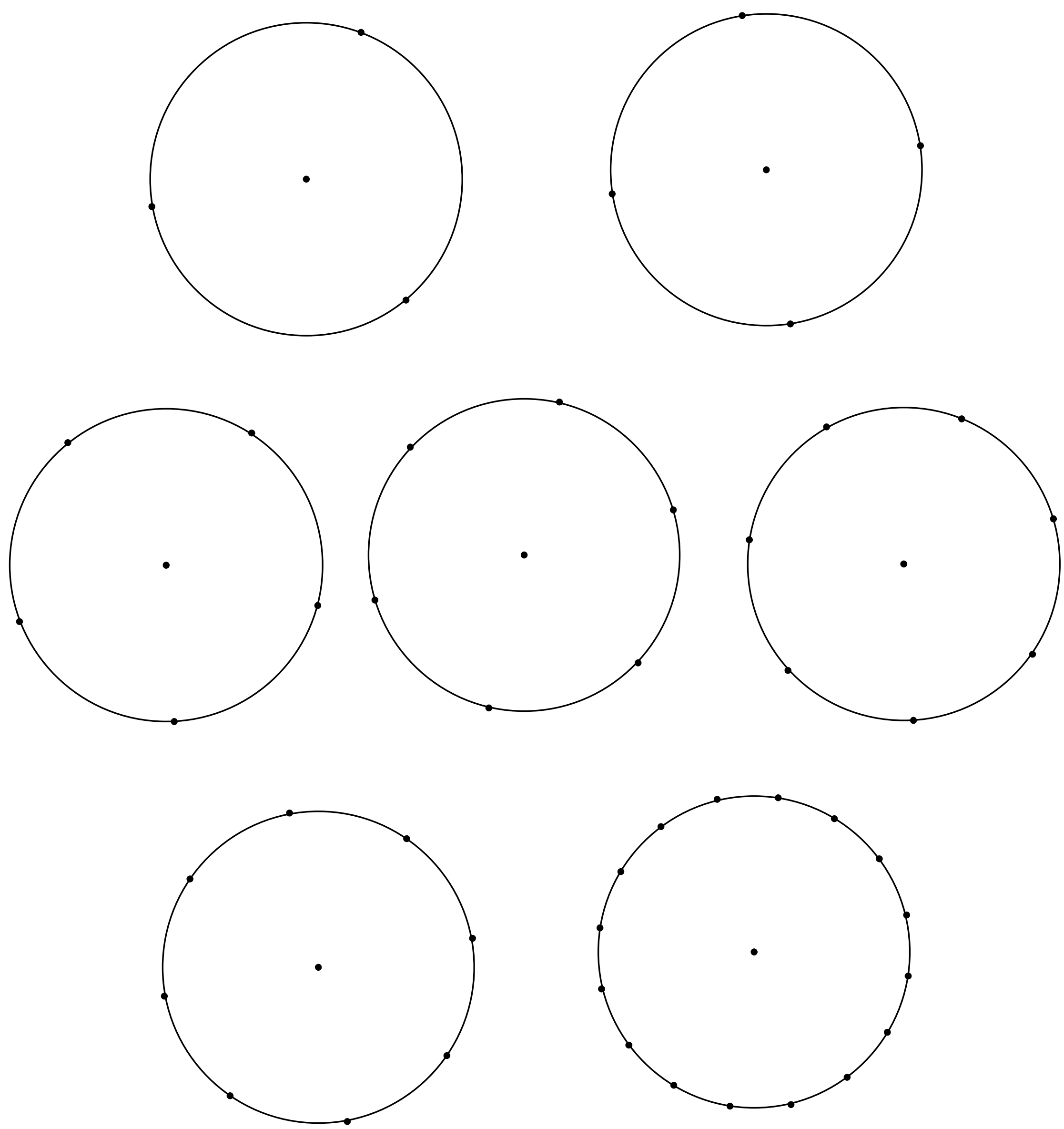
Note: For the sample data, the lengths of the apothem for the 16-gon and for the radius of the circle are 22 mm. Increasing the number of sides of the polygon to numbers greater than sixteen will not change the length of the apothem within measurement tolerance. The factors which affect the area for large values of n are the value of n and the length of s. As n increases, s decreases. The area will continue to increase until the limiting value, the area of the circle, is reached.

Participants are performing at the van Hiele Deductive Level in this activity, because properties of polygons and parts of polygons are intrarelated and interrelated with the corresponding properties and parts of circles, and because formulas and conclusions are determined deductively.

Polygons in Circles

Each group member should complete the activity for one or more different polygons so that all the polygons are considered. Share your findings with others in your group as you complete the activity.

1. To construct a regular polygon with *n* sides, connect *n* equally-spaced points on the circumference of a circle. Connect the points on the following congruent circles with a straightedge to complete the regular polygons. Name each polygon.



2. Draw segments connecting the center of each circle to the vertices of its inscribed polygon. The segments are called the radii of the polygons.

   The isosceles triangles formed by the radii and the sides of the polygon can be used to find the area of each polygon. In each polygon draw an altitude to the base of one of the isosceles triangles. This segment is called the apothem of the polygon.

   Find the area of a regular polygon in terms of the number of sides, n, the length of a side, s, and the length of the apothem, a.

3. Find the area of a regular polygon in terms of its perimeter P and the length of the apothem, a.

4. Use a ruler to measure, to the nearest millimeter, the length of a side and the length of an apothem for each of the regular polygons. Calculate the area of your polygon(s). Complete the following table by sharing your measurements and calculations with members of your group.

Number of sides ***n***	**Side Length** ***s*** **(mm)**	**Apothem Length** ***a*** **(mm)**	**Polygon Area** **(mm^2)**
3			
4			
5			
6			
7			
8			
16			

5. Based on observation, complete the following statements for polygons and circles with congruent radii.
   - As n increases, the side length s __________.
   - As n increases, the apothem length a __________.
   - As n increases, the area of the polygon __________.
   - As the chord length s decreases, the distance a of the chord from the center of the circle __________.
   - As the chord length s decreases, the length of its intercepted arc __________.

As n approaches infinity ($n \rightarrow \infty$), determine the limits of P, the perimeter, and a, the apothem. Then determine the limit of the area of the polygon in terms of the limits of P and a.

Angles Associated with a Circle

Overview: This activity illustrates the properties of angles, chords, and tangents of a circle.

Objective: **TExES Mathematics Competencies**
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
V.018.A. The beginning teacher recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of a circle as a quadratic function of the radius, probability as the ratio of two areas, area of a plane region as a definite integral).
V.018.C. The beginning teacher uses inductive reasoning to make conjectures and uses deductive methods to evaluate the validity of conjectures.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

Geometry TEKS
b.2.A. The student uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships.
b.2.B. The student makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.
b.3.B. The student constructs and justifies statements about geometric figures and their properties.
b.3.D. The student uses inductive reasoning to formulate a conjecture.
b.3.E. The student uses deductive reasoning to prove a statement.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

Background: Participants need to know circle terms such as central and inscribed angle.

Materials: protractor, centimeter ruler

New Terms: cyclic quadrilateral, major arc, minor arc

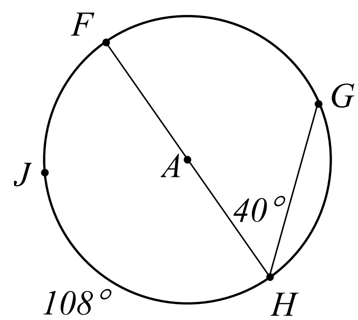
Procedures:

Have participants recall the definitions of a central angle and an inscribed angle and that the measure of an inscribed angle is equal to half the measure of the central angle that intercepts the same arc. Also participants should recall what is meant by a secant line and a tangent line to a circle.

