Some topics in algebra

Stephen Semmes
Rice University

Preface

These informal notes deal with some topics in algebra that seem to be related to various areas. The reader is expected to have some familiarity with abstract algebra, although a fair amount of details are often given in the arguments being considered. At the same time, much of the discussion may be somewhat more algebraic than in some related matters involving interactions between algebra and analysis.

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Part I

Algebras, modules, and tensor products

Chapter 1

Modules and tensor products

1.1 Sums and products of modules

Let k be a commutative ring with a multiplicative identity element, and let I be a nonempty set. Suppose that for each $j \in I$, V_j is a module over k. It is easy to see that the Cartesian product

$$\prod_{j \in I} V_j \tag{1.1.1}$$

is a module over k too, with respect to coordinatewise addition and scalar multiplication. This is the *direct product* of the V_j's, $j \in I$.

If $v \in \prod_{j \in I} V_j$ and $l \in I$, then let v_l be the lth coordinate of v in V_l. Put

$$\bigoplus_{j \in I} V_j = \left\{ v \in \prod_{j \in I} V_j : v_l = 0 \text{ for all but finitely many } l \in I \right\}. \tag{1.1.2}$$

This is the *direct sum* of the V_j's, $j \in I$. Of course, this is a submodule of $\prod_{j \in I} V_j$, as a module over k. If I has only finitely many elements, then (1.1.2) is the same as $\prod_{j \in I} V_j$.

Let Z be another module over k, and suppose that ϕ_j is a homomorphism from V_j in Z, as modules over k, for each $j \in I$. If $v \in \bigoplus_{j \in I} V_j$, then

$$\phi(v) = \sum_{j \in I} \phi_j(v_j) \tag{1.1.3}$$

defines an element of Z, because $\phi_j(v_j) = 0$ for all but finitely many $j \in I$. It is easy to see that

$$\phi \text{ is a homomorphism from } \bigoplus_{j \in I} V_j \text{ into } Z, \tag{1.1.4}$$

as modules over k. Conversely, if ϕ is any homomorphism from $\bigoplus_{j \in I} V_j$ into Z, as modules over k, then ϕ corresponds to a unique family of module homomorphisms ϕ_j from V_j into Z, $j \in I$, in this way.

If V is any module over k, then let

$$\operatorname{Hom}(V, Z) = \operatorname{Hom}_k(V, Z) \tag{1.1.5}$$

be the space of homomorphisms from V into Z, as modules over k. This is a module over k too, with respect to pointwise addition and scalar multiplication of mappings from V into Z. The remarks in the previous paragraph show that there is a natural isomorphism

$$\text{from } \operatorname{Hom}_k\Big(\bigoplus_{j \in I} V_j, Z\Big) \text{ onto } \prod_{j \in I} \operatorname{Hom}_k(V_j, Z), \tag{1.1.6}$$

as modules over k.

Similarly, let V be a module over k, and suppose that Z_j is a module over k for each $j \in I$. Under these conditions, a module homomorphism from V into $\prod_{j \in I} Z_j$ corresponds exactly to a family of module homomorphisms from V into Z_j, $j \in I$. This leads to a natural isomorphism

$$\text{from } \operatorname{Hom}_k\Big(V, \prod_{j \in I} Z_j\Big) \text{ onto } \prod_{j \in I} \operatorname{Hom}_k(V, Z_j), \tag{1.1.7}$$

as modules over k.

Suppose that ϕ_j is a homomorphism from V into Z_j, as modules over k, for each $j \in I$. Let ϕ be the element of $\prod_{j \in I} \operatorname{Hom}_k(V, Z_j)$ whose jth component is ϕ_j for each $j \in I$. This corresponds to the homomorphism

$$\text{from } V \text{ into } \prod_{j \in I} Z_j, \tag{1.1.8}$$

as modules over k, whose jth coordinate is equal to ϕ_j, as in the preceding paragraph.

Suppose for the moment that

$$\phi \in \bigoplus_{j \in I} \operatorname{Hom}_k(V, Z_j), \tag{1.1.9}$$

so that

$$\phi_j = 0 \text{ for all but finitely many } j \in I. \tag{1.1.10}$$

In this case, we get a homomorphism

$$\text{from } V \text{ into } \bigoplus_{j \in I} Z_j, \tag{1.1.11}$$

as modules over k.

Suppose now that ϕ corresponds to a homomorphism as in (1.1.11), as modules over k. This means that for each $v \in V$,

$$\phi_j(v) = 0 \text{ for all but finitely many } j \in I. \tag{1.1.12}$$

If V is finitely generated as a module over k, then one can verify that (1.1.10) holds, so that (1.1.9) holds.

Of course, k may be considered as a module over itself. If $z \in Z$, then

$$t \mapsto t \cdot z \tag{1.1.13}$$

is a homomorphism from k into Z, as modules over k. It is easy to see that every homomorphism from k into Z, as modules over k, is of this form for a unique $z \in Z$. This defines an isomorphism

$$\text{between } \mathrm{Hom}_k(k, Z) \text{ and } Z, \tag{1.1.14}$$

as modules over k.

A module W over k is said to be *free* if there is a family $\{w_j\}_{j \in I}$ of elements of W such that every element of W can be expressed in a unique way as a linear combination of the w_j's, $j \in I$, with coefficients in k, and where all but finitely many of the coefficients are equal to 0. In this case, $\{w_j\}_{j \in I}$ may be called a *basis* for W as a module over k, and one may say that W is *freely generated* by the w_j's, $j \in I$, as a module over k. Equivalently, a free module over k can be expressed as a direct sum of copies of k, as a module over itself.

1.2 Module-valued functions

Let k be a commutative ring with a mutliplicative identity element, and let Z be a module over k. Also let X be a nonempty set, and let

$$c(X, Z) \tag{1.2.1}$$

be the space of all Z-valued functions on X. This is a module over k, with respect to pointwise addition and scalar multiplication of functions. This is the same as the direct product of copies of Z indexed by X.

If $f \in c(X, Z)$, then the *support* of f is defined to be the set of $x \in X$ such that $f(x) \neq 0$. One may consider X to be equipped with the discrete topology, to be consistent with the analogous definition for functions on topological spaces. Let

$$c_{00}(X, Z) \tag{1.2.2}$$

be the set of such functions whose support has only finitely many elements. This is a submodule of $c(X, Z)$, as a module over k, which corresponds to the direct sum of copies of Z indexed by X. If X has only finitely many elements, then this is the same as $c(X, Z)$.

In particular, $c_{00}(X, k)$ is a free module over k. If $x \in X$, then let δ_x be the k-valued function on X equal to 1 at x and to 0 at every other element of X. Thus $\delta_x \in c_{00}(X, k)$, and $c_{00}(X, k)$ is freely generated by δ_x, $x \in X$.

If $f \in c_{00}(X, Z)$, then

$$\sum_{x \in X} f(x) \tag{1.2.3}$$

reduces to a finite sum, and defines an element of Z. This defines a homomorphism from $c_{00}(X, Z)$ into Z, as modules over k.

Let ζ be a Z-valued function on X. If $f \in c_{00}(X, k)$, then $f \cdot \zeta \in c_{00}(X, Z)$, so that

$$\sum_{x \in X} f(x) \cdot \zeta(x) \tag{1.2.4}$$

defines an element of Z, as in the preceding paragraph. This defines a homomorphism from $c_{00}(X, k)$ into Z, as modules over k. One can check that every homomorphism ϕ from $c_{00}(X, k)$ into Z, as modules over k, corresponds to a unique $\zeta \in c(X, Z)$ in this way, with

$$\zeta(x) = \phi(\delta_x) \tag{1.2.5}$$

for every $x \in X$. This defines an isomorphism between $\mathrm{Hom}_k(c_{00}(X, k), Z)$ and $c(X, Z)$, as modules over k.

Let W be another module over k, and suppose that for each $x \in X$, ϕ_x is a homomorphism from Z into W, as modules over k. Thus

$$x \mapsto \phi_x \tag{1.2.6}$$

is an element of $c(X, \mathrm{Hom}_k(Z, W))$. If $z \in Z$, then

$$x \mapsto \phi_x(z) \tag{1.2.7}$$

is an element of $c(X, W)$. The mapping from $z \in Z$ to (1.2.7) defines a homomorphism from Z into $c(X, W)$, as modules over k. Every homomorphism from Z into $c(X, W)$, as modules over k, corresponds to a unique element of $c(X, \mathrm{Hom}_k(Z, W))$ in this way, as in the previous section, which leads to a natural isomorphism

$$\text{from } \mathrm{Hom}_k(Z, c(X, W)) \text{ onto } c(X, \mathrm{Hom}_k(Z, W)), \tag{1.2.8}$$

as modules over k, as before.

Let (1.2.6) be an element of $c(X, \mathrm{Hom}_k(Z, W))$ again. If $f \in c_{00}(X, k)$ and $z \in Z$, then

$$f(x) \cdot \phi_x(z) \tag{1.2.9}$$

defines an element of $c_{00}(X, W)$, so that

$$\sum_{x \in X} f(x) \cdot \phi_x(z) \tag{1.2.10}$$

defines an element of W, as before. It is easy to see that this defines a mapping from $c_{00}(X,k) \times Z$ into W that is bilinear over k. Conversely, if μ is any mapping from $c_{00}(X,k) \times Z$ into W that is bilinear over k, then

$$\phi_x = \mu(\delta_x, \cdot) \tag{1.2.11}$$

is a homomorphism from Z into W, as modules over k, for each $x \in X$. One can check that μ is the same as the mapping from $c_{00}(X,k) \times Z$ into W defined by (1.2.10) under these conditions.

Similarly, if (1.2.6) is an element of $c(X, \mathrm{Hom}_k(Z,W))$ and $F \in c_{00}(X,Z)$, then

$$\phi_x(F(x)) \tag{1.2.12}$$

defines an element of $c_{00}(X,W)$, so that

$$\sum_{x \in X} \phi_x(F(x)) \tag{1.2.13}$$

defines an element of W. In particular, if $f \in c_{00}(X,k)$ and $z \in Z$, then

$$F(x) = f(x) \cdot z \tag{1.2.14}$$

defines an element of $c_{00}(X,Z)$, in which case (1.2.12) and (1.2.13) are the same as (1.2.9) and (1.2.10), respectively. The mapping from F to (1.2.13) defines a homomorphism from $c_{00}(X,Z)$ into W, as modules over k. One can verify that every homomorphism Φ from $c_{00}(X,Z)$ into W, as modules over k, corresponds to a unique element (1.2.6) of $c(X, \mathrm{Hom}_k(Z,W))$ in this way. More precisely, if $x \in X$ and $z \in Z$, then $\delta_x\, z$ defines an element of $c_{00}(X,Z)$, and one can take

$$\phi_x(z) = \Phi(\delta_x\, z). \tag{1.2.15}$$

1.3 Tensor products over commutative rings

Let k be a commutative ring with a multiplicative identity element, and let V_1, V_2 be modules over k. A *tensor product* of V_1 and V_2 is a module

$$V_1 \bigotimes_k V_2 = V_1 \bigotimes V_2 \tag{1.3.1}$$

over k with the following two properties. First, $V_1 \bigotimes V_2$ should be equipped with a mapping from $V_1 \times V_2$ into $V_1 \bigotimes V_2$ that is bilinear over k. The image of $(v_1, v_2) \in V_1 \times V_2$ under this mapping may be expressed as $v_1 \otimes v_2$.

Second, let W be a module over k, and let μ be a mapping from $V_1 \times V_2$ into W that is bilinear over k. Under these conditions, μ can be expressed in a unique way as the composition of the mapping from $V_1 \times V_2$ into $V_1 \bigotimes V_2$ mentioned in the preceding paragraph with a homomorphism from $V_1 \bigotimes V_2$ into W, as modules over k. This means that there is a unique module homomorphism $\widetilde{\mu}$ from $V_1 \bigotimes V_2$ into W such that

$$\widetilde{\mu}(v_1 \otimes v_2) = \mu(v_1, v_2) \tag{1.3.2}$$

for every $v_1 \in V_1$ and $v_2 \in V_2$.

It is well known and not too difficult to show that such a tensor product is unique up to a suitable isomorphic equivalence. We also have that $V_1 \bigotimes V_2$ is generated by the associated image of $V_1 \times V_2$, as a module over k, or simply as a commutative group with respect to addition. This is easily seen using the standard construction of tensor products, or the uniqueness of tensor product up to isomorphism.

Let X be a nonempty set, and let Z be a module over k. If $f \in c_{00)}(X, k)$ and $z \in Z$, then $f\, z \in c_{00}(X, Z)$. This defines a mapping

$$\text{from } c_{00}(X, k) \times Z \text{ into } c_{00}(X, Z) \tag{1.3.3}$$

that is bilinear over k. One can check that $c_{00}(X, Z)$ satisfies the requirements of a tensor product of $c_{00}(X, k)$ and Z, as modules over k, with respect to this mapping. This uses the description of bilinear mappings from $c_{00}(X, k) \times Z$ into other modules over k mentioned in the previous section.

Let W_1, W_2 be another pair of modules over k, and let ϕ_j be a homomorphism from V_j into W_j for $j = 1, 2$, as modules over k. This leads to a mapping

$$(v_1, v_2) \mapsto \phi_1(v_1) \otimes \phi_2(v_2) \tag{1.3.4}$$

from $V_1 \times V_2$ into $W_1 \bigotimes W_2$ that is bilinear over k. Thus we get a unique homomorphism ϕ

$$\text{from } V_1 \bigotimes V_2 \text{ into } W_1 \bigotimes W_2, \tag{1.3.5}$$

as modules over k, such that

$$\phi(v_1 \otimes v_2) = \phi_1(v_1) \otimes \phi_2(v_2) \tag{1.3.6}$$

for every $v_1 \in V_1$, $v_2 \in V_2$.

If $\phi_j(V_j) = W_j$ for $j = 1, 2$, then

$$\phi\Big(V_1 \bigotimes V_2\Big) = W_1 \bigotimes W_2. \tag{1.3.7}$$

This follows from the fact that $W_1 \bigotimes W_2$ is generated by the associated image of $W \times W_2$, as a module over k.

Suppose for the moment that $V_j = W_j$ for $j = 1, 2$, and that ϕ_j is the identity mapping on V_j, $j = 1, 2$. In this case, the identity mapping on $V_1 \bigotimes V_2$ satisfies (1.3.6), so that ϕ is the identity mapping, by uniqueness.

Let Z_1, Z_2 be a pair of modules over k, and let ψ_j be a homomorphism from W_j into Z_j, $j = 1, 2$, as modules over k. Also let ψ be the corresponding homomorphism

$$\text{from } W_1 \bigotimes W_2 \text{ into } Z_1 \bigotimes Z_2, \tag{1.3.8}$$

as modules over k, as in (1.3.6). Of course, $\psi_j \circ \phi_j$ is a module homomorphism from V_j into Z_j, $j = 1, 2$. One can verify that $\psi \circ \phi$ is the same as the module homomorphism

$$\text{from } V_1 \bigotimes V_2 \text{ into } Z_1 \bigotimes Z_2 \tag{1.3.9}$$

associated to $\psi_j \circ \phi_j$, $j = 1, 2$, as before.

Suppose now that $V_j = Z_j$ for $j = 1, 2$, and that $\psi_j \circ \phi_j$ is the identity mapping on V_j, $j = 1, 2$. Under these conditions, we get that

$$\psi \circ \phi \text{ is the identity mapping on } V_1 \bigotimes V_2, \tag{1.3.10}$$

as before.

Similarly, if $\phi_j \circ \psi_j$ is the identity mapping on W_j for $j = 1, 2$, then $\phi \circ \psi$ is the identity mapping on $W_1 \bigotimes W_2$. If ϕ_j is a module isomorphism from V_j onto W_j for $j = 1, 2$, then it follows that

$$\phi \text{ is a module isomorphism from } V_1 \bigotimes V_2 \text{ onto } W_1 \bigotimes W_2. \tag{1.3.11}$$

1.4 Some properties of tensor products

Let k be a commutative ring with a multiplicative identity element, and let V_1, V_2 be modules over k. Observe that

$$(v_1, v_2) \mapsto v_2 \otimes v_2 \tag{1.4.1}$$

defines a mapping from $V_1 \times V_2$ into $V_2 \bigotimes V_1$ that is bilinear over k, which leads to a homomorphism

$$\text{from } V_1 \bigotimes V_2 \text{ into } V_2 \bigotimes V_1, \tag{1.4.2}$$

as modules over k. Similarly,

$$(v_2, v_1) \mapsto v_1 \otimes v_2 \tag{1.4.3}$$

defines a mapping from $V_2 \times V_1$ into $V_1 \otimes V_2$ that is bilinear over k, which leads to a homomorphism

$$\text{from } V_2 \bigotimes V_1 \text{ into } V_1 \bigotimes V_2, \tag{1.4.4}$$

as modules over k. One can check that the compositions of these two homomorphisms in either order is equal to the identity mapping on $V_1 \bigotimes V_2$ or $V_2 \bigotimes V_1$, as appropriate. Thus these homomorphisms are isomorphisms.

Note that scalar multiplication on V_1 defines a mapping from $V_1 \times k$ into V_1 that is bilinear over k, where k is considered as a module over itself. One can verify that

$$V_1 \text{ satisfies the requirements of } V_1 \bigotimes k, \tag{1.4.5}$$

using this mapping. Similarly,

$$V_2 \text{ satisfies the requirements of } k \bigotimes V_2, \tag{1.4.6}$$

with respect to the bilinear mapping from $k \times V_2$ into V_2 corresponding to scalar multiplication on V_2.

If V_3 is another module over k, then it is well known that there is a unique isomorphism

$$\text{from } (V_1 \bigotimes V_2) \bigotimes V_3 \text{ onto } V_1 \bigotimes (V_2 \bigotimes V_3), \tag{1.4.7}$$

as modules over k, with

$$(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3) \tag{1.4.8}$$

for every $v_1 \in V_1$, $v_2 \in V_2$, and $v_3 \in V_3$. Alternatively, one can define tensor products of finitely many modules over k directly, and get unique isomorphisms with $V_1 \bigotimes V_2 \bigotimes V_3$ such that $(v_1 \otimes v_2) \otimes v_3$ and $v_1 \otimes (v_2 \otimes v_3)$ correspond to $v_1 \otimes v_2 \otimes v_3$ for every $v_1 \in V_1$, $v_2 \in V_2$, and $v_3 \in V_3$.

Let I be a nonempty set, let V_j be a module over k for every $j \in I$, and let W, Z be modules over k too. It is easy to see that a bilinear mapping

$$\text{from } \Big(\bigoplus_{j \in I} V_j\Big) \times W \text{ into } Z \tag{1.4.9}$$

corresponds exactly to a family of bilinear mappings from $V_j \times W$ into Z, $j \in I$. In particular, if Z_j is a module over k for each $j \in I$, then a family of bilinear mappings from $V_j \times W$ into Z_j, $j \in I$, leads to a bilinear mapping

$$\text{from } \Big(\bigoplus_{j \in I} V_j\Big) \times W \text{ into } \bigoplus_{j \in I} Z_j. \tag{1.4.10}$$

It is well known that there is a natural isomorphism

$$\text{from } \Big(\bigoplus_{j \in I} V_j\Big) \bigotimes W \text{ onto } \bigoplus_{j \in I} (V_j \bigotimes W), \tag{1.4.11}$$

as modules over k. To see this, one can define natural mappings in both directions, as follows.

If $j \in I$, then $V_j \bigotimes W$ comes with a bilinear mapping from $V_j \times W$ into $V_j \bigotimes W$, as in the previous section. This leads to a bilinear mapping

$$\text{from } \Big(\bigoplus_{j \in I} V_j\Big) \times W \text{ into } \bigoplus_{j \in I} (V_j \bigotimes W), \tag{1.4.12}$$

as before. Using this, we get a homomorphism

$$\text{from } \Big(\bigoplus_{j \in I} V_j\Big) \bigotimes W \text{ into } \bigoplus_{j \in I} (V_j \bigotimes W), \tag{1.4.13}$$

as modules over k, as in the previous section. If $v \in \bigoplus_{j \in I} V_j$ and $w \in W$, then this mapping sends $v \otimes w$ to the element of $\bigoplus_{j \in I} (V_j \bigotimes W)$ whose jth coordinate is equal to $v_j \otimes w$ for every $j \in I$. Note that $v_j = 0$ for all but finitely many j, so that $v_j \otimes w = 0$ for all but finitely many j.

If $l \in I$, then there is an obvious inclusion mapping from V_l into $\bigoplus_{j \in I} V_j$, which sends $v_l \in V_l$ to the element of $\bigoplus_{j \in I} V_j$ whose lth coordinate is equal to v_l, and whose other coordinates are equal to 0. This leads to a homomorphism

$$\text{from } V_l \bigotimes W \text{ into } \Big(\bigoplus_{j \in I} V_j\Big) \bigotimes W, \tag{1.4.14}$$

as modules over k, using the identity mapping on W. We may combine these mappings to get a homomorphism

$$\text{from } \bigoplus_{j \in I} (V_j \bigotimes W) \text{ into } \Big(\bigoplus_{j \in I} V_j\Big) \bigotimes W, \tag{1.4.15}$$

as modules over k, as in Section 1.1. More precisely, if $l \in I$, $v_l \in V_l$, and $w \in W$, then $v_l \otimes w \in V_l \bigotimes W$, and we get an element of $\bigoplus_{j \in I} (V_j \bigotimes W)$ by taking the jth coordinate in $V_j \bigotimes W$ to be 0 when $j \neq l$. The mapping just described sends this element of $\bigoplus_{j \in I} (V_j \bigotimes W)$ to an element of $\big(\bigoplus_{j \in I} V_j\big) \bigotimes W$ of the form $v \otimes w$, where $v \in \bigoplus_{j \in I} V_j$ has lth coordinate equal to v_l, and jth coordinate equal to 0 when $j \neq l$.

One can verify that the mappings described in the previous two paragraphs are inverses of each other, so that we get a module isomorphism, as before. In particular, if V is a free module over k, then $V \bigotimes W$ corresponds to a direct sum of copies of W. If W is also free as a module over k, then $V \bigotimes W$ is free as well.

1.5 Modules over associative algebras

Let k be a commutative ring with a multiplicative identity element, let A_1 be an associative algebra over k with a multiplicative identity element $e_1 = e_{A_1}$, and let V_1 be a module over k. Suppose that for every $a_1 \in A_1$ and $v_1 \in V_1$, $a_1 \cdot v_1$ is defined as an element of V_1, and that the corresponding mapping from $A_1 \times V_1$ into V_1 is bilinear over k. If we also have that

$$a_1 \cdot (b_1 \cdot v_1) = (a_1\, b_1) \cdot v_1 \tag{1.5.1}$$

and

$$e_1 \cdot v_1 = v_1 \tag{1.5.2}$$

for every $a_1, b_1 \in A_1$ and $v_1 \in V_1$, then V_1 is said to be a *left module over* A_1.

Let W_1 be a submodule of V_1, as a module over k. If

$$a_1 \cdot w_1 \in W_1 \tag{1.5.3}$$

for every $a_1 \in A_1$ and $w_1 \in W_1$, then W_1 is said to be a *submodule* of V_1, as a left module over A_1. Of course, this means that W_1 is a left module over A_1 too.

Let Z_1 be another module over k that is also a left module over A_1. A mapping ϕ_1 from V_1 into Z_1 is said to be a *homomorphism* from V_1 into Z_1 as left modules over A_1 if ϕ is linear over k and

$$\phi_1(a_1 \cdot v_1) = a_1 \cdot \phi(v_1) \tag{1.5.4}$$

for every $a_1 \in A_1$ and $v_1 \in V_1$. We may also say that ϕ_1 is *linear over A_1 on the left* in this case. Note that the kernel of ϕ_1 is a submodule of V_1, as a left module over A_1, under these conditions. If W_1 is a submodule of V_1, as a left module over A_1, then the obvious inclusion mapping from W_1 into V_2 may be considered as a homomorphism from W_1 into V_1, as left modules over A_1.

Let W_1 be a submodule of V_1 again, as a left module over A_1, and consider the quotient V_1/W_1, defined initially as a module over k. If $a_1 \in A_1$, then we would like to define the action of a_1 on V_1/W_1 on the left by putting

$$a_1 \cdot q_1(v_1) = q_1(a_1 \cdot v_1) \tag{1.5.5}$$

for every $v_1 \in V_1$, where q_1 is the natural quotient mapping from V_1 onto V_1/W_1. One can check that the right side depends only on a_1 and $q_1(v_1)$, and that V_1/W_1 becomes a left module over A_1 in this way. By construction, q_1 is a homomorphism from V_1 onto V_1/W_1, as left modules over A_1.

Similarly, let A_2 be an associative algebra over k with a multiplicative identity element $e_2 = e_{A_2}$, and let V_2 be a module over k. Suppose that for every $a_2 \in A_2$ and $v_2 \in V_2$, $v_2 \cdot a_2$ is defined as an element of V_2, and that the corresponding mapping from $V_2 \times A_2$ into V_2 is bilinear over k. If

$$(v_2 \cdot a_2) \cdot b_2 = v_2 \cdot (a_2\, b_2) \tag{1.5.6}$$

and

$$v_2 \cdot e_2 = v_2 \tag{1.5.7}$$

for every $a_2, b_2 \in A_2$ and $v_2 \in V_2$, then V_2 is a *right module over* A_2. Of course, left and right modules over commutative algebras are basically the same. One can define submodules of right modules, homomorphisms between right modules, and quotients of right modules in essentially the same ways as for left modules.

1.6 Sums, products, and compositions

Let k be a commutative ring with a multiplicative identity element, let A be an associative algebra over k with a multiplicative identity element $e = e_A$, and let V, W, and Z be modules over k. Suppose that ϕ is a homomormophism from V into W, and that ψ is a homomorphism from W into Z, as modules over k, so that the composition $\psi \circ \phi$ of ϕ and ψ is a homomorphism from V into Z, as modules over k. If V, W, and Z are left modules over A, ϕ is a homomorphism from V into W as left modules over A, and ψ is a homomorphism from W into Z as left modules over A, then $\psi \circ \phi$ is a homomorphism from V into

Z, as left modules over A. Similarly, if V, W, and Z are right modules over A, ϕ is a homomorphism from V into W as right modules over A, and ψ is a homomorphism from W into Z as right modules over A, then $\psi \circ \phi$ is a homomorphism from V into Z, as right modules over A.

Suppose that ϕ is a one-to-one homomorphism from V onto W, as modules over k, so that the inverse ϕ^{-1} of ϕ is a homomorphism from W into V, as modules over k. Equivalently, this means that ϕ is an isomorphism from V onto W, as modules over k. If V, W are left modules over A, and ϕ is a homomorphism from V onto W, as left modules over A, then ϕ^{-1} is a homomorphism from W onto V, as left modules over A. Similarly, if V and W are right modules over A, and ϕ is a homomorphism from V onto W, as right modules over A, then ϕ^{-1} is a homomorphism from W onto V, as right modules over A. In each case, ϕ is said to be an *isomorphism* from V onto W, as left or right modules over A, as appropriate.

Note that A may be considered as both a left and right module over itself, in the obvious way. Let $w \in W$ be given, and suppose for the moment that W is a left module over A. In this case,

$$a \mapsto a \cdot w \tag{1.6.1}$$

defines a homomorphism from A into W, as left modules over A. It is easy to see that every homomorphism ϕ from A into W as left modules over A corresponds to a unique $w \in W$ in this way, with $w = \phi(e)$.

Similarly, if W is a right module over A, then

$$a \mapsto w \cdot a \tag{1.6.2}$$

defines a homomorphism from A into W, as right modules over A. As before, every homomorphism ϕ from A into W as right modules over A corresponds to a unique $w \in W$ in this way, with $w = \phi(e)$.

Let I be a nonempty set, and let V_j be a module over k for each $j \in I$, so that the direct sum and product of the V_j's may be defined as modules over k as in Section 1.1. If V_j is a left module over A for every $j \in I$, then $\prod_{j \in I} V_j$ is a left module over A too, where the action of A on the left is defined coordinatewise. Under these conditions, $\bigoplus_{j \in I} V_j$ is a submodule of $\prod_{j \in I} V_j$, as a left module over A. Similarly, if V_j is a right module over A for each $j \in I$, then $\prod_{j \in I} V_j$ is a right module over A as well, where the action of A on the right is defined coordinatewise, and $\bigoplus_{j \in I} V_j$ is a submodule of $\prod_{j \in I} V_j$, as a right module over A.

A left module over A is said to be *free* as a left module over A if it is isomorphic to the direct sum of a family of copies of A, as a left module over itself. Similarly, a right module over A is said to be *free* as a right module over A if it is isomorphic to the direct sum of a family of copies of A, as a right module over itself.

If X is a nonempty set, and V is a left or right module over A, then the space $c(X, V)$ of V-valued functions on X is a left or right module over A, as appropriate, where the action of A is defined pointwise. This corresponds to the direct product of copies of V indexed by X. Similarly, the space $c_{00}(X, V)$ of V-valued functions on X with finite support is a submodule of $c(X, V)$, as a left or right module over A, as appropriate. This corresponds to the direct sum of copies of V indexed by X.

1.7 Spaces of homomorphisms

Let k be a commutative ring with a multiplicative identity element, let A be an associative algebra over k with a multiplicative identity element $e = e_A$, and let V, W be modules over k. If V and W are both left moduules over A, then the space of homomorphisms from V into W as left modules over A may be denoted

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_A(V, W). \tag{1.7.1}$$

Similarly, if V and W are both right modules over A, then the space of homomorphisms from V into W as right modules over A may be denoted in the same way. In both cases, $\mathrm{Hom}_A(V, W)$ is a module over k, with respect to pointwise addition and scalar multiplication of mappings from V into W.

If W is a left module over A, and A is considered as a left module over itself, then the characterization of homomorphisms from A into W, as left modules over A, mentioned in the previous section defines an isomorphism

$$\text{between } \mathrm{Hom}_A(A, W) \text{ and } W, \tag{1.7.2}$$

as modules over k. Similarly, if W is a right module over A, and A is considered as a right module over itself, then the characterization of homomorphisms from A into W, as right modules over A, discussed in the previous section defines an isomorphism as in (1.7.2) again, as modules over k.

Let I be a nonempty set, and let V_j be a module over k for each $j \in I$. If V_j is a left module over A for every $j \in I$, and W is a left module over A, then there is a natural isomorphism

$$\text{from } \mathrm{Hom}_A\Big(\bigoplus_{j \in I} V_j, W\Big) \text{ onto } \prod_{j \in I} \mathrm{Hom}_A(V_j, W), \tag{1.7.3}$$

as modules over k, as in Section 1.1. Similarly, if V_j is a right module over A for every $j \in I$, and W is a right module over A, then there is a natural isomorphism as in (1.7.3), as modules over k.

Let V be a module over k again, and let W_j be a module over k for every $j \in I$. If V is a left module over A, and W_j is a left module over A for every $j \in I$, then there is a natural isomorphism

$$\text{from } \mathrm{Hom}_A\Big(V, \prod_{j \in I} W_j\Big) \text{ onto } \prod_{j \in I} \mathrm{Hom}_A(V, W_j), \tag{1.7.4}$$

as modules over k, as in Section 1.1. Similarly, if V is a right module over A, and W_j is a right module over A for every $j \in I$, then there is a natural isomorphism as in (1.7.4), as modules over k.

Let V, W, Z be modules over k, and let ϕ_0 be a homomorphism from V into W, as modules over k. Observe that

$$\psi \mapsto \psi \circ \phi_0 \tag{1.7.5}$$

defines a homomorphism

$$\text{from } \mathrm{Hom}_k(W, Z) \text{ into } \mathrm{Hom}_k(V, Z), \tag{1.7.6}$$

as modules over k. If

$$\phi_0(V) = W, \tag{1.7.7}$$

then this homomorphism is injective. If ϕ_0 is an isomorphism from V onto W, as modules over k, then (1.7.5) defines an isomorphism as in (1.7.6), as modules over k.

If ψ_0 is a homomorphism from W into Z, as modules over k, then

$$\phi \mapsto \psi_0 \circ \phi \tag{1.7.8}$$

defines a homomorphism

$$\text{from } \mathrm{Hom}_k(V, W) \text{ into } \mathrm{Hom}_k(V, Z), \tag{1.7.9}$$

as modules over k. If

$$\psi_0 \text{ is injective,} \tag{1.7.10}$$

then this homomorphism is injective. If ψ_0 is an isomorphism from W onto Z, as modules over k, then (1.7.5) defines an isomorphism as in (1.7.9), as modules over k.

Suppose now that V, W, and Z are left modules over A. If ϕ_0 is a homomorphism from V into W, as left modules over A, then (1.7.5) defines a homomorphism

$$\text{from } \mathrm{Hom}_A(W, Z) \text{ into } \mathrm{Hom}_A(V, Z), \tag{1.7.11}$$

as modules over k. This homomorphism is injective when ϕ_0 is surjective, as before. If ϕ_0 is an isomorphism from V onto W, as left modules over A, then (1.7.5) defines an isomorphism as in (1.7.11), as modules over k.

Similarly, if ψ_0 is a homomorphism from W into Z as left modules over A, then (1.7.8) defines a homomorphism

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V, Z), \tag{1.7.12}$$

as modules over k. This homomorphism is injective when ψ_0 is injective. If ψ_0 is an isomorphism from W onto Z, as modules over A, then (1.7.8) defines an isomorphism as in (1.7.12), as modules over k. Of course, there are analogous statements when V, W, and Z are right modules over A, ϕ_0 is a homomorphism from V into W as right modules over A, and ψ_0 is a homomorphism from W into Z as right modules over A.

1.8 Additional structure on $\mathrm{Hom}(V, W)$

Let k be a commutative ring with a multiplicative identity element, let A, B be associative algebras over k with multiplicative identity elements e_A, e_B, respectively, and let V, W be modules over k. If W is a left or right module over B, then $\mathrm{Hom}_k(V, W)$ is a left or right module over B as well, respectively, where the action of B is defined pointwise. Similarly, if V is a left or right module over B, then $\mathrm{Hom}_k(V, W)$ is a right or left module over B, respectively, in a natural way. More precisely, if $\phi \in \mathrm{Hom}_k(V, W)$, then the action of an element b of B on ϕ is defined by composing ϕ with the action of b on V.

Suppose now that V and W are left modules over A, and that W is a left or right module over B, where the actions of A and B on W commute with each other. In this case, one may say that W is a *bimodule* over A and B. Equivalently, this means that the action of $a \in A$ on W is a homomorphism from W into itself, as a module over B. This is the same as saying that the action of $b \in B$ on W is a homomorphim from W into itself, as a module over A. Under these conditions, the space $\mathrm{Hom}_A(V, W)$ of homomorphisms from V into W, as left modules over A, is a left or right module over B, where the action of B is defined pointwise, as usual. Of course, there is an analogous statement when V and W are right modules over A. This corresponds to some remarks on p22 of [3].

Suppose that V and W are left modules over A again, and now that V is a left or right module over B, and that the actions of A and B on V commute with each other. Under these conditions, the space $\mathrm{Hom}_A(V, W)$ of homomorphisms from V into W, as left modules over A, is a right or left module over B, respectively. As before, the action of $b \in B$ on $\phi \in \mathrm{Hom}_A(V, W)$ is defined by composing ϕ with the action of b on V. There is an analogous statement when V and W are right modules over A. This corresponds to some more remarks on p22 of [3].

The left and right actions of A on itself commute, by associativity, so that A is a bimodule as a left and right module over itself. If W is a left module over A, then the space $\mathrm{Hom}_A(A, W)$ of homomorphisms from A into W, as left modules over A, is a left module over A too, with respect to the action determined by the action of A on itself on the right. Remember that there is a natural isomorphism between $\mathrm{Hom}_A(A, W)$ and W, as modules over k, as in the previous section. One can check that this is an isomorphism

$$\text{between } \mathrm{Hom}_A(A, W) \text{ and } W, \text{ as left modules over } A, \tag{1.8.1}$$

as mentioned on p21 of [3]. Similarly, if W is a right module over A, then the space $\mathrm{Hom}_A(A, W)$ of homomorphisms from A into W, as right modules over A, is a right module over A, with respect to the action determined by the action of A on itself on the left. One can verify that the isomorphism between $\mathrm{Hom}_A(A, W)$ and W, as modules over k, mentioned in the previous section is an isomorphism

$$\text{between } \mathrm{Hom}_A(A, W) \text{ and } W, \text{ as right modules over } A. \tag{1.8.2}$$

Let Z be another module over k, and let ϕ_0 be a homomorphism from V into W, as modules over k. Suppose for the moment that V, W, and Z are left modules over A, and that ϕ_0 is a homomorphism from V into W, as left modules over A. Suppose also that V and W are both left or both right modules over B, where the actions of A and B on each of V and W commute, and that ϕ_0 is a homomorphism from V into W as left or right modules over B too. Under these conditions, one can check that composition with ϕ_0 defines a homomorphism

$$\text{(1.8.3) from } \mathrm{Hom}_A(W, Z) \text{ into } \mathrm{Hom}_A(V, Z), \text{ as right or left modules over } B,$$

as appropriate, as in (1.7.5).

Suppose now that W and Z are both left or both right modules over B, where the actions of A and B on each of W and Z commute. Let ψ_0 be a homomorphism from W into Z, as modules over k, left modules over A, and left or right modules over B, as appropriate. In this case, composition with ψ_0 defines a homomorphism

$$\text{(1.8.4) from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V, Z), \text{ as left or right modules over } B,$$

as appropriate, as in (1.7.8).

As usual, there are analogous statements when V, W, and Z are right modules over A, and ϕ_0, ψ_0 are homomorphisms from V into W and W into Z, respectively, as right modules over A.

1.9 Tensor products over algebras

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element $e = e_A$. Also let V_1, V_2 be modules over k, with V_1 a right module over A, and V_2 a left module over A. A *tensor product* of V_1 and V_2 over A is a module $V_1 \bigotimes_A V_2$ over k with the following two properties.

First, $V_1 \bigotimes_A V_2$ should be equipped with a mapping from $V_1 \times V_2$ into $V_1 \bigotimes_A V_2$ that is bilinear over k. If $v_1 \in V_1$ and $v_2 \in V_2$, then the image of (v_1, v_2) under this mapping may be denoted $v_1 \otimes v_2$, or $v_1 \otimes_A v_2$ to indicate the role of A. This mapping should also satisfy

$$(v_1 \cdot a) \otimes v_2 = v_1 \otimes (a \cdot v_2) \tag{1.9.1}$$

for every $a \in A$, $v_1 \in V_1$, and $v_2 \in V_2$.

Second, let W be a module over k, and let b be a mapping from $V_1 \times V_2$ into W that is bilinear over k. Suppose that

$$b(v_1 \cdot a, v_2) = b(v_1, a \cdot v_2) \tag{1.9.2}$$

for every $a \in A$, $v_1 \in V$, and $v_2 \in V_2$. Under these conditions, there should be a unique homomorphism c from $V_1 \bigotimes_A V_2$ into W, as modules over k, such that

$$b(v_1, v_2) = c(v_1 \otimes v_2) \tag{1.9.3}$$

for every $v_1 \in V_1$ and $v_2 \in V_2$.

Of course, this reduces to the definition in Section 1.3 when $A = k$. As before, $V_1 \bigotimes_A V_2$ is unique up to a suitable isomorphic equivalence, and $V_1 \bigotimes_A V_2$ is generated by the associated image of $V_1 \times V_2$, as a module over k, or even as a commutative group with respect to addition. The latter follows from the standard construction of tensor products, or their uniqueness up to isomorphism, as in the previous case. Note that the bilinear mapping from $V_1 \times V_2$ into $V_1 \bigotimes_A V_2$ mentioned earlier leads to a homomorphism

$$\text{from } V_1 \bigotimes_k V_2 \text{ into } V_1 \bigotimes_A V_2, \tag{1.9.4}$$

as modules over k. This homomorphism is a surjection, by the preceding remark.

Let W_1, W_2 be another pair of modules over k, with W_1 a right module over A, and W_2 and left module over k. Suppose that ϕ_1 is a homomorphism from V_1 into V_2, as right modules over A, and that ϕ_2 is a homomorphism from V_2 into W_2, as left modules over A. Note that

$$(v_1, v_2) \mapsto \phi_1(v_1) \otimes \phi_2(v_2) \tag{1.9.5}$$

is bilinear over k as a mapping from $V_1 \times V_2$ into $W_1 \bigotimes_A W_2$. If $a \in A$, $v_1 \in V_1$, and $v_2 \in V_2$, then

$$\begin{aligned} \phi_1(v_1 \cdot a) \otimes \phi_2(v_2) &= (\phi_1(v_1) \cdot a) \otimes \phi_2(v_2) \\ &= \phi_1(v_1) \otimes (a \cdot \phi_2(v_2)) = \phi_1(v_1) \otimes \phi_2(a \cdot v_2). \end{aligned} \tag{1.9.6}$$

Using this, we get a unique homomorphism ϕ

$$\text{from } V_1 \bigotimes_A V_2 \text{ into } W_1 \bigotimes_A W_2, \tag{1.9.7}$$

as modules over k, such that

$$\phi(v_1 \otimes v_2) = \phi_1(v_1) \otimes \phi_2(v_2) \tag{1.9.8}$$

for every $v_1 \in V_1$ and $v_2 \in V_2$.

If $\phi_j(V_j) = W_j$ for $j = 1, 2$, then

$$\phi\Big(V_1 \bigotimes_A V_2\Big) = W_1 \bigotimes_A W_2, \tag{1.9.9}$$

because $W_1 \bigotimes_A W_2$ is generated by the associated image of $W_1 \times W_2$, as a module over k.

Let Z_1, Z_2 be a pair of modules over k, with Z_1 a right module over A, and Z_2 a left module over A. Also let ψ_1 be a homomorphism from W_1 into Z_1, as right modules over A, and let ψ_2 be a homomorphism from W_2 into Z_2, as left modules over A. This leads to a homomorphism ψ

$$\text{from } W_1 \bigotimes_A W_2 \text{ into } Z_1 \bigotimes_A Z_2, \tag{1.9.10}$$

as modules over k, as before. Note that $\psi_1 \circ \phi_1$ is a homomorphism from V_1 into Z_1, as right modules over A, and that $\psi_2 \circ \phi_2$ is a homomorphism from V_2 into Z_2, as left modules over A. One can check that $\psi \circ \phi$ is the same as the homomorphism

$$\text{from } V_1 \bigotimes_A V_2 \text{ into } Z_1 \bigotimes_A Z_2, \tag{1.9.11}$$

as modules over k, associated to $\psi_1 \circ \phi_1$, $\psi_2 \circ \phi_2$, as before.

Suppose that ϕ_1 is an isomorphism from V_1 onto W_1, as right modules over A, and that ϕ_2 is an isomorphism from V_2 onto W_2, as left modules over A. Under these conditions,

$$\phi \text{ is an isomorphism from } V_1 \bigotimes_A V_2 \text{ onto } W_1 \bigotimes_A W_2, \tag{1.9.12}$$

as modules over k. More precisely, ϕ^{-1} can be obtained from ϕ_1^{-1}, ϕ_2^{-1} as a mapping from $W_1 \bigotimes_A W_2$ into $V_1 \bigotimes_A V_2$ in the same way as before.

1.10 Additional properties and structure

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element $e = e_A$. Remember that A may be considered as both a left module and a right module over itself. If V is a module over k that is a right module over A, then one can check that

$$V \text{ satisfies the requirements of } V \bigotimes_A A, \tag{1.10.1}$$

as a module over k, using the mapping $(v, a) \mapsto v \cdot a$ from $V \times A$ into V. Similarly, if W is a module over k that is a left module over A, then

$$W \text{ satisfies the requirements of } A \bigotimes_A W, \tag{1.10.2}$$

as a module over k, using the mapping $(a, w) \mapsto a \cdot w$ from $A \times W$ into W.

Let I be a nonempty set, and let V_j be a module over k for each $j \in I$. Suppose that V_j is a right module over A for each $j \in I$, and that W is a module over k that is a left module over A. Under these conditions, one can get a natural isomorphism

$$\text{from } \Big(\bigoplus_{j \in I} V_j\Big) \bigotimes_A W \text{ onto } \bigoplus_{j \in I} (V_j \bigotimes_A W), \tag{1.10.3}$$

as modules over k, as in Section 1.4. Similarly, if V_j is a left module over A for every $j \in I$, and W is a right module over A, then there is a natural isomorphism

$$\text{from } W \bigotimes_A \Big(\bigoplus_{j \in I} V_j\Big) \text{ onto } \bigoplus_{j \in I} (W \bigotimes_A V_j), \tag{1.10.4}$$

as modules over k.

If V is a free right module over A, and W is any left module over A, then it follows that $V \bigotimes_A W$ corresponds to a direct sum of copies of W, as a module over k. Similarly, if V is any right module over A, and W is a free left module over A, then $V \bigotimes_A W$ corresponds to a direct sum of copies of V, as a module over k.

Let V be a right module over A, let W be a left module over A, and let B be another associative algebra over k with a multiplicative identity element e_B. If V or W is also a left or right module over B, where the actions of A and B commute, then

$$V \bigotimes_A W \text{ is a left or right module over } B, \tag{1.10.5}$$

as appropriate, as on p23 of [3].

More precisely, suppose that V is a left or right module over B, where the actions of A and B on V commute. If $b \in B$, then the action of b on V is a homomorphism from V into itself, as a right module over A. Of course, the identity mapping on W is a homomorphism from W into itself, as a left module over A. One can use these two module homomorphisms to get a homomorphism from $V \bigotimes_A W$ into itself, as a module over k, as in the previous section. This is used to define the action of b on $V \bigotimes_A W$. If $v \in V$ and $w \in W$, then the action of b on $v \otimes_A w$ is obtained by combining the action of b on v with w using $\otimes_A$, and the action of b on $V \bigotimes_A W$ is uniquely determined by this property. One can check that this makes $V \bigotimes_A W$ a left or right module over B, as appropriate. If W is a left or right module over B, where the actions of A and B on W commute, then the action of B on $V \bigotimes_A W$ can be defined analogously.

In particular, $V \bigotimes_A A$ is a right module over A in a natural way, because A is a left and right module over itself, and the actions of A on itself on the left and right commute. One can check that

$$V \bigotimes_A A \text{ corresponds to } V \text{ as a right module over } A, \tag{1.10.6}$$

as in (1.10.1). Similarly,

$$A \bigotimes_A W \text{ corresponds to } W \text{ as a left module over } A, \tag{1.10.7}$$

as in (1.10.2).

Let V_1, V_2, W_1, W_2 be modules over k, with V_1, W_1 right modules over A, and V_2, W_2 left modules over A. Also let ϕ_1 be a homomorphism from V_1 into W_1, as right modules over A, and let ϕ_2 be a homomorphism from V_2 into W_2, as left modules over A. This leads to a unique homomorphism ϕ from $V_1 \bigotimes_A V_2$ into $W_1 \bigotimes_A W_2$, as modules over k, that satisfies (1.9.8). Suppose that V_1, W_1 are both left modules over B, or both right modules over B, where the actions of A and B on V_1 and W_1 commute. Thus $V_1 \bigotimes_A V_2$ and $W_1 \bigotimes_A W_2$ may be considered as left or right modules over B, as appropriate. If

$$\phi_1 \text{ is a homomorphism from } V_1 \text{ into } W_1, \text{ as modules over } B, \tag{1.10.8}$$

then one can check that

$$\phi \text{ is a homomorphism from } V_1 \bigotimes_A V_2 \text{ into } W_1 \bigotimes_A W_2, \text{ as modules over } B. \tag{1.10.9}$$

Similarly, suppose instead that V_2, W_2 are both left or both right modules over B, where the actions of A and B on V_2, W_2 commute, so that $V_1 \bigotimes_A V_2$ and $W_1 \bigotimes_A W_2$ may be considered as left or right modules over B, as appropriate. If

$$\phi_2 \text{ is a homomorphism from } V_2 \text{ into } W_2, \text{ as modules over } B, \tag{1.10.10}$$

then one can verify that (1.10.9) holds again.

1.11 Opposite algebras

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k. The corresponding *opposite algebra* A^{op} is defined to be the same as A as a module over k. The product of $x, y \in A^{op}$ in A^{op} is defined to be the same as the product $y\,x$ of y and x in A. It is easy to see that A^{op} is an associative algebra over k too. If A has a multiplicative identity element e_A, then e_A is the multiplicative identity element in A^{op} as well.

Of course, if A is a commutative algebra, then A^{op} is the same as A. Note that the opposite algebra $(A^{op})^{op}$ of A^{op} is always the same as A.

If $x \in A$, then it may be helpful to use x^{op} to indicate that a is being considered as an element of A^{op}. This is similar to the definition on p109 of [3], with slightly different notation. Using this, multiplication in A^{op} is given by

$$x^{op}\,y^{op} = (y\,x)^{op}. \tag{1.11.1}$$

Let B be another associative algebra over k, and let ϕ be a homomorphism from A into B, as modules over k. Note that ϕ is a homomorphism from A into B as algebras over k if and only if ϕ is a homomorphism from A^{op} into B^{op}, as algebras over k. Let us say that ϕ is an *opposite algebra homomorphism* from A into B if

$$\phi(x\,y) = \phi(y)\,\phi(x) \tag{1.11.2}$$

for every $x, y \in A$. This is equivalent to asking that ϕ be a homomorphism from A into B^{op}, or from A^{op} into B, as algebras over k. If A, B have multiplicative identity elements e_A, e_B, respectively, then one may wish to ask that

$$\phi(e_A) = e_B. \tag{1.11.3}$$

If ϕ is a one-to-one opposite algebra homomorphism from A onto B, then the inverse mapping ϕ^{-1} is an opposite algebra homomorphism from B onto A. In this case, we may say that ϕ is an *opposite algebra isomorphism* from A onto B. If A has a multiplicative identity element e_A, then it follows that $\phi(e_A)$ is the multiplicative identity element in B. An opposite algebra isomorphism from A onto itself may also be called an *opposite algebra automorphism* of A.

An algebra *involution* on A is an opposite algebra homomorphism from A into itself whose composition with itself is the identity mapping on A. If k is the field of complex numbers, then one may also be interested in the analogous notion for conjugate-linear mappings.

Let n be a positive integer, so that the space $M_n(A)$ of $n \times n$ matrices with entries in A is an associative algebra over k with respect to matrix multiplication. Of course, $M_n(A^{op})$ is an associative algebra over k too. One can check that the usual mapping from a matrix to its transpose defines an opposite algebra isomorphism from $M_n(A)$ onto $M_n(A^{op})$.

Suppose that A has a multiplicative identity element e_A, and let V be a module over k. If V is a left module over A, then V may be considered as a right module over A^{op}, with

$$v \cdot a^{op} = a \cdot v \tag{1.11.4}$$

for every $a \in A$ and $v \in V$, as on p109 of [3]. Similarly, if V is a right module over A, then V may be considered as a left module over A^{op}.

Suppose that V is a right module over A, and that W is a module over k that is a left module over A. Thus V may be considered as a left module over A^{op}, and W may be considered as a right module over A^{op}, as in the preceding paragraph. Let $V \bigotimes_A W$ be a tensor product of V and W over A, and let $W \bigotimes_{A^{op}} V$ be a tensor product of W and V over A^{op}. If $v \in V$ and $w \in W$, then let $v \otimes_A w$ and $w \otimes_{A^{op}} v$ be the corresponding elements of $V \bigotimes_A W$ and $W \bigotimes_{A^{op}} V$, respectively.

Of course,

$$(v, w) \mapsto w \otimes_{A^{op}} v \tag{1.11.5}$$

defines a mapping from $V \times W$ into $W \bigotimes_{A^{op}} V$ that is bilinear over k. If $a \in A$, $v \in V$, and $w \in W$, then this mapping sends

$$(v \cdot a, w) = (a^{op} \cdot v, w) \tag{1.11.6}$$

and

$$(v, a \cdot w) = (v, w \cdot a^{op}) \tag{1.11.7}$$

to

$$(w \cdot a^{op}) \otimes_{A^{op}} v = w \otimes_{A^{op}} (a^{op} \cdot v). \tag{1.11.8}$$

It follows that there is a unique homomorphism from $V \bigotimes_A W$ into $W \bigotimes_{A^{op}} V$, as modules over k, with

$$v \otimes_A w \mapsto w \otimes_{A^{op}} v \tag{1.11.9}$$

for every $v \in V$ and $w \in W$.

Similarly, there is a unique homomorphism from $W \bigotimes_{A^{op}} V$ into $V \bigotimes_A W$, as modules over k, with

$$w \otimes_{A^{op}} v \mapsto v \otimes_A w \tag{1.11.10}$$

for every $v \in V$ and $w \in W$. It is easy to see that this is the inverse of the mapping mentioned in the preceding paragraph. Thus we get an isomorphism

$$\text{from } V \bigotimes_A W \text{ onto } W \bigotimes_{A^{op}} V, \tag{1.11.11}$$

as on p109 of [3].

1.12 Some double tensor products

Let k be a commutative ring with a multiplicative identity element, and let A, B be associative algebras over k, with multiplicative identity elements e_A, e_B, respectively. Also let V, W, and Z be modules over k, and suppose that

(1.12.1) V is a right module over A, W is a left module over A, W is a right module over B, and Z is a left module over B.

More precisely, the actions of A and B on W should commute, so that W is a bimodule.

Let $V \bigotimes_A W$ be a tensor product of V and W over A, and let $W \bigotimes_B Z$ be a tensor product of W and Z over B. Thus

(1.12.2) $V \bigotimes_A W$ may be considered as a right module over B, and $W \bigotimes_B Z$ may be considered as a left module over A,

as in Section 1.10. Let

$$(V \bigotimes_A W) \bigotimes_B Z \tag{1.12.3}$$

be a tensor product of $V \bigotimes_A W$ and Z over B, and let

$$V \bigotimes_A (W \bigotimes_B Z) \tag{1.12.4}$$

be a tensor product of V and $W \bigotimes_B Z$ over A.

Under these conditions, it is well known that there is a unique homomorphism from (1.12.3) into (1.12.4), as modules over k, with

$$(v \otimes_A w) \otimes_B z \mapsto v \otimes_A (w \otimes_B z) \tag{1.12.5}$$

for every $v \in V$, $w \in W$, and $z \in Z$. To see this, let $z \in Z$ be given, and consider the mapping from $V \times W$ into (1.12.4) with

$$(v, w) \mapsto v \otimes_A (w \otimes_B z) \tag{1.12.6}$$

for every $v \in V$ and $w \in W$. This mapping is bilinear over k, and we have that

$$(v \cdot a) \otimes_A (w \otimes_B z) = v \otimes_A (a \cdot (w \otimes_B z)) = v \otimes_A ((a \cdot w) \otimes_B z) \tag{1.12.7}$$

for every $a \in A$, $v \in V$, and $w \in W$. This leads to a unique homomorphism from $V \bigotimes_A W$ into (1.12.4), as modules over k, with

$$v \otimes_A w \mapsto v \otimes_A (w \otimes_B z) \tag{1.12.8}$$

for every $v \in V$ and $w \in W$. It is easy to see that this homomorphism is linear in z over k, by uniqueness.

If $b \in B$, then

$$(v \otimes_A w) \cdot b = v \otimes_A (w \cdot b) \tag{1.12.9}$$

is sent to

$$v \otimes_A ((w \cdot b) \otimes_B z) = v \otimes_A (w \otimes_B (b \cdot z)) \tag{1.12.10}$$

for every $v \in V$ and $w \in W$. Thus we get a mapping from $\big(V \bigotimes_A W\big) \times Z$ into (1.12.4) that is bilinear over k, and for which the action of $b \in B$ on $\big(V \bigotimes_A W\big)$ on the right corresponds to the action of b on Z on the left. This leads to a unique homomorphism from (1.12.3) into (1.12.4), as modules over k, associated to the bilinear mapping on $\big(V \bigotimes_A W\big) \times Z$ just mentioned. In particular, this homomorphism satisfies (1.12.5), and one can check that it is uniquely determined by this property.

Similarly, there is a unique homomorphism from (1.12.4) into (1.12.3), as modules over k, with

$$v \otimes_A (w \otimes_B z) \mapsto (v \otimes_A w) \otimes_B z \tag{1.12.11}$$

for every $v \in V$, $w \in W$, and $z \in Z$. One can verify that these homomorphisms are inverses of each other, and thus isomorphisms. This corresponds to Proposition 5.1 on p27 of [3]. This was also mentioned in Section 1.4 when $A = B = k$.

1.13 Homomorphisms and tensor products

Let k be a commutative ring with a multiplicative identity element, and let A, B be associative algebras over k with multiplicative identity elements e_A, e_B, respectively, again. Also let V, W, and Z be modules over k, and suppose now that

$$\begin{aligned} &V \text{ is a left module over } A, W \text{ is a right module over } A,\\ &W \text{ is a left module over } B, \text{ and } Z \text{ is a left module over } B. \end{aligned} \tag{1.13.1}$$

The actions of A and B on W should commute, so that W is a bimodule, as usual.

Let $W \bigotimes_A V$ be a tensor product of W and V over A, which may be considered as a left module over B, as in Section 1.10. Similarly, the space $\mathrm{Hom}_B(W, Z)$ of homomorphisms from W into Z, as left modules over B, may be considered as a left module over A, as in Section 1.8. Thus the space

$$\mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W, Z)\big) \tag{1.13.2}$$

of homomorphisms from V into $\mathrm{Hom}_B(W, Z)$, as left modules over A, can be defined as a module over k in the usual way. The space

$$\mathrm{Hom}_B\Big(\Big(W \bigotimes_A V\Big), Z\Big) \tag{1.13.3}$$

of homomorphisms from $W \bigotimes_A V$ into Z, as left modules over B, is defined as a module over k as well.

An element of (1.13.2) can be evaluated at $v \in V$ to get an element of $\mathrm{Hom}_B(W, Z)$, which can be evaluated at $w \in W$ to get an element $\beta(w, v)$ of Z. This defines a mapping β from $W \times V$ into Z, which is bilinear over k. One can check that

$$\beta(w \cdot a, v) = \beta(w, a \cdot v) \tag{1.13.4}$$

for every $a \in A$, $v \in V$, and $w \in W$. Similarly,

$$\beta(b \cdot w, v) = b \cdot \beta(w, v) \tag{1.13.5}$$

for every $b \in B$, $v \in V$, and $w \in W$. Conversely, if β is a mapping from $W \times V$ into Z that is bilinear over k and satsfies these two conditions, then β determines an element of (1.13.2) in this way.

If β is a mapping from $W \times V$ into Z that is bilinear over k and satisfies (1.13.4), then there is a unique homomorphism from $W \bigotimes_A V$ into Z, as modules over k, with

$$w \otimes_A v \mapsto \beta(w, v) \tag{1.13.6}$$

for every $v \in V$ and $w \in W$, as usual. If β satisfies (1.13.5) too and $b \in B$, then we get that

$$b \cdot (w \otimes_A v) = (b \cdot w) \otimes_A v \mapsto \beta(b \cdot w, v) = b \cdot \beta(w, v) \tag{1.13.7}$$

for every $v \in V$ and $w \in W$. This implies that β corresponds to a homomorphism from $W \bigotimes_A V$ into Z, as modules over B. Using this and the remarks in the preceding paragraph, we get a homomorphism from (1.13.2) into (1.13.3), as modules over k.

Conversely, if $v \in V$ and $w \in W$, then an element of (1.13.3) can be evaluated at $w \otimes_A v$ to get an element $\beta(w, v)$ of Z. This defines a mapping β from $W \times V$ into Z that is bilinear over k, and one can check that it satisfies (1.13.4) and (1.13.5). Thus β determines an element of (1.13.2), as before. This defines a homomorphism from (1.13.3) into (1.13.2), as modules over k.

It is easy to see that the homomorphisms between (1.13.2) and (1.13.3) described in the previous two paragraphs are inverses of each other. This corresponds to Proposition 5.2 on p28 of [3].

Similarly, suppose instead that

(1.13.8) V is a right module over A, W is a left module over A,
W is a right module over B, and Z is a right module over B,

where the actions of A and B on W commute. Let $V \bigotimes_A W$ be a tensor product of V and W over A, which may be considered as a right module over B. The space $\mathrm{Hom}_B(W, Z)$ of homomorphisms from W into Z, as right modules over B, may be considered as a right module over A. This means that the space (1.13.2) of homomorphisms from V into $\mathrm{Hom}_B(W, Z)$, as right modules over A, is defined as a module over k. The space

$$\mathrm{Hom}_B\big((V \bigotimes_A W), Z\big) \tag{1.13.9}$$

of homomorphisms from $V \bigotimes_A W$ into Z, as right modules over B, is defined as a module over k too.

As before, an element of (1.13.2) can be evaluated at $v \in V$ and $w \in W$ to get an element $\beta(v, w)$ of Z. This defines β as a mapping from $V \times W$ into Z that is bilinear over k. One can verify that

$$\beta(v \cdot a, w) = \beta(v, a \cdot w) \tag{1.13.10}$$

for every $a \in A$, $v \in V$, and $w \in W$, and that

$$\beta(v, w \cdot b) = \beta(v, w) \cdot b \tag{1.13.11}$$

for every $b \in B$, $v \in V$, and $w \in W$. Conversely, a mapping β from $V \times W$ into Z that is bilinear over k and satisfies (1.13.10) and (1.13.11) determines an element of (1.13.2) in this way.

Of course, if β is a mapping from $V \times W$ into Z that is bilinear over k and satisfies (1.13.10), then there is a unique homomorphism from $V \bigotimes_A W$ into Z, as modules over k, such that

$$v \otimes_A w \mapsto \beta(v, w) \tag{1.13.12}$$

for every $v \in V$ and $w \in W$. If β satisfies (1.13.11) as well and $b \in B$, then

$$(v \otimes_A w) \cdot b = v \otimes_A (w \cdot b) \mapsto \beta(v, w \cdot b) = \beta(v, w) \cdot b \tag{1.13.13}$$

for every $v \in V$ and $w \in W$. This implies that β corresponds to a homomorphism from $V \bigotimes_A W$ into Z as right modules over B, as before. This leads to a homomorphism from (1.13.2) into (1.13.9), as modules over k.

Conversely, an element of (1.13.9) corresponds to a mapping β from $V \times W$ into Z that is bilinear over k, and one can verify that β satisfies (1.13.12) and (1.13.13). This determines an element of (1.13.2), as before. Using this, we get a homomorphism from (1.13.9) into (1.13.2), as modules over k. One can check that this is the inverse of the homomorphism mentioned in the preceding paragraph. This corresponds to Proposition 5.2' on p28 of [3].

1.14 Bilinearity over A, B

Let k, A, and B be as in the previous section, and let V, W, and Z be modules over k again. In this section, we suppose that

(1.14.1) V is a left module over A, W is a right module over B,
Z is a left module over A, and Z is a right module over B,

where the actions of A and B on Z commute, as usual.

Let $V \bigotimes_k W$ be a tensor product of V and W, as modules over k. The actions of A on V and B on W lead to actions of A and B on $V \bigotimes_k W$, so that $V \bigotimes_k W$ is a left module over A and a right module over B, as before. One can check that these actions of A and B on $V \bigotimes_k W$ commute.

Consider the space

$$\mathrm{Hom}_{A,B}\big((V \bigotimes_k W), Z\big) \tag{1.14.2}$$

of homomorphisms from $V \bigotimes_k W$ into Z, as both left modules over A, and right modules over B. This is a module over k, with respect to pointwise addition and scalar multiplication of mappings, as usual.

Let β be a mapping from $V \times W$ into Z that is bilinear over k. This leads to a unique homomorphism $\widetilde{\beta}$ from $V \bigotimes_k W$ into Z, as modules over k, such that

$$\widetilde{\beta}(v \otimes w) = \beta(v, w) \tag{1.14.3}$$

for every $v \in V$ and $w \in W$. Suppose that

(1.14.4) for each $w \in W$, $\beta(\cdot, w)$ is a homomorphism from V into Z, as left modules over A.

In this case, one can check that $\widetilde{\beta}$ is a homomorphism from $V \bigotimes_k W$ into Z, as left modules over A. Similarly, if

(1.14.5) for each $v \in V$, $\beta(v, \cdot)$ is a homomorphism from W into Z, as right modules over B,

then $\widetilde{\beta}$ is a homomorphism from $V \bigotimes_k W$ into Z, as right modules over B.

Thus, if (1.14.4) and (1.14.5) both hold, then $\widetilde{\beta}$ is an element of (1.14.2). Conversely, every element of (1.14.2) corresponds to a mapping from $V \times W$ into Z that is bilinear over k and satisfies (1.14.4) and (1.14.5) in this way. This can be seen by composing with the natural mapping from $V \times W$ into $V \bigotimes_k W$.

Note that the space $\mathrm{Hom}_B(W, Z)$ of homomorphisms from W into Z, as right modules over B, may be considered as a left module over A, as in Section 1.8. This means that the space

$$\mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W, Z)\big) \tag{1.14.6}$$

of homomorphisms from V into $\mathrm{Hom}_B(W, Z)$, as left modules over A, can be defined in the usual way. Similarly, the space $\mathrm{Hom}_A(V, Z)$ of homomorphisms from V into Z, as left modules over A, may be considered as a right module over B. This permits us to define the space

$$\mathrm{Hom}_B\big(W, \mathrm{Hom}_A(V, Z)\big) \tag{1.14.7}$$

of homomorphisms from W into $\mathrm{Hom}_A(V, Z)$, as right modules over B, in the usual way.

It is easy to see that the elements of both (1.14.6) and (1.14.7) correspond exactly to mappings from $V \times W$ into Z that are bilinear over k and satisfy (1.14.4) and (1.14.5). This defines a natural isomorphism between (1.14.6) and (1.14.7), as modules over k, as in Exercise 4 on p32 of [3]. More precisely, (1.14.6) and (1.14.7) are both isomorphic to (1.14.2), as modules over k, in a natural way.

1.15 Direct families of homomorphisms

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose for the moment that W and Z are modules over k, that are either both left modules over A, or both right modules over A. Suppose also that ϕ is a homomorphism from W into Z, and that ψ is a homomorphism from Z into W, as modules over A, such that

$$\psi \circ \phi \text{ is the identity mapping on } W. \tag{1.15.1}$$

This implies that ϕ is injective on W, $\psi(Z) = W$, and

$$\phi \circ \psi \text{ is the identity mapping on } \phi(W). \tag{1.15.2}$$

Under these conditions, one can check that

$$Z \text{ corresponds to the direct sum of } \phi(W) \text{ and } \ker \psi, \tag{1.15.3}$$

as a module over A.

Now let V be a module over k, let I be a nonempty set, and let V_j be a module over k for each $j \in I$. Suppose that these are either all left modules over A, or all right modules over A. Suppose also that for each $j \in I$, i_j is a homomorphism from V_j into V, and that π_j is a homomorphism from V into V_j, as modules over A, with

$$\pi_j \circ i_j \text{ equal to the identity mapping on } V_j. \tag{1.15.4}$$

If $j, l \in I$ and $j \neq l$, then we suppose in addition that

$$\pi_l \circ i_j = 0. \tag{1.15.5}$$

Under these conditions, the family of i_j, π_j, $j \in I$, is said to be a *direct family of homomorphisms*, as on p4 of [3].

Suppose for the moment that

$$\bigoplus_{j \in I} V_j \subseteq V \subseteq \prod_{j \in I} V_j, \tag{1.15.6}$$

with V a submodule of $\prod_{j \in I} V_j$, as a module over A. If $l \in I$, then let i_l be the mapping from V_l into V that sends $v_l \in V_l$ to the element of $\bigoplus_{j \in I} V_j$ with lth coordinate equal to v_l, and all other coordinates equal to 0. If π_l is the restriction of the lth coordinate projection on $\prod_{j \in I} V_j$ to V, then it is easy to see that we get a direct family of homomorphisms.

Let V be any left or right module over A again, and suppose that we have a direct family of homomorphisms i_j, π_j, $j \in I$, as before. Using the i_j's, we get a homomorphism

$$\text{from } \bigoplus_{j \in I} V_j \text{ into } V, \tag{1.15.7}$$

as modules over A. We can also use the π_j's to get a homomorphism

(1.15.8) $$\text{from } V \text{ into } \prod_{j \in I} V_j,$$

as modules over A. It is easy to see that

(1.15.9) $$\text{the composition of these two homomorphisms is the identity mapping on } \bigoplus_{j \in I} V_j,$$

by (1.15.4) and (1.15.5). In particular, this means that the homomorphism mentioned in (1.15.7) is an injection.

Suppose that I has only finitely many elements, so that $\bigoplus_{j \in I} V_j$ and $\prod_{j \in I} V_j$ are the same, and the homomorphism mentioned in (1.15.8) is a surjection. In this case, it is easy to see that the homomorphism mentioned in (1.15.7) is a surjection if and only if the homomorphism mentioned in (1.15.8) is an injection. This happens if and only if

(1.15.10) $$\sum_{j \in I} i_j \circ \pi_j \text{ is the identity mapping on } V,$$

as on p4 of [3].

Chapter 2

Modules and tensor products, 2

2.1 Compositions and $\mathrm{Hom}(\cdot, W)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that V_1, V_2, V_3, and W are modules over k that are either all left modules over A, or all right modules over A. Let ϕ_1 be a homomorphism from V_1 into V_2, and let ϕ_2 be a homomorphism from V_2 into V_3, as modules over A.

If ψ_2 is a homomorphism from V_2 into W, as modules over A, then

$$\Phi_1(\psi_2) = \psi_2 \circ \phi_1 \tag{2.1.1}$$

is a homomorphism from V_1 into W, as modules over A. Similarly, if ψ_3 is a homomorphism from V_3 into W, as modules over A, then

$$\Phi_2(\psi_3) = \psi_3 \circ \phi_2 \tag{2.1.2}$$

is a homomorphism from V_2 into W, as modules over A. Note that Φ_1 defines a homomorphism from $\mathrm{Hom}_A(V_2, W)$ into $\mathrm{Hom}_A(V_1, W)$, and that Φ_2 defines a homomorphism from $\mathrm{Hom}_A(V_3, W)$ into $\mathrm{Hom}_A(V_2, W)$, as modules over k.

Thus $\Phi_1 \circ \Phi_2$ is a homomorphism from $\mathrm{Hom}_A(V_3, W)$ into $\mathrm{Hom}_A(V_1, W)$, as modules over k. If $\psi_3 \in \mathrm{Hom}_A(V_3, W)$, then

$$(\Phi_1 \circ \Phi_2)(\psi_3) = \Phi_1(\Phi_2(\psi_3)) = (\Phi_2(\psi_3)) \circ \phi_1 = \psi_3 \circ (\phi_2 \circ \phi_1). \tag{2.1.3}$$

Of course, $\phi_2 \circ \phi_1$ is a homomorphism from V_1 into V_3, as modules over A. If

$$\phi_2 \circ \phi_1 = 0, \tag{2.1.4}$$

then we get that

$$\Phi_1 \circ \Phi_2 = 0. \tag{2.1.5}$$

Note that (2.1.4) is the same as saying that

$$\phi_1(V_1) \subseteq \ker \phi_2. \tag{2.1.6}$$

Similarly, (2.1.5) is the same as saying that

$$\Phi_2(\mathrm{Hom}_A(V_3, W)) \subseteq \ker \Phi_1. \tag{2.1.7}$$

Alternatively,

$$\ker \Phi_1 = \{\psi_2 \in \mathrm{Hom}_A(V_2, W) : \phi_1(V_1) \subseteq \ker \psi_2\}, \tag{2.1.8}$$

by construction. If $\psi_3 \in \mathrm{Hom}_A(V_3, W)$, then

$$\ker \Phi_2(\psi_3) = \ker \psi_3 \circ \phi_2 \supseteq \ker \phi_2. \tag{2.1.9}$$

Thus (2.1.6) implies that

$$\phi_1(V_1) \subseteq \ker \Phi_2(\psi_3), \tag{2.1.10}$$

so that $\Phi_2(\psi_3) \in \ker \Phi_1$.

Suppose now that

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \longrightarrow 0 \tag{2.1.11}$$

is an exact sequence, so that

$$\phi_1(V_1) = \ker \phi_2, \quad \phi_2(V_2) = V_3. \tag{2.1.12}$$

It is well known that

$$0 \longrightarrow \mathrm{Hom}_A(V_3, W) \xrightarrow{\Phi_2} \mathrm{Hom}_A(V_2, W) \xrightarrow{\Phi_1} \mathrm{Hom}_A(V_1, W) \tag{2.1.13}$$

is exact under these conditions, so that Φ_2 is injective and

$$\Phi_2(\mathrm{Hom}_A(V_3, W)) = \ker \Phi_1. \tag{2.1.14}$$

This is known as *left exactness* of $\mathrm{Hom}_A(\cdot, W)$, and is part of Proposition 4.4 on p26 of [3].