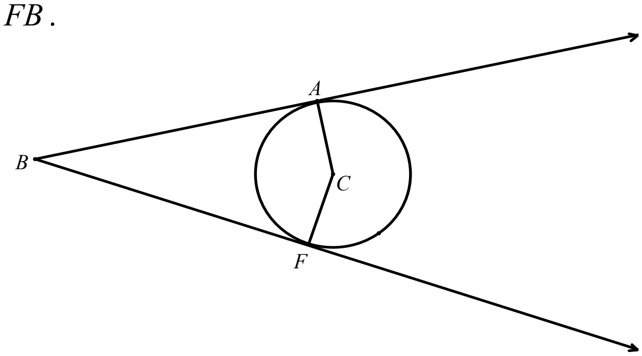
We will define two new terms. A *minor arc* is an arc of a circle that is smaller than a semicircle. A *major arc* is an arc of a circle that is larger than a semicircle. Remind participants to add these terms to their glossaries.

Participants will use deduction to prove relationships among angles formed by secant and tangent lines to a circle and the associated arcs that are intercepted by these arcs. For that reason many of the solutions are written as proofs.

1. Determine $m\overset{\frown}{FJ}$ and $m\overset{\frown}{GH}$ in circle *A* below. Explain.
   Angle H intercepts $\overset{\frown}{FG}$.
   Since $m\angle H = 40^\circ$, *then* $m\overset{\frown}{FG} = 80^o$
   Together, $\overset{\frown}{FG}$ *and* $\overset{\frown}{GH}$ *form a semicircle.*
   Therefore, $m\overset{\frown}{GH}$ *is* $180^\circ - 80^\circ$, *or* 100°.
   $\overset{\frown}{FJ}$ *and* $\overset{\frown}{FH}$ *also form a semicircle.*
   Therefore, $m\overset{\frown}{FJ}$ *is* $180^\circ - 108^\circ$, *or* 72°.



2. In the figure, which is drawn to scale, $\overrightarrow{BA}$ is tangent to circle *C* at *A*; $\overrightarrow{BF}$ is tangent to circle *C* at *F*. Confirm that $\angle CAB$ and $\angle CFB$ are right angles. Use this fact to prove that $\overline{AB} \cong \overline{FB}$.

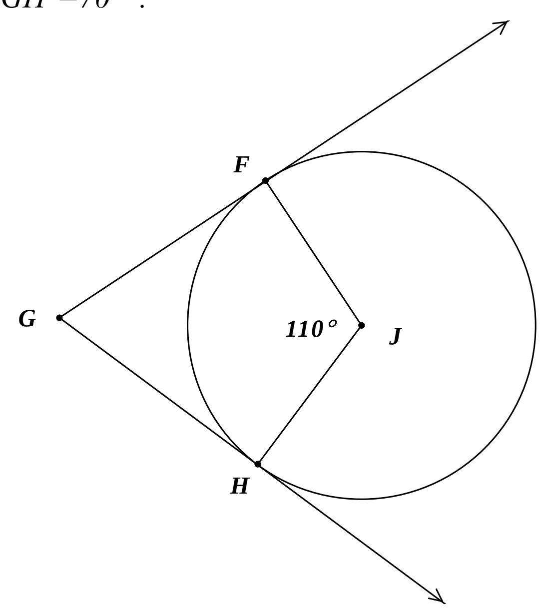


Participants may use the corner of a sheet of paper or a protractor to confirm that $\angle CAB$ and $\angle CFB$ are right angles.

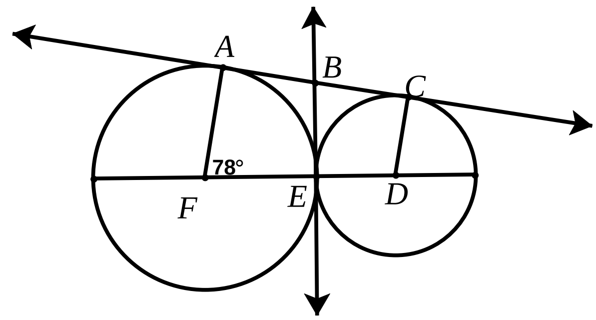
Draw $\overline{BC}$	(*Through any two points a unique line can be drawn.*)
ΔCFB *and* ΔCAB *are right triangles*	(*Definition of right triangles*)
$\overline{AC} \cong \overline{CF}$	(*All radii in a circle are congruent.*)
$\overline{BC} \cong \overline{BC}$	(*Reflexive Property of Equality*)
$\Delta CFB \cong \Delta CAB$	(*If the hypotenuse and a leg of one right triangle are congruent to the corresponding hypotenuse and leg of another right triangle, then the triangles are congruent.*)
$\overline{AB} \cong \overline{FB}$	(*Corresponding sides of congruent triangles are congruent.*)

This problem has two important results that need to be emphasized: 1) the tangent line is perpendicular to the radius at the point of tangency and 2) tangent segments to a circle from a point outside the circle are congruent.

3. $\overrightarrow{GF}$ and $\overrightarrow{GH}$ are tangent to circle *J*. Determine $m\angle FGH$.
   Angle GFJ *and* $\angle GHJ$ *are right angles. Since the sum of the measures of the interior angles of a quadrilateral is* 360^o, $\angle FGH$ *and* $\angle FJH$ *are supplementary. Therefore,* $m\angle FGH = 70^\circ$.



4. $\overleftrightarrow{AC}$ is tangent to circle *F* at *A* and to circle *D* at *C*. $\overleftrightarrow{EB}$ is tangent to circles *D* and *F* at *E*. Determine $m\angle ABE$ and $m\angle CDE$. Explain.

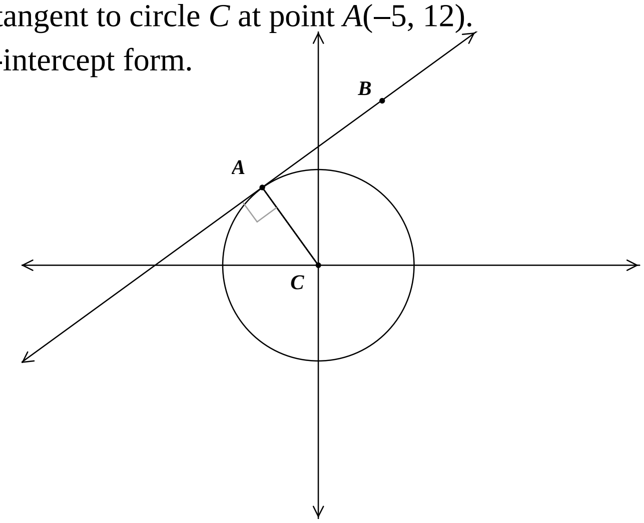


$m\angle ABE$ and $m\angle CDE$ are equal. $\angle ABE$ and $\angle CDE$ are each angles in two quadrilaterals that have a 78^o angle and two 90^o angles. Since the sum of the measures of the interior angles equals to 360^o in each quadrilateral then,

$m\angle ABE = 360^o - (2\cdot 90^o + 78^o)$ *and* $m\angle CDE = 360^o - (2\cdot 90^o + 78^o)$

$m\angle ABE = 102^O$ $m\angle CDE = 102^o$

5. Circle *C* is centered at the origin. $\overleftrightarrow{AB}$ is tangent to circle *C* at point *A*(–5, 12). Determine the equation of $\overleftrightarrow{AB}$ in slope-intercept form.



Draw the radius from the origin to the point of tangency, A. $\overline{CA}$ is perpendicular to $\overleftrightarrow{AB}$. Determine the slope of $\overline{CA}$.

$$m = \frac{12-0}{-5-0}$$

$$m = \frac{-12}{5}$$

The slope of tangent $\overleftrightarrow{AB}$ is the negative reciprocal of this slope. Write the equation of $\overleftrightarrow{AB}$ in slope-intercept form. Use point $A(-5,12)$ to determine the y-intercept.

$$y = \frac{5}{12}x + b$$

$$12 = \frac{5}{12}(-5) + b$$

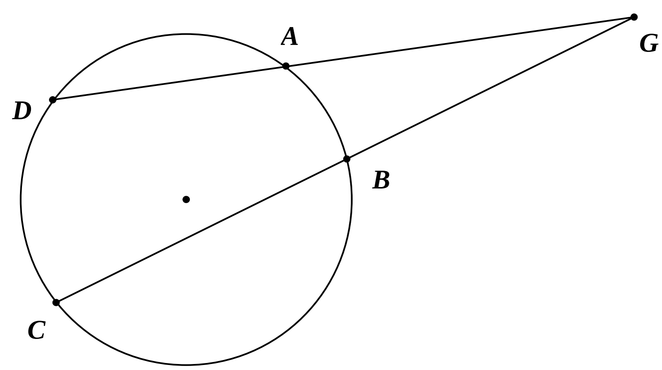
$$12 = \frac{-25}{12} + b$$

$$b = \frac{169}{12}$$

The equation of $\overleftrightarrow{AB}$ *is* $y = \frac{5}{12}x + \frac{169}{12}$.

In 6 and 7, determine $m\angle AGB$ in terms of $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$.

6. Hint: Construct $\overline{BD}$.
   $\angle ADB$ *and* $\angle CBD$ *are inscribed angles.*



$$m\angle ADB = \frac{1}{2} \cdot m\overset{\frown}{AB}$$
$$\text{and } m\angle DBC = \frac{1}{2} \cdot m\overset{\frown}{CD}$$

(*The measure of an inscribed angle is half of the measure of its intercepted arc.*)

$$m\angle DBC = m\angle ADB + m\angle G$$

(*The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.*)

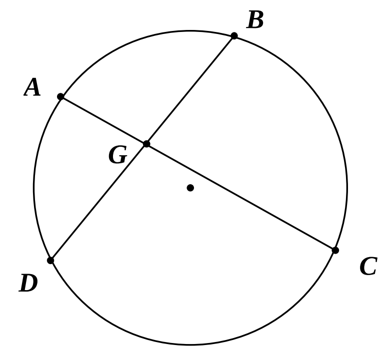
$$\frac{1}{2} \cdot m\overset{\frown}{CD} = \frac{1}{2} \cdot m\overset{\frown}{AB} + m\angle G$$

(*Substitution*)

$$m\angle G = \frac{1}{2} \cdot m\overset{\frown}{CD} - \frac{1}{2} \cdot m\overset{\frown}{AB}$$
$$\text{or } m\angle G = \frac{1}{2}\left(m\overset{\frown}{CD} - m\overset{\frown}{AB}\right)$$

(*Subtraction*)

7. Hint: Construct $\overline{BC}$.



$m\angle ACB = \frac{1}{2} \cdot m\overset{\frown}{AB}$ (*The measure of an inscribed angle is half of the measure of its intercepted arc.*)

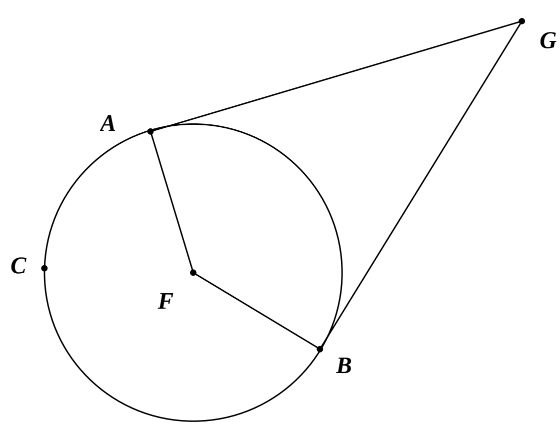
$m\angle CBD = \frac{1}{2} \cdot m\overset{\frown}{CD}$

$m\angle AGB = m\angle ACB + m\angle DBC$ (*The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles.*)

$m\angle AGB = \frac{1}{2} \cdot m\overset{\frown}{AB} + \frac{1}{2} \cdot m\overset{\frown}{CD}$ (*Substitution*)

or $m\angle AGB = \frac{1}{2}\left(m\overset{\frown}{AB} + m\overset{\frown}{CD}\right)$

8. $\overline{AG}$ and $\overline{BG}$ are tangent to circle *F*. Determine $m\angle G$ in terms of $\overset{\frown}{AB}$ alone and then in terms of $\overset{\frown}{AB}$ and $\overset{\frown}{ACB}$.



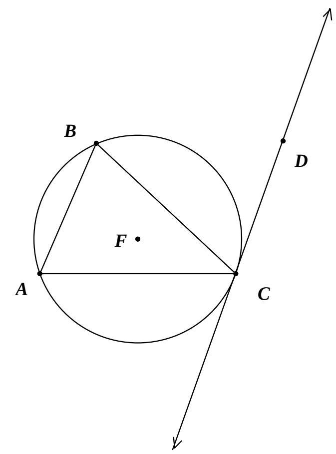
$FB \perp BG\ ;\ FA \perp AG$	(*A tangent line is perpendicular to the radius at the point of tangency.*)
$\angle A$ *and* $\angle B$ *are right angles*	(*Definition of perpendicular lines*)
$m\angle A = m\angle B = 90^o$	(*Definition of right angles*)
$m\angle F = m\overset{\frown}{AB}$	(*A central angle has a measure equal to that of its intercepted arc.*)
$m\angle A + m\angle F + m\angle G + m\angle B = 360^o$	(*The sum of the measures of the angles in a quadrilateral is* 360^o.)
$90^o + m\angle G + 90^o + m\overset{\frown}{AB} = 360^o$	(*Substitution Property*)
$m\angle G = 180^o - m\overset{\frown}{AB}$	(*Subtraction*)

This leads to the conclusion that $\angle G$ *and* $\angle F$ *are supplementary.*

In addition,

$m\overset{\frown}{AB} = 360^o - m\overset{\frown}{ACB}$	(*The sum of the arcs in a circle is* 360^o.)
$m\angle G = 180^o - \left(360^o - m\overset{\frown}{ACB}\right)$ $m\angle G = m\overset{\frown}{ACB} - 180^o$	(*Substitution*)
$m\angle G + m\angle G = m\overset{\frown}{ACB} - 180^o + 180^o - m\overset{\frown}{AB}$ $2 \cdot m\angle G = m\overset{\frown}{ACB} - m\overset{\frown}{AB}$	(*Addition*)
$m\angle G = \frac{1}{2}(m\overset{\frown}{ACB} - m\overset{\frown}{AB})$	(*Division*)

9. $\overleftrightarrow{CD}$ is tangent to circle F at C. Determine the relationship between the measures of $\angle BCD$ and $\angle BAC$.



$\overline{FC} \perp \overline{CD}$	(*A tangent line is perpendicular to the radius at the point of tangency.*)
$\angle FCD$ *is a right angle*	(*Definition of perpendicular lines*)
$m\angle FCD = 90^o$	(*Definition of right angle*)
$m\angle FCD = m\angle FCB + m\angle BCD$	(*Angle Addition*)
$m\angle FCB + \angle BCD = 90^o$	(*Substitution*)
$m\angle FCB = 90^o - m\angle BCD$	(*Subtraction*)
$\overline{FC} \cong \overline{FB}$	(*All radii in a circle are congruent.*)
$\angle FBC \cong \angle FCB$	(*If two sides of a triangle are congruent, then the angles opposite those sides are congruent.*)
$m\angle FBC = 90^o - m\angle BCD$	(*Substitution*)
$m\angle FBC + m\angle FCB + m\angle CFB = 180^o$	(*The sum of the measures of the angles in a triangle is* 180^o.)
$90^o - m\angle BCD + 90^o - m\angle BCD + m\angle CFB = 180^o$	(*Substitution*)
$m\angle CFB = 2 \cdot m\angle BCD$	(*Subtraction*)

$m\angle CFB = m\overset{\frown}{BC}$ (*A central angle has a measure equal to that of its intercepted arc.*)

$m\angle BAC = \frac{1}{2} \cdot m\overset{\frown}{BC}$ (*The measure of an inscribed angle is half of the measure of its intercepted arc.*)