The injectivity of Φ_2 follows from the surjectivity of ϕ_2, as in Section 1.7. We also have that (2.1.7) holds when (2.1.6) holds, as before.

Suppose that $\psi_2 \in \mathrm{Hom}_A(V_2, W)$ is an element of the kernel of Φ_1. We would like to find $\psi_3 \in \mathrm{Hom}_A(V_3, W)$ such that

$$\psi_2 = \Phi_2(\psi_3) = \psi_3 \circ \phi_2 \tag{2.1.15}$$

when (2.1.12) holds. Equivalently, this means that

$$\psi_2(v_2) = \psi_3(\phi_2(v_2)) \tag{2.1.16}$$

for every $v_2 \in V_2$.

If $\ker \phi_2 \subseteq \phi_1(V)$, then

$$\ker \phi_2 \subseteq \ker \psi_2, \tag{2.1.17}$$

by (2.1.8). If $v_2, v_2' \in V_2$ and $\phi_2(v_2) = \phi_2(v_2')$, then $v_2 - v_2' \in \ker \phi_2$, so that $v_2 - v_2' \in \ker \psi_2$, and thus $\psi_2(v_2) = \psi_2(v_2')$. This implies that ψ_3 is uniquely determined on $\phi_2(V_2)$ by (2.1.16). It follows that ψ_3 is uniquely determined on V_3 by (2.1.16), because $\phi_2(V_2) = V_3$, by hypothesis. It is easy to see that ψ_3 is a homomorphism from V_3 into W, as modules over A, because of the analogous properties of ϕ_2 and ψ_2, as desired.

2.2 Some converse statements

Let us continue with the same notation and hypotheses as in the previous section. The remarks in the previous section correspond to the "only if" part (i) of Proposition 2.9 on p22 of [1]. The "if" part of the proposition deals with the necessity of the various conditions on ϕ_1, ϕ_2, in order that the corresponding conditions on Φ_1, Φ_2 hold for all W.

If ϕ_2 is not surjective, for instance, then $\phi_2(V_2)$ is a proper submodule of V_3, as a module over A. This means that the quotient

$$W = V_3/\phi_2(V_2) \tag{2.2.1}$$

is a nontrivial module over A. Let ψ_3 be the natural quotient mapping from V_3 onto W, which is a nonzero homomorphism from V_3 onto W, as modules over A. In this case,

$$\Phi_2(\psi_3) = \psi_3 \circ \phi_2 = 0, \tag{2.2.2}$$

so that Φ_2 is not injective. This shows that the surjectivity of ϕ_2 is necessary to get the injectivity of Φ_2 for all W.

Similarly, if (2.1.5) holds for every W, then (2.1.4) holds. To see this, one can take $W = V_3$, and ψ_3 to be the identity mapping on V_3, considered as a homomorphism from V_3 into W, as modules over A. Thus

$$(\Phi_1 \circ \Phi_2)(\psi_3) = \psi_3 \circ (\phi_2 \circ \phi_1) = \phi_2 \circ \phi_1, \tag{2.2.3}$$

as in (2.1.3). If (2.1.5) holds, then we get that (2.1.4) holds, as desired. Equivalently, this shows that if (2.1.7) holds for every W, then (2.1.6) holds.

Suppose now that

$$\ker \Phi_1 \subseteq \Phi_2(\mathrm{Hom}_A(V_3, W)) \tag{2.2.4}$$

for every W, and let us check that

$$\ker \phi_2 \subseteq \phi_1(V_1). \tag{2.2.5}$$

Of course, $\phi_1(V_1)$ is a submodule of V_2, as a module over A, so that

$$W = V_2/\phi_1(V_1) \tag{2.2.6}$$

is a module over A as well. Let ψ_2 be the natural quotient mapping from V_2 onto W, which is a homomorphism from V_2 onto W, as modules over A. Observe that

$$\Phi_1(\psi_2) = \psi_2 \circ \phi_1 = 0, \tag{2.2.7}$$

by construction. This means that $\psi_2 \in \ker \Phi_1$.

Using (2.2.4), we get that there is a $\psi_3 \in \mathrm{Hom}_A(V_3, W)$ such that $\psi_2 = \Phi_2(\psi_3) = \psi_3 \circ \phi_2$. In particular, it follows that the kernel of ϕ_2 is contained in the kernel of ψ_2. This implies (2.2.5), because

$$\ker \psi_2 = \phi_1(V_1), \tag{2.2.8}$$

by construction.

2.3 Compositions and $\mathrm{Hom}(V, \cdot)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A again. In this section, we suppose that V, W_1, W_2, and W_3 are modules over k that are either all left modules over A, or all right modules over A. Let ψ_1 be a homomorphism from W_1 into W_2, and let ψ_2 be a homomorphism from W_2 into W_3, as modules over A.

If ϕ_1 is a homomorphism from V into W_1, as modules over A, then

$$\Psi_1(\phi_1) = \psi_1 \circ \phi_1 \tag{2.3.1}$$

is a homomorphism from V into W_2, as modules over A. Similarly, if ϕ_2 is a homomorphism from V into W_2, as modules over A, then

$$\Psi_2(\phi_2) = \psi_2 \circ \phi_2 \tag{2.3.2}$$

is a homomorphism from V into W_3, as modules over A. As usual, Ψ_1 defines a homomorphism from $\mathrm{Hom}_A(V, W_1)$ into $\mathrm{Hom}_A(V, W_2)$, and Ψ_2 defines a homomorphism from $\mathrm{Hom}_A(V, W_2)$ into $\mathrm{Hom}_A(V, W_3)$, as modules over k.

Thus $\Psi_2 \circ \Psi_1$ is a homomorphism from $\mathrm{Hom}_A(V, W_1)$ into $\mathrm{Hom}_A(V, W_3)$, as modules over k. If $\phi_1 \in \mathrm{Hom}_A(V, W_1)$, then

$$(\Psi_2 \circ \Psi_1)(\phi_1) = \Psi_2(\Psi_1(\phi_1)) = \psi_2 \circ (\Psi_1(\phi_1)) = (\psi_2 \circ \psi_1) \circ \phi_1. \tag{2.3.3}$$

Note that $\psi_2 \circ \psi_1$ is a homomorphism from W_1 into W_3, as modules over A. If

$$\psi_2 \circ \psi_1 = 0, \tag{2.3.4}$$

then it follows that

$$\Psi_2 \circ \Psi_1 = 0. \tag{2.3.5}$$

Of course, (2.3.4) is the same as saying that

$$\psi_1(W_1) \subseteq \ker \psi_2, \tag{2.3.6}$$

and (2.3.5) is the same as saying that

$$\Psi_1(\mathrm{Hom}_A(V, W_1)) \subseteq \ker \Psi_2. \tag{2.3.7}$$

Observe that

$$\ker \Psi_2 = \{\phi_2 \in \mathrm{Hom}_A(V, W_2) : \phi_2(V) \subseteq \ker \psi_2\}. \tag{2.3.8}$$

If $\phi_1 \in \mathrm{Hom}_A(V, W_1)$, then

$$(\Psi_1(\phi_1))(V) = (\psi_1(\phi_1(V))) \subseteq \psi_1(W_1). \tag{2.3.9}$$

This implies that

$$(\Psi_1(\phi_1))(V) \subseteq \ker \psi_2 \tag{2.3.10}$$

when (2.3.6) holds. This is another way to get that $\Psi_1(\phi_1) \in \ker \Psi_2$, using (2.3.8).

Suppose that

$$0 \longrightarrow W_1 \xrightarrow{\psi_1} W_2 \xrightarrow{\psi_2} W_3 \tag{2.3.11}$$

is an exact sequence, so that ψ_1 is injective, and

$$\psi_1(W_1) = \ker \psi_2. \tag{2.3.12}$$

It is well known that

$$0 \longrightarrow \mathrm{Hom}_A(V, W_1) \xrightarrow{\Psi_1} \mathrm{Hom}_A(V, W_2) \xrightarrow{\Psi_2} \mathrm{Hom}_A(V, W_3) \tag{2.3.13}$$

is exact in this case, so that Ψ_1 is injective, and

$$\Psi_1(\mathrm{Hom}_A(V, W_1)) = \ker \Psi_2. \tag{2.3.14}$$

This means that $\mathrm{Hom}_A(V, \cdot)$ is *left exact*, which is another part of Proposition 4.4 on p26 of [3].

The injectivity of Ψ_1 follows immediately from the injectivity of ψ_1, as in Section 1.7. We have also seen that (2.3.7) holds when (2.3.6) holds.

Let $\phi_2 \in \mathrm{Hom}_A(V, W_2)$ be an element of the kernel of Ψ_2. We would like to find $\phi_1 \in \mathrm{Hom}_A(V, W_1)$ such that

$$\phi_2 = \Psi_1(\phi_1) = \psi_1 \circ \phi_1 \tag{2.3.15}$$

when ψ_1 is injective and (2.3.12) holds. More precisely,

$$\phi_2(V) \subseteq \psi_1(W_1) \tag{2.3.16}$$

when $\ker \psi_2 \subseteq \psi_1(W_1)$, by (2.3.8). Using this, we can get $\phi_1 \in \mathrm{Hom}_A(V, W_1)$ as in (2.3.15), because ψ_1 is injective.

2.4 Some more converse statements

Let us continue with the same notation and hypotheses as in the previous section. The remarks in the previous section correspond to the "only if" part of part (ii) of Proposition 2.9 on p23 of [1]. As before, the "if" part of the proposition concerns the necessity of the conditions on ψ_1, ψ_2, for the corresponding conditions on Ψ_1, Ψ_2 to hold for all V.

If ψ_1 is not injective, then the kernel of ψ_1 is a nontrivial submodule of W_1, as a module over A. Let us take

$$V = \ker \psi_1, \tag{2.4.1}$$

and let ϕ_1 be the natural inclusion mapping from V into W_1. Thus ϕ_1 is a nonzero homomorphism from V into W_1, as modules over A. Clearly

$$\Psi_1(\phi_1) = \psi_1 \circ \phi_1 = 0, \tag{2.4.2}$$

by construction, so that Ψ_1 is not injective. It follows that the injectivity of ψ_1 is needed for Ψ_1 to be injective for any V.

Suppose that (2.3.5) holds for every V, and let us check that (2.3.4) holds. To do this, we take $V = W_1$, and ϕ_1 to be the identity mapping on W_1, considered as a homomorphism from V into W_1, as a module over A. In this case,

$$(\Psi_2 \circ \Psi_1)(\phi_1) = (\psi_2 \circ \psi_1) \circ \phi_1 = \psi_2 \circ \psi_1, \tag{2.4.3}$$

as in (2.3.3). Thus (2.3.5) implies (2.3.4), as desired. Equivalently, if (2.3.7) holds for every V, then (2.3.6) holds.

If

$$\ker \Psi_2 \subseteq \Psi_1(\mathrm{Hom}_A(V, W_1)) \tag{2.4.4}$$

for every V, then we would like to verify that

$$\ker \psi_2 \subseteq \psi_1(W_1). \tag{2.4.5}$$

Let us take

$$V = \ker \psi_2, \tag{2.4.6}$$

considered as a module over A. More precisely, V is a submodule of W_2, and we take ϕ_2 to be the natural inclusion mapping from V into W_2. This is a homomorphism from V into W_2, as modules over A, with

$$\Psi_2(\phi_2) = \psi_2 \circ \phi_2 = 0. \tag{2.4.7}$$

Thus $\phi_2 \in \ker \Psi_2$.

If (2.4.4) holds, then there is a $\phi_1 \in \mathrm{Hom}_A(V, W_1)$ such that $\phi_2 = \Psi_1(\phi_1) = \psi_1 \circ \phi_1$. This implies that

$$\ker \psi_2 = V = \phi_2(V) \subseteq \psi_1(W_1), \tag{2.4.8}$$

as desired.

2.5 Compositions and tensor products

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that V_1, V_2, and V_3 are right modules over A, and that W is a left module over A. Let $V_1 \bigotimes_A W$, $V_2 \bigotimes_A W$, and $V_3 \bigotimes_A W$ be tensor products of V_1, V_2, and V_3, respectively, with W over A.

Suppose that ϕ_1 is a homomorphism from V_1 into V_2, and that ϕ_2 is a homomorphism from V_2 into V_3, as right modules over A. Using ϕ_1, we get a unique homomorphism Φ_1 from $V_1 \bigotimes_A W$ into $V_2 \bigotimes_A W$, as modules over k, such that

$$\Phi_1(v_1 \otimes w) = \phi_1(v_1) \otimes w \tag{2.5.1}$$

for every $v_1 \in V_1$ and $w \in W$. More precisely, Φ_1 is obtained from ϕ_1 and the identity mapping on W as in Section 1.9. Similarly, there is a unique homomorphism Φ_2 from $V_2 \bigotimes_A W$ into $V_3 \bigotimes_A W$, as modules over k, such that

$$\Phi_2(v_2 \otimes w) = \phi_2(v_2) \otimes w \tag{2.5.2}$$

for every $v_2 \in V_2$ and $w \in W$.

Of course, $\phi_2 \circ \phi_1$ is a homomorphism from V_1 into V_3, as right modules over A. This leads to a unique homomorphism from $V_1 \bigotimes_A W$ into $V_3 \bigotimes_A W$, as modules over k, with

$$v_1 \otimes w \mapsto \phi_2(\phi_1(v_1)) \otimes w \tag{2.5.3}$$

for every $v_1 \in V_1$ and $w \in W$, as before. This homomorphism is the same as

$$\Phi_2 \circ \Phi_1, \tag{2.5.4}$$

as in Section 1.9. In particular, if

$$\phi_2 \circ \phi_1 = 0, \tag{2.5.5}$$

then

$$\Phi_2 \circ \Phi_1 = 0. \tag{2.5.6}$$

Equivalently, if

$$\phi_1(V_1) \subseteq \ker \phi_2, \tag{2.5.7}$$

then

$$\Phi_1\big(V_1 \bigotimes_A W\big) \subseteq \ker \Phi_2. \tag{2.5.8}$$

Suppose that

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \longrightarrow 0 \tag{2.5.9}$$

is an exact sequence, so that ϕ_2 is surjective, and

$$\phi_1(V_1) = \ker \phi_2. \tag{2.5.10}$$

Under these conditions, it is well known that

$$V_1 \bigotimes_A W \xrightarrow{\Phi_1} V_2 \bigotimes_A W \xrightarrow{\Phi_2} V_3 \bigotimes_A W \longrightarrow 0 \tag{2.5.11}$$

is an exact sequence, so that Φ_2 is surjective and

$$\Phi_1\big(V_1 \bigotimes_A W\big) = \ker \Phi_2. \tag{2.5.12}$$

This is known as *right exactness* of $V \mapsto V \bigotimes_A W$, which is part of Proposition 4.5 on p26 of [3]. More precisely, the surjectivity of Φ_2 follows from the surjectivity of ϕ_2, as in Section 1.9.

To show that (2.5.12) holds, note first that (2.5.8) holds, as before. Of course, $\Phi_1\big(V_1 \bigotimes_A W\big)$ is a submodule of $V_2 \bigotimes_A W$, as a module over k. Thus the quotient

$$Y = \big(V_2 \bigotimes_A W\big)/\Phi_1\big(V_1 \bigotimes_A W\big) \tag{2.5.13}$$

is defined as a module over k. Let q_Y be the natural quotient mapping from $V_2 \bigotimes_A W$ onto Y.

Using (2.5.8), we get that there is a unique homomorphism $\widetilde{\Phi}_2$ from Y into $V_3 \bigotimes_A W$, as modules over k, such that

$$\widetilde{\Phi}_2 \circ q_Y = \Phi_2. \tag{2.5.14}$$

We would like to show that $\widetilde{\Phi}_2$ is an isomorphism.

If $v_2 \in V_2$ and $w \in W$, then

$$q_Y(v_2 \otimes w) \tag{2.5.15}$$

is an element of Y. Observe that

$$q_Y(\phi_1(v_1) \otimes w) = 0 \tag{2.5.16}$$

for every $v_1 \in V_1$ and $w \in W$, by (2.5.1) and the definition of Y. This implies that (2.5.15) only depends on $\phi_2(v_2)$ and w, because of (2.5.10). This leads to a mapping from $V_3 \times W$ into Y, with

$$(\phi_2(v_2), w) \mapsto q_Y(v_2 \otimes w) \tag{2.5.17}$$

for every $v_2 \in V_2$ and $w \in W$, because $\phi_2(V_2) = V_3$, by hypothesis. It is easy to see that this mapping is bilinear over k.

If $a \in A$, $v_2 \in V_2$, and $w \in W$, then $\phi_2(v_2 \cdot a) = \phi_2(v_2) \cdot a$ and $(v_2 \cdot a) \otimes w = v_2 \otimes (a \cdot w)$ in $V_2 \bigotimes_A W$. This implies that

$$(\phi_2(v_2) \cdot a, w) \text{ and } (\phi_2(v_2), a \cdot w) \tag{2.5.18}$$

are mapped to the same element of Y by the mapping described in the preceding paragraph. This leads to a unique homomorphism from $V_3 \bigotimes_A W$ into Y, as modules over k, with

$$\phi_2(v_2) \otimes w \mapsto q_Y(v_2 \otimes w) \tag{2.5.19}$$

for every $v_2 \in V_2$ and $w \in W$.

One can check that the composition of the mapping from $V_3 \bigotimes_A W$ into Y defined in the previous paragraph with $\widetilde{\Phi}_2$ is the identity mapping on $V_3 \bigotimes_A W$. Similarly, one can verify that the composition of $\widetilde{\Phi}_2$ with the mapping from $V_3 \bigotimes_A W$ into Y defined in the previous paragraph is the identity mapping on Y. Thus $\widetilde{\Phi}_2$ is an isomorphism from Y onto $V_3 \bigotimes_A W$, as modules over k.

In particular, the kernel of $\widetilde{\Phi}_2$ is trivial. This implies (2.5.12), as desired.

Of course, there is an analogous right exactness property for $W \mapsto V \bigotimes_A W$, which is also part of Proposition 4.5 on p26 of [3].

2.6 Another approach

Let us continue with the same notation and hypotheses as in the previous section. The fact that the exactness of (2.5.9) implies the exactness of (2.5.11) corresponds to Proposition 2.18 on p28 of [1]. The argument in [1] uses the characterization of exactness of sequences like these in Sections 2.1 and 2.2, as follows.

Let Z be an arbitrary module over k. Thus, for each $j = 1, 2, 3$, the space

$$\operatorname{Hom}_k\big(V_j \bigotimes_A W, Z\big) \tag{2.6.1}$$

of homomorphisms from $V_j \bigotimes_A W$ into Z, as modules over k, is a module over k with respect to pointwise addition and scalar multiplication of mappings into Z, as usual. Let $\widehat{\Phi}_1$ be the homomorphism

$$\text{from } \operatorname{Hom}_k\big(V_2 \bigotimes_A W, Z\big) \text{ into } \operatorname{Hom}_k\big(V_1 \bigotimes_A W, Z\big), \tag{2.6.2}$$

as modules over k, that sends an element of $\operatorname{Hom}_k\big(V_2 \bigotimes_A W, Z\big)$ to its composition with Φ_1. Similarly, let $\widehat{\Phi}_2$ be the homomorphism

$$\text{from } \operatorname{Hom}_k\big(V_3 \bigotimes_A W, Z\big) \text{ into } \operatorname{Hom}_k\big(V_2 \bigotimes_A W, Z\big), \tag{2.6.3}$$

as modules over k, that sends an element of $\operatorname{Hom}_k\big(V_3 \bigotimes_A W, Z\big)$ to its composition with Φ_2.

The exactness of (2.5.11) implies that

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_k\big(V_3 \bigotimes_A W, Z\big) &\xrightarrow{\widehat{\Phi}_2} \operatorname{Hom}_k\big(V_2 \bigotimes_A W, Z\big) \\ &\xrightarrow{\widehat{\Phi}_1} \operatorname{Hom}_k\big(V_1 \bigotimes_A W, Z\big) \end{aligned} \tag{2.6.4}$$

is exact, as in Section 2.1. Conversely, in order to show that (2.5.11) is exact, it sufficies to verify that (2.6.4) is exact for every Z, as in Section 2.2.

Remember that the space $\operatorname{Hom}_k(W, Z)$ of homomorphisms from W into Z, as modules over k, may be considered as a right module over A in a natural way, because W is a left module over A, as in Section 1.8. We have also seen that (2.6.1) is isomorphic to

$$\operatorname{Hom}_A\big(V_j, \operatorname{Hom}_k(W, Z)\big), \tag{2.6.5}$$

as modules over k, in a natural way for $j = 1, 2, 3$, as in Section 1.13.

Let Φ_1' be the homomorphism

$$\text{from } \operatorname{Hom}_A\big(V_2, \operatorname{Hom}_k(W, Z)\big) \text{ into } \operatorname{Hom}_A\big(V_1, \operatorname{Hom}_k(W, Z)\big), \tag{2.6.6}$$

as modules over k, that sends an element of $\operatorname{Hom}_A\big(V_2, \operatorname{Hom}_k(W, Z)\big)$ to its composition with ϕ_1. Similarly, let Φ_2' be the homomorphism

$$\text{from } \operatorname{Hom}_A\big(V_3, \operatorname{Hom}_k(W, Z)\big) \text{ into } \operatorname{Hom}_A\big(V_2, \operatorname{Hom}_k(W, Z)\big), \tag{2.6.7}$$

as modules over k, that sends an element of $\operatorname{Hom}_A\big(V_3, \operatorname{Hom}_k(W, Z)\big)$ to its composition with ϕ_2.

One can check that Φ_1', Φ_2' correspond exactly to $\widehat{\Phi}_1$, $\widehat{\Phi}_2$, respectively, with respect to the isomorphisms between (2.6.1) and (2.6.5), $j = 1, 2, 3$, mentioned in the preceding paragraph. This uses the way that Φ_1, Φ_2 are obtained from ϕ_1, ϕ_2, respectively, as in the previous section.

It follows that the exactness of (2.6.4) is equivalent to the exactness of

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A\big(V_3, \operatorname{Hom}_k(W, Z)\big) &\stackrel{\Phi_2'}{\longrightarrow} \operatorname{Hom}_A\big(V_2, \operatorname{Hom}_k(W, Z)\big) \\ &\stackrel{\Phi_1'}{\longrightarrow} \operatorname{Hom}_A\big(V_1, \operatorname{Hom}_k(W, Z)\big). \end{aligned} \tag{2.6.8}$$

The exactness of (2.5.9) implies the exactness of (2.6.8), as in Section 2.1. Thus the exactness of (2.5.9) implies the exactness of (2.6.4). More precisely, this works for every module Z over k. This shows that the exactness of (2.5.9) implies the exactness of (2.5.11), as before.

2.7 Projective modules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. A left module V over A is said to be *projective* if it has the following property. Let W and Z be left modules over A, and let ψ be a homomorphism from W onto Z, as left modules over A. If ϕ_Z is a homomorphism from V into Z, as left modules over A, then there should be a homomorphism ϕ_W from V into W, as left modules over A, such that

$$\phi_Z = \psi \circ \phi_W. \tag{2.7.1}$$

Of course, projectivity of right modules is defined analogously.

It is easy to see that A is projective as a left module over itself. Indeed, suppose that W, Z, ψ, and ϕ_Z are as in the preceding paragraph, with $V =$

A. Note that ϕ_Z is uniquely determined by $\phi_Z(e_A)$, and similarly that any homomorphism ϕ_W from A into W, as left modules over A, is determined by $\phi_W(e_A)$. Thus it suffices to choose $\phi_W(e_A) \in W$ such that

$$\phi_Z(e_A) = \psi(\phi_W(e_A)), \tag{2.7.2}$$

and to take ϕ_W to be the corresponding module homomorphism from A into W.

Let I be a nonempty set, and let V_j be a left module over A for each $j \in I$. It is well known that

$$V_j \text{ is projective for every } j \in I \tag{2.7.3}$$

if and only if

$$V = \bigoplus_{j\in I} V_j \text{ is projective as a left module over } A, \tag{2.7.4}$$

as in Proposition 2.1 on p6 of [3]. To show that (2.7.3) implies (2.7.4), let W, Z, ψ, and ϕ_Z be given as before. If $l \in I$, then one can ge a homomorphism $\phi_{l,Z}$ from V_l into Z, as left modules over A, by composing ϕ_Z with the natural inclusion mapping from V_l into V. Using the projectivity of V_l, we get a homomorphism $\phi_{l,W}$ from V_l into W, as left modules over A, such that

$$\phi_{l,Z} = \psi \circ \phi_{l,W}. \tag{2.7.5}$$

One can combine the homomorphisms $\phi_{l,W}$, $l \in I$, to get a homomorphism ϕ_W from V into W, as left modules over A, that satisfies (2.7.1). In particular, if V is free as a left module over A, then it follows that V is projective.

Conversely, suppose that (2.7.4) holds, and let us check that (2.7.3) holds. Let $l \in I$ be given, let W, Z, and ψ be as before, and let $\phi_{l,Z}$ be a homomorphism from V_l into Z, as left modules over A. Using $\phi_{l,Z}$, we can get a homomorphism ϕ_Z from V into Z, as left modules over A, whose composition with the natural inclusion mapping from V_l into V is equal to $\phi_{l,Z}$, and whose composition with the natural inclusion mapping from V_j into V is equal to 0 when $j \neq l$. If V is projective, then there is a homomorphism ϕ_W from V into W, as left modules over A, that satisfies (2.7.1). Let $\phi_{l,W}$ be the composition of ϕ_W with the natural inclusion mapping from V_l into V, which is a homomorphism from V_l into W, as left modules over A. Under these conditions, (2.7.5) follows by composing both sides of (2.7.1) with the natural inclusion mapping from V_l into V. This implies that V_l is projective, as desired.

If V is any left module over A, then it is well known that

$$\text{there is a homomorphism from a free left module over } A \text{ onto } V, \tag{2.7.6}$$

as left modules over A. Indeed, if X is a free left module over A, then any mapping from the generators of X, as a free left module over A, into V can be extended to a homomorphism from X into V, as left modules over A. This homomorphism maps X onto the submodule of V, as a left module over A, generated by the images of the generators of X. If X is freely generated by sufficiently many elements, then one can use this to get a homomorphism from X onto V.

Let Y be a left module over A, and let η be a homomorphism from Y onto V, as left modules over A. If V is projective, then there is a homomorphism ϕ from V into Y, as modules over A, such that

(2.7.7) $\eta \circ \phi$ is the identity mapping on V.

This corresponds to taking $W = Y$, $Z = V$, $\psi = \eta$, and ϕ_Z equal to the identity mapping on V, in the definition of projectivity. In particular, (2.7.7) implies that ϕ is injective, and that

(2.7.8) $\phi \circ \eta$ is the identity mapping on $\phi(V)$.

Note that

(2.7.9) $$\ker \eta = \ker(\phi \circ \eta)$$

is a submodule of Y. Under these conditions,

(2.7.10) Y corresponds to the direct sum of $\phi(V)$ and $\ker \eta$,

as a left module over A, by standard arguments. This corresponds to part of Proposition 2.4 on p7 of [3].

It is well known that a left module V over A is projective if and only if

(2.7.11) there is a free left module over A that is isomorphic to the direct sum of V and another left module over A,

as left modules over A, as in Theorem 2.2 on p6 of [3]. Of course, the "if" part follows from the fact that (2.7.4) implies (2.7.3). The "only if" part follows from (2.7.6) and (2.7.10). Using the "if" part and (2.7.6), we get that projectivity of V is necessary for the property mentioned in the preceding paragraph, as in Proposition 2.4 on p7 of [3].

Let V be a left module over A, let U_1, U_2 be right modules over A, and let θ be a homomorphism from U_1 into U_2, as right modules over A. Also let $U_1 \bigotimes_A V$, $U_2 \bigotimes_A V$ be tensor products of U_1, U_2 with V over A, respectively. Thus we get a homomorphism

(2.7.12) Θ from $U_1 \bigotimes_A V$ into $U_2 \bigotimes_A V$,

as modules over k, using θ and the identity mapping on V, as in Section 1.9. If V is projective and θ is injective, then it is well known that

(2.7.13) Θ is injective.

More precisely, let U_3 be another right module over A, let θ' be a homomorphism from U_2 into U_3, as right modules over A, and let $U_3 \bigotimes_A V$ be a tensor product of U_3 and V over A. As before, we get a homomorphism

$$\Theta' \text{ from } U_2 \bigotimes_A V \text{ into } U_3 \bigotimes_A V, \tag{2.7.14}$$

as modules over k, using θ' and the identity mapping on V. Suppose that

$$U_1 \xrightarrow{\theta} U_2 \xrightarrow{\theta'} U_3 \tag{2.7.15}$$

is exact, so that $\theta(U_1) = \ker \theta'$. If V is projective, then

$$U_1 \bigotimes_A V \xrightarrow{\Theta} U_2 \bigotimes_A V \xrightarrow{\Theta'} U_3 \bigotimes_A V \tag{2.7.16}$$

is exact, so that $\Theta\big(U_1 \bigotimes_A V\big) = \ker \Theta'$.

This corresponds to Proposition 1.1a on p106 of [3], where "left balanced" is defined on p97 of [3]. If V is a free module over A, then this can be verified directly. Otherwise, one can use (2.7.11) to reduce to the case of free modules. This is related to Exercise 4 on p31 of [1]. There are analogous statements with the roles of right and left modules interchanged, as usual.

2.8 Injective modules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. A left module W over A is said to be *injective* if it has the following property. Let V be a left module over A, let V_0 be a submodule of V, and let ϕ_0 be a homomorphism from V_0 into W, as left modules over A. Under these conditions, there should be a homomorphism ϕ from V into W, as left modules over A, such that

$$\phi = \phi_0 \text{ on } V_0. \tag{2.8.1}$$

Injectivity of a right module over A is defined in the same way.

Let I be a nonempty set, and let W_j be a left module over A for each $j \in I$. It is well known that

$$W_j \text{ is injective for every } j \in I \tag{2.8.2}$$

if and only if

$$\prod_{j \in I} W_j \text{ is injective,} \tag{2.8.3}$$

as in Proposition 3.1 on p8 of [3]. To see this, let V be a left module over A, let V_0 be a submodule of V, and suppose that (2.8.2) holds. If ϕ_0 is a homomorphism from V_0 into $\prod_{j \in I} W_j$, as left modules over A, then for each $l \in I$ we get a homomorphism $\phi_{0,l}$ from V_0 into W_l, as left modules over A, in the usual way. If (2.8.2) holds, then for every $l \in I$, $\phi_{0,l}$ can be extended to a homomorphism ϕ_l from V into W_l, as left modules over A. This leads to a homomorphism ϕ from V into $\prod_{j\in I} W_j$, as left modules over A. The restriction of ϕ to V_0 is equal to ϕ_0, by construction.

Conversely, suppose that (2.8.3) holds, and let us check that (2.8.2) holds. Let $l \in I$ be given, let V be a left module over A again, let V_0 be a submodule of V, and let $\phi_{0,l}$ be a homomorphism from V_0 into W_l, as left modules over A. We can get a homomorphism ϕ_0 from V_0 into $\prod_{j\in I} W_j$, as left modules over A, using the zero homomorphism from V_0 into W_j when $j \neq l$. If (2.8.3) holds, then there is an extension ϕ of ϕ_0 to a homomorphism from V into $\prod_{j\in I} W_j$, as left modules over A. The lth component ϕ_l of ϕ is an extension of $\phi_{0,l}$ to a homomorphism from V into W_l, as left modules over A, as desired.

Let W be a left module over A again, and consider the following condition:

(2.8.4) if $\mathcal{I}$ is a left ideal in A, and $\phi_{\mathcal{I}}$ is a homomorphism from $\mathcal{I}$ into W, then there is a $w_0 \in W$ such that $\phi_{\mathcal{I}}(a) = a \cdot w_0$ for every $a \in \mathcal{I}$.

More precisely, $\mathcal{I}$ may be considered as a left module over A, and $\phi_{\mathcal{I}}$ is supposed to be a homomorphism from $\mathcal{I}$ into W, as left modules over A. It is well known that W is injective as a left module over A if and only if this condition holds, as in Theorem 3.2 on p8 of [3].

To get the necessity of this condition, let $\mathcal{I}$ and $\phi_{\mathcal{I}}$ be given as before. Note that A may be considered as a left module over itself, so that $\mathcal{I}$ is a submodule of A. If W is injective, then there is an extension ϕ of $\phi_{\mathcal{I}}$ to a homomorphism from A into W, as left modules over A. In this case, the condition holds with

$$w_0 = \phi(e_A). \tag{2.8.5}$$

To show the sufficiency of the condition, let V be a left module over A, let V_0 be a submodule of V, and let ϕ_0 be a homomorphism from V_0 into W, as left modules over A. Also let $v_1 \in V$ be given, and observe that

$$V_1 = \{a \cdot v_1 + v_0 : a \in A,\, v_0 \in V_0\} \tag{2.8.6}$$

is a submodule of V that contains V_0. Put

$$\mathcal{I}_1 = \{a \in A : a \cdot v_1 \in V_0\}, \tag{2.8.7}$$

which is a left ideal in A. If $a \in \mathcal{I}_1$, then put

$$\phi_{\mathcal{I}_1}(a) = \phi_0(a \cdot v_1). \tag{2.8.8}$$

It is easy to see that this defines a homomorphism from $\mathcal{I}_1$ into W, as left modules over A.

By hypothesis, there is a $w_1 \in W$ such that

$$\phi_0(a \cdot v_1) = \phi_{\mathcal{I}_1}(a) = a \cdot w_1 \tag{2.8.9}$$

for every $a \in \mathcal{I}_1$. If $a \in A$ and $v_0 \in V_0$, then we would like to put

$$\phi_1(a \cdot v_1 + v_0) = a \cdot w_1 + \phi_0(v_0). \tag{2.8.10}$$

One can check that ϕ_1 is well-defined as a mapping from V_1 into W, using (2.8.9). More precisely, ϕ_1 is a homomorphism from V_1 into W, as left modules over A. Of course,

$$\phi_1 = \phi_0 \text{ on } V_0, \tag{2.8.11}$$

by construction.

If $V = V_1$, then there is nothing more to do. If V can be generated by V_0 and finitely many other elements as a left module over A, then one can extend ϕ_0 to V by repeating this argument finitely many times. Similarly, if V can be generated by V_0 and an infinite sequence of other elements, as a left module over A, then one can repeat this argument to extend ϕ_0 to an increasing sequence of submodules of V, whose union is V.

Otherwise, let $\mathcal{E}$ be the collection of all ordered pairs (U, ϕ_U), where U is a submodule of V that contains V_0, and ϕ_U is a homomorphism from U into W, as left modules over A, that extends ϕ_0. If $(U_1, \phi_{U_1}), (U_2, \phi_{U_2}) \in \mathcal{E}$, then put

$$(U_1, \phi_{U_1}) \preceq (U_2, \phi_{U_2}) \tag{2.8.12}$$

when $U_1 \subseteq U_2$ and $\phi_{U_2} = \phi_{U_1}$ on U_1. This defines a partial ordering on $\mathcal{E}$, and one can use Zorn's lemma or Hausdorff's maximality principle to get that $\mathcal{E}$ has a maximal element with respect to this partial order. If a maximal element of $\mathcal{E}$ did not correspond to a homomorphism from V into W, then the earlier construction could be used to get an extension to a larger submodule of V.

It is well known that every left module over A is isomorphic to a submodule of an injective module, as in Theorem 3.3 on p9 of [3]. We shall return to this in Sections 3.14 and 3.15.

Let W and Z be left modules over A, and let ζ be an injective homomorphism from W into Z, as left modules over A. If W is injective as a left module over A, then there is a homomorphism ϕ from Z into W, as left modules over A, such that

$$\phi \circ \zeta \text{ is the identity mapping on } W. \tag{2.8.13}$$

More precisely, one can use the injectivity of W to extend the inverse of ζ on $\zeta(W)$ to a homomorphism ϕ from Z into W. Note that $\phi(Z) = W$, and that

$$\zeta \circ \phi \text{ is the identity mapping on } \zeta(W). \tag{2.8.14}$$

Of course,

$$\ker \phi = \ker(\zeta \circ \phi) \tag{2.8.15}$$

is a submodule of Z. One can check that

$$Z \text{ corresponds to the direct sum of } \zeta(W) \text{ and } \ker \phi \tag{2.8.16}$$

as a left module over A, using standard arguments. This corresponds to part of Proposition 3.4 on p10 of [3].

In fact, the injectivity of W is necessary for this property to hold, as in Proposition 3.4 on p10 of [3]. Indeed, if W is any left module over A, then there is an injective homomorphism from W into an injective module Z, as mentioned earlier. In this case, (2.8.16) implies that $\zeta(W)$ is injective, as before, so that W is injective.

Of course, there are analogous statements for right modules over A.

2.9 Covariant ϕ-extensions

Let k be a commutative ring with a multiplicative identity element, and let A, B be associative algebras over k with multiplicative identity elements e_A, e_B, respectively. Suppose that ϕ is an algebra homomorphism from A into B, with $\phi(e_A) = e_B$.

Let V be a module over k. If V is a left module over B, then V may be considered as a left module over A, where the action of $a \in A$ on V is defined to be the given action of $\phi(a) \in B$ on V. Similarly, if V is a right module over B, then V may be considered as a right module over A. This corresponds to some remarks on p28f of [3]. If A and B are commutative, then this is called *restriction of scalars*, as on p27 of [1].

Of course, B may be considered as a left and right module over itself. Thus B may be considered as a left and right module over A, as in the preceding paragraph. Note that the left action on B by A or B commutes with the right action by A or B. We also have that

(2.9.1) ϕ may be considered as a homomorphism from A into B, as left and right modules over A.

If V is a right module over A, then let

$$V_{(\phi)} = V \bigotimes\nolimits_A B \tag{2.9.2}$$

be a tensor product of V and B over A, where B is considered as a left module over A, as in the preceding paragraph. More precisely, B may be considered as a right module over itself too, and the left action of A on B commutes with the right action of B on itself. This means that (2.9.2) may be considered as a right module over B, as in Section 1.10.

Similarly, if V is a left module over A, then let

$$_{(\phi)}V = B \bigotimes\nolimits_A V \tag{2.9.3}$$

be a tensor product of B and V over A, where B is considered as a right module over A. This may be considered as a left module over B, as before. In each of these two cases, the resulting module over B is called the *covariant ϕ-extension* of V, as on p29 of [3]. This corresponds to *extension of scalars* when A and B are commutative, as on p28 of [1].

Let V be a right module over A again, and remember that $V \bigotimes_A A$ is isomorphic to V, as a right module over A, in a natural way, as in Section 1.10. If we consider ϕ as a homomorphism from A into B, as left modules over A, then we get a unique homomorphism from $V \bigotimes_A A$ into (2.9.2), as modules over k, with

$$v \otimes_A a \mapsto v \otimes_A \phi(a) \tag{2.9.4}$$

for every $a \in A$ and $v \in V$, as in Section 1.9. More precisely, one can check that this is a homomorphism from $V \bigotimes_A A$ into (2.9.2), as right modules over A. This may be identified with a homomorphism

$$\text{from } V \text{ into } V_{(\phi)}, \tag{2.9.5}$$

as right modules over A, as on p29 of [3].

If V is a left module over A, then $A \bigotimes_A V$ is isomorphic to V as a left module over A in a natural way, as before. If we consider ϕ as a homomorphism from A into B, as right modules over A, then we get a unique homomorphism from $A \bigotimes_A V$ into (2.9.3) with

$$a \otimes_A v \mapsto \phi(a) \otimes_A v \tag{2.9.6}$$

for every $a \in A$ and $v \in V$. One can verify that this is a homomorphism from $A \bigotimes_A V$ into (2.9.3), as left modules over A. This may be identified with a homomorphism

$$\text{from } V \text{ into } {}_{(\phi)}V, \tag{2.9.7}$$

as left modules over A, as before.

2.10 More on covariant ϕ-extensions

Let us continue with the same notation and hypotheses as at the beginning of the previous section. Suppose that V is a right module over A, and let Z be a left module over B. Thus $B \bigotimes_B Z$ corresponds to Z, as a left module over B, in a natural way, which may be considered as a left module over A, as before. This means that

$$V \bigotimes_A (B \bigotimes_B Z) \tag{2.10.1}$$

corresponds to

$$V \bigotimes_A Z \tag{2.10.2}$$

in a natural way, as modules over k. We also have that (2.10.1) is isomorphic to

$$V_{(\phi)} \bigotimes_B Z = \big(V \bigotimes_A B\big) \bigotimes_B Z, \tag{2.10.3}$$

as modules over k, as in Section 1.12. Here B is considered as a left module over A, and a right module over itself. Thus (2.10.2) is isomorphic to the left side of (2.10.3), as modules over k, as in (1) on p29 of [3].

Similarly, suppose that V is a right module over B, and that Z is a left module over A. One can check that there is a natural isomorphism between (2.10.2) and

$$V \bigotimes_{B^{(\phi)}} Z, \tag{2.10.4}$$

as modules over k, using the remarks in Section 1.12, as before. This corresponds to (2) on p29 of [3].

Suppose now that V is a left module over A, and that Z is a left module over B. There is a natural isomorphism between

$$\mathrm{Hom}_B({}_{(\phi)}V, Z) = \mathrm{Hom}_B\big(B \bigotimes_A V, Z\big) \tag{2.10.5}$$

and

$$\mathrm{Hom}_A\big(V, \mathrm{Hom}_B(B, Z)\big), \tag{2.10.6}$$

as modules over k, as in Section 1.13. We also have that $\mathrm{Hom}_B(B, Z)$ corresponds to Z in a natural way, as a left module over B, as in Section 1.8. This leads to a natural isomorphism between the left side of (2.10.5) and

$$\mathrm{Hom}_A(V, Z), \tag{2.10.7}$$

as modules over k. This corresponds to (3) on p29 of [3].

Similarly, suppose that V is a right module over A, and that Z is a right module over B. One can get a natural isomorphism between

$$\mathrm{Hom}_B(V_{(\phi)}, Z) \tag{2.10.8}$$

and (2.10.7), as modules over k, using the remarks in Section 1.13, as in the preceding paragraph. This corresponds to (3') on p30 of [3].

If V is a right module over A that is projective as a module over A, then Proposition 6.1 on p30 of [3] states that $V_{(\phi)}$ is projective as a right module over B. More precisely, one can use the isomorphism between (2.10.8) and (2.10.7) to reduce the projectivity condition for $V_{(\phi)}$ to the projectivity condition for V. Similarly, if V is a left module over A that is projective as a module over A, then ${}_{(\phi)}V$ is projective as a left module over B.

2.11 Contravariant ϕ-extensions

Let us continue with the same notation and hypotheses as at the beginning of Section 2.9 again.

If V is a right module over A again, then let

$$V^{(\phi)} = \mathrm{Hom}_A(B, V) \tag{2.11.1}$$

be the space of homomorphisms from B into V, as right modules over A. This may be considered as a right module over B, as in Section 1.8, because B is a left module over itself, and the left action commutes with the right action by A. Similarly, if V is a left module over A, then let

$$^{(\phi)}V = \mathrm{Hom}_A(B, V) \tag{2.11.2}$$

be the space of homomorphisms from B into V, as left modules over A. This may be considered as a left module over B, because B is a right module over itself, and the right action commutes with the left action by A, as before. In both cases, the resulting module over B is called the *contravariant ϕ-extension* of V, as on p29 of [3].

Suppose that V is a right module over A again, and remember that the space $\mathrm{Hom}_A(A, V)$ of homomorphisms from A into V, as right modules over A, is isomorphic to V as a right module over A in a natural way, as in Section 1.8. There is a natural homomorphism

$$\text{from } \mathrm{Hom}_A(B, V) \text{ into } \mathrm{Hom}_A(A, V), \tag{2.11.3}$$

as modules over k, defined by

$$\psi \mapsto \psi \circ \phi, \tag{2.11.4}$$

as in Section 1.7. In fact, this is a homomorphism as in (2.11.3), as right modules over A, because ϕ is a homomorphism from A into B, as both left and right modules over A. This leads to a homomorphism

$$\text{from } V^{(\phi)} \text{ into } V, \tag{2.11.5}$$

as right modules over A, using the isomorphism between $\mathrm{Hom}_A(A, V)$ and V mentioned earlier. This corresponds to a remark on p29 of [3].

If V is a left module over A, then the space $\mathrm{Hom}_A(A, V)$ of homomorphisms from A into V, as left modules over A, is isomorphic to V as a left module over A, as in Section 1.8 again. As before, there is a natural homomorphism as in (2.11.3), as modules over k, defined by (2.11.4). More precisely, this is a homomorphism as in (2.11.3), as left modules over A, because ϕ is a homomorphism from A into B, as both left and right modules over A. This leads to a homomorphism

$$\text{from } {}^{(\phi)}V \text{ into } V, \tag{2.11.6}$$

as left modules over A, using the isomorphism between $\mathrm{Hom}_A(A, V)$ and V.

Suppose now that V is a left module over B, and that Z is a left module over A. There is a natural isomorphism between

$$\mathrm{Hom}_B(V, {}^{(\phi)}Z) = \mathrm{Hom}_B\big(V, \mathrm{Hom}_A(B, Z)\big) \tag{2.11.7}$$

and

$$\mathrm{Hom}_A\Big(\big(B \bigotimes_B V\big), Z\Big), \tag{2.11.8}$$

as modules over k, as in Section 1.13. Remember that $B \bigotimes_B V$ corresponds to V in a natural way, as a left module over B, as in Section 1.10. Thus (2.11.8) is isomorphic to

$$\mathrm{Hom}_A(V, Z) \tag{2.11.9}$$

in a natural way, as modules over k. This means that the left side of (2.11.7) is isomorphic to (2.11.9) in a natural way, as modules over k, as in (4) on p29 of [3].

Similarly, suppose that V is a right module over B, and that Z is a right module over A. One can use the remarks in Section 1.13 to get a natural isomorphism between

$$\mathrm{Hom}_B(V, Z^{(\phi)}) \tag{2.11.10}$$

and (2.11.9), as modules over k, as before. This corresponds to (4') on p30 of [3].

If Z is a right module over A that is injective as a module over A, then Proposition 6.1a on p30 of [3] states that $Z^{(\phi)}$ is injective as a right module over B. This uses the isomorphism between (2.11.10) and (2.11.9) to reduce the injectivity of $Z^{(\phi)}$ to the injectivity of Z. Similarly, if Z is a left module over A that is injective as a module over A, then ${}^{(\phi)}Z$ is injective as a left module over B.

2.12 ϕ-Projectivity

We continue with the same notation and hypotheses as at the beginning of Section 2.9.

Let V be a right module over B. Thus V may be considered as a right module over A, and we use $V(A)$ to refer to V as a module over A. As a right module over A, we may define its covariant ϕ-extension

$$V(A)_{(\phi)} = V(A) \bigotimes_A B, \tag{2.12.1}$$

as before.

Consider the mapping from $V(A) \times B$ into V defined by

$$(v, b) \mapsto v \cdot b \tag{2.12.2}$$

for every $v \in V(A)$ and $b \in B$, using the action of B on V on the right. If $a \in A$, then a acts on $V(A)$ on the right by the action of $\phi(a) \in B$ on V on the right, as a module over B. Similarly, a acts on B on the left, as a left module over A, by multiplication on the left by $\phi(a)$. If we let a act on the right on $v \in V(A)$ in this way, then we get the same result in (2.12.2) as when we let a act on the left on $b \in B$. More precisely, the result is

$$(v \cdot \phi(a)) \cdot b = v \cdot (\phi(a)\, b), \tag{2.12.3}$$

in terms of multiplication on B and the action of B on V on the right, as a module over B.

It follows that there is a unique homomorphism g from (2.12.1) into V, as modules over k, such that

$$g(v \otimes b) = v \cdot b \tag{2.12.4}$$

for every $v \in V(A)$ and $b \in B$. More precisely, g is a homomorphism from $V(A)_{(\phi)}$ into V, as right modules over B, using the action of B on $V(A)_{(\phi)}$ as in Section 2.9. This corresponds to the homomorphism g defined on p30 of [3].

We may consider $V(A)_{(\phi)}$ as a right module over A, as usual. Remember that there is a natural homomorphism from $V(A)$ into $V(A)_{(\phi)}$, as right modules over A, as in Section 2.9. This mapping was obtained from the homomorphism from $V(A) \bigotimes_A A$ into $V(A) \bigotimes_A B$, as right modules over A, that corresponds to the identity mapping on $V(A)$ and ϕ, as a homomorphism from A into B, as left modules over A. One can check that

(2.12.5) the composition of the natural mapping from $V(A)$ into $V(A)_{(\phi)}$ with g is equal to the identity mapping on $V(A)$,

as on p30 of [3]. In particular, this implies that

$$g(V(A)_{(\phi)}) = V. \tag{2.12.6}$$

Of course, the kernel of g is a submodule of $V(A)_{(\phi)}$, as a right module over B. Thus $\ker g$ may be considered as a submodule of $V(A)_{(\phi)}$, as a right module over A. In fact, $V(A)_{(\phi)}$ corresponds to the direct sum of $\ker g$ and the image of the natural mapping from $V(A)$ into $V(A)_{(\phi)}$, as a right module over A, by the remarks in the preceding paragraph.

Note that the natural mapping from $V(A)$ in $V(A)_{(\phi)}$ is injective, by (2.12.5). This and the statement in the previous paragraph are mentioned in Exercise 13 on p32 of [1], for modules over commutative rings. More precisely, the natural mapping from $V(A)$ into $V(A)_{(\phi)}$ is called g in [1], and the mapping g here is called p in [1].

If V is projective as a right module over B, then

(2.12.7) $V(A)_{(\phi)}$ corresponds to the direct sum of $\ker g$ and another submodule of $V(A)_{(\phi)}$, as a right module over B,

as in Section 2.7. If (2.12.7) holds, then V is said to be *ϕ-projective* as a right module over B, as on p30 of [3]. In this case, the restriction of g to the other submodule of $V(A)_{(\phi)}$ is an isomorphism from that submodule onto V, as right modules over B.

If $V(A)$ is projective as a right module over A, then $V(A)_{(\phi)}$ is projective as a right module over B, as in Section 2.10. If V is also ϕ-projective, as a right module over B, then it follows that

(2.12.8) V is projective as a right module over B,

as in Section 2.7. This corresponds to the first part of Proposition 6.2 on p30 of [3].

If B is projective as a right module over A, and V is projective as a right module over B, then the second part of Proposition 6.2 on p30 of [3] states that

(2.12.9) $V(A)$ is projective as a right module over A.

To see this, let W and Z be right modules over A, and let ψ be a homomorphism from W onto Z, as right modules over A. Consider the contravariant ϕ-extensions $W^{(\phi)}$ and $Z^{(\phi)}$ of W and Z, respectively, as in the previous section. Using ψ, we get a homomorphism Ψ from $W^{(\phi)}$ into $Z^{(\phi)}$, as right modules over B. More precisely, Ψ sends a homomorphism from B into W, as right modules over A, to its composition with ψ, to get a homomorphism from B into Z, as right modules over A.

If B is projective as a right module over A, then

$$\Psi(W^{(\phi)}) = Z^{(\phi)}. \tag{2.12.10}$$

Using Ψ, we get a homomorphism $\widehat{\Psi}$

$$\text{from } \operatorname{Hom}_B(V, W^{(\phi)}) \text{ into } \operatorname{Hom}_B(V, Z^{(\phi)}), \tag{2.12.11}$$

as modules over k. As before, $\widehat{\Psi}$ sends a homomorphism from V into $W^{(\phi)}$, as right modules over B, to its composition with Ψ, to get a homomorphism from V into $Z^{(\phi)}$, as right modules over B. If V is projective, as a right module over B, then

$$\widehat{\Psi}\big(\operatorname{Hom}_B(V, W^{(\phi)})\big) = \operatorname{Hom}_B(V, Z^{(\phi)}). \tag{2.12.12}$$

Similarly, we can use ψ to get a homomorphism $\widetilde{\Psi}$

$$\text{from } \operatorname{Hom}_A(V(A), W) \text{ into } \operatorname{Hom}_A(V(A), Z), \tag{2.12.13}$$

as modules over k. As usual, $\widetilde{\Psi}$ sends a homomorphism from $V(A)$ into W, as right modules over A, to its composition with ψ, to get a homomorphism from $V(A)$ into Z, as right modules over A. To show that V is projective, as a right module over A, we would like to check that

$$\widetilde{\Psi}\big(\operatorname{Hom}_A(V(A), W)\big) = \operatorname{Hom}_A(V(A), Z). \tag{2.12.14}$$

Remember that there are natural isomorphisms between $\operatorname{Hom}_A(V(A), W)$, $\operatorname{Hom}_A(V(A), Z)$ and $\operatorname{Hom}_B(V, W^{(\phi)})$, $\operatorname{Hom}_B(V, Z^{(\phi)})$, respectively, as modules over k, as in the previous section. Using this, we get that (2.12.14) follows from (2.12.12), as desired.

Of course, there are analogous statements for left modules.

2.13 ϕ-Injectivity

We continue with the same notation and hypotheses as at the beginning of Section 2.9 again.

Let Z be a right module over B, which may be considered as a right module over A. To be precise, we use $Z(A)$ to refer to Z as a module over A, whose contravariant ϕ-extension is defined by

$$Z(A)^{(\phi)} = \operatorname{Hom}_A(B, Z(A)), \tag{2.13.1}$$

as before.

If $z \in Z$, then

$$b \mapsto z \cdot b \tag{2.13.2}$$

defines a homomorphism from B into Z, as right modules over B. This may also be considered as a homomorphism from B into $Z(A)$, as right modules over A, which is to say an element of (2.13.1). This defines a mapping h from Z into (2.13.1), with

$$h(z) = (2.13.2) \tag{2.13.3}$$

for every $z \in Z$, as on p30 of [3]. One can check that h is a homomorphism from Z into $Z(A)^{(\phi)}$, as right modules over B, where the action of B on $Z(A)^{(\phi)}$ on the right is defined using the action of B on itself on the left, as in Section 2.11.

Remember that there is a natural homomorphism from $Z(A)^{(\phi)}$ into $Z(A)$, as right modules over A, as in Section 2.11. This mapping was obtained from the homomorphism from $\mathrm{Hom}_A(B, Z(A))$ into $\mathrm{Hom}_A(A, Z(A))$ defined by composition with ϕ, by identifying $\mathrm{Hom}_A(A, Z(A))$ with $Z(A)$ in the usual way. This is the same as evaluating an element of $Z(A)^{(\phi)}$ at e_B, as a homomorphism from B into $Z(A)$.

Observe that

(2.13.4) the composition of h with the natural mapping from $Z(A)^{(\phi)}$ into $Z(A)$ is equal to the identity mapping on $Z(A)$,

as on p30 of [3]. In particular, this implies that

(2.13.5) h is injective on Z.

Note that $h(Z)$ is a submodule of $Z(A)^{(\phi)}$, as a right module over B, and thus as a right module over A. We also have that $Z(A)^{(\phi)}$ corresponds to the direct sum of $h(Z)$ and the kernel of the natural mapping from $Z(A)^{(\phi)}$ into $Z(A)$, as a right module over A, because of (2.13.4).

If Z is injective as a right module over B, then

(2.13.6) $Z(A)^{(\phi)}$ corresponds to the direct sum of $h(Z)$ and another submodule of $Z(A)^{(\phi)}$, as a right module over B,

as in Section 2.8. If (2.13.6) holds, then Z is said to be *ϕ-injective* as a right module over Z, as on p31 of [3].

If $Z(A)$ is injective as a right module over A, then $Z(A)^{(\phi)}$ is injective as a right module over B, as in Section 2.11. If Z is also ϕ-injective, as a right module over B, then it follows that

(2.13.7) Z is injective as a right module over B,

as in Section 2.8. This corresponds to the first part of Proposition 6.2a on p31 of [3].

Suppose that B is projective as a left module over A, and that Z is injective as a right module over B. Under these conditions, the second part of Proposition 6.2a on p31 of [3] states that

(2.13.8) $$Z(A) \text{ is injective as a right module over } A.$$

Let V be a right module over A, and let V_0 be a submodule of V. Of course, there is a natural mapping

(2.13.9) $$\text{from } \mathrm{Hom}_A(V, Z(A)) \text{ into } \mathrm{Hom}_A(V_0, Z(A)),$$

which sends a homomorphism from V into $Z(A)$, as right modules over A, to its restriction to V_0. This defines a homomorphism as in (2.13.9), as a mapping between modules over k, and we would like to show that this homomorphism is a surjection.

Let $V_{(\phi)}$, $(V_0)_{(\phi)}$ be the covariant ϕ-extensions of V, V_0, respectively, as in Section 2.9. The natural inclusion mapping from V_0 into V leads to a homomorphism

(2.13.10) $$\text{from } (V_0)_{(\phi)} \text{ into } V_{(\phi)},$$

as right modules over B, using the identity mapping on B. This homomorphism is injective, because B is projective as a left module over A, as in Section 2.7.

There are natural isomorphisms between $\mathrm{Hom}_A(V, Z(A))$, $\mathrm{Hom}_A(V_0, Z(A))$ and $\mathrm{Hom}_B(V_{(\phi)}, Z)$, $\mathrm{Hom}_B((V_0)_{(\phi)}, Z)$, respectively, as modules over k, as in Section 2.10. There is also a natural homomorphism

(2.13.11) $$\text{from } \mathrm{Hom}_B(V_{(\phi)}, Z) \text{ into } \mathrm{Hom}_B((V_0)_{(\phi)}, Z),$$

as modules over k, which sends a homomorphism from $V_{(\phi)}$ into Z, as right modules over B, to its composition with the homomorphism as in (2.13.10). This homomorphism maps $\mathrm{Hom}_B(V_{(\phi)}, Z)$ onto $\mathrm{Hom}_B((V_0)_{(\phi)}, Z)$, because Z is injective as a right module over B, and the homomorphism as in (2.13.10) is injective, as before.

It follows that the natural homomorphism as in (2.13.9) is surjective, as desired.

2.14 A ϕ-projectivity property

We continue with the notation and hypotheses mentioned at the beginning of Section 2.9.

Let V be a right module over A, so that $V_{(\phi)}$ is a right module over B, as in Section 2.9. The first part of Proposition 6.3 on p31 of [3] states that

(2.14.1) $$V_{(\phi)} \text{ is } \phi\text{-projective},$$

in the sense described in Section 2.12.

Let $V_{(\phi)}(A)$ be $V_{(\phi)}$, considered as a right module over A, as in Section 2.12. The covariant ϕ-extension of $V_{(\phi)}(A)$ is defined by

$$(V_{(\phi)}(A))_{(\phi)} = V_{(\phi)}(A) \bigotimes_A B, \tag{2.14.2}$$

as before. Let $g_{V_{(\phi)}}$ be the homomorphism from (2.14.2) into $V_{(\phi)}$, as right modules over B, defined in Section 2.12. In fact, $g_{V_{(\phi)}}$ maps (2.14.2) onto $V_{(\phi)}$, as before. We would like to show that (2.14.2) corresponds to the direct sum of the kernel of $g_{V_{(\phi)}}$ and another submodule of (2.14.2), as a right module over B.

Let $B \bigotimes_A B$ be a tensor product of B with itself over A, where the first B is considered as a right module over A, and the second B is considered as a left module over A. Note that $B \bigotimes_A B$ may be considered as a left and right module over B, by considering the first B as a left module over itself, and the second B as a right module over itself. It is easy to see that these two actions of B on $B \bigotimes_A B$ on the left and right commute with each other. It follows that $B \bigotimes_A B$ may be considered as a left and right module over A as well.

Let α be the mapping from B into $B \bigotimes_A B$ defined by

$$\alpha(b) = e_B \otimes b \tag{2.14.3}$$

for every $b \in B$. One can check that α is a homomorphism from B into $B \bigotimes_A B$, as left modules over A, and right modules over B, as on p31 of [3].

It is easy to see that there is a unique homomorphism β from $B \bigotimes_A B$ into B, as modules over k, such that

$$\beta(b_1 \otimes b_2) = b_1\, b_2 \tag{2.14.4}$$

for every $b_1, b_2 \in B$. More precisely, this is a homomorphism from $B \bigotimes_A B$ into B as left and right modules over B. In particular, β may be considered as a homomorphism from $B \bigotimes_A B$ into B, as left modules over A, as on p31 of [3]. Note that

$$\beta \circ \alpha \text{ is the identity mapping on } B, \tag{2.14.5}$$

by construction.

Using the identity mapping on V and α on B, we get a homomorphism α'

$$\text{from } V_{(\phi)} = V \bigotimes_A B \text{ into } V \bigotimes_A \big(B \bigotimes_A B\big), \tag{2.14.6}$$

as modules over k. More precisely, this is a homomorphism as in (2.14.6), as right modules over B, as on p31 of [3]. Similarly, we can use the identity mapping on V and β on $B \bigotimes_A B$ to get a homomorphism β'

$$\text{from } V \bigotimes_A \big(B \bigotimes_A B\big) \text{ into } V \bigotimes_A B = V_{(\phi)}, \tag{2.14.7}$$

as right modules over B. It is easy to see that

$$\beta' \circ \alpha' \text{ is the identity mapping on } V_{(\phi)}, \tag{2.14.8}$$

because of (2.14.5).

As in Section 1.12, there is a natural isomorphism

(2.14.9) between $V \bigotimes_A (B \bigotimes_A B)$ and $(V \bigotimes_A B) \bigotimes_A B = (V_{(\phi)}(A))_{(\phi)}$,

as modules over k. More precisely, this is an isomorphism as in (2.14.9), as right modules over B. Thus β' corresponds to a homomorphism

(2.14.10) from $(V_{(\phi)}(A))_{(\phi)}$ into $V_{(\phi)}$,

as right modules over B. In fact, β' corresponds to the homomorphism $g_{V_{(\phi)}}$ mentioned earlier, as on p31 of [3].

Similarly, α' corresponds to a homomorphism

(2.14.11) from $V_{(\phi)}$ into $(V_{(\phi)}(A))_{(\phi)}$,

as right modules over B. Using (2.14.8), we get that

(2.14.12) the composition of this homomorphism with $g_{V_{(\phi)}}$ is the identity mapping on $V_{(\phi)}$.

This implies that

(2.14.13) $(V_{(\phi)}(A))_{(\phi)}$ corresponds to the direct sum of $\ker g_{V_{(\phi)}}$ and the image of $V_{(\phi)}$ under the homomorphism corresponding to α' as in (2.14.11),

as a right module over B, as desired.

Alternatively, there is a natural homomorphism from V into $V_{(\phi)}$, as right modules over A, as in Section 2.9. This homomorphism may be described equivalently by

(2.14.14) $$v \mapsto v \otimes \phi(e_a) = v \otimes e_B$$

for each $v \in V$. This leads to a homomorphism from $V_{(\phi)} = V \bigotimes_A B$ into (2.14.2), as right modules on B, using the identity mapping on B. This is the same as the homomorphism corresponding to α' as in the preceding paragraph. The composition of this homomorphism with $g_{V_{(\phi)}}$ is the identity mapping on $V_{(\phi)}$, as before.

There are analogous statements for left modules over A, as in Proposition 6.3 on p31 of [3].

2.15 A ϕ-injectivity property

We continue with the notation and hypotheses mentioned at the beginning of Section 2.9 again.

Let V be a right module over A. The second part of Proposition 6.3 on p31 of [3] states that

(2.15.1) $$V^{(\phi)} \text{ is } \phi\text{-injective}$$

as a right module over B, in the sense described in Section 2.13. Remember that $V^{(\phi)}$ is defined as in Section 2.11, and let $V^{(\phi)}(A)$ be $V^{(\phi)}$, considered as a right module over A, as before. Thus the contravariant ϕ-extension of $V^{(\phi)}(A)$ is defined by

$$(V^{(\phi)}(A))^{(\phi)} = \mathrm{Hom}_A(B, V^{(\phi)}(A)). \tag{2.15.2}$$

Let $h_{V^{(\phi)}}$ be the homomorphism from $V^{(\phi)}$ into (2.15.2), as right modules over B, defined in Section 2.13. Remember that this homomorphism is injective, as before. We would like to show that (2.15.2) corresponds to the direct sum of the image of $h_{V^{(\phi)}}$ and another submodule of (2.15.2), as a right module over B.

Observe that

$$(V^{(\phi)}(A))^{(\phi)} = \mathrm{Hom}_A\big(B, \mathrm{Hom}_A(B, V)\big) \tag{2.15.3}$$

is isomorphic to

$$\mathrm{Hom}_A\big(B \bigotimes\nolimits_A B, V\big), \tag{2.15.4}$$

as modules over k, in a natural way, as in Section 1.13. More precisely, this is an isomorphism between right modules over B. Here $B \bigotimes_A B$ is considered as a right module over A, and as a left module over B, so that the space (2.15.4) of homomorphisms from $B \bigotimes_A B$ into V, as right modules over A, may be considered as a right module over B.

Let $\widetilde{\alpha}$ be the mapping from B into $B \bigotimes_A B$ defined by

$$\widetilde{\alpha}(b) = b \otimes e_B \tag{2.15.5}$$

for every $b \in B$. This is a homomorphism from B into $B \bigotimes_A B$, as left modules over B, and right modules over A.

Let $\widetilde{\alpha}'$ be the mapping

$$\text{from (2.15.4) into } V^{(\phi)} = \mathrm{Hom}_A(B, V) \tag{2.15.6}$$

defined by composing an element of (2.15.4) with $\widetilde{\alpha}$, to get a homomorphism from B into V, as right modules over A. This defines a homomorphism from (2.15.4) into $V^{(\phi)}$, as right modules over B. Using the isomorphism between (2.15.3) and (2.15.4) mentioned earlier, we get that $\widetilde{\alpha}'$ corresponds to a homomorphism

$$\text{from } (V^{(\phi)}(A))^{(\phi)} \text{ into } V^{(\phi)}, \tag{2.15.7}$$

as right modules over B.

Remember that β is the homomorphism from $B \bigotimes_A B$ into B determined by (2.14.4). Let $\widetilde{\beta}'$ be the mapping

$$\text{from } V^{(\phi)} = \mathrm{Hom}_A(B, V) \text{ into (2.15.4)} \tag{2.15.8}$$

defined by composing an element of $\mathrm{Hom}_A(B, V)$ with β. This defines a homomorphism as in (2.15.8), as right modules over B. One can verify that $\widetilde{\beta}'$

corresponds to $h_{V^{(\phi)}}$, using the isomorphism between (2.15.3) and (2.15.4) mentioned earlier. This uses the way that the action of B on $V^{(\phi)}$ on the right is defined.

Of course,

$$\beta \circ \widetilde{\alpha} \text{ is the identity mapping on } B, \tag{2.15.9}$$

by construction. This implies that

$$\widetilde{\alpha}' \circ \widetilde{\beta}' \text{ is the identity mapping on } V^{(\phi)}. \tag{2.15.10}$$

Equivalently, this means that the composition of $h_{V^{(\phi)}}$ with the homomorphism from $(V^{(\phi)}(A))^{(\phi)}$ into $V^{(\phi)}$ corresponding to $\widetilde{\alpha}'$ is the identity mapping on $V^{(\phi)}$. It follows that $(V^{(\phi)}(A))^{(\phi)}$ corresponds to the direct sum of the image of $h_{V^{(\phi)}}$ and the kernel of the homomorphism corresponding to $\widetilde{\alpha}'$, as a right module over B, as desired.

Alternatively, there is a natural homomorphism from $V^{(\phi)}$ into V, as right modules over A, as in Section 2.11. Equivalently, this homomorphism sends an element of $V^{(\phi)} = \mathrm{Hom}_A(B, V)$ to its value at e_B. This leads to a natural mapping from (2.15.2) into $V^{(\phi)}$, by composing an element of $\mathrm{Hom}_A(B, V^{(\phi)}(A))$ with the homomorphism just mentioned to get an element of $\mathrm{Hom}_A(B, V)$. This defines a homomorphism from $(V^{(\phi)}(A))^{(\phi)}$ into $V^{(\phi)}$, as right modules over B. This is the same as the homomorphism corresponding to $\widetilde{\alpha}'$ mentioned earlier.

One can check that that composition of $h_{V^{(\phi)}}$ with the homomorphism just mentioned is the same as the identity mapping on $V^{(\phi)}$, as before. This basically reduces to evaluating an element of $V^{(\phi)} = \mathrm{Hom}_A(B, V)$ at $b\, e_B = b$ for each $b \in B$, because of the way that the action of $b \in B$ on $V^{(\phi)}$ on the right is defined.

There are analogous statements for left modules over A, as usual.

Chapter 3

Modules and tensor products, 3

3.1 Semisimple modules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a module over k that is a left or right module over A. We say that V is *simple* as a module over A if $A \neq \{0\}$, and V does not have any proper nonzero submodules, as a module over A. We say that V is *semisimple* as a module over A if it is isomorphic to the direct sum of a family of simple modules, as a module over A.

Let V_0 be a submodule of V, as a module over A. It is well known that

(3.1.1) V corresponds to the direct sum of V_0 and another submodule, as a module over A

if and only if

(3.1.2) there is a homomorphism ϕ_0 from V onto V_0, as modules over A, that is equal to the identity mapping on V_0.

More precisely, if (3.1.2) holds, then V corresponds to the direct sum of V_0 and the kernel of ϕ_0, as a module over A.

It is well known that V is semisimple as a module over A if and only if (3.1.1) holds for every submodule V_0 of V, as a module over A, as in Proposition 4.1 on p11 of [3]. To show the "only if" part, suppose that V corresponds to the direct sum of a family $\{V_j\}_{j \in I}$ of simple submodules, as a module over A. If I_0 is a subset of I, then let $V(I_0)$ be the submodule of V generated by V_j, $j \in I_0$.

Let W be a submodule of V, as a module over A. Let us say that $I_0 \subseteq I$ is *admissible* with respect to W if

$$V(I_0) \cap W = \{0\}. \tag{3.1.3}$$

We would like to use a subset I_1 of I that is admissible with respect to W and maximal with respect to inclusion. If I has only finitely or countably many elements, then one can get a maximal admissible subset of I using fairly straightforward arguments. Otherwise, one can use Zorn's lemma or Hausdorff's maximality principle to get a maximal admissible subset of I.

If $j \in I \setminus I_1$, then $I_1 \cup \{j\}$ is not admissible with respect to W, so that

$$V(I_1 \cup \{j\}) \cap W \neq \{0\}. \tag{3.1.4}$$

This means that there is a nonzero element of W that can be expressed as the sum of elements of $V(I_1)$ and V_j. This element of V_j has to be nonzero, because I_1 is admissible with respect to W. It follows that there is a nonzero element of V_j that can be expressed as the sum of elements of $V(I_1)$ and W. Thus

$$V_j \cap (V(I_1) + W) \neq \{0\}, \tag{3.1.5}$$

which implies that

$$V_j \subseteq V(I_1) + W, \tag{3.1.6}$$

because $V_j \cap (V(I_1) + W)$ is a submodule of V_j, as a module over A, and V_j is simple, by hypothesis. This shows that

$$V = V(I_1) + W, \tag{3.1.7}$$

because (3.1.6) holds for every $j \in I \setminus I_1$. This means that V corresponds to the direct sum of $V(I_1)$ and W, as a module over A, because I_1 is admissible with respect to W.

Conversely, suppose that (3.1.1) holds for every submodule V_0 of V, as a module over A, so that (3.1.2) holds for every submodule V_0 of V. Let W be a submodule of V, and let W_0 be a submodule of W, as modules over A. In particular, W_0 is a submodule of V, so that there is a homomorphism from V onto W_0, as modules over A, that is equal to the identity mapping on W_0. The restriction of this homomorphism to W is a homomorphism from W onto W_0, as modules over A, that is equal to the identity mapping on W_0. It follows that

(3.1.8) W corresponds to the direct sum of W_0 and another submodule, as a module over A,

as before.

If k is a field, and V has finite dimension as a vector space over k, then one can use this repeatedly to express V as the direct sum of finitely many simple submodules, as a module over A. Otherwise, one can use the following argument.

Let Z be a nonzero submodule of V, as a module over A. We would like to show that Z contains a nonzero simple submodule, as a module over A. Let z be a nonzero element of Z. One can use Zorn's lemma or Hausdorff's maximality principle to show that there is a submodule Z_0 of Z, as a module over A, such that

$$z \notin Z_0, \tag{3.1.9}$$

and Z_0 is maximal with respect to inclusion. By hypothesis, there is a submodule Z_1 of Z such that Z corresponds to the direct sum of Z_0 and Z_1, as a module over A, as in (3.1.8). Note that $Z_1 \neq \{0\}$, because of (3.1.9). We would like to check that Z_1 is simple, as a module over A.

If Z_1 is not simple, then there is a nonzero proper submodule Z_2 of Z_1. This means that Z_1 corresponds to the direct sum of Z_2 and another nonzero proper submodule Z_3, as a module over A, by hypothesis. Thus Z corresponds to the direct sum of Z_0, Z_2, and Z_3, as a module over A. It is easy to see that z cannot be an element of both $Z_0 + Z_2$ and $Z_0 + Z_3$, because of (3.1.9). This contradicts the maximality of Z_0, as desired.

Let us say that a collection C of submodules of V is *admissble* if every element of C is simple as a module over A, and the submodule of V generated by C corresponds to the direct sum of the elements of C, as a module over A. One can use Zorn's lemma or Hausdorff's maximality principle to show that there is an admissble collection C_0 that is maximal with respect to inclusion. Let W be the submodule of V generated by the elements of C_0. If $W \neq V$, then V corresponds to the direct sum of W and a nonzero submodule Z, as a module over A, by hypothesis. Let Z_0 be a submodule of Z that is simple, as a module over A, as in the previous paragraphs. Under these conditions, $C_0 \cup \{Z_0\}$ is admissible, contradicting the maximality of C_0. It follows that $V = W$, so that V is semisimple as a module over A.

3.2 Direct systems of modules

Let I be a set. A binary relation $\preceq$ on I is said to be a *pre-order* on I if it is reflexive and transitive on I. In this case, the binary relation on I defined by

$$j \simeq l \quad \text{when } j \preceq l \text{ and } l \preceq j \tag{3.2.1}$$

defines an equivalence relation on I. If (3.2.1) holds if and only if $j = l$, then $\preceq$ is said to be a *partial ordering* on I.

Let us say that $(I, \preceq)$ is a *pre-directed set* or a *pre-directed system* if $\preceq$ is a pre-ordering on I, and for every $j, l \in I$ there is an $r \in I$ such that $j, l \preceq r$. Thus a directed set or system is the same as a pre-directed set or system, where the pre-ordering is a partial ordering. Of course, slightly different conventions are sometimes used.

One can define direct limits using pre-directed sets instead of directed sets, and we shall do this now for modules. Let k be a commutative ring with a multiplicative identity element, and let $(I, \preceq)$ be a nonempty pre-directed set. Also let V_j be a module over k for every $j \in I$. Suppose that for every $j, l \in I$ with $j \preceq l$ we have a homomorphism $\nu_{j,l}$ from V_j into V_l, as modules over k, with the following two properties. First, $\nu_{j,j}$ is the identity mapping on V_j for every $j \in I$. Second, if $j, l, r \in I$ and $j \preceq l \preceq r$, then

$$\nu_{l,r} \circ \nu_{j,l} = \nu_{j,r}. \tag{3.2.2}$$

Under these conditions, the family of modules V_j and homomorphisms $\nu_{j,l}$ is said to form a *direct* or *inductive system* over $(I, \preceq)$, as before. If $j, l \in I$ satisfy (3.2.1), then it follows that $\nu_{j,l}$ and $\nu_{l,j}$ are inverses of each other.

If $l \in I$, then let ι_l be the natural injection from V_l into the direct sum $\bigoplus_{j \in I} V_j$, so that for each $v_l \in V_l$, the jth coordinate of $\iota_l(v_l)$ is equal to v_l when $j = l$, and to 0 when $j \neq l$. Let U be the subset of $\bigoplus_{j \in I} V_j$ consisting of finite sums of the form

$$\iota_l(v_l) - \iota_r(\nu_{l,r}(v_l)), \tag{3.2.3}$$

where $l, r \in I$, $l \preceq r$, and $v_l \in V_l$. It is easy to see that U is a submodule of $\bigoplus_{j \in I} V_j$, as a module over k. Put

$$\varinjlim V_j = \Big(\bigoplus_{j \in I} V_j\Big)/U, \tag{3.2.4}$$

where the quotient on the right defines a module over k. This is the *direct* or *inductive limit* of the direct system of V_j's, $j \in I$, as a module over k.

Let q be the natural quotient mapping from $\bigoplus_{j \in I} V_j$ into (3.2.4), and for each $l \in I$, put

$$\nu_l = q \circ \iota_l. \tag{3.2.5}$$

This is a homomorphism from V_l into (3.2.4), as modules over k. Note that

$$\nu_l = \nu_r \circ \nu_{l,r} \tag{3.2.6}$$

for every $l, r \in I$ with $l \preceq r$, by construction. The direct limit consists more precisely of the module (3.2.4) together with the homomorphisms ν_l, as in Exercise 14 on p32f of [1].

Every element of (3.2.4) can be expressed as $\nu_l(v_l)$ for some $l \in I$ and $v_l \in V_l$, as before. This uses the hypothesis that $(I, \preceq)$ be a pre-directed set. If

$$\nu_l(v_l) = 0, \tag{3.2.7}$$

then there is an $r \in I$ such that $l \preceq r$ and

$$\nu_{l,r}(v_l) = 0. \tag{3.2.8}$$

This corresponds to Exercise 15 on p33 of [1].

Let Z be another module over k, and let ζ_l be a homomorphism from V_l into Z, as modules over k, for each $l \in I$. Suppose that for every $l, r \in I$ with $l \preceq r$, we have that

$$\zeta_l = \zeta_r \circ \nu_{l,r}. \tag{3.2.9}$$

Under these conditions, there is a unique homomorphism ζ from (3.2.4) into Z, as modules over k, such that

$$\zeta \circ \nu_l = \zeta_l \tag{3.2.10}$$

for every $l \in I$. The direct limit is uniquely determined up to isomorphism by this property, as in Exercise 16 on p33 of [1].

Let A be an associative algebra over k, with a multiplicative identity element e_A. Suppose either that V_j is a left module over A for every $j \in I$, or a right module over A for every $j \in I$. Suppose also that for every $j, l \in I$, $\nu_{j,l}$ is a homomorphism from V_j into V_l, as modules over A. Of course, $\bigoplus_{j \in I} V_j$ is a left or right module over A too, as appropriate. It is easy to see that the set U defined earlier is a submodule of $\bigoplus_{j \in I} V_j$, as a module over A, in this case. Thus (3.2.4) may be considered as a left or right module over A, as appropriate. Similarly, (3.2.5) defines a homomorphism from V_l into (3.2.4), as modules over A, for each $l \in I$. There are analogues of the remarks in the previous paragraph for modules over A as well.

3.3 Direct systems of submodules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a left or right module over A, and let $(I, \preceq)$ be a nonempty pre-directed set. Suppose that V_j is a submodule of V for each $j \in I$, with

$$V_l \subseteq V_r \tag{3.3.1}$$

when $l, r \in I$ and $l \preceq r$. Note that

$$\bigcup_{j \in I} V_j \tag{3.3.2}$$

is a submodule of V, because $(I, \preceq)$ is a pre-directed set. We can get a direct system by taking $\nu_{j,l}$ to be the natural inclusion mapping from V_j into V_l when $j, l \in I$ and $j \preceq l$. The corresponding direct limit is isomorphic to (3.3.2), as in Exercise 17 on p33 of [1]. Of course, any collection of submodules of V is partially ordered by inclusion, and one can simpy ask that such a collection be a directed set with respect to inclusion.

Now let I be a nonempty set, and let V_j be a left module over A for each $j \in I$. Thus the direct sum $V = \bigoplus_{j \in I} V_j$ is a left module over A as well. Let $\mathcal{I}$ be the collection of nonempty finite subsets of I. It is easy to see that $\mathcal{I}$ is a directed set with respect to the partial ordering defined by inclusion.

If $I_1 \in \mathcal{I}$, then let $V(I_1)$ be the submodule of V consisting of elements whose jth coordinate is equal to 0 when $j \in I \setminus I_1$. If $I_2 \in \mathcal{I}$ and $I_1 \subseteq I_2$, then

$$V(I_1) \subseteq V(I_2). \tag{3.3.3}$$

Clearly

$$V = \bigcup_{I_1 \in \mathcal{I}} V(I_1). \tag{3.3.4}$$

As before, we may consider the family of $V(I_1)$'s, $I_1 \in \mathcal{I}$, as a direct system of modules, using the natural inclusion mapping from $V(I_1)$ into $V(I_2)$ when $I_1, I_2 \in \mathcal{I}$ and $I_1 \subseteq I_2$. The corresponding direct limit is isomorphic to V, as a left module over A, as in the preceding paragraph.

Alternatively, if $I_1 \in \mathcal{I}$, then put

$$V_{I_1} = \prod_{j \in I_1} V_j, \tag{3.3.5}$$

which is a left module over A. If $I_2 \in \mathcal{I}$ and $I_1 \subseteq I_2$, then there is a natural injective homomorphism from V_{I_1} into V_{I_2}, as left modules over A. This homomorphism sends an element of V_{I_1} to the element of V_{I_2} with the same jth coordinates when $j \in I_1$, and with all other coordinates equal to 0. Using these homomorphisms, the family of V_{I_1}'s, $I_1 \in \mathcal{I}$, becomes a direct system of modules. Of course, V_{I_1} is isomorphic to $V(I_1)$, as left modules over A, in a natural way for each $I_1 \in \mathcal{I}$, and the homomorphism from V_{I_1} into V_{I_2} just mentioned corresponds exactly to the inclusion of $V(I_1)$ into $V(I_2)$ when $I_1 \subseteq I_2$. This leads to an isomorphism between the direct limit of the V_{I_1}'s and the direct limit of the $V(I_1)$'s, as left modules over A. There are analogous statements for right modules over A.

3.4 Homomorphisms between direct systems

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k, with a multiplicative identity element e_A. Also let $(I, \preceq)$ be a nonempty directed set, and let V_j, W_j be left modules over A for every $j \in I$. Suppose that for every $j, l \in I$ with $j \preceq l$, we have homomorphisms $\nu_{j,l}$, $\mu_{j,l}$ from V_j, W_j into V_l, W_l, respectively, as left modules over A, and that these homomorphisms satisfy the two conditions mentioned in Section 3.2. Thus the direct limits of the V_j's and W_j's can be defined as left modules over A, as before. If $l \in I$, then we let ν_l, μ_l be the corresponding homomorphisms from V_l, W_l into $\varinjlim V_j$, $\varinjlim W_j$, respectively.

Let ϕ_j be a homomorphism from V_j into W_j, as left modules over A, for each $j \in I$. If $j, l \in I$ and $j \preceq l$, then we ask that

$$\phi_l \circ \nu_{j,l} = \mu_{j,l} \circ \phi_j. \tag{3.4.1}$$

Under these conditions, the family of ϕ_j's, $j \in I$, is considered to define a *homomorphism* between the direct systems of V_j's and W_j's, $j \in I$, as in Exercise 18 on p33 of [1].

In this case, there is a unique homomorphism $\phi = \varinjlim \phi_j$ from $\varinjlim V_j$ into $\varinjlim W_j$, as left modules over A, such that

$$\phi \circ \nu_l = \mu_l \circ \phi_l \tag{3.4.2}$$

for every $l \in I$, as in [1]. Indeed, $\mu_l \circ \phi_l$ defines a homomorphism from V_l into $\varinjlim W_j$, as left modules over A, for each $l \in I$. If $l, r \in I$ and $l \preceq r$, then

$$(\mu_r \circ \phi_r) \circ \nu_{l,r} = \mu_r \circ \mu_{l,r} \circ \phi_l = \mu_l \circ \phi_l. \tag{3.4.3}$$

Using this, we can get the homomorphism ϕ from the homomorphisms $\mu_l \circ \phi_l$, $l \in I$, as in Section 3.2.

Let Z_j be a left module over A for every $j \in I$, and suppose that for every $j, l \in I$ with $j \preceq l$, we have a homomorphism $\zeta_{j,l}$ from Z_j into Z_l, as left modules over A, that satisfy the two conditions mentioned in Section 3.2. This means that the direct limit of the Z_j's can be defined as a left module over A as before, and we let ζ_l be the corresponding homomorphism from Z_l into $\varinjlim Z_j$ for each $l \in I$.

Let ψ_j be a homomorphism from W_j into Z_j, as left modules over A, for each $j \in I$, and suppose that

$$\psi_l \circ \mu_{j,l} = \zeta_{j,l} \circ \psi_j \tag{3.4.4}$$

for every $j, l \in$ with $j \preceq l$. This leads to a unique homomorphism $\psi = \varinjlim \psi_j$ from $\varinjlim W_j$ into $\varinjlim Z_j$, as left modules over A, such that

$$\psi \circ \mu_l = \zeta_l \circ \psi_l \tag{3.4.5}$$

for every $l \in I$, as before.

Of course, $\psi_j \circ \phi_j$ is a homomorphism from V_j into Z_j, as left modules over A, for every $j \in I$. If $j, l \in I$ and $j \preceq l$, then

$$(\psi_l \circ \phi_l) \circ \nu_{j,l} = \psi_l \circ \mu_{j,l} \circ \phi_j = \zeta_{j,l} \circ (\psi_j \circ \phi_j). \tag{3.4.6}$$

We also have that

$$(\psi \circ \phi) \circ \nu_l = \psi \circ \mu_l \circ \phi_l = \zeta_l \circ (\psi_l \circ \phi_l) \tag{3.4.7}$$

for every $l \in I$. This means that

$$\varinjlim(\psi_j \circ \phi_j) = \psi \circ \phi, \tag{3.4.8}$$

as homomorphisms from $\varinjlim V_j$ into $\varinjlim Z_j$, as left modules over A.

Suppose that

$$\psi_j \circ \phi_j = 0 \tag{3.4.9}$$

for every $j \in I$, and observe that

$$\psi \circ \phi = 0. \tag{3.4.10}$$

In fact, suppose that

$$V_j \xrightarrow{\phi_j} W_j \xrightarrow{\psi_j} Z_j \tag{3.4.11}$$

is exact for each $j \in I$, so that we have an exact sequence of homomorphisms between direct systems, as in Exercise 19 on p33 of [1]. We would like to check that

$$\varinjlim V_j \xrightarrow{\phi} \varinjlim W_j \xrightarrow{\psi} \varinjlim Z_j \tag{3.4.12}$$

is exact, as in [1].

Remember that an arbitary element of $\varinjlim W_j$ can be expressed as $\mu_l(w_l)$ for some $l \in I$ and $w_l \in W_l$, as in Section 3.2. If

$$\psi(\mu_l(w_l)) = 0, \tag{3.4.13}$$

then

$$\zeta_l(\psi_l(w_l)) = 0, \tag{3.4.14}$$

by (3.4.5). This implies that there is an $r \in I$ such that $l \preceq r$ and

$$\zeta_{l,r}(\psi_l(w_l)) = 0, \tag{3.4.15}$$

as in Section 3.2. It follows that

$$\psi_r(\mu_{l,r}(w_l)) = 0, \tag{3.4.16}$$

by (3.4.4).

Using our exactness hypothesis, we get that there is a $v_r \in V_r$ such that

$$\mu_{l,r}(w_l) = \phi_r(v_r). \tag{3.4.17}$$

Remember that

$$\mu_l(w_l) = \mu_r(\mu_{l,r}(w_l)), \tag{3.4.18}$$

as in (3.2.6). Thus

$$\mu_l(w_l) = \mu_r(\phi_r(v_r)) = \phi(\nu_r(v_r)), \tag{3.4.19}$$

using (3.4.2) in the second step. This shows that $\mu_l(w_l)$ is in the image of $\varinjlim V_j$ under ϕ, as desired.

Of course, there are analogous statements for right modules.

3.5 Limits of bilinear mappings

Let k be a commutative ring with a multiplicative identity element, and let $(I, \preceq)$ be a nonempty directed set. Also let V_j be a module over k for each $j \in I$, and suppose that for every $j, l \in I$ with $j \preceq l$ we have a homomorphism $\nu_{j,l}$ from V_j into V_l, as modules over k, that satisfies the two conditions mentioned in Section 3.2. This means that the direct limit $\varinjlim V_j$ can be defined as a module over k as in Section 3.2, and we let ν_l be the corresponding homomorphism from V_l into the direct limit, as modules over k, for each $l \in I$, as before.

Let W and Z be modules over k, and suppose that for each $j \in I$, b_j is a mapping from $V_j \times W$ into Z that is bilinear over k. If $w \in W$, then

$$b_{j,w}(v) = b_j(v, w) \tag{3.5.1}$$

defines a homomorphism from V_j into Z, as modules over k. Suppose that for every $l, r \in I$ with $l \preceq r$ and $w \in W$ we have that

$$b_{l,w} = b_{r,w} \circ \nu_{l,r}. \tag{3.5.2}$$

This implies that for each $w \in W$ there is a unique homomorphism b_w from $\varinjlim V_j$ into Z such that

$$b_w \circ \nu_l = b_{l,w} \tag{3.5.3}$$

for every $l \in I$, as in Section 3.2. Of course, (3.5.2) is the same as saying that

$$b_l(v_l, w) = b_r(\nu_{l,r}(v_l), w) \tag{3.5.4}$$

for every $l, r \in I$ with $l \preceq r$, $v_l \in V_l$, and $w \in W$.

Consider the mapping b

$$\text{from } \big(\varinjlim V_j\big) \times W \text{ into } Z \tag{3.5.5}$$

defined by

$$b(v, w) = b_w(v) \tag{3.5.6}$$

for every $v \in \varinjlim V_j$ and $w \in W$. Of course, $b(v, w)$ is linear over k in v, as in the preceding paragraph. One can check that $b(v, w)$ is linear over k in w too, because b_w is uniquely determined by (3.5.3). Note that b is uniquely determined by the condition that

$$b(\nu_l(v_l), w) = b_l(v_l, w) \tag{3.5.7}$$

for every $l \in I$, $v_l \in V_l$, and $w \in W$.

Let A be an associative algebra over k with a multiplicative identity element e_A. Suppose for the moment that V_j is a right module over A for every $j \in I$, and that $\nu_{j,l}$ is a homomorphism from V_j into V_l, as right modules over A, for every $j, l \in I$ with $j \preceq l$. This implies that $\varinjlim V_j$ is a right module over A, and that ν_l is a homomorphism from V_l into $\varinjlim V_j$, as right modules over A, for every $l \in I$. Suppose also that W is a left module over A, and that

$$b_j(v_j \cdot a, w) = b_j(v_j, a \cdot w) \tag{3.5.8}$$

for every $j \in I$, $a \in A$, $v_j \in V_j$, and $w \in W$. If $l \in I$, $a \in A$, $v_l \in V_l$, and $w \in W$, then it follows that

$$\begin{aligned} b(\nu_l(v_l) \cdot a, w) = b(\nu_l(v_l \cdot a), w) &= b_l(v_l \cdot a, w) \\ &= b_l(v_l, a \cdot w) = b(\nu_l(v_l), a \cdot w). \end{aligned} \tag{3.5.9}$$

This means that

$$b(v \cdot a, w) = b(v, a \cdot w) \tag{3.5.10}$$

for every $a \in A$, $v \in \varinjlim V_j$, and $w \in W$, because every element of $\varinjlim V_j$ can be expressed as $\nu_l(v_l)$ for some $l \in I$ and $v_l \in V_l$. This leads to a homomorphism β

$$\text{from } \big(\varinjlim V_j\big) \bigotimes_A W \text{ into } Z, \tag{3.5.11}$$

as modules over k, in the usual way. More precisely,

$$\beta(v \otimes w) = b(v, w) \tag{3.5.12}$$

for every $v \in \varinjlim V_j$ and $w \in W$, and β is uniquely determined by this property. Equivalently,

$$\beta(\nu_l(v_l) \otimes w) = b(\nu_l(v_l), w) = b_l(v_l, w) \tag{3.5.13}$$

for every $l \in I$, $v_l \in V_l$, and $w \in W$. Of course, there are analogous statements when V_j is a left module over A for every $j \in I$, W is a right module over A, and so on.

3.6 Direct limits and tensor products

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let $(I, \preceq)$ be a nonempty pre-directed set, and let V_j be a right module over A for each $j \in I$. Suppose that for each $j, l \in I$ with $j \preceq l$, we have a homomorphism $\nu_{j,l}$ from V_j into V_l, as right modules over A, that satisfies the two conditions mentioned in Section 3.2. Thus the direct limit of the V_j's can be defined as a right module over A as before.

Let W be a left module over A, and for each $j \in I$, let $V_j \bigotimes_A W$ be a tensor product of V_j and W over A. If $j, l \in I$ and $j \preceq l$, then we get a homomorphism $\theta_{j,l}$

$$\text{from } V_j \bigotimes_A W \text{ into } V_l \bigotimes_A W, \tag{3.6.1}$$

as modules over k, using $\nu_{j,l}$ and the identity mapping on W. Note that $\theta_{j,j}$ is the identity mapping on $V_j \bigotimes_A W$ for every $j \in I$, because $\nu_{j,j}$ is the identity mapping on V_j. If $j, l, r \in I$ and $j \preceq l \preceq r$, then

$$\theta_{l,r} \circ \theta_{j,l} = \theta_{j,r}, \tag{3.6.2}$$

because of the analogous property of $\nu_{j,r}$. Thus the family of $V_j \bigotimes_A W$, $j \in I$, is a direct system of modules over k with respect to these homomorphisms.

This means that the direct limit

$$\varinjlim \big(V_j \bigotimes_A W\big) \tag{3.6.3}$$

may be defined in the usual way, as a module over k. If $l \in I$, then we let θ_l be the corresponding homomorphism from $V_l \bigotimes_A W$ into (3.6.3), as modules over k. If $l, r \in I$ and $l \preceq r$, then

$$\theta_l = \theta_r \circ \theta_{l,r}, \tag{3.6.4}$$

as before.

If $l \in I$, then let ν_l be the natural homomorphism from V_l into $\varinjlim V_j$, as right modules over A. This leads to a homomorphism ρ_l from $V_l \bigotimes_A W$ into

$$(3.6.5) \qquad \big(\varinjlim V_j\big) \bigotimes_A W,$$

as modules over k, using the identity mapping on W. If $r \in I$ satisfies $l \preceq r$, then

$$(3.6.6) \qquad \rho_r \circ \theta_{l,r} = \rho_l,$$

because of the analogous property for $\nu_{l,r}$. It follows that there is a unique homomorphism ρ from (3.6.3) into (3.6.5), as modules over k, such that

$$(3.6.7) \qquad \rho \circ \theta_l = \rho_l$$

for every $l \in I$. We would like to show that

(3.6.8) ρ is an isomorphism from (3.6.3) onto (3.6.5),

as modules over k, as in Exercise 20 on p33f of [1].

If $l \in I$, $v_l \in V_l$, and $w \in W$, then put

$$(3.6.9) \qquad b_l(v_l, w) = \theta_l(v_l \otimes w)$$

This defines a mapping from $V_l \times W$ into (3.6.3) that is bilinear over k. This is the same as the composition of the natural bilinear mapping from $V_l \times W$ into $V_l \bigotimes_A W$ with θ_l. In particular,

$$(3.6.10) \qquad b_l(v_l \cdot a, w) = b_l(v_l, a \cdot w)$$

for every $a \in A$. If $l, r \in I$, $l \preceq r$, $v_l \in V_l$, and $w \in W$, then

$$(3.6.11) \qquad \begin{aligned} b_r(\nu_{l,r}(v_l), w) = \theta_r(\nu_{l,r}(v_l) \otimes w) &= \theta_r(\theta_{l,r}(v_l \otimes w)) \\ &= \theta_l(v_l \otimes w) = b_l(v_l, w). \end{aligned}$$

This leads to a mapping b from

$$(3.6.12) \qquad \big(\varinjlim V_j\big) \times W$$

into (3.6.3) that is bilinear over k, as in the previous section. More precisely, b satisfies (3.5.10), which leads to a homomorphism β from (3.6.5) into (3.6.3), as modules over k, as before. If $l \in I$, $v_l \in V_l$, and $w \in W$, then

$$(3.6.13) \qquad \beta(\nu_l(v_l) \otimes w) = b_l(v_l, w) = \theta_l(v_l \otimes w).$$

In this case, we also have that

$$(3.6.14) \qquad \rho(\theta_l(v_l \otimes w)) = \rho_l(v_l \otimes w) = \nu_l(v_l) \otimes w.$$

One can use this to check that β and ρ are inverses of each other.

There are analogous statements when the limit is taken in the second factor in the tensor product.

3.7 Double limits and bilinear mappings

Let $(I_1, \preceq_1)$, $(I_2, \preceq_2)$ be nonempty pre-directed sets, and put $I = I_1 \times I_2$. Consider the binary relation $\preceq$ defined on I by

$$(j_1, j_2) \preceq (l_1, l_2) \text{ when } j_1 \preceq_1 l_1 \text{ and } j_2 \preceq_2 l_2, \tag{3.7.1}$$

where $j_1, l_1 \in I_1$ and $j_2, l_2 \in I_2$. It is easy to see that $(I, \preceq)$ is a pre-directed set.

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V_{j_1} be a right module over A for every $j_1 \in I_1$, and let W_{j_2} be a left module over A for every $j_2 \in I_2$. Suppose that for every $j_1, l_1 \in I_1$ with $j_1 \preceq_1 l_1$ we have a homomorphism ν_{j_1,l_1} from V_{j_1} into V_{j_2}, as right modules over A, and that these homomorphisms satisfy the two conditions mentioned in Section 3.2. Similarly, suppose that for every $j_2, l_2 \in I_2$ with $j_2 \preceq_2 l_2$ we have a homomorphism μ_{j_2,l_2} from W_{j_2} into W_{l_2}, as left modules over A, that satisfy the same two conditions.

The direct limits of the V_{j_1}'s and W_{j_2}'s may be defined as right and left modules over A, respectively, as before. If $l_1 \in I_1$ and $l_2 \in I_2$, then we let ν_{l_1}, μ_{l_2} be the corresponding homomorphisms from V_{l_1}, W_{l_2} into $\varinjlim V_{j_1}$, $\varinjlim W_{j_2}$, respectively.

Let Z be a module over k, and for each $j_1 \in I_1$ and $j_2 \in I_2$, let b_{j_1,j_2} be a mapping from $V_{j_1} \times W_{j_2}$ into Z that is bilinear over k. If $l_1, r_1 \in I_1$, $l_2, r_2 \in I_2$, $l_1 \preceq_1 r_1$, $l_2 \preceq_2 r_2$, $v_{l_1} \in V_{l_1}$, and $w_{l_2} \in W_{l_2}$, then we ask that

$$b_{l_1,l_2}(v_{l_1}, w_{l_2}) = b_{r_1,r_2}(\nu_{l_1,r_1}(v_{l_1}), \mu_{l_2,r_2}(w_{l_2})). \tag{3.7.2}$$

We also ask that

$$b_{j_1,j_2}(v_{j_1} \cdot a, w_{j_2}) = b_{j_1,j_2}(v_{j_1}, a \cdot w_{j_2}) \tag{3.7.3}$$

for every $j_1 \in I_1$, $j_2 \in I_2$, $a \in A$, $v_{j_1} \in V_{j_1}$, and $w_{j_2} \in W_{j_2}$.

Let $j_2 \in I_2$ be given. As in Section 3.5, there is a unique mapping b_{j_2}

$$\text{from } \big(\varinjlim V_{j_1}\big) \times W_{j_2} \text{ into } Z \tag{3.7.4}$$

such that

$$b_{j_2}(\nu_{l_1}(v_{l_1}), w_{j_2}) = b_{l_1,j_2}(v_{l_1}, w_{j_2}) \tag{3.7.5}$$

for every $l_1 \in I_1$, $v_{l_1} \in V_{l_1}$, and $w_{j_2} \in W_{j_2}$. This mapping is bilinear over k, and satisfies

$$b_{j_2}(v \cdot a, w_{j_2}) = b_{j_2}(v, a \cdot w_{j_2}) \tag{3.7.6}$$

for every $a \in A$, $v \in \varinjlim V_{j_1}$, and $w_{j_2} \in W_{j_2}$.

Similarly, there is a unique mapping b

$$\text{from } \big(\varinjlim V_{j_1}\big) \times \big(\varinjlim W_{j_2}\big) \text{ into } Z \tag{3.7.7}$$

such that

(3.7.8) $$b(v, \mu_{l_2}(w_{l_2})) = b_{l_2}(v, w_{l_2})$$

for every $v \in \varinjlim V_{j_1}$, $l_2 \in I_2$, and $w_{l_2} \in W_{l_2}$. This mapping is bilinear over k, and satisfies

(3.7.9) $$b(v \cdot a, w) = b(v, a \cdot w)$$

for every $a \in A$, $v \in \varinjlim V_{j_1}$, and $w \in \varinjlim W_{j_2}$. Note that (3.7.8) is equivalent to saying that

(3.7.10) $$b(\nu_{l_1}(v_{l_1}), \mu_{l_2}(w_{l_2})) = b_{l_1,l_2}(v_{l_1}, w_{l_2})$$

for every $l_1 \in I_1$, $l_2 \in I_2$, $v_{l_1} \in V_{l_1}$, and $w_{l_2} \in W_{l_2}$.

Using b, we get a unique homomorphism β

(3.7.11) $$\text{from } \big(\varinjlim V_{j_1}\big) \bigotimes\nolimits_A \big(\varinjlim W_{j_2}\big) \text{ into } Z,$$

as modules over k, such that

(3.7.12) $$\beta(v \otimes w) = b(v, w)$$

for every $v \in \varinjlim V_{j_1}$ and $w \in \varinjlim W_{j_2}$. Equivalently, this means that

(3.7.13) $$\beta(\nu_{l_1}(v_{l_1}) \otimes \mu_{l_2}(w_{l_2})) = b(\nu_{l_1}(v_{l_1}), \mu_{l_2}(w_{l_2})) = b_{l_1,l_2}(v_{l_1}, w_{l_2})$$

for every $l_1 \in I_1$, $l_2 \in I_2$, $v_{l_1} \in V_{l_1}$, and $w_{l_2} \in W_{l_2}$.

3.8 Double limits and tensor products

Let us continue with the same notation and hypotheses as in the first three paragraphs of the previous section.

If $j_1 \in I_1$ and $j_2 \in I_2$, then let $V_{j_1} \bigotimes_A W_{j_2}$ be a tensor product of V_{j_1} and W_{j_2} over A. If $j = (j_1, j_2), l = (l_1, l_2) \in I$ and $j \preceq l$, then we get a homomorphism $\theta_{j,l}$

(3.8.1) $$\text{from } V_{j_1} \bigotimes\nolimits_A W_{j_2} \text{ into } V_{l_1} \bigotimes\nolimits_A W_{l_2},$$

as modules over k, using ν_{j_1,l_1} and μ_{j_2,l_2}. In particular, $\theta_{j,j}$ is the identity mapping on $V_{j_1} \bigotimes_A W_{j_2}$, because ν_{j_1,j_1} is the identity mapping on V_{j_1}, and μ_{j_2,j_2} is the identity mapping on W_{j_2}. If $j, l, r \in I$ and $j \preceq l \preceq r$, then

(3.8.2) $$\theta_{l,r} \circ \theta_{j,l} = \theta_{j,r},$$

beause of the analogous properties of ν_{j_1,r_1} and μ_{j_2,r_2}. This shows that the family of $V_{j_1} \bigotimes_A W_{j_2}$, $j = (j_1, j_2) \in I$, is a direct system of modules over k with respect to these homomorphisms.

Thus the direct limit

(3.8.3) $$\varinjlim \big(V_{j_1} \bigotimes\nolimits_A W_{j_2}\big)$$

may be defined as a module over k in the usual way. If $l = (l_1, l_2) \in I$, then let θ_l be the corresponding homomorphism from $V_{l_1} \bigotimes_A W_{l_2}$ into (3.8.3), as modules over k. If $l, r \in I$ and $l \preceq r$, then

$$\theta_l = \theta_r \circ \theta_{l,r}, \tag{3.8.4}$$

as usual.

If $l = (l_1, l_2) \in I$, then let ρ_l be the homomorphism from $V_{l_1} \bigotimes_A W_{l_2}$ into

$$\big(\varinjlim V_{j_1}\big) \bigotimes_A \big(\varinjlim W_{j_2}\big), \tag{3.8.5}$$

as modules over k, corresponding to ν_{l_1} and μ_{l_2}. If $r = (r_1, r_2) \in I$ and $l \preceq r$, then

$$\rho_r \circ \theta_{l,r} = \rho_l, \tag{3.8.6}$$

because of the analogous properties for ν_{l_1,r_1} and μ_{l_2,r_2}. This implies that there is a unique homomorphism ρ from (3.8.3) into (3.8.5), as modules over k, such that

$$\rho \circ \theta_l = \rho_l \tag{3.8.7}$$

for every $l \in I$. We would like to show that

$$\rho \text{ is an isomorphism from (3.8.3) onto (3.8.5),} \tag{3.8.8}$$

as modules over k, as in Proposition 9.2* on p99 of [3].

If $l = (l_1, l_2) \in I$, $v_{l_1} \in V_{l_1}$, and $w_{l_2} \in W_{l_2}$, then put

$$b_{l_1,l_2}(v_{l_1}, w_{l_2}) = \theta_l(v_{l_1} \otimes w_{l_2}), \tag{3.8.9}$$

which defines a mapping from $V_{l_1} \times W_{l_2}$ into (3.8.3) that is bilinear over k. We also have that

$$b_{l_1,l_2}(v_{l_1} \cdot a, w_{l_2}) = b_{l_1,l_2}(v_{l_1}, a \cdot w_{l_2}) \tag{3.8.10}$$

for every $a \in A$, because b_{l_1,l_2} is the same as the composition of the natural bilinear mapping from $V_{l_1} \times W_{l_2}$ into $V_{l_1} \bigotimes_A W_{l_2}$ with θ_l. If $r = (r_1, r_2) \in I$ and $l \preceq r$, then

$$\begin{aligned} b_{r_1,r_2}(\nu_{l_1,r_1}(v_{l_1}), \mu_{l_2,r_2}(w_{l_2})) &= \theta_r(\nu_{l_1,r_1}(v_{l_1}) \otimes \mu_{l_2,r_2}(w_{l_2})) \\ &= \theta_r(\theta_{l,r}(v_{l_1} \times w_{l_2})) \\ &= \theta_l(v_{l_1} \otimes w_{l_2}) = b_{l_1,l_2}(v_{l_1}, w_{l_2}). \end{aligned} \tag{3.8.11}$$

Using this, we get a mapping b from

$$\big(\varinjlim V_{j_1}\big) \times \big(\varinjlim W_{j_2}\big) \tag{3.8.12}$$

into (3.8.3) that is bilinear over k, as in the previous section. Remember that b satisfies (3.7.9), which leads to a homomorphism β from (3.8.5) into (3.8.3), as before. If $l = (l_1, l_2) \in I$, $v_{l_1} \in V_{l_1}$, and $w_{l_2} \in W_{l_2}$, then

$$\beta(\nu_{l_1}(v_{l_1}) \otimes \mu_{l_2}(w_{l_2})) = b_{l_1,l_2}(v_{l_1}, w_{l_2}) = \theta_l(v_{l_1} \otimes w_{l_2}). \tag{3.8.13}$$

Observe that

$$\rho(\theta_l(v_{l_1} \otimes w_{l_2})) = \rho_l(v_{l_1} \otimes w_{l_2}) = \nu_{l_1}(v_{l_1}) \otimes \mu_{l_2}(w_{l_2}). \tag{3.8.14}$$

One can use this to verify that β and ρ are inverses of each other.

3.9 Iterated direct limits

Let $(I_1, \preceq_1)$, $(I_2, \preceq_2)$ be nonempty pre-directed sets again, and let $I = I_1 \times I_2$ be equipped with the corresponding pre-order $\preceq$, as in Section 3.7. Also let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A.

Suppose that for each $j = (j_1, j_2) \in I$,

$$V_j = V_{(j_1,j_2)} \tag{3.9.1}$$

is a left module over A. If $j, l \in I$, then we ask that $\nu_{j,l}$ be a homomorphism from V_j into V_l, as left modules over A, and that these homomorphisms satisfy the two conditions mentioned in Section 3.2. Thus the direct limit

$$\varinjlim V_j = \varinjlim^I V_j \tag{3.9.2}$$

of the V_j's, $j \in I$, can be defined in the usual way, as a left module over A. If $l = (l_1, l_2) \in I$, then we get a natural homomorphism

$$\nu_l = \nu_{(l_1,l_2)} \text{ from } V_l = V_{(l_1,l_2)} \text{ into (3.9.2)}, \tag{3.9.3}$$

as left modules over A, as before. If $r = (r_1, r_2) \in I$ and $l \preceq r$, then

$$\nu_{(l_1,l_2)} = \nu_l = \nu_r \circ \nu_{l,r} = \nu_{(r_1,r_2)} \circ \nu_{l,r}, \tag{3.9.4}$$

by construction.

Let $j_2 \in I_2$ be given. If $j_1, l_1 \in I_1$ and $j_1 \preceq_1 l_1$, then put

$$j = (j_1, j_2) \text{ and } l = (l_1, j_2) \tag{3.9.5}$$

for the moment, and observe that $j \preceq l$. In this case,

$$\nu^{j_2}_{j_1,l_1} = \nu_{j,l} \tag{3.9.6}$$

is a homomorphism from $V_j = V_{(j_1,j_2)}$ into $V_l = V_{(l_1,j_2)}$, as left modules over A. It is easy to see that these homomorphisms satisfy the usual two conditions in Section 3.2, so that the $V_{(j_1,j_2)}$'s, $j_1 \in I_1$, form a direct system of modules. Let

$$\varinjlim^{I_1} V_{(j_1,j_2)} \tag{3.9.7}$$

be the corresponding direct limit, which is a left module over A.

If $l_1 \in I_1$, then we get a natural homomorphism

$$\nu^{j_2}_{l_1} \text{ from } V_{(l_1,j_2)} \text{ into (3.9.7)}, \tag{3.9.8}$$

as left modules over A, as in Section 3.2. If $r_1 \in I_1$ and $l_1 \preceq_1 r_1$, then

$$\nu^{j_2}_{l_1} = \nu^{j_2}_{r_1} \circ \nu^{j_2}_{l_1,r_1}, \tag{3.9.9}$$

as before.

Let $l_2 \in I_2$ be given, with $j_2 \preceq_2 l_2$. If $j_1 \in I_1$, then put

$$j = (j_1, j_2) \text{ and } l = (j_1, l_2) \tag{3.9.10}$$

for the moment, and observe that $j \preceq l$. Under these conditions,

$$\phi_{j_1}^{j_2,l_2} = \nu_{j,l} \tag{3.9.11}$$

is a homomorphism from $V_j = V_{(j_1,j_2)}$ into $V_l = V_{(j_1,l_2)}$, as left modules over A. If $l_1 \in I_1$ and $j_1 \preceq_1 l_1$, then one can check that

$$\phi_{l_1}^{j_2,l_2} \circ \nu_{j_1,l_1}^{j_2} = \nu_{j_1,l_1}^{l_2} \circ \phi_{j_1}^{j_2,l_2}. \tag{3.9.12}$$

More precisely, both sides are the same as the homomorphism

$$\nu_{(j_1,j_2),(l_1,l_2)} \tag{3.9.13}$$

from $V_{(j_1,j_2)}$ into $V_{(l_1,l_2)}$.

Of course, $V_{(j_1,l_2)}$, $j_1 \in I_1$, forms a direct system of left modules over A, as before. As in Section 3.4, there is a unique homomorphism

$$\phi^{j_2,l_2} = \varinjlim^{I_1} \phi_{j_1}^{j_2,l_2} \quad \text{from } \varinjlim^{I_1} V_{(j_1,j_2)} \text{ into } \varinjlim^{I_1} V_{(j_1,l_2)} \tag{3.9.14}$$

such that

$$\phi^{j_2,l_2} \circ \nu_{l_1}^{j_2} = \nu_{l_1}^{l_2} \circ \phi_{l_1}^{j_2,l_2} \tag{3.9.15}$$

for every $l_1 \in I_1$.

One can use the uniqueness of ϕ^{j_2,l_2} to get that these homomorphisms satisfy the two conditions mentioned in Section 3.2. This means that the family of modules of the form (3.9.7), with $j_2 \in I_2$, is a direct system. Thus the corresponding direct limit

$$\varinjlim^{I_2} \big(\varinjlim^{I_1} V_{(j_1,j_2)}\big) \tag{3.9.16}$$

can be defined as a left module over A in the usual way.

If $l_2 \in I_2$, then there is a natural homomorphism

$$\phi^{l_2} \text{ from } \varinjlim^{I_1} V_{(j_1,l_2)} \text{ into (3.9.16)}, \tag{3.9.17}$$

as before. If $r_2 \in I_2$ and $l_2 \preceq r_2$, then

$$\phi^{l_2} = \phi^{r_2} \circ \phi^{l_2,r_2}, \tag{3.9.18}$$

as usual.

Let $l = (l_1, l_2) \in I$ be given, and remember that $\nu_{l_1}^{l_2}$ is a homomorphism from $V_{(l_1,l_2)}$ into $\varinjlim^{I_1} V_{(j_1,l_2)}$, as left modules over A, as in (3.9.8). Thus

$$\xi_l = \phi^{l_2} \circ \nu_{l_1}^{l_2} \tag{3.9.19}$$

defines a homomorphism from $V_l = V_{(l_1,L_2)}$ into (3.9.16), as left modules over A. Suppose that $r = (r_1, r_2) \in I$ satisfies $l \preceq r$, so that

$$\nu_{l,r} = \nu_{l_1,r_2}^{r_2} \circ \phi_{l_1}^{l_2,r_2}, \tag{3.9.20}$$

as in the equality between (3.9.12) and (3.9.13). It follows that

$$\xi_r \circ \nu_{l,r} = \phi^{r_2} \circ \nu_{r_1}^{r_2} \circ \nu_{l_1,r_2}^{r_2} \circ \phi_{l_1}^{l_2,r_2} = \phi^{r_1} \circ \nu_{l_1}^{r_2} \circ \phi_{l_1}^{l_2,r_2}, \tag{3.9.21}$$

where the second step is as in (3.9.9). This implies that

$$\xi_r \circ \nu_{l,r} = \phi^{r_2} \circ \phi^{l_2,r_2} \circ \nu_{l_1}^{l_2} = \phi^{l_2} \circ \nu_{l_1}^{l_2} = \xi_l, \tag{3.9.22}$$

using (3.9.15) in the first step, and (3.9.18) in the second step.

As in Section 3.2, there is a unique homomorphism

$$\xi \text{ from (3.9.2) into (3.9.16),} \tag{3.9.23}$$

as left modules over A, such that

$$\xi \circ \nu_l = \xi_l \tag{3.9.24}$$

for every $l \in I$. Here ν_l is the natural homomorphism from V_l into (3.9.2), as before.

Let $l_2 \in I_2$ be given. If $l_1 \in I_1$, so that $l = (l_1, l_2) \in I$, then $\nu_l = \nu_{(l_1,l_2)}$ maps $V_l = V_{(l_1.l_2)}$ into (3.9.2), as before. If $r_1 \in I_1$ and $l_1 \preceq r_1$, then $\nu_{l_1,r_1}^{l_2}$ maps $V_{(l_1,l_2)}$ into $V_{(r_1,l_2)}$, as in (3.9.6). Note that

$$\nu_{(l_1,l_2)} = \nu_{(r_1,l_2)} \circ \nu_{l_1,r_1}^{l_2}, \tag{3.9.25}$$

as in (3.9.4).

Using $\nu_{(l_1,l_2)}$, $l_1 \in I_1$, we get a homomorphism

$$\eta^{l_2} \text{ from } \varinjlim^{I_1} V_{(j_1,l_2)} \text{ into (3.9.2),} \tag{3.9.26}$$

as left modules over A, as in Section 3.2. This homomorphism is characterized by the property that

$$\eta^{l_2} \circ \nu_{l_1}^{l_2} = \nu_{(l_1,l_2)} \tag{3.9.27}$$

for every $l_1 \in I_1$. Here $\nu_{l_1}^{l_2}$ is the natural homomorphism from $V_{(l_1,l_2)}$ into $\varinjlim^{I_1} V_{(j_1,l_2)}$, as in (3.9.8).

Suppose that $r_2 \in I_2$ satisfies $l_2 \preceq_2 r_2$, and let ϕ^{l_2,r_2} be as in (3.9.14). We would like to verify that

$$\eta^{r_2} \circ \phi^{l_2,r_2} = \eta^{l_2}. \tag{3.9.28}$$

To do this, observe that

$$\eta^{r_2} \circ \phi^{l_2,r_2} \circ \nu_{l_1}^{l_2} = \eta^{r_2} \circ \nu_{l_1}^{r_2} \circ \phi_{l_1}^{l_2,r_2}, \tag{3.9.29}$$

by (3.9.15). This implies that

$$\eta^{r_2} \circ \phi^{l_2,r_2} \circ \nu_{l_1}^{l_2} = \nu_{(l_1,r_2)} \circ \phi_{l_1}^{l_2,r_2}, \tag{3.9.30}$$

by (3.9.27). Using the definition (3.9.11) of $\phi_{l_1}^{l_2,r_2}$, we get that

$$\eta^{r_2} \circ \phi^{l_2,r_2} \circ \nu_{l_1}^{l_2} = \nu_{(l_1,r_2)} \circ \nu_{(l_1,l_2),(l_1,r_2)} = \nu_{(l_1,l_2)}. \tag{3.9.31}$$

Here the second step is as in (3.9.4). It follows that (3.9.28) holds, because η^{l_2} is uniquely determined by (3.9.27).

Thus we can use the η^{l_2}'s, $l_2 \in I_2$, to get a homomorphism

$$\eta \text{ from (3.9.16) into (3.9.2)}, \tag{3.9.32}$$

as left modules over A, as in Section 3.2. This homomorphism is uniquely determined by the property that

$$\eta \circ \phi^{l_2} = \eta^{l_2} \tag{3.9.33}$$

for each $l_2 \in I_2$, where ϕ^{l_2} is as in (3.9.17). We would like to check that the homomorphism ξ from (3.9.23) and η are inverses of each other.

To show that $\eta \circ \xi$ is the identity mapping on (3.9.2), it suffices to verify that

$$\eta \circ \xi \circ \nu_l = \nu_l \tag{3.9.34}$$

on V_l for every $l = (l_1, l_2) \in I$. Note that

$$\eta \circ \xi \circ \nu_l = \eta \circ \xi_l = \eta \circ \phi^{l_2} \circ \nu_{l_1}^{l_2}, \tag{3.9.35}$$

using (3.9.24) in the first step, and (3.9.19) in the second step. Thus

$$\eta \circ \xi \circ \nu_l = \eta^{l_2} \circ \nu_{l_1}^{l_2} = \nu_{(l_1,l_2)} = \nu_l, \tag{3.9.36}$$

as desired, using (3.9.33) in the first step, and (3.9.27) in the second step.

To show that $\xi \circ \eta$ is the identity mapping on (3.9.16), it suffices to check that

$$\xi \circ \eta \circ \phi^{l_2} = \phi^{l_2} \tag{3.9.37}$$

on $\varinjlim^{I_1} V_{(j_1,l_2)}$ for every $l_2 \in I_2$, where ϕ^{l_2} is as in (3.9.17). Equivalently, this means that

$$\xi \circ \eta^{l_2} = \phi^{l_2} \tag{3.9.38}$$

on $\varinjlim^{I_1} V_{(j_1,l_2)}$, because of (3.9.33). In order to show this, it suffices to verify that

$$\xi \circ \eta^{l_2} \circ \nu_{l_1}^{l_2} = \phi^{l_2} \circ \nu_{l_1}^{l_2} \tag{3.9.39}$$

on $V_{(l_1,l_2)}$ for every $l_1 \in I_1$, where $\nu_{l_1}^{l_2}$ is as in (3.9.8). Observe that

$$\xi \circ \eta^{l_2} \circ \nu_{l_1}^{l_2} = \xi \circ \nu_l = \xi_l, \tag{3.9.40}$$

using (3.9.27) in the first step, and (3.9.24) in the second step. Thus (3.9.39) follows from the definition (3.9.19) of ξ_l, as desired.

Of course, there are analogous statements for right modules over A.

3.10 Another property of compositions

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V_1, V_2, and V_3 be right modules over A, and let W_1, W_2, and W_3 be left modules over A. Suppose that $V_j \bigotimes_A W_l$ is a tensor product of V_j and W_l over A for each $j, l = 1, 2, 3$.

Let ϕ_1, ψ_1 be homomorphisms from V_1, W_1 into V_2, W_2, respectively, and let ϕ_2, ψ_2 be homomorphisms from V_2, W_2 into V_3, W_3, respectively, as modules over A. If $j = 1$ or 2 and $l = 1, 2$, or 3, then there is a unique homomorphism

$$\Phi_{j,l} \text{ from } V_j \bigotimes_A W_l \text{ into } V_{j+1} \bigotimes_A W_l, \tag{3.10.1}$$

as modules over k, such that

$$\Phi_{j,l}(v_j \otimes w_l) = \phi_j(v_j) \otimes w_l \tag{3.10.2}$$

for every $v_j \in V_j$ and $w_l \in W_l$. Equivalently, $\Phi_{j,l}$ is obtained from ϕ_j and the identity mapping on W_l as in Section 1.9. Similarly, if $j = 1, 2$, or 3 and $l = 1$ or 2, then there is a unique homomorphism

$$\Psi_{j,l} \text{ from } V_j \bigotimes_A W_l \text{ into } V_j \bigotimes_A W_{l+1}, \tag{3.10.3}$$

as modules over k, such that

$$\Psi_{j,l}(v_j \otimes w_l) = v_j \otimes \psi_l(w_l) \tag{3.10.4}$$

for every $v_j \in V_j$ and $w_l \in W_l$. This is the same as the homomorphism obtained from ψ_l and the identity mapping on V_j, as before.

In the same way, we can use ϕ_2 and ψ_2 to get a unique homomorphism

$$\Theta_2 \text{ from } V_2 \bigotimes_A W_2 \text{ into } V_3 \bigotimes_A W_3, \tag{3.10.5}$$

as modules over k, such that

$$\Theta_2(v_2 \otimes w_2) = \phi_2(v_2) \otimes \psi_2(w_2) \tag{3.10.6}$$

for every $v_2 \in V_2$ and $w_2 \in W_2$. Note that

$$\Theta_2 = \Psi_{3,2} \circ \Phi_{2,2} = \Phi_{2,3} \circ \Psi_{2,2}. \tag{3.10.7}$$

If

$$\phi_2 \circ \phi_1 = 0, \tag{3.10.8}$$

then

$$\Phi_{2,2} \circ \Phi_{1,2} = 0. \tag{3.10.9}$$

This is because $\Phi_{2,2} \circ \Phi_{1,2}$ is the same as the homomorphism from $V_1 \bigotimes_A W_2$ into $V_3 \bigotimes_A W_2$ obtained from $\phi_2 \circ \phi_1$ and the identity mapping on W_2. This implies that

$$\Theta_{2,2} \circ \Phi_{1,2} = 0, \tag{3.10.10}$$

by (3.10.7). It follows that

$$\Phi_{1,2}\big(V_1 \bigotimes_A W_2\big) \subseteq \ker \Theta_{2,2} \tag{3.10.11}$$

in this case.

Similarly, if

$$\psi_2 \circ \psi_1 = 0, \tag{3.10.12}$$

then

$$\Psi_{2,2} \circ \Psi_{2,1} = 0. \tag{3.10.13}$$

This implies that

$$\Theta_{2,2} \circ \Psi_{2,1} = 0, \tag{3.10.14}$$

so that

$$\Psi_{1,2}\big(V_2 \bigotimes_A W_1\big) \subseteq \ker \Theta_{2,2}, \tag{3.10.15}$$

as before.

Suppose now that

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \longrightarrow 0 \tag{3.10.16}$$

and

$$W_1 \xrightarrow{\psi_1} W_2 \xrightarrow{\psi_2} W_3 \longrightarrow 0 \tag{3.10.17}$$

are exact sequences, so that ϕ_2, ψ_2 are surjective, and

$$\phi_1(V_1) = \ker \phi_2, \quad \psi_1(W_1) = \ker \psi_2. \tag{3.10.18}$$

In this case, it is well known that we get an exact sequence

$$\begin{aligned} &\big(V_1 \bigotimes_A W_2\big) \bigoplus \big(V_2 \bigotimes_A W_1\big) \longrightarrow V_2 \bigotimes_A W_2 \\ &\xrightarrow{\Theta_{2,2}} V_3 \bigotimes_A W_3 \longrightarrow 0, \end{aligned} \tag{3.10.19}$$

using the mapping obtained from $\Phi_{1,2}$ and $\Psi_{2,1}$ in the first step. This corresponds to part of Proposition 4.3 on p24f of [3], which is stated more abstractly and under slightly different conditions, as mentioned at the top of p26 of [3].

The exactness of (3.10.19) means that $\Theta_{2,2}$ is surjective, and that

$$\Phi_{1,2}\big(V_1 \bigotimes_A W_2\big) + \Psi_{2,1}\big(V_2 \bigotimes_A W_1\big) = \ker \Theta_{2,2}. \tag{3.10.20}$$

Of course,

$$\Phi_{1,2}\big(V_1 \bigotimes_A W_2\big) + \Psi_{2,1}\big(V_2 \bigotimes_A W_1\big) \subseteq \ker \Theta_{2,2}, \tag{3.10.21}$$

by (3.10.11) and (3.10.15).

To get the exactness of (3.10.19), the statement and argument in [3] uses the right exactness of the tensor product in each factor, as in Section 2.5. The surjectivity of $\Theta_{2,2}$ can be obtained from the surjectivity of $\Phi_{2,2}$ and $\Psi_{3,2}$, or from the surjectivity of $\Phi_{2,3}$ and $\Psi_{2,2}$, by (3.10.7). The surjectivity of these mappings follows from the surjectivity of ϕ_2 and ψ_2, as before. If an element $u_{2,2}$ of $V_2 \bigotimes_A W_2$ is in the kernel of $\Theta_{2,2}$, then

$$\Psi_{2,2}(u_{2,2}) \in \ker \Phi_{2,3}, \tag{3.10.22}$$

by (3.10.7). This implies that there is an element $u_{1,3}$ of $V_1 \bigotimes_A W_3$ such that

$$\Psi_{2,2}(u_{2,2}) = \Phi_{1,3}(u_{1,3}), \tag{3.10.23}$$

as in Section 2.5.

Note that $\Psi_{1,2}$ is surjective, because ψ_2 is surjective, as in Section 1.9. This means that there is an element $u_{1,2}$ of $V_1 \bigotimes_A W_2$ such that

$$\Psi_{1,2}(u_{1,2}) = u_{1,3}. \tag{3.10.24}$$

It is easy to see that

$$\Phi_{1,3} \circ \Psi_{1,2} = \Psi_{2,2} \circ \Phi_{1,2}, \tag{3.10.25}$$

and more precisely that both sides correspond to the mapping from $V_1 \bigotimes_A W_2$ into $V_2 \bigotimes_A W_3$ obtained from ϕ_1 and ψ_2 in the usual way. Thus

$$\Phi_{1,3}(u_{1,3}) = \Phi_{1,3}(\Psi_{1,2}(u_{1,2})) = \Psi_{2,2}(\Phi_{1,2}(u_{1,2})). \tag{3.10.26}$$

This means that

$$\Psi_{2,2}(u_{2,2}) = \Psi_{2,2}(\Phi_{1,2}(u_{1,2})), \tag{3.10.27}$$

by (3.10.23).

Equivalently,

$$u_{2,2} - \Phi_{1,2}(u_{1,2}) \in \ker \Psi_{2,2}. \tag{3.10.28}$$

It follows that there is an element $u_{2,1}$ of $V_2 \bigotimes_A W_1$ such that

$$u_{2,2} - \Phi_{1,2}(u_{1,2}) = \Psi_{2,1}(u_{2,1}), \tag{3.10.29}$$

as in Section 2.5. This shows that $u_{2,2}$ is contained in the left side of (3.10.20), as desired.

Alternatively, we can use an argument like the one in Section 2.5. The surjectivity of $\Theta_{2,2}$ follows from the surjectivity of ϕ_2 and ψ_2, as in Section 1.9. The left side of (3.10.20) is a submodule of $V_2 \bigotimes_A W_2$, as a module over k, so that the quotient

$$Y = \big(V_2 \bigotimes_A W_2\big) \Big/ \Big(\Phi_{1,2}\big(V_1 \bigotimes_A W_2\big) + \Psi_{2,1}\big(V_2 \bigotimes_A W_1\big)\Big) \tag{3.10.30}$$

is defined as a module over k. Let q_Y be the natural quotient mapping from $V_2 \bigotimes_A W_2$ onto Y.

Because of (3.10.21), there is a unique homomorphism $\widetilde{\Theta}_{2,2}$ from Y into $V_3 \bigotimes_A W_3$, as modules over k, such that

$$\widetilde{\Theta}_{2,2} \circ q_Y = \Theta_{2,2}. \tag{3.10.31}$$

It suffices to show that $\widetilde{\Theta}_{2,2}$ is an isomorphism.

If $v_2 \in V_2$ and $w_2 \in W_2$, then

$$q_Y(v_2 \otimes w_2) \tag{3.10.32}$$

is an element of Y. One can check that (3.10.32) only depends on $\phi_2(v_2)$ and $\psi_2(w_2)$, because of (3.10.18) and (3.10.30). This leads to a mapping from $V_3 \times W_3$ into Y, with

$$(\phi_2(v_2), \psi_2(w_2)) \mapsto q_Y(v_2 \otimes w_2) \tag{3.10.33}$$

for every $v_2 \in V_2$ and $w_2 \in W_2$, because ϕ_2 and ψ_2 are surjective, by hypothesis. One can also verify that this mapping is bilinear over k.

If $a \in A$, $v_2 \in V_2$, and $w_2 \in W_2$, then it is easy to see that

$$(\phi_2(v_2) \cdot a, \psi_2(w_2)) \text{ and } (\phi_2(v_2), a \cdot \psi_2(w_2)) \tag{3.10.34}$$

are mapped to the same element of Y by the mapping described in the preceding paragraph. This implies that there is a unique homomorphism from $V_3 \bigotimes_A W_3$ into Y, as modules over k, with

$$\phi_2(v_2) \otimes \psi_2(w_2) \mapsto q_Y(v_2 \otimes w_2) \tag{3.10.35}$$

for every $v_2 \in V_2$ and $w_2 \in W_2$.

One can check that this homomorphism is the inverse of $\widetilde{\Theta}_{2,2}$. In particular, $\widetilde{\Theta}_{2,2}$ is an isomorphism, as desired.

3.11 Compositions and $\mathrm{Hom}(\cdot, \cdot)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V_j and W_l be all left or all right modules over A, for $j, l = 1, 2, 3$. Suppose that ϕ_j, ψ_j are homomorphisms from V_j, W_j into V_{j+1}, W_{j+1}, respectively, as modules over A, for $j = 1, 2$.

If $j = 1$ or 2 and $l = 1, 2$, or 3, then

$$\Phi_{j,l}(\alpha_{j+1,l}) = \alpha_{j+1,l} \circ \phi_j \tag{3.11.1}$$

defines a homomorphism

$$\text{from } \mathrm{Hom}_A(V_{j+1}, W_l) \text{ into } \mathrm{Hom}_A(V_j, W_l), \tag{3.11.2}$$

as modules over k. Similarly, if $j = 1, 2$, or 3 and $l = 1$ or 2, then

$$\Psi_{j,l}(\alpha_{j,l}) = \psi_l \circ \alpha_{j,l} \tag{3.11.3}$$

defines a homomorphism

$$\text{from } \mathrm{Hom}_A(V_j, W_l) \text{ into } \mathrm{Hom}_A(V_j, W_{l+1}), \tag{3.11.4}$$

as modules over k.

If $\alpha_{3,1}$ is a homomorphism from V_3 into W_1, as modules over A, then

$$\Theta(\alpha_{3,1}) = \psi_1 \circ \alpha_{3,1} \circ \phi_2 \tag{3.11.5}$$

is a homomorphism from V_2 into W_2, as modules over A. This defines a homomorphism

$$\text{from } \mathrm{Hom}_A(V_3, V_1) \text{ into } \mathrm{Hom}_A(V_2, W_2), \tag{3.11.6}$$

as modules over k. Equivalently,

$$\Theta = \Psi_{2,1} \circ \Phi_{2,1} = \Phi_{2,2} \circ \Psi_{3,1}. \tag{3.11.7}$$

If

$$\phi_2 \circ \phi_1 = 0, \tag{3.11.8}$$

then

$$\Phi_{1,2} \circ \Phi_{2,2} = 0, \tag{3.11.9}$$

because $\Phi_{1,2} \circ \Phi_{2,2}$ is the same as the homomorphism from $\mathrm{Hom}_A(V_3, W_2)$ into $\mathrm{Hom}_A(V_1, W_2)$ corresponding to composition with $\phi_2 \circ \phi_1$. This implies that

$$\Phi_{2,1} \circ \Theta = 0, \tag{3.11.10}$$

by (3.11.7). This means that

$$\Theta\big(\mathrm{Hom}_A(V_3, W_1)\big) \subseteq \ker \Phi_{1,2} \tag{3.11.11}$$

when (3.11.8) holds.

Similarly, if

$$\psi_2 \circ \psi_1 = 0, \tag{3.11.12}$$

then

$$\Psi_{2,2} \circ \Psi_{2,1} = 0, \tag{3.11.13}$$

because $\Psi_{2,2} \circ \Psi_{2,1}$ is the same as the homomorphism from $\mathrm{Hom}_A(V_2, W_1)$ into $\mathrm{Hom}_A(V_2, W_3)$ corresponding to composition with $\psi_2 \circ \psi_1$. This implies that

$$\Psi_{2,2} \circ \Theta = 0, \tag{3.11.14}$$

by (3.11.7). Thus

$$\Theta\big(\mathrm{Hom}_A(V_3, W_1)\big) \subseteq \ker \Psi_{2,2} \tag{3.11.15}$$

when (3.11.12) holds.

Suppose from now on in this section that

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \longrightarrow 0 \tag{3.11.16}$$

and

$$0 \longrightarrow W_1 \xrightarrow{\psi_1} W_2 \xrightarrow{\psi_1} W_3 \tag{3.11.17}$$

are exact sequences. This means that ϕ_2 is surjective, ψ_1 is injective, and

$$\phi_1(V_1) = \ker \phi_2, \quad \psi_1(W_1) = \ker \psi_2. \tag{3.11.18}$$

It is well known that

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_A(V_3, V_1) &\stackrel{\Theta}{\longrightarrow} \mathrm{Hom}_A(V_2, W_2) \\ &\longrightarrow \mathrm{Hom}_A(V_1, W_2) \bigoplus \mathrm{Hom}_A(V_2, W_3) \end{aligned} \tag{3.11.19}$$

is an exact sequence in this case, using the mapping obtained from $\Phi_{1,2}$ and $\Psi_{2,2}$ in the last step. This corresponds to part of Proposition 4.3a on p25 of [3], which is stated more abstractly and under slightly different conditions, as mentioned at the top of p26 of [3].

The exactness of (3.11.19) means that Θ is injective, and that

$$\Theta\big(\mathrm{Hom}_A(V_3, W_1)\big) = (\ker \Phi_{1,2}) \cap (\ker \Psi_{2,2}). \tag{3.11.20}$$

Note that

$$\Theta\big(\mathrm{Hom}_A(V_3, W_1)\big) \subseteq (\ker \Phi_{1,2}) \cap (\ker \Psi_{2,2}), \tag{3.11.21}$$

by (3.11.11) and (3.11.15).

The statement and argument in [3] use the left exactness of $\mathrm{Hom}_A(\cdot, \cdot)$ in each variable, as in Sections 2.1 and 2.3, to get the exactness of (3.11.19). We can also use arguments like those in Sections 2.1 and 2.3 more directly here. In particular, the injectivity of Θ can be obtained from the injectivity of $\Phi_{2,1}$ and $\Psi_{2,1}$, or from the injectivity of $\Phi_{2,2}$ and $\Psi_{3,1}$, by (3.11.7). The injectivity of these mappings follows from the surjectivity of ϕ_2 and the injectivity of ψ_1, as before. The injectivity of Θ can be obtained directly from the surjectivity of ϕ_2 and the injectivity of ψ_1 as well, using the definition (3.11.5) of Θ.

Let $\alpha_{2,2}$ be a homomorphism from V_2 into W_2, as modules over A, with

$$\alpha_{2,2} \in (\ker \Phi_{1,2}) \cap (\ker \Psi_{2,2}). \tag{3.11.22}$$

The condition that $\alpha_{2,2} \in \ker \Phi_{1,2}$ means that

$$\alpha_{2,2} \circ \phi_1 = \Phi_{1,2}(\alpha_{2,2}) = 0, \tag{3.11.23}$$

which is the same as saying that

$$\ker \phi_2 = \phi_1(V_1) \subseteq \ker \alpha_{2,2}. \tag{3.11.24}$$

This implies that there is a homomorphism $\alpha_{3,2}$ from V_3 into W_2, as modules over A, such that

$$\Phi_{2,2}(\alpha_{3,2}) = \alpha_{3,2} \circ \phi_2 = \alpha_{2,2}, \tag{3.11.25}$$

because $\phi_2(V_2) = V_3$, by hypothesis.

Similarly, the condition that $\alpha_{2,2} \in \ker \Psi_{2,2}$ means that

$$\psi_2 \circ \alpha_{2,2} = \Psi_{2,2}(\alpha_{2,2}) = 0, \tag{3.11.26}$$

which is the same as saying that

$$\alpha_{2,2}(V_2) \subseteq \ker \psi_2 = \psi_1(W_1). \tag{3.11.27}$$

This implies that

$$\alpha_{3,2}(V_3) = \alpha_{2,3}(\phi_2(V_2)) = \alpha_{2,2}(V_2) \subseteq \psi_1(W_1). \tag{3.11.28}$$

It follows that there is a homomorphism $\alpha_{3,1}$ from V_3 into W_1, as modules over A, such that

$$\Psi_{3,1}(\alpha_{3,1}) = \psi_1 \circ \alpha_{3,1} = \alpha_{3,2}, \tag{3.11.29}$$

because ψ_1 is injective, by hypothesis. Combining this with (3.11.25), we obtain that

$$\Theta(\alpha_{3,1}) = \psi_1 \circ \alpha_{3,1} \circ \phi_2 = \alpha_{3,2} \circ \phi_2 = \alpha_{2,2}, \tag{3.11.30}$$

as desired.

3.12 Finitely-generated submodules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a left or right module over A. If V_1, V_2 are submodules of V, then it is easy to see that

$$V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1,\, v_2 \in V_2\} \tag{3.12.1}$$

is a submodule of V as well. If V_1, V_2 are finitely generated as modules over A, then $V_1 + V_2$ is finitely generated as a module over A too.

The collection of all finitely generated submodules of V is partially ordered by inclusion, and in fact it is a directed set, by the remarks in the preceding paragraph. Note that every element of V is contained in a finitely-generated submodule of V, because the action of A on any element of V defines a submodule of V. Thus V is the union of all of its finitely-generated submodules, so that V may be considered as the direct limit of its finitely-generated submodules. This corresponds to part of Exercise 17 on p33 of [1].

Suppose now that V is a right module over A, let W be a left module over A, and let $V \bigotimes_A W$ be a tensor product of V and W. Also let V_0, W_0 be submodules of V, W, respectively, as modules over A, and let $V_0 \bigotimes W_0$ be a tensor product of V_0 and W_0 over A. If $v \in V$ and $w \in W$, then we let $v \otimes w$ be the corresponding element of $V \bigotimes_A W$, as usual. Similarly, if $v \in V_0$ and $w \in W_0$, then let $v \otimes_0 w$ be the corresponding element of $V_0 \bigotimes_A W_0$. As in Section 1.9, there is a unique homomorphism from $V_0 \bigotimes_A W_0$ into $V \bigotimes_A W$, as modules over k, with

$$v \otimes_0 w \mapsto v \otimes w \tag{3.12.2}$$

for every $v \in V_0$ and $w \in W_0$.

Let n be a positive integer, and suppose that $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n$ in W satisfy

$$\sum_{j=1}^{n} v_j \otimes w_j = 0 \tag{3.12.3}$$

in $V \bigotimes_A W$. If

$$v_1, \ldots, v_n \in V_0 \text{ and } w_1, \ldots, w_n \in W_0, \tag{3.12.4}$$

then

$$\sum_{j=1}^{n} v_j \otimes_0 w_j \tag{3.12.5}$$

defines an element of $V_0 \bigotimes_A W_0$. Of course, (3.12.3) holds when (3.12.5) is equal to 0 in $V_0 \bigotimes_A W_0$.

In fact, there are finitely-generated submodules V_0, W_0 of V, W, respectively, as modules over A, such that (3.12.4) holds and (3.12.5) is equal to 0 in $V_0 \bigotimes_A W_0$, as in Corollary 2.13 on p25 of [1]. The proof uses the standard way of constructing tensor products as quotients. Thus (3.12.3) says that a certain formal expression involving $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ can be expressed in terms of finitely many elements of V and W in a suitable way. It suffices to use finitely-generated submodules V_0, W_0 of V, W, respectively, such that (3.12.4) holds, and which contain the finitely many elements of V and W just mentioned, to get that (3.12.5) is equal to 0 in $V_0 \bigotimes_A W_0$.

Alternatively, $V \bigotimes_A W$ may be obtained as a direct limit of tensor products of finitely-generated submodules of V and W, as in Section 3.8. This uses the fact that V and W may be obtained as direct limits of finitely-generated submodules, as before. One can use this to get that there are finitely-generated submodules V_0, W_0 of V, W, respectively, such that (3.12.4) holds and (3.12.5) is equal to 0 in $V_0 \bigotimes_A W_0$, as in Section 3.2.

One could obtain the same conclusion using direct limits in each factor separately, as in Section 3.6. That is, one can first obtain $V \otimes_A W$ as a direct limit of tensor products of finitely-generated submodules of V with W, as in Section 3.6. This implies that there is a finitely-generated submodule V_0 of V such that $v_1, \ldots, v_n \in V_0$ and (3.12.5) is equal to 0 in $V_0 \bigotimes_A W_0$, with $W_0 = W$. Similarly, one can obtain $V_0 \bigotimes_A W$ as a direct limit of tensor products of V_0 with finitely-generated submodules of W. This implies that there is a finitely-generated submodule W_0 of W such that (3.12.4) holds, and (3.12.5) is equal to 0 in $V_0 \bigotimes_A W_0$.

3.13 An interesting homomorphism

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, W be both left or both right modules over A, and let V_0, W_0 be submodules of V, W, respectively. Suppose that ϕ is a homomorphism from V into W, as modules over A, with

$$\phi(V_0) \subseteq W_0. \tag{3.13.1}$$

Thus the restriction ϕ_0 of ϕ to V_0 may be considered as a homomorphism from V_0 into W_0, as modules over A. Note that

$$\ker \phi_0 = (\ker \phi) \cap V_0. \tag{3.13.2}$$

Of course, the quotients V/V_0, W/W_0 may be defined as modules over A in the usual way. Let q_V, q_W be the natural quotient mappings from V, W onto V/V_0, W/W_0, respectively. Because of (3.13.1), there is a unique homomorphism ϕ_q from V/V_0 into W/W_0, as modules over A, such that

$$\phi_q \circ q_V = q_W \circ \phi, \tag{3.13.3}$$

as homomorphisms from V into W/W_0. In particular,

$$q_V(\ker \phi) \subseteq \ker \phi_q. \tag{3.13.4}$$

Because $\phi(V)$ is a submodule of W, the quotient

$$W/\phi(V) \tag{3.13.5}$$

is defined as a module over A. This is known as the *cokernel* of ϕ, as on p19 of [1], and p3 of [3]. Let q_ϕ be the natural quotient mapping from W onto $W/\phi(V)$. Similarly,

$$W_0/\phi_0(V_0) = W_0/\phi(V_0) \tag{3.13.6}$$

is the cokernel of ϕ_0, as a homomorphism from V_0 into W_0, and we let q_{ϕ_0} be the natural quotient mapping from W_0 onto $W_0/\phi_0(V_0)$.