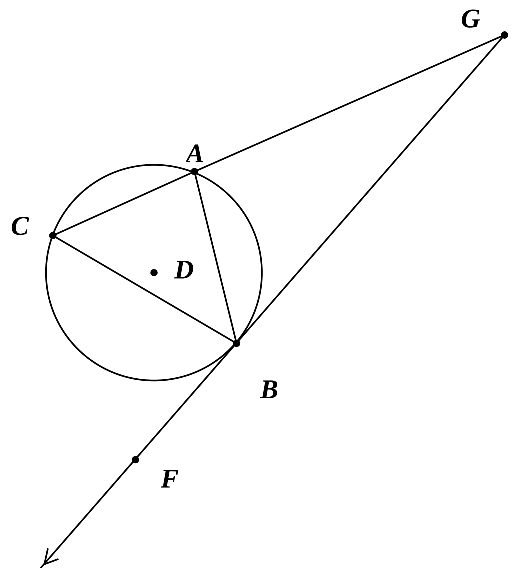
$m\overset{\frown}{BC} = 2 \cdot m\angle BAC$ (*Multiplication*)

$2 \cdot m\angle BAC = 2 \cdot m\angle BCD$ (*Substitution*)

$m\angle BAC = m\angle BCD$ (*Division*)

In addition, we can see that $m\angle BCD = \frac{1}{2} \cdot m\overset{\frown}{BC}$.

10. $\overline{BG}$ is tangent to circle *D* at *B*. Determine the measure of $\angle G$ in terms of the intercepted arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CB}$.



$\angle ABG \cong \angle ACB$ (*The angle between a tangent and a chord is congruent to the inscribed angle intercepted by the arc on the same side as the tangent.*)

$m\angle ACB = \frac{1}{2} m\overset{\frown}{AB}$ *and* $m\angle BAC = \frac{1}{2} m\overset{\frown}{BC}$ (*The measure of an inscribed angle is equal to half of the measure of its intercepted arc.*)

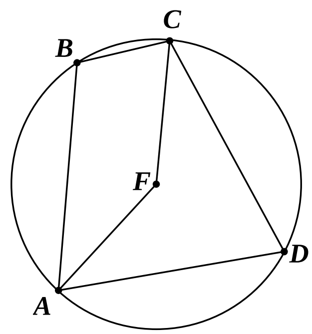
$m\angle BAC = m\angle G + m\angle ABG$ (*The measure of the exterior angle of a triangle is equal to the sum of the measures of the interior remote angles.)*

$m\angle BAC = m\angle G + m\angle ACB$ (*Substitution*)

$\frac{1}{2} m\overarc{BC} = m\angle G + \frac{1}{2} m\overarc{AB}$ (*Substitution*)

$m\angle G = \frac{1}{2} m\overarc{BC} - \frac{1}{2} m\overarc{AB} = \frac{1}{2}\left(m\overarc{BC} - m\overarc{AB}\right)$ (*Subtraction*)

11. A quadrilateral inscribed in a circle is a *cyclic quadrilateral*. Determine the relationships among the various arcs and inscribed angles.



Draw $\overline{FC}$ *and* $\overline{FA}$.

$m\angle CDA = \frac{1}{2} m\overarc{ABC}$

and $m\angle ABC = \frac{1}{2} m\overarc{CDA}$

(*The measure of an inscribed angle is equal to half of the measure of its intercepted arc.*)

$m\overarc{ABC} + m\overarc{CDA} = 360^o$. (*The sum of the measures of the arcs in a circle is* 360^o)

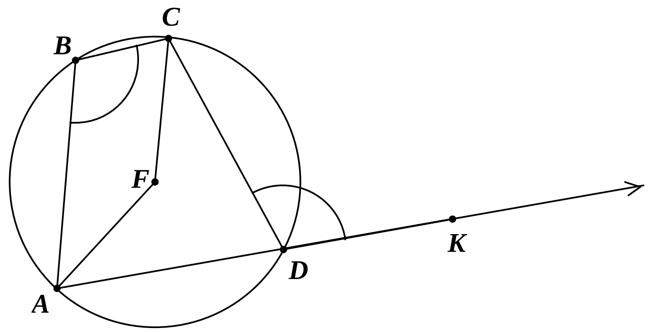
$m\angle CDA + m\angle ABC = \frac{1}{2}\left(m\overarc{ABC} + m\overarc{CDA}\right)$ (*Substitution*)

$m\angle CDA + m\angle ABC = 180^O$ (*Substitution*)

Similarly, $m\angle BCD + m\angle BAD = 180^O$

Opposite angles in a cyclic quadrilateral are supplementary.

Determine the relationship between the exterior angle to the diagonally opposite interior angle of a cyclic quadrilateral.



$m\angle CDA + m\angle ABC = 180^O$ *and* *(Linear pair)*

$m\angle CDA + m\angle CDK = 180^O$

$m\angle CDK = m\angle ABC$ *(Substitution)*

An exterior angle of a cyclic quadrilateral is congruent to the diagonally opposite interior angle.

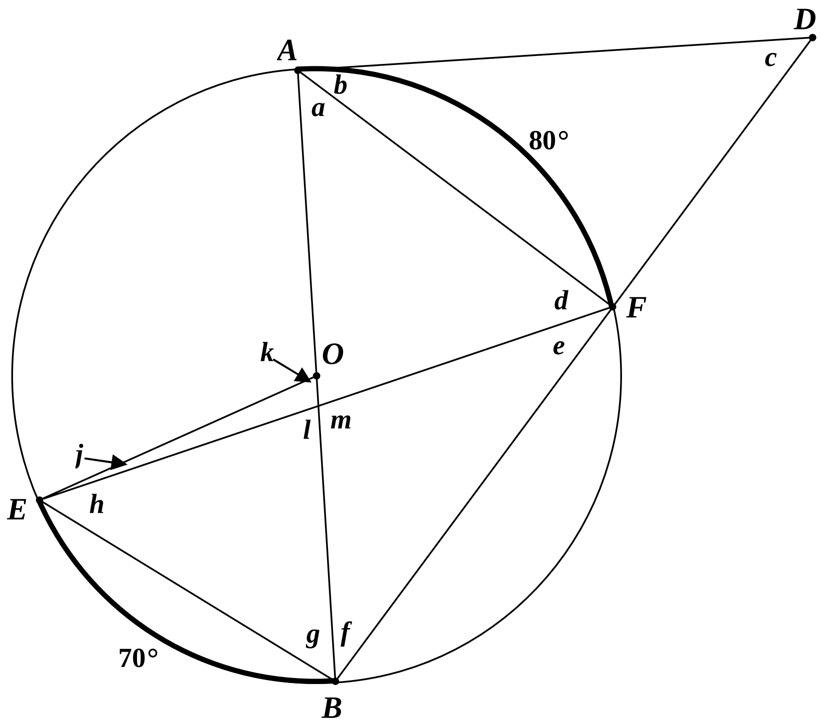
Remind participants to add the term cyclic quadrilateral to their glossaries.

12. $\overline{AD}$ is a tangent segment to circle *O* at *A*. $\overline{AB}$ is a diameter. $\overline{BD}$ intersects the circumference at *F*. $m\overset{\frown}{AF} = 80^o$. *E* is a point on $\overset{\frown}{AB}$, on the opposite arc to $\overset{\frown}{AFB}$, such that $m\overset{\frown}{EB} = 70^o$. Find the measures of the marked angles.

a = $\underline{50^o}$ *b* = $\underline{45^o}$ *c*= $\underline{50^o}$ *d*= $\underline{55^0}$ *e*= $\underline{35^o}$ *f* = $\underline{40^o}$

g= $\underline{55^o}$ *h*=$\underline{50^o}$ *j*= $\underline{5^o}$ *k*= $\underline{70^o}$ *l*= $\underline{75^o}$ *m* = $\underline{105^o}$

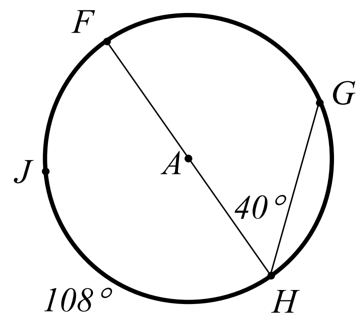
One possible starting point is to determine m $\overset{\frown}{AE}$ and m $\overset{\frown}{BF}$ using diameter $\overline{AB}$. Then, determine the measures of central $\angle BOE$ and inscribed angles FAB, AFE, BFE, ABE, ABF, and BEF. Then, determine $m\angle FAD$. The remaining angles may be determined using the sum of the measures of the interior angles of a triangle as well as the definition of a linear pair.