Suppose that $v \in V$ and

$$q_V(v) \in \ker \phi_q. \tag{3.13.7}$$

This implies that

$$q_W(\phi(v)) = \phi_q(q_V(v)) = 0, \tag{3.13.8}$$

so that

$$\phi(v) \in W_0. \tag{3.13.9}$$

We would like to put

$$\Delta(q_V(v)) = q_{\phi_0}(\phi(v)), \tag{3.13.10}$$

which is an element of $W_0/\phi_0(V_0)$. If $u \in V$ and $u - v \in V_0$, so that $q_V(u) = q_V(v)$, then

$$\phi(u) - \phi(v) = \phi(u - v) \in \phi_0(V_0), \tag{3.13.11}$$

and thus $q_{\phi_0}(\phi(u)) = q_{\phi_0}(\phi(v))$. This shows that the right side of (3.13.10) does not depend on the choice of v.

It is easy to see that Δ defines a homomorphism from $\ker \phi_q$ into $W_0/\phi_0(V_0)$, as modules over A. Let us check that

$$\ker \Delta = q_V(\ker \phi). \tag{3.13.12}$$

If $v \in \ker \phi$, then (3.13.7) holds, by (3.13.4), and (3.13.10) is equal to 0. This shows that $q_V(\ker \phi)$ is contained in the kernel of Δ. If $v \in V$ satisfies (3.13.7), and (3.13.10) is equal to 0, then

$$\phi(v) \in \phi_0(V_0) = \phi(V_0). \tag{3.13.13}$$

This means that there is a $u \in V_0$ such that $\phi(v) = \phi(u)$, and thus $v - u \in \ker \phi$. It follows that

$$q_V(v) = q_V(v - u) \in q_V(\ker \phi), \tag{3.13.14}$$

which shows that the kernel of Δ is contained in $q_V(\ker \phi)$.

Let us verify that

$$\Delta(\ker \phi_q) = q_{\phi_0}(\phi(V) \cap W_0). \tag{3.13.15}$$

The fact that Δ maps $\ker \phi_q$ into the right side follows from the definition of Δ. If $v \in V$ and $\phi(v) \in W_0$, then

$$\phi_q(q_V(v)) = q_W(\phi(v)) = 0, \tag{3.13.16}$$

so that $q_V(v) \in \ker \phi_q$. This means that $\Delta(q_V(v))$ is as in (3.13.10), so that the right side of (3.13.15) is contained in the image of Δ.

There is a natural homomorphism

$$\text{from } W_0/\phi_0(V_0) \text{ into } W/\phi(V), \tag{3.13.17}$$

as modules over A, obtained from the obvious inclusion mapping from W_0 into W. This also uses the fact that $\phi_0(V_0) = \phi(V_0)$ is contained in $\phi(V)$. It is easy to see that the kernel of this homomorphism is equal to the right side of (3.13.15).

The quotient

$$(W/W_0)/\phi_q(V/V_0) \tag{3.13.18}$$

is defined as a module over A, because $\phi_q(V/V_0)$ is a submodule of W/W_0. Of course,

$$\phi_q(V/V_0) = \phi_q(q_V(V)) = q_W(\phi(V)), \tag{3.13.19}$$

using (3.13.3) in the second step. Thus (3.13.18) is the same as

$$(W/W_0)/q_W(\phi(V)). \tag{3.13.20}$$

There is a natural homomorphism

$$\text{from } W/\phi(V) \text{ onto } (W/W_0)/q_W(\phi(V)), \tag{3.13.21}$$

as modules over A, which is induced by q_W. The kernel of this homomorphism is equal to

$$q_\phi(W_0). \tag{3.13.22}$$

This is the same as the image of $W_0/\phi_0(V)$ in $W/\phi(V)$ under the natural homomorphism as in (3.13.17).

The remarks in this section correspond to Proposition 2.10 and its proof on p23 of [1], and are related to Lemmas 3.2 and 3.3 on p40 of [3].

3.14 An abstract construction

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. If V is a left module over A, then we would like to find a left module $D(V)$ over V that contains V as a submodule, and has the following property:

(3.14.1) if $\mathcal{I}$ is a left ideal in A, and ϕ is a homomorphism from $\mathcal{I}$ into V, then there is a $w \in D(V)$ such that $\phi(a) = a \cdot w$ for every $a \in \mathcal{I}$.

Here $\mathcal{I}$ may be considered as a left module over A, and ϕ is supposed to be a homomorphism from $\mathcal{I}$ into V, as left modules over A. This corresponds to the first part of the proof of Theorem 3.3 on p9 of [3].

Let $\mathcal{L}_A$ be the collection of all left ideals in A, and for each $\mathcal{I} \in \mathcal{L}_A$, let $\mathrm{Hom}_A(\mathcal{I}, V)$ be the space of all homomorphisms from $\mathcal{I}$ into V, as left modules over A. This is a module over k, and we let

(3.14.2) $\Phi_0(\mathcal{I})$ be a subset of $\mathrm{Hom}_A(\mathcal{I}, V)$ that generates $\mathrm{Hom}_A(\mathcal{I}, V)$,

as a module over k. Put

$$\Phi_0 = \bigcup_{\mathcal{I} \in \mathcal{L}_A} (\{\mathcal{I}\} \times \Phi_0(\mathcal{I})), \tag{3.14.3}$$

and let Z_0 be a free left module over A with a basis consisting of elements denoted $z_{(\mathcal{I},\phi)}$, with $(\mathcal{I}, \phi) \in \Phi_0$. Thus

$$Z_1 = V \bigoplus Z_0 \tag{3.14.4}$$

is a left module over A.

Let $(\mathcal{I}, \phi) \in \Phi_0$ be given. If $x \in \mathcal{I}$, then $\phi(x) \in V$, and

$$(\phi(x), -x\, z_{(\mathcal{I},\phi)}) \tag{3.14.5}$$

is an element of Z_1. If $a \in A$, then $a\, x \in \mathcal{I}$, and

$$a \cdot (\phi(x), -x\, z_{(\mathcal{I},\phi)}) = (a \cdot \phi(x), -a\, x\, z_{(\mathcal{I},\phi)}) = (\phi(a\, x), -a\, x\, z_{(\mathcal{I},\phi)}). \tag{3.14.6}$$

This implies that the collection of elements of Z_1 of the form (3.14.5) with $x \in \mathcal{I}$ is a submodule of Z_1, as a left module over A.

Let Y be the submodule of Z_1, as a left module over A, generated by elements of the form (3.14.5), with $(\mathcal{I}, \phi) \in \Phi_0$ and $x \in \mathcal{I}$. We would like to take

$$D(V) = Z_1/Y, \tag{3.14.7}$$

as a left module over A.

Suppose that $v \in V$ and

$$(v, 0) \in Y. \tag{3.14.8}$$

This means that there are finitely many distinct elements $(\mathcal{I}_j, \phi_j)$, $1 \le j \le n$, of Φ_0, and $x_j \in \mathcal{I}_j$ for each $j = 1, \ldots, n$, such that

$$(v, 0) = \sum_{j=1}^{n} (\phi_j(x_j), -x_j\, z_{(\mathcal{I}_j, \phi_j)}). \tag{3.14.9}$$

It follows that

$$\sum_{j=1}^{n} \phi_j(x_j) = v \tag{3.14.10}$$

and

$$\sum_{j=1}^{n} x_j\, z_{(\mathcal{I}_j, \phi_j)} = 0 \tag{3.14.11}$$

in Z_0. Because the $(\mathcal{I}_j, \phi_j)$'s are distinct elements of Φ_0, we get that $x_j = 0$ for each $j = 1, \ldots, n$. This implies that $v = 0$, by (3.14.10).

Consider the mapping from V into $D(V)$ that sends $v \in V$ to the image of $(v, 0)$ under the quotient mapping from Z_1 onto $D(V)$. This is a homomorphism from V into $D(V)$, as left modules over A, which is injective, by the remarks in the preceding paragraph. Thus we may identify V with its image in $D(V)$ under this mapping.

Let $(\mathcal{I}, \phi) \in \Phi_0$ be given, so that $(0, z_{(\mathcal{I},\phi)})$ is an element of Z_1. Let w be the image of $(0, z_{(\mathcal{I},\phi)})$ under the quotient mapping from Z_1 onto $D(V)$. If $a \in \mathcal{I}$, then $\phi(a) \in V$ is identified with the image of $(\phi(a), 0)$ in the quotient, as before. This is the same as the image of

$$(0, a\, z_{(\mathcal{I},\phi)}) = a \cdot (0, z_{(\mathcal{I},\phi)}) \tag{3.14.12}$$

in the quotient, by construction. Thus $\phi(a)$ is identified with $a \cdot w$ in $D(V)$.

This shows that (3.14.1) holds when $\phi \in \Phi_0(\mathcal{I})$. If ϕ is any homomorphism from $\mathcal{I}$ into V, as left modules over A, then ϕ can be expressed as a linear combination of elements of $\Phi_0(\mathcal{I})$ with coefficients in k. Using this, one can get the analogous statement for ϕ, by reducing to the previous case.

3.15 Another abstract construction

Let us continue with the same notation and hypotheses as in the previous section. We would like to show that there is an injective left module over A that contains V as a submodule, as in Theorem 3.3 on p9 of [3].

Let B be an infinite set whose cardinality is strictly larger than the cardinality of A, and which is as small as possible with these two properties. Suppose that B is well ordered by $\preceq$. We would like B to be "minimal" as a well-ordered set with this cardinality, in the sense that for each $\beta \in B$, the cardinality of

$$\{\alpha \in B : \alpha \preceq \beta,\ \alpha \ne \beta\} \tag{3.15.1}$$

is strictly less than the cardinality of B. If B does not have this property already, then there is a smallest $\beta \in B$ such that (3.15.1) has the same cardinality as B. In this case, we can replace B with (3.15.1), using the restriction of $\preceq$ to this set.

If $\beta \in B$, then we would like to define a left module $Q_\beta(V)$ over A as follows. If β is the smallest element of B, then we take $Q_\beta(V) = V$. Otherwise, (3.15.1) is not empty, and we suppose that $Q_\alpha(V)$ has been defined for every element α of this set. Suppose for the moment that (3.15.1) has a maximal element α, which means that β is the minimal element of B with $\alpha \preceq \beta$ and $\alpha \neq \beta$. Under these conditions, we take

$$Q_\beta(V) = D(Q_\alpha(V)). \tag{3.15.2}$$

If (3.15.1) has no maximal element, then we take

$$Q_\beta(V) = \bigcup \{Q_\alpha(V) : \alpha \in B,\ \alpha \preceq \beta,\ \alpha \neq \beta\}. \tag{3.15.3}$$

More precisely, if $\alpha_1, \alpha_2 \in B$, $\alpha_1 \preceq \alpha_2$, and $Q_{\alpha_1}(V)$, $Q_{\alpha_2}(V)$ have already been defined, then $Q_{\alpha_1}(V)$ should be a submodule of $Q_{\alpha_2}(V)$. This implies that (3.15.3) is a left module over A too, that contains $Q_\alpha(V)$ as a submodule when α is an element of (3.15.1). Thus we can define $Q_\beta(V)$ for every $\beta \in B$, and we put

$$Q(V) = \bigcup_{\beta \in B} Q_\beta(V). \tag{3.15.4}$$

This is a left module over A that contains $Q_\beta(V)$ as a submodule for every $\beta \in B$, as before.

In particular, $Q(V)$ contains V as a submodule, and we would like to show that $Q(V)$ is injective as a left module over A. To do this, let $\mathcal{I}$ be a left ideal in A, and let ϕ be a homomorphism from $\mathcal{I}$ into $Q(V)$, as left modules over A. If $x \in \mathcal{I}$, then we can choose $\alpha(x) \in B$ such that

$$\phi(x) \in Q_{\alpha(x)}(V). \tag{3.15.5}$$

One can check that the cardinality of

$$\bigcup_{x \in \mathcal{I}} \{\alpha \in B : \alpha \preceq \alpha(x)\} \tag{3.15.6}$$

is strictly less than the cardinality of B. More precisely, for each $x \in \mathcal{I}$, the cardinality of

$$\{\alpha \in B : \alpha \preceq \alpha(x)\} \tag{3.15.7}$$

is strictly less than the cardinality of B, by construction. If $\mathcal{I}$ has only finitely many elements, then one can choose $x \in \mathcal{I}$ so that $\alpha(x)$ is maximal, and use this condition on (3.15.7). If $\mathcal{I}$ is an infinite set, then $\mathcal{A}$ is an infinite set, and the cardinality of (3.15.7) is less than or equal to the cardinality of A for every $x \in \mathcal{I}$. This implies that the cardinality of (3.15.6) is less than or equal to the cardinality of $\mathcal{I} \times A$, which is equal to the cardinality of A.

It follows that (3.15.6) is a proper subset of B. If $\beta \in B$ is not in (3.15.6), then it is easy to see that

$$\alpha(x) \preceq \beta \tag{3.15.8}$$

for every $x \in \mathcal{I}$. This implies that

$$\phi(\mathcal{I}) \subseteq Q_\beta(V). \tag{3.15.9}$$

Observe that B has no maximal element. Let β_1 be the minimal element of B such that $\beta \preceq \beta_1$ and $\beta \neq \beta_1$. Thus $Q_{\beta_1}(V) = D(Q_\beta(V))$, by construction. It follows that there is a

$$w \in Q_{\beta_1}(V) \subseteq Q(V) \tag{3.15.10}$$

such that

$$\phi(a) = a \cdot w \tag{3.15.11}$$

for every $a \in \mathcal{I}$, as in the previous section. This implies that $Q(V)$ is injective as a left module over A, as in Section 2.8.

Chapter 4

Some associative algebras

4.1 Tensor products of algebras

Let k be a commutative ring with a multiplicative identity element, and let A_1, A_2 be associative algebras over k. In particular, A_1 and A_2 are modules over k, and we take $A = A_1 \bigotimes_k A_2$ to be a tensor product of A_1 and A_2, as modules over k. We would like to define multiplication on A in such a way that

$$(a_1 \otimes a_2)\,(b_1 \otimes b_2) = (a_1\,b_1) \otimes (a_2\,b_2) \tag{4.1.1}$$

for every $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$.

Note that

$$(a_1, a_2, b_1, b_2) \mapsto (a_1\,b_1) \otimes (a_2\,b_2) \tag{4.1.2}$$

is a multilinear mapping from $A_1 \times A_2 \times A_1 \times A_2$ into A, which is to say that it is linear over k in each variable. If tensor products of finitely many modules over k are defined in terms of multilinear mappings, then it follows that there is a unique homomorphism from a tensor product

$$A_1 \bigotimes_k A_2 \bigotimes_k A_1 \bigotimes_k A_2 \tag{4.1.3}$$

into A, as modules over k, with

$$a_1 \otimes a_2 \otimes b_1 \otimes b_2 \mapsto (a_1\,b_1) \otimes (a_2\,b_2) \tag{4.1.4}$$

for every $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$.

As in Section 1.4, there is a natural isomorphism between (4.1.3) and a tensor product

$$\Big(A_1 \bigotimes_k A_2\Big) \bigotimes_k \Big(A_1 \bigotimes_k A_2\Big) = A \bigotimes_k A, \tag{4.1.5}$$

where $a_1 \otimes a_2 \otimes b_1 \otimes b_2$ corresponds to $(a_1 \otimes a_2) \otimes (b_1 \otimes b_2)$ for every $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. Thus the module homomorphism from (4.1.3) into A mentioned in the preceding paragraph corresponds to a unique homomorphism from (4.1.5) into A, as modules over k, such that

$$(a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \mapsto (a_1\, b_1) \otimes (a_2\, b_2) \tag{4.1.6}$$

for every $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$.

Of course, $A \bigotimes_k A$ comes with a mapping from $A \times A$ that is bilinear over k. The composition of this mapping with the module homomorphism from (4.1.5) into A just mentioned is a bilinear mapping from $A \times A$ into A that can be used to define multiplication on A, and which satisfies (4.1.1), by construction.

Alternatively, if $b_1 \in A_1$ and $b_2 \in A_2$, then

$$(a_1, a_2) \mapsto (a_1\, b_1) \otimes (a_2\, b_2) \tag{4.1.7}$$

defines a mapping from $A_1 \times A_2$ into A that is bilinear over k. This leads to a unique homomorphism from $A = A_1 \bigotimes_k A_2$ into itself, as a module over k, such that

$$a_1 \otimes a_2 \mapsto (a_1\, b_1) \otimes (a_2\, b_2) \tag{4.1.8}$$

for every $a_1 \in A_1$ and $a_2 \in A_2$. It is easy to see that this homomorphism depends linearly on b_1 and b_2, by uniqueness.

If $a \in A$, then the value of the homomorphism just mentioned at a defines a mapping from $A_1 \times A_2$ into A, as a function of $b_1 \in A_1$ and $b_2 \in A_2$, that is bilinear over k. This leads to a unique homomorphism from A into itself, as a module over k, that sends $b_1 \otimes b_2$ to the value of the previous homomorphism depending on b_1 and b_2 at a for each $b_1 \in A_1$ and $b_2 \in A_2$. This homomorphism depends linearly on a, by uniqueness.

This defines a mapping from $A \times A$ into A that is bilinear over k and can be used to define multiplication on A, where (4.1.1) holds by construction. One can check that multiplication on A is associative, because multiplication is associative on A_1 and A_2, by hypothesis. Similarly, if multiplication is commutative on A_1 and A_2, then multiplication is commutative on A.

If A_1 and A_2 have multiplicative identity elements e_1 and e_2, respectively, then one can verify that $e_1 \otimes e_2$ is the multiplicative identity element in A. In this case,

$$a_1 \mapsto a_1 \otimes e_2 \tag{4.1.9}$$

and

$$a_2 \mapsto e_1 \otimes a_2 \tag{4.1.10}$$

are algebra homomorphisms from A_1 and A_2 into A, respectively. Note that

$$(a_1 \otimes e_2)\, (e_1 \otimes a_2) = (e_1 \otimes a_2)\, (a_1 \otimes e_2) = a_1 \otimes a_2 \tag{4.1.11}$$

for every $a_1 \in A_1$ and $a_2 \in A_2$.

Let C be another associative algebra over k, and let ϕ_1, ϕ_2 be algebra homomorphisms from A_1, A_2 into C, respectively. Suppose that

$$\phi_1(a_1)\, \phi_2(a_2) = \phi_2(a_2)\, \phi_1(a_1) \tag{4.1.12}$$

for every $a_1 \in A_1$ and $a_2 \in A_2$. Observe that

$$(a_1, a_2) \mapsto \phi_1(a_1)\,\phi_2(a_2) \tag{4.1.13}$$

is bilinear over k as a mapping from $A_1 \times A_2$ into C. This leads to a unique homomorphism ϕ from A into C, as modules over k, such that

$$\phi(a_1 \otimes a_2) = \phi_1(a_1) \otimes \phi_2(a_2) \tag{4.1.14}$$

for every $a_1 \in A_1$ and $a_2 \in A_2$. One can check that ϕ is an algebra homomorphism from A into C under these conditions.

4.2 Modules over tensor products

Let k be a commutative ring with a multiplicative identity element, and let A_1, A_2 be associative algebras over k, with multiplicative identity elements e_1, e_2, respectively. Also let V be a module over k. Suppose for the moment that V is a left module over each of A_1 and A_2, and that the actions of A_1 and A_2 on V commute with each other. This means that

$$a_1 \cdot_{A_1} (a_2 \cdot_{A_2} v) = a_2 \cdot_{A_2} (a_1 \cdot_{A_1} v) \tag{4.2.1}$$

for every $a_1 \in A_1$, $a_2 \in A_2$, and $v \in V$, where $\cdot_{A_1}$ and $\cdot_{A_2}$ refer to the actions of A_1 and A_2 on V.

If $v \in V$, then the mapping from $(a_1, a_2) \in A_1 \times A_2$ to (4.2.1) is bilinear over k. Let $A = A_1 \bigotimes_k A_2$ be a tensor product of A_1 and A_2, as modules over k. It follows that there is a unique homomorphism from A into V, as modules over k, that maps $a_1 \otimes a_2$ to (4.2.1) for every $a_1 \in A_1$ and $a_2 \in A_2$. If $a \in A$, then let $a \cdot v = a \cdot_A v$ be the image of a under this mapping. Using this notation, we have that

$$(a_1 \otimes a_2) \cdot_A v = a_1 \cdot_{A_1} (a_2 \cdot_{A_2} v) = a_2 \cdot_{A_2} (a_1 \cdot_{A_1} v) \tag{4.2.2}$$

for every $a_1 \in A_1$ and $a_2 \in A_2$.

Of course, one can do this for every $v \in V$, so that $a \cdot_A v$ is defined as an element of V for every $a \in A$ and $v \in V$. Remember that A may be considered as an associative algebra over k, as in the previous section. One can check that this makes V into a left module over A, as on p163 of [3].

Conversely, if V is a left module over A, then one can get commuting actions of A_1 and A_2 on V, as on p163 of [3]. More precisely, if $a_1 \in A_1$, $a_2 \in A_2$, and $v \in V$, then the actions of a_1 and a_2 on v are defined to be the actions of $a_1 \otimes e_2$ and $e_1 \otimes a_2$ on V, using the given action of A on V.

There are analogous statements for commuting right actions of A_1 and A_2 on V, as on p163 of [3].

Suppose now that V is a left module over A_1 and a right module over A_2, and that the actions of A_1 and A_2 on V commute with each other. Thus

$$a_1 \cdot_{A_1} (v \cdot_{A_2} a_2) = (a_1 \cdot_{A_1} v) \cdot_{A_2} a_2 \tag{4.2.3}$$

for every $a_1 \in A_1$, $a_2 \in A_2$, and $v \in V$. Let A_2^{op} be the opposite algebra associated to A_2, as in Section 1.11. Remember that we may use a_2^{op} to indicate that $a_2 \in A_2$ is being considered as an element of A_2^{op}. We may consider V as a left module over A_2^{op}, with

$$a_2^{op} \cdot_{A_2^{op}} v = v \cdot_{A_2} a_2 \tag{4.2.4}$$

for every $a_2 \in A_2$ and $v \in V$, as before.

Under these conditions, we may consider V to be a left module over a tensor product $A_1 \bigotimes_k A_2^{op}$ of A_1 and A_2^{op}, as on p163 of [3] again. Similarly, we may consider V to be a right module over a tensor product of A_1^{op} and A_2, as in [3].

4.3 Formal power series

Let k be a commutative ring with a multiplicative identity element, and let V be a module over k. Also let n be a positive integer, and let $T_1, \ldots, T_n$ be commuting indeterminates. As on p93 of [9], we normally try to use upper-case letters for indeterminates, and lower-case letters for elements of rings or modules.

By a *multi-index* of length n we mean an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, and we put

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{4.3.1}$$

The corresponding formal monomial

$$T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n} \tag{4.3.2}$$

in $T_1, \ldots, T_n$ has degree $|\alpha|$. We use $\mathbf{Z}_+$ for the set of all positive integers, so that $(\mathbf{Z}_+ \cup \{0\})^n$ is the set of all multi-indices of length n.

A *formal power series* in $T_1, \ldots, T_n$ with coefficients in V can be expressed as

$$f(T) = f(T_1, \ldots, T_n) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha, \tag{4.3.3}$$

where $f_\alpha \in V$ for every multi-index α. The space $V[[T_1, \ldots, T_n]]$ of all such formal power series can be defined as the space of all V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$. This is a module over k with respect to pointwise addition and scalar multiplication of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$, which corresponds to termwise addition and scalar multiplication of formal power series as in (4.3.3).

The space $V[T_1, \ldots, T_n]$ of *formal polynomials* in $T_1, \ldots, T_n$ with coefficients in V may be defined as the subset of $V[[T_1, \ldots, T_n]]$ consisting of formal power series as in (4.3.3) such that $f_\alpha = 0$ for all but finitely many multi-indices α. More precisely, this may be defined as the space of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ with finite support. This is a submodule of $V[[T_1, \ldots, T_n]]$, as a module over k. We may identify V with the submodule of $V[T_1, \ldots, T_n]$ consisting of $f(T)$ as in (4.3.3) such that $f_\alpha = 0$ when $\alpha \neq 0$.

Let A be an associative algebra over k, let $f(T) \in A[[T_1, \ldots, T_n]]$ be as in (4.3.3), and let

$$g(T) = \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} g_\beta \, T^\beta \tag{4.3.4}$$

be another element of $A[[T_1, \ldots, T_n]]$. If γ is a multi-index of length n, then put

$$h_\gamma = \sum_{\substack{\alpha, \beta \in (\mathbf{Z}_+ \cup \{0\})^n \\ \alpha + \beta = \gamma}} f_\alpha \, g_\beta. \tag{4.3.5}$$

Note that the sum on the right has only finitely many terms, and thus defines an element of A. This means that

$$h(T) = \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} h_\gamma \, T^\gamma \tag{4.3.6}$$

defines an element of $A[[T_1, \ldots, T_n]]$, and we put

$$f(T) \, g(T) = h(T). \tag{4.3.7}$$

One can check that $A[[T_1, \ldots, T_n]]$ is an associative algebra over k with respect to this definition of multiplication. If multiplication on A is commutative, then $A[[T_1, \ldots, T_n]]$ is a commutative algebra too. It is easy to see that $A[T_1, \ldots, T_n]$ is a subalgebra of $A[[T_1, \ldots, T_n]]$, which contains A as a subalgebra. If A has a multiplicative identity element e, then the corresponding formal power series is the multiplicative identity element in $A[[T_1, \ldots, T_n]]$. In particular, $k[[T_1, \ldots, T_n]]$ is a commutative associative algebra over k.

If $f(T) \in k[[T_1, \ldots, T_n]]$ is as in (4.3.3), and $g(T) \in V[[T_1, \ldots, T_n]]$ is as in (4.3.4), then (4.3.5) defines an element of V for each multi-index γ, using scalar multiplication on V. Under these conditions, we can define $h(T)$ as an element of $V[[T_1, \ldots, T_n]]$ as in (4.3.6), so that $f(T) \, g(T)$ can be defined as in (4.3.7). One can verify that $V[[T_1, \ldots, T_n]]$ is a module over $k[[T_1, \ldots, T_n]]$ in this way. If $f(T) \in k[T_1, \ldots, T_n]$ and $g(T) \in V[T_1, \ldots, T_n]$, then $h(T)$ is an element of $V[T_1, \ldots, T_n]$, which makes $V[T_1, \ldots, T_n]$ a module over $k[T_1, \ldots, T_n]$.

Similarly, let A be an associative algebra over k with a multiplicative identity element e, and suppose that V is a left or right module over A. As before, the action of A on V can be extended to an action of $A[[T_1, \ldots, T_n]]$ on $V[[T_1, \ldots, T_n]]$, so that $V[[T_1, \ldots, T_n]]$ becomes a left or right module over $A[[T_1, \ldots, T_n]]$. One may also consider $V[T_1, \ldots, T_n]$ as a left or right module over $A[T_1, \ldots, T_n]$, as appropriate, in the same way.

Consider the mapping from $A[[T_1, \ldots, T_n]]$ onto A defined by

$$f(T) \mapsto f_0, \tag{4.3.8}$$

where $f(T)$ is as in (4.3.3). It is easy to see that this defines a homomorphism from $A[[T_1, \ldots, T_n]]$ onto A, as algebras over k.

4.4 Polynomial functions

Let k be a commutative ring with a multiplicative identity element, and let n be a positive integer. If V is a module over k, then let V^n be the space of n-tuples of elements of V, which is a module over k with respect to coordinatewise addition and scalar multiplication.

If $t = (t_1, \ldots, t_n) \in k^n$ and α is a multi-index of length n, then t^α is defined as an element of k by

$$t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}, \tag{4.4.1}$$

where $t_j^{\alpha_j}$ is interpreted as being the multiplicative identity element 1 in k when $\alpha_j = 0$, as usual. If β is another multi-index of length n, then

$$t^{\alpha+\beta} = t^\alpha \, t^\beta. \tag{4.4.2}$$

Let $T_1, \ldots, T_n$ be n commuting indeterminates, and let V be a module over k again. Also let $f(T)$ be a formal polynomial in $T_1, \ldots, T_n$ with coefficients in V, as in (4.3.3). If $t \in k^n$, then

$$f(t) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, t^\alpha \tag{4.4.3}$$

defines an element of V, where all but finitely many terms in the sum on the right are equal to 0, by hypothesis. Of course,

$$f(T) \mapsto f(t) \tag{4.4.4}$$

is a homomorphism from $V[T_1, \ldots, T_n]$ into V, as modules over k.

Let A be an associative algebra over k, so that $A[T_1, \ldots, T_n]$ is an associative algebra over k too, as in the previous section. If $f(T), g(T) \in A[T_1, \ldots, T_n]$, $h(T) = f(T) \, g(T)$, and $t \in k^n$, then one can check that

$$h(t) = f(t) \, g(t). \tag{4.4.5}$$

Let V be a module over k again, and remember that $V[T_1, \ldots, T_n]$ may be considered as a module over $k[T_1, \ldots, T_n]$. Let $f(T) \in k[T_1, \ldots, T_n]$ and $g(T) \in V[T_1, \ldots, T_n]$ be given, so that $h(T) = f(T) \, g(T) \in V[T_1, \ldots, T_n]$. If $t \in k^n$, then $f(t)$ is defined as an element of k, $g(t)$ and $h(t)$ are defined as elements of V, and one can verify that (4.4.5) holds.

Let A be an associative algebra over k with a multiplicative identity element e, and suppose that V is a left or right module over A. If one of $f(T)$, $g(T)$ is in $A[T_1, \ldots, T_n]$ and the other is in $V[T_1, \ldots, T_n]$, as appropriate, then $h(T) = f(T) \, g(T) \in V[T_1, \ldots, T_n]$, as in the previous section. If $t \in k^n$, then $f(t)$, $g(t)$ are defined as elements of A or V, as appropriate, $h(t) \in V$, and (4.4.5) holds.

Suppose that $a = (a_1, \ldots, a_n) \in A$ has commuting coordinates, so that

$$a_j \, a_l = a_l \, a_j \tag{4.4.6}$$

for every $j, l = 1, \ldots, n$, which holds automatically when $n = 1$. If α is a multi-index of length n, then

$$a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \tag{4.4.7}$$

defines an element of A, where $a_j^{\alpha_j}$ is interpreted as being equal to e when $\alpha_j = 0$. If β is another multi-index of length n, then

$$a^{\alpha+\beta} = a^\alpha \, a^\beta. \tag{4.4.8}$$

Let $f(T)$ be a formal polynomial in $T_1, \ldots, T_n$ with coefficients in k, as in (4.3.3) again. Under these coditions, $f(a)$ is defined as an element of A by

$$f(a) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, a^\alpha, \tag{4.4.9}$$

where all but finitely many terms in the sum on the right are equal to 0, by hypothesis. If $g(T) \in k[T_1, \ldots, T_n]$ and $h(T) = f(T)\, g(T)$, then one can check that

$$h(a) = f(a)\, g(a). \tag{4.4.10}$$

Thus $f(T) \mapsto f(a)$ defines an algebra homomorphism from $k[T_1, \ldots, T_n]$ into A.

4.5 Invertibility in $A[[T_1, \ldots, T_n]]$

Let k be a commutative ring with a multiplicative identity element, let n be a positive integer, and let $T_1, \ldots, T_n$ be n commuting indeterminates. Also let V be a module over k, let $f(T) \in V[[T_1, \ldots, T_n]]$ be as in (4.3.3), and let N be a nonnegative integer. Let us say that $f(T)$ *vanishes to order* N if

$$f_\alpha = 0 \tag{4.5.1}$$

for every multi-index α of length n with $|\alpha| \leq N$. One may consider this to hold trivially when $N = -1$.

Suppose that $f(T) \in k[[T_1, \ldots, T_n]]$ and $g(T) \in V[[T_1, \ldots, T_n]]$ vanish to order $N_1, N_2 \geq -1$, respectively. Under these conditions, one can check that

$$f(T)\, g(T) \text{ vanishes to order } N_1 + N_2 + 1. \tag{4.5.2}$$

Let A be an associative algebra over k with a multiplicative identity element e. If $f(T), g(T) \in A[[T_1, \ldots, T_n]]$ vanish to order $N_1, N_2 \geq -1$, respectively. then (4.5.2) holds, as before.

Suppose that $a(T) \in A[[T_1, \ldots, T_n]]$ vanishes to order 0, and let j be a nonnegative integer. Observe that

$$a(T)^j \text{ vanishes to order } j - 1. \tag{4.5.3}$$

Let l be a nonnegative integer, and consider

$$\sum_{j=0}^{l} a(T)^j. \tag{4.5.4}$$

If α is a multi-index of length n, then the coefficient of T^α in (4.5.4) does not depend on l when

$$l \ge |\alpha|, \tag{4.5.5}$$

because of (4.5.3). In this case, we can define

$$\sum_{j=0}^{\infty} a(T)^j \tag{4.5.6}$$

as an element of $A[[T_1, \ldots, T_n]]$ by saying that for each multi-index α of length n, the coefficient of T^α in (4.5.6) is the same as in (4.5.4) when (4.5.5) holds.

It is easy to see that

$$(e - a(T)) \sum_{j=0}^{l} a(T)^j = \Big(\sum_{j=0}^{l} a(T)^j\Big) (e - a(T)) = e - a(T)^{l+1} \tag{4.5.7}$$

for every $l \ge 0$, by a standard argument. One can use this to check that

$$(e - a(T)) \sum_{j=0}^{\infty} a(T)^j = \Big(\sum_{j=0}^{\infty} a(T)^j\Big) (e - a(T)) = e. \tag{4.5.8}$$

This shows that $e - a(T)$ is invertible in $A[[T_1, \ldots, T_n]]$, with inverse equal to (4.5.6).

Let $f(T) \in A[[T_1, \ldots, T_n]]$ be as in (4.3.3), and suppose that f_0 is invertible in A. This implies that $f(T)$ can be expressed as

$$f(T) = f_0\,(e - a(T)), \tag{4.5.9}$$

where $a(T) \in A[[T_1, \ldots, T_n]]$ vanishes to order 0. It follows that $f(T)$ is invertible in $A[[T_1, \ldots, T_n]]$, by the remarks in the preceding paragraph.

Conversely, if $f(T)$ is invertible in $A[[T_1, \ldots, T_n]]$, then f_0 is invertible in A. This follows from the fact that $f(T) \mapsto f_0$ is an algebra homomorphism from $A[[T_1, \ldots, T_n]]$ onto A, as in Section 4.3.

4.6 Partial derivatives

Let n be a positive integer, let α be a multi-index of length n, and let l be an integer with $1 \le l \le n$. Under these conditions, we define the multi-index $\alpha(l)$ of length n by

$$\begin{aligned} \alpha_j(l) &= \alpha_j && \text{when } j \ne l \\ &= \alpha_l - 1 && \text{when } j = l \text{ and } \alpha_l \ge 1 \\ &= 0 && \text{when } j = l \text{ and } \alpha_l = 0. \end{aligned} \tag{4.6.1}$$

Similarly, the multi-index $\alpha^+(l)$ of length n is defined by

$$\begin{aligned} \alpha_j^+(l) &= \alpha_j && \text{when } j \neq l \\ &= \alpha_l + 1 && \text{when } j = l. \end{aligned} \tag{4.6.2}$$

Let k be a commutative ring with a multiplicative identity element, let V be a module over k, and let $T_1, \ldots, T_n$ be n commuting indeterminates. If $f(T) \in V[[T_1, \ldots, T_n]]$ is as in (4.3.3), then the formal *partial derivative* of $f(T)$ in T_l can be defined as an element of $V[[T_1, \ldots, T_n]]$ by

$$\partial_l f(T) = \frac{\partial}{\partial T_l} f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} (\alpha_l + 1) \cdot f_{\alpha^+(l)} \, T^\alpha. \tag{4.6.3}$$

This is essentially the same as

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \alpha_l \cdot f_\alpha \, T^{\alpha(l)} = \sum_{\substack{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \\ \alpha_l \geq 1}} \alpha_l \cdot f_\alpha \, T^{\alpha(l)}. \tag{4.6.4}$$

Clearly $f(T) \mapsto \partial_l f(T)$ is a homomorphism from $V[[T_1, \ldots, T_n]]$ into itself, as a module over k, which maps $V[T_1, \ldots, T_n]$ into itself. One can verify that

$$\partial_l(\partial_m f(T)) = \partial_m(\partial_l f(T)) \tag{4.6.5}$$

for every $l, m = 1, \ldots, n$ and $f(T) \in V[[T_1, \ldots, n]]$.

Let A be an associative algebra over k, so that $A[[T_1, \ldots, T_n]]$ is an associative algebra over k too, as in Section 4.3. If $f(T), g(T) \in A[[T_1, \ldots, T_n]]$, then one can check that

$$\partial_l(f(T)\, g(T)) = (\partial_l f(T))\, g(T) + f(T)\, \partial_l g(T) \tag{4.6.6}$$

for every $l = 1, \ldots, n$. If $f(T) \in k[[T_1, \ldots, T_n]]$ and $g(T) \in V[[T_1, \ldots, T_n]]$, then $f(T)\, g(T) \in V[[T_1, \ldots, T_n]]$, as in Section 4.3, and satisfies (4.6.6). If A has a multiplicative identity element e, and V is a left or right module over A, then $V[[T_1, \ldots, T_n]]$ is a left or right module over A too, as before. One can verify that the analogue of (4.6.6) holds in this case as well.

Let $t \in k^n$ be given, and suppose that $u \in k^n$ satisfies

$$u_j\, u_l = 0 \tag{4.6.7}$$

for each $j, l = 1, \ldots, n$. Let α be a multi-index of length n, so that t^α and $(t+u)^\alpha$ are defined as elements of k as in Section 4.4. It is easy to see that

$$(t_l + u_l)^{\alpha_l} = t_l^{\alpha_l} + \alpha_l \cdot t_l^{\alpha_l - 1}\, u_l \tag{4.6.8}$$

for each $l = 1, \ldots, n$ with $\alpha_l \geq 1$, because $u_l^2 = 0$. This implies that

$$(t_l + u_l)^{\alpha_l} = t_l^{\alpha_l} + \alpha_l \cdot t_l^{\alpha_l(l)}\, u_l \tag{4.6.9}$$

for every $l = 1, \ldots, n$, which is automatic when $\alpha_l = 0$. It follows that

$$(t+u)^\alpha = t^\alpha + \sum_{l=1}^n \alpha_l \cdot t^{\alpha(l)}\, u_l. \tag{4.6.10}$$

If V is a module over k and $f(T) \in V[T_1, \ldots, T_n]$, then we get that

$$f(t+u) = f(t) + \sum_{l=1}^n (\partial_l f)(t)\, u_l. \tag{4.6.11}$$

Here $f(t)$, $f(t+u)$, and $(\partial_l f)(t)$ are defined as elements of V as in Section 4.4.

Let A be an associative algebra over k with a multiplicative identity element e, and suppose that $a \in A^n$ has commuting coordinates. Also let u be an element of A^n that satisfies (4.6.7), and whose coordinates commute with the coordinates of a, so that

$$a_j\, u_l = u_l\, a_j \tag{4.6.12}$$

for every $j, l = 1, \ldots, n$. Note that $a + u$ has commuting coordinates, because u has commuting coordinates. Let α be a multi-index of length n again, so that a^α and $(a+u)^\alpha$ are defined as elements of A. As in (4.6.9), we have that

$$(a_l + u_l)^{\alpha_l} = a_l^{\alpha_l} + \alpha_l \cdot a_l^{\alpha_l(l)}\, u_l \tag{4.6.13}$$

for each $l = 1, \ldots, n$. This means that

$$(a+u)^\alpha = a^\alpha + \sum_{l=1}^n \alpha_l \cdot a^{\alpha(l)}\, u_l, \tag{4.6.14}$$

as before. If $f(T) \in k[T_1, \ldots, T_n]$, then it follows that

$$f(a+u) = f(a) + \sum_{l=1}^n (\partial_l f)(a)\, u_l, \tag{4.6.15}$$

where $f(a)$, $f(a+u)$, and $(\partial_l f)(a)$ are defined as elements of A as in Section 4.4.

4.7 Algebras of module homomorphisms

Let k be a commutative ring with a multiplicative identity element, and let V be a module over k. The space $\mathrm{Hom}_k(V, V)$ of homomorphisms from V into itself, as a module over k, is an associative algebra over k, with respect to composition of mappings. Of course, the identity mapping on V is the multiplicative identity element in $\mathrm{Hom}_k(V, V)$.

Let A be an associative algebra over k with a multiplicative identity element e_A. If V is a left module over A, then the space $\mathrm{Hom}_A(V, V)$ of homomorphisms from V into itself, as a left module over A, is a subalgebra of $\mathrm{Hom}_k(V, V)$, as an algebra over k. Similarly, if V is a right module over A, then the space $\mathrm{Hom}_A(V,V)$ of homomorphisms from V into itself, as a right module over A, is a subalgebra of $\mathrm{Hom}_k(V,V)$, as an algebra over k. Note that the identity mapping on V is an element of $\mathrm{Hom}_A(V,V)$ in both cases.

A *representation* of A on a module V over k may be defined as an algebra homomorphism from A into $\mathrm{Hom}_k(V,V)$ that sends e_A to the identity mapping on V. Thus a representation of A on V corresponds exactly to an action of A on V that makes V a left module over A. Similarly, an opposite algebra homomorphism from A into $\mathrm{Hom}_k(V,V)$ that sends e_A to the identity mapping on V corresponds exactly to an action of A on V that makes V a right module over A.

Let n be a positive integer, and let A^n be the space of n-tuples of elements of A. This is a module over k with respect to coordinatewise addition and scalar multiplication, and both a left and right module over A with respect to left and right multiplication by elements of A coordinatewise on A^n. Note that these actions of A on A^n on the left and the right commute with each other, by associativity.

Let $\alpha = (\alpha_{j,l})$ be an $n \times n$ matrix with entries in A. If $x \in A^n$, then let $T^L_\alpha(x)$ and $T^R_\alpha(x)$ be the elements of A^n whose jth coordinates are given by

$$(T^L_\alpha(x))_j = \sum_{l=1}^{n} \alpha_{j,l}\, x_l \tag{4.7.1}$$

and

$$(T^R_\alpha(x))_j = \sum_{l=1}^{n} x_l\, \alpha_{j,l}, \tag{4.7.2}$$

respectively, for every $j = 1, \ldots, n$. These define homomorphisms from A^n into itself as a module over k, which are the same when A is commutative. More precisely, T^L_α is a homomorphism from A^n into itself, as a right module over A, and T^R_α is a homomorphism from A^n into itself, as a left module over A.

One can check that every homomorphism from A^n into itself, as a right module over A, can be expressed as T^L_α for a unique matrix α. Similarly, every homomorphism from A^n, as a left module over itself, can be expressed as T^R_α for a unique matrix α. This uses the fact that A^n is freely generated by the elements with one coordinate equal to e_A and the other coordinates equal to 0, as a left or right module over A.

One can verify that this defines an isomorphism from the algebra $M_n(A)$ of $n \times n$ matrices with entries in A onto the algebra of homomorphisms from A^n onto itself, as a right module over A. Similarly, we get an isomorphism from the algebra $M_n(A^{op})$ of $n \times n$ matrices with entries in the opposite algebra A^{op} of A onto the algebra of homomorphisms from A^n into itself, as a left module over A.

4.8 Modules over semigroups

Let k be a commutative ring with a multiplicative identity element, and let Σ be a semigroup, with the semigroup operation expressed multiplicatively, and with an identity element e_Σ. Also let V be a module over k. Suppose that for every $x \in \Sigma$ and $v \in V$, $x \cdot v$ is defined as an element of V, and that

$$v \mapsto x \cdot v \tag{4.8.1}$$

is a homomorphism from V into itself, as a module over k. Suppose too that

$$x \cdot (y \cdot v) = (x\, y) \cdot v \tag{4.8.2}$$

for every $x, y \in \Sigma$ and $v \in V$, and that

$$e_\Sigma \cdot v = v \tag{4.8.3}$$

for every $v \in V$. Under these conditions, V is said to be a *left module over* Σ.

Equivalently, the action of Σ on V on the left corresponds to a mapping from Σ into the algebra $\mathrm{Hom}_k(V, V)$ of all homomorphisms from V into itself, as a module over k. The condition (4.8.2) says that this mapping is a homomorphism from Σ into $\mathrm{Hom}_k(V, V)$, as a semigroup with respect to composition of mappings. Similarly, (4.8.3) says that this mapping sends e_Σ to the identity mapping on V. A mapping from Σ into $\mathrm{Hom}_k(V, V)$ with these properties is also known as a *representation* of Σ on V.

Suppose now that for every $x \in \Sigma$ and $v \in V$, $v \cdot x$ is defined as an element of V, and that

$$v \mapsto v \cdot x \tag{4.8.4}$$

is a homomorphism from V into itself, as a module over k. Suppose in addition that

$$(v \cdot x) \cdot y = v \cdot (x\, y) \tag{4.8.5}$$

for every $x, y \in \Sigma$ and $v \in V$, and that

$$v \cdot e_\Sigma = v \tag{4.8.6}$$

for every $v \in V$. In this case, V is said to be a *right module over* Σ.

The *opposite semigroup* Σ^{op} associated to Σ is defined to be the same as Σ as a set, with the product of $x, y \in \Sigma^{op}$ equal to the product $y\, x$ of y and x in Σ. If $x \in \Sigma$, then it may be helpful to use x^{op} to indicate that x is being considered as an element of Σ^{op}, as in Section 1.11. Thus multiplication in Σ^{op} is given by

$$x^{op}\, y^{op} = (y\, x)^{op}, \tag{4.8.7}$$

as before. Note that e_Σ is the identity element in Σ^{op} as well. A homomorphism from Σ^{op} into another semigroup may be called an *opposite semigroup homomorphism* from Σ into the other semigroup.

A mapping from Σ into $\mathrm{Hom}_k(V, V)$ can be used to define an action of Σ on V on the right, as in (4.8.4). Observe that (4.8.5) is the same as saying that this mapping is an opposite semigroup homomorphism, with respect to composition of mappings on $\mathrm{Hom}_k(V, V)$. Of course, (4.8.6) is the same as saying that this mapping sends e_Σ to the identity mapping on V. This means that a right module over Σ corresponds to a representation of Σ^{op}. A left or right module over Σ may be considered as a right or left module over Σ^{op}, respectively.

4.9 Semigroup algebras

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let Σ be a semigroup, with the semigroup operation expressed multiplicatively, and with an identity element e_Σ. The corresponding *semigroup algebra* $A(\Sigma)$ of Σ over A basically consists of finite formal sums of the form

$$a_1\, x_1 + \cdots + a_n\, x_n, \tag{4.9.1}$$

where $a_j \in A$ and $x_j \in \Sigma$ for every $j = 1, \ldots, n$.

More precisely, $A(\Sigma)$ can be defined as the space $c_{00}(\Sigma, A)$ of A-valued functions on Σ with finite support. This is a module over k with respect to pointwise addition and scalar multiplication, and both a left and right module over A with respect to pointwise multiplication on the left and right by elements of A. An element x of Σ may be identified with the A-valued function on Σ equal to e_A at x, and equal to 0 at every other element of Σ.

Multiplication on Σ can be extended to $A(\Sigma)$ in such a way that it is bilinear over k, and

$$(a\, x)\, (b\, y) = (a\, b)\, (x\, y) \tag{4.9.2}$$

for every $a, b \in A$ and $x, y \in \Sigma$, as on p148 of [3]. It is easy to see that $A(\Sigma)$ is an associative algebra over k with respect to this definition of multiplication.

The element of $A(\Sigma)$ corresponding to e_Σ is the multiplicative identity element of $A(\Sigma)$. If we identify $a \in A$ with $a\, e_\Sigma \in A(\Sigma)$, then A corresponds to a subalgebra of $A(\Sigma)$.

Let V be a module over k, and suppose for the moment that V is a left module over $A(\Sigma)$. In particular, this means that V is a left module over A, because A corresponds to a subalgebra of $A(\Sigma)$. If $x \in \Sigma$, then $v \mapsto x \cdot v$ defines a homomorphism from V into itself, as a module over k, and as a left module over A, by identifying x with an element of $A(\Sigma)$ as before. This makes V a left module over Σ, as in the previous section. The condition that the actions of elements of Σ be homomorphisms from V into itself, as a left module over A, means that the actions of Σ and A on V commute with each other.

Conversely, suppose that V is a left module over Σ and A, and that the actions of Σ and A commute with each other. Under these conditions, one can define an action of $A(\Sigma)$ on V, so that V becomes a left module over $A(\Sigma)$, as an associative algebra over k. This corresponds to some remarks on p149 of [3].

Similarly, if V is a right module over $A(\Sigma)$, then V is a right module over Σ and A, and the actions of Σ and A on V commute with each other. Conversely, if V is a right module over Σ and A, and the actions of Σ and A commute with each other, then one can define an action of $A(\Sigma)$ on V, so that V becomes a right module over $A(\Sigma)$, as an associative algebra over k.

Suppose that V is a left module over A, so that the space $\mathrm{Hom}_A(V, V)$ of homomorphisms from V into itself, as a left module over A, is a subalgebra of $\mathrm{Hom}_k(V, V)$, as an algebra over k. An action of Σ on V by elements of $\mathrm{Hom}_A(V, V)$ that makes V a left module over Σ corresponds exactly to a homomorphism from Σ into $\mathrm{Hom}_A(V, V)$, as a semigroup with respect to composition, that sends e_Σ to the identity mapping on V.

Similarly, if V is a right module over A, then the space $\mathrm{Hom}_A(V, V)$ of homomorphisms from V into itself, as a right module over A, is a subalgebra of $\mathrm{Hom}_k(V, V)$. An action of Σ on A by elements of $\mathrm{Hom}_A(V, V)$ that makes V a right module over Σ corresponds exactly to an opposite semigroup homomorphism from Σ into $\mathrm{Hom}_A(V, V)$ that sends e_Σ to the identity mapping on V.

Let B be another associative algebra over k, with a multiplicative identity element e_B. Suppose that ϕ_A is a homomorphism from A into B, as algebras over k, such that $\phi_A(e_A) = e_B$. Let ϕ_Σ be a homomorphism from Σ into B, as a semigroup with respect to multiplication, such that $\phi_\Sigma(e_\Sigma) = e_B$. Suppose also that

$$\phi_A(a)\,\phi_\Sigma(x) = \phi_\Sigma(x)\,\phi_A(a) \tag{4.9.3}$$

for every $a \in A$ and $x \in \Sigma$. Under these conditions, there is a unique homomorphism $\phi_{A(\Sigma)}$ from $A(\Sigma)$ into B, as algebras over k, such that

$$\phi_{A(\Sigma)}(a\,x) = \phi_A(a)\,\phi_\Sigma(x) \tag{4.9.4}$$

for every $a \in A$ and $x \in \Sigma$.

Note that $A(\Sigma)$ satisfies the conditions on B mentioned in the preceding paragraph. This uses the natural mappings from A and Σ into $A(\Sigma)$ as ϕ_A and ϕ_Σ, respectively.

4.10 Free semigroups

Let E be a nonempty set, and for each positive integer n, let E^n be the nth Cartesian power of E, consisting of n-tuples of elements of E. We may consider the elements of E^n as formal products of n elements of E, which may be expressed as strings of n elements of E. We take E^0 to be a set with a single element, denoted e_Σ, representing the empty string.

Thus

$$\Sigma = \Sigma(E) = \bigcup_{n=0}^{\infty} E^n \tag{4.10.1}$$

consists of all finite formal products of elements of E, or finite strings of elements of E, including the empty string. This is a semigroup with respect to formal products, which corresponds to concatenation of finite strings of elements of E.

By construction, e_Σ is the identity element in Σ. This is the *free semigroup* generated by E.

Let Σ_1 be a semigroup with identity element e_{Σ_1}. Any mapping from E into Σ_1 has a unique extension to a semigroup homomorphism from Σ into Σ_1 that sends e_Σ to e_{Σ_1}.

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k, with a multiplicative identity element e_A. One can use Σ to get the corresponding semigroup algebra $A(\Sigma)$ over A, as in the previous section. This is mentioned on p148 of [3], when E is finite. In this case, $A(\Sigma)$ corresponds to the *free ring* mentioned on p146 of [3].

Let B be another associative algebra over k, with a multiplicative identity element e_B. Also let ϕ_A be a homomorphism from A into B, as algebras over k, with $\phi_A(e_A) = e_B$. Suppose that ϕ_E is a mapping from E into B such that

$$\phi_A(a)\,\phi_E(x) = \phi_E(x)\,\phi_A(a) \tag{4.10.2}$$

for every $a \in A$ and $x \in E$. Let ϕ_Σ be the unique extension of ϕ_E to a homomorphism from Σ into B, as a semigroup with respect to multiplication, such that $\phi_\Sigma(e_\Sigma) = e_B$. It is easy to see that (4.9.3) holds in this case. This implies that there is a unique homomorphism $\phi_{A(\Sigma)}$ from $A(\Sigma)$ into B, as algebras over k, that satisfies (4.9.4). As before, $A(\Sigma)$ satisfies these conditions on B, with respect to the natural mappings from A and E into $A(\Sigma)$.

4.11 Tensor algebras using bimodules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a module over k that is a bimodule over A, which is to say that A is both a left and right module over A, and that the actions of A on V on the left and right commute with each other.

Let $V \bigotimes_A V$ be a tensor product of V with itself over A. This is a module over k, and in fact a left and right module over A, using the action of A on the left on the left factor of V, and the action of A on the right on the the right factor of V. Note that the actions of A on $V \bigotimes_A V$ on the left and on the right commute with each other.

Put $T^1 V = V$, and $T^2 V = V \bigotimes_A V$. Similarly, if $n \geq 3$ is an integer, then we take $T^n V$ to be a tensor product of n V's over A. The order of the n factors of V is important here, but the way that the n-fold tensor product is arranged into products of pairs does not really matter, up to natural isomorphisms, as in Section 1.12.

More precisely, $T^n V$ is a left module over A, using the action of A on the left on the left-most factor of V. Similarly, $T^n V$ is a right module over A, using the action of A on the right on the right-most factor of A. These actions of A on $T^n V$ on the left and right commute with each other.

If n_1 and n_2 are positive integers, then a tensor product $(T^{n_1} V) \bigotimes_A (T^{n_2} V)$ of $T^{n_1} V$ and $T^{n_2} V$ over A is a left and right module over A, using the action of A on the left on $T^{n_1}V$, and the action of A on the right on $T^{n_2}V$. These actions of A on $(T^{n_1}V)\bigotimes_A(T^{n_2}V)$ on the left and the right commute with each other, and $(T^{n_1}V)\bigotimes_A(T^{n_2}V)$ is isomorphic to $T^{n_1+n_2}V$ in a natural way, as a bimodule over A.

The natural bilinear mapping from $(T^{n_1}V)\times(T^{n_2}V)$ into $(T^{n_1}V)\bigotimes_A(T^{n_2}V)$ leads to a mapping

$$\text{from } (T^{n_1}V)\times(T^{n_2}V) \text{ into } T^{n_1+n_2}V \tag{4.11.1}$$

that is bilinear over k. If $u_1 \in T^{n_1}V$ and $u_2 \in T^{n_2}V$, then let $u_1\,u_2$ be the image of (u_1, u_2) into $T^{n_1+n_2}V$ under this mapping. If $a \in A$, then

$$(u_1 \cdot a)\,u_2 = u_1\,(a \cdot u_2), \tag{4.11.2}$$

by construction. We also have that

$$(a \cdot u_1)\,u_2 = a \cdot (u_1\,u_2) \tag{4.11.3}$$

and

$$u_1\,(u_2 \cdot a) = (u_1\,u_2) \cdot a. \tag{4.11.4}$$

Let us take $T^0V = A$, which may be considered as a bimodule over A too. The mapping as in (4.11.1) mentioned earlier can be extended to the cases where $n_1 = 0$ or $n_2 = 0$, using the action of A on $T^{n_2}V$ on the left when $n_1 = 0$, the action of A on $T^{n_1}V$ on the right when $n_2 = 0$, and multiplication on A when $n_1 = n_2 = 0$. This extension satisfies (4.11.2), (4.11.3), and (4.11.4) when $n_1 = 0$ or $n_2 = 0$ as well.

Let n_1, n_2, and n_3 be nonnegative integers, and let $u_j \in T^{n_j}V$, $j = 1, 2, 3$, be given. One can check that

$$(u_1\,u_2)\,u_3 = u_1\,(u_2\,u_3) \tag{4.11.5}$$

in $T^{n_1+n_2+n_3}V$. This uses the natural isomorphisms between

$$\big((T^{n_1}V)\bigotimes_A(T^{n_2}V)\big)\bigotimes_A(T^{n_3}V), \tag{4.11.6}$$

$$(T^{n_1}V)\bigotimes_A\big((T^{n_2}V)\bigotimes_A(T^{n_3}V)\big), \tag{4.11.7}$$

and $T^{n_1+n_2+n_3}V$, as in Section 1.12.

Put

$$TV = \bigoplus_{n=0}^{\infty} T^nV, \tag{4.11.8}$$

which may be considered initially as a module over k, and a bimodule over A. There is a natural mapping from $(TV)\times(TV)$ into TV, using the mappings as in (4.11.1) for every $n_1, n_2 \geq 0$. If $u, w \in TV$, then let $u\,w$ be the image of (u, w) in TV under this mapping. This makes TV an associative algebra over k.

Let us identify $A = T^0 V$ with the corresponding subset of TV, which is a subalgebra of TV. Multiplication of elements of TV by elements of A on the left or right is the same as the action of A on TV on the left and right, respectively, as a bimodule over A. In particular, e_A is the multiplicative identity element in TV.

Let E be a nonempty set, and suppose that V is freely generated, as a bimodule over A, by E. Thus V corresponds to the direct sum of a family of copies of A indexed by E, as a bimodule over A. If n is a positive integer, then we can take $T^n V$ to be freely generated by E^n, as a bimodule over A. This also works with $n = 0$, where E^0 is interpreted as consisting of a single element, as in the previous section. If Σ is the free semigroup generated by E, then TV corresponds to the semigroup algebra $A(\Sigma)$.

4.12 Tensor algebras and homomorphisms

Let us continue with the same notation and hypotheses as in the previous section. Also let B be an associative algebra over k, with a multiplicative identity element e_B. Suppose that ϕ_A is a homomorphism from A into B, as algebras over k, with $\phi_A(e_A) = e_B$. Using this, B may be considered as a bimodule over A.

Let ϕ_V be a homomorphism from V into B, as modules over k, and as bimodules over A. This means that

$$\phi_V(a \cdot v) = \phi_A(a)\, \phi_V(v) \tag{4.12.1}$$

and

$$\phi_V(v \cdot a) = \phi_V(v)\, \phi_A(a) \tag{4.12.2}$$

for every $a \in A$ and $v \in V$. If $v_1, v_2 \in V$ and $a \in A$, then it follows that

$$\phi_V(v_1 \cdot a)\, \phi_V(v_2) = \phi_V(v_1)\, \phi_A(a)\, \phi_V(v_2) = \phi_V(v_1)\, \phi_V(a \cdot v_2). \tag{4.12.3}$$

Of course,

$$(v_1, v_2) \mapsto \phi_V(v_1)\, \phi_V(v_2) \tag{4.12.4}$$

defines a mapping from $V \times V$ into B that is bilinear over k. Using (4.12.3), we get that there is a unique homomorphism $\phi_{T^2 V}$ from $T^2 V$ into B, as modules over k, such that

$$\phi_{T^2 V}(v_1 \otimes v_2) = \phi_V(v_1)\, \phi_V(v_2) \tag{4.12.5}$$

for every $v_1, v_2 \in V$. More precisely, $\phi_{T^2 V}$ is a homomorphism from $T^2 V$ into B, as bimodules over A.

Similarly, we can get a natural homomorphism

$$\phi_{T^n V} \text{ from } T^n V \text{ into } B \tag{4.12.6}$$

for every $n \geq 0$, as modules over k, and as bimodules over A. We use ϕ_A when $n = 0$, and ϕ_V when $n = 1$. If n_1, n_2 are nonnegative integers and $u_j \in T^{n_j} V$ for $j = 1, 2$, then

$$\phi_{T^{n_1+n_2} V}(u_1\, u_2) = \phi_{T^{n_1} V}(u_1)\, \phi_{T^{n_2} V}(u_2). \tag{4.12.7}$$

Let ϕ_{TV} be the mapping from TV into B that corresponds to $\phi_{T^n V}$ on $T^n V$ for every $n \geq 0$. This is a homomorphism from TV into B, as algebras over k.

Note that TV is generated, as an algebra over k, by A and $V = T^1 V$, considered as subsets of TV. This implies that any algebra homomorphism from TV into B is uniquely determined by its restriction to A and V. In particular, ϕ_{TV} is uniquely determined by ϕ_A and ϕ_V.

If ϕ is any algebra homomorphism from TV into B, then we can take ϕ_A to be the restriction of ϕ to A, and ϕ_V to be the restriction of ϕ to $V = T^1 V$.

Observe that TV satisfies the conditions on B here, using the natural mappings from A and V into TV.

4.13 Local rings

Let A be a ring, with a nonzero multiplicative identity element e_A. Put

(4.13.1) $$\mathcal{I}_0 = \{a \in A : a \text{ does not have a left inverse in } A\}.$$

We say that A is a *local ring* if

(4.13.2) $$\mathcal{I}_0 \text{ is a left ideal in } A.$$

This is the same as the condition (LC) on p147 in [3].

Suppose that A is a local ring. If $\mathcal{I}$ is a proper left ideal in A, then

(4.13.3) $$\mathcal{I} \subseteq \mathcal{I}_0,$$

as in Proposition 2.1 on p147 of [3]. Indeed, if $\mathcal{I}$ is a proper left ideal in A, then no element of $\mathcal{I}$ can have a left inverse in A, so that (4.13.3) holds.

Suppose for the sake of a contradiction that $a \in \mathcal{I}_0$ and $b \in A$ satisfy

(4.13.4) $$a\, b = e_A.$$

This implies that

(4.13.5) $$(e_A - b\, a)\, b = b - b\, a\, b = b - b = 0.$$

Note that $b\, a \in \mathcal{I}_0$, because $\mathcal{I}_0$ is a left ideal in A. It follows that $e_A - b\, a \notin \mathcal{I}_0$, because $e_A \notin \mathcal{I}_0$. Thus $e_A - b\, b$ has a left inverse c in A. However, this means that

(4.13.6) $$b = c\, (e_a - b\, a)\, b = 0,$$

which is a contradiction. This shows that

(4.13.7) $$\text{no element of } \mathcal{I}_0 \text{ has a right inverse in } A,$$

as in Proposition 2.1 on p147 of [3].

If $b \in A$, then $\mathcal{I}_0\, b$ is a left ideal in A, because $\mathcal{I}_0$ is a left ideal in A. We also have that $e_A \notin \mathcal{I}_0\, b$, as in the preceding paragraph, so that $\mathcal{I}_0\, b \neq A$. This implies that

(4.13.8) $$\mathcal{I}_0\, b \subseteq \mathcal{I}_0,$$

as in (4.13.3). This means that

$$\mathcal{I}_0 \text{ is a right ideal in } A \tag{4.13.9}$$

too, as in Proposition 2.1 on p147 of [3].

Let $a \in A \setminus \mathcal{I}_0$ be given, so that a has a left inverse $b \in A$. This means that a is a right inverse of b, so that $b \notin \mathcal{I}_0$. It follows that b has a left inverse $c \in A$. This implies that

$$c = c\,b\,a = a, \tag{4.13.10}$$

so that

$$a\,b = c\,b = e_A. \tag{4.13.11}$$

Thus

$$\text{every element of } A \setminus \mathcal{I}_0 \text{ has a two-sided inverse in } A, \tag{4.13.12}$$

as in Proposition 2.1 on p147 of [3].

Using the remarks in the previous paragraphs, we get that

$$\mathcal{I}_0 = \{a \in A : a \text{ does not have a right inverse in } A\}. \tag{4.13.13}$$

This implies that $\mathcal{I}_0$ contains every proper right ideal in A, as before. This is another part of Proposition 2.1 on p147 of [3].

Because $\mathcal{I}_0$ is a two-sided ideal in A, the quotient $A/\mathcal{I}_0$ is defined as a ring. Of course, this ring is nontrivial, because $\mathcal{I}_0 \neq A$. In fact, $A/\mathcal{I}_0$ is a division ring, which is to say that every nonzero element of $A/\mathcal{I}_0$ has a multiplicative inverse in $A/\mathcal{I}_0$, as in Proposition 2.1 on p147 of [3].

One might consider the previous definition of a local ring as being the definition of a "left local ring", and define a notion of "right local ring" analogously. The earlier remarks show that left local rings are right local rings, and the converse holds similarly. Thus one may simply define local rings in this way, as on p148 of [3].

Suppose for the moment that A is commutative. In this case, the property of being a local ring is often defined in terms of the uniqueness of a maximal proper ideal in A, as on p4 of [1]. If A is a local ring in the sense of (4.13.2), then $\mathcal{I}_0$ is the unique maximal ideal in A. Conversely, if A has a unique maximal proper ideal, then A is a local ring in the sense considered in this section. Indeed, every non-invertible element of A generates a proper ideal in A. There are also well-known arguments for showing that every proper ideal in A is contained in a maximal proper ideal, using Zorn's lemma or Hausdorff's maximality principle. If there is a unique maximal proper ideal in A, then it consists exactly of the noninvertible elements of A.

4.14 Local rings and power series

Let A be any ring with a nonzero multiplicative identity element e_A again, let n be a positive integer, and let $T_1, \ldots, T_n$ be commuting indeterminates. Consider the corresponding ring $A[[T_1, \ldots, T_n]]$ of formal power series in $T_1, \ldots, T_n$ with coefficients in A, as in Section 4.3. Remember that A may be identified with the subring of $A[[T_1, \ldots, T_n]]$ consisting of formal power series for which the coefficient of T^α is equal to 0 when $\alpha \neq 0$.

We have also seen that the mapping from an element of $A[[T_1, \ldots, T_n]]$ to its coefficient of T^0 defines a ring homomorphism from $A[[T_1, \ldots, T_n]]$ onto A. If $a \in A$ has a left or right inverse in $A[[T_1, \ldots, T_n]]$, then it follows that a has a left or right inverse in A, as appropriate.

Let $f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha$ be an element of $A[[T_1, \ldots, T_n]]$. If $f(T)$ has a left or right inverse in $A[[T_1, \ldots, T_n]]$, then f_0 has a left or right inverse in A, as appropriate, because of the homomorphism mentioned in the preceding paragraph. Conversely, if f_0 has a left or right inverse in A, then $f(T)$ has a left or right inverse in $A[[T_1, \ldots, T_n]]$, as appropriate. This follows from the invertibility of the elements of $A[[T_1, \ldots, T_n]]$ for which the coefficient of T^0 is e_A, as in Section 4.5.

Let $\mathcal{I}_0$ be as in (4.13.1), and consider

$$\{f(T) \in A[[T_1, \ldots, T_n]] : f_0 \in \mathcal{I}_0\}. \tag{4.14.1}$$

This is the same as

$$\{f(T) \in A[[T_1, \ldots, T_n]] : f(T) \text{ does not have a left inverse in } A[[T_1, \ldots, T_n]]\}, \tag{4.14.2}$$

by the remarks in the preceding paragraph. It is easy to see that (4.14.1) is a left ideal in $A[[T_1, \ldots, T_n]]$ if and only if $\mathcal{I}_0$ is a left ideal in A. This implies that

$$A[[T_1, \ldots, T_n]] \text{ is a local ring} \tag{4.14.3}$$

if and only if A is a local ring, as in Exercise 6 on p159 of [3]. In particular, (4.14.3) holds when A is a division ring, as on p148 of [3].

4.15 Exterior rings

Let A be a ring with a nonzero multiplicative identity element, let n be a positive integer, and let $x_1, \ldots, x_n$ be indeterminates. We would like to consider the corresponding *exterior* or *Grassmann* ring $E_A(x_1, \ldots, x_n)$ in $x_1, \ldots, x_n$ with coefficients in A.

Let $I = \{j_1, \ldots, j_r\}$ be a subset of $\{1, \ldots, n\}$, where

$$1 \leq j_1 < j_2 < \cdots < j_r \leq n. \tag{4.15.1}$$

Put

$$x_I = x_{j_1} \wedge x_{j_2} \wedge \cdots \wedge x_{j_r}, \tag{4.15.2}$$

which may be considered as a formal expression, for the moment. We can define $E_A(x_1, \ldots, x_n)$ initially as the space of formal sums of the form

$$\sum_{I \subseteq \{1, \ldots, n\}} a_I \, x_I, \tag{4.15.3}$$

where $a_I \in A$ for each $I \subseteq \{1, \ldots, n\}$. More precisely, this can be defined as the set of all A-valued functions on the set of all subsets of $\{1, \ldots, n\}$, which are identified with formal sums as in (4.15.3). If $I = \emptyset$, then we may identify x_I with e_A, so that $a \in A$ is identified with $a\, x_I$.

Of course, $E_A(x_1, \ldots, x_n)$ is a left and right module over A, with respect to termwise addition and multiplication on the left or right by elements of A. Note that $E_A(x_1, \ldots, x_n)$ is freely generated by x_I, $I \subseteq \{1, \ldots, n\}$, as a left or right module over A. If $I_1, I_2 \subseteq \{1, \ldots, n\}$, then we would like to define the product of x_{I_1} and x_{I_2}. Put

$$x_{I_1} \wedge x_{I_2} = 0 \quad \text{when } I_1 \cap I_2 \neq \emptyset. \tag{4.15.4}$$

Suppose now that I_1 and I_2 are disjoint sets, with r_1, r_2 elements, respectively. Let σ be the permutation on $I_1 \cup I_2$ that sends the elements of I_1 to the first r_1 elements of $I_1 \cup I_2$, in order, and send I_2 to the next r_2 elements of $I_1 \cup I_2$, in order. This can be extended to a permutation on $\{1, \ldots, n\}$, by sending any element of $\{1, \ldots, n\} \setminus (I_1 \cup I_2)$ to itself. Put

$$\begin{aligned} x_{I_1} \wedge x_{I_2} &= x_{I_1 \cup I_2} && \text{when } \sigma \text{ is an even permutation,} \\ &= -x_{I_1 \cup I_2} && \text{when } \sigma \text{ is an odd permutation.} \end{aligned} \tag{4.15.5}$$

We can use this to define multiplication on $E_A(x_1, \ldots, x_n)$ in an obvious way, where x_I commutes with every element of A for each $I \subseteq \{1, \ldots, n\}$. It is well known that $E_A(x_1, \ldots, x_n)$ a ring with respect to this definition of multiplication, which contains A as a subring, and with e_A as the multiplicative identity element in $E_A(x_1, \ldots, x_n)$. If A is an algebra over a commutative ring k with a multiplicative identity element, then $E_A(x_1, \ldots, x_n)$ is an algebra over k as well.

Let V be a left or right module over A, and let $\mathcal{E}_V(x_1, \ldots, x_n)$ be the space of formal sums

$$\sum_{I \subseteq \{1, \ldots, n\}} v_I\, x_I, \tag{4.15.6}$$

with $v_I \in V$ for each $I \subseteq \{1, \ldots, n\}$. As before, this can be defined more precisely as the space of V-valued functions on the set of all subsets if $\{1, \ldots, n\}$, which are identified with formal sums as in (4.15.6). This is a left or right module over A too, as appropriate, with respect to termwise addition, and where A acts termwise on the coefficients v_I in (4.15.6).

In fact, $\mathcal{E}_V(x_1, \ldots, x_n)$ is a left or right module over $E_A(x_1, \ldots, x_n)$, as appropriate. If $I_0 \subseteq \{1, \ldots, n\}$, then we put

$$x_{I_0} \cdot \Big(\sum_{I \subseteq \{1, \ldots, n\}} v_I\, x_I \Big) = \sum_{I \subseteq \{1, \ldots, n\}} v_I\, x_{I_0} \wedge x_I \tag{4.15.7}$$

when V is a left module over A, and

$$\Big(\sum_{I \subseteq \{1, \ldots, n\}} v_I\, x_I \Big) \cdot x_{I_0} = \sum_{I \subseteq \{1, \ldots, n\}} v_I\, x_I \wedge x_{I_0} \tag{4.15.8}$$

when V is a right module over A. One can use this and the action of A on V to define the action of $E_A(x_1, \ldots, x_n)$ on $\mathcal{E}_V(x_1, \ldots, x_n)$ on the left or right, as appropriate, in an obvious way.

Part II

Differentiation and complexes

Chapter 5

Differentiation and dual numbers

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, W be modules over k that are either both left modules over A, or both right modules over A. If ϕ is a homomorphism from V into W, as modules over A, then

(5.0.1) $$\text{the } \textit{cokernel} \text{ of } \phi \text{ is } W/\phi(V),$$

as on p19 of [1] and p3 of [3]. Similarly,

(5.0.2) $$\text{the } \textit{coimage} \text{ of } \phi \text{ is } V/\ker\phi.$$

Note that ϕ induces an isomorphism from its coimage onto its image $\phi(V)$, as modules over A.

5.1 Modules with differentiation

Let k and A be as before, and let V be a module over k that is a left or right module over A. Supose that d_V is a homomorphism from V into itself, as a module over A, such that

(5.1.1) $$d_V \circ d_V = 0.$$

Under these conditions, V is said to be a *module with differentiation*, as on p53 of [3].

Put

(5.1.2) $$Z(V) = \ker d_V, \quad B(V) = d_V(V),$$

which are submodules of V, as a module over A. Note that

(5.1.3) $$B(V) \subseteq Z(V),$$

by (5.1.1).

Similarly, put

$$Z'(V) = V/d_V(V) \tag{5.1.4}$$

and

$$B'(V) = V/\ker d_V, \tag{5.1.5}$$

which are also modules over A, as quotients of V. Equivalently, $Z'(V)$ is the cokernel of d_V, and $B'(V)$ is the coimage of d_V, as on p53 of [3].

The natural quotient mapping from V onto $B'(V)$ can be expressed as the composition of the natural quotient mapping from V onto $Z'(V)$ with a unique homomorphism

$$\text{from } Z'(V) \text{ onto } B'(V), \tag{5.1.6}$$

as modules over A, because $d_V(V) \subseteq \ker d_V$. We also have that d_V can be expressed as the composition of the natural quotient mapping from V onto $B'(V)$ with a unique isomorphism

$$\delta \text{ from } B'(V) \text{ onto } B(V), \tag{5.1.7}$$

as modules over A, as on p53 of [3].

Thus d_V can be factored into a composition of module homomorphisms,

$$V \longrightarrow Z'(V) \longrightarrow B'(V) \xrightarrow{\delta} B(V) \longrightarrow Z(V) \longrightarrow V, \tag{5.1.8}$$

as on p53 of [3]. The first step uses the natural quotient mapping, and the second step uses the mapping obtained from the natural quotient mapping as in (5.1.6). The last two steps use the natural inclusion mappings.

This leads to a homomorphism

$$\widetilde{d}_V \text{ from } Z'(V) \text{ into } Z(V), \tag{5.1.9}$$

as modules over A, as on p54 of [3]. More precisely, $\widetilde{d}_V$ is obtained by composing the second, third, and fourth homomorphisms indicated in (5.1.8). Equivalently, $\widetilde{d}_V$ is induced by d_V in an obvious way.

Observe that

$$\widetilde{d}_V(Z'(V)) = B(V), \tag{5.1.10}$$

so that

$$\text{the cokernel of } \widetilde{d}_V \text{ is equal to } Z(V)/B(V). \tag{5.1.11}$$

The kernel of $\widetilde{d}_V$ is the same as the kernel of the mapping in (5.1.6), because the third and fourth homomorphisms indicated in (5.1.8) are injections. This means that

$$\ker \widetilde{d}_V = \ker d_V/d_V(V) = Z(V)/B(V), \tag{5.1.12}$$

as on p54 of [3]. Put

$$H(V) = Z(V)/B(V), \tag{5.1.13}$$

which is the *homology module* of V with respect to d_V, as on p54 of [3]. This may be considered as a submodule of $Z'(V)$, as a module over A.

If W is any left or right module over A, then W may be considered as a module with differentiation, by taking $d_W = 0$. In this case,

$$(5.1.14) \qquad Z(W) = Z'(W) = H(W) = W, \quad B(W) = B'(W) = \{0\}.$$

If V is any module with differentiation, then $Z(V)$, $Z'(V)$, $H(V)$, $B(V)$, and $B'(V)$ may be considered as modules with differentiation equal to 0, as on p55 of [3].

5.2 Mappings and differentiation

Let k be a commutative ring with a multiplicative identity element again, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that (V, d_V), (W, d_W) are both left modules or both right modules over A, with differentiation. Let ϕ be a homomorphism from V into W, as modules over A. We say that ϕ is a *homomorphism* from V into W as modules with differentiation if

$$(5.2.1) \qquad d_W \circ \phi = \phi \circ d_V,$$

as on p54 of [3]. One may refer to ϕ as a map or mapping from V into W, as modules with differentiation, as well, as in [3].

In this case, we get that

$$(5.2.2) \qquad \phi(Z(V)) \subseteq Z(W)$$

and

$$(5.2.3) \qquad \phi(B(V)) \subseteq B(W).$$

We also obtain induced homomorphisms

$$(5.2.4) \qquad \phi_{Z'} \text{ from } Z'(V) \text{ into } Z'(W)$$

and

$$(5.2.5) \qquad \phi_{B'} \text{ from } B'(V) \text{ into } B'(W),$$

as modules over A. Note that

$$(5.2.6) \qquad \phi_{Z'}(H(V)) \subseteq H(W),$$

because of (5.2.2). The restriction of $\phi_{Z'}$ to $H(V)$ may be denoted ϕ_H, considered as a homomorphism from $H(V)$ into $H(W)$, as modules over A.

Remember that $\widetilde{d}_V$ is the homomorphism from $Z'(V)$ into $Z(V)$ induced by d_V, and let $\widetilde{d}_W$ be the analogous homomorphism for W. It is easy to see that

$$(5.2.7) \qquad \phi \circ \widetilde{d}_V = \widetilde{d}_W \circ \phi_{Z'},$$

as homomorphisms from $Z'(V)$ into $Z(W)$. This corresponds to the commutativity of the diagram at the bottom of p54 of [3].

If ϕ is injective, then it is easy to see that

$$\phi(Z(V)) = \phi(V) \cap Z(W). \tag{5.2.8}$$

Similarly, we get that

$$\phi_{B'} \text{ is injective.} \tag{5.2.9}$$

Suppose for the moment that

$$\phi(V) = W. \tag{5.2.10}$$

Observe that

$$\phi_{Z'}(Z'(V)) = Z'(W) \tag{5.2.11}$$

and

$$\phi_{B'}(B'(V)) = B'(W). \tag{5.2.12}$$

One can check that

$$\phi(B(V)) = B(W). \tag{5.2.13}$$

Let ψ be another homomorphism from V into W, as modules over A, and as modules with differentiation. A homomorphism σ from V into W, as modules over A, is said to be a *homotopy* between ϕ and ψ if

$$d_W \circ \sigma + \sigma \circ d_V = \phi - \psi, \tag{5.2.14}$$

as on p54 of [3]. This implies that

$$(\phi - \psi)(Z(V)) \subseteq B(W). \tag{5.2.15}$$

It follows that

$$\phi_H = \psi_H, \tag{5.2.16}$$

as on p54 of [3].

Let (W_1, d_{W_1}) be another left or right module over A with differentiation, depending on whether V, W are left or right modules over A. Also let ϕ_1 be a homomorphism from W into W_1, as modules over A with differentiation. This implies that $\phi_1 \circ \phi$ is a homomorphism from V into W_1, as modules over A with differentiation. One can verify that

$$(\phi_1 \circ \phi)_{Z'} = (\phi_1)_{Z'} \circ \phi_{Z'} \tag{5.2.17}$$

and

$$(\phi_1 \circ \phi)_{B'} = (\phi_1)_{B'} \circ \phi_{B'}. \tag{5.2.18}$$

In particular,

$$(\phi_1 \circ \phi)_H = (\phi_1)_H \circ \phi_H. \tag{5.2.19}$$

Note that

$$d_V(\ker \phi) \subseteq \ker \phi, \tag{5.2.20}$$

by (5.2.1). Similarly,

$$d_W(\phi(V)) \subseteq \phi(V). \tag{5.2.21}$$

5.3 Submodules and quotients

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a left or right module over A with differentiation, and let V_0 be a submodule of V, as a module over A. Let us say that V_0 is a *submodule* of V as a module with differentiation if

$$d_V(V_0) \subseteq V_0. \tag{5.3.1}$$

In this case, V_0 is a module with differentiation too, with d_{V_0} equal to the restriction of d_V to V_0. Of course, the natural inclusion mapping from V_0 into V is a homomorphism from V_0 into V, as modules with differentiation.

Let q_0 be the natural quotient mapping from V onto V/V_0. Note that $q_0 \circ d_V$ is equal to 0 on V_0, by (5.3.1), so that there is a unique homomorphism d_{V/V_0} from V/V_0 into itself, as a module over A, such that

$$d_{V/V_0} \circ q_0 = q_0 \circ d_V. \tag{5.3.2}$$

It is easy to see that

$$d_{V/V_0} \circ d_{V/V_0} = 0, \tag{5.3.3}$$

so that V/V_0 is a module with differentiation with respect to d_{V/V_0}. By construction, q_0 is a homomorphism from V onto V/V_0, as modules with differentiation.

Observe that

$$Z(V) \cap (\ker q_0) = Z(V) \cap V_0 = Z(V_0). \tag{5.3.4}$$

This is basically the same as (5.2.8), and it corresponds to the exactness of the bottom row of the diagram on the bottom of p54 of [3].

Remember that

$$q_0(B(V)) = B(V/V_0), \tag{5.3.5}$$

as in (5.2.13). Suppose that $v \in V$ satisfies

$$q_0(v) \in B(V/V_0). \tag{5.3.6}$$

This means that there is a $u \in V$ such that

$$q_0(v) = q_0(d_V(u)), \tag{5.3.7}$$

by (5.3.5). Equivalently, there is a $v_0 \in V_0$ such that

$$v - d_V(u) = v_0. \tag{5.3.8}$$

It follows that

(5.3.9) the images of v and v_0 under the natural quotient mapping from V onto $Z'(V)$ are the same.

This corresponds to the exactness of the top row of the diagram on the bottom of p54 of [3]. Similarly, if we also have that

$$v \in Z(V), \tag{5.3.10}$$

then

$$v_0 \in Z(V_0), \tag{5.3.11}$$

by (5.3.8). This corresponds to the exactness of the first part of the sequence in (4) on p55 of [3].

There is a natural homomorphism

$$\Delta \text{ from } H(V/V_0) \text{ into } H(V_0), \tag{5.3.12}$$

as modules over A, as on p55 of [3]. To see this, let an element of

$$H(V/V_0) = Z(V/V_0)/B(V/V_0) \tag{5.3.13}$$

be given. This can be represented by the image of $q_0(v) \in Z(V/V_0)$ in this quotient, where $v \in V$. Observe that

$$q_0(d_V(v)) = d_{V/V_0}(q_0(v)) = 0, \tag{5.3.14}$$

so that $d_V(v) \in V_0$. More precisely,

$$d_V(v) \in Z(V_0). \tag{5.3.15}$$

We would like to take Δ of the image of $q_0(v)$ in (5.3.13) to be the image of $d_V(v)$ in

$$H(V_0) = Z(V_0)/B(V_0). \tag{5.3.16}$$

To show that this is well defined, suppose that $v' \in V$ and the image of $q_0(v')$ in (5.3.13) is the same as the image of $q_0(v)$, so that

$$q_0(v - v') = q_0(v) - q_0(v') \in B(V/V_0). \tag{5.3.17}$$

This means that there is a $u \in V$ such that

$$q_0(v - v') = d_{V/V_0}(q_0(u)) = q_0(d_V(u)), \tag{5.3.18}$$

and thus

$$v - v' - d_V(u) \in V_0. \tag{5.3.19}$$

It follows that

$$d_V(v) - d_V(v') = d_V(v - v' - d_V(u)) \in B(V_0), \tag{5.3.20}$$

so that the images of $d_V(v)$ and $d_V(v')$ in (5.3.16) are the same. It is easy to see that Δ is a homomorphism from $H(V/V_0)$ into $H(V_0)$, as modules over A.

Let $(q_0)_{Z'}$ be the homomorphism from $Z'(V)$ into $Z'(V/V_0)$, as modules over A, induced by q_0, as in the previous section. The restriction of $(q_0)_{Z'}$ to $H(V)$ is denoted $(q_0)_H$, and maps $H(V)$ into $H(V/V_0)$, as before. Let us check that

$$\Delta \circ (q_0)_H = 0. \tag{5.3.21}$$

If $v \in Z(V)$, so that $d_V(v) = 0$, then Δ of the image of $q_0(v)$ in $H(V/V_0)$ is equal to 0, by construction. This means that $\Delta \circ (q_0)_H$ of the image of v in $H(V) = Z(V)/B(v)$ is equal to 0, as desired.

Suppose that $v \in V$, $q_0(v) \in Z(V/V_0)$, and that Δ maps the image of $q_0(v)$ in $H(V/V_0)$ to 0 in $H(V_0)$. This means that

$$d_V(v) \in B(V_0), \tag{5.3.22}$$

so that $d_V(v) = d_V(u)$ for some $u \in V_0$. It follows that

$$q_0(v) = q_0(v-u) \text{ and } v-u \in Z(V). \tag{5.3.23}$$

Thus the image of $q_0(v)$ in $H(V/V_0)$ is the same as the image of $q_0(v-u)$ in $H(V/V_0)$, and the image of $q_0(v-u)$ in $H(V/V_0)$ is the same as $(q_0)_H$ of the image of $v-u$ in $H(V)$. This shows that

$$(q_0)_H(H(V)) = \ker \Delta, \tag{5.3.24}$$

which corresponds to the exactness of part of the sequence (4) on p55 of [3].

Let ι_0 be the natural inclusion mapping from V_0 into V. This leads to an induced homomorphism $(\iota_0)_{Z'}$ from $Z'(V_0)$ into $Z'(V)$, as in the previous section. The restriction of $(\iota_0)_{Z'}$ to $H(V_0)$ is denoted $(\iota_0)_H$, as usual, and maps $H(V_0)$ into $H(V)$. It is easy to see that

$$(\iota_0)_H \circ \Delta = 0, \tag{5.3.25}$$

by construction. This is because the element of $Z(V_0)$ in (5.3.15) is automatically in $B(V)$.

Suppose that $w \in Z(V_0)$, and that the image of w in $H(V_0)$ is in the kernel of $(\iota_0)_H$. This means that there is a $v \in V$ such that

$$w = d_V(v). \tag{5.3.26}$$

This implies that

$$d_{V/V_0}(q_0(v)) = q_0(d_V(v)) = q_0(w) = 0, \tag{5.3.27}$$

so that $q_0(v) \in Z(V/V_0)$. It follows that the image of w in $H(V_0)$ is the same as Δ of the image of $q_0(v)$ in $H(V/V_0)$. Thus

$$\ker(\iota_0)_H = \Delta(H(V/V_0)), \tag{5.3.28}$$

which corresponds to the exactness of another part of the sequence (4) on p55 of [3].

5.4 Dual numbers

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let d be a formal symbol, which is taken to commute with all elements of k and A, and satisfy $d^2 = 0$. The algebra $A[d]$ of *dual numbers* associated to A consists of expressions of the form $a_1 + a_2\, d$, with $a_1, a_2 \in A$, as on p56 of [3].

More precisely, $A[d]$ is free as a left and right module over A, with generators e_A and d. If $a_1, a_2, b_1, b_2 \in A$, then we put

$$(a_1 + a_2\, d)\,(b_1 + b_2\, d) = a_1\, b_1 + (a_1\, b_2 + a_2\, b_1)\, d. \tag{5.4.1}$$

One can check that this makes $A[d]$ into an associative algebra over k that contains A as a subalgebra, and with e_A as the multiplicative identity element.

A left or right module (V, d_V) over A with differentiation corresponds exactly to a left or right module over $A[d]$, as appropriate, as on p56 of [3]. In this case, d_V corresponds to the action of d on V, as a module over $A[d]$. Similarly, a homomorphism between left or right modules over A with differentiation corresponds exactly to a homomorphism between left or right modules over $A[d]$, as appropriate.

Remember that any module V over A may be considered as a module with differentiation, by taking $d_V = 0$. Equivalently, V may be considered as a module over $A[d]$, where the action of d on V is equal to 0. In particular, A may be considered as a left and right module over $A[d]$ in this way, as on p56 of [3].

Let (V, d_V) be a left module over A with differentiation, which may be considered as a left module over $A[d]$. If $v \in V$, then

$$a \mapsto a \cdot v \tag{5.4.2}$$

defines a homomorphism from A into V, as left modules over A. Every homomorphism from A into V, as left modules over A, corresponds to a unique $v \in V$ in this way. If $v \in Z(V)$, then (5.4.2) defines a homomorphism from A into V, as left modules over A with differentiation, or equivalently as left modules over $A[d]$. Conversely, if (5.4.2) defines a homomorphism from A into V as left modules over A with differentiation, or equivalently as left modules over $A[d]$, then one can check that $v \in Z(V)$.

This defines an isomorphism

$$\text{between } \mathrm{Hom}_{A[d]}(A, V) \text{ and } Z(V), \tag{5.4.3}$$

as modules over k, as on p56 of [3]. More precisely, $\mathrm{Hom}_{A[d]}(A, V)$ may be considered as a left module over A, by considering A as a right module over itself. In fact, we have an isomorphism as in (5.4.3) as left modules over A. This may also be considered as an isomorphism between left modules over A with differentiation, or left modules over $A[d]$, with differentiation equal to 0, which corresponds to considering A as a right module over itself with differentiation equal to 0. There are analogous statements for right modules, as usual.

Let us check that

$$A \bigotimes_{A[d]} V = Z'(V) = V/d_V(V), \tag{5.4.4}$$

as on p56 of [3]. More precisely, this means that $Z'(V)$ satisfies the requirements of a tensor product, initially as a module over k.

Let W be a module over k, and let b be a mapping from $A \times V$ into W that is bilinear over k. We may be interested in situations where

$$b(a \cdot \alpha, v) = b(a, \alpha \cdot v) \tag{5.4.5}$$

for every $a \in A$, $\alpha \in A[d]$, and $v \in V$. This happens if and only if

$$b(a\,a_1, v) = b(a, a_1 \cdot v) \tag{5.4.6}$$

and

$$0 = b(a \cdot d, v) = b(a, d \cdot v) = b(a, d_V(v)) \tag{5.4.7}$$

for every $a, a_1 \in A$ and $v \in V$.

There is an obvious mapping from $A \times V$ into $Z'(V)$, defined by

$$(a, v) \mapsto \text{ the image of } a \cdot v \text{ in } Z'(V) = V/d_V(V). \tag{5.4.8}$$

It is easy to see that this mapping satisfies (5.4.6), (5.4.7), and thus (5.4.5).

Let W be a module over k again, and let b be a mapping from $A \times V$ into W that is bilinear over k and satisfies (5.4.5). Observe that

$$b(a, v) = b(e_A, a \cdot v) \tag{5.4.9}$$

and

$$b(e_A, d_V(v)) = 0 \tag{5.4.10}$$

for every $a \in A$ and $v \in V$. Because of (5.4.10), there is a unique homomorphism c from $Z'(V)$ into W such that

(5.4.11) the composition of the natural quotient mapping from V onto $Z'(V)$ with c

is the same as

$$v \mapsto b(e_A, v). \tag{5.4.12}$$

It follows that

(5.4.13) b is the same as the composition of (5.4.8) with c,

by (5.4.9). It is easy to see that c is uniquely determined by this property as well.

This shows that $Z'(V)$ satisfies the requirements of a tensor product of A and V over $A[d]$, as a module over k, and using the mapping (5.4.8) from $A \times V$ into $Z'(V)$. Such a tensor product is also a left module over A in a natural way, because A is a left module over itself. It is easy to see that this corresponds to $Z'(V)$ as a left module over A, as a quotientof V.

Similarly, if V is a right module over A with differentiation, and thus a right module over $A[d]$, then $Z'(V)$ satisfies the requirements of a tensor product of V and A over $A[d]$, as a module over k. Such a tensor product is a right module over A in a natural way, because A is a right module over itself, which corresponds to $Z'(V)$ as a right module over A, as a quotient of V.

5.5 Easy differentiation

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let W_1, W_2 be modules over k that are both left or both right modules over A, and let $\psi_{1,2}$ be a homomorphism from W_1 into W_2, as modules over A. Put

$$W_{1,2} = W_1 \times W_2, \tag{5.5.1}$$

considered as a left or right module over A, as appropriate, where the module operations are defined coordinatewise. Equivalently, $W_{1,2}$ is the direct sum of W_1 and W_2, as a module over A.

If $w_1 \in W_1$ and $w_2 \in W_2$, then put

$$d_{W_{1,2}}((w_1, w_2)) = (0, \psi_{1,2}(w_1)). \tag{5.5.2}$$

This defines a homomorphism from $W_{1,2}$ into itself, as a module over A, with

$$d_{W_{1,2}} \circ d_{W_{1,2}} = 0. \tag{5.5.3}$$

This makes $W_{1,2}$ a module with differentiation.

Observe that

$$Z(W_{1,2}) = (\ker \psi_{1,2}) \times W_2 \tag{5.5.4}$$

and

$$B(W_{1,2}) = \{0\} \times \psi_{1,2}(W_1). \tag{5.5.5}$$

It follows that

$$Z'(W_{1,2}) \text{ is isomorphic to } W_1 \times (W_2/\psi_{1,2}(W_1)) \tag{5.5.6}$$

and

$$B'(W_{1,2}) \text{ is isomorphic to } (W_1/(\ker \psi_{1,2})) \times \{0\}, \tag{5.5.7}$$

as modules over A. Similarly,

$$H(W_{1,2}) \text{ is isomorphic to } (\ker \psi_{1,2}) \times (W_2/\psi_{1,2}(W_1)), \tag{5.5.8}$$

as a module over A.

Let W be a left or right module over A, and put

$$W^x = W \times W, \tag{5.5.9}$$

following the notation on p57 of [3]. This is the same as $W_{1,2}$, with $W_1 = W_2 = W$. More precisely, W^x is a module with differentiation, with

$$d_{W^x}((w_1, w_2)) = (0, w_1) \tag{5.5.10}$$

for every $w_1, w_2 \in W$, as on p57 of [3]. This is the same as $d_{W_{1,2}}$, with $\psi_{1,2}$ equal to the identity mapping on W.

In this case,

$$Z(W^x) = B(W^x) = \{0\} \times W. \tag{5.5.11}$$

Similarly,

$$Z'(W^x) = B'(W^x) = W^x/(\{0\} \times W), \tag{5.5.12}$$

which is isomorphic to W in an obvious way, as a module over A. Of course,

$$H(W^x) = \{0\}. \tag{5.5.13}$$

Let d and $A[d]$ be as in the previous section, and let η be the obvious inclusion mapping from A into $A[d]$, as on p57 of [3]. If W is a left module over A, then

$$_{(\eta)}W = A[d] \bigotimes\nolimits_A W \tag{5.5.14}$$

is a left module over $A[d]$, which is the covariant η-extension of W, as in Section 2.9. Similarly,

$$^{(\eta)}W = \mathrm{Hom}_A(A[d], W) \tag{5.5.15}$$

is a left module over $A[d]$, which is the contravariant η-extension of W, as in Section 2.11.

Consider the mapping

$$(w_1, w_2) \mapsto e_A \otimes w_1 + (e_A\, d) \otimes w_2 \tag{5.5.16}$$

from W^x into (5.5.14). This defines an isomorphism from W^x onto (5.5.14), as modules over k, and left modules over A. More precisely, this defines an isomorphism from W^x onto (5.5.14), as left modules over $A[d]$. This corresponds to a remark on p57 of [3].

If $w_1, w_2 \in W$, then

$$a_1 + a_2\, d \mapsto a_1 \cdot w_2 + a_2 \cdot w_1 \tag{5.5.17}$$

defines a homomorphism from $A[d]$ into W, as left modules over A. Thus

$$(w_1, w_2) \mapsto (5.5.17) \tag{5.5.18}$$

defines a mapping from W^x into (5.5.15). It is easy to see that this is an isomorphism from W^x onto (5.5.15) as modules over k, and left modules over A.

If $w_1, w_2 \in W$, then the analogue of (5.5.17) for $d_{W^x}((w_1, w_2)) = (0, w_1)$ is

$$a_1 + a_2\, d \mapsto a_1 \cdot w_1. \tag{5.5.19}$$

Equivalently, this is the image of $d_{W^x}((w_1, w_2))$ in (5.5.15) under the mapping (5.5.18). If $a_1, a_2 \in A$, then

$$(a_1 + a_2\, d)\, d = a_1\, d \tag{5.5.20}$$

is mapped to

$$a_1 \cdot w_1 \tag{5.5.21}$$

by (5.5.17). This implies that (5.5.18) defines an isomorphism from W^x onto (5.5.15), as left modules over $A[d]$. This corresponds to another remark on p57 of [3].

Of course, there are analogous statements when W is a right module over A.

5.6 Some isomorphisms with W^x

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra with a multiplicative identity element e_A. Also let d and $A[d]$ be as in Section 5.4, and let η be the obvious inclusion mapping from A into $A[d]$, as before. Suppose that (V, d_V) is a left module over A with differentiation, which may be considered as a left module over $A[d]$. Let σ be a homomorphism from V into itself, as a module over A, and suppose that

$$d_V \circ \sigma + \sigma \circ d_V \text{ is the identity mapping on } V. \tag{5.6.1}$$

Equivalently, this means that σ is a homotopy between the identity mapping and the zero mapping on V, as homomorphisms from V into itself as a module with differentiation, as in Section 5.2.

If $v \in Z(V)$, then (5.6.1) implies that

$$v = d_V(\sigma(v)) \in B(V). \tag{5.6.2}$$

This means that

$$B(V) = Z(V) \tag{5.6.3}$$

in this case. Put

$$W = B(V), \tag{5.6.4}$$

which is a left module over A. Thus W^x may be defined as in the previous section.

Consider the mapping ϕ from V into W^x defined by

$$\phi(v) = (d_V(v), d_V(\sigma(v))) \tag{5.6.5}$$

for every $v \in V$. Note that this is a homomorphism from V into W^x, as modules over A. If $v \in V$, then

$$\begin{aligned} \phi(d_V(v)) &= (d_V(d_V(v)), d_V(\sigma(d_V(v)))) \\ &= (0, d_V(v) - d_V(d_V(\sigma(v)))) = (0, d_V(v)) = d_{W^x}(\phi(v)), \end{aligned} \tag{5.6.6}$$

using (5.6.1) in the second step. This shows that ϕ is a homomorphism from V into W^x, as modules with differentation. If $\phi(v) = 0$, then

$$v = \sigma(d_V(v)) + d_V(\sigma(v)) = 0, \tag{5.6.7}$$

using (5.6.1) in the second step again.

Let $w \in W$ be given, and observe that

$$d_V(\sigma(w)) = w, \tag{5.6.8}$$

by (5.6.1). We also have that

$$\begin{aligned} d_V(\sigma(\sigma(w))) &= \sigma(w) - \sigma(d_V(\sigma(w))) \\ &= \sigma(w) - \sigma(w) + \sigma(\sigma(d_V(w))) = 0, \end{aligned} \tag{5.6.9}$$

using (5.6.1) in the first and second steps. It follows that

$$\phi(w) = (0, w) \tag{5.6.10}$$

and

$$\phi(\sigma(w)) = (w, 0). \tag{5.6.11}$$

This implies that $\phi(V) = W^x$. This shows that ϕ is an isomorphism from V onto W^x, as modules with differentiation.

Using this, we get that

$$V \text{ is isomorphic to (5.5.14)} \tag{5.6.12}$$

and

$$V \text{ is isomorphic to (5.5.15),} \tag{5.6.13}$$

as modules over $A[d]$. This corresponds to the fact that (b) implies (c), (c') in Proposition 2.3 on p57 of [3].

5.7 Some related properties of (V, d_V)

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let d and $A[d]$ be as in Section 5.4, and let η be the obvious inclusion mapping from A into $A[d]$, as in the previous sections. Suppose that (V, d_V) is a left module over A with differentiation, which may be considered as a left module over $A[d]$, as before.

If there is a left module W over A such that (5.6.12) holds, then

$$V \text{ is } \eta\text{-projective,} \tag{5.7.1}$$

as in Section 2.14. Similarly, if there is a left module W over A such that (5.6.13) holds, then

$$V \text{ is } \eta\text{-injective,} \tag{5.7.2}$$

as in Section 2.15. These statements correspond to (c) implies (a) and (c') implies (a') in Proposition 2.3 on p57 of [3].

Suppose that (5.7.1) holds, and let us show that there is a homomorphism σ from V into itself, as a module over A, that satisfies (5.6.1). This corresponds to the fact that (a) implies (b) in Proposition 2.3 on p57 of [3]. Let us use $V(A)$ to refer to V as a left module over A, so that

$$_{(\eta)}V(A) = A[d] \bigotimes_A V(A), \tag{5.7.3}$$

as in Section 2.9. This is isomorphic to $V(A)^x$ in a natural way, as left modules over $A[d]$, as in Section 5.5.

There is a natural homomorphism from (5.7.3) into V, as left modules over $A[d]$, obtained from the action of $A[d]$ on V on the left, as in Section 2.12. If we identify (5.7.3) with $V(A)^x$, as in the preceding paragraph, then this mapping corresponds to the mapping f from $V(A)^x$ into V defined by

$$f((v_1, v_2)) = v_1 + d_V(v_2) \tag{5.7.4}$$

for every $v_1, v_2 \in V$. Our hypothesis (5.7.1) is the same as saying that $V(A)^x$ corresponds to the direct sum of the kernel of f and another submodule of $V(A)^x$, as a left module over $A[d]$, as in Section 2.12. This means that there is a homomorphism g from V into $V(A)^x$, as left modules over $A[d]$, such that

$$f \circ g \text{ is the identity mapping on } V. \tag{5.7.5}$$

We can express g as

$$g(v) = (\tau(v), \sigma(v)) \tag{5.7.6}$$

for every $v \in V$, where σ, τ are mappings from V into itself, by definition of $V(A)^x$. More precisely, σ and τ are homomorphisms from V into itself, as a left module over A, because g is a homomorphism from V into $V(A)^x$ as left modules over $A[d]$, and thus over A. The condition that g be a homomorphism from V into $V(A)^x$ as left modules over $A[d]$ implies that

$$(0, \tau(v)) = d_{V(A)^x}(g(v)) = g(d_V(v)) = (\tau(d_V(v)), \sigma(d_V(v))) \tag{5.7.7}$$

for every $v \in V$. We also have that

$$v = f(g(v)) = \tau(v) + d_V(\sigma(v)) \tag{5.7.8}$$

for every $v \in V$, by (5.7.5). It is easy to obtain (5.6.1) from (5.7.7) and (5.7.8), as desired.

Suppose now that (5.7.2) holds, and let us check that there is a homomorphism σ from V into itself, as a module over A, that satisfies (5.6.1). This corresponds to (a') implies (b) in Proposition 2.3 on p57 of [3]. Remember that

$$^{(\eta)}V(A) = \mathrm{Hom}_A(A[d], V(A)), \tag{5.7.9}$$

as in Section 2.11. This is isomorphic to $V(A)^x$ in a natural way, as left modules over $A[d]$, as in Section 5.5 again.

There is a natural homomorphism from V into (5.7.9), as left modules over $A[d]$, obtained from the action of $A[d]$ on V on the left, as in Section 2.13. Using the identification of (5.7.9) with $V(A)^x$ mentioned in the preceding paragraph, we get that this mapping corresponds to the mapping f from V into $V(A)^x$ defined by

$$f(v) = (d_V(v), v) \tag{5.7.10}$$

for every $v \in V$.

As in Section 2.13, (5.7.2) means that $V(A)^x$ corresponds to the direct sum of $f(V)$ and another submodule of $V(A)^x$, as a left module over $A[d]$. This implies that there is a homomorphism g from $V(A)^x$ into V, as left modules over $A[d]$, such that

$$g \circ f \text{ is the identity mapping on } V. \tag{5.7.11}$$

We may express g as

$$g((v_1, v_2)) = \sigma(v_1) + \tau(v_2) \tag{5.7.12}$$

for some homomorphisms σ, τ from V into itself, as a left module over A, and every $v_1, v_2 \in V$.

Because g is a homomorphism from $V(A)^x$ into V as left modules over $A[d]$, we have that

$$\begin{aligned} d_V(\sigma(v_1)) + d_V(\tau(v_2)) &= d_V(g((v_1, v_2))) \\ &= g(d_{V(A)^x}((v_1, v_2))) = g((0, v_1)) = \tau(v_1) \end{aligned} \tag{5.7.13}$$

for every $v_1, v_2 \in V$. We also have that

$$\sigma(d_V(v)) + \tau(v) = v \tag{5.7.14}$$

for every $v \in V$, by (5.7.11). One can get (5.6.1) by taking $v_1 = v$ and $v_2 = 0$ in (5.7.13), and combining the result with (5.7.14).

If V satisfies (5.7.1) or (5.7.2), then it follows that

$$H(V) = \{0\}, \tag{5.7.15}$$

by (5.6.3). This corresponds to Corollary 2.4 on p58 of [3].

5.8 Some more properties of (V, d_V)

Let us continue with the same notation and hypotheses as at the beginning of the previous section.

Suppose for the moment that

$$V \text{ is projective as a left module over } A[d]. \tag{5.8.1}$$

This automatically implies that V is η-projective, as in Section 2.12. Put $W = B(V)$, considered as a left module over A, as in Section 5.6. The remarks in the previous two sections imply that

$$V \text{ is isomorphic to } W^x, \text{ as a left module over } A[d]. \tag{5.8.2}$$

Equivalently,

(5.8.3) $\qquad V$ is isomorphic to ${}_{(\eta)}W,$ as a left module over $A[d]$.

Using (5.8.1), we also get that

(5.8.4) $\qquad V$ is projective as a left module over A,

as in Section 2.12. More precisely, this uses the fact that $A[d]$ is projective as a left module over A, by construction. Note that V is isomorphic to W^x, as a left module over A, by (5.8.2). Of course, W^x is isomorphic to the direct sum of two copies of W, as a left module over A, by construction. It follows that

(5.8.5) $\qquad W$ is projective as a left module over A.

Conversely, suppose that there is a left module W over A such that (5.8.3) and (5.8.5) hold. Using (5.8.3), we get that V is η-projective, as in Section 2.14. As before, (5.8.3) is equivalent to (5.8.2), which implies that V is isomorphic to W^x, as a left module over A. Of course, W^x is projective as a left module over A, by (5.8.5). This means that (5.8.4) holds. It follows that (5.8.1) holds, because V is η-projective, as in Section 2.12. This corresponds to the first part of Proposition 2.5 on p58 of [3].

Similarly, suppose for the moment that

(5.8.6) $\qquad V$ is injective as a left module over $A[d]$.

This implies that V is η-injective, as in Section 2.13. If $W = B(V)$, as a left module over A again, then (5.8.2) holds, as in the previous two sections. This is equivalent to saying that

(5.8.7) $\qquad V$ is isomorphic to ${}^{(\eta)}W,$ as a left module over $A[d]$,

as before.

Note that $A[d]$ is projective as a right module over A. Using this and (5.8.6), we obtain that

(5.8.8) $\qquad V$ is injective as a left module over A,

as in Section 2.13. As before, (5.8.2) implies that V is isomorphic to W^x, as a left module over A. It follows that

(5.8.9) $\qquad W$ is injective as a left module over A,

because W^x is isomorphic to the direct sum of two copies of W, as a left module over A, by construction.

Conversely, suppose that (5.8.7) and (5.8.9) hold for some left module W over A. As in Section 2.15, (5.8.7) implies that V is η-injective. Because (5.8.7) is equivalent to (5.8.2), as before, we get that V is isomorphic to W^x, as a left module over A. Note that W^x is injective as a left module over A, by (5.8.9). Thus (5.8.8) holds. This implies that (5.8.6) holds, because V is η-injective, as in Section 2.13. This corresponds to the second part of Proposition 2.5 on p58 of [3].

5.9 Graded modules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a left or right module over A. Of course, $\mathbf{Z}$ denotes the set of integers.

As on p58 of [3], a *grading* of V is defined by a family of submodules V^j, $j \in \mathbf{Z}$, such that V corresponds to the direct sum of the V^j's, as a module over A. One may refer to this as a $\mathbf{Z}$-grading of V, to be more precise.

In this case, if $v \in V$, then v can be expressed in a unique way as

$$v = \sum_{j=-\infty}^{\infty} v^j, \tag{5.9.1}$$

where $v^j \in V^j$ for each j, and $v^j = 0$ for all but finitely many j. If $j \in \mathbf{Z}$, then v^j is called the *homogeneous component* of v of degree j, as in [3]. Every element of V^j is said to be *homogeneous of degree* j, so that 0 is homogeneous of every degree.

If $V^j = \{0\}$ when $j < 0$, then V is said to be *positive* as a graded module, as on p58 of [3]. Similarly, if $V^j = \{0\}$ when $j > 0$, then V is said to be *negative* as a graded module.

It is sometimes convenient to put

$$V_j = V^{-j} \tag{5.9.2}$$

for each $j \in \mathbf{Z}$, particularly when V is negative as a graded module.

Let U be a submodule of V, as a module over A. Thus

$$U^j = U \cap V^j \tag{5.9.3}$$

is a submodule of U, as a module over A, for every $j \in \mathbf{Z}$. If every element of U can be expressed as the sum of elements of finitely many U^j's, then U is said to be *homogeneous*, as a submodule of V, as on p58 of [3]. This means that U corresponds to the direct sum of the U^j's, $j \in \mathbf{Z}$, as a module over A, so that U is graded too.

Of course, the quotient V/U is a module over A as well. If $j \in \mathbf{Z}$, then the natural quotient mapping leads to a homomorphism

$$\text{from } V^j/U^j \text{ onto } (V^j + U)/U, \tag{5.9.4}$$

as modules over A. More precisely, the kernel of this homomorphism is trivial, by definition of U^j, and so we get an isomorphism as in (5.9.4). Note that every element of V/U may be expressed as the sum of elements of finitely many of the submodules $(V^j + U)/U$, because every element of V can be expressed as the sum of elements of finitely many V^j's.

If U is a homogeneous submodule of V, then it is easy to see that V/U corresponds to the direct sum of the submodules $(V^j + U)/U$, $j \in \mathbf{Z}$. This implies that V/U is graded as a module over A, with

$$(V/U)^j = (V^j + U)/U \tag{5.9.5}$$

for each $j \in \mathbf{Z}$, as on p58 of [3]. It is sometimes convenient to identify V/U with the direct sum of V^j/U^j, $j \in \mathbf{Z}$, under these conditions.

Let W be another left or right module over A, depending on whether V is a left or right module over A, and suppose that W is graded too. A homomorphism ϕ from V into W, as modules over A, is said to have *degree* $m \in \mathbf{Z}$ if

$$\phi(V^j) \subseteq W^{j+m} \tag{5.9.6}$$

for every $j \in \mathbf{Z}$, as on p58 of [3]. If $j \in \mathbf{Z}$, then the restriction of ϕ to V^j, considered as a mapping from V^j into W^{j+m}, is denoted ϕ^j, and called the j*th component* of ϕ, as in [3].

Observe that

$$\ker \phi \text{ is a homogeneous submodule of } V \tag{5.9.7}$$

and

$$\phi(V) \text{ is a homogeneous submodule of } W \tag{5.9.8}$$

in this case. More precisely, if $j \in \mathbf{Z}$, then

$$(\ker \phi)^j = (\ker \phi) \cap V^j = \ker \phi^j \tag{5.9.9}$$

and

$$(\phi(V))^j = \phi(V) \cap W^j = \phi^{j-m}(V^{j-m}). \tag{5.9.10}$$

Thus the coimage $V/\ker \phi$ and cokernel $W/\phi(V)$ of ϕ may be considered as graded modules over A, as on p58 of [3]. If $j \in \mathbf{Z}$, then

$$(V/\ker \phi)^j \text{ is isomorphic to } V^j/(\ker \phi)^j = V^j/\ker \phi^j \tag{5.9.11}$$

and

$$(W/\phi(V))^j \text{ is isomorphic to } W^j/(\phi(V))^j = W^j/\phi^{j-m}(V^{j-m}), \tag{5.9.12}$$

as modules over A, as before.

Note that the isomorphism from $V/\ker \phi$ onto $\phi(V)$, as modules over A, induced by ϕ has degree m. As a result, this should not normally be used to identify the coimage of ϕ with $\phi(V)$, as mentioned on p58 of [3].

If U is a homogeneous submodule of V, then the natural quotient mapping from V onto V/U has degree 0, with respect to the grading on V/U defined earlier.

Let Z be another left or right module over A, depending on whether V, W are left or right modules over A, and suppose that Z is graded as well. Also let ϕ be a homomorphism from V into W of degree m again, and let ψ be a homomorphism from W into Z, as modules over A, or degree $n \in \mathbf{Z}$. Under these conditions,

$$\psi \circ \phi \text{ has degree } m + n, \tag{5.9.13}$$

as a homomorphism from V into Z.

5.10 Complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. A left or right module (V, d_V) with differentiation is said to be a *complex* if V is graded and d_V has degree 1, as on p58 of [3].

Equivalently, suppose that V is a graded left or right module over A, and that for each $j \in \mathbf{Z}$, d_V^j is a homomorphism from V^j into V^{j+1}, as modules over A. This leads to a unique homomorphism d_V from V into itself, as a module over A, of degree 1, with jth component equal to d_V^j for each $j \in \mathbf{Z}$. The condition that $d_V \circ d_V = 0$ on V is the same as saying that

$$d_V^{j+1} \circ d_V^j = 0 \tag{5.10.1}$$

on V_j for each $j \in \mathbf{Z}$, as on p58 of [3].

The term "cochain complex" is often used for a complex in this sense, as mentioned on p58 of [3]. Similarly, a "chain complex" may be obtained by lowering indices, as in (5.9.2).

If (V, d_V) is a complex, then $Z(V) = \ker d_V$ and $B(V) = d_V(V)$ are homogeneous submodules of V, as in the previous section. If $j \in \mathbf{Z}$, then

$$Z(V)^j = (\ker d_V)^j = (\ker d_V) \cap V^j = \ker d_V^j \tag{5.10.2}$$

and

$$B(V)^j = (d_V(V))^j = d_V(V) \cap V^j = d_V^{j-1}(V^{j-1}), \tag{5.10.3}$$

as in the previous section.

This leads to natural gradings on $Z'(V) = V/d_V(V)$, $B'(V) = V/\ker d_V$, and $H(V) = Z(V)/B(V)$, as before. If $j \in \mathbf{Z}$, then

$$\begin{aligned} &Z'(V)^j = (V/d_V(V))^j \text{ is isomorphic to} \\ &V^j/(d_V(V))^j = V^j/d_V^{j-1}(V^{j-1}) \end{aligned} \tag{5.10.4}$$

and

$$\begin{aligned} &B'(V)^j = (V/\ker d_V)^j \text{ is isomorphic to} \\ &V^j/(\ker d_V)^j = V^j/\ker d_V^j, \end{aligned} \tag{5.10.5}$$

as modules over A. Similarly,

$$\begin{aligned} &H(V)^j = (Z(V)/B(V))^j \text{ is isomorphic to} \\ &Z(V)^j/B(V)^j = (\ker d_V^j)/d_V^{j-1}(V^{j-1}), \end{aligned} \tag{5.10.6}$$

as modules over A. Note that

$$H(V) \text{ is homogeneous as a submodule of } Z'(V). \tag{5.10.7}$$

Remember that $\widetilde{d}_V$ is the homomorphism from $Z'(V)$ into $Z(V)$, as modules over A, induced by d_V in the obvous way. It is easy to see that $\widetilde{d}_V$ has degree 1, because of the same property of d_V.

Let U be a submodule of V, as a module over A with differentiation. Thus U may be considered as a module over A with differentiation too, with d_U equal to the restriction of d_V to U. Suppose that U is also homogeneous as a submodule of V, so that U is graded as well. Of course, d_U has degree 1 on U, so that (U, d_U) is a complex.

As in Section 5.3, V/U is a module over A with differentiation, where $d_{V/U}$ is induced on V/U by d_V in the usual way. Similarly, V/U is graded in a natural way, because U is homogeneous, as in the previous section. It is easy to see that $d_{V/U}$ has degree 1 on V/U, so that $(V/U, d_{V/U})$ is a complex too.

5.11 Maps between complexes

Let k be a commutative ring with a mutliplcative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V, d_V), (W, d_W) be both left or both right modules over A with differentiation that are complexes. Let ϕ be a homomorphism from V into W, as modules over A with differentiation. If ϕ has degree 0, then ϕ is said to be a *map* from V into W as complexes, as on p59 of [3]. Note that $d_W \circ \phi = \phi \circ d_V$ is the same as saying that

(5.11.1) $$d_W^j \circ \phi^j = \phi^{j+1} \circ d_V^j$$

for every integer j.

As in Section 5.2, we have that $\phi(Z(V)) \subseteq Z(W)$, or equivalently

(5.11.2) $$\phi^j(Z(V)^j) \subseteq Z(W)^j$$

for each j. Similarly, $\phi(B(V)) \subseteq B(W)$, which means that

(5.11.3) $$\phi^j(B(V)^j) \subseteq B(W)^j$$

for every j.

Remember that $\phi_{Z'}$, $\phi_{B'}$ are the homomorphisms from $Z'(V)$, $B'(V)$ into $Z'(W)$, $B'(W)$, respectively, induced by ϕ. It is easy to see that $\phi_{Z'}$ and $\phi_{B'}$ have degree 0. Thus the restriction ϕ_H of $\phi_{Z'}$ to $H(V)$ has degree 0, as a homomorphism into $H(W)$. We also have that

(5.11.4) $$\phi^{j+1} \circ \widetilde{d}_V^j = \widetilde{d}_W^j \circ \phi_{Z'}^j$$

for each j, because of the analogous statement for ϕ in Section 5.2. This corresponds to some remarks on p59 of [3].

Let ψ be another map from V into W, as complexes. Suppose that σ is a homomorphism from V into W, as modules over A, of degree -1. If

(5.11.5) $$d_W \circ \sigma + \sigma \circ d_V = \phi - \psi,$$

then σ is said to be a *homotopy* between ϕ and ψ, as maps between complexes, as on p59 of [3]. Equivalently, this means that

(5.11.6) $$d_W^{j-1} \circ \sigma^j + \sigma^{j+1} \circ d_V^j = \phi^j - \psi^j$$

for every j. This implies that

$$(\phi^j - \psi^j)(Z(V)^j) \subseteq B(W)^j \tag{5.11.7}$$

for each j, so that

$$\phi_H^j = \psi_H^j \tag{5.11.8}$$

for each j, as before.

Note that ϕ is injective on V if and only if

$$\phi^j \text{ is injective on } V^j \tag{5.11.9}$$

for each j. In this case, ϕ maps $Z(V)$ onto the intersection of $\phi(V)$ and $Z(W)$, as in Section 5.2. Equivalently,

$$\phi^j(Z(V)^j) = \phi^j(V^j) \cap Z(W)^j \tag{5.11.10}$$

for each j. It follows that $\phi_{B'}$ is injective on $B'(V)$, as before, which means that

$$\phi_{B'}^j \text{ is injective on } B'(V)^j \tag{5.11.11}$$

for each j.

Similarly, ϕ maps V onto W if and only if

$$\phi^j(V^j) = W^j \tag{5.11.12}$$

for each j. Under these conditions, $\phi_{Z'}$ is surjective, which is the same as saying that

$$\phi_{Z'}^j(Z'(V)^j) = Z'(W)^j \tag{5.11.13}$$

for each j. We also have that $\phi_{B'}$ is surjective, or equivalently

$$\phi_{B'}^j(B'(V)^j) = B'(W)^j \tag{5.11.14}$$

for every j.

The surjectivity of ϕ implies that ϕ maps $B(V)$ onto $B(W)$, as in Section 5.2. This means that

$$\phi^j(B(V)^j) = B(W)^j \tag{5.11.15}$$

for each j.

If ϕ is any map from V into W as complexes, then the kernel of ϕ is a submodule of V, as a module over A with differentiation, and $\ker \phi$ is homogeneous as a submodule of V. Similarly, $\phi(V)$ is a submodule of W, as a module over A with differentiation, and $\phi(V)$ is homogeneous as a submodule of W. This means that $\ker \phi$ and $\phi(V)$ are complexes, with respect to the restriction of d_V to $\ker \phi$, and the restriction of d_W to $\phi(V)$, respectively.

5.12 Some related homomorphisms

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V, d_V) be a left or right module over A with differentiation that is a complex. Suppose that U is a submodule of V, as a module over A with differentiation, so that U is a module with differentiation too, with d_U equal to the restriction of d_V to U. Suppose that U is homogeneous as a submodule of V as well, so that (U, d_U) is a complex. Note that the natural inclusion mapping from U into V is a map between complexes.

Remember that V/U is a module over A with differentiation, where $d_{V/U}$ is induced by d_V in the usual way, and that V/U is a complex. The natural quotient mapping q from V onto V/U is a map between complexes.

Observe that

$$Z(V)^j \cap (\ker q^j) = Z(V)^j \cap U^j = Z(U)^j \tag{5.12.1}$$

for each j. This corresponds to a remark in Section 5.3, and is basically the same as (5.11.10). This is also related to some remarks on p59 of [3].

As in (5.11.15), we have that

$$q^j(B(V)^j) = B(V/U)^j \tag{5.12.2}$$

for every integer j. If $v^j \in V^j$ for some j and

$$q^j(v^j) \in B(V/U)^j, \tag{5.12.3}$$

then there is a $w^{j-1} \in V^{j-1}$ such that

$$q^j(v^j) = q^j(d_V^{j-1}(w^{j-1})), \tag{5.12.4}$$

by (5.12.2). This means that there is a $u^j \in U^j$ such that

$$v^j - d_V^{j-1}(w^{j-1}) = u^j. \tag{5.12.5}$$

This implies that

(5.12.6) the images of v^j and u^j in $Z'(V)^j$, under the natural quotient mapping from V onto $Z'(V)$, are the same.

This corresponds to a remark in Section 5.3, and is related to some remarks on p59 of [3].

If we also have that

$$v^j \in Z(V)^j, \tag{5.12.7}$$

then we get that

$$u^j \in Z(U)^j, \tag{5.12.8}$$

by (5.12.5). This corresponds to another remark in Section 5.3, and is related to some remarks on p59 of [3].

Remember that there is a natural homomorphism Δ from $H(V/U)$ into $H(U)$, as modules over A, as in Section 5.3. In fact, this homomorphism has degree 1. Indeed, let an integer j and an element of $H(V/U)^j$ be given. This element can be represented as the image of

$$q^j(v^j) \in Z(V/U)^j \tag{5.12.9}$$

under the natural quotient mapping from $Z(V/U)$ onto $H(V/U)$, where $v^j \in V^j$. Using (5.12.9), we get that

$$q^{j+1}(d_V^j(v^j)) = d_{V/U}^j(q^j(v^j)) = 0, \tag{5.12.10}$$

so that $d_V^j(v^j) \in U^{j+1}$.

More precisely,

$$d_V^j(v^j) \in Z(U)^{j+1}. \tag{5.12.11}$$

Remember that Δ of the image of $q^j(v^j)$ in $H(V/U)^j$ is equal to the image of $d_V^j(v^j)$ in $H(U)$, under the natural quotient mapping from $Z(U)$ onto $H(U)$. It follows that Δ of the image of $q^j(v^j)$ in $H(V/U)$ is an element of $H(U)^{j+1}$, because of (5.12.11). This means that

$$\Delta(H(V/U)^j) \subseteq H(U)^{j+1}, \tag{5.12.12}$$

as desired.

Let $q_{Z'}$ be the homomorphism from $Z'(V)$ into $Z'(V/U)$ induced by q, as usual. The restriction q_H of $q_{Z'}$ to $H(V)$ maps $H(V)$ into $H(V/U)$. Remember that $\Delta \circ q_H = 0$, as in Section 5.3. This implies that

$$\Delta^j \circ q_H^j = 0 \tag{5.12.13}$$

for every j.

Let an integer j and $v^j \in V^j$ be given, with $q^j(v^j) \in Z(V/U)^j$, and suppose that Δ^j maps the image of $q^j(v^j)$ in $H(V/U)^j$ to 0 in $H(U)^{j+1}$. This means that

$$d_V^j(v^j) \in B(U)^{j+1}, \tag{5.12.14}$$

which is to say that $d_V^j(v^j) = d_V^j(u^j)$ for some $u^j \in U^j$. Thus

$$q^j(v^j) = q^j(v^j - u^j) \text{ and } v^j - u^j \in Z(V)^j. \tag{5.12.15}$$

This implies that q_H^j of the image of $v^j - u^j$ in $H(V)^j$ is the same as the image of $q^j(v^j)$ in $H(V/U)^j$. It follows that

$$q_H^j(H(V)^j) = \ker \Delta^j, \tag{5.12.16}$$

as in Section 5.3, and which is related to some remarks on p59 of [3].

Let ι be the natural inclusion mapping from U into V, and let $\iota_{Z'}$ be the induced mapping from $Z'(U)$ into $Z'(V)$, as usual. The restriction of $\iota_{Z'}$ to

$H(U)$ is denoted ι_H, as before, and maps $H(U)$ into $H(V)$. Remember that $\iota_H \circ \Delta = 0$, as in Section 5.3, so that

$$\iota_H^{j+1} \circ \Delta^j = 0 \tag{5.12.17}$$

for every j.

Let an integer j and $w^{j+1} \in Z(U)^{j+1}$ be given, and suppose that the image of w^{j+1} in $H(U)^{j+1}$ is in the kernel of ι_H^{j+1}. This implies that there is a $v^j \in V^j$ such that

$$w^{j+1} = d_V^j(v^j). \tag{5.12.18}$$

It follows that

$$d_{V/U}^j(q^j(v^j)) = q^{j+1}(d_V^j(v^j)) = q^{j+1}(w^{j+1}) = 0, \tag{5.12.19}$$

so that $q^j(v^j) \in Z(V/U)^j$. This means that the image of w^{j+1} in $H(U)^{j+1}$ is the same as Δ^j of the image of $q^j(v^j)$ in $H^j(V/U)$. This shows that

$$\ker \iota_H^{j+1} = \Delta^j(H(V/U)^j), \tag{5.12.20}$$

as in Section 5.3, and which is related to some remarks on p59 of [3].

5.13 Double gradings

Let k be a commutative ring with a multiplicative identity element, let A be an associative algebra over k with a multiplicative identity element e_A, and let V be a left or right module over A. A *double grading* or *bi-grading* of V is defined by a family of submodules $V^{j,l}$, $j, l \in \mathbf{Z}$, such that V corresponds to the direct sum of the $V^{j,l}$'s, as a module over A, as on p60 of [3]. One may also refer to this as a $\mathbf{Z}^2$-grading of V, as a module over A. The elements of $V^{j,l}$ are said to be *bihomogeneous of bidegree* (j, l).

If $r \in \mathbf{Z}$, then put

$$V^r = \sum_{j+l=r} V^{j,l}, \tag{5.13.1}$$

which is the subset of V consisting of finite sums of elements of $V^{j,l}$, with $j, l \in \mathbf{Z}$ and $j + l = r$. This is a submodule of V, which corresponds to the direct sum of the $V^{j,l}$'s with $j + l = r$, as a module over A. Note that V corresponds to the direct sum of the V^r's, $r \in \mathbf{Z}$, as a module over A. This is the *associated grading* of V by the given double grading, as on p60 of [3]. A bihomogeneous element of V of bidegree (j, l) is thus homogeneous of degree $j + l$ with respect to the associated grading.

If $V^{j,l} = \{0\}$ when $j < 0$ or $l < 0$, then V is said to be *positive* as a doubly-graded module, as on p60 of [3]. Similarly, if $V^{j,l} = \{0\}$ when $j > 0$ or $l > 0$, then V is said to be *negative* as a doubly-graded module. As before, it is sometimes convenient to put

$$V_{j,l} = V^{-j,-l} \tag{5.13.2}$$

for every $j, l \in \mathbf{Z}$, particularly when V is negative as a doubly-graded module.

Let U be a submodule of V, as a module over A, so that

$$U^{j,l} = U \cap V^{j,l} \tag{5.13.3}$$

is a submodule of U for every $j, l \in \mathbf{Z}$. If every element of U can be expressed as the sum of elements of finitely many $U^{j,l}$'s, then U is said to be *bihomogeneous*, as a submodule of V, as on p60 of [3]. In this case, U corresponds to the direct sum of the $U^{j,l}$'s, $j, l \in \mathbf{Z}$, as a module over A, so that U is doubly-graded as well. Note that this implies that U is a homogeneous submodule of V with respect to the associated grading.

If $j, l \in \mathbf{Z}$, then the natural quotient mapping from V onto V/U leads to a homomorphism

$$\text{from } V^{j,l}/U^{j,l} \text{ onto } (V^{j+l} + U)/U, \tag{5.13.4}$$

as modules over A. The kernel of this homomorphism is trivial, by the definition of $U^{j,l}$, so that we get an isomorphism as in (5.13.4). Every element of V/U can be expressed as the sum of elements of finitely many of the submodules $(V^{j,l}+U)/U$, because every element of V can be expressed as the sum of finitely many $V^{j,l}$'s.

If U is a bihomogeneous submodule of V, then one can check that V/U corresponds to the direct sum of the submodules $(V^{j,l} + U)/U$, $j, l \in \mathbf{Z}$. This means that V/U is doubly-graded as a module over A, with

$$(V/U)^{j,l} = (V^{j,l} + U)/U \tag{5.13.5}$$

for every $j, l \in \mathbf{Z}$. In this case, it is sometimes convenient to identify V/U with the direct sum of $V^{j,l}/U^{j,l}$, as on p60 of [3].

Let W be another doubly-graded left or right module over A, depending on whether V is a left or right module over A. Also let ϕ be a homomorphism from V into W, as modules over A, and let p, q be integers. We say that ϕ has *bidegree* (p, q) if

$$\phi(V^{j,l}) \subseteq W^{j+p,l+q} \tag{5.13.6}$$

for every $j, l \in \mathbf{Z}$, as on p60 of [3]. Under these conditions, if $j, l \in \mathbf{Z}$, then the restriction of ϕ to $V^{j,l}$, considered as a mapping from $V^{j,l}$ into $W^{j+p,l+q}$, is denoted $\phi^{j,l}$, and called the (p, q)-*component* of ϕ, as in [3]. A homomorphism of bidegree (p, q) from V into W may be considered as a homomorphism of degree $p + q$ with respect to the associated gradings on V, W.

In this case, it is easy to see that

$$\ker \phi \text{ is a bihomogeneous submodule of } V \tag{5.13.7}$$

and

$$\phi(V) \text{ is a bihomogeneous submodule of } W. \tag{5.13.8}$$

If $j, l \in \mathbf{Z}$, then

$$(\ker \phi)^{j,l} = (\ker \phi) \cap V^{j,l} = \ker \phi^{j,l} \tag{5.13.9}$$

and

$$(\phi(V))^{j,l} = \phi(V) \cap W^{j,l} = \phi^{j-p,l-q}(V^{j-p,l-q}). \tag{5.13.10}$$

The coimage $V/\ker\phi$ and cokernel $W/\phi(V)$ of ϕ may be considered as doubly-graded modules over A. If $j, l \in \mathbf{Z}$, then

$$(V/\ker\phi)^{j,l} \text{ is isomorphic to } V^{j,l}/(\ker\phi)^{j,l} = V^{j,l}/\ker\phi^{j,l} \tag{5.13.11}$$

and

$$\begin{aligned}(W/\phi(V))^{j,l} &\text{ is isomorphic to} \\ W^{j,l}/\phi(V)^{j,l} &= W^{j,l}/\phi^{j-p,l-q}(V^{j-p,l-q}),\end{aligned} \tag{5.13.12}$$

as modules over A. The isomorphism from $V/\ker\phi$ onto $\phi(V)$, as modules over A, induced by ϕ has bidegree (p, q) as well. This is related to some remarks on p60 of [3].

If U is a bihomogeneous submodule of V, then the natural quotient mapping from V onto V/U has bidegree $(0, 0)$, with respect to the double grading on V/U defined previously.

Let Z be another doubly-graded left or right module over A, depending on whether V, W are left or right modules over A. If ψ is a homomorphism from W into Z, as modules over A, of bidegree (p', q'), then

$$\psi \circ \phi \text{ has bidegree } (p + p', q + q'), \tag{5.13.13}$$

as a homomorphism from V into Z.

One may also consider n-graded modules over A for any positive integer n, as on p62 of [3].

5.14 Double complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a doubly-graded left or right module over A.

Suppose that $d_1 = d_{V,1}$ and $d_2 = d_{V,2}$ are two differentiation operators on V, which is to say that d_1, d_2 are homomorphisms from V into itself, as a module over A, such that

$$d_1 \circ d_1 = d_2 \circ d_2 = 0. \tag{5.14.1}$$

Suppose as well that d_1 has bidegree $(1, 0)$ on V, d_2 has bidegree $(0, 1)$, and that d_1, d_2 anticommute on V, so that

$$d_2 \circ d_1 + d_1 \circ d_2 = 0. \tag{5.14.2}$$

Under these conditions, V is said to be a *double complex* with respect to d_1, d_2, as on p60 of [3].

If $j, l \in \mathbf{Z}$, then we get homomorphisms $d_1^{j,l}$, $d_2^{j,l}$ from $V^{j,l}$ into $V^{j+1,l}$, $V^{j,l+1}$, respectively, as modules over A, as in the previous section. As before, (5.14.1) is the same as saying that

$$d_1^{j+1,l} \circ d_1^{j,l} = 0 \tag{5.14.3}$$

nad

$$d_2^{j,l+1} \circ d_2^{j,l} = 0 \tag{5.14.4}$$

on $V^{j,l}$ for every $j, l \in \mathbf{Z}$. Similarly, (5.14.2) is the same as asking that

$$d_2^{j+1,l} \circ d_1^{j,l} + d_1^{j,l+1} \circ d_2^{j,l} = 0 \tag{5.14.5}$$

on $V^{j,l}$ for every $j, l \in \mathbf{Z}$.

Remember that V may be considered as a singly-graded module over A, with V^r as in (5.13.1) for every $r \in \mathbf{Z}$. Put

$$d = d_V = d_1 + d_2, \tag{5.14.6}$$

which is a homomorphism from V into itself of degree 1, as a singly-graded module over A. Observe that $d \circ d = 0$ on V, by (5.14.1) and (5.14.2). Thus V is a single complex with respect to d, as on p61 of [3]. One calls d the *total differentiation operator* on V, as in [3]. If $j, l \in \mathbf{Z}$, then the restriction of d to $V^{j,l}$ is equal to

$$d_1^{j,l} + d_2^{j,l}. \tag{5.14.7}$$

In particular,

$$d(V^{j,l}) \subseteq V^{j+1,l} + V^{j,l+1}. \tag{5.14.8}$$

Conversely, suppose that d is a homomorphism from V into itself of degree 0, as a singly-graded module over A, such that $d \circ d = 0$ and (5.14.8) holds for every $j, l \in \mathbf{Z}$. If $j, l \in \mathbf{Z}$, then there are unique homomorphisms $d_1^{j,l}$, $d_2^{j,l}$ from $V^{j,l}$ into $V^{j+1,l}$, $V^{j,l+1}$, respectively, as modules over A, such that the restriction of d to $V^{j,l}$ is equal to (5.14.7). One can check that (5.14.3), (5.14.4), and (5.14.5) hold for every $j, l \in \mathbf{Z}$ in this case. Let d_1, d_2 be the homomorphisms from V into itself whose restrictions to $V^{j,l}$ are equal to $d_1^{j,l}$, $d_2^{j,l}$, respectively, for every $j, l \in \mathbf{Z}$. It follows that V is a double complex with respect to d_1, d_2, and that d is the corresponding total differentiation operator, as on p61 of [3].

If V is a double complex, then the corresponding modules $Z(V)$, $B(V)$, $Z'(V)$, $B'(V)$, and $H(V)$ are defined by considering V as a single complex, as before. More precisely, these are singly-graded modules over A.

If n is any positive integer, then one may consider n-complexes over A, as on p62 of [3].

5.15 Maps between double complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with multiplicative identity element e_A. Also let

V and W be both left or both right modules over A that are double complexes, with differentiation operators $d_{V,1}$, $d_{V,2}$ and $d_{W,1}$, $d_{W,2}$, respectively.

Suppose that ϕ is a homomorphism from V into W, as modules over A, of bidegree $(0,0)$. If

$$d_{W,1} \circ \phi = \phi \circ d_{V,1} \tag{5.15.1}$$

and

$$d_{W,2} \circ \phi = \phi \circ d_{V,2}, \tag{5.15.2}$$

then ϕ is said to be a *map* from V into W, as double complexes, as on p61 of [3]. In this case, it is easy to see that ϕ is a map from V into W, considered as single complexes, as in the previous section.

Note that (5.15.1) is the same as saying that

$$d_{W,1}^{j,l} \circ \phi^{j,l} = \phi^{j+1,l} \circ d_{V,1}^{j,l} \tag{5.15.3}$$

for every $j, l \in \mathbf{Z}$. Similarly, (5.15.2) is the same as saying that

$$d_{W,2}^{j,l} \circ \phi^{j,l} = \phi^{j,l+1} \circ d_{V,2}^{j,l} \tag{5.15.4}$$

for every j, l.

Let ϕ and ψ be maps from V into W, as double complexes. Also let σ_1, σ_2 be homomorphisms from V into W, as modules over A, of bidegrees $(-1,0)$, $(0,-1)$, respectively. Suppose that

$$d_{W,1} \circ \sigma_1 + \sigma_1 \circ d_{V,1} + d_{W,2} \circ \sigma_2 + \sigma_2 \circ d_{V,2} = \phi - \psi. \tag{5.15.5}$$

Suppose in addition that

$$\sigma_1 \circ d_{V,2} + d_{W,2} \circ \sigma_1 = 0 \tag{5.15.6}$$

and

$$\sigma_2 \circ d_{V,1} + d_{W,1} \circ \sigma_2 = 0. \tag{5.15.7}$$

Under these conditions, (σ_1, σ_2) is said to define a *homotopy* between ϕ and ψ, as on p61 of [3].

Of course, (5.15.5) is the same as saying that

$$\begin{aligned} d_{W,1}^{j-1,l} \circ \sigma_1^{j,l} + \sigma_1^{j+1,l} \circ d_{V,1}^{j,l} + d_{W,2}^{j,l-1} \circ \sigma_2^{j,l} + \sigma_2^{j,l+1} \circ d_{V,2}^{j,l} \\ = \phi^{j,l} - \psi^{j,l} \end{aligned} \tag{5.15.8}$$

for every $j, l \in \mathbf{Z}$. Similarly, (5.15.6) is the same as saying that

$$\sigma_1^{j,l+1} \circ d_{V,2}^{j,l} + d_{W,2}^{j-1,l} \circ \sigma_1^{j,l} = 0 \tag{5.15.9}$$

for every j, l. We also have that (5.15.7) is the same as saying that

$$\sigma_2^{j+1,l} \circ d_{V,1}^{j,l} + d_{W,1}^{j,l-1} \circ \sigma_2^{j,l} = 0 \tag{5.15.10}$$

for every j, l.

Put

$$\sigma = \sigma_1 + \sigma_2, \tag{5.15.11}$$

which is a homomorphism from V into W of degree -1, as singly-graded modules over A. Let d_V, d_W be the total differentiation operators on V, W, respectively. One can check that

$$d_W \circ \sigma + \sigma \circ d_V = \phi - \psi, \tag{5.15.12}$$

using (5.15.8), (5.15.9), and (5.15.10). This means that σ is a homotopy between ϕ and ψ, as maps from V into W, as single complexes over A, as on p62 of [3]. If $j, l \in \mathbf{Z}$, then the restriction of σ to $V^{j,l}$ is equal to

$$\sigma_1^{j,l} + \sigma_2^{j,l}, \tag{5.15.13}$$

so that

$$\sigma(V^{j,l}) \subseteq V^{j-1,l} + V^{j,l-1}, \tag{5.15.14}$$

as in [3].

Conversely, suppose that σ is a homotopy between ϕ and ψ, as maps from V into W, as single complexes over A, that satisfies (5.15.14) for every $j, l \in \mathbf{Z}$. In this case, if $j, l \in \mathbf{Z}$, then there are unique homomorphisms $\sigma_1^{j,l}$, $\sigma_2^{j,l}$ from $V^{j,l}$ into $V^{j-1,l}$, $V^{j,l-1}$, respectively, as modules over A, such that the restriction of σ to $V^{j,l}$ is equal to (5.15.13). This leads to homomorphisms σ_1, σ_2 from V into W, as modules over A, of bidegrees $(-1, 0)$, $(0, -1)$, respectively, whose restrictions to $V^{j,l}$ are equal to $\sigma_1^{j,l}$, $\sigma_2^{j,l}$, respectively, for each $j, l \in \mathbf{Z}$. One can check that (5.15.8), (5.15.9), and (5.15.10) hold for every $j, l \in \mathbf{Z}$, because of (5.15.12). This implies that (5.15.5), (5.15.6), and (5.15.7) hold, so that (σ_1, σ_2) is a homotopy between ϕ and ψ, as maps from V into W as double complexes, as on p62 of [3].

Chapter 6

More on differentiation

6.1 Tensor products and gradings

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a graded right module over A, and let W be a graded left module over A.

Suppose that $V \bigotimes_A W$ is a tensor product of V and W, as right and left modules over A, respectively. Thus $V \bigotimes_A W$ is a module over k that is isomorphic to the direct sum of a family of tensor products $V^j \bigotimes_A W^l$, $j, l \in \mathbf{Z}_+$. This means that $V \bigotimes_A W$ is a doubly-graded module over k, with

$$(6.1.1) \qquad \Big(V \bigotimes_A W\Big)^{j,l} = V^j \bigotimes_A W^l$$

for every $j, l \in \mathbf{Z}$, considered as a submodule of $V \bigotimes_A W$. This corresponds to a remark on p63 of [3].

If $r \in \mathbf{Z}$, then put

$$(6.1.2) \qquad \Big(V \bigotimes_A W\Big)^r = \sum_{j+l=r} \Big(V \bigotimes_A W\Big)^{j,l} = \sum_{j+l=r} V^j \bigotimes_A W^l,$$

as in Section 5.13. This is the subset of $V \bigotimes_A W$ consisting of finite sums of elements of (6.1.1), with $j, l \in \mathbf{Z}$ and $j + l = r$, as before. This is a submodule of $V \bigotimes_A W$, as a module over k, which corresponds to the direct sum of (6.1.1) with $j + l = r$, as a module over k. We also have that $V \bigotimes_A W$ corresponds to the direct sum of (6.1.2) with $r \in \mathbf{Z}$, as a module over k. This defines the grading on $V \bigotimes_A W$ associated to the double grading defined in the preceding paragraph, as in Section 5.13.

Let V_1 be another graded right module over A, let W be another graded left module over A, and let $V_1 \bigotimes_A W_1$ be a tensor product of V_1 and W_1 over A. This is a doubly-graded module over k, as before.

Let ϕ_1, ψ_1 be homomorphisms from V, W into V_1, W_1, as modules over A, of degrees $p_1, q_1 \in \mathbf{Z}$, respectively. The corresponding mapping from $V \bigotimes_A W$ into

$V_1 \bigotimes_A W_1$ is defined a bit differently from before, without gradings. Namely, there is a unique homomorphism

$$\phi_1 \otimes \psi_1 \text{ from } V \bigotimes_A W \text{ into } V_1 \bigotimes_A W_1, \tag{6.1.3}$$

as modules over k, such that for every $j, l \in \mathbf{Z}$, $v^j \in V^j$, and $w^l \in W_l$, we have that

$$(\phi_1 \otimes \psi_1)(v^j \otimes w^l) = (-1)^{j\,q_1} (\phi_1(v^j) \otimes \psi_1(w^l)), \tag{6.1.4}$$

as on p64 of [3]. Note that

$$\phi_1 \otimes \psi_1 \text{ has bidegree } (p_1, q_1). \tag{6.1.5}$$

Equivalently, if $j, l \in \mathbf{Z}$, then the restriction $(\phi_1 \otimes \psi_1)^{j,l}$ of $\phi_1 \otimes \psi_1$ to $\big(V \bigotimes_A W\big)^{j,l}$ is a homomorphism into $\big(V_1 \bigotimes_A W_1\big)^{j+p_1, l+q_1}$, as modules over k. This homomorphism corresponds to

$$\begin{aligned} &(-1)^{j\,q_1} \text{ times the homomorphism from } V^j \bigotimes_A W^l \text{ into} \\ &V_1^{j+p_1} \bigotimes_A W_1^{l+q_1}, \text{ as modules over } k, \text{ associated to } \phi_1^j \text{ and } \psi_1^l \end{aligned} \tag{6.1.6}$$

in the usual way, as on p63 of [3].