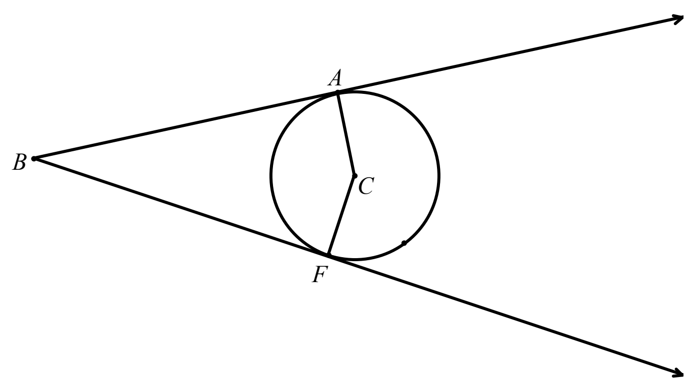
Participants are performing at the van Hiele Deductive Level in this activity, because they are asked to provide deductive arguments.

Angles Associated with a Circle

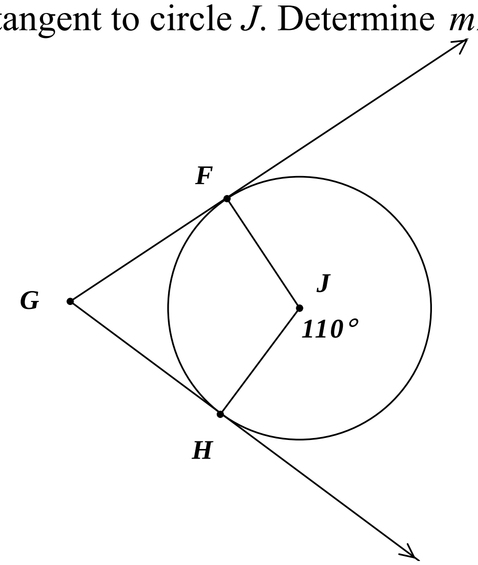
1. Determine $m\widehat{FJ}$ and $m\widehat{GH}$ in circle *A* below. Explain.



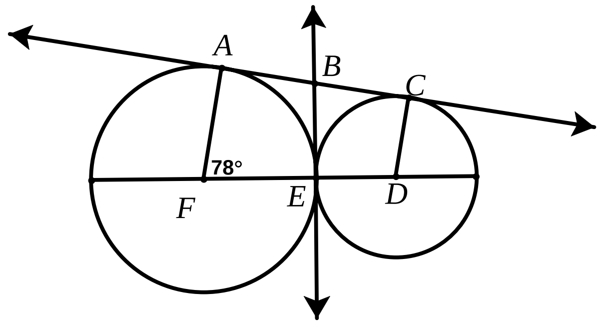
2. In the figure, which is drawn to scale $\overrightarrow{BA}$ is tangent to circle *C* at *A*; $\overrightarrow{BF}$ is tangent to circle *C* at *F*. Confirm that $\angle CAB$ and $\angle CFB$ are right angles. Use this fact to prove that $\overline{AB} \cong \overline{FB}$.



3. $\overrightarrow{GF}$ and $\overrightarrow{GH}$ are tangent to circle *J*. Determine $m\angle FGH$.



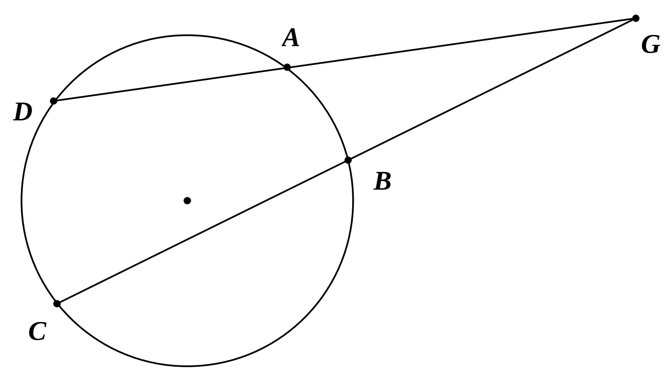
4. $\overleftrightarrow{AC}$ is tangent to circle F at A and to circle D at C. $\overleftrightarrow{EB}$ is tangent to circles D and F at E. Determine $m\angle ABE$ and $m\angle CDE$. Explain.



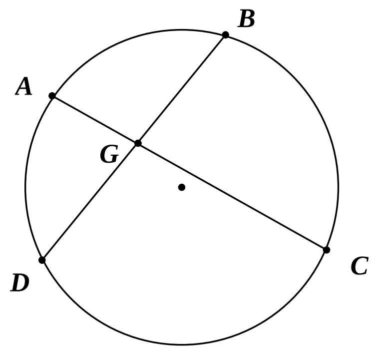
5. Circle C is centered at the origin. $\overleftrightarrow{AB}$ is tangent to circle C at point $A(-5, 12)$. Determine the equation of $\overleftrightarrow{AB}$ in slope-intercept form.

In 6 and 7, determine $m\angle AGB$ in terms of $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$.

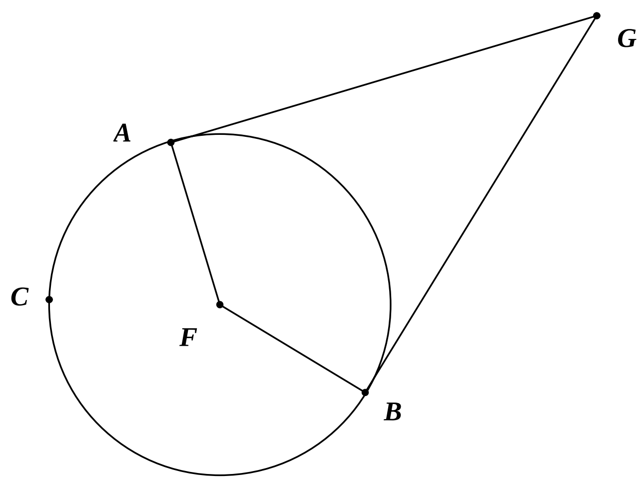
6. Hint: Construct $\overline{BD}$.



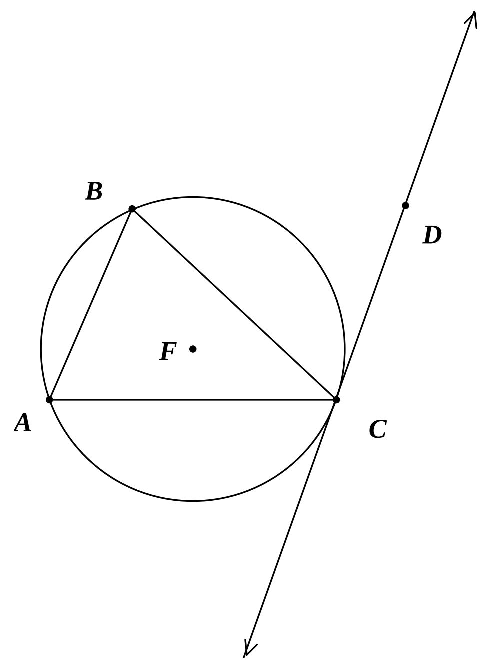
7. Hint: Construct $\overline{BC}$.