Let V_2 be a third graded right module over A, let W_2 be a third graded left module over A, and let $V_2 \bigotimes_A W_2$ be a tensor product of V_2 and W_2 over A, which is a doubly-graded module over k. Also let ϕ_2, ψ_2 be homomorphisms from V_1, W_2 into V_2, W_2, as modules over A, of degrees $p_2, q_2 \in \mathbf{Z}$, respectively. This leads to a homomorphism $\phi_2 \otimes \psi_2$ from $V_1 \bigotimes_A W_1$ into $V_2 \bigotimes_A W_2$, as modules over k, of bidegree (p_2, q_2), as before.

Similarly, $\phi_2 \circ \phi_1$, $\psi_2 \circ \psi_1$ are homomorphisms from V, W into V_2, W_2, as modules over A, of degrees $p_1 + p_2$, $q_1 + q_2$, respectively. This leads to a homomorphism $(\phi_2 \circ \phi_1) \otimes (\psi_2 \circ \psi_1)$ from $V \bigotimes_A W$ into $V_2 \bigotimes_A W_2$, as modules over k, of bidegree $(p_1 + p_2, q_1 + q_2)$. One can check that

$$(\phi_2 \circ \phi_1) \otimes (\psi_2 \circ \psi_1) = (-1)^{p_1\,q_2} (\phi_2 \otimes \psi_2) \circ (\phi_1 \otimes \psi_1), \tag{6.1.7}$$

as on p63 of [3].

6.2 Tensor products of complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V, d_V) be a graded right module over A with differentiation that is a complex, and let (W, d_W) be a graded left module over A with differentiation that is a complex. Suppose that $V \bigotimes_A W$ is a tensor product of V and W over A, which is doubly-graded as a module over k, as in the previous section.

The identity mappings I_V, I_W on V, W are homomorphisms from V, W into themselves, respectively, as modules over A, of degree 0. Let

$$\delta_1 = d_V \otimes I_W \tag{6.2.1}$$

and

$$\delta_2 = I_V \otimes d_W \tag{6.2.2}$$

be the homomorphisms from $V \bigotimes_A W$ into itself, as a module over k, defined as in the previous section. If $j, l \in \mathbf{Z}$, $v^j \in V_j$, and $w^l \in W^l$, then

$$\delta_1(v^j \otimes w^l) = d_V(v^j) \otimes w^l \tag{6.2.3}$$

and

$$\delta_2(v^j \otimes w^l) = (-1)^j \, (v^j \otimes d_W(w^l)), \tag{6.2.4}$$

and δ_1, δ_2 are uniquely determined by these properties. Of course, δ_1 has bidegree $(1, 0)$, and δ_2 has bidegree $(0, 1)$.

Observe that

$$\delta_1 \circ \delta_1 = \delta_2 \circ \delta_2 = 0 \tag{6.2.5}$$

on $V \bigotimes_A W$, because $d_V \circ d_V = 0$ on V and $d_W \circ d_W = 0$ on W. We also have that

$$\delta_1 \circ \delta_2 + \delta_2 \circ \delta_1 = 0 \tag{6.2.6}$$

on $V \bigotimes_A W$, by (6.1.7). This shows that

(6.2.7) $V \bigotimes_A W$ is a double complex with respect to δ_1 and δ_2,

as on p63f of [3].

Let (V_0, d_{V_0}) be another graded right module over A with differentiation that is a complex, let (W_0, d_{W_0}) be another graded left module over A with differentiation that is a complex, and let $V_0 \bigotimes_A W_0$ be a tensor product of V_0 and W_0 over A. This is a double complex with respect to the corresponding differentiation operators $\delta_{1,0}$ and $\delta_{2,0}$, as before.

Let ϕ, ψ be mappings from V, W into V_0, W_0, respectively, as complexes. In this case, $\phi \otimes \psi$ is the same as the mapping from $V \bigotimes_A W$ into $V_0 \bigotimes_A W_0$ defined without using gradings, because ψ has degree 0, by hypothesis. One can check that

(6.2.8) $\phi \otimes \psi$ is a map from $V \bigotimes_A W$ into $V_0 \bigotimes_A W_0$,

as double complexes, as on p63 of [3].

Let ϕ_0, ψ_0 be another pair of mappings from V, W into V_0, W_0, respectively, as complexes. Suppose that σ, τ are homotopies between ϕ, ψ and ϕ_0, ψ_0, respectively. Thus σ, τ are homomorphisms from V, W into V_0, W_0, respectively, as modules over A, and with degree -1, such that

$$d_{V_0} \circ \sigma + \sigma \circ d_V = \phi - \phi_0 \tag{6.2.9}$$

and

$$d_{W_0} \circ \tau + \tau \circ d_W = \psi - \psi_0. \tag{6.2.10}$$

Put

$$\sigma_1 = \sigma \otimes \psi \tag{6.2.11}$$

and

$$\sigma_2 = \phi_0 \otimes \tau, \tag{6.2.12}$$

which are homomorphisms from $V \bigotimes_A W$ into $V_0 \bigotimes_A W_0$, as modules over k, with bidegrees $(-1, 0)$ and $(0, -1)$, respectively.

Observe that

$$\begin{aligned} \sigma_1 \circ \delta_2 = (\sigma \otimes \psi) \circ (I_V \otimes d_W) &= \sigma \otimes (\psi \circ d_W) = \sigma \otimes (d_{W_0} \circ \psi) \\ &= -(I_{V_0} \otimes d_{W_0}) \circ (\sigma \otimes \psi) = -\delta_{2,0} \circ \sigma_1, \end{aligned} \tag{6.2.13}$$

by (6.1.7). Similarly,

$$\begin{aligned} \sigma_2 \circ \delta_1 = (\phi_0 \otimes \tau) \circ (d_V \otimes I_W) &= -(\phi_0 \circ d_V) \otimes \tau = -(d_{V_0} \circ \phi_0) \otimes \tau \\ &= -(d_{V_0} \otimes I_{W_0}) \circ (\phi_0 \otimes \tau) = -\delta_{1,0} \circ \sigma_2. \end{aligned} \tag{6.2.14}$$

We also have that

$$\delta_{1,0} \circ \sigma_1 = (d_{V_0} \otimes I_{W_0}) \circ (\sigma \otimes \psi) = (d_{V_0} \circ \sigma) \otimes \psi \tag{6.2.15}$$

and

$$\sigma_1 \circ \delta_1 = (\sigma \otimes \psi) \circ (d_V \otimes I_W) = (\sigma \circ d_V) \otimes \psi. \tag{6.2.16}$$

This implies that

$$\delta_{1,0} \circ \sigma_1 + \sigma_1 \circ \delta_1 = (d_{V_0} \circ \sigma + \sigma \circ d_V) \otimes \psi = (\phi - \phi_0) \otimes \psi. \tag{6.2.17}$$

Similarly,

$$\delta_{2,0} \circ \sigma_2 = (I_{V_0} \otimes d_{W_0}) \circ (\phi_0 \otimes \tau) = \phi_0 \otimes (d_{W_0} \circ \tau) \tag{6.2.18}$$

and

$$\sigma_2 \circ \delta_2 = (\phi_0 \otimes \tau) \circ (I_V \otimes d_W) = \phi_0 \otimes (\tau \circ d_W). \tag{6.2.19}$$

It follows that

$$\delta_{2,0} \circ \sigma_2 + \sigma_2 \circ \delta_2 = \phi_0 \otimes (d_{W_0} \circ \tau + \tau \circ d_W) = \phi_0 \otimes (\psi - \psi_0). \tag{6.2.20}$$

Combining (6.2.17) and (6.2.20), we obtain that

$$\begin{aligned} &\delta_{1,0} \circ \sigma_1 + \sigma_1 \circ \delta_1 + \delta_{2,0} \circ \sigma_2 + \sigma_2 \circ \delta_2 \\ &\quad = (\phi - \phi_0) \otimes \psi + \phi_0 \otimes (\psi - \psi_0) = \phi \otimes \psi - \phi_0 \otimes \psi_0. \end{aligned} \tag{6.2.21}$$

This shows that

(6.2.22) (σ_1, σ_2) defines a homotopy between $\phi \otimes \psi$ and $\phi_0 \otimes \psi_0$,

as maps from $V \bigotimes_A W$ into $V_0 \bigotimes_A W_0$, as double complexes. This corresponds to a remark on p63 of [3].

6.3 Gradings and homomorphisms

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose for the moment that V, W are both left or both right modules over A, so that the corresponding space $\mathrm{Hom}_A(V, W)$ of homomorphisms from V into W, as modules over A, is a module over k.

Let V_1, W_1 be left or right modules over A, depending on whether V, W are left or right modules over A. Also let ϕ_1 be a homomorphism from V_1 into V, and let ψ_1 be a homomorphism from W into W_1, as modules over A. If $\alpha \in \mathrm{Hom}(V, W)$, then put

$$(\mathrm{Hom}(\phi_1, \psi_1))(\alpha) = \psi_1 \circ \alpha \circ \phi_1, \tag{6.3.1}$$

as on p20 of [3]. Note that $\mathrm{Hom}(\phi_1, \psi_1)$ defines a homomorphism

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V_1, W_1), \tag{6.3.2}$$

as modules over k, as in [3].

Let V_2, W_2 be left or right modules over A as well, depending on whether V, W are left or right modules over A. Suppose that ϕ_2 is a homomorphism from V_2 into V_1, and that ψ_2 is a homomorphism from W_1 into W_2, as modules over A. Thus $\mathrm{Hom}(\phi_2, \psi_2)$ defines a homomorphism

$$\text{from } \mathrm{Hom}_A(V_1, W_1) \text{ into } \mathrm{Hom}_A(V_2, W_2), \tag{6.3.3}$$

as modules over k, as before.

Under these conditions, $\phi_1 \circ \phi_2$ is a homomorphism from V_2 into V, and $\psi_2 \circ \psi_1$ is a homomorphism from W into W_2, as modules over A. This means that $\mathrm{Hom}(\phi_1 \circ \phi_2, \psi_2 \circ \psi_1)$ defines a homomorphism

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V_2, W_2), \tag{6.3.4}$$

as usual. One can check that

$$\mathrm{Hom}(\phi_1 \circ \phi_2, \psi_2 \circ \psi_1) = \mathrm{Hom}(\phi_2, \psi_2) \circ \mathrm{Hom}(\phi_1, \psi_1). \tag{6.3.5}$$

Suppose now that V, W are graded modules over A. We would like to define $\mathrm{Hom}_A^{gr}(V, W)$ as a doubly-graded module over k, with

$$\big(\mathrm{Hom}_A^{gr}(V, W)\big)^{j,l} = \mathrm{Hom}_A(V^{-j}, W^l) \tag{6.3.6}$$

for every $j, l \in \mathbf{Z}$, as on p62 of [3]. We can simply take $\mathrm{Hom}_A^{gr}(V, W)$ to be the direct sum of (6.3.6), as a module over k, but it is sometimes convenient to realize it in a slightly different way.

Let $|V|$, $|W|$ be the underlying modules over A corresponding to V, W, respectively, without gradings, as on p63 of [3]. If $\alpha \in \mathrm{Hom}_A^{gr}(V, W)$ and $j, l \in \mathbf{Z}$, then let $\alpha^{j,l}$ be the corresponding element of (6.3.6), as usual. This leads to a unique homomorphism from $|V|$ into $|W|$, that we shall also denote by α, with

$$\alpha = \sum_{l=-\infty}^{\infty} \alpha^{j,l} \quad \text{on } V^{-j} \tag{6.3.7}$$

for every $j \in \mathbf{Z}$. More precisely, $\alpha^{j,l} = 0$ for all but finitely many $(j,l) \in \mathbf{Z}^2$, by hypothesis. This implies that all but finitely many terms in the sum on the right side of (6.3.7) are equal to 0 for any j, and that all of the terms in the sum are equal to 0 for all but finitely many j.

If $j, l \in \mathbf{Z}$, then $\alpha^{j,l}$ is uniquely determined by this homomorphism from $|V|$ into $|W|$, by restricting the homomorphism to V^{-j}, and taking the component of this restriction that maps into W^l. This defines an injective homomorphism

$$\text{from } \mathrm{Hom}^{gr}_A(V,W) \text{ into } \mathrm{Hom}_A(|V|,|W|), \tag{6.3.8}$$

as modules over k, and we may identify $\mathrm{Hom}^{gr}_A(V,W)$ with the corresponding submodule of $\mathrm{Hom}_A(|V|,|W|)$, as a module over k. Note that $\mathrm{Hom}^{gr}_A(V,W)$ normally corresponds to a proper submodule of $\mathrm{Hom}_A(|V|,|W|)$, as on p63 of [3].

If $r \in \mathbf{Z}$, then put

$$\big(\mathrm{Hom}^{gr}_A(V,W)\big)^r = \sum_{j+l=r} \big(\mathrm{Hom}^{gr}_A(V,W)\big)^{j,l}, \tag{6.3.9}$$

as in Section 5.13. This is a submodule of $\mathrm{Hom}^{gr}_A(V,W)$, as a module over k, which corresponds to the direct sum of (6.3.6) over all $j, l \in \mathbf{Z}$ with $j + l = r$. As before, $\mathrm{Hom}^{gr}_A(V,W)$ corresponds to the direct sum of (6.3.9) over $r \in \mathbf{Z}$, as a module over k, which defines the grading on $\mathrm{Hom}^{gr}_A(V,W)$ associated to the double grading already defined.

Let $M^r(V,W)$ be the space of homomorphisms from V into W, as modules over A, of degree r, as in Exercise 5 on p73 of [3]. This is a submodule of $\mathrm{Hom}_A(|V|,|W|)$, as a module over k. Observe that

$$\big(\mathrm{Hom}^{gr}_A(V,W)\big)^r = \big(\mathrm{Hom}^{gr}_A(V,W)\big) \cap M^r(V,W). \tag{6.3.10}$$

Put

$$M(V,W) = \sum_{r=-\infty}^{\infty} M^r(V,W), \tag{6.3.11}$$

as in Exercise 5 on p73 of [3]. This is a submodule of $\mathrm{Hom}_A(|V|,|W|)$, as a module over k, which corresponds to the direct sum of $M^r(V,W)$, $r \in \mathbf{Z}$. We also have that

$$\mathrm{Hom}^{gr}_A(V,W) \subseteq M(V,W), \tag{6.3.12}$$

by (6.3.10).

6.4 Induced homomorphisms on $\mathrm{Hom}_A^{gr}(V,W)$

Let us continue with the same notation and hypotheses as in the previous section. Let V_1, W_1 be left or right graded modules over A, depending on whether V, W are left or right modules over A. Also let ϕ_1 be a homomorphism from V_1 into V, and let ψ_1 be a homomorphism from W into W_1, as modules over A, of degrees $p_1, q_1 \in \mathbf{Z}$, respectively. The corresponding homomorphism $\mathrm{Hom}^{gr}(\phi_1,\psi_1)$

(6.4.1) $$\text{from } \mathrm{Hom}_A^{gr}(V,W) \text{ into } \mathrm{Hom}_A^{gr}(V_1,W_1),$$

as modules over k, is defined a bit differently from before, without gradings. More precisely,

(6.4.2) $$\mathrm{Hom}^{gr}(\phi_1,\psi_1) \text{ has bidegree } (p_1,q_1).$$

If $j, l \in \mathbf{Z}$, then $\mathrm{Hom}^{gr}(\phi_1,\psi_1)$ should map

(6.4.3) $$\big(\mathrm{Hom}_A^{gr}(V,W)\big)^{j,l} \text{ into } \big(\mathrm{Hom}_A^{gr}(V_1,W_1)\big)^{j+p_1,l+q_1}.$$

This corresponds to a mapping

(6.4.4) $$\text{from } \mathrm{Hom}_A(V^{-j},W^l) \text{ into } \mathrm{Hom}_A(V_1^{-j-p_1},W_1^{l+q_1}),$$

as in (6.3.6). Note that

(6.4.5) $$\phi_1^{-j-p_1}(V_1^{-j-p_1}) \subseteq V^{-j}$$

and

(6.4.6) $$\psi_1^l(W^l) \subseteq W_1^{l+q_1},$$

by hypothesis. Thus $\mathrm{Hom}(\phi_1^{-j-p_1},\psi_1^l)$ defines a homomorphism as in (6.4.4), as modules over k.

We would like to take

(6.4.7) $$\begin{gathered}\text{the restriction of } \mathrm{Hom}^{gr}(\phi_1,\psi_1) \text{ to } \big(\mathrm{Hom}_A^{gr}(V,W)\big)^{j,l} \text{ to be}\\ \text{the mapping corresponding to } (-1)^{j\,q_1}\,\mathrm{Hom}(\phi_1^{-j-p_1},\psi_1^l),\end{gathered}$$

as on p63 of [3]. Of course, there is a unique homomorphism as in (6.4.1), as modules over k, with this property.

Let V_2, W_2 be another pair of left or right graded modules over A, depending on whether V, W are left or right modules over A. Also let ϕ_2 be a homomorphism from V_2 into V_1, and let ψ_2 be a homomorphism from W_1 into W_2, as modules over A, of degrees $p_2, q_2 \in \mathbf{Z}$, respectively. This leads to a homomorphism $\mathrm{Hom}^{gr}(\phi_2,\psi_2)$

(6.4.8) $$\text{from } \mathrm{Hom}_A^{gr}(V_1,W_1) \text{ into } \mathrm{Hom}_A^{gr}(V_2,W_2),$$

as modules over k, as before. Of course, this homomorphism has bidegree (p_2,q_2).

It follows that $\phi_1 \circ \phi_2$ is a homomorphism from V_2 into V, and that $\psi_2 \circ \psi_1$ is a homomorphism from W into W_2, as modules over A, of degrees $p_1 + p_2$, $q_1 + q_2$, respectively. Thus we get a homomorphism $\mathrm{Hom}^{gr}(\phi_1 \circ \phi_2, \psi_2 \circ \psi_1)$

$$\text{from } \mathrm{Hom}^{gr}_A(V, W) \text{ into } \mathrm{Hom}^{gr}_A(V_2, W_2), \tag{6.4.9}$$

as modules over k, of bidegree $(p_1 + p_2, q_1 + q_2)$. One can verify that

$$\mathrm{Hom}^{gr}(\phi_1 \circ \phi_2, \psi_2 \circ \psi_1) = (-1)^{p_1\, q_2}\, \mathrm{Hom}^{gr}(\phi_2, \psi_2) \circ \mathrm{Hom}^{gr}(\phi_1, \psi_1), \tag{6.4.10}$$

as on p63 of [3]. More precisely, it suffices to check that this holds on (6.3.6) for each $j, l \in \mathbf{Z}$. This uses (6.3.5), and the way that these homomorphisms are defined.

6.5 Complexes and homomorphisms

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V, d_V) and (W, d_W) be both left or both right graded modules over A with differentiation that are complexes. Thus $\mathrm{Hom}^{gr}_A(V, W)$ is a doubly-graded module over k, as in Section 6.3.

The identity mappings I_V, I_W on V, W are homomorphisms from V, W into themselves, respectively, of degree 0, as before. Let

$$\delta_1 = \mathrm{Hom}^{gr}(d_V, I_W) \tag{6.5.1}$$

and

$$\delta_2 = \mathrm{Hom}^{gr}(I_V, d_W) \tag{6.5.2}$$

be the homomorphisms from $\mathrm{Hom}^{gr}_A(V, W)$ into itself, as a module over k, defined as in the previous section. Remember that δ_1 has bidegree $(1, 0)$, and δ_2 has bidegree $(0, 1)$.

It is easy to see that

$$\delta_1 \circ \delta_1 = \delta_2 \circ \delta_2 = 0 \tag{6.5.3}$$

on $\mathrm{Hom}^{gr}_A(V, W)$, because $d_V \circ d_V = 0$ on V and $d_W \circ d_W = 0$ on W, and using (6.4.10). One can also use (6.4.10) to get that

$$\delta_1 \circ \delta_2 + \delta_2 \circ \delta_1 = 0 \tag{6.5.4}$$

on $\mathrm{Hom}^{gr}_A(V, W)$. This means that

$$\mathrm{Hom}^{gr}_A(V, W) \text{ is a double complex with respect to } \delta_1 \text{ and } \delta_2, \tag{6.5.5}$$

as on p63 of [3].

Let (V_0, d_{V_0}), (W_0, d_{W_0}) be another pair of left or right graded modules over A with differentiation that are complexes, depending on whether V, W are left or right modules over A. As before, $\mathrm{Hom}^{gr}_A(V_0, W_0)$ is a doubly-graded module over k, and we let $\delta_{1,0}$, $\delta_{2,0}$ be the corresponding differentiation operators on it.

Let ϕ be a map from V_0 into V, and let ψ be a map from W into W_0, as complexes. This leads to a homomorphism $\mathrm{Hom}^{gr}(\phi, \psi)$ from $\mathrm{Hom}^{gr}_A(V, W)$ into $\mathrm{Hom}^{gr}_A(V_0, W_0)$, as modules over k, of bidegree $(0, 0)$, as in the previous section. This is the same as the homomorphism $\mathrm{Hom}(\phi, \psi)$ defined as in Section 6.3 without using gradings, restricted to $\mathrm{Hom}^{gr}_A(V, W)$, because ψ has degree 0. One can verify that

(6.5.6) $\quad \mathrm{Hom}^{gr}(\phi, \psi)$ is a map from $\mathrm{Hom}^{gr}_A(V, W)$ into $\mathrm{Hom}^{gr}_A(V_0, W_0)$,

as double complexes, as on p63 of [3].

Let ϕ_0 be another map from V_0 into V, and let ψ_0 be another map from W into W_0, as complexes. Also let σ, τ be homotopies between ϕ, ψ and ϕ_0, ψ_0, respectively. This means that σ is a homomorphism from V_0 into V, as modules over A, with degree -1, such that

$$d_V \circ \sigma + \sigma \circ d_{V_0} = \phi - \phi_0. \tag{6.5.7}$$

Similarly, τ is a homomorphism from W into W_0, as modules over A, of degree -1, with

$$d_{W_0} \circ \tau + \tau \circ d_W = \psi - \psi_0. \tag{6.5.8}$$

Note that

$$\sigma_1 = \mathrm{Hom}^{gr}(\sigma, \psi) \tag{6.5.9}$$

and

$$\sigma_2 = \mathrm{Hom}^{gr}(\phi_0, \tau) \tag{6.5.10}$$

are homomorphisms from $\mathrm{Hom}^{gr}_A(V, W)$ into $\mathrm{Hom}^{gr}_A(V_0, W_0)$, as modules over k, of bidegrees $(-1, 0)$ and $(0, -1)$, respectively.

It is easy to see that

$$\begin{aligned} \sigma_1 \circ \delta_2 &= \mathrm{Hom}^{gr}(\sigma, \psi) \circ \mathrm{Hom}^{gr}(I_V, d_W) \\ &= \mathrm{Hom}^{gr}(\sigma, \psi \circ d_W) = \mathrm{Hom}^{gr}(\sigma, d_{W_0} \circ \psi) \\ &= -\mathrm{Hom}^{gr}(I_{V_0}, d_{W_0}) \circ \mathrm{Hom}^{gr}(\sigma, \psi) = -\delta_{2,0} \circ \sigma_1, \end{aligned} \tag{6.5.11}$$

using (6.4.10). Similarly,

$$\begin{aligned} \sigma_2 \circ \delta_1 &= \mathrm{Hom}^{gr}(\phi_0, \tau) \circ \mathrm{Hom}^{gr}(d_V, I_W) \\ &= -\mathrm{Hom}^{gr}(d_V \circ \phi_0, \tau) = -\mathrm{Hom}^{gr}(\phi_0 \circ d_{V_0}, \tau) \\ &= -\mathrm{Hom}^{gr}(d_{V_0}, I_{W_0}) \circ \mathrm{Hom}^{gr}(\phi_0, \tau) = -\delta_{1,0} \circ \sigma_2. \end{aligned} \tag{6.5.12}$$

Observe that

$$\delta_{1,0} \circ \sigma_1 = \mathrm{Hom}^{gr}(d_{V_0}, I_{W_0}) \circ \mathrm{Hom}^{gr}(\sigma, \psi) = \mathrm{Hom}^{gr}(\sigma \circ d_{V_0}, \psi) \tag{6.5.13}$$

and

$$\sigma_1 \circ \delta_1 = \mathrm{Hom}^{gr}(\sigma, \psi) \circ \mathrm{Hom}^{gr}(d_V, I_W) = \mathrm{Hom}^{gr}(d_V \circ \sigma, \psi). \tag{6.5.14}$$

This shows that

$$\begin{aligned}(6.5.15)\qquad \delta_{1,0}\circ\sigma_1+\sigma_1\circ\delta_1 &= \mathrm{Hom}^{gr}(d_{V_0}\circ\sigma+\sigma\circ d_V,\psi)\\ &= \mathrm{Hom}^{gr}(\phi-\phi_0,\psi).\end{aligned}$$

Similarly,

$$(6.5.16)\qquad \delta_{2,0}\circ\sigma_2=\mathrm{Hom}^{gr}(I_{V_0},d_{W_0})\circ\mathrm{Hom}^{gr}(\phi_0,\tau)=\mathrm{Hom}^{gr}(\phi_0,d_{W_0}\circ\tau)$$

and

$$(6.5.17)\qquad \sigma_2\circ\delta_2=\mathrm{Hom}^{gr}(\phi_0,\tau)\circ\mathrm{Hom}^{gr}(I_V,d_W)=\mathrm{Hom}^{gr}(\phi_0,\tau\circ d_W).$$

This implies that

$$\begin{aligned}(6.5.18)\qquad \delta_{2,0}\circ\sigma_2+\sigma_2\circ\delta_2 &= \mathrm{Hom}^{gr}(\phi_0,d_{W_0}\circ\tau+\tau\circ d_W)\\ &= \mathrm{Hom}^{gr}(\phi_0,\psi-\phi_0).\end{aligned}$$

It follows that

$$\begin{aligned}(6.5.19)\qquad &\delta_{1,0}\circ\sigma_1+\sigma_1\circ\delta_1+\delta_{2,0}\circ\sigma_2+\sigma_2\circ\delta_2\\ &= \mathrm{Hom}^{gr}(\phi-\phi_0,\psi)+\mathrm{Hom}^{gr}(\phi_0,\psi-\psi_0)\\ &= \mathrm{Hom}^{gr}(\phi,\psi)+\mathrm{Hom}^{gr}(\phi_0,\psi_0).\end{aligned}$$

This means that

$$(6.5.20)\qquad \begin{aligned}&(\sigma_1,\sigma_2)\text{ defines a homotopy between}\\ &\mathrm{Hom}^{gr}(\phi,\psi)\text{ and }\mathrm{Hom}^{gr}(\phi_0,\psi_0),\end{aligned}$$

as maps from $\mathrm{Hom}^{gr}_A(V,W)$ into $\mathrm{Hom}^{gr}_A(V_0,W_0)$, as double complexes. This corresponds to a remark on p63 of [3], as before.

6.6 Maps of any degree

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V,d_V), (W,d_W) be both left or both right graded modules over A with differentiation that are complexes.

Suppose that ϕ is a homomorphism from V into W, as modules over A, of degree $r\in\mathbf{Z}$ such that

$$(6.6.1)\qquad d_W\circ\phi=(-1)^r\,\phi\circ d_V.$$

Under these conditions, ϕ is considered to be a *map of degree* r from V into W, as complexes, as in Exercise 1 on p72 of [3]. If $r=0$, then this reduces to the definition of a map between complexes in Section 5.11. Of course, (6.6.1) is the same as saying that

$$(6.6.2)\qquad d_W^{j+r}\circ\phi^j=(-1)^r\,\phi^{j+1}\circ d_V^j$$

for each integer j.

It is easy to see that $\phi(Z(V)) \subseteq Z(W)$ in this case, as in Section 5.2. More precisely, we have that

$$\phi^j(Z(V)^j) \subseteq Z(W)^{j+r} \tag{6.6.3}$$

for every integer j, as in Section 5.11. Similarly, $\phi(B(V)) \subseteq B(W)$, and in fact

$$\phi^j(B(V)^j) \subseteq B(W)^{j+r} \tag{6.6.4}$$

for every j.

As in Section 5.2, ϕ induces homomorphisms $\phi_{Z'}$, $\phi_{B'}$ from $Z'(V)$, $B'(V)$ into $Z'(W)$, $B'(W)$, respectively, as modules over A. We also have that $\phi_{Z'}$ maps $H(V)$ into $H(W)$, and we let ϕ_H be the restriction of $\phi_{Z'}$ to H, as before. Note that $\phi_{Z'}$, $\phi_{B'}$, and ϕ_H have degree r, as in Section 5.11.

Remember that $\widetilde{d}_V$, $\widetilde{d}_W$ are the homomorphisms from $Z'(V)$, $Z'(W)$ into $Z(V)$, $Z(W)$ induced by d_V, d_W, respectively, as in Section 5.1. One can check that

$$\phi \circ \widetilde{d}_V = (-1)^r \, \widetilde{d}_W \circ \phi_{Z'}, \tag{6.6.5}$$

as in Section 5.2. Equivalently, this means that

$$\phi^{j+1} \circ \widetilde{d}_V^j = (-1)^r \, \widetilde{d}_W^{j+r} \circ \phi_{Z'}^j \tag{6.6.6}$$

for every j, as in Section 5.11.

Let ψ be another map of degree r from V into W, as complexes, and let σ be a homomorphism from V into W, as modules over A, of degree $r-1$. If

$$d_W \circ \sigma + (-1)^r \, \sigma \circ d_V = \phi - \psi, \tag{6.6.7}$$

then σ is said to be a *homotopy* between ϕ and ψ, as maps of degree r between complexes, as in Exercise 1 on p72 of [3]. This reduces to the definition of a homotopy between maps between complexes when $r = 0$, as in Section 5.11. This is the same as saying that

$$d_W^{j+r-1} \circ \sigma^j + (-1)^r \, \sigma^{j+1} \circ d_V^j = \phi^j - \psi^j \tag{6.6.8}$$

for every j, as usual. This implies that

$$(\phi^j - \psi^j)(Z(V)^j) \subseteq B(W)^{j+r} \tag{6.6.9}$$

for every j, so that $\phi_H^j = \psi_H^j$ for each j, as before.

Let (W_1, d_{W_1}) be another left or right graded module over A with differentiation that is a complex, depending on whether V, W are left or right modules over A. Also let ϕ_1 be a map of degree $r_1 \in \mathbf{Z}$ from W into W_1, as complexes. Thus $\phi_1 \circ \phi$ is a homomorphism from V into W_1, as modules over A, of degree $r + r_1$. Observe that

$$d_{W_1} \circ \phi_1 \circ \phi = (-1)^{r_1} \, \phi_1 \circ d_W \circ \phi = (-1)^{r+r_1} \, \phi_1 \circ \phi \circ d_V, \tag{6.6.10}$$

so that $\phi_1 \circ \phi$ is a map of degree $r + r_1$ from V into W_1, as complexes. This corresponds to part of Exercise 1 on p72f of [3].

Let ψ_1 be another map of degree r_1 from W into W_1, as complexes, and let σ_1 be a homotopy between ϕ_1 and ψ_1, as maps of degree r_1 between complexes. Put

$$\tau = \sigma_1 \circ \phi + (-1)^{r_1} \, \psi_1 \circ \sigma, \tag{6.6.11}$$

where ϕ, ψ, and σ are as before. This is a homomorphism from V into W_1, as modules over A, of degree $r + r_1 - 1$. We would like to verify that

$$\tau \text{ is a homotopy between } \phi_1 \circ \phi \text{ and } \psi_1 \circ \psi, \tag{6.6.12}$$

as maps of degree $r + r_1$ between complexes, as in Exercise 1 on p72f of [3].

More precisely,

$$\sigma_1 \circ \phi \text{ is a homotopy between } \phi_1 \circ \phi \text{ and } \psi_1 \circ \phi, \tag{6.6.13}$$

and

$$(-1)^{r_1} \, \psi_1 \circ \sigma \text{ is a homotopy between } \psi_1 \circ \phi \text{ and } \psi_1 \circ \psi, \tag{6.6.14}$$

as maps of degree $r + r_1$ between complexes. Of course, these are homomorphisms from V into W_1, as modules over A, of degree $r + r_1 - 1$, as before. Observe that

$$\begin{aligned} & d_{W_1} \circ \sigma_1 \circ \phi + (-1)^{r+r_1} \, \sigma_1 \circ \phi \circ d_V \\ & \quad = \quad d_{W_1} \circ \sigma_1 \circ \phi + (-1)^{r_1} \, \sigma_1 \circ d_W \circ \phi \\ & \quad = \quad (\phi_1 - \psi_1) \circ \phi, \end{aligned} \tag{6.6.15}$$

which implies (6.6.13). Similarly,

$$\begin{aligned} & (-1)^{r_1} \, d_{W_1} \circ \psi_1 \circ \sigma + (-1)^{r+r_1} \, (-1)^{r_1} \, \psi_1 \circ \sigma \circ d_V \\ & \quad = \quad \psi_1 \circ d_{W_1} \circ \sigma + (-1)^r \, \psi_1 \circ \sigma \circ d_V \\ & \quad = \quad \psi_1 \circ (\phi - \psi), \end{aligned} \tag{6.6.16}$$

which implies (6.6.14). Combining these two equations, we get that

$$d_{W_1} \circ \tau + (-1)^{r+r_1} \, \tau \circ d_V = \phi_1 \circ \phi - \psi_1 \circ \psi, \tag{6.6.17}$$

which implies (6.6.12).

6.7 Double complexes and any bidegree

Let k be a commutative ring with a mutliplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, W be both left or both right modules over A that are double complexes, with differentiation operators $d_{V,1}$, $d_{V,2}$ and $d_{W,1}$, $d_{W,2}$, respectively.

Suppose that ϕ is a homomorphism from V into W, as modules over A, of bidegree (p, q) for some $p, q \in \mathbf{Z}$. If

$$d_{W,1} \circ \phi = (-1)^{p+q}\, \phi \circ d_{V,1} \tag{6.7.1}$$

and

$$d_{W,2} \circ \phi = (-1)^{p+q}\, \phi \circ d_{V,2}, \tag{6.7.2}$$

then ϕ is considered to be a *map of bidegree* (p, q) from V into W, as double complexes, as in Exercise 2 on p73 of [3]. This reduces to the definition of a map between double complexes in Section 5.15 when $p, q = 0$. As usual, (6.7.1) and (6.7.2) are the same as saying that

$$d_{W,1}^{j+p,l+q} \circ \phi^{j,l} = (-1)^{p+q}\, \phi^{j+1,l} \circ d_{V_1} \tag{6.7.3}$$

and

$$d_{W,2}^{j+p,l+q} \circ \phi^{j,l} = (-1)^{p+q}\, \phi^{j,l+1} \circ d_{V,2} \tag{6.7.4}$$

for all integers j, l.

Remember that V, W may be considered as single complexes, using the total differentiation operators

$$d_V = d_{V,1} + d_{V,2}, \quad d_W = d_{W,1} + d_{W,2}, \tag{6.7.5}$$

as in Section 5.14. If ϕ is a map of bidegree (p, q) from V into W, as double complexes, then it is easy to see that ϕ is a map of degree $p+q$ from V into W, as single complexes, as in the previous section.

Let ψ be another map of bidegree (p, q) from V into W, as double complexes, and let σ_1, σ_2 be homomomorphisms from V into W, as modules over A, of bidegrees $(p-1, q)$, $(p, q-1)$, respectively. Suppose that

$$\begin{aligned} &d_{W,1} \circ \sigma_1 + (-1)^{p+q}\, \sigma_1 \circ d_{V,1} + d_{W,2} \circ \sigma_2 + (-1)^{p+q}\, \sigma_2 \circ d_{V,2} \\ &= \phi - \psi. \end{aligned} \tag{6.7.6}$$

Suppose also that

$$(-1)^{p+q}\, \sigma_1 \circ d_{V,2} + d_{W,2} \circ \sigma_1 = 0 \tag{6.7.7}$$

and

$$(-1)^{p+q}\, \sigma_2 \circ d_{V,1} + d_{W,1} \circ \sigma_2 = 0. \tag{6.7.8}$$

In this case, (σ_1, σ_2) defines a *homotopy* between ϕ and ψ, as maps of bidegree (p, q) between double complexes, as in Exercise 2 on p73 of [3]. This reduces to the definition of a homotopy between maps between double complexes in Section 5.15 when $p = q = 0$.

Note that (6.7.6) is the same as saying that

$$\begin{aligned} &d_{W,1}^{j+p-1,l+q} \circ \sigma_1^{j,l} + (-1)^{p+q}\, \sigma_1^{j+1,l} \circ d_{V,1}^{j,l} \\ &+ d_{W,2}^{j+p,l+q-1} \circ \sigma_2^{j,l} + (-1)^{p+q}\, \sigma_2^{j,l+1} \circ d_{V,2}^{j,l} \\ &= \phi^{j,l} - \psi^{j,l} \end{aligned} \tag{6.7.9}$$

for all integers j, l. Similarly, (6.7.7) is the same as saying that

$$(-1)^{p+q}\,\sigma_1^{j,l+1}\circ d_{V,2}^{j,l} + d_{W,2}^{j+p-1,l+q}\circ\sigma_1^{j,l} = 0 \tag{6.7.10}$$

for every j, l, and (6.7.8) is the same as saying that

$$(-1)^{p+q}\,\sigma_2^{j+1,l}\circ d_{V,1}^{j,l} + d_{W,1}^{j+p,l+q-1}\circ\sigma_2^{j,l} = 0 \tag{6.7.11}$$

for every j, l.

Put

$$\sigma = \sigma_1 + \sigma_2, \tag{6.7.12}$$

which is a homomorphism from V into W of degree $p+q-1$, as singly-graded modules over A. One can verify that

$$d_W\circ\sigma + (-1)^{p+q}\,\sigma\circ d_V = \phi - \psi, \tag{6.7.13}$$

using (6.7.6), (6.7.7), and (6.7.8). This implies that σ is a homotopy between ϕ and ψ, as maps of degree $p+q$ from V into W as single complexes over A, as in the previous section. This reduces to the analogous statement in Section 5.15 when $p=q=0$.

Let $\widetilde{W}$ be another left or right module over A that is a double complex, depending on whether V, W are left or right modules over A, and let $d_{\widetilde{W},1}$, $d_{\widetilde{W},2}$ be the corresponding differentiation operators on $\widetilde{W}$. Also let $\widetilde{\phi}$ be a map of bidegree $(\widetilde{p},\widetilde{q})$ from W into $\widetilde{W}$, as double complexes, for some $\widetilde{p},\widetilde{q}\in\mathbf{Z}$. It is easy to see that $\widetilde{\phi}\circ\phi$ is a map of bidegree $(p+\widetilde{p}, q+\widetilde{q})$ from V into $\widetilde{W}$, as double complexes.

Let $\widetilde{\psi}$ be another map of bidegree $(\widetilde{p},\widetilde{q})$ from W into $\widetilde{W}$, as double complexes, and let $(\widetilde{\sigma}_1,\widetilde{\sigma}_2)$ be a homotopy between $\widetilde{\phi}$ and $\widetilde{\psi}$, as maps of bidegree $(\widetilde{p},\widetilde{q})$ between double complexes. One can check that

$$(\widetilde{\sigma}_1\circ\phi,\widetilde{\sigma}_2\circ\phi) \text{ is a homotopy between } \widetilde{\phi}\circ\phi \text{ and } \widetilde{\psi}\circ\phi, \tag{6.7.14}$$

as maps of bidegree $(p+\widetilde{p}, q+\widetilde{q})$ between double complexes. Similarly,

$$((-1)^{\widetilde{p}+\widetilde{q}}\,\widetilde{\psi}\circ\sigma_1, (-1)^{\widetilde{p}+\widetilde{q}}\,\widetilde{\psi}\circ\sigma_2) \text{ is a homotopy between } \widetilde{\psi}\circ\phi \text{ and } \widetilde{\psi}\circ\psi, \tag{6.7.15}$$

as maps of bidegree $(p+\widetilde{p}, q+\widetilde{q})$ between double complexes. This is analogous to the corresponding statements in the previous section.

Put

$$\tau_j = \widetilde{\sigma}_j\circ\phi + (-1)^{\widetilde{p}+\widetilde{q}}\,\widetilde{\psi}\circ\sigma_j, \tag{6.7.16}$$

for $j=1,2$. The remarks in the preceding paragraph imply that

$$(\tau_1,\tau_2) \text{ is a homotopy between } \widetilde{\phi}\circ\phi \text{ and } \widetilde{\psi}\circ\psi, \tag{6.7.17}$$

as maps of bidegree $(p+\widetilde{p}, q+\widetilde{q})$ between double complexes. This corresponds to part of Exercise 2 on p73 of [3].

6.8 Tensor products and any degrees

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that (V, d_V), (V_0, d_{V_0}) are graded right modules over A with differentiation that are complexes, and that (W, d_W), (W_0, d_{W_0}) are graded left modules over A with differentiation that are complexes. Let $V \bigotimes_A W$, $V_0 \bigotimes_A W_0$ be tensor products of V, W and V_0, W_0 over A, respectively. These are double complexes over k with respect to the corresponding differentiation operators δ_1, δ_2 and $\delta_{1,0}$, $\delta_{2,0}$, respectively, discussed in Section 6.2.

Let ϕ, ψ be maps of degrees $p, q \in \mathbf{Z}$ from V, W into V_0, W_0, respectively, as complexes. This leads to a homomorphism $\phi \otimes \psi$ from $V \bigotimes_A W$ into $V_0 \bigotimes_A W_0$, as modules over k, of bidegree (p, q), as in Section 6.1. Let us check that

$$\phi \otimes \psi \text{ is a map of bidegree } (p,q) \text{ from } V \bigotimes_A W \text{ into } V_0 \bigotimes_A W_0, \tag{6.8.1}$$

as double complexes. This is part of Exercise 2 on p73 of [3], and the analogous statement for $p = q = 0$ was mentioned in Section 6.2.

We shall use (6.1.7) repeatedly, as before. Observe that

$$\begin{aligned}
\delta_{1,0} \circ (\phi \otimes \psi) &= (d_{V_0} \otimes I_{W_0}) \circ (\phi \otimes \psi) = (d_{V_0} \circ \phi) \otimes \psi \\
&= (-1)^p \, (\phi \circ d_V) \otimes \psi = (-1)^{p+q} \, (\phi \otimes \psi) \circ (d_V \otimes I_W) \\
&= (-1)^{p+q} \, (\phi \otimes \psi) \circ \delta_1.
\end{aligned} \tag{6.8.2}$$

Similarly,

$$\begin{aligned}
\delta_{2,0} \circ (\phi \otimes \psi) &= (I_{V_0} \otimes d_{W_0}) \circ (\phi \otimes \psi) = (-1)^p \, \phi \otimes (d_{W_0} \circ \psi) \\
&= (-1)^{p+q} \, \phi \otimes (\psi \circ d_W) = (-1)^{p+q} \, (\phi \otimes \psi) \circ (I_V \otimes d_W) \\
&= (-1)^{p+q} \, (\phi \otimes \psi) \circ \delta_2,
\end{aligned} \tag{6.8.3}$$

as desired.

Let ϕ_0, ψ_0 be another pair of mappings of degrees p, q from V, W into V_0, W_0, respectively, as complexes. Also let σ, τ be homotopies between ϕ, ψ and ϕ_0, ψ_0, as maps of degrees p, q between complexes, respectively. This means that σ, τ are homomorphisms from V, W into V_0, W_0, as modules over A, with degrees $p - 1$, $q - 1$, respectively, such that

$$d_{V_0} \circ \sigma + (-1)^p \, \sigma \circ d_V = \phi - \phi_0 \tag{6.8.4}$$

and

$$d_{W_0} \circ \tau + (-1)^q \, \tau \circ d_W = \psi - \psi_0, \tag{6.8.5}$$

as in Section 6.6. Put

$$\sigma_1 = \sigma \otimes \psi \tag{6.8.6}$$

and

$$\sigma_2 = (-1)^p \, \phi_0 \otimes \tau. \tag{6.8.7}$$

These are homomorphisms from $V \bigotimes_A W$ into $V_0 \bigotimes_A W_0$, as modules over k, of bidegrees $(p-1, q)$ and $(p, q-1)$, respectively.

The basic properties of σ_1 and σ_2 are analogous to those in Section 6.2. First,

$$\begin{aligned} \sigma_1 \circ \delta_2 &= (\sigma \otimes \psi) \circ (I_V \otimes d_W) = \sigma \otimes (\psi \circ d_W) \\ &= (-1)^q \, \sigma \otimes (d_{W_0} \circ \psi) = (-1)^{p-1+q} \, (I_{V_0} \otimes d_{W_0}) \circ (\sigma \otimes \psi) \\ &= -(-1)^{p+q} \, \delta_{2,0} \circ \sigma_1. \end{aligned} \tag{6.8.8}$$

Similarly,

$$\begin{aligned} \sigma_2 \circ \delta_1 &= (-1)^p \, (\phi_0 \otimes \tau) \circ (d_V \otimes I_W) = (-1)^{p+q-1} \, (\phi_0 \circ d_V) \otimes \tau \\ &= (-1)^{q-1} \, (d_{V_0} \circ \phi_0) \otimes \tau = (-1)^{q-1} \, (d_{V_0} \otimes I_{W_0}) \circ (\phi_0 \otimes \tau) \\ &= -(-1)^{p+q} \, \delta_{1,0} \circ \sigma_2. \end{aligned} \tag{6.8.9}$$

Next,

$$\delta_{1,0} \circ \sigma_1 = (d_{V_0} \otimes I_{W_0}) \circ (\sigma \otimes \psi) = (d_{V_0} \circ \sigma) \otimes \psi \tag{6.8.10}$$

and

$$\sigma_1 \circ \delta_1 = (\sigma \otimes \psi) \circ (d_V \otimes I_W) = (-1)^q \, (\sigma \circ d_V) \otimes \psi. \tag{6.8.11}$$

It follows that

$$\begin{aligned} \delta_{1,0} \circ \sigma_1 + (-1)^{p+q} \, \sigma_1 \circ \delta_1 &= (d_{V_0} \circ \sigma) \otimes \psi + (-1)^p \, (\sigma \circ d_V) \otimes \psi \\ &= (d_{V_0} \circ \sigma + (-1)^p \, \sigma \circ d_V) \otimes \psi = (\phi - \phi_0) \otimes \psi. \end{aligned} \tag{6.8.12}$$

Similarly,

$$\delta_{2,0} \circ \sigma_2 = (-1)^p \, (I_{V_0} \otimes d_{W_0}) \circ (\phi_0 \otimes \tau) = \phi_0 \otimes (d_{W_0} \circ \tau) \tag{6.8.13}$$

and

$$\sigma_2 \circ \delta_2 = (-1)^p \, (\phi_0 \otimes \tau) \circ (I_V \otimes d_W) = (-1)^p \, \phi_0 \otimes (\tau \circ d_W). \tag{6.8.14}$$

This implies that

$$\begin{aligned} \delta_{2,0} \circ \sigma_2 + (-1)^{p+q} \, \sigma_2 \circ \delta_2 &= \phi_0 \otimes (d_{W_0} \circ \tau) + (-1)^q \, \phi_0 \otimes (\tau \circ d_W) \\ &= \phi_0 \otimes (d_{W_0} \circ \tau + (-1)^q \, \tau \circ d_W) \\ &= \phi_0 \otimes (\psi - \psi_0). \end{aligned} \tag{6.8.15}$$

We can combine (6.8.12) and (6.8.15) to get that

$$\begin{aligned} &\delta_{1,0} \circ \sigma_1 + (-1)^{p+q} \, \sigma_1 \circ \delta_1 + \delta_{2,0} \circ \sigma_2 + (-1)^{p+q} \, \sigma_2 \circ \delta_2 \\ &= \phi \otimes \psi - \phi_0 \otimes \psi_0. \end{aligned} \tag{6.8.16}$$

This shows that

$$(\sigma_1, \sigma_2) \text{ is a homotopy between } \phi \otimes \psi \text{ and } \phi_0 \otimes \psi_0, \tag{6.8.17}$$

as maps of bidegree (p, q) from $V \bigotimes_A W$ into $V_0 \bigotimes_A W_0$, as double complexes. This is another part of Exercise 2 on p73 of [3].

6.9 Homomorphisms and any degrees

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that (V, d_V), (V_0, d_{V_0}), (W, d_W), and (W_0, d_{W_0}) are all left or all right graded modules over A with differentiation that are complexes. Thus $\mathrm{Hom}^{gr}_A(V, W)$ and $\mathrm{Hom}^{gr}_A(V_0, W_0)$ are double complexes over k with respect to the corresponding differentiation operators δ_1, δ_2 and $\delta_{1,0}$, $\delta_{2,0}$ discussed in Section 6.5.

Let ϕ be a map of degree $p \in \mathbf{Z}$ from V_0 into V, and let ψ be a map of degree $q \in \mathbf{Z}$ from W into W_0, as complexes. This leads to a homomorphism $\mathrm{Hom}^{gr}(\phi, \psi)$ from $\mathrm{Hom}^{gr}_A(V, W)$ into $\mathrm{Hom}^{gr}_A(V_0, W_0)$, as modules over k, of bidegree (p, q), as in Section 6.4. We would like to verify that

$$\text{(6.9.1)} \qquad \begin{array}{l} \mathrm{Hom}^{gr}(\phi, \psi) \text{ is a map of bidegree } (p, q) \\ \text{from } \mathrm{Hom}^{gr}_A(V, W) \text{ into } \mathrm{Hom}^{gr}_A(V_0, W_0), \end{array}$$

as double complexes. This is part of Exercise 2 on p73 of [3], and the analogous statement for $p = q = 0$ was mentioned in Section 6.5.

We shall use (6.4.10) repeatedly in this section, as before. In particular, we have that

$$\text{(6.9.2)} \qquad \begin{aligned} \delta_{1,0} \circ \mathrm{Hom}^{gr}(\phi, \psi) &= \mathrm{Hom}^{gr}(d_{V_0}, I_{W_0}) \circ \mathrm{Hom}^{gr}(\phi, \psi) \\ &= \mathrm{Hom}^{gr}(\phi \circ d_{V_0}, \psi) = (-1)^p \, \mathrm{Hom}^{gr}(d_V \circ \phi, \psi) \\ &= (-1)^{p+q} \, \mathrm{Hom}^{gr}(\phi, \psi) \circ \mathrm{Hom}^{gr}(d_V, I_W) \\ &= (-1)^{p+q} \, \mathrm{Hom}^{gr}(\phi, \psi) \circ \delta_1. \end{aligned}$$

Similarly,

$$\text{(6.9.3)} \qquad \begin{aligned} \delta_{2,0} \circ \mathrm{Hom}^{gr}(\phi, \psi) &= \mathrm{Hom}^{gr}(I_{V_0}, d_{W_0}) \circ \mathrm{Hom}^{gr}(\phi, \psi) \\ &= (-1)^p \, \mathrm{Hom}^{gr}(\phi, d_{W_0} \circ \psi) = (-1)^{p+q} \, \mathrm{Hom}^{gr}(\phi, \psi \circ d_W) \\ &= (-1)^{p+q} \, \mathrm{Hom}^{gr}(\phi, \psi) \circ \mathrm{Hom}^{gr}(I_V, d_W) \\ &= (-1)^{p+q} \, \mathrm{Hom}^{gr}(\phi, \psi) \circ \delta_2. \end{aligned}$$

Let ϕ_0 be another map of degree p from V_0 into V, and let ψ_0 be another map of degree q from W into W_0, as complexes. Suppose that σ, τ are homotopies between ϕ, ψ and ϕ_0, ψ_0, as maps of degree p, q between complexes, respectively. Thus σ is a homomorphism from V_0 into V, as modules over A, with degree $p-1$, such that

$$\text{(6.9.4)} \qquad d_V \circ \sigma + (-1)^p \, \sigma \circ d_{V_0} = \phi - \phi_0.$$

Similarly, τ is a homomorphism from W into W_0, as modules over A, with degree $q - 1$, such that

$$\text{(6.9.5)} \qquad d_{W_0} \circ \tau + (-1)^q \, \tau \circ d_W = \psi - \psi_0.$$

Put

$$\text{(6.9.6)} \qquad \sigma_1 = (-1)^p \, \mathrm{Hom}^{gr}(\sigma, \psi)$$

and

$$\sigma_2 = (-1)^p \operatorname{Hom}^{gr}(\phi_0, \tau). \tag{6.9.7}$$

These are homomorphisms from $\operatorname{Hom}^{gr}_A(V, W)$ into $\operatorname{Hom}^{gr}_A(V_0, W_0)$, as modules over k, of bidegrees $(p-1, q)$ and $(p, q-1)$, respectively. The basic properties of these homomorphisms are analogous to those in Section 6.5. Observe first that

$$\begin{aligned} \sigma_1 \circ \delta_2 &= (-1)^p \operatorname{Hom}^{gr}(\sigma, \psi) \circ \operatorname{Hom}^{gr}(I_V, d_W) \\ &= (-1)^p \operatorname{Hom}^{gr}(\sigma, \psi \circ d_W) = (-1)^{p+q} \operatorname{Hom}^{gr}(\sigma, d_{W_0} \circ \psi) \\ &= (-1)^{q-1} \operatorname{Hom}^{gr}(I_{V_0}, d_{W_0}) \circ \operatorname{Hom}^{gr}(\sigma, \psi) = -(-1)^{p+q} \delta_{2,0} \circ \sigma_1. \end{aligned} \tag{6.9.8}$$

Similarly,

$$\begin{aligned} \sigma_2 \circ \delta_1 &= (-1)^p \operatorname{Hom}^{gr}(\phi_0, \tau) \circ \operatorname{Hom}^{gr}(d_V, I_W) \\ &= (-1)^{p+q-1} \operatorname{Hom}^{gr}(d_V \circ \phi_0, \tau) = (-1)^{q-1} \operatorname{Hom}^{gr}(\phi_0 \circ d_{V_0}, \tau) \\ &= (-1)^{q-1} \operatorname{Hom}^{gr}(d_{V_0}, I_{W_0}) \circ \operatorname{Hom}^{gr}(\phi_0, \tau) = -(-1)^{p+q} \delta_{1,0} \circ \sigma_2. \end{aligned}$$

We also have that

$$\begin{aligned} \delta_{1,0} \circ \sigma_1 &= (-1)^p \operatorname{Hom}^{gr}(d_{V_0}, I_{W_0}) \circ \operatorname{Hom}^{gr}(\sigma, \psi) \\ &= (-1)^p \operatorname{Hom}^{gr}(\sigma \circ d_{V_0}, \psi) \end{aligned} \tag{6.9.9}$$

and

$$\begin{aligned} \sigma_1 \circ \delta_1 &= (-1)^p \operatorname{Hom}^{gr}(\sigma, \psi) \circ \operatorname{Hom}^{gr}(d_V, I_W) \\ &= (-1)^{p+q} \operatorname{Hom}^{gr}(d_V \circ \sigma, \psi). \end{aligned} \tag{6.9.10}$$

This implies that

$$\begin{aligned} \delta_{1,0} \circ \sigma_1 + (-1)^{p+q} \sigma_1 \circ \delta_1 &= \operatorname{Hom}^{gr}((-1)^p \sigma \circ d_{V_0} + d_V \circ \sigma, \psi) \\ &= \operatorname{Hom}^{gr}(\phi - \phi_0, \psi). \end{aligned} \tag{6.9.11}$$

Similarly,

$$\begin{aligned} \delta_{2,0} \circ \sigma_2 &= (-1)^p \operatorname{Hom}^{gr}(I_{V_0}, d_{W_0}) \circ \operatorname{Hom}^{gr}(\phi_0, \tau) \\ &= \operatorname{Hom}^{gr}(\phi_0, d_{W_0} \circ \tau) \end{aligned} \tag{6.9.12}$$

and

$$\begin{aligned} \sigma_2 \circ \delta_2 &= (-1)^p \operatorname{Hom}^{gr}(\phi_0, \tau) \circ \operatorname{Hom}^{gr}(I_V, d_W) \\ &= (-1)^p \operatorname{Hom}^{gr}(\phi_0, \tau \circ d_W). \end{aligned} \tag{6.9.13}$$

It follows that

$$\begin{aligned} \delta_{2,0} \circ \sigma_2 + (-1)^{p+q} \sigma_2 \circ \delta_2 &= \operatorname{Hom}^{gr}(\phi_0, d_{W_0} \circ \tau + (-1)^q \tau \circ d_W) \\ &= \operatorname{Hom}^{gr}(\phi_0, \psi - \psi_0). \end{aligned} \tag{6.9.14}$$

Combining (6.9.11) and (6.9.14), we get that

$$\begin{aligned}(6.9.15)\qquad &\delta_{1,0}\circ\sigma_1+(-1)^{p+q}\,\sigma_1\circ\delta_1+\delta_{2,0}\circ\sigma_2+(-1)^{p+2}\,\sigma_2\circ\delta_2\\ &=\mathrm{Hom}^{gr}(\phi,\psi)+\mathrm{Hom}^{gr}(\phi_0,\psi_0).\end{aligned}$$

This shows that

(6.9.16) (σ_1,σ_2) is a homotopy between $\mathrm{Hom}^{gr}(\phi,\psi)$ and $\mathrm{Hom}^{gr}(\phi_0,\psi_0)$,

as maps of bidegree (p,q) from $\mathrm{Hom}^{gr}_A(V,W)$ into $\mathrm{Hom}^{gr}_A(V_0,W_0)$, as double complexes. This is another part of Exercise 2 on p73 of [3].

6.10 The complex $M(V,W)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V,d_V), (W,d_W) be both left or both right modules over A with differentiation that are complexes. As in Section 6.3, we let $|V|$, $|W|$ be the underlying modules over A corresponding to V, W, without gradings.

If $r\in\mathbf{Z}$, then $M^r(V,W)$ is the space of homomorphisms from V into W, as modules over A, of degree r, as in Section 6.3. This is a submodule of $\mathrm{Hom}_A(|V|,|W|)$, as a module over k, and $M(V,W)$ is the submodule of $\mathrm{Hom}_A(V,W)$ spanned by the $M^r(V,W)$'s, $r\in\mathbf{Z}$, as before. More precisely, $M(V,W)$ corresponds to the direct sum of the $M^r(V,W)$'s, $r\in\mathbf{Z}$, as a module over k, so that $M(V,W)$ may be considered as a graded module over k.

If $\phi\in M^r(V,W)$ for some $r\in\mathbf{Z}$, then put

$$(6.10.1)\qquad d_1^r(\phi)=\phi\circ d_V$$

and

$$(6.10.2)\qquad d_2^r(\phi)=(-1)^{r+1}\,d_W\circ\phi,$$

which are elements of $M^{r+1}(V,W)$. These define unique homomorphisms d_1, d_2 from $M(V,W)$ into itself, as a module over k, of degree 1, whose restrictions to $M^r(V,W)$ are equal to d_1^r, d_2^r, respectively, for each $r\in\mathbf{Z}$. Observe that

$$(6.10.3)\qquad d_1\circ d_1=d_2\circ d_2=0$$

on $M(V,W)$, because $d_V\circ d_V=0$ on V and $d_W\circ d_W=0$ on W, by hypothesis. One can also verify that

$$(6.10.4)\qquad d_1\circ d_2+d_2\circ d_1=0$$

on $M(V,W)$.

Put

$$(6.10.5)\qquad d=d_1+d_2,$$

which is a homomorphism from $M(V,W)$ into itself, as a module over k, of degree 1. It is easy to see that

$$(6.10.6)\qquad d\circ d=0$$

on $M(V,W)$. This means that $M(V,W)$ is a module with differentiation with respect to d, and in fact a complex. This corresponds to part of Exrcise 5 on p73 of [3].

If $r \in \mathbf{Z}$, then let $\mathrm{Map}^r(V,W)$ be the space of maps of degree r from V into W, as complexes. This is a submodule of $M^r(V,W)$, as a module over k. It is easy to see that

$$Z(M(V,W))^r = \mathrm{Map}^r(V,W), \tag{6.10.7}$$

where the left side is defined by considering $M(V,W)$ as a complex with respect to d. This is part of Exercise 5 on p73f of [3].

Let $\mathrm{Map}_0^r(V,W)$ be the subset of $\mathrm{Map}^r(V,W)$ consisting of maps of degree r that are homotopic to 0, as maps of degree r between complexes. This is a submodule of $\mathrm{Map}^r(V,W)$, as a module over k. One can check that

$$B(M(V,W))^r = \mathrm{Map}_0^r(V,W), \tag{6.10.8}$$

where the left side is defined by considering $M(V,W)$ as a complex with respect to d again. This is another part of Exercise 5 on p73f of [3].

Let $\mathrm{Hom}_A^{gr}(V,W)$ be the doubly-graded module over k defined in Section 6.3. Remember that $\mathrm{Hom}_A^{gr}(V,W)$ corresponds to a submodule of $M(V,W)$, as a module over k, and that the single grading on $\mathrm{Hom}_A^{gr}(V,W)$ obtained from the double grading is compatible with the grading on $M(V,W)$. We also have that $\mathrm{Hom}_A^{gr}(V,W)$ is a double complex with respect to the homomorphisms δ_1, δ_2 defined in Section 6.5.

One can verify that δ_1 is the same as the restriction of d_1 to $\mathrm{Hom}_A^{gr}(V,W)$. More precisely, if $j,l \in \mathbf{Z}$, then the restriction of δ_1 to $\left(\mathrm{Hom}_A^{gr}(V,W)\right)^{j,l}$ is defined by composition with d_V, which is the same as d_1. Similarly,

(6.10.9) the restriction of δ_2 to $\left(\mathrm{Hom}_A^{gr}(V,W)\right)^{j,l}$ is $(-1)^j$ times the mapping defined by composition with d_W.

However,

(6.10.10) the restriction of d_2 to $\left(\mathrm{Hom}_A^{gr}(V,W)\right)^{j,l}$ is $(-1)^{j+l+1}$ times the mapping defined by composition with d_W,

because $\left(\mathrm{Hom}_A^{gr}(V,W)\right)^{j,l}$ corresponds to a submodule of $M^r(V,W)$, with $r = j+l$.

6.11 Tensor products and homology

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V,d_V) be a graded right module over A with differentiation that is a complex, and let (W,d_W) be a graded left module over A with differentiation that is a complex. Suppose that $V \bigotimes_A W$ is a tensor product of V and W over

A, which is a double complex over k with respect to the double grading and differentiation operators δ_1, δ_2 defined in Sections 6.1 and 6.2, respectively.

Remember that $V \bigotimes_A W$ may be considered as a single complex over k, with respect to the single grading obtained from the double grading, and the corresponding total differentiation operator

$$\delta = \delta_1 + \delta_2, \tag{6.11.1}$$

as in Section 5.14. In particular,

$$Z\big(V \bigotimes_A W\big),\ B\big(V \bigotimes_A W\big),\ \text{and } H\big(V \bigotimes_A W\big) \tag{6.11.2}$$

are defined as singly-graded modules over k in the usual way, by considering $V \bigotimes_A W$ as a single complex.

Let $j, l \in \mathbf{Z}$ be given. Of course,

$$(v^j, w^l) \mapsto v^j \otimes w^l \tag{6.11.3}$$

defines a mapping from $V^j \times W^l$ into $V^j \bigotimes_A W^l = \big(V \bigotimes_A W\big)^{j,l}$ that is bilinear over k and satisfies

$$(v^j \cdot a) \otimes w^l = v^j \otimes (a \cdot w^l) \tag{6.11.4}$$

for every $a \in A$, $v^j \in V^j$, and $w^l \in W^l$. Observe that

$$\delta_1(v^j \otimes w^l) = 0 \tag{6.11.5}$$

when $d_V(v^j) = 0$, and

$$\delta_2(v^j \otimes w^l) = 0 \tag{6.11.6}$$

when $d_W(w^l) = 0$. This means that

$$\delta(v^j \otimes w^l) = 0 \tag{6.11.7}$$

when $d_V(v^j), d_W(w^l) = 0$.

Equivalently,

$$v^j \otimes w^l \in Z\big(V \bigotimes_A W\big)^{j+l} \tag{6.11.8}$$

when $v^j \in Z(V)^j$ and $w^l \in Z(W)^l$. Let $Z(V) \bigotimes_A Z(W)$ be a tensor product of $Z(V)$ and $Z(W)$ over A, which is a doubly-graded module over k. In particular,

$$\big(Z(V) \bigotimes_A Z(W)\big)^{j,l} = Z(V)^j \bigotimes_A Z(W)^l \tag{6.11.9}$$

is a tensor product of $Z(V)^j$ and $Z(W)^l$ over A. The restriction of (6.11.3) to $Z(V)^j \times Z(W)^l$ is a bilinear mapping over k into $\big(V \bigotimes_W W\big)^{j+l}$ that satisfies (6.11.4). This leads to a homomorphism

$$\text{from } Z(V)^j \bigotimes_A Z(W)^l \text{ into } Z\big(V \bigotimes_A W\big)^{j+l}, \tag{6.11.10}$$

as modules over k, in the usual way.

This defines a homomorphism

$$\text{from } Z(V) \bigotimes_A Z(W) \text{ into } Z\big(V \bigotimes_A W\big), \tag{6.11.11}$$

as modules over k. This homomorphism has degree 0, with respect to the single grading on $Z(V) \bigotimes_A Z(W)$ obtained from the double grading mentioned earlier.

This leads to a homomorphism

$$\eta \text{ from } Z(V) \bigotimes_A Z(W) \text{ into } H\big(V \bigotimes_A W\big), \tag{6.11.12}$$

as modules over k, by composing the homomorphism as in (6.11.11) with the natural quotient mapping

$$\text{from } Z\big(V \bigotimes_A W\big) \text{ onto } H\big(V \bigotimes_A W\big). \tag{6.11.13}$$

This corresponds to the homomorphism η in the diagram (1) on p64 of [3], under slightly different conditions. This homomorphism has degree 0 too, with respect to the single grading on $Z(V) \bigotimes_A Z(W)$ obtained from the double grading. More precisely, the restriction of this homomorphism to (6.11.9) is the homomorphism

$$\text{from } Z(V)^j \bigotimes_A Z(W)^l \text{ into } H\big(V \bigotimes_A W\big)^{j+l}, \tag{6.11.14}$$

as modules over k, obtained by composing the homomorphism as in (6.11.10) with the natural quotient mapping

$$\text{from } Z\big(V \bigotimes_A W\big)^{j+l} \text{ onto } H\big(V \bigotimes_A W\big)^{j+l}. \tag{6.11.15}$$

Similarly, one can check that

$$v^j \otimes w^l \in B\big(V \bigotimes_A W\big)^{j+l} \tag{6.11.16}$$

when

$$v^j \in B(V)^j \text{ and } w^l \in Z(W)^l, \tag{6.11.17}$$

and when

$$v^j \in Z(V)^j \text{ and } w^l \in B(W)^l. \tag{6.11.18}$$

This means that

$$\begin{gathered} v^j \otimes w^l \text{ is mapped to 0 in } H\big(V \bigotimes_A W\big)^{j+l} \\ \text{by the homomorphism in (6.11.14)} \end{gathered} \tag{6.11.19}$$

when either (6.11.17) or (6.11.18) holds.

Consider the mapping

$$\text{from } Z(V)^j \times Z(W)^l \text{ into } H\big(V \bigotimes_A W\big)^{j+l}, \tag{6.11.20}$$

with

$$(v^j, w^l) \mapsto \text{the image of } v^j \otimes w^l \text{ under the quotient mapping (6.11.15)} \tag{6.11.21}$$

for every $v^j \in V^j$ and $w^l \in W^l$. This mapping is equal to 0 on $B(V)^j \times Z(W)^l$, and on $Z(V)^j \times B(W)^l$, as in the preceding paragraph. If $v^j \in V^j$ and $w^l \in W^l$, then it follows the image of (v^j, w^l) under this mapping only depends on the images of v^j, w^l under the natural quotient mappings from $Z(V)^j$, $Z(W)^l$ onto $H(V)^j$, $H(W)^l$, respectively. This leads to a mapping

$$\text{from } H(V)^j \times H(W)^l \text{ into } H\big(V \bigotimes_A W\big)^{j+l}, \tag{6.11.22}$$

that is bilinear over k.

Remember that $H(V)$ is a graded right module over A, and that $H(W)$ is a graded left module over A. Let $H(V) \bigotimes_A H(W)$ be a tensor product of $H(V)$ and $H(W)$ over A, which is a doubly-graded module over k. Thus

$$\big(H(V) \bigotimes_A H(W)\big)^{j,l} = H(V)^j \bigotimes_A H(W)^l \tag{6.11.23}$$

is a tensor product of $H(V)^j$ and $H(W)^l$ over A. Using (6.11.4), we get that the mapping in (6.11.22) satisfies the analogous property with respect to the actions of A on $H(V)^j$ on the right and on $H(W)^l$ on the left. This leads to a homomorphism

$$\text{from } H(V)^j \bigotimes_A H(W)^l \text{ into } H\big(V \bigotimes_A W\big)^{j+l}, \tag{6.11.24}$$

as modules over k.

This defines a homomorphism

$$\alpha \text{ from } H(V) \bigotimes_A H(W) \text{ into } H\big(V \bigotimes_A W\big), \tag{6.11.25}$$

as modules over k. This homomorphism has degree 0, with respect to the single grading on $H(V) \bigotimes_A H(W)$ obtained from the double grading. This corresponds to the homomorphism in Proposition 6.1 on p64 of [3], under slightly different conditions.

6.12 Some homomorphisms related to α

Let us continue with the same notation and hypotheses as in the previous section.

Using the natural quotient mappings from $Z(V)$, $Z(W)$ onto $H(V)$, $H(W)$, respectively, we get a homomorphism

$$\xi \text{ from } Z(V) \bigotimes_A Z(W) \text{ onto } H(V) \bigotimes_A H(W), \tag{6.12.1}$$

as modules over k. This corresponds to the homomorphism ξ in the diagram (1) on p64 of [3], under slightly different conditions. More precisely, the surjectivity of ξ follows from the surjectivity of the quotient mappings, as in Section 1.9. Note that ξ has bidegee $(0,0)$ with respect to the usual double gradings on $Z(V) \bigotimes_A Z(W)$ and $H(V) \bigotimes_A H(W)$, because the quotient mappings have degree 0. In particular, ξ has degree 0 with respect to the corresponding single gradings.

By construction,

$$\alpha \circ \xi = \eta, \tag{6.12.2}$$

where η is as in (6.11.12). This corresponds to part of Proposition 6.1 on p64 of [3], and in fact α is uniquely determined by this property.

Remember that $Z'(V) = V/d_V(V)$, $Z'(W) = W/d_W(W)$, as in Section 5.1. Let $Z'(V) \bigotimes_A Z'(W)$ be a tensor product of $Z'(V)$ and $Z'(W)$ over A, which is a doubly-graded module over k. Using the natural quotient mappings from V, W onto $Z'(V)$, $Z'(W)$, respectively, we get a homomorphism

$$\text{from } V \bigotimes_A W \text{ onto } Z'(V) \bigotimes_A Z'(W), \tag{6.12.3}$$

as modules over k. This homomorphism has bidegree $(0,0)$ with respect to the induced double gradings, and thus degree 0 with respect to the associated single gradings.

The natural inclusion mappings from $Z(V)$, $Z(W)$ into V, W, respectively, lead to a natural homomorphism

$$\text{from } Z(V) \bigotimes_A Z(W) \text{ into } V \bigotimes_A W, \tag{6.12.4}$$

as modules over k. This homomorphism has bidegree $(0,0)$ with respect to the appropriate double gradings, and thus degree 0 with respect to the associated single gradings. We can compose this homomorphism with the one in (6.12.3) to get a natural homomorphism

$$\text{from } Z(V) \bigotimes_A Z(W) \text{ into } Z'(V) \bigotimes_A Z'(W), \tag{6.12.5}$$

as modules over k. This homomorphism ha bidegree $(0,0)$ with respect to the usual double gradings, and thus degree 0 with respect to the associated single gradings.

Alternatively, the restrictions of the natural quotient mappings from V, W onto $Z'(V)$, $Z'(W)$ to $Z(V)$, $Z(W)$ define natural homomorphisms

$$\text{from } Z(V), Z(W) \text{ into } Z'(V), Z'(W), \tag{6.12.6}$$

respectively, as modules over A. The homomorphism in (6.12.5) is the same as the one obtained from these homomorphisms in the usual way.