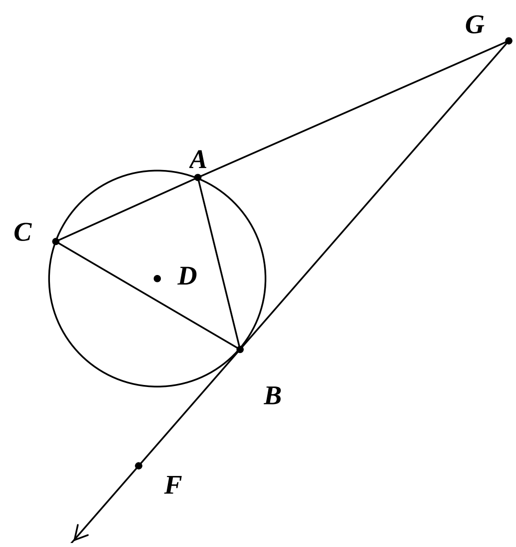
8. $\overline{AG}$ and $\overline{BG}$ are tangent to circle *F*. Determine $m\angle G$ in terms of $\overset{\frown}{AB}$ alone and then in terms of $\overset{\frown}{AB}$ and $\overset{\frown}{ACB}$.



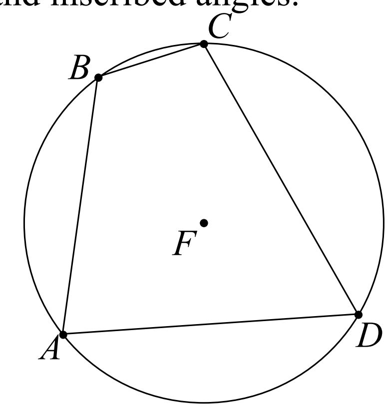
9. $\overleftrightarrow{CD}$ is tangent to circle *F* at *C*. Determine the relationship between the measures of $\angle BCD$ and $\angle BAC$.



10. $\overline{BG}$ is tangent to circle *D* at *B*. Determine the measure of $\angle G$ in terms of the intercepted arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CB}$.



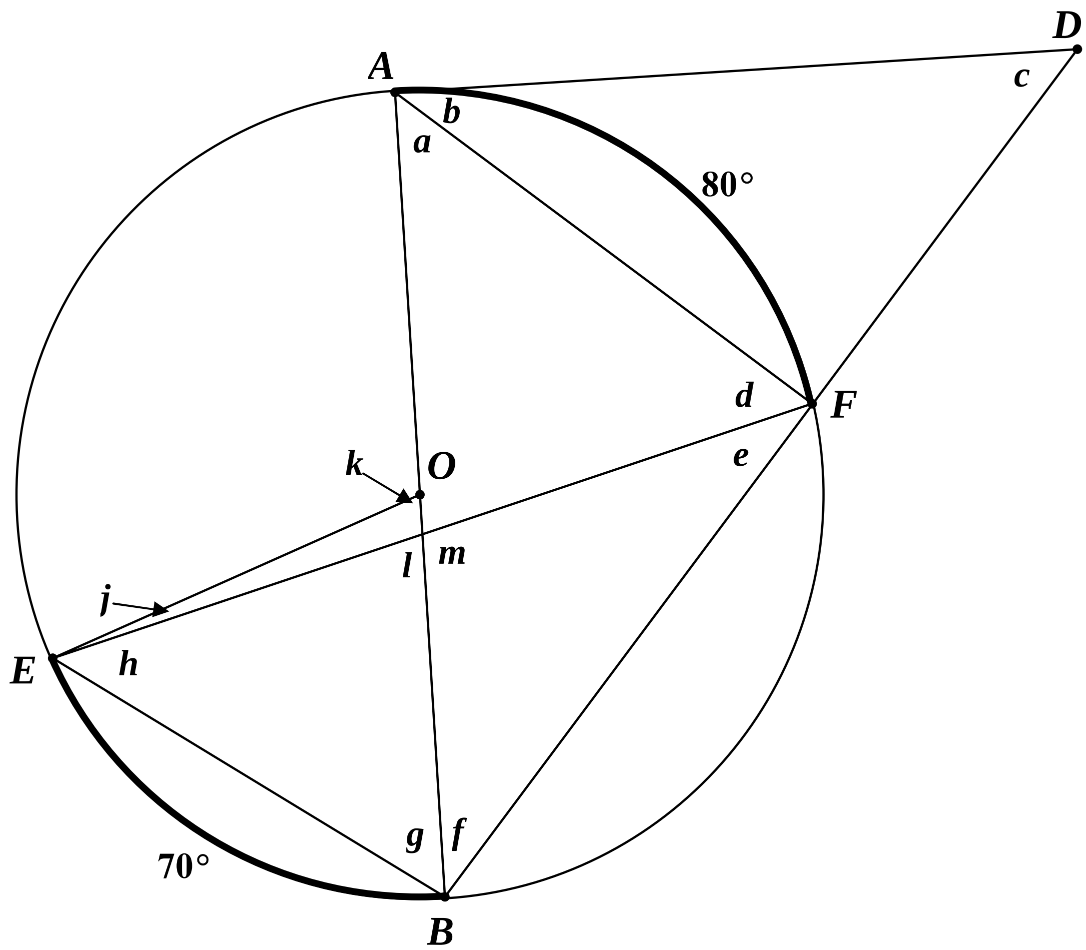
11. A quadrilateral inscribed in a circle is a cyclic quadrilateral. Determine the relationships among the various arcs and inscribed angles.



12. $\overline{AD}$ is a tangent segment to circle O at A. $\overline{AB}$ is a diameter. $\overline{BD}$ intersects the circumference at F. $m\overset{\frown}{AF} = 80^{o}$. E is a point on $\overset{\frown}{AB}$, on the opposite arc to $\overset{\frown}{AFB}$, such that $m\overset{\frown}{EB} = 70^{o}$. Find the measures of the marked angles.

$a =$ ____ $b =$ ____ $c =$ ____ $d =$ ____ $e =$ ____ $f =$ ____

$g =$ ____ $h =$ ____ $j =$ ____ $k =$ ____ $l =$ ____ $m =$ ____



Parts of a Circle

Overview: In this activity, participants determine the areas of sectors as proportional parts of the area of the whole circle, areas of segments as parts of the areas of sectors, and the areas of annuli as the difference between concentric circles.

Objective: **TExES Mathematics Competencies**

III.011.A. The beginning teacher applies dimensional analysis to derive units and formulas in a variety of situations (e.g., rates of change of one variable with respect to another) and to find and evaluate solutions to problems.
III.011.D. The beginning teacher applies the Pythagorean theorem, proportional reasoning, and right triangle trigonometry to solve measurement problems.
III.012.B. The beginning teacher uses properties of points, lines, planes, angles, lengths, and distances to solve problems.
III.013.B. The beginning teacher analyzes the properties of circles and the lines that intersect them.
III.013.D. The beginning teacher computes the perimeter, area, and volume of figures and shapes created by subdividing and combining other figures and shapes.
V.018.D. The beginning teacher uses formal and informal reasoning to justify mathematical ideas.

Geometry TEKS

b.3.B. The student constructs and justifies statements about geometric figures and their properties.
c.1. The student uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.
d.2.C. The student develops and uses formulas including distance and midpoint.
e.1.A. The student finds area of regular polygons and composite figures.
e.1.B. The student finds area of sectors and arc lengths of circles using proportional reasoning.
e.2.C. Based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.
f.2. The student uses ratios to solve problems involving similar figures.

Background: Participants need to know circle terms, such as the various angles of a circle, properties of chords, and tangent properties, and the formula for the area of a circle.