Of course, $H(V) = Z(V)/d_V(V)$, $H(W) = Z(W)/d_W(W)$ are homogeneous submodules of $Z'(V)$, $Z'(W)$, as graded modules over A, as in Section 5.10. Using the corresponding inclusion mappings, we get a natural homomorphism

$$\tau \text{ from } H(V) \bigotimes_A H(W) \text{ into } Z'(V) \bigotimes_A Z'(W), \tag{6.12.7}$$

as modules over k. This corresponds to the homomorphism τ in the diagram (1) on p64 of [3], under slightly different conditions. This homomorphism has bidegree $(0, 0)$ with respect to the appropriate double gradings, and thus degree 0 with respect to the associated single gradings.

The homomorphisms as in (6.12.6) are the same as the compositions of the natural quotient mappings from $Z(V)$, $Z(W)$ onto $H(V)$, $H(W)$ with the natural inclusion mappings into $Z'(V)$, $Z'(W)$, respectively. This implies that

(6.12.8) $\tau \circ \xi$ is the same as the homomorphism as in (6.12.5).

More precisely, the homomorphism as in (6.12.4)

(6.12.9) $$\text{maps } Z(V) \bigotimes_A Z(W) \text{ into } Z\big(V \bigotimes_A W\big),$$

as in the previous section. Equivalently, the homomorphism as in (6.12.4) is essentially the same as the one as in (6.11.11), because $Z\big(V \bigotimes_A W\big)$ is a submodule of $V \bigotimes_A W$.

The restriction of the homomorphism as in (6.12.3) to $Z\big(V \bigotimes_A W\big)$ defines a homomorphism

(6.12.10) $$\text{from } Z\big(V \bigotimes_A W\big) \text{ into } Z'(V) \bigotimes_A Z'(W),$$

as modules over k. This homomorphism has degree 0 with respect to the corresponding single gradings.

One can check that

(6.12.11) the homomorphism as in (6.12.10) is equal to 0 on $B\big(V \bigotimes_A W\big)$.

This leads to a homomorphism

(6.12.12) $$\zeta \text{ from } H\big(V \bigotimes_A W\big) \text{ into } Z'(V) \bigotimes_A Z'(W),$$

as modules over k. This corresponds to the homomorphism ζ in the diagram (1) on p64 of [3], under slightly different conditions. This homomorphism has degree 0 with respect to the appropriate single gradings.

It is easy to see that

(6.12.13) $\zeta \circ \eta$ is the same as the homomorphism as in (6.12.5),

where η is as in (6.11.12), by construction. This means that

(6.12.14) $$\tau \circ \xi = \zeta \circ \eta,$$

by (6.12.8). This corresponds to the commutativity of the diagram (1) on p64 of [3], under slightly different conditions.

Observe that

(6.12.15) $$\zeta \circ \alpha \circ \xi = \zeta \circ \eta = \tau \circ \xi,$$

using (6.12.2) in the first step. This implies that

$$\zeta \circ \alpha = \tau, \tag{6.12.16}$$

because ξ is surjective, as in (6.12.1). This corresponds to part of Proposition 6.1 on p64 of [3], under slightly different conditions.

6.13 Homology and $\mathrm{Hom}_A^{gr}(V, W)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let (V, d_V) and (W, d_W) be both left or both right graded modules over A with differentiation that are complexes. Remember that $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a doubly-graded module over k as in Section 6.3, and as a double complex with respect to the differentiation operators δ_1, δ_2 discussed in Section 6.5. It follows that $\mathrm{Hom}_A^{gr}(V, W)$ may be considered as a single complex over k, with respect to the single grading obtained from the double grading, and the total differentiation operator

$$\delta = \delta_1 + \delta_2. \tag{6.13.1}$$

Using this,

$$Z\big(\mathrm{Hom}_A^{gr}(V, W)\big),\ B\big(\mathrm{Hom}_A^{gr}(V, W)\big),\ \text{and } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \tag{6.13.2}$$

may be defined as singly-graded modules over k in the usual way.

Let $j, l \in \mathbf{Z}$ and

$$\phi^{j,l} \in \big(\mathrm{Hom}_A^{gr}(V, W)\big)^{j,l} = \mathrm{Hom}_A(V^{-j}, W^l) \tag{6.13.3}$$

be given. Of course, $\phi^{j,l}$ may be considered as an element of $\mathrm{Hom}_A^{gr}(V, W)$ as well. By construction,

$$\delta_1(\phi^{j,l}) = \phi^{j,l} \circ d_V^{-j-1}, \tag{6.13.4}$$

which is an element of $\mathrm{Hom}_A^{gr}(V, W)^{j+1,l}$, and

$$\delta_2(\phi^{j,l}) = (-1)^j\, d_W^l \circ \phi^{j,l}, \tag{6.13.5}$$

which is an element of $\mathrm{Hom}_A^{gr}(V, W)^{j,l+1}$. This means that

$$\delta_1(\phi^{j-1,l}) = \phi^{j-1,l} \circ d_V^{-j} \tag{6.13.6}$$

and

$$\delta_2(\phi^{j,l-1}) = (-1)^j\, d_W^{l-1} \circ \phi^{j,l-1} \tag{6.13.7}$$

are elements of $\big(\mathrm{Hom}_A\, gr(V, W)\big)^{j,l}$.

Let $\phi \in \mathrm{Hom}_A^{gr}(V, W)$ be given, and let $\phi^{j,l}$ be the component of ϕ in $\big(\mathrm{Hom}_A^{gr}(V, W)\big)^{j,l}$, as usual. Observe that

$$(\delta_1(\phi))^{j,l} = \phi^{j-1,l} \circ d_V^{-j} \tag{6.13.8}$$

and

$$(\delta_2(\phi))^{j,l} = (-1)^j \, d_W^{l-1} \circ \phi^{j,l-1}, \tag{6.13.9}$$

as in the preceding paragraph. Thus

$$(\delta(\phi))^{j,l} = \phi^{j-1,l} \circ d_V^{-j} + (-1)^j \, d_W^{l-1} \circ \phi^{j,l-1}. \tag{6.13.10}$$

It follows that $\phi \in Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ if and only if

$$\phi^{j-1,l} \circ d_V^{-j} + (-1)^j \, d_W^{l-1} \circ \phi^{j,l-1} = 0 \tag{6.13.11}$$

for every j, l. Equivalently, this means that

$$\phi^{j-1,l+1} \circ d_V^{-j} + (-1)^j \, d_W^{l} \circ \phi^{j,l} = 0 \tag{6.13.12}$$

for every j, l. This is also the same as saying that

$$\phi^{j,l} \circ d_V^{-j-1} + (-1)^{j+1} \, d_W^{l-1} \circ \phi^{j+1,l-1} = 0 \tag{6.13.13}$$

for every j, l.

Let $j, l \in \mathbf{Z}$ be given, and remember that $\phi^{j,l}$ is a homomorphism from V^{-j} into W^l, as modules over A. If $\phi \in Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$, then

$$\phi^{j,l}(Z(V)^{-j}) \subseteq Z(W)^l, \tag{6.13.14}$$

by (6.13.12). In this case, we also have that

$$\phi^{j,l}(B(V)^{-j}) \subseteq B(W)^l, \tag{6.13.15}$$

by (6.13.13). This implies that $\phi^{j,l}$ induces a homomorphism

$$\text{from } H(V)^{-j} \text{ into } H(W)^l, \tag{6.13.16}$$

as modules over A. Note that this induced homomorphism is equal to 0 for all but finitely many (j,l), because $\phi^{j,l} = 0$ for all but finitely many (j,l), by the definition of $\mathrm{Hom}_A^{gr}(V,W)$.

Let $\mathrm{Hom}_A^{gr}(H(V),H(W))$ be the doubly-graded module over k obtained from $H(V)$, $H(W)$ as singly-graded modules over A as in Section 6.3. Using the induced homomorphisms in (6.13.16) for all j, l, we get an element of $\mathrm{Hom}_A^{gr}(H(V),H(W))$ from ϕ. This defines a homomorphism

$$\text{from } Z\big(\mathrm{Hom}_A^{gr}(V,W)\big) \text{ into } \mathrm{Hom}_A^{gr}(H(V),H(W)), \tag{6.13.17}$$

as modules over k. Remember that $Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ is a singly-graded module over k, and that $\mathrm{Hom}_A^{gr}(H(V),H(W))$ has a single grading induced by the double grading. It is easy to see that the homomorphism in (6.13.17) has degree 0 with respect to these single gradings.

Suppose for the moment that $\phi \in B\big(\mathrm{Hom}_A^{gr}(V,W)\big)$, so that

$$\phi = \delta(\psi) \tag{6.13.18}$$

for some $\psi \in \mathrm{Hom}_A^{gr}(V, W)$. This means that

$$\phi^{j,l} = (\delta(\psi))^{j,l} = \psi^{j-1,l} \circ d_V^{-j} + (-1)^j \, d_W^{l-1} \circ \psi^{j,l-1} \tag{6.13.19}$$

for every j, l, as in (6.13.10). It follows that

$$\phi^{j,l}(Z(V)^{-j}) \subseteq B(W)^l \tag{6.13.20}$$

for every j, l. This implies that the induced homomorphism as in (6.13.16) is equal to 0 for every j, l. Equivalently, ϕ is mapped to 0 by the homomorphism as in (6.13.17).

This shows that the homomorphism as in (6.13.17) leads to a homomorphism

$$\alpha' \text{ from } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ into } \mathrm{Hom}_A^{gr}(H(V), H(W)). \tag{6.13.21}$$

This homomorphism has degree 0 with respect to the appropriate single gradings. This corresponds to the homomorphism in Proposition 6.1a on p65f of [3], under slightly different conditions.

6.14 Some properties of $\mathrm{Hom}_A^{gr}(Z'(V), Z(W))$

We continue with the same notation and hypotheses as in the previous section. Remember that $Z'(V) = V/d_V(V)$ and $Z(W)$ may be considered as singly-graded modules over A, as in Section 5.10. Thus $\mathrm{Hom}_A^{gr}(Z'(V), Z(W))$ may be defined as a doubly-graded module over k, as in Section 6.3.

Using the natural quotient mapping from V onto $Z'(V)$, and the natural inclusion mapping from $Z(W)$ into W, we get a natural homomorphism

$$\text{from } \mathrm{Hom}_A^{gr}(Z'(V), Z(W)) \text{ into } \mathrm{Hom}_A^{gr}(V, W), \tag{6.14.1}$$

as modules over k, as in Section 6.4. This homomorphism has bidegree $(0, 0)$, because the quotient and inclusion mappings mentioned before have degree 0. This homomorphism is also injective.

Let $\phi \in \mathrm{Hom}_A^{gr}(V, W)$ be given. Suppose that

$$\phi^{j,l}(V^{-j}) \subseteq Z(W)^l \tag{6.14.2}$$

for every j, l. Suppose too that

$$\phi^{j,l} = 0 \text{ on } B(V)^{-j} \tag{6.14.3}$$

for every j, l. It is easy to see that ϕ is in the image of the homomorphism as in (6.14.1) under the conditions. Conversely, every element of the image of the homomorphism as in (6.14.1) has these two properties.

Observe that (6.14.3) is the same as saying that

$$\delta_1(\phi^{j,l}) = 0 \tag{6.14.4}$$

in $\big(\mathrm{Hom}_A^{gr}(V,W)\big)^{j+1,l}$, by (6.13.4). Similarly, (6.14.2) is the same as saying that

$$\delta_2(\phi^{j,l}) = 0 \tag{6.14.5}$$

in $\mathrm{Hom}_A^{gr}(V,W)^{j,l+1}$, by (6.13.5). It follows that (6.14.3) holds for every j, l if and only if

$$\delta_1(\phi) = 0. \tag{6.14.6}$$

Similarly, (6.14.2) holds for every j, l if and only if

$$\delta_2(\phi) = 0. \tag{6.14.7}$$

In particular, the natural homomorphism as in (6.14.1) may be considered as a homomorphism

$$\text{from } \mathrm{Hom}^{gr}(Z'(V), Z(W)) \text{ into } Z\big(\mathrm{Hom}_A^{gr}(V,W)\big), \tag{6.14.8}$$

as modules over k. This homomorphism has degree 0, with respect to the appropriate single gradings. This leads to a natural homomorphism

$$\eta \text{ from } \mathrm{Hom}_A^{gr}(Z'(V), Z(W)) \text{ into } H\big(\mathrm{Hom}_A^{gr}(V,W)\big), \tag{6.14.9}$$

as modules over k, by composing the homomorphism as in (6.14.8) with the natural quotient mapping from $Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ onto $H\big(\mathrm{Hom}_A^{gr}(V,W)\big)$. This homomorphism has degree 0 with respect to the appropriate single gradings as well. This corresponds to the homomorphism η in the diagram (1) on p64 of [3], under slightly different consitions.

There is also a natural homomorphism

$$\xi \text{ from } \mathrm{Hom}_A^{gr}(Z'(V), Z(W)) \text{ into } \mathrm{Hom}_A^{gr}(H(V), H(W)), \tag{6.14.10}$$

as modules over k. This uses the natural inclusion mapping from $H(V)$ into $Z'(V)$, and the natural quotient mapping from $Z(W)$ onto $H(W)$, as in Section 6.4 again. This homomorphism has bidegree $(0,0)$, because these inclusion and quotient mappings have degree 0. This corresponds to the homomorphism ξ in the diagram (1) on p64 of [3], under slightly different conditions.

One can check that

$$\alpha' \circ \eta = \xi, \tag{6.14.11}$$

as homomorphisms from $\mathrm{Hom}_A^{gr}(Z'(V), Z(W))$ into $\mathrm{Hom}_A^{gr}(H(V), H(W))$. To see this, observe first that $\alpha' \circ \eta$ is the same as the composition of the homomorphisms as in (6.14.8) and (6.13.17). It is easy to see that this composition is the same as ξ. This corresponds to part of Proposition 6.1a on p65f of [3], under slightly different conditions.

6.15 Some properties of $\mathrm{Hom}_A^{gr}(Z(V), Z'(W))$

Let us continue with the same notation and hypotheses as in the previous two sections. As before, $Z(V)$ and $Z'(W) = W/d_W(W)$ may be considered as singly-graded modules over A, so that $\mathrm{Hom}_A^{gr}(Z(V), Z'(W))$ may be defined as a doubly-graded module over k.

We can use the natural inclusion mapping from $Z(V)$ into V, and the natural quotient mapping from W onto $Z'(W)$, to get a natural homomorphism

$$\text{from } \mathrm{Hom}_A^{gr}(V, W) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), Z'(W)), \tag{6.15.1}$$

as modules over k, as in Section 6.4. This homomorphism has bidegree $(0, 0)$, because the inclusion and quotient maps being used have degree 0.

Of course, there are natural homomorphisms

$$\text{from } Z(V), Z(W) \text{ into } Z'(V), Z'(W), \tag{6.15.2}$$

respectively, as modules over A. These homomorphisms are obtained by composing the natural inclusion mappings from $Z(V)$, $Z(W)$ into V, W with the natural quotient mappings from V, W onto $Z'(V)$, $Z'(W)$, respectively. Note that these homomorphisms have degree 0.

Using these homomorphisms, we get a natural homomorphism

$$\text{from } \mathrm{Hom}_A^{gr}(Z'(V), Z(W)) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), Z'(W)), \tag{6.15.3}$$

as modules over k, as in Section 6.4 again. This homomorphism has bidegree $(0, 0)$, as before. This homomorphism is the same as the composition of the homomorphism as in (6.14.1) with the homomorphism as in (6.15.1).

If $\phi \in B\big(\mathrm{Hom}_A^{gr}(V, W)\big)$, then ϕ maps $Z(V)^{-j}$ into $B(W)^l$ for every j, l, as in (6.13.20). This means that ϕ is mapped to 0 in $\mathrm{Hom}_A^{gr}(Z(V), Z'(W))$ by the homomorphism as in (6.15.1). In particular, this leads to a natural homomorphism

$$\zeta \text{ from } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), Z'(W)), \tag{6.15.4}$$

as modules over k. This homomorphism has degree 0 with respect to the appropriate single gradings. This corresponds to the homomorphism ζ in the diagram (1) on p64 of [3], under slightly different conditions.

Remember that η is the homomorphism as in (6.14.9). It is easy to see that

$$\zeta \circ \eta \text{ is the same as the homomorphism as in (6.15.3).} \tag{6.15.5}$$

Using the natural quotient mapping from $Z(V)$ onto $H(V)$, and the natural inclusion mapping from $H(W)$ into $Z'(W)$, we get a natural homomorphism

$$\tau \text{ from } \mathrm{Hom}_A^{gr}(H(V), H(W)) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), Z'(W)), \tag{6.15.6}$$

as modules over k, as in Section 6.4. Note that

$$\tau \text{ is injective,} \tag{6.15.7}$$

as has bidegree $(0, 0)$, as usual. This corresponds to the homomorphism τ in diagram (1) on p64 of [3], under slightly different conditions.

Remember that ξ is the homomorphism in (6.14.10). It is easy to see that

(6.15.8) $\tau \circ \xi$ is the same as the homomorphism as in (6.15.3).

It follows that

$$\tau \circ \xi = \zeta \circ \eta, \tag{6.15.9}$$

by (6.15.5). This corresponds to the commutativity of the diagram (1) on p64 of [3], under slightly different conditions.

Remember that α' is the homomorphism as in (6.13.21). One can check that

$$\tau \circ \alpha' = \zeta, \tag{6.15.10}$$

as homomorphisms from $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ into $\mathrm{Hom}_A^{gr}(Z(V), Z'(W))$. In fact, α' is uniquely determined by this property, because τ is injective. This corresponds to part of Proposition 6.1a on p65f of [3], under slightly different conditions. Note that (6.14.11) could also be obtained from (6.15.10), by composing both sides with η, and using (6.15.9).

Chapter 7

More on differentiation, 2

7.1 Some splitting and α

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. As in Section 6.11, we let (V, d_V) be a graded right module over A with differentiation that is a complex, and (W, d_W) be a graded left module over A with differentiation that is a complex. We also let $V \bigotimes_A W$ be a tensor product of V and W over A, which is a double complex over k with respect to the double grading and differentiation operators δ_1, δ_2 defined in Sections 6.1 and 6.2. Thus $V \bigotimes_A W$ may be considered as a single complex over k too, with respect to the single grading obtained from the double grading, and the total differentiation operator $\delta = \delta_1 + \delta_2$, as in Section 5.14.

Remember that

$$Z'(V) = V/d_V(V), \ Z'(W) = W/d_W(W) \tag{7.1.1}$$

are graded modules over A, and that $H(V)$, $H(W)$ are homogeneous submodules of $Z'(V)$, $Z'(W)$, respectively, as in Section 5.10. In this section, we suppose that

(7.1.2) $Z'(V)$ corresponds to the direct sum of $H(V)$ and another homogeneous submodule of $Z'(V)$,

as a right module over A, and similarly that

(7.1.3) $Z'(W)$ corresponds to the direct sum of $H(W)$ and another homogeneous submodule of $Z'(W)$,

as a left module over A. This means that $Z'(V)$, $Z'(W)$ correspond to the direct sums of $H(V)$, $H(W)$ and other submodules, respectively, as graded modules over A.

Equivalently,

(7.1.4) there are homomorphisms from $Z'(V), Z'(W)$ onto $H(V), H(W)$, as modules over A, that are equal to the identity mappings on $H(V), H(W)$, respectively, and that have degree 0.

We can compose the natural quotient mappings from V, W onto $Z'(V)$, $Z'(W)$, respectively, with the homomorphisms as in (7.1.4) to get homomorphisms

(7.1.5) $$\beta, \gamma \text{ from } V, W \text{ onto } H(V), H(W),$$

respectively, as modules over A, of degree 0.

Of course, the natural quotient mappings from V, W onto $Z'(V)$, $Z'(W)$ are equal to 0 on $d_V(V)$, $d_W(W)$, respectively. This means that the compositions of these quotient mappings with d_V, d_W, respectively, are equal to 0. It follows that

(7.1.6) $$\beta \circ d_V,\ \gamma \circ d_W = 0.$$

Let $H(V) \bigotimes_A H(W)$ be a tensor product of $H(V)$ and $H(W)$, which is a doubly-graded module over k. Using β and γ, we get a homomorphism $\beta \otimes \gamma$ from $V \bigotimes_A W$ onto $H(V) \bigotimes_A H(W)$, as modules over k, of bidegree $(0, 0)$, as in Section 6.1. It is easy to see that

(7.1.7) $$(\beta \otimes \gamma) \circ \delta_1 = (\beta \otimes \gamma) \circ \delta_2 = 0,$$

using (7.1.6). This implies that

(7.1.8) $$(\beta \otimes \gamma) \circ \delta = 0.$$

This leads to a homomorphism

(7.1.9) $$\text{from } H\big(V \bigotimes_A W\big) \text{ into } H(V) \bigotimes_A H(W),$$

as modules over k, in a natural way. More precisely, the composition of the natural quotient mapping from $Z\big(V \bigotimes_A W\big)$ onto $H\big(V \bigotimes_A W\big)$ with this homomorphism is the same as the restriction of $\beta \otimes \gamma$ to $Z\big(V \bigotimes_A W\big)$. This homomorphism has degree 0, with respect to the usual single grading on $H\big(V \bigotimes_A W\big)$, and the single grading on $H(V) \bigotimes_A H(W)$ obtained from the double grading in the usual way.

Let α be the homormophism from $H(V) \bigotimes_A H(W)$ into $H\big(V \bigotimes_A W\big)$ discussed in Section 6.11. One can check that

(7.1.10) the composition of α with the homomorphism as in (7.1.9) is the identity mapping on $H(V) \bigotimes_A H(W)$.

This implies that

(7.1.11) $$\alpha \text{ is injective on } H(V) \bigotimes_A H(W).$$

We also get that

$$H\big(V \bigotimes_A W\big) \text{ corresponds to the direct sum of } \alpha\big(H(V) \bigotimes_A H(W)\big) \text{ and another homogeneous submodule of } H\big(V \bigotimes_A W\big), \tag{7.1.12}$$

as a singly-graded module over k. This corresponds to Proposition 6.2 on p66 of [3], under slightly different conditions.

7.2 More splitting and α

Let us return to the same notation and hypotheses as at the beginning of the previous section. Put

$$V_1 = B(V) = d_V(V),\ W_1 = B(W) = d_W(W), \tag{7.2.1}$$

which are homogeneous submodules of V, W, respectively. Suppose that

$$V_2, V_3 \text{ are homogeneous submodules of } V \tag{7.2.2}$$

and

$$W_2, W_3 \text{ are homogeneous submodules of } W, \tag{7.2.3}$$

with the following two properties. First,

$$V \text{ corresponds to the direct sum of } V_1, V_2, \text{ and } V_3, \tag{7.2.4}$$

as a right module over A, and

$$W \text{ corresponds to the direct sum of } W_1, W_2, \text{ and } W_3, \tag{7.2.5}$$

as a left module over A. Second,

$$Z(V) = V_1 + V_2 \tag{7.2.6}$$

and

$$Z(W) = W_1 + W_2. \tag{7.2.7}$$

More precisely, $Z(V)$ and $Z(W)$ correspond to the direct sums of V_1, V_2 and W_1, W_2, respectively, as graded modules over A. It follows that the restrictions of the natural quotient mappings from $Z(V)$, $Z(W)$ onto $H(V)$, $H(W)$ to V_2, W_2 are isomorphisms

$$\text{from } V_2, W_2 \text{ onto } H(V), H(W), \tag{7.2.8}$$

respectively, as graded modules over A.

Similarly, the restrictions of the natural quotient mappings from V, W onto $Z'(V)$, $Z'(W)$ to $V_2 + V_3$, $W_2 + W_3$ are isomorphisms

$$\text{from } V_2 + V_3, W_2 + W_3 \text{ onto } Z'(V), Z'(W), \tag{7.2.9}$$

respectively, as modules over A. The images of V_2, W_2 under these quotient mappings are the same as the images of $Z(V)$, $Z(W)$, by hypothesis. Thus these images correspond to $H(V)$, $H(W)$, respectively. Of course, $V_2 + V_3$ and $W_2 + W_3$ correspond to the direct sums of V_2, V_3 and W_2, W_3, respectively, as modules over A. It follows that (7.1.2) and (7.1.3) hold under these conditions.

Let $V_a \bigotimes_A W_b$ be a tensor product of V_a and W_b over A for each $a, b = 1, 2, 3$. Of course, $V \bigotimes_A W$ is isomorphic to the direct sum of $V_a \bigotimes_A W_b$, $a, b = 1, 2, 3$, as a doubly-graded module over k. Thus we may identify $V_a \bigotimes_A W_b$ with a bihomogeneous submodule of $V \bigotimes_A W$, as a doubly-graded module over k, for each $a, b = 1, 2, 3$.

If $a = 1, 2$, or 3, then we may take $V_a \bigotimes_A Z(W)$ to be the bihomogeneous submodule of $V \bigotimes_A W$ corresponding to the direct sum of $V_a \bigotimes_A W_b$, $b = 1, 2$. Similarly, if $b = 1, 2$, or 3, then we may take $Z(V) \bigotimes_A W_b$ to be the bihomogeneous submodule of $V \bigotimes_A W$ corresponding to the direct sum of $V_a \bigotimes_A W_b$, $a = 1, 2$. We may take $Z(V) \bigotimes_A Z(W)$ to be the bihomogeneous submodule of $V \bigotimes_A W$ corresponding to the direct sum of $V_a \bigotimes_A W_b$, $a, b = 1, 2$, as well.

The restrictions of d_V, d_W to V_3, W_3 are one-to-one mappings onto V_1, W_1, respectively. If $a = 1, 2$, or 3, then it follows that

$$\begin{gathered}\text{the restriction of } \delta_2 \text{ to } V_a \bigotimes_A W_3 \\ \text{is a one-to-one mapping onto } V_a \bigotimes_A W_1.\end{gathered} \tag{7.2.10}$$

If $a = 1$ or 2, then $\delta_1 = 0$ on $V_a \bigotimes_A W_3$, so that $\delta = \delta_2$ on $V_a \bigotimes_A W_3$, and thus

$$\begin{gathered}\text{the restriction of } \delta \text{ to } V_a \bigotimes_A W_3 \\ \text{is a one-to-one mapping onto } V_a \bigotimes_A W_1.\end{gathered} \tag{7.2.11}$$

Similarly, if $b = 1, 2$, or 3, then

$$\begin{gathered}\text{the restriction of } \delta_1 \text{ to } V_3 \bigotimes_A W_b \\ \text{is a one-to-one mapping onto } V_1 \bigotimes_A W_b.\end{gathered} \tag{7.2.12}$$

If $b = 1$ or 2, then $\delta_2 = 0$ on $V_3 \bigotimes_A W_b$, so that $\delta = \delta_1$ on $V_3 \bigotimes_A W_b$, and

$$\begin{gathered}\text{the restriction of } \delta \text{ to } V_3 \bigotimes_A W_b \\ \text{is a one-to-one mapping onto } V_1 \bigotimes_A W_b.\end{gathered} \tag{7.2.13}$$

Observe that

$$\begin{gathered}\text{the restriction of } \delta \text{ to } V_3 \bigotimes_A W_3 \text{ is a one-to-one} \\ \text{mapping into } \big(V_1 \bigotimes_A W_3\big) + \big(V_3 \bigotimes_A W_1\big),\end{gathered} \tag{7.2.14}$$

by (7.2.10) and (7.2.12), with $a = b = 3$. Of course,

$$Z(V) \bigotimes_A Z(W) \subseteq Z\big(V \bigotimes_A W\big). \tag{7.2.15}$$

One can check that

$$Z\big(V \bigotimes_A W\big) \subseteq \big(Z(V) \bigotimes_A Z(W)\big) + \big(V_1 \bigotimes_A W_3\big) + \big(V_3 \bigotimes_A W_1\big), \tag{7.2.16}$$

using the earlier remarks.

It is easy to see that

$$\begin{aligned} B\big(V \bigotimes_A W\big) &= \big(V_1 \bigotimes_A W_1\big) + \big(V_2 \bigotimes_A W_1\big) + \big(V_1 \bigotimes_A W_2\big) \\ &\quad + \delta\big(V_3 \bigotimes_A W_3\big), \end{aligned} \tag{7.2.17}$$

using (7.2.11) and (7.2.13). Note that

$$\delta\big(V_3 \bigotimes_A W_3\big) \subseteq Z\big(V \bigotimes_A W\big) \cap \big(\big(V_1 \bigotimes_A W_3\big) + \big(V_3 \bigotimes_A W_1\big)\big). \tag{7.2.18}$$

We would like to verify that

$$\delta\big(V_3 \bigotimes_A W_3\big) = Z\big(V \bigotimes_A W\big) \cap \big(\big(V_1 \bigotimes_A W_3\big) + \big(V_3 \bigotimes_A W_1\big)\big). \tag{7.2.19}$$

Every element of $V_1 \bigotimes_A W_3$ can be expressed as $\delta_1(y)$ for a unique y in $V_3 \bigotimes_A W_3$, by (7.2.12), with $b = 3$. Similarly, every element of $V_3 \bigotimes_A W_1$ can be expressed as $\delta_2(z)$ for a unique $z \in V_3 \bigotimes W_3$, by (7.2.10), with $a = 3$. We also have that

$$\delta(\delta_1(y) + \delta_2(z)) = \delta_2(\delta_1(y)) + \delta_1(\delta_2(z)). \tag{7.2.20}$$

This is equal to 0 when $y = z$, as in Section 6.2.

Conversely, if (7.2.20) is equal to 0, then one can check that $y = z$, using (7.2.10) and (7.2.12). This implies that (7.2.19) holds.

It follows that

$$Z\big(V \bigotimes_A W\big) = \big(Z(V) \bigotimes_A Z(W)\big) + \delta\big(V_3 \bigotimes_A W_3\big). \tag{7.2.21}$$

Using this, we get that

$$H\big(V \bigotimes_A W\big) \text{ is isomorphic to } V_2 \bigotimes_A W_2, \tag{7.2.22}$$

as graded modules over k. Equivalently,

$$H\big(V \bigotimes_A W\big) \text{ is isomorphic to } H(V) \bigotimes_A H(W), \tag{7.2.23}$$

as graded modules over k. More precisely,

$$\alpha \text{ is an isomorphism from } H(V) \bigotimes_A H(W) \text{ onto } H\big(V \bigotimes_A W\big), \tag{7.2.24}$$

as graded modules over k under these conditions, where α is as in Section 6.11. This corresponds to Proposition 7.4 on p70 of [3].

7.3 Some splitting and α'

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. As in Section 6.13, we let (V, d_V) and (W, d_W) be both left or both right graded modules over A with differentiation that are complexes. Thus $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a doubly-graded module over k, and in fact as a double complex with respect to the appropriate differentiation operators δ_1, δ_2, as in Sections 6.3 and 6.5. This implies that $\mathrm{Hom}_A^{gr}(V, W)$ is a single complex over k with respect to the single grading obtained from the double grading and the total differentation operator $\delta = \delta_1 + \delta_2$, as before.

Suppose that $Z'(V)$ satisfies (7.1.2), as a left or right module over A, as appropriate. In this section, we also ask that

(7.3.1) $$Z(W) \text{ corresponds to the direct sum of } B(W) \text{ and another homogeneous submodule of } Z(W),$$

as a module over A. Equivalently, this means that $Z(W)$ corresponds to the direct sum of $B(W)$ and another submodule, as a graded module over A.

As before, the condition on $Z'(V)$ is the same as saying that there is a homomorphism from $Z'(V)$ onto $H(V)$, as modules over A, that is equal to the identity mapping on $H(V)$, and has degree 0. This leads to a homomorphism

(7.3.2) $$\beta \text{ from } V \text{ onto } H(V),$$

as modules over A, with degree 0, by composing the natural quotient mapping from V onto $Z'(V)$ with the previous homomorphism from $Z'(V)$ onto $H(V)$. Note that

(7.3.3) $$\beta \circ d_V = 0.$$

Using (7.3.1), we get that

(7.3.4) $$\begin{gathered}\text{there is a homomorphism from } H(W) \text{ into } Z(W),\\ \text{as modules over } A, \text{ of degree } 0, \text{ whose composition}\\ \text{with the natural quotient mapping from } Z(W) \text{ onto } H(W)\\ \text{is the identity mapping on } H(W).\end{gathered}$$

More precisely, $Z(W)$ corresponds to the direct sum of $B(W)$ and the image of $H(W)$ under this homomorphism, as a graded module over A. Let $\widetilde{\gamma}$ be the homomorphism as in (7.3.4), considered as a homomorphism

(7.3.5) $$\text{from } H(W) \text{ into } W.$$

Alternatively, $\widetilde{\gamma}$ may be considered as the composition of the homomorphism in (7.3.4) with the natural inclusion mapping from $Z(W)$ into W. Of course,

(7.3.6) $$d_W \circ \widetilde{\gamma} = 0,$$

because $\widetilde{\gamma}(H(W)) \subseteq Z(W)$.

Note that $\mathrm{Hom}^{gr}_A(H(V), H(W))$ may be defined as a doubly-graded module over k, as in Section 6.3. As in Section 6.4, we can use β and $\widetilde{\gamma}$ to get a homomorphism

(7.3.7) $\mathrm{Hom}^{gr}(\beta, \widetilde{\gamma})$ from $\mathrm{Hom}^{gr}_A(H(V), H(W))$ into $\mathrm{Hom}^{gr}_A(V, W)$,

as modules over k, of bidegree $(0, 0)$.

One can check that

$$\delta_1 \circ \mathrm{Hom}^{gr}(\beta, \widetilde{\gamma}) = 0, \tag{7.3.8}$$

using (7.3.3). Similarly,

$$\delta_2 \circ \mathrm{Hom}^{gr}(\beta, \widetilde{\gamma}) = 0, \tag{7.3.9}$$

because of (7.3.6). This implies that

$$\delta \circ \mathrm{Hom}^{gr}(\beta, \widetilde{\gamma}) = 0. \tag{7.3.10}$$

Equivalently,

$$\mathrm{Hom}^{gr}(\beta, \widetilde{\gamma})\big(\mathrm{Hom}^{gr}_A(H(V), H(W))\big) \subseteq Z\big(\mathrm{Hom}^{gr}_A(V, W)\big). \tag{7.3.11}$$

This leads to a homomorphism

(7.3.12) from $\mathrm{Hom}^{gr}_A(H(V), H(W))$ into $H\big(\mathrm{Hom}^{gr}_A(V, W)\big)$,

as modules over k, by composing $\mathrm{Hom}^{gr}(\beta, \widetilde{\gamma})$ with the natural quotient mapping

(7.3.13) from $Z\big(\mathrm{Hom}^{gr}_A(V, W)\big)$ onto $H\big(\mathrm{Hom}^{gr}_A(V, W)\big)$.

This homomorphism has degree 0, with respect to the single grading associated to the double grading on $\mathrm{Hom}^{gr}_A(H(V), H(W))$, and the appropriate single grading on $H\big(\mathrm{Hom}^{gr}_A(V, W)\big)$.

Let α' be the homomorphism

(7.3.14) from $H\big(\mathrm{Hom}^{gr}_A(V, W)\big)$ into $\mathrm{Hom}^{gr}_A(H(V), H(W))$

discussed in Section 6.13. One can check that

(7.3.15) the composition of the homomorphism as in (7.3.12) with α' is the identity mapping on $\mathrm{Hom}^{gr}_A(H(V), H(W))$.

It follows that

$$\alpha'\Big(H\big(\mathrm{Hom}^{gr}_A(V, W)\big)\Big) = \mathrm{Hom}^{gr}_A(H(V), H(W)). \tag{7.3.16}$$

We also get that

(7.3.17) $H\big(\mathrm{Hom}^{gr}_A(V, W)\big)$ corresponds to the direct sum of $\ker \alpha'$ and another homogeneous submodule of $H\big(\mathrm{Hom}^{gr}_A(V, W)\big)$,

as a singly-graded module over k. This corresponds to Proposition 6.2a on p66 of [3], under slightly different conditions.

7.4 More splitting and α'

We return to the same notation and hypotheses as at the beginning of the previous section. As in Section 7.2, we would like to consider stronger splitting conditions on V and W now.

As before, we put

$$V_1 = B(V),\ W_1 = B(W), \tag{7.4.1}$$

which are homogeneous submodules of V, W, respectively. We suppose again that V_2, V_3 are homogeneous submodules of V, and that V corresponds to the direct sum of V_1, V_2, and V_3, as a module over A. Similarly, we suppose that W_2, W_3 are homogeneous submodules of W, and that W corresponds to the direct sum of W_1, W_2, and W_3, as a module over A. We also ask that

$$Z(V) = V_1 + V_2,\ Z(W) = W_1 + W_2, \tag{7.4.2}$$

as before, so that $Z(V)$ and $Z(W)$ correspond to the direct sums of V_1, V_2 and W_1, W_2, respectively, as graded modules over A. This implies that the restrictions of the natural quotient mappings from $Z(V)$, $Z(W)$ onto $H(V)$, $H(W)$ to V_2, W_2 are isomorphisms onto $H(V)$, $H(W)$, respectively, as graded modules over A, as before.

It follows that $Z'(V)$ corresponds to the direct sum of $H(V)$ and another homogeneous submodules of $Z'(V)$, as a module over A, as in Section 7.2. Of course, (7.3.1) holds, by hypothesis. This means that V and W satisfy the conditions mentioned in the previous section.

Note that $\mathrm{Hom}_A^{gr}(V_a, W_b)$ may be defined as a doubly-graded module over k as in Section 6.3 for each $a, b = 1, 2, 3$. It is easy to see that

(7.4.3) $\mathrm{Hom}_A^{gr}(V, W)$ is isomorphic to the direct sum of $\mathrm{Hom}_A^{gr}(V_a, W_b), a, b = 1, 2, 3,$

as a doubly-graded module over k, in a natural way. We shall use this to identify $\mathrm{Hom}_A^{gr}(V_a, W_b)$ with a bihomogeneous submodule of $\mathrm{Hom}_A^{gr}(V, W)$, as a doubly-graded module over k, for each $a, b = 1, 2, 3$.

Similarly, if $a = 1, 2$, or 3, then

(7.4.4) $\mathrm{Hom}_A^{gr}(V_a, Z(W))$ is isomorphic to the direct sum of $\mathrm{Hom}_A^{gr}(V_a, W_b), b = 1, 2,$

as a doubly-graded module over k in a natural way. If $b = 1, 2$, or 3, then

(7.4.5) $\mathrm{Hom}_A^{gr}(Z(V), W_b)$ is isomorphic to the direct sum of $\mathrm{Hom}_A^{gr}(V_a, W_b), a = 1, 2,$

as a doubly-graded module over k in a natural way. We also have that

(7.4.6) $\mathrm{Hom}_A^{gr}(Z(V), Z(W))$ is isomorphic to the direct sum of $\mathrm{Hom}_A^{gr}(V_a, W_b), a, b = 1, 2,$

as a doubly-graded module over k in a natural way. These modules may be identified with bihomogeneous submodules of $\mathrm{Hom}_A^{gr}(V, W)$, as a doubly-graded module over k, as before.

If $a = 2$ or 3, then one can check that

(7.4.7) $$\delta_1 = 0 \text{ on } \mathrm{Hom}_A^{gr}(V_a, W_b)$$

for each $b = 1, 2, 3$. This uses the way that $\mathrm{Hom}_A^{gr}(V_a, W_b)$ is identified with a submodule of $\mathrm{Hom}_A^{gr}(V, W)$, so that an element of $\mathrm{Hom}_A^{gr}(V_a, W_b)$ corresponds to an element of $\mathrm{Hom}_A^{gr}(V, W)$ whose components in $\mathrm{Hom}_A^{gr}(V_{a'}, W_{b'})$ are equal to 0 when $a \neq a'$ or $b \neq b'$. This also uses the fact that $V_1 = d_V(V)$, by construction. Of course,

(7.4.8) $$\delta_2 = 0 \text{ on } \mathrm{Hom}_A^{gr}(V_a, W_b)$$

for each $a = 1, 2, 3$ and $b = 1, 2$, because $d_W = 0$ on W_1, W_2.

The restrictions of d_V, d_W to V_3, W_3 are one-to-one mappings onto V_1, W_1, respectively, as in Section 7.2. If $a = 1, 2$, or 3, then it follows that

(7.4.9) the restriction of δ_2 to $\mathrm{Hom}_A^{gr}(V_a, W_3)$ is a one-to-one mapping onto $\mathrm{Hom}_A^{gr}(V_a, W_1)$.

If $a = 2$ or 3, then $\delta_1 = 0$ on $\mathrm{Hom}_A^{gr}(V_a, W_3)$, as in (7.4.7), so that $\delta = \delta_2$ on $\mathrm{Hom}_A^{gr}(V_a, W_3)$. This implies that

(7.4.10) the restriction of δ to $\mathrm{Hom}_A^{gr}(V_a, W_3)$ is a one-to-one mapping onto $\mathrm{Hom}_A^{gr}(V_a, W_1)$

when $a = 2$ or 3.

If $b = 1, 2$, or 3, then it is easy to see that

(7.4.11) the restriction of δ_1 to $\mathrm{Hom}_A^{gr}(V_1, W_b)$ is a one-to-one mapping onto $\mathrm{Hom}_A^{gr}(V_3, W_b)$.

If $b = 1$ or 2, then $\delta_2 = 0$ on $\mathrm{Hom}_A^{gr}(V_1, W_b)$, so that $\delta = \delta_1$ on $\mathrm{Hom}_A^{gr}(V_1, W_b)$, and thus

(7.4.12) the restriction of δ to $\mathrm{Hom}_A^{gr}(V_1, W_b)$ is a one-to-one mapping onto $\mathrm{Hom}_A^{gr}(V_3, W_b)$.

Note that

(7.4.13) the restriction of δ to $\mathrm{Hom}_A^{gr}(V_1, W_3)$ is a one-to-one mapping into $\mathrm{Hom}_A^{gr}(V_3, W_3) + \mathrm{Hom}_A^{gr}(V_1, W_1)$,

by (7.4.9) with $a = 1$, and (7.4.11) with $b = 3$.

Clearly

(7.4.14) $$\delta = 0 \text{ on } \mathrm{Hom}_A^{gr}(V_a, W_b)$$

when $a = 2$ or 3 and $b = 1$ or 2, by (7.4.7) and (7.4.8). This means that

$$\mathrm{Hom}^{gr}_A(V_2, Z(W)) + \mathrm{Hom}^{gr}_A(V_3, Z(W)) \subseteq Z\big(\mathrm{Hom}^{gr}_A(V, W)\big). \tag{7.4.15}$$

Observe that

$$\delta\big(\mathrm{Hom}^{gr}_A(V_1, W_1)\big) = \delta\big(\mathrm{Hom}^{gr}_A(V_3, W_3)\big) = \mathrm{Hom}^{gr}_A(V_3, W_1), \tag{7.4.16}$$

by (7.4.10) with $a = 3$, and (7.4.12) with $b = 1$. One can verify that

$$\begin{aligned} Z\big(\mathrm{Hom}^{gr}_A(V, W)\big) \subseteq{}& \mathrm{Hom}^{gr}_A(V_2, Z(W)) + \mathrm{Hom}^{gr}_A(V_3, Z(W)) \\ &+\mathrm{Hom}^{gr}_A(V_1, W_1) + \mathrm{Hom}^{gr}_A(V_3, W_3). \end{aligned} \tag{7.4.17}$$

This uses (7.4.10) with $a = 2$, (7.4.12) with $b = 2$, and (7.4.13).

We also have that

$$\begin{aligned} B\big(\mathrm{Hom}^{gr}_A(V, W)\big) ={}& \mathrm{Hom}^{gr}_A(V_2, W_1) + \mathrm{Hom}^{gr}_A(V_3, W_1) + \mathrm{Hom}^{gr}_A(V_3, W_2) \\ &+\delta\big(\mathrm{Hom}^{gr}_A(V_1, W_3)\big), \end{aligned} \tag{7.4.18}$$

by (7.4.10), (7.4.12), and (7.4.14). Of course,

$$\begin{aligned} &\delta\big(\mathrm{Hom}^{gr}_A(V_1, W_3)\big) \\ &\subseteq Z\big(\mathrm{Hom}^{gr}_A(V, W)\big) \cap \big(\mathrm{Hom}^{gr}_A(V_1, W_1) + \mathrm{Hom}^{gr}_A(V_3, W_3)\big), \end{aligned} \tag{7.4.19}$$

by (7.4.13). We would like to check that

$$\begin{aligned} &\delta\big(\mathrm{Hom}^{gr}_A(V_1, W_3)\big) \\ &= Z\big(\mathrm{Hom}^{gr}_A(V, W)\big) \cap \big(\mathrm{Hom}^{gr}_A(V_1, W_1) + \mathrm{Hom}^{gr}_A(V_3, W_3)\big). \end{aligned} \tag{7.4.20}$$

Every element of $\mathrm{Hom}^{gr}_A(V_1, W_1)$ can be expressed as

$$\delta_2(\phi) \tag{7.4.21}$$

for a unique $\phi \in \mathrm{Hom}^{gr}_A(V_1, W_3)$, by (7.4.9), with $a = 1$. Similarly, every element of $\mathrm{Hom}^{gr}_A(V_3, W_3)$ can be expressed as

$$\delta_1(\psi) \tag{7.4.22}$$

for a unique $\psi \in \mathrm{Hom}^{gr}_A(V_1, W_3)$, by (7.4.11) with $b = 3$. Note that

$$\delta(\delta_2(\phi) + \delta_1(\psi)) = \delta_1(\delta_2(\phi)) + \delta_2(\delta_1(\psi)). \tag{7.4.23}$$

This is equal to 0 when $\phi = \psi$, as in Section 6.5. Conversely, if (7.4.23) is equal to 0, then one can check that $\phi = \psi$, using (7.4.9) and (7.4.11).

This implies (7.4.20). It follows that

$$\begin{aligned} Z\big(\mathrm{Hom}^{gr}_A(V, W)\big) ={}& \mathrm{Hom}^{gr}_A(V_2, Z(W)) + \mathrm{Hom}^{gr}_A(V_3, Z(W)) \\ &+\delta\big(\mathrm{Hom}^{gr}_A(V_1, W_3)\big), \end{aligned} \tag{7.4.24}$$

by (7.4.15) and (7.4.17).

Combining this with (7.4.18), we obtain that

$$H\big(\mathrm{Hom}^{gr}_A(V,W)\big) \text{ is isomorphic to } \mathrm{Hom}^{gr}_A(V_2,W_2), \tag{7.4.25}$$

as graded modules over k. This means that

$$H\big(\mathrm{Hom}^{gr}_A(V,W)\big) \text{ is isomorphic to } \mathrm{Hom}^{gr}_A(H(V),H(W)), \tag{7.4.26}$$

as graded modules over k. In fact,

$$\begin{aligned} &\alpha' \text{ is an isomorphism from } H\big(\mathrm{Hom}^{gr}_A(V,W)\big) \\ &\text{onto } \mathrm{Hom}^{gr}_A(H(V),H(W)) \end{aligned} \tag{7.4.27}$$

as graded modules over k in this case, where α' is as in Section 6.13. This corresponds to Proposition 7.4 on p70 of [3] again.

7.5 Transforming modules and complexes

We have seen a couple of basic ways in which pairs of singly-graded modules or complexes can be combined to get doubly-graded modules or complexes, as in the previous chapter. This is discussed more broadly in Section 5 of Chapter IV of [3], for combining r singly-graded modules of complexes into r-graded modules and r-tuple complexes for any r. In particular, one can take $r = 1$, and consider transformations of singly-graded modules or complexes into other singly-graded modules or complexes. We shall consider some instances of this here, and in later sections.

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a right module over A, let W be a left module over A, and let $V \bigotimes_A W$ be a tensor product of V and W over A. In this section, we shall normally consider W as fixed, and consider $V \bigotimes_A W$ as a transform of V. Of course, one could just as well consider $V \bigotimes_A W$ as a transform of W, with V fixed.

If V is a graded module over A, then $V \bigotimes_A W$ is graded as a module over k in a natural way. More precisely, if $V^j \bigotimes_A W$ is a tensor product of V^j with W over A for each j, then $V \bigotimes_A W$ is isomorphic to the direct sum of $V^j \bigotimes_A W$, $j \in \mathbf{Z}$. Thus we can identify $V^j \bigotimes_A W$ with a submodule of $V \bigotimes_A W$ for each j, as a module over k, and take

$$\big(V \bigotimes_A W\big)^j = V^j \bigotimes_A W. \tag{7.5.1}$$

This corresponds to some remarks on p62 of [3], with $r = 1$.

We may consider W as a graded module over A, with

$$\begin{aligned} W^l &= W \quad \text{when } l = 0 \\ &= \{0\} \quad \text{when } l \neq 0, \end{aligned} \tag{7.5.2}$$

as on p75 of [3]. If V is a graded module over A, then $V \bigotimes_A W$ may be considered as a doubly-graded module over k, as in Section 6.1, with

$$(7.5.3) \qquad \big(V \bigotimes_A W\big)^{j,l} = V^j \bigotimes_A W^l \begin{array}{ll} = V^j \bigotimes_A W & \text{when } l = 0 \\ = \{0\} & \text{when } l \neq 0. \end{array}$$

In this case, the single grading on $V \bigotimes_A W$ associated to the double grading is the same as the single grading in the preceding paragraph.

Suppose now that (V, d_V) is a right module over A with differentiation. Let

$$(7.5.4) \qquad d = d_{V \bigotimes_A W}$$

be the homomorphism from $V \bigotimes_A W$ into itself, as a module over k, obtained from d_V and the identity mapping on W in the usual way. Thus

$$(7.5.5) \qquad d(v \otimes w) = d_V(v) \otimes w$$

for every $v \in V$ and $w \in W$, and d is uniquely determined by this property. It is easy to see that

$$(7.5.6) \qquad d \circ d = 0 \quad \text{on } V \bigotimes_A W,$$

so that $V \bigotimes_A W$ is a module with differentiation with respect to d.

Suppose that (V, d_V) is in fact a graded right module over A with differentiation that is a complex. We may consider W as a graded left module over A with differentiation that is a complex, with the grading as in (7.5.2), and differentiation operator $d_W = 0$, as on p75 of [3]. In this case, $V \bigotimes_A W$ is a double complex, with differentiation operators

$$(7.5.7) \qquad \delta_1 = d, \ \delta_2 = 0,$$

as in Section 6.2. This means that $V \bigotimes_A W$ is a single complex with respect to the total differentiation operator

$$(7.5.8) \qquad \delta = \delta_1 + \delta_2 = d,$$

and the single gradng associated to the double grading on $V \bigotimes_A W$, as in Section 5.14.

Alternatively, if $V \bigotimes_A W$ is considered as a singly-graded module over k as in (7.5.1), then it is easy to see that d has degree 1. This makes $V \bigotimes_A W$ into a single complex, as on p63 of [3]. It is easy to see that this is the same as considering $V \bigotimes_A W$ as a single complex as in the preceding paragraph.

7.6 Induced mappings and homotopies

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a right module over A, let W be a left module over A, and let $V \bigotimes_A W$ be a tensor product of V and W over A, as before.

Suppose for the moment that (V, d_V) is a right module over A with differentiation, and let $d = d_{V \bigotimes_A W}$ be defined on $V \bigotimes_A W$ as in (7.5.4). Let (V_0, d_{V_0}) be another right module over A with differentiation, and let $V_0 \bigotimes_A W$ be a tensor product of V_0 and W over A. We can define

$$d_0 = d_{V_0 \bigotimes_A W} \tag{7.6.1}$$

on $V_0 \bigotimes_A W$ in the same way as before, so that $V_0 \bigotimes_A W$ is a module with differentiation with respect to d_0.

Let ϕ be a homomorphism from V into V_0, as right modules over A with differentiation. This leads to a homomorphism Φ

$$\text{from } V \bigotimes_A W \text{ into } V_0 \bigotimes_A W, \tag{7.6.2}$$

as modules over k, using the identity mapping on W. It is easy to see that Φ is a homomorphism as in (7.6.2), as modules with differentiation.

Let ψ be another homomorphism from V into V_0, as right modules over A with differentiation, and let Ψ be the corresponding homomorphism as in (7.6.2), as modules over k with differentiation. Suppose that σ is a homotopy between ϕ and ψ, so that σ is a homomorphism from V into V_0, as right modules over A, such that

$$d_{V_0} \circ \sigma + \sigma \circ d_V = \phi - \psi. \tag{7.6.3}$$

Let Σ be the homomorphism as in (7.6.2), as modules over k, corresponding to σ and the identity mapping on W. Observe that

$$d_0 \circ \Sigma + \Sigma \circ d = \Phi - \Psi, \tag{7.6.4}$$

so that Σ is a homotopy between Φ and Ψ.

Suppose now that V, V_0 are graded right modules over A, so that $V \bigotimes_A W$, $V_0 \bigotimes_A W$ may be considered as graded modules over k, as in (7.5.1). Let ϕ be a homomorphism from V into V_0, as right modules over A, and let Φ be the homomorphism as in (7.6.2) corresponding to ϕ and the identity mapping on W, as usual. Suppose that ϕ has degree $p \in \mathbf{Z}$, and note that Φ has degree p as well.

If we consider W as a graded module as in (7.5.2), then $V \bigotimes_A W$, $V_0 \bigotimes_A W$ may be considered as doubly-graded modules over k, as in (7.5.3). It is easy to see that Φ is the same as the homomorphism as in (7.6.2) of bidegree $(p, 0)$ associated to ϕ and the identity mapping on W as in Section 6.1 in this case.

Now let (V, d_V), (V_0, d_{V_0}) be graded right modules over A with differentiation that are complexes, so that $V \bigotimes_A W$, $V_0 \bigotimes_A W$ may be considered as single complexes too, as in the previous section. Suppose that ϕ is a map of degree p from V into V_0, as complexes, as in Section 6.6. Let Φ be the homomorphism as in (7.6.2), as modules over k, corresponding to ϕ and the identity mapping on W, as usual. It is easy to see that Φ is a map of degree p as in (7.6.2), as complexes.

Let ψ be another map of degree p from V into V_0, as complexes, and let Ψ be the corresponding map of degree p as in (7.6.2), as complexes, as in the preceding paragraph. Suppose that σ is a homotopy between ϕ and ψ, as maps of degree p between complexes, as in Section 6.6. Let Σ be the homomorphism as in (7.6.2), as modules over k, corresponding to σ and the identity mapping on W, as before. One can check that Σ is a homotopy between Φ and Ψ, as maps of degree p between complexes.

As in the previous section, we may consider W as a complex, with $d_W = 0$. Using this, we may consider $V \bigotimes_A W$, $V_0 \bigotimes_A W$ as double complexes, as before. We may also consider Φ, Ψ as maps of bidegree $(p, 0)$ as in (7.6.2), as double complexes, as in Section 6.8. Under these conditions, $(\Sigma, 0)$ defines a homotopy between Φ and Ψ, as maps of bidegree $(p, 0)$ as in (7.6.2), as double complexes, as in Section 6.8.

Of course, if $p = 0$, then the remarks in the preceding paragraph correspond to some in Section 6.2.

7.7 An analogue of α

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. As before, we let V be a right module over A, W be a left module over A, and $V \bigotimes_A W$ be a tensor product of V and W over A. Suppose that (V, d_V) is a right module over A with differentiation again, so that $V \bigotimes_A W$ is a module over k with differentiation operator $d = d_{V \bigotimes_A W}$ as in (7.5.4).

If $v \in Z(V)$ and $w \in W$, then

(7.7.1) $$d(v \otimes w) = d_V(v) \otimes w = 0,$$

so that $v \otimes w \in Z\big(V \bigotimes_A W\big)$. Thus

(7.7.2) $$(v, w) \mapsto v \otimes w$$

defines a mapping from $Z(V) \times W$ into $Z\big(V \bigotimes_A W\big)$ that is bilinear over k. If $a \in A$, $v \in Z(V)$, and $w \in W$, then the values of this mapping at $(v \cdot a, w)$ and $(v, a \cdot w)$ are the same. Let $Z(V) \bigotimes_A W$ be a tensor product of $Z(V)$ and W over A. This leads to a natural homomorphism

(7.7.3) $$\text{from } Z(V) \bigotimes_A W \text{ into } Z\big(V \bigotimes_A W\big),$$

as modules over k. Equivalently, there is a natural homomorphism

(7.7.4) $$\text{from } Z(V) \bigotimes_A W \text{ into } V \bigotimes_A W,$$

as modules over k, associated to the obvious inclusion mapping from $Z(V)$ into V and the identity mapping on W. One can check that this homomorphism maps $Z(V) \bigotimes_A W$ into $Z\big(V \bigotimes_A W\big)$, because of (7.7.1), so that we get a homomorphism as in (7.7.3).

Similarly, if $v \in B(V)$ and $w \in W$, then $v = d_V(u)$ for some $u \in V$, so that

$$v \otimes w = d_V(u) \otimes w = d(u \otimes w) \tag{7.7.5}$$

is an element of $B\big(V \bigotimes_A W\big)$. If $B(V) \bigotimes_A W$ is a tensor product of $B(V)$ and W over A, then one can use this to get a natural homomorphism

$$\text{from } B(V) \bigotimes_A W \text{ into } B\big(V \bigotimes_A W\big), \tag{7.7.6}$$

as modules over k, as before. Equivalently, there is a natural homomorphism

$$\text{from } B(V) \bigotimes_A W \text{ into } V \bigotimes_A W, \tag{7.7.7}$$

as modules over k, associated to the obvious inclusion mapping from $B(V)$ into V and the identity mapping on W. This homomorphism maps $B(V) \bigotimes_A W$ into $B\big(V \bigotimes_A W\big)$, because of (7.7.5), and thus may be considered as a homomorphism as in (7.7.6).

Of course, there is a natural homomorphism

$$\text{from } B(V) \bigotimes_A W \text{ into } Z(V) \bigotimes_A W, \tag{7.7.8}$$

obtained from the obvious inclusion mapping from $B(V)$ into $Z(V)$ and the identity mapping on W. The composition of this homomorphism with the one as in (7.7.3) is the same as the composition of the homomorphism as in (7.7.6) with the obvious inclusion mapping from $B\big(V \bigotimes_A W\big)$ into $Z\big(V \bigotimes_A W\big)$. Equivalently, the composition of the homomorphism as in (7.7.8) with the one as in (7.7.4) is the same as the homomorphism as in (7.7.7).

Consider the natural quotient mapping

$$\text{from } Z\big(V \bigotimes_A W\big) \text{ onto } H\big(V \bigotimes_A W\big). \tag{7.7.9}$$

The composition of the homomorphism as in (7.7.3) with this quotient mapping leads to a natural homomorphism

$$\eta \text{ from } Z(V) \bigotimes_A W \text{ into } H\big(V \bigotimes_A W\big), \tag{7.7.10}$$

as modules over k.

If $v \in Z(V)$ and $w \in W$, then $v \otimes w \in Z\big(V \bigotimes_A W\big)$, as before, which can be mapped into $H\big(V \bigotimes_A W\big)$ by the natural quotient mapping. It is easy to see that the image in $H\big(V \bigotimes_A W\big)$ depends only on w and the image of v under the natural quotient mapping from $Z(V)$ onto $H(V)$. This defines a mapping from $H(V) \times W$ into $H\big(V \bigotimes_A W\big)$ that is bilinear over k, and has the usual compatibility property for the actions of A on $H(V)$ on the right, and on W on the left. If $H(V) \bigotimes_A W$ is a tensor product of $H(V)$ and W over A, then we get a natural homomorphism

$$\alpha \text{ from } H(V) \bigotimes_A W \text{ into } H\big(V \bigotimes_A W\big), \tag{7.7.11}$$

as modules over k.

Using the natural quotient mapping from $Z(V)$ onto $H(V)$ and the identity mapping on W, we get a homomorphism

(7.7.12) $$\xi \text{ from } Z(V) \bigotimes_A W \text{ onto } H(V) \bigotimes_A W,$$

as modules over k. The surjectivity of ξ follows from a remark in Section 1.9, as usual. It is easy to see that

(7.7.13) $$\alpha \circ \xi = \eta,$$

by construction. This determines α uniquely, because ξ is surjective.

7.8 Analogues of τ, ζ

Let us continue with the same notation and hypotheses as in the previous section.

Let $Z'(V) \bigotimes_A W$ be a tensor product of $Z'(V) = V/d_V(V)$ and W over A. We can use the natural quotient mapping from V onto $Z'(V)$ and the identity mapping on W to get a homomorphism

(7.8.1) $$\text{from } V \bigotimes_A W \text{ onto } Z'(V) \bigotimes_A W,$$

as modules over k. The composition of the homomorphism as in (7.7.4) with this one defines a homomorphism

(7.8.2) $$\text{from } Z(V) \bigotimes_A W \text{ into } Z'(V) \bigotimes_A W,$$

as modules over k. Alternatively, the restriction to $Z(V)$ of the natural quotient mapping from V onto $Z'(V)$ defines a homomorphism

(7.8.3) $$\text{from } Z(V) \text{ into } Z'(V),$$

as modules over k. The homomorphism as in (7.8.2) is the same as the one obtained from the homomorphism as in (7.8.3) and the identity mapping on W in the usual way.

We can use the inclusion mapping from $H(V)$ into $Z'(V)$ and the identity mapping on W to get a natural homomorphism

(7.8.4) $$\tau \text{ from } H(V) \bigotimes_A W \text{ into } Z'(V) \bigotimes_A W,$$

as modules over k. The homomorphism as in (7.8.3) is the same as the composition of the natural quotient mapping from $Z(V)$ onto $H(V)$ with the natural inclusion mapping into $Z'(V)$. This implies that

(7.8.5) $\tau \circ \xi$ is the same as the homomorphism as in (7.8.2).

The restriction of the homomorphism as in (7.8.1) to $Z\big(V \bigotimes_A W\big)$ defines a homomorphism

$$\text{from } Z\big(V \bigotimes_A W\big) \text{ into } Z'(V) \bigotimes_A W, \tag{7.8.6}$$

as modules over k. It is easy to see that this homomorphism is equal to 0 on $B\big(V \bigotimes_A W\big)$. This leads to a homomorphism

$$\zeta \text{ from } H\big(V \bigotimes_A W\big) \text{ into } Z'(V) \bigotimes_A W, \tag{7.8.7}$$

as modules over k.

Observe that

$$\zeta \circ \eta \text{ is the same as the homomorphism as in (7.8.2)}, \tag{7.8.8}$$

by construction. Thus

$$\tau \circ \xi = \zeta \circ \eta, \tag{7.8.9}$$

by (7.8.5).

This implies that

$$\zeta \circ \alpha \circ \xi = \zeta \circ \eta = \tau \circ \xi, \tag{7.8.10}$$

by (7.7.13). It follows that

$$\zeta \circ \alpha = \tau, \tag{7.8.11}$$

because ξ is surjective, as in (7.7.12). This could also be verified more directly from the definitions.

Suppose now that (V, d_V) is a graded right module over A with differentiation that is a complex. Thus $V \bigotimes_A W$ may be considered as a graded module over k, as in (7.5.1), and in fact as a complex with respect to d, as in Section 7.5. Similarly,

$$B(V) \bigotimes_A W, \ Z(V) \bigotimes_A W, \ H(V) \bigotimes_A W, \text{ and } Z'(V) \bigotimes_A W \tag{7.8.12}$$

may be considered as graded modules over k. In this case, it is easy to see that the various homomorphisms mentioned earlier have degree 0. The properties of these homomorphisms correspond to the commutativity of the diagram (1) and parts of Proposition 6.1 on p64 of [3], under slightly different conditions.

We may also consider W as a graded left module over A with differentiation that is a complex, with the grading as in (7.5.2) and $d_W = 0$, as in Section 7.5. This means that $B(W) = \{0\}$, and $Z(W) = H(W) = Z'(W) = W$. Under these conditions, the remarks in this and the previous section correspond to those in Sections 6.11 and 6.12.

7.9 Splitting $Z'(V)$ and α

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a right module over A, let W be a left module over A, and let $V \bigotimes_A W$ be a tensor product of V and W over A, as before. We suppose that (V, d_V) is a right module over A with differentiation again, so that $V \bigotimes_A W$ is a module over k with differentiation operator $d = d_{V \bigotimes_A W}$ as in (7.5.4).

Suppose that

$$\text{(7.9.1)} \qquad \begin{array}{l} Z'(V) \text{ corresponds to the direct sum} \\ \text{of } H(V) \text{ and another submodule of } Z'(V), \end{array}$$

as a right module over A. Equivalently, this means that

$$\text{(7.9.2)} \qquad \begin{array}{l} \text{there is a homomorphism from } Z'(V) \text{ onto } H(V), \text{ as modules} \\ \text{over } A, \text{ that is equal to the identity mapping on } H(V). \end{array}$$

We can compose the natural quotient mapping from V onto $Z'(V) = V/d_V(V)$ with this homomorphism to get a homomorphism

$$\text{(7.9.3)} \qquad \beta \text{ from } V \text{ onto } H(V),$$

as modules over A. Note that

$$\text{(7.9.4)} \qquad \beta \circ d_V = 0$$

on V.

Using β and the identity mapping on W, we get a homomorphism

$$\text{(7.9.5)} \qquad \text{from } V \bigotimes_A W \text{ onto } H(V) \bigotimes_A W,$$

as modules over k. Remember that d is the homomorphism from $V \bigotimes_A W$ into itself associated to d_V and the identity mapping on W. It is easy to see that

$$\text{(7.9.6)} \qquad \begin{array}{l} \text{the composition of } d \text{ with the homomorphism as in (7.9.5)} \\ \text{is equal to 0 on } V \bigotimes_A W. \end{array}$$

This implies that the homomorphism as in (7.9.5) is equal to 0 on $B\big(V \bigotimes_A W\big)$.

In particular, this leads to a homomorphism

$$\text{(7.9.7)} \qquad \text{from } H\big(V \bigotimes_A W\big) \text{ into } H(V) \bigotimes_A W,$$

as modules over k. Namely, the composition of the natural quotient mapping from $Z\big(V \bigotimes_A W\big)$ onto $H\big(V \bigotimes_A W\big)$ with this homomorphism is the same as the restriction of the homomorphism as in (7.9.5) to $Z\big(V \bigotimes_A W\big)$. One can verify that

$$\text{(7.9.8)} \qquad \begin{array}{l} \text{the composition of } \alpha \text{ with the homomorphism as in (7.9.7)} \\ \text{is the identity mapping on } H(V) \bigotimes_A W, \end{array}$$

where α is as in (7.7.11).

It follows that

$$\alpha \text{ is injective on } H(V) \bigotimes_A W. \tag{7.9.9}$$

This also implies that

(7.9.10) $H\big(V \bigotimes_A W\big)$ corresponds to the direct sum of

$\alpha\big(H(V) \bigotimes_A W\big)$ and another submodule of $H\big(V \bigotimes_A W\big)$,

as a module over k.

If (V, d_V) is a complex, then $V \bigotimes_A W$ may be considered as a complex with respect to d, as in Section 7.5. In this case, one can ask that (7.9.1) hold for $Z'(V)$ as a graded right module over A, which is to say that the other submodule of $Z'(V)$ is homogeneous too. This implies that the homomorphisms mentioned in the previous paragraphs have degree 0. It follows that the other submodule of $H\big(V \bigotimes_A W\big)$ mentioned in (7.9.10) is homogeneous as well. This corresponds to Proposition 6.2 on p66 of [3], under slightly different conditions.

We may also consider W as a complex, with grading as in (7.5.2) and $d_W = 0$, as before. This permits one to consider the remarks in this section as a simpler version of those in Section 7.1.

7.10 An injectivity condition and α

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. As usual, we let V be a right module over A, W be a left module over A, and $V \bigotimes_A W$ be a tensor product of V and W over A. Suppose again that (V, d_V) is a right module over A with differentiation, so that $V \bigotimes_A W$ is a module over k with differentiation operator $d = d_{V \bigotimes_A W}$ as in (7.5.4). In this section, we shall consider an additional injectivity condition related to V and W, and its consequences for $V \bigotimes_A W$.

Let $B(V) \bigotimes_A W$ be a tensor product of $B(V)$ and W over A again. As before, there is a natural homomorphism

$$\text{from } B(V) \bigotimes_A W \text{ into } V \bigotimes_A W, \tag{7.10.1}$$

as modules over k, corresponding to the obvious inclusion mapping from $B(V)$ into V and the identity mapping on W. More precisely, this homomorphism maps $B(V) \bigotimes_A W$ into the submodule $B\big(V \bigotimes_A W\big)$ of $V \bigotimes_A W$, as in Section 7.7. In fact, one can check that

(7.10.2) the homomorphism as in (7.10.1)

maps $B(V) \bigotimes_A W$ onto $B\big(V \bigotimes_A W\big)$.

Let us suppose now that

(7.10.3) the homomorphism as in (7.10.1) is injective on $B(V) \bigotimes_A W$.

There is a natural homomorphism

$$\text{from } V \bigotimes_A W \text{ onto } B(V) \bigotimes_A W, \tag{7.10.4}$$

as modules over k, corresponding to d_V as a mapping from V onto $B(V)$, and the identity mapping on W. Observe that

(7.10.5) d is the same as the composition of the homomorphisms as in (7.10.4) and (7.10.1),

as a homomorphism from $V \bigotimes_A W$ into itself, as a module over k. In particular,

(7.10.6) the kernel of the homomorphism as in (7.10.4) is contained in $\ker d$.

Using (7.10.3), we get that

(7.10.7) the kernel of the homomorphism as in (7.10.4) is equal to $\ker d$,

as submodules of $B \bigotimes_A W$.

Let $Z(V) \bigotimes_A W$ be a tensor product of $Z(V)$ and W over A again too. As usual, there is a natural homomorphism

$$\text{from } Z(V) \bigotimes_A W \text{ into } V \bigotimes_A W, \tag{7.10.8}$$

as modules over k, corresponding to the obvious inclusion mapping from $Z(V)$ into V, and the identity mapping on W. This homomorphism maps $Z(V) \bigotimes_A W$ into $Z\big(V \bigotimes_A W\big)$, as in Section 7.7.

Of course,

$$0 \longrightarrow Z(V) \longrightarrow V \xrightarrow{d_V} B(V) \longrightarrow 0 \tag{7.10.9}$$

is an exact sequence of right modules over A, using the obvious inclusion mapping from $Z(V)$ into V. It follows that

$$Z(V) \bigotimes_A W \longrightarrow V \bigotimes_A W \longrightarrow B(V) \bigotimes_A W \longrightarrow 0 \tag{7.10.10}$$

is an exact sequence of modules over k, using the homomorphism as in (7.10.8) in the first step, and the homomorphism as in (7.10.4) in the second step, as in Section 2.5. This means that

(7.10.11) the homomorphism as in (7.10.8) maps $Z(V) \bigotimes_A W$ onto the kernel of the homomorphism as in (7.10.4).

Combining this with (7.10.7), we obtain that

(7.10.12) the homomorphism as in (7.10.8) maps $Z(V)\bigotimes_A W$ onto $Z\big(V\bigotimes_A W\big)$,

under these conditions.

Remember that the homomorphism η from $Z(V) \otimes_A W$ into $H\big(V \otimes_A W\big)$ as in (7.7.10) is defined by taking the composition of the homomorphism as in (7.10.8), considered as a homomorphism from $Z(V) \otimes_A W$ into $Z\big(V \otimes_A W\big)$, with the natural quotient mapping from $Z\big(V \otimes_A W\big)$ onto $H\big(V \otimes_A W\big)$. Using (7.10.12), we get that

(7.10.13) η maps $Z(V)\bigotimes_A W$ onto $H\big(V\bigotimes_A W\big)$.

Let $H(V) \otimes_A W$ be a tensor product of $H(V)$ and W over A again, and remember that α is the homomorphism from $H(V) \otimes_A W$ into $H\big(V \otimes_A W\big)$ as in (7.7.11). As before, we let ξ be the homomorphism from $Z(V) \otimes_A W$ onto $H(A) \otimes_A W$, as modules over k, corresponding to the natural quotient mapping from $Z(V)$ onto $H(V)$, and the identity mapping on W. We have seen that $\alpha \circ \xi = \eta$, as in (7.7.13), which determines α uniquely, because ξ is surjective. This implies that

(7.10.14) α maps $H(V)\bigotimes_A W$ onto $H\big(V\bigotimes_A W\big)$,

because of (7.10.13). This is related to Theorem 7.2 on p68 of [3], and some remarks on p70 of [3].