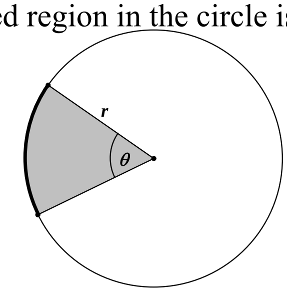
Materials: compass, centimeter ruler, calculator, easel paper, colored markers

New Terms: annulus (plural is annuli), sector of a circle, segment of a circle

Procedures:

Participants will define and determine methods for finding areas of sectors, segments of circles, and annuli. Participants complete 1 – 5 in their groups (about ten minutes). During whole-group discussion, ask participants to share their strategies for completing 1-5. The remaining items may be completed in groups and presented on easel paper if desired. These problems are also suitable for a homework assignment.

1. The shaded region in the circle is called a sector. Define *sector of a circle*.



A sector is the region bounded by an arc and the two radii to the endpoints of the arc.

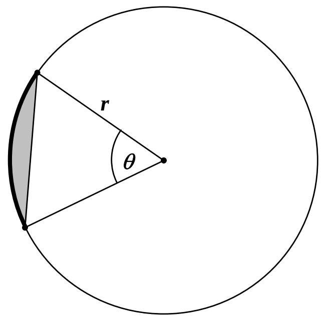
2. The length of the arc and the area of the sector may be related to the length of the circumference and the area of the circle, respectively. Find the length of the arc and the area of the sector in terms of the radius of the circle, *r*, and the central angle, *θ*, measured in degrees, which determines the sector.
The ratio of the measure of the central angle to 360º is equal to the ratio of the arc length to the length of the circumference, and also to the ratio of the area of the sector to the area of the circle.

$$\frac{\theta}{360^o} = \frac{arc\ \ length}{2\pi r} = \frac{area\ \ of\ \ sector}{\pi r^2}.$$

$$Arc\ length = \frac{\theta}{360^o} 2\pi r.$$

$$Area\ of\ sector = \frac{\theta}{360^o} \pi r^2.$$

3. The shaded region is a segment of a circle. Define *segment of a circle.*

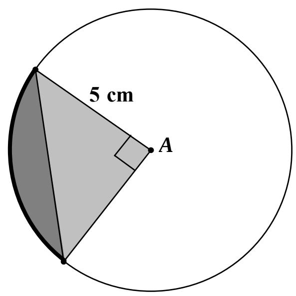


A segment of a circle is the region between a chord of a circle and the included arc.

Explain how to find the area of the segment of a circle.

The area of a segment can be determined by subtracting the area of the triangle bounded by the two radii and the chord from the area of the sector.

4. Determine the areas of the shaded segment and the sector for each circle.

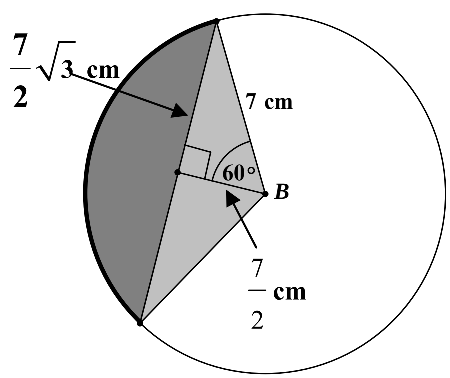


Circle A:

Area of sector $= \frac{90^o}{360^o}\pi \cdot 5^2 = \frac{25\pi}{4} cm^2$.

Area of segment = Area of sector – Area of triangle.

Area of segment $= \frac{25\pi}{4} - \frac{25}{2} cm^2$.



Circle B:

Area of sector $= \frac{120^o}{360^o}\pi \cdot 7^2 = \frac{49\pi}{3} cm^2$.

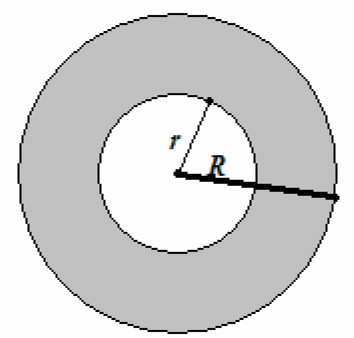
Area of segment =Area of sector – Area of triangle.

Area of triangle $= \frac{7}{2}\left(\frac{7}{2}\sqrt{3}\right) = \frac{49}{4}\sqrt{3}\, cm^2$.

Area of segment $= \frac{49\pi}{3} - \frac{7}{2}\left(\frac{7}{2}\sqrt{3}\right)$

$= \frac{49\pi}{3} - \frac{49\sqrt{3}}{4} cm^2$.

5. The cross section of a doughnut or bagel is a real-world example of an annulus (plural, annuli). Using your compass, draw an annulus. Label the radius of the smaller (inner) circle *r* and the radius of the larger (outer) circle *R*. Define *annulus*. Explain how to find the area of an annulus.



An annulus is the region bounded by two concentric circles.

Area of annulus $= \pi R^2 - \pi r^2$

Area of annulus $= \pi (R^2 - r^2)$.

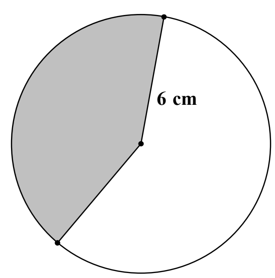
Remind participants to add the new terms sector of a circle, segment of a circle, and annulus to their glossaries.

6. Determine the area of an annulus with the radius, *r*, of the smaller circle 6 cm and the radius, *R*, of the larger circle 10 cm.

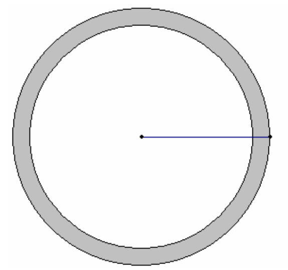
$$\begin{aligned} \textit{Area of annulus} &= \pi (R^2 - r^2) \\ &= \pi (10^2 - 6^2) \\ &= 64\pi \text{ cm}^2. \end{aligned}$$

7. Determine the measure of the central angle in the shaded region given that the area of the shaded sector is 15π cm^2 and the radius is 6 cm.

$$\begin{aligned} \textit{Area of sector} &= \frac{\theta}{360} \cdot \pi r^2 \\ 15\pi &= \frac{\theta}{360} \cdot \pi 6^2 \\ 15\pi &= \frac{\pi\theta}{10} \\ \theta &= 150^o \end{aligned}$$

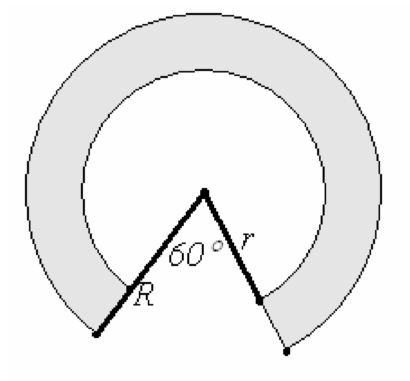


8. Determine r, the radius of the smaller circle, given the shaded area is 30π cm^2 and the radius of the larger circle is 18 cm.



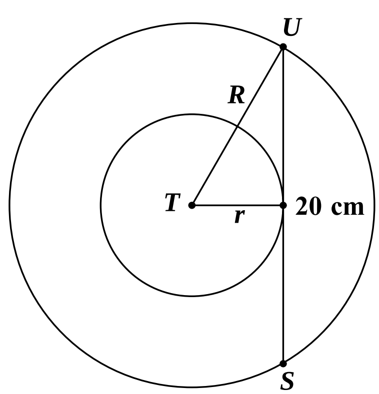
$$\begin{aligned} \text{Area of annulus} &= \pi R^2 - \pi r^2 \\ 30\pi &= \pi 18^2 - \pi r^2 \\ 30\pi &= 324\pi - \pi r^2 \\ \pi r^2 &= 294\pi \\ r &\approx 17.15 \text{ cm} \end{aligned}$$

9. Determine the radius of each circle given that the area of the shaded region is 15π cm^2 and $r = \frac{2}{3}R$.



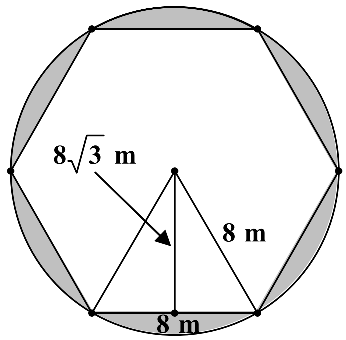
$$\begin{aligned} \text{Area of annulus} &= \frac{5}{6}\left(\pi R^2 - \pi r^2\right) \\ 150\pi &= \frac{5\pi}{6}\left[R^2 - \left(\frac{2}{3}R\right)^2\right] \\ 180 &= R^2 - \frac{4R^2}{9} \\ 180 &= \frac{5R^2}{9} \\ 324 &= R^2 \\ R &= 18 \text{ cm} \\ r &= \frac{2}{3}\cdot 18 \text{ cm} \\ r &= 12 \text{ cm} \end{aligned}$$

10. In the figure, $\overline{US}$ is a chord of the larger concentric circle and tangent to the smaller concentric circle. US = 20 cm. Find the area of the annulus.



Chord $\overline{US}$ is bisected at the point of tangency, since ΔSTU is an isosceles triangle.
Area of annulus = $\pi(R^2 - r^2)$.
Using the Pythagorean Theorem,
$R^2 - r^2 = 10^2$ or $R^2 - r^2 = 100$.
Area of annulus = 100π cm^2.

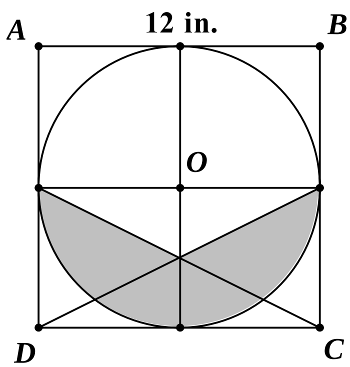
11. A circle circumscribes a hexagon with side length of 8 meters. Find the area of the region between the hexagon and the circle.



Area of shaded region = Area of circle – Area of hexagon.

$$\text{Area of shaded region } = \pi r^2 - \frac{1}{2}Pa$$
$$= 64\pi - \frac{1}{2}48(8\sqrt{3})$$
$$= 64\pi - 192\sqrt{3}\ m^2.$$

12. Circle *O* is inscribed in the square *ABCD* with sides of length 12 inches. The diameters shown are perpendicular bisectors of $\overline{AB}$ and $\overline{AD}$. Find the area of the *unshaded* region inside the square.



The length of the radius of O is 6 in.

Shaded area = *Area of semi-circle – Area of* ΔPQR

$$\text{Shaded area } = \frac{\pi \cdot 6^2}{2} - \frac{1}{2}(12)(3)$$
$$= 18\pi - 18\, in^2$$

$$\text{Area of square } ABCD = (12)^2 = 144\, in^2$$
$$\text{Area of unshaded region } = 144 - (18\pi - 18)$$
$$= 162 - 18\pi\ in^2.$$

Participants are performing at the van Hiele Relational Level in this activity since they are required to select from the various properties and formulas among different figures in order to solve the problems.

Parts of a Circle

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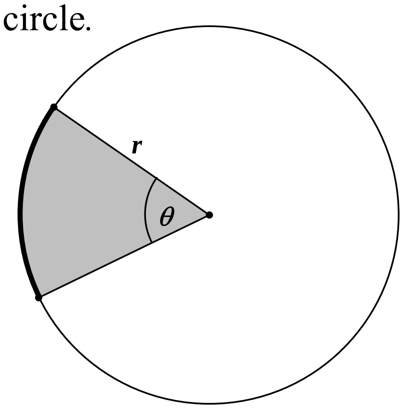

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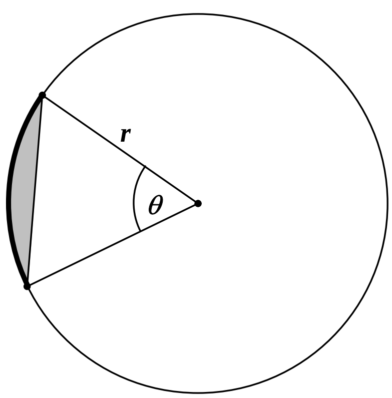
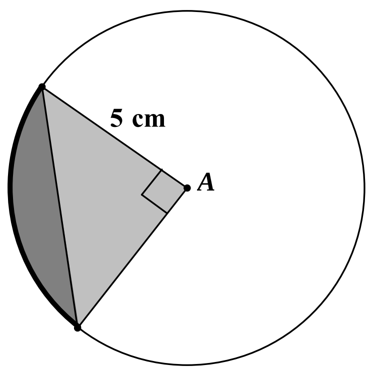
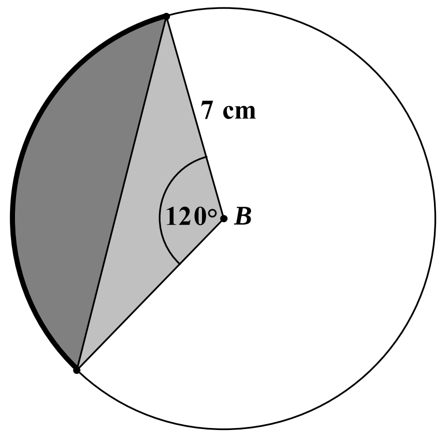


Explain how to find the area of the segment of a circle.

4. Determine the areas of the shaded segment and the sector for each circle.





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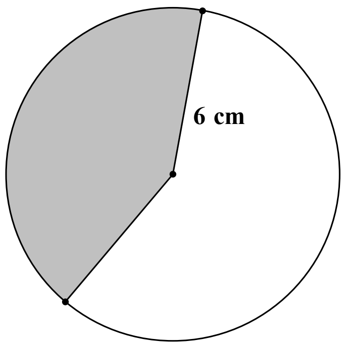

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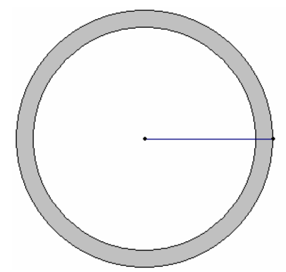

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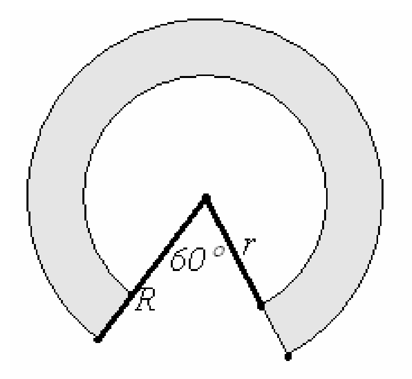

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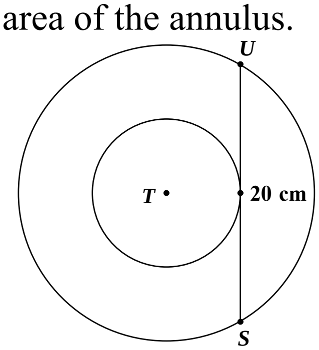

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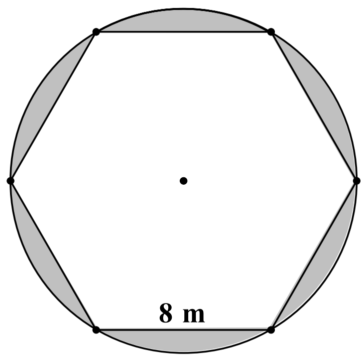

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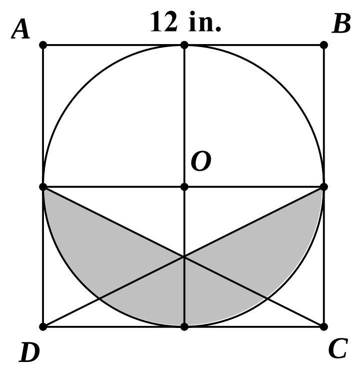

References and Additional Resources

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