Suppose that V corresponds to the direct sum of $B(V)$ and another submodule of V, as a right module over A. This implies that $V \otimes_A W$ is isomorphic to the direct sum of $B(V) \otimes_A W$ and a tensor product of the other submodule of V with W over A, as modules over k. In particular, (7.10.3) holds in this case. This is related to Proposition 7.4 on p70 of [3].

7.11 Injectivity of α

Let us return to the same notation and hypotheses as at the beginning of the previous section. As usual, we let $B(V) \otimes_A W$, $Z(V) \otimes_A W$, and $H(V) \otimes_A W$ be tensor products of $B(V)$, $Z(V)$, and $H(V)$ with W over A, respectively. Although we do not ask that (7.10.3) hold for the moment, some of the remarks in the previous section can still be used here.

There is a natural homomorphism

(7.11.1) from $B(V)\bigotimes_A W$ into $Z(V)\bigotimes_A W$,

as modules over k, corresponding to the obvious inclusion mapping from $B(V)$ into $Z(V)$, and the identity mapping on W. Remember that

(7.11.2) the homomorphism as in (7.10.1) is the same as

the composition of the homomorphism as in (7.11.1)
with the homomorphism as in (7.10.8),

as in Section 7.7.

Clearly

$$0 \longrightarrow B(V) \longrightarrow Z(V) \longrightarrow H(V) \longrightarrow 0 \tag{7.11.3}$$

is an exact sequence of right modules over A, using the appropriate inclusion and quotient mappings. This implies that

$$B(V) \bigotimes_A W \longrightarrow Z(V) \bigotimes_A W \xrightarrow{\xi} H(V) \bigotimes_A W \longrightarrow 0 \tag{7.11.4}$$

is an exact sequence of modules over k, as in Section 2.5. This uses the homomorphism as in (7.11.1) in the first step, and ξ is the homomorphism corresponding to the natural quotient mapping from $Z(V)$ onto $H(V)$ and the identity mapping on W, as before.

Remember that the homomorphism as in (7.10.8) maps $Z(V) \otimes_A W$ into $Z\big(V \otimes_A W\big)$. Observe that

(7.11.5) $B\big(V \bigotimes_A W\big)$ is contained in the image of $Z(V) \bigotimes_A W$
under the homomorphism as in (7.10.8),

by (7.10.2) and (7.11.2).

Remember that η is the composition of the homomorphism as in (7.10.8), considered as a homomorphism from $Z(V) \otimes_A W$ into $Z\big(V \otimes_A W\big)$, with the natural quotient mapping from $Z\big(V \otimes_A W\big)$ onto $H\big(V \otimes_A W\big)$. Thus

(7.11.6) $\ker \eta$ is the same as the inverse image of $B\big(V \bigotimes_A W\big)$
under the homomorphism as in (7.10.8).

In fact,

(7.11.7) $\ker \eta$ is the submodule of $Z(V) \bigotimes_A W$ spanned by
the kernel of the homomorphism as in (7.10.8)
and the image of $B(V) \bigotimes_A W$
under the homomorphism as in (7.11.1),

because of (7.10.2) and (7.11.2).

The exactness of (7.11.4) implies that

(7.11.8) $\ker \xi$ is the same as the image of $B(V) \bigotimes_A W$
under the homomorphism as in (7.11.1).

This means that

(7.11.9) $\ker \eta$ is the submodule of $Z(V) \bigotimes_A W$ spanned by
the kernel of the homomorphism as in (7.10.8) and $\ker \xi$.

Remember that α is the homomorphism from $H(V) \bigotimes_A W$ into $H\big(V \bigotimes_A W\big)$ as in (7.7.11). We have seen that $\alpha \circ \xi = \eta$, as in (7.7.13), and that α is uniquely determined by this property, because ξ is surjective. Observe that

(7.11.10) α is injective if and only if $\ker \eta = \ker \xi$.

This happens if and only if

(7.11.11) the kernel of the homomorphism as in (7.10.8) is contained in the image of $B(V) \bigotimes_A W$ under the homomorphism as in (7.11.1).

In particular, α is injective when

(7.11.12) the homomorphism as in (7.10.8) is injective on $Z(V) \bigotimes_A W$.

Note that (7.10.3) implies that

(7.11.13) the homomorphism as in (7.11.1) is injective on $B(V) \bigotimes_A W$.

We also have that (7.10.3) follows from (7.11.12) and (7.11.13).

Let $Z'(V) \bigotimes_A W$ be a tensor product of $Z'(V) = V/d_V(V)$ and W over A, and let τ be the natural homomorphism from $H(V) \bigotimes_A W$ into $Z'(V) \bigotimes_A W$, as modules over k, corresponding to the obvious inclusion mapping from $H(V)$ into $Z'(V)$ and the identity mapping on W, as in Section 7.8. Remember that τ is equal to the composition of α with the homomorphism ζ defined in Section 7.8, as in (7.8.11). This implies that

(7.11.14) α is injective when τ is injective.

These criteria for the injectivity of α are related to Theorem 7.2 on p68 of [3], and some remarks on p70 of [3].

If V corresponds to the direct sum of $Z(V)$ and another submodule, as a right module over A, then $V \bigotimes_A W$ is isomorphic to the direct sum of $Z(V) \bigotimes_A W$ and a tensor product of the other submodule of V with W over A, as modules over k. This implies that the homomorphism as in (7.10.8) is injective on $Z(V) \bigotimes_A W$, so that α is injective, as before. Similarly, if $Z'(V)$ corresponds to the direct sum of $H(V)$ and another submodule of $Z'(V)$, as a right module over A, then α is injective, as in Section 7.9, which could also be obtained from the injectivity of τ in this case. This is related to Proposition 7.4 on p70 of [3].

7.12 Differentiation and $\mathrm{Hom}(V, \cdot)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, W be both left or both right modules over A.

Suppose for the moment that (W, d_W) is a module with differentiation. Let

$$d = d_{\mathrm{Hom}_A(V,W)} \tag{7.12.1}$$

be the homomorphism from $\mathrm{Hom}_A(V, W)$ into itself, as a module over k, defined by composition with d_W. Thus, if $\phi \in \mathrm{Hom}_A(V, W)$, then we put

$$d(\phi) = d_W \circ \phi. \tag{7.12.2}$$

Observe that

$$d \circ d = 0 \quad \text{on } \mathrm{Hom}_A(V, W), \tag{7.12.3}$$

so that $\mathrm{Hom}_A(V, W)$ is a module with differentiation with respect to d.

Suppose now that W is a graded module over A. We would like to define $\mathrm{Hom}^{gr}_A(V, W)$ as a graded module over k, with

$$\big(\mathrm{Hom}^{gr}_A(V, W)\big)^l = \mathrm{Hom}_A(V, W^l) \tag{7.12.4}$$

for every $l \in \mathbf{Z}$, as on p62 of [3], with $r = 1$. Although we can simply take $\mathrm{Hom}^{gr}_A(V, W)$ to be the direct sum of (7.12.4) over l, it is also sometimes convenient to consider it as a submodule of related modules over k.

Let $|W|$ be the underlying module over A corresponding to W, without a grading, as on p63 of [3]. If $\phi \in \mathrm{Hom}^{gr}_A(V, W)$ and $l \in \mathbf{Z}$, then let ϕ^l be the corresponding element of (7.12.4), as usual, so that $\phi^l = 0$ for all but finitely many l. It follows that

$$\sum_{l=-\infty}^{\infty} \phi^l \tag{7.12.5}$$

defines a homomorphism from V into $|W|$, as modules over A. It is easy to see that ϕ is uniquely determined by (7.12.5), and so we shall often identify ϕ with (7.12.5).

This defines an injective homomorphism

$$\text{from } \mathrm{Hom}^{gr}_A(V, W) \text{ into } \mathrm{Hom}_A(V, |W|), \tag{7.12.6}$$

as modules over k, and we may identify $\mathrm{Hom}^{gr}_A(V, W)$ with the corresponding submodule of $\mathrm{Hom}_A(V, |W|)$, as a module over k. Note that $\mathrm{Hom}^{gr}_A(V, W)$ may correspond to a proper submodule of $\mathrm{Hom}_A(V, |W|)$, as on p63 of [3]. If V is finitely generated as a module over A, then one can check that the homomorphism as in (7.12.6) maps $\mathrm{Hom}^{gr}_A(V, W)$ onto $\mathrm{Hom}_A(V, |W|)$.

Of course, $|W|$ may be considered as a submodule of $\prod_{l=-\infty}^{\infty} W_l$, as a module over A, so that $\mathrm{Hom}_A(V, |W|)$ corresponds to a submodule of

$$\mathrm{Hom}_A\Big(V, \prod_{l=-\infty}^{\infty} W_l\Big), \tag{7.12.7}$$

as a module over k. Remember that there is a natural isomorphism from (7.12.7) onto

$$\prod_{l=-\infty}^{\infty} \mathrm{Hom}_A(V, W_l), \tag{7.12.8}$$

as modules over k, as in Section 1.7. Thus $\mathrm{Hom}_A^{gr}(V, W)$ corresponds to the submodule of $\mathrm{Hom}_A(V, |W|)$, as a module over k, that correesponds to the submodule of (7.12.8) given by the direct sum.

As in Section 7.5, V may be considered as a graded module over A, with

$$\begin{aligned} V^j &= V \quad \text{when } j = 0 \\ &= \{0\} \quad \text{when } j \neq 0. \end{aligned} \tag{7.12.9}$$

Using this, $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a doubly-graded module over k, as in Section 6.3, with

$$\begin{aligned} \big(\mathrm{Hom}_A^{gr}(V, W)\big)^{j,l} = \mathrm{Hom}_A(V^{-j}, W^l) &= \mathrm{Hom}_A(V, W^l) \quad \text{when } j = 0 \\ &= \{0\} \quad \text{when } j \neq 0. \end{aligned} \tag{7.12.10}$$

The single grading on $\mathrm{Hom}_A^{gr}(V, W)$ associated to this double grading is the same as in (7.12.4).

Suppose that (W, d_W) is a graded module over A that is a complex. In particular, $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a graded module over k as in (7.12.4). We can define

$$d = d_{\mathrm{Hom}_A^{gr}(V,W)} \tag{7.12.11}$$

on $\mathrm{Hom}_A^{gr}(V, W)$ as follows. If $\phi \in \mathrm{Hom}_A^{gr}(V, W)$, then $d(\phi)$ may be defined as an element of $\mathrm{Hom}_A^{gr}(V, W)$ by putting

$$d(\phi)^{l+1} = d_W^l \circ \phi^l \tag{7.12.12}$$

for each l. It is easy to see that $\mathrm{Hom}_A^{gr}(V, W)$ is a complex with respect to d, as on p63 of [3].

The underlying module $|W|$ over A corresponding to W may be considered as a module with differentiation with respect to d_W, so that $d = d_{\mathrm{Hom}_A(V,|W|)}$ may be defined on $\mathrm{Hom}_A(V, |W|)$ as in (7.12.1). If we consider $\mathrm{Hom}_A^{gr}(V, W)$ as a submodule of $\mathrm{Hom}_A(V, |W|)$, as a module over k, then the restriction of d on $\mathrm{Hom}_A(V, |W|)$ to $\mathrm{Hom}_A^{gr}(V, W)$ is the same as the definition of d on $\mathrm{Hom}_A^{gr}(V, W)$ in the preceding paragraph. Thus $\mathrm{Hom}_A^{gr}(V, W)$ may be considered as a submodule of $\mathrm{Hom}_A(V, |W|)$, as a module over k with differentiation.

We may consider V as a complex, using the grading in (7.12.9), and differentiation operator $d_V = 0$, as on p75 of [3]. If we consider $\mathrm{Hom}_A^{gr}(V, W)$ as a doubly-graded module over k as in (7.12.10), then $\mathrm{Hom}_A^{gr}(V, W)$ is a double complex, with

$$\delta_1 = 0, \quad \delta_2 = d = d_{\mathrm{Hom}_A^{gr}(V,W)}, \tag{7.12.13}$$

as in Section 6.5. This implies that $\mathrm{Hom}_A^{gr}(V, W)$ is a single complex with respect to the total differentiation operator

$$\delta = \delta_1 + \delta_2 = d = d_{\mathrm{Hom}_A^{gr}(V,W)}, \tag{7.12.14}$$

and the single grading associated to the double grading on $\mathrm{Hom}_A^{gr}(V, W)$, as in Section 5.14. This is the same as considering $\mathrm{Hom}_A^{gr}(V, W)$ as a single complex as before.

7.13 Induced mappings and $\mathrm{Hom}(V, \cdot)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, W, and W_0 be all left or all right modules over A.

Suppose that (W, d_W) and (W_0, d_{W_0}) are modules with differentiation. Let $d = d_{\mathrm{Hom}_A(V,W)}$ be defined on $\mathrm{Hom}_A(V, W)$ as in (7.12.1), and define

$$d_0 = d_{\mathrm{Hom}_A(V,W_0)} \tag{7.13.1}$$

on $\mathrm{Hom}_A(V, W_0)$ in the same way.

Let ϕ be a homomorphism from W into W_0, as modules over A. Consider the homomorphism Φ

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V, W_0), \tag{7.13.2}$$

as modules over k, defined by composing an element of $\mathrm{Hom}_A(V, W)$ with ϕ to get an element of $\mathrm{Hom}_A(V, W_0)$. Suppose now that ϕ is a homomorphism from W into W_0, as modules over A with differentiation. It is easy to see that Φ is a homomorphism as in (7.13.2), as modules over k with differentiation, in this case.

Let ψ be another homomorphism from W into W_0, as modules over A with differentiation, and let Ψ be the corresponding homomorphism as in (7.13.2), as modules over k with differentiation. Suppose that σ is a homotopy between ϕ and ψ, which means that σ is a homomorphism from W into W_0, as modules over A, such that

$$d_{W_0} \circ \sigma + \sigma \circ d_W = \phi - \psi. \tag{7.13.3}$$

Let Σ be the homomorphism as in (7.13.2), as modules over k, that corresponds to σ as before. It is easy to see that

$$d_0 \circ \Sigma + \Sigma \circ d = \Phi - \Psi, \tag{7.13.4}$$

so that Σ is a homotopy between Φ and Ψ.

Suppose now that W, W_0 are graded modules over A, so that $\mathrm{Hom}_A^{gr}(V, W)$, $\mathrm{Hom}_A^{gr}(V, W_0)$ may be defined as graded modules over k as in (7.12.4). Let ϕ be a homomorphism from W into W_0, as modules over A, of degree $p \in \mathbf{Z}$. This leads to a homomorphism Φ

$$\text{from } \mathrm{Hom}_A^{gr}(V, W) \text{ into } \mathrm{Hom}_A^{gr}(V, W_0), \tag{7.13.5}$$

as modules over k, of degree p too. More precisely, if $l \in \mathbf{Z}$, then Φ acts on (7.12.4) by composition with ϕ, as on p63 of [3].

Let $|W|$, $|W_0|$ be the underlying modules over A corresponding to W, W_0, respectively, without gradings, as on p63 of [3]. Remember that $\mathrm{Hom}_A^{gr}(V, W)$ and $\mathrm{Hom}_A^{gr}(V, W_0)$ may be considered as submodules of $\mathrm{Hom}_A(V, |W|)$ and $\mathrm{Hom}_A(V, |W_0|)$, respectively, as modules over k, as in the previous section. Using this, we have that the homomorphism Φ as in (7.13.5) is the same as the restriction of the homomorphism Φ

$$\text{from } \mathrm{Hom}_A(V, |W|) \text{ into } \mathrm{Hom}_A(V, |W_0|) \tag{7.13.6}$$

as in (7.13.2) to $\mathrm{Hom}_A^{gr}(V, W)$.

If V is considered as a graded module over A as in (7.12.9), then we can define $\mathrm{Hom}_A^{gr}(V, W)$ and $\mathrm{Hom}_A^{gr}(V, W_0)$ as doubly-graded modules over k, as in Section 6.3. In this case,

$$\text{the homomorphism } \Phi \text{ as in (7.13.5) corresponds to } \mathrm{Hom}^{gr}(I_V, \phi), \tag{7.13.7}$$

in the notation of Section 6.4, where I_V is the identity mapping on V.

Suppose that (W, d_W), (W_0, d_{W_0}) are graded modules over A with differentiation that are complexes, so that $\mathrm{Hom}_A^{gr}(V, W)$, $\mathrm{Hom}_A^{gr}(V, W_0)$ may be considered as single complexes, as in the previous section. Let ϕ be a map of degree $p \in \mathbf{Z}$ from W into W_0, as complexes, as in Section 6.6. If Φ is the corresponding homomorphism as in (7.13.5), then Φ is a map of degree p as in (7.13.5), as complexes.

Let ψ be another map of degree p from W into W_0, as complexes, and let Ψ be the corresponding map of degree p as in (7.13.5), as complexes. Also let σ be a homotopy between ϕ and ψ, as maps of degree p between complexes, as in Section 6.6. In particular, σ is a homomorphism from W into W_0 of degree $p - 1$, as modules over A. Let Σ be the corresponding homomorphism as in (7.13.5), as modules over k, which has degree $p - 1$ as well. One can verify that Σ is a homotopy between ϕ and ψ, as maps of degree p between complexes.

Let us consider V as a complex, with $d_V = 0$, and graded as in (7.12.9). Thus we may consider $\mathrm{Hom}_A^{gr}(V, W)$ and $\mathrm{Hom}_A^{gr}(V, W_0)$ as double complexes, as in the previous section. In this case, Φ and Ψ may be considered as maps of bidegree $(0, p)$ as in (7.13.5), as double complexes, as in Section 6.9. We also get that $(0, \Sigma)$ defines a homotopy between Φ and Ψ, as maps of bidegree $(0, p)$ as in (7.13.5), as double complexes, as in Section 6.9. If $p = 0$, then these remarks correspond to some of those in Section 6.5.

7.14 An analogue of α'

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V and W be both left or both right modules over A, and suppose that (W, d_W) is a module over A with differentiation. Thus $\mathrm{Hom}_A(V, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,W)}$ as in (7.12.1).

By definition, $Z\big(\mathrm{Hom}_A(V, W)\big)$ consists of $\phi \in \mathrm{Hom}_A(V, W)$ such that

$$d(\phi) = d_W \circ \phi = 0 \tag{7.14.1}$$

on V. This is the same as saying that

$$\phi(V) \subseteq Z(W), \tag{7.14.2}$$

so that

$$Z\big(\mathrm{Hom}_A(V, W)\big) = \mathrm{Hom}_A(V, Z(W)), \tag{7.14.3}$$

where the latter is considered as a submodule of $\mathrm{Hom}_A(V, W)$.

Similarly, $\phi \in B\big(\mathrm{Hom}_A(V, W)\big)$ if and only if

$$\phi = d(\psi) = d_W \circ \psi \tag{7.14.4}$$

for some $\psi \in \mathrm{Hom}_A(V, W)$. In this case,

$$\phi(V) = d_W(\psi(V)) \subseteq d_W(W) = B(W). \tag{7.14.5}$$

This means that

$$B\big(\mathrm{Hom}_A(V, W)\big) \subseteq \mathrm{Hom}_A(V, B(W)), \tag{7.14.6}$$

where the latter is considered as a submodule of $\mathrm{Hom}_A(V, W)$.

There is a natural homomorphism

$$\xi \text{ from } \mathrm{Hom}_A(V, Z(W)) \text{ into } \mathrm{Hom}_A(V, H(W)), \tag{7.14.7}$$

as modules over k, defined by sending an element of $\mathrm{Hom}_A(V, Z(W))$ to its composition with the natural quotient mapping from $Z(W)$ onto $H(W)$. Equivalently, ξ may be considered as a homomorphism

$$\text{from } Z\big(\mathrm{Hom}_A(V, W)\big) \text{ into } \mathrm{Hom}_A(V, H(W)), \tag{7.14.8}$$

by (7.14.3). Note that

$$\ker \xi = \mathrm{Hom}_A(V, B(W)), \tag{7.14.9}$$

considered as a submodule of (7.14.3). This leads to a natural homomorphism

$$\alpha' \text{ from } H\big(\mathrm{Hom}_A(V, W)\big) \text{ into } \mathrm{Hom}_A(V, H(W)), \tag{7.14.10}$$

because of (7.14.6).

The natural quotient mapping from $Z\big(\mathrm{Hom}_A(V, W)\big)$ onto $H\big(\mathrm{Hom}_A(V, W)\big)$ may be considered as a homomorphism

$$\eta \text{ from } \mathrm{Hom}_A(V, Z(W)) \text{ onto } H\big(\mathrm{Hom}_A(V, W)\big), \tag{7.14.11}$$

as modules over k, by (7.14.3). By construction,

$$\alpha' \circ \eta = \xi. \tag{7.14.12}$$

This determines α' uniquely, because η is surjective.

There is a natural homomorphism

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V, Z'(W)), \tag{7.14.13}$$

as modules over k, which sends an element of $\mathrm{Hom}_A(V, W)$ to its composition with the natural quotient mapping from W onto $Z'(W) = W/d_W(W)$. This leads to a natural homomorphism

$$\text{from } \mathrm{Hom}_A(V, Z(W)) \text{ into } \mathrm{Hom}_A(V, Z'(W)), \tag{7.14.14}$$

as modules over k, which is defined by restricting the homomorphism as in (7.14.13) to $\mathrm{Hom}_A(V, Z(W))$, considered as a submodule of $\mathrm{Hom}_A(V, W)$. This homomorphism could also be described by saying that it sends an element of $\mathrm{Hom}_A(V, Z(W))$ to its composition with the restriction to $Z(W)$ of the natural quotient mapping from W onto $Z'(W)$.

Of course, $H(W)$ may be considered as a submodule of $Z'(W)$, as a module over A, so that $\mathrm{Hom}_A(V, H(W))$ may be considered as a submodule of $\mathrm{Hom}_A(V, Z'(W))$, as a module over k. The inclusion mapping

(7.14.15) τ from $\mathrm{Hom}_A(V, H(W))$ into $\mathrm{Hom}_A(V, Z'(W))$

is an injective module homomorphism, and we get that

(7.14.16) $\tau \circ \xi$ is the same as the homomorphism as in (7.14.14).

Note that

(7.14.17) the kernel of the homomorphism as in (7.14.14) is equal to $\mathrm{Hom}_A(V, B(W))$,

considered as a submodule of $\mathrm{Hom}_A(V, Z(W))$, as a module over k. This can be seen directly, or using (7.14.9) and (7.14.16). This leads to a natural homomorphism

(7.14.18) ζ from $H\big(\mathrm{Hom}_A(V, W)\big)$ into $\mathrm{Hom}_A(V, Z'(W))$,

because of (7.14.3) and (7.14.6).

It is easy to see that

(7.14.19) $\zeta \circ \eta$ is the same as the homomorphism as in (7.14.14),

by construction. This means that

$$\tau \circ \xi = \zeta \circ \eta, \tag{7.14.20}$$

by (7.14.16).

One can verify that

$$\tau \circ \alpha' = \zeta, \tag{7.14.21}$$

as homomorphisms from $H\big(\mathrm{Hom}_A(V, W)\big)$ into $\mathrm{Hom}_A(V, Z'(W))$. Of course, α' is uniquely determined by this property, because τ is injective. Each of (7.14.12) and (7.14.21) can be obtained from the other using (7.14.20), by composing with η or τ, as appropriate.

7.15 Complexes and α'

Let us continue with the same notation and hypotheses as in the previous section, and suppose now that (W, d_W) is a graded module over A that is a complex. Thus $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a graded module over k as in (7.12.4), and as a complex, with differentiation operator $d = d_{\mathrm{Hom}_A^{gr}(V,W)}$ as in (7.12.11).

The underlying module $|W|$ over A corresponding to W may be considered as a module with differentiation with respect to d_W, so that $\mathrm{Hom}_A(V, |W|)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,|W|)}$ defined as in (7.12.1). Remember that $\mathrm{Hom}^{gr}_A(V, W)$ may be considered as a submodule of $\mathrm{Hom}_A(V, |W|)$, as a module over k with differentiation, as in Section 7.12.

Note that

$$\text{(7.15.1)} \qquad \mathrm{Hom}^{gr}_A(V, B(W)),\ \mathrm{Hom}^{gr}_A(V, Z(W)),\ \mathrm{Hom}^{gr}_A(V, H(W)), \text{ and } \mathrm{Hom}^{gr}_A(V, Z'(W))$$

may be defined as graded modules over k as in (7.12.4) too. More precisely,

$$\text{(7.15.2)} \qquad \mathrm{Hom}^{gr}_A(V, B(W)) \text{ and } \mathrm{Hom}^{gr}_A(V, Z(W)) \text{ are homogeneous submodules of } \mathrm{Hom}^{gr}_A(V, W),$$

and

$$\text{(7.15.3)} \qquad \mathrm{Hom}^{gr}_A(V, H(W)) \text{ is a homogeneous submodule of } \mathrm{Hom}^{gr}_A(V, Z'(W)).$$

The remarks in the previous section can be applied to $|W|$, and we would like to consider the analogous statements for $\mathrm{Hom}^{gr}_A(V, \cdot)$ in place of $\mathrm{Hom}_A(V, \cdot)$. We have that

$$\text{(7.15.4)} \qquad Z\big(\mathrm{Hom}_A(V, |W|)\big) = \mathrm{Hom}_A(V, Z(|W|)),$$

as in (7.14.3), for instance, and similarly

$$\text{(7.15.5)} \qquad Z\big(\mathrm{Hom}^{gr}_A(V, W)\big) = \mathrm{Hom}^{gr}_A(V, Z(W)).$$

We also have that

$$\text{(7.15.6)} \qquad B\big(\mathrm{Hom}_A(V, |W|)\big) \subseteq \mathrm{Hom}_A(V, B(|W|)),$$

as in (7.14.6), and

$$\text{(7.15.7)} \qquad B\big(\mathrm{Hom}^{gr}_A(V, W)\big) \subseteq \mathrm{Hom}^{gr}_A(V, B(W)).$$

One can check that

$$\text{(7.15.8)} \qquad B\big(\mathrm{Hom}^{gr}_A(V, W)\big) = \mathrm{Hom}^{gr}_A(V, W) \cap B\big(\mathrm{Hom}_A(V, |W|)\big).$$

Using this,

$$\text{(7.15.9)} \qquad \text{we may consider } H\big(\mathrm{Hom}^{gr}_A(V, W)\big) \text{ as a submodule of } H\big(\mathrm{Hom}_A(V, |W|)\big),$$

as a module over k. Similarly,

$$\text{(7.15.10)} \qquad \text{we may consider } Z'\big(\mathrm{Hom}^{gr}_A(V, W)\big) \text{ as a submodule of } Z'\big(\mathrm{Hom}_A(V, |W|)\big),$$

as a module over k.

There is a natural homomorphism

(7.15.11) $$\xi \text{ from } \mathrm{Hom}_A(V, Z(|W|)) \text{ into } \mathrm{Hom}_A(V, H(|W|)),$$

as modules over k, as in (7.14.7). This may be considered as a homomorphism

(7.15.12) $$\text{from } Z\big(\mathrm{Hom}_A(V, |W|)\big) \text{ into } \mathrm{Hom}_A(V, H(|W|)),$$

by (7.15.4), as before. It is easy to see that

(7.15.13) $$\xi \text{ maps } \mathrm{Hom}_A^{gr}(V, Z(W)) \text{ into } \mathrm{Hom}_A^{gr}(V, H(W)),$$

and that

(7.15.14) $$\text{the restriction of } \xi \text{ to } \mathrm{Hom}_A^{gr}(V, Z(W)) \text{ has degree } 0.$$

Remember that

(7.15.15) $$\ker \xi = \mathrm{Hom}_A(V, B(|W|)),$$

as in (7.14.9). This implies that

(7.15.16) the kernel of the restriction of ξ to $\mathrm{Hom}_A^{gr}(V, Z(W))$ is $\mathrm{Hom}_A^{gr}(V, B(W))$.

Using (7.15.15), we get a natural homomorphism

(7.15.17) $$\alpha' \text{ from } H\big(\mathrm{Hom}_A(V, |W|)\big) \text{ into } \mathrm{Hom}_A(V, H(|W|)),$$

as in (7.14.10). One can check that

(7.15.18) $$\alpha' \text{ maps } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ into } \mathrm{Hom}_A^{gr}(V, H(W)),$$

and that

(7.15.19) $$\text{the restriction of } \alpha' \text{ to } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ has degree } 0.$$

Because of (7.15.4), the natural quotient mapping from $Z\big(\mathrm{Hom}_A(V, |W|)\big)$ onto $H\big(\mathrm{Hom}_A(V, |W|)\big)$ may be considered as a homomorphism

(7.15.20) $$\eta \text{ from } \mathrm{Hom}_A(V, Z(|W|)) \text{ onto } H\big(\mathrm{Hom}_A(V, |W|)\big),$$

as modules over k, as in (7.14.11). One can verify that

(7.15.21) $$\eta \text{ maps } \mathrm{Hom}_A^{gr}(V, Z(W)) \text{ onto } H\big(\mathrm{Hom}_A^{gr}(V, W)\big),$$

and that

(7.15.22) $$\text{the restriction of } \eta \text{ to } \mathrm{Hom}_A^{gr}(V, Z(W)) \text{ has degree } 0.$$

More precisely, the restriction of η to $\mathrm{Hom}_A^{gr}(V, Z(W))$ corresponds to the natural quotient mapping from $Z\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ onto $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$, because of (7.15.5) and (7.15.8). We also have that $\alpha' \circ \eta = \xi$, as in (7.14.12). In particular, this holds on $\mathrm{Hom}_A^{gr}(V, Z(W))$.

There is a natural homomorphism

(7.15.23) from $\mathrm{Hom}_A(V, |W|)$ into $\mathrm{Hom}_A(V, Z'(|W|))$,

as modules over k, as in (7.14.13). This leads to a natural homomorphism

(7.15.24) from $\mathrm{Hom}_A(V, Z(|W|))$ into $\mathrm{Hom}_A(V, Z'(|W|))$,

as modules over k, as in (7.14.14), by restricting the homomorphism as in (7.15.23) to $\mathrm{Hom}_A(V, Z(|W|))$. It is easy to see that

(7.15.25) the homomorphism as in (7.15.23) maps $\mathrm{Hom}_A^{gr}(V, W)$ into $\mathrm{Hom}_A^{gr}(V, Z'(W))$,

and that

(7.15.26) the restriction of the homomorphism as in (7.15.23) to $\mathrm{Hom}_A^{gr}(V, W)$ has degree 0.

Thus the restriction of the homomorphism as in (7.15.24) to $\mathrm{Hom}_A^{gr}(V, Z(W))$ defines a natural homomorphism

(7.15.27) from $\mathrm{Hom}_A^{gr}(V, Z(W))$ into $\mathrm{Hom}_A^{gr}(V, Z'(W))$

of degree 0.

The inclusion mapping

(7.15.28) τ from $\mathrm{Hom}_A(V, H(|W|))$ into $\mathrm{Hom}_A(V, Z'(|W|))$

is an injective module homomorphism, as in (7.14.15). Of course, τ maps $\mathrm{Hom}_A^{gr}(V, H(W))$ into $\mathrm{Hom}_A^{gr}(V, Z'(W))$. Remember that $\tau \circ \xi$ is the same as the homomorphism as in (7.15.24), as in (7.14.16).

Remember that the kernel of the homomorphism as in (7.15.24) is equal to $\mathrm{Hom}_A(V, B(|W|))$, as in (7.14.17). This leads to a natural homomorphism

(7.15.29) ζ from $H\big(\mathrm{Hom}_A(V, |W|)\big)$ into $\mathrm{Hom}_A(V, Z'(|W|))$,

as in (7.14.18). It is easy to see that

(7.15.30) $$\zeta\big(H\big(\mathrm{Hom}_A^{gr}(V, W)\big)\big) \subseteq \mathrm{Hom}_A^{gr}(V, Z'(W))$$

and

(7.15.31) the restriction of ζ to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ has degree 0,

because of the analogous properties of the homomorphism as in (7.15.27).

As in (7.14.19), $\zeta \circ \eta$ is the same as the homomorphism as in (7.15.24). This means that $\tau \circ \xi = \zeta \circ \eta$, as in (7.14.20). In particular, this holds on $\mathrm{Hom}_A^{gr}(V, Z(W))$.

Similarly, $\tau \circ \alpha' = \zeta$ on $H\big(\mathrm{Hom}_A(V, |W|)\big)$, as in (7.14.21). In particular, this holds on $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$.

These homomorphisms and their properties correspond to the commutative diagram (1) on p64 of [3], and parts of Proposition 6.1a on p65 of [3], under slightly different conditions.

Remember that V may be considered as a graded module over A with differentiation that is a complex, with grading as in (7.12.9) and $d_V = 0$, as in Section 7.12. Of course, $B(V) = \{0\}$, and $Z(V) = H(V) = Z'(V) = V$ in this case. Using this, the remarks in this and the previous section correspond to those in Sections 6.13 – 6.15.

Chapter 8

More on differentiation, 3

8.1 Splitting $Z(W)$ and α'

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V and W be both left or both right modules over A, and suppose that (W, d_W) is a module over A with differentiation. Remember that $\mathrm{Hom}_A(V, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,W)}$ as in (7.12.1).

Suppose that

(8.1.1) $Z(W)$ corresponds to the direct sum of $B(W)$ and another submodule of $Z(W)$,

as a module over A. This means that

(8.1.2) there is a homomorphism from $H(W)$ into $Z(W)$, as modules over A, whose composition with the natural quotient mapping from $Z(W)$ onto $H(W)$ is the identity mapping on $H(W)$.

Indeed, $Z(W)$ corresponds to the direct sum of $B(W)$ and the image of $H(W)$ under this homomorphism in this case, as a module over A.

Let $\widetilde{\gamma}$ be the homomorphism as in (8.1.2), considered as a homomorphism

(8.1.3) from $H(W)$ into W.

Equivalently, $\widetilde{\gamma}$ may be considered as the composition of the homomorphism in (8.1.2) with the natural inclusion mapping from $Z(W)$ into W. Note that

(8.1.4) $$d_W \circ \widetilde{\gamma} = 0,$$

because $\widetilde{\gamma}(H(W)) \subseteq Z(W)$.

Using $\widetilde{\gamma}$, we get a homomorphism

(8.1.5) $$\text{from } \mathrm{Hom}_A(V, H(W)) \text{ into } \mathrm{Hom}_A(V, W),$$

as modules over k. This homomorphism sends an element of $\mathrm{Hom}_A(V, H(W))$ to its composition with $\widetilde{\gamma}$. This may be considered as a homomorphism

(8.1.6) $$\text{from } \mathrm{Hom}_A(V, H(W)) \text{ into } Z\big(\mathrm{Hom}_A(V, W)\big),$$

or equivalently into $\mathrm{Hom}_A(V, Z(W))$.

This leads to a homomorphism

(8.1.7) $$\text{from } \mathrm{Hom}_A(V, H(W)) \text{ into } H\big(\mathrm{Hom}_A(V, W)\big),$$

as modules over k, by composing the previous homomorphism as in (8.1.6) with the natural quotient mapping

(8.1.8) $$\text{from } Z\big(\mathrm{Hom}_A(V, W)\big) \text{ onto } H\big(\mathrm{Hom}_A(V, W)\big).$$

Let α' be the homomorphism from $H\big(\mathrm{Hom}_A(V, W)\big)$ into $\mathrm{Hom}_A(V, H(W))$ defined in Section 7.14. One can verify that

(8.1.9) the composition of the homomorphism as in (8.1.7) with α' is the identity mapping on $\mathrm{Hom}_A(V, H(W))$.

This implies that

(8.1.10) $$\alpha'\big(H\big(\mathrm{Hom}_A(V, W)\big)\big) = \mathrm{Hom}_A(V, H(W)).$$

We also obtain that

(8.1.11) $H\big(\mathrm{Hom}_A(V, W)\big)$ corresponds to the direct sum of $\ker \alpha'$ and another submodule of $H\big(\mathrm{Hom}_A(V, W)\big)$,

as a module over k.

Suppose now that (W, d_W) is a graded module over A with differentiation that is a complex, and that

(8.1.12) $Z(W)$ corresponds to the direct sum of $B(W)$ and another homogeneous submodule of $Z(W)$,

as a module over A. This means that $Z(W)$ corresponds to the direct sum of $B(W)$ and another submodule, as a graded module over A. It follows that

(8.1.13) the homomorphism as in (8.1.2) has degree 0,

so that

(8.1.14) the homomorphism $\widetilde{\gamma}$ as in (8.1.3) has degree 0.

Remember that $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a graded module over k as in (7.12.4), and as a complex, with differentiation operator $d = d_{\mathrm{Hom}_A^{gr}(V,W)}$ as in (7.12.11). We may also consider the underlying module $|W|$ over A corresponding to W, without a grading, as a module with differentiation with respect to d_W, so that $\mathrm{Hom}_A(V, |W|)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,|W|)}$ defined as in (7.12.1). As in Section 7.12, we may consider $\mathrm{Hom}_A^{gr}(V, W)$ as a submodule of $\mathrm{Hom}_A(V, |W|)$, as a module over k with differentiation.

We can use $\widetilde{\gamma}$ to get a homomorphism

(8.1.15) $$\text{from } \mathrm{Hom}_A(V, H(|W|)) \text{ into } \mathrm{Hom}_A(V, |W|),$$

as modules over k, as in (8.1.5). This may be considered as a homomorphism

(8.1.16) $$\text{from } \mathrm{Hom}_A(V, H(|W|)) \text{ into } Z\big(\mathrm{Hom}_A(V, |W|)\big),$$

as in (8.1.6), or equivalently into $\mathrm{Hom}_A(V, Z(|W|))$. It is easy to see that

(8.1.17) $$\begin{gathered}\text{the homomorphism as in (8.1.15)}\\ \text{maps } \mathrm{Hom}_A^{gr}(V, H(W)) \text{ into } \mathrm{Hom}_A^{gr}(V, W).\end{gathered}$$

The restriction of this homomorphism to $\mathrm{Hom}_A^{gr}(V, H(W))$ may be considered as a homomorphism

(8.1.18) $$\text{from } \mathrm{Hom}_A^{gr}(V, H(W)) \text{ into } Z\big(\mathrm{Hom}_A^{gr}(V, W)\big),$$

or equivalently into $\mathrm{Hom}_A^{gr}(V, Z(W))$, as before. Note that

(8.1.19) $$\text{the homomorphism as in (8.1.18) has degree 0,}$$

because of (8.1.14).

As before, we get a homomorphism

(8.1.20) $$\text{from } \mathrm{Hom}_A(V, H(|W|)) \text{ into } H\big(\mathrm{Hom}_A(V, |W|)\big),$$

as modules over k, by composing the homomorphism as in (8.1.16) with the natural quotient mapping

(8.1.21) $$\text{from } Z\big(\mathrm{Hom}_A(V, |W|)\big) \text{ onto } H\big(\mathrm{Hom}_A(V, |W|)\big).$$

One can check that the homomorphism as in (8.1.20) maps $\mathrm{Hom}_A^{gr}(V, H(W))$ into $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$. More precisely, the restriction of this homomorphism to $\mathrm{Hom}_A^{gr}(V, H(W))$ is the same as the homomorphism

(8.1.22) $$\text{from } \mathrm{Hom}_A^{gr}(V, H(W)) \text{ into } H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$$

obtained by composing the homomorphism as in (8.1.18) with the natural quotient mapping

(8.1.23) $$\text{from } Z\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ onto } H\big(\mathrm{Hom}_A^{gr}(V, W)\big).$$

It is easy to see that

(8.1.24) the homomorphism as in (8.1.22) has degree 0,

because of (8.1.19).

Let α' be the homomorphism

$$\text{from } H\big(\mathrm{Hom}_A(V, |W|)\big) \text{ into } \mathrm{Hom}_A(V, H(|W|)) \tag{8.1.25}$$

discussed in Sections 7.14 and 7.15. As in (8.1.9),

(8.1.26) the composition of the homomorphism as in (8.1.20) with α' is the identity mapping on $\mathrm{Hom}_A(V, H(|W|))$.

It follows that

$$\alpha'\Big(H\big(\mathrm{Hom}_A(V, |W|)\big)\Big) = \mathrm{Hom}_A(V, H(|W|)) \tag{8.1.27}$$

and

(8.1.28) $H\big(\mathrm{Hom}_A(V, |W|)\big)$ corresponds to the direct sum of $\ker \alpha'$ and another submodule of $H\big(\mathrm{Hom}_A(V, |W|)\big)$,

as a module over k, as in (8.1.10) and (8.1.11).

Under these conditions, α' maps $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ into $\mathrm{Hom}_A^{gr}(V, H(W))$, and the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ has degree 0, as in (7.15.18) and (7.15.19). Using (8.1.26), we get that

(8.1.29) the composition of the homomorphism as in (8.1.22) with the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is the identity mapping on $\mathrm{Hom}_A^{gr}(V, H(W))$.

In particular, this implies that

$$\alpha'\Big(H\big(\mathrm{Hom}_A^{gr}(V, W)\big)\Big) = \mathrm{Hom}_A^{gr}(V, H(W)). \tag{8.1.30}$$

Of course,

(8.1.31) the kernel of the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is a homogeneous submodule of $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$,

as a graded module over k, because the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ has degree 0. Similarly,

(8.1.32) the image of $\mathrm{Hom}_A^{gr}(V, H(W))$ under the homomorphism as in (8.1.22) is a homogeneous submodule of $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$, because of (8.1.24). In fact,

(8.1.33) $H\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ corresponds to the direct sum of the homogeneous submodules in (8.1.31) and (8.1.32),

as a graded module over k, because of (8.1.29). This corresponds to Proposition 6.2a on p66 of [3], under slightly different conditions.

We may consider V as a complex, with the grading as in (7.12.9), and $d_V = 0$, as in Section 7.12. In this case, the previous remarks correspond to those in Section 7.3.

8.2 Surjectivity conditions and α'

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V and W be both left or both right modules over A again, and suppose that (W, d_W) is a module over A with differentiation. Thus $\mathrm{Hom}_A(V,W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,W)}$ as in (7.12.1), as before.

Remember that $Z\big(\mathrm{Hom}_A(V,W)\big)$ is the same as $\mathrm{Hom}_A(V, Z(W))$, and that $B\big(\mathrm{Hom}_A(V,W)\big)$ is contained in $\mathrm{Hom}_A(V, B(W))$, as in Section 7.14. Consider the condition that

(8.2.1) $$B\big(\mathrm{Hom}_A(V,W)\big) = \mathrm{Hom}_A(V, B(W)),$$

which is to say that d maps $\mathrm{Hom}_A(V,W)$ onto $\mathrm{Hom}_A(V, B(W))$. In particular, this holds when V is projective as a module over A.

Alternatively, let W_0 be a submodule of W, as a module over A. If

(8.2.2) W corresponds to the direct sum of $Z(W)$ and W_0,

as a module over A, then

(8.2.3) the restriction of d_W to W_0 is a one-to-one mapping onto $B(W)$.

Conversely, one can check that (8.2.3) implies (8.2.2). It is easy to see that (8.2.3) implies (8.2.1).

Let α' be the homomorphism from $H\big(\mathrm{Hom}_A(V,W)\big)$ into $\mathrm{Hom}_A(V, H(W))$ defined in Section 7.14. One can verify that the kernel of α' is the same as the image of $\mathrm{Hom}_A(V, B(W))$ under the natural quotient homomorphism from $Z\big(\mathrm{Hom}_A(V,W)\big)$ onto $H\big(\mathrm{Hom}_A(V,W)\big)$, by construction. In particular,

(8.2.4) α' is injective on $H\big(\mathrm{Hom}_A(V,W)\big)$ if and only if (8.2.1) holds.

Remember that ξ is the natural homomorphism from $\mathrm{Hom}_A(V, Z(W))$ into $\mathrm{Hom}_A(V, H(W))$ defined by composing an element of $\mathrm{Hom}_A(V, Z(W))$ with the natural quotient mapping from $Z(W)$ onto $H(W)$, as in Section 7.14. Observe that

(8.2.5) $$\alpha'\big(H\big(\mathrm{Hom}_A(V,W)\big)\big) = \xi\big(\mathrm{Hom}_A(V, Z(W))\big),$$

by construction. In particular,

$$\alpha'\big(H\big(\mathrm{Hom}_A(V,W)\big)\big) = \mathrm{Hom}_A(V, H(W)) \tag{8.2.6}$$

if and only if

$$\xi\big(\mathrm{Hom}_A(V, Z(W))\big) = \mathrm{Hom}_A(V, H(W)). \tag{8.2.7}$$

If V is projective as a module over A, then (8.2.7) holds. This implies that (8.2.6) holds, as in the preceding paragraph.

If $Z(W)$ corresponds to the direct sum of $B(W)$ and another submodule, as a module over A, then one can check directly that (8.2.7) holds. This means that (8.2.6) holds, which could also be obtained as in (8.1.10).

8.3 Surjectivity conditions and complexes

Let us continue with the same notation and hypotheses as in the previous section, and suppose in addition that (W, d_W) is a graded module over A that is a complex. This means that $\mathrm{Hom}_A^{gr}(V,W)$ may be defined as a graded module over k as in (7.12.4), and as a complex, with differentiation operator $d = d_{\mathrm{Hom}_A^{gr}(V,W)}$ as in (7.12.11).

As in Section 7.15, $Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ is the same as $\mathrm{Hom}_A^{gr}(V, Z(W))$, and $B\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ is contained in $\mathrm{Hom}_A^{gr}(V, B(W))$. Consider the condition that

$$B\big(\mathrm{Hom}_A^{gr}(V,W)\big) = \mathrm{Hom}_A^{gr}(V, B(W)), \tag{8.3.1}$$

which means that d maps $\mathrm{Hom}_A^{gr}(V,W)$ onto $\mathrm{Hom}_A^{gr}(V, B(W))$. One can check that this holds when V is projective as a module over A, as before.

Let W_0 be a homogeneous submodule of W, as a graded module over A. Note that

$$W \text{ corresponds to the direct sum of } Z(W) \text{ and } W_0, \tag{8.3.2}$$

as a module over A, if and only if this holds with for W as a graded module over A. Of course, the restriction of d_W to W_0 is a homomorphism into $B(W)$, as modules over A, of degree 1. As in the previous section, (8.3.2) holds if and only if

$$\text{the restriction of } d_W \text{ to } W_0 \text{ is a one-to-one mapping onto } B(W). \tag{8.3.3}$$

One can verify that (8.3.1) holds when (8.3.3) holds.

Remember that the underlying module $|W|$ over A corresponding to W, without a grading, may be considered as a module over A with differentiation with respect to d_W, so that $\mathrm{Hom}_A(V, |W|)$ may be considered as a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,W)}$ as in (7.12.1). We may also consider $\mathrm{Hom}_A^{gr}(V,W)$ as a submodule of $\mathrm{Hom}_A(V, |W|)$, as a module over k with differentiation, as in Section 7.12. Similarly, $\mathrm{Hom}_A^{gr}(V, B(W))$ may be considered as a submodule of $\mathrm{Hom}_A(V, B(|W|))$, as a module over k, which may be considered as a submodule of $\mathrm{Hom}_A(V, |W|)$. It is easy to see that

$$\mathrm{Hom}_A^{gr}(V, B(W)) = \mathrm{Hom}_A^{gr}(V, W) \cap \mathrm{Hom}_A(V, B(|W|)), \tag{8.3.4}$$

as submodules of $\mathrm{Hom}_A(V, |W|)$.

Let α' be the homomorphism

$$\text{from } H\big(\mathrm{Hom}_A(V, |W|)\big) \text{ into } \mathrm{Hom}_A(V, H(|W|)) \tag{8.3.5}$$

defined in Sections 7.14 and 7.15. As in the previous section, the kernel of α' is the same as the image of $\mathrm{Hom}_A(V, B(|W|))$ under the natural quotient homomorphism

$$\text{from } Z\big(\mathrm{Hom}_A(V, |W|)\big) \text{ onto } H\big(\mathrm{Hom}_A(V, |W|)\big). \tag{8.3.6}$$

Remember that the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is a homomorphism into $\mathrm{Hom}_A^{gr}(V, H(W))$ of degree 0. as in Section 7.15. The kernel of the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is the same as the image of $\mathrm{Hom}_A^{gr}(V, B(W))$ under the natural quotient homomorphism

$$\text{from } Z\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ onto } H\big(\mathrm{Hom}_A^{gr}(V, W)\big), \tag{8.3.7}$$

because of (8.3.4). It follows that

$$\begin{aligned}&\text{the restriction of } \alpha' \text{ to } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ is injective}\\ &\text{if and only if (8.3.1) holds.}\end{aligned} \tag{8.3.8}$$

Let ξ be the natural homomorphism

$$\text{from } \mathrm{Hom}_A(V, Z(|W|)) \text{ into } \mathrm{Hom}_A(V, H(|W|)) \tag{8.3.9}$$

defined by composing an element of $\mathrm{Hom}_A(V, Z(|W|))$ with the natural quotient mapping from $Z(|W|)$ onto $H(|W|)$, as in Sections 7.14 and 7.15. Thus

$$\alpha'\big(H\big(\mathrm{Hom}_A(V, |W|)\big)\big) = \xi\big(\mathrm{Hom}_A(V, Z(|W|))\big), \tag{8.3.10}$$

as in (8.2.5). The restriction of ξ to $\mathrm{Hom}_A^{gr}(V, Z(W))$ is a homomorphism into $\mathrm{Hom}_A^{gr}(V, H(W))$ of degree 0, as in Section 7.15. We also have that

$$\alpha'\big(H\big(\mathrm{Hom}_A^{gr}(V, W)\big)\big) = \xi\big(\mathrm{Hom}_A^{gr}(V, Z(W))\big), \tag{8.3.11}$$

by construction. This means that

$$\alpha'\big(H\big(\mathrm{Hom}_A^{gr}(V, W)\big)\big) = \mathrm{Hom}_A^{gr}(V, H(W)) \tag{8.3.12}$$

if and only if

$$\xi\big(\mathrm{Hom}_A^{gr}(V, Z(W))\big) = \mathrm{Hom}_A^{gr}(V, H(W)). \tag{8.3.13}$$

One can verify that (8.3.13) holds when V is projective as a module over A. Thus (8.3.12) holds in this case.

Suppose that $Z(W)$ corresponds to the direct sum of $B(W)$ and another homogeneous submodule of $Z(W)$, as a module over A, and thus as a graded module over A. One can check directly that (8.3.13) under these conditions, so that (8.3.12) holds. This could also be obtained as in (8.1.30).

These properties of α' are related to Propositions 7.2 and 7.4 on p68, 70 of [3], and some remarks on p70 of [3].

8.4 Differentiation and $\mathrm{Hom}(\cdot, W)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, W be both left or both right modules over A.

Suppose for the moment that (V, d_V) is a module with differentiation. Let

$$d = d_{\mathrm{Hom}_A(V,W)} \tag{8.4.1}$$

be the homomorphism from $\mathrm{Hom}_A(V, W)$ into itself, as a module over A, defined by composing d_V with elements of $\mathrm{Hom}_A(V, W)$. That is to say, if ϕ is an element of $\mathrm{Hom}_A(V, W)$, then we put

$$d(\phi) = \phi \circ d_V. \tag{8.4.2}$$

Clearly

$$d \circ d = 0 \quad \text{on } \mathrm{Hom}_A(V, W), \tag{8.4.3}$$

so that $\mathrm{Hom}_A(V, W)$ is a module with differentiation with respect to d.

Suppose now that V is a graded module over A. We would like to define $\mathrm{Hom}_A^{gr}(V, W)$ as a graded module over k, with

$$\big(\mathrm{Hom}_A^{gr}(V, W)\big)^j = \mathrm{Hom}_A(V^{-j}, W) \tag{8.4.4}$$

for every $j \in \mathbf{Z}$, as on p62 of [3], with $r = 1$. As in previous situations like this, we can simply take $\mathrm{Hom}_A^{gr}(V, W)$ to be the direct sum of (8.4.4) over $j \in \mathbf{Z}$, as a module over k, but it is sometimes convenient to consider it as a submodule of related modules over k.

Let $|V|$ be the underlying module over A corresponding to V, without a grading, as on p63 of [3]. If $\phi \in \mathrm{Hom}_A^{gr}(V, W)$, then let ϕ^j be the corresponding element of (8.4.4) for each $j \in \mathbf{Z}$, as usual, so that $\phi^j = 0$ for all but finitely many j. This leads to a unique homomorphism from $|V|$ into W, as modules over A, that we shall also denote by ϕ, with

$$\phi = \phi^j \quad \text{on } V^{-j} \tag{8.4.5}$$

for each $j \in \mathbf{Z}$.

This defines an injective homomorphism

$$\text{from } \mathrm{Hom}_A^{gr}(V, W) \text{ into } \mathrm{Hom}_A(|V|, W), \tag{8.4.6}$$

as modules over k, so that we may identify $\mathrm{Hom}^{gr}_A(V, W)$ with the corresponding submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k. More precisely, $\mathrm{Hom}^{gr}_A(V, W)$ corresponds to the submodule of $\mathrm{Hom}_A(V, W)$ consisting of the homomorphisms from $|V|$ into W, as modules over A, whose restrictions to V^{-j} are equal to 0 for all but finitely many $j \in \mathbf{Z}$. This is related to a remark on p63 of [3]. Remember that $\mathrm{Hom}_A(|V|, W)$ corresponds to a direct product of $\mathrm{Hom}_A(V^{-j}, W)$, $j \in \mathbf{Z}$, as a module over k, as in Section 1.7.

Suppose for the moment that W is considered as a graded module over A, with

$$W^l = W \quad \text{when } l = 0 \qquad = \{0\} \quad \text{when } l \neq 0, \tag{8.4.7}$$

as in Section 7.5. This permits us to consider $\mathrm{Hom}^{gr}_A(V, W)$ as a doubly-graded module over k, as in Section 6.3, with

$$\begin{aligned} \big(\mathrm{Hom}^{gr}_A(V, W)\big)^{j,l} = \mathrm{Hom}_A(V^{-j}, W_l) &= \mathrm{Hom}_A(V^{-j}, W) \quad \text{when } l = 0 \\ &= \{0\} \qquad\qquad\qquad\quad \text{when } l \neq 0. \end{aligned} \tag{8.4.8}$$

Note that the single grading associated to this double grading is the same as in (8.4.4).

Suppose for the rest of the section that (V, d_V) is a graded module over A that is a complex. Thus $\mathrm{Hom}^{gr}_A(V, W)$ may be defined as a graded module over k as in (8.4.4). Let us define

$$d = d_{\mathrm{Hom}^{gr}_A(V,W)} \tag{8.4.9}$$

on $\mathrm{Hom}^{gr}_A(V, W)$ as follows. If $\phi \in \mathrm{Hom}^{gr}_A(V, W)$, then $d(\phi)$ is defined as an element of $\mathrm{Hom}^{gr}_A(V, W)$ by putting

$$d(\phi)^{j+1} = \phi^j \circ d_V^{-j-1} \tag{8.4.10}$$

for each j. One can check that $\mathrm{Hom}^{gr}_A(V, W)$ is a complex with respect to d, as on p63 of [3].

The underlying module $|V|$ over A corresponding to V may be considered as a module with differentiation with respect to d_V, so that $d = d_{\mathrm{Hom}_A(|V|,W)}$ may be defined on $\mathrm{Hom}_A(|V|, W)$ as in (8.4.1). It is easy to see that the restriction of d on $\mathrm{Hom}_A(|V|, W)$ to $\mathrm{Hom}^{gr}_A(V, W)$, considered as a submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k, is the same as the definition of d on $\mathrm{Hom}^{gr}_A(V, W)$ in the preceding paragraph. This means that $\mathrm{Hom}^{gr}_A(V, W)$ may be considered as a submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k with differentiation.

Let us consider W as a complex, with the grading as in (8.4.7), and differentiation operator $d_W = 0$, as on p75 of [3]. This permits us to consider $\mathrm{Hom}^{gr}_A(V, W)$ as a double complex, with double grading as in (8.4.8), and

$$\delta_1 = d = d_{\mathrm{Hom}^{gr}_A(V,W)}, \quad \delta_2 = 0, \tag{8.4.11}$$

as in Section 6.5. It follows that $\mathrm{Hom}^{gr}_A(V, W)$ is a single complex with respect to the total differentiation operator

$$\delta = \delta_1 + \delta_2 = d = d_{\mathrm{Hom}^{gr}_A(V,W)}, \tag{8.4.12}$$

and the single grading associated to the double grading on $\mathrm{Hom}^{gr}_A(V, W)$, as in Section 5.14. This is equivalent to considering $\mathrm{Hom}^{gr}_A(V, W)$ as a single complex as before.

8.5 Induced mappings and $\mathrm{Hom}(\cdot, W)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, V_0, and W be all left or all right modules over A.

Suppose for the moment that (V, d_V) and (V_0, d_{V_0}) are modules with differentiation. Let $d = d_{\mathrm{Hom}_A(V,W)}$ be defined on $\mathrm{Hom}_A(V, W)$ as in (8.4.1), and let

$$d_0 = d_{\mathrm{Hom}_A(V_0,W)} \tag{8.5.1}$$

be defined on $\mathrm{Hom}_A(V_0, W)$ in the same way.

Let ϕ be a homomorphism from V_0 into V, as modules over A. Consider the homomorphism Φ

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V_0, W), \tag{8.5.2}$$

as modules over k, defined by composing ϕ with an element of $\mathrm{Hom}_A(V, W)$ to get an element of $\mathrm{Hom}_A(V_0, W)$. Suppose now that ϕ is a homomorphism from V_0 into V, as modules over A with differentiation. In this case, one can check that Φ is a homomorphism as in (8.5.2), as modules over k with differentiation.

Let ψ be another homomorphism from V_0 into V, as modules over A with differentiation, and let Ψ be the corresponding homomorphism as in (8.5.2), as modules over k with differentiation. Also let σ be a homotopy between ϕ and ψ, so that σ is a homomorphism from V_0 into V, as modules over A, such that

$$d_V \circ \sigma + \sigma \circ d_{V_0} = \phi - \psi. \tag{8.5.3}$$

Suppose that Σ is the homomorphism as in (8.5.2), as modules over k, that corresponds to σ in the same way as before. One can check that

$$d_0 \circ \Sigma + \Sigma \circ d = \Phi - \Psi, \tag{8.5.4}$$

so that Σ is a homotopy between Φ and Ψ.

Suppose now that V, V_0 are graded modules over A, so that $\mathrm{Hom}^{gr}_A(V, W)$ and $\mathrm{Hom}^{gr}_A(V_0, W)$ may be defined as graded modules over k as in (8.4.4). Let ϕ be a homomorphism from V_0 into V, as modules over A, of degree $p \in \mathbf{Z}$. This leads to a homomorphism Φ

$$\text{from } \mathrm{Hom}^{gr}_A(V, W) \text{ into } \mathrm{Hom}^{gr}_A(V_0, W), \tag{8.5.5}$$

as modules over k, of degree p as well. More precisely, if $j \in \mathbf{Z}$, then Φ acts on (8.4.4) by composing the restriction of ϕ to V_0^{-j-p} with an element of (8.4.4) to get an element of

$$(8.5.6) \qquad \big(\mathrm{Hom}_A^{gr}(V_0, W)\big)^{j+p} = \mathrm{Hom}_A(V_0^{-j-p}, W),$$

as on p63 of [3].

Let $|V|$, $|V_0|$ be the underlying modules over A corresponding to V, V_0, respectively, without gradings, as on p63 of [3]. We may consider $\mathrm{Hom}_A^{gr}(V, W)$ and $\mathrm{Hom}_A^{gr}(V_0, W)$ as submodules of $\mathrm{Hom}_A(|V|, W)$ and $\mathrm{Hom}_A(|V_0|, W)$, respectively, as modules over k, as in the previous section. Using this identification, the homomorphism Φ as in (8.5.5) is the same as the restriction of the homomorphism Φ

$$(8.5.7) \qquad \text{from } \mathrm{Hom}_A(|V|, W) \text{ into } \mathrm{Hom}_A(|V_0|, W)$$

as in (8.5.2) to $\mathrm{Hom}_A^{gr}(V, W)$.

If we consider W as a graded module as in (8.4.7), then $\mathrm{Hom}_A^{gr}(V, W)$ and $\mathrm{Hom}_A^{gr}(V_0, W)$ may be defined as doubly-graded modules over k, as in Section 6.3. Under these conditions,

$$(8.5.8) \qquad \begin{array}{l} \text{the homomorphism } \Phi \text{ as in } (8.5.5) \\ \text{corresponds to } \mathrm{Hom}^{gr}(\phi, I_W). \end{array}$$

This uses the notation in Section 6.4, with I_W being the identity mapping on W.

Suppose that (V, d_V), (V_0, d_{V_0}) are graded modules over A with differentiation that are complexes, so that $\mathrm{Hom}_A^{gr}(V, W)$ and $\mathrm{Hom}_A^{gr}(V_0, W)$ may be considered as single complexes too, as in the previous section. Let ϕ be a map of degree $p \in \mathbf{Z}$ from V_0 into V, as complexes, as in Section 6.6. If Φ is the corresponding homomorphism as in (8.5.5), then it is easy to see that Φ is a map of degree p as in (8.5.5), as complexes.

Let ψ be another map of degree p from V_0 into V, as complexes, and let Ψ be the corresponding map of degree p as in (8.5.5), as complexes. Suppose that σ is a homotopy between ϕ and ψ, as maps of degree p between complexes, as in Section 6.6. Remember that this means in particular that σ is a homomorphism from V_0 into V of degree $p-1$, as modules over A. Thus we get a homomorphism Σ as in (8.5.5), as modules over k, of degree $p-1$. One can check that $(-1)^p\,\Sigma$ is a homotopy between ϕ and ψ, as maps of degree p between complexes.

We may consider W as a complex, with $d_W = 0$, and graded as in (8.4.7). This permits us to consider $\mathrm{Hom}_A^{gr}(V, W)$ and $\mathrm{Hom}_A^{gr}(V_0, W)$ as double complexes, as in the previous section. We may also consider Φ and Ψ as maps of bidegree $(p, 0)$ as in (8.5.5), as double complexes, as in Section 6.9. Under these conditions, $((-1)^p\,\Sigma, 0)$ defines a homotopy between Φ and Ψ, as maps of bidgree $(p, 0)$ as in (8.5.5), as double complexes, as in Section 6.9. These remarks correspond to some of those in Section 6.5 when $p = 0$.

8.6 Another analogue of α'

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V and W be both left or both right modules over A, and suppose that (V, d_V) is a module over A with differentiation. This means that $\mathrm{Hom}_A(V, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,W)}$ as in (8.4.1).

In this case, $Z\big(\mathrm{Hom}_A(V, W)\big)$ consists of $\phi \in \mathrm{Hom}_A(V, W)$ such that

$$d(\phi) = \phi \circ d_V = 0 \tag{8.6.1}$$

on V. Equivalently, this means that

$$B(V) \subseteq \ker \phi. \tag{8.6.2}$$

Of course, if $\phi \in \mathrm{Hom}_A(V, W)$, then the restriction of ϕ to $Z(V)$ defines a homomorphism from $Z(V)$ into W, as modules over A. If $\phi \in Z\big(\mathrm{Hom}_A(V, W)\big)$, then the restriction of ϕ to $Z(V)$ induces a homomorphism

$$\text{from } H(V) \text{ into } W, \tag{8.6.3}$$

as modules over A, because of (8.6.2). This defines a homomorphism

$$\text{from } Z\big(\mathrm{Hom}_A(V, W)\big) \text{ into } \mathrm{Hom}_A(H(V), W), \tag{8.6.4}$$

as modules over k.

Suppose for the moment that $\phi \in B\big(\mathrm{Hom}_A(V, W)\big)$, so that

$$\phi = d(\psi) = \psi \circ d_V \tag{8.6.5}$$

for some $\psi \in \mathrm{Hom}_A(V, W)$. This implies that

$$Z(V) \subseteq \ker \phi. \tag{8.6.6}$$

This means that the induced homomorphism as in (8.6.3) is equal to 0. This is the same as saying that ϕ is mapped to 0 by the homomorphism as in (8.6.4).

Thus $B\big(\mathrm{Hom}_A(V, W)\big)$ is contained in the kernel of the homomorphism as in (8.6.4). It follows that the homomorphism as in (8.6.4) induces a homomorphism

$$\alpha' \text{ from } H\big(\mathrm{Hom}_A(V, W)\big) \text{ into } \mathrm{Hom}_A(H(V), W), \tag{8.6.7}$$

as modules over k.

There is a natural homomorphism

$$\text{from } \mathrm{Hom}_A(Z'(V), W) \text{ into } \mathrm{Hom}_A(V, W), \tag{8.6.8}$$

as modules over k. This is defined by composing the natural quotient mapping from V onto $Z'(V) = V/d_V(V)$ with an element of $\mathrm{Hom}_A(Z'(V), W)$ to get an element of $\mathrm{Hom}_A(V, W)$. Note that

$$\text{the homomorphism as in (8.6.8) is injective.} \tag{8.6.9}$$

We also have that

$$\text{the homomorphism as in (8.6.8) maps } \mathrm{Hom}_A(Z'(V), W) \text{ onto } Z\big(\mathrm{Hom}_A(V, W)\big). \tag{8.6.10}$$

This leads to a natural homomorphism

$$\eta \text{ from } \mathrm{Hom}_A(Z'(V), W) \text{ onto } H\big(\mathrm{Hom}_A(V, W)\big), \tag{8.6.11}$$

as modules over k. More precisely, η is the composition of the homomorphism as in (8.6.8), considered as a homomorphism onto $Z\big(\mathrm{Hom}_A(V, W)\big)$, with the natural quotient mapping from $Z\big(\mathrm{Hom}_A(V, W)\big)$ onto $H\big(\mathrm{Hom}_A(V, W)\big)$.

There is a natural homomorphism

$$\xi \text{ from } \mathrm{Hom}_A(Z'(V), W) \text{ into } \mathrm{Hom}_A(H(V), W), \tag{8.6.12}$$

as modules over k, which sends an element of $\mathrm{Hom}_A(Z'(V), W)$ to its restriction to $H(V)$, considered as a submodule of $Z'(V)$. It is easy to see that

$$\alpha' \circ \eta = \xi, \tag{8.6.13}$$

as homomorphisms from $\mathrm{Hom}_A(Z'(V), W)$ into $\mathrm{Hom}_A(H(V), W)$. More precisely, $\alpha' \circ \eta$ is the same as the composition of the homomorphism as in (8.6.8), considered as a homomorphism onto $Z\big(\mathrm{Hom}_A(v, W)\big)$, with the homomorphism as in (8.6.4). This composition is the same as ξ, by construction. Note that α' is uniquely determined by (8.6.13), because η is surjective.

There is a natural homomorphism

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(Z(V), W), \tag{8.6.14}$$

as modules over k, which sends an element of $\mathrm{Hom}_A(V, W)$ to its restriction to $Z(V)$. There is also a natural homomorphism

$$\text{from } Z(V) \text{ into } Z'(V), \tag{8.6.15}$$

as modules over A, which is the restriction to $Z(V)$ of the natural quotient mapping from V onto $Z'(V)$. Equivalently, this is the composition of the natural quotient mapping from $Z(V)$ onto $H(V)$ with the natural inclusion mapping from $H(V)$ into $Z'(V)$.

Using the homomorphism as in (8.6.15), we get a natural homomorphism

$$\text{from } \mathrm{Hom}_A(Z'(V), W) \text{ into } \mathrm{Hom}_A(Z(V), W), \tag{8.6.16}$$

as modules over k. This homomorphism sends an element of $\mathrm{Hom}_A(Z'(V), W)$ to its composition with the homomorphism as in (8.6.15). This is the same as the composition of the homomorphism as in (8.6.8) with the homomorphism as in (8.6.14).

If $\phi \in B\big(\operatorname{Hom}_A(V, W)\big)$, then ϕ is mapped to 0 by the homomorphism as in (8.6.14), because of (8.6.6). Using this and the restriction of the homomorphism as in (8.6.14) to $Z\big(\operatorname{Hom}_A(V, W)\big)$, we get a natural homomorphism

(8.6.17) ζ from $H\big(\operatorname{Hom}_A(V, W)\big)$ into $\operatorname{Hom}_A(Z(V), W)$,

as modules over k. One can check that

(8.6.18) $\zeta \circ \eta$ is the same as the homomorphism as in (8.6.16).

More precisely, $\zeta \circ \eta$ is the same as the composition of the homomorphism as in (8.6.8) with the homomorphism as in (8.6.14), which is the same as the homomorphism as in (8.6.16).

Using the natural quotient mapping from $Z(V)$ onto $H(V)$, we get a natural homomorphism

(8.6.19) τ from $\operatorname{Hom}_A(H(V), W)$ into $\operatorname{Hom}_A(Z(V), W)$,

as modules over k. That is to say, τ sends an element of $\operatorname{Hom}_A(H(V), W)$ to its composition with the natural quotient mapping from $Z(V)$ onto $H(V)$. In particular,

(8.6.20) τ is injective.

One can verify that

(8.6.21) $\tau \circ \xi$ is the same as the homomorphism as in (8.6.16).

More precisely, $\tau \circ \xi$ is the same as the composition of the homomorphism as in (8.6.8) with the homomorphism as in (8.6.14), which is the same as the homomorphism as in (8.6.16). This implies that

(8.6.22) $$\tau \circ \xi = \zeta \circ \eta,$$

by (8.6.18).

We also have that

(8.6.23) $$\tau \circ \alpha' = \zeta,$$

as homomorphisms from $H\big(\operatorname{Hom}_A(V, W)\big)$ into $\operatorname{Hom}_A(Z(V), W)$. This can be seen directly from the definitions. Alternatively, each of (8.6.13) and (8.6.23) can be obtained from the other using (8.6.22), by composing with η or τ, as appropriate. Of course, α' is uniquely determined by (8.6.23), because τ is injective.

8.7 Another α' and complexes

We continue with the same notation and hypotheses as in the previous section, and suppose now that (V, d_V) is a graded module over A that is a complex. This means that $\operatorname{Hom}_A^{gr}(V, W)$ may be defined as a graded module over k as in

(8.4.4), and as a complex, with differentiation operator $d = d_{\mathrm{Hom}_A^{gr}(V,W)}$ as in (8.4.9). Similarly,

$$\text{(8.7.1)} \qquad \mathrm{Hom}_A^{gr}(B(V), W),\ \mathrm{Hom}_A^{gr}(Z(V), W),\ \mathrm{Hom}_A^{gr}(H(V), W), \text{ and } \mathrm{Hom}_A^{gr}(Z'(V), W)$$

may be defined as graded modules over k as in (8.4.4).

The underlying module $|V|$ over A corresponding to V may be considered as a module with differentiation operator d_V, and $\mathrm{Hom}_A(|V|, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(|V|,W)}$ defined as in (8.4.1). We have also seen that $\mathrm{Hom}_A^{gr}(V, W)$ may be considered as a submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k with differentiation, as in Section 8.4.

Of course, $B(|V|)$ and $Z(|V|)$ are the same as $B(V)$ and $Z(V)$, respectively, without the gradings. Thus $B(|V|)$ and $Z(|V|)$ may be considered as the same as $|B(V)|$ and $|Z(V)|$, respectively. In particular,

$$\text{(8.7.2)} \qquad \begin{array}{l} \mathrm{Hom}_A^{gr}(B(V), W) \text{ and } \mathrm{Hom}_A^{gr}(Z(V), W) \text{ may be considered} \\ \text{as submodules of } \mathrm{Hom}_A(B(|V|), W) \text{ and } \mathrm{Hom}_A(Z(|V|), W), \end{array}$$

as modules over k, respectively.

Similarly, $H(|V|)$ and $Z'(|V|)$ are the same as $H(V)$ and $Z'(V)$, without the gradings. This means that $H(|V|)$ and $Z'(|V|)$ may be considered as the same as $|H(V)|$ and $|Z'(V)|$, respectively. It follows that

$$\text{(8.7.3)} \qquad \begin{array}{l} \mathrm{Hom}_A^{gr}(H(V), W) \text{ and } \mathrm{Hom}_A^{gr}(Z'(V), W) \text{ may be considered} \\ \text{as submodules of } \mathrm{Hom}_A(H(|V|), W) \text{ and } \mathrm{Hom}_A(Z'(|V|), W), \end{array}$$

as modules over k, respectively.

The remarks in the previous section can be applied to $|V|$, and we would like to consider analogous statements for $\mathrm{Hom}_A^{gr}(\cdot, W)$ instead of $\mathrm{Hom}_A(\cdot, W)$ here. Note that

$$\text{(8.7.4)} \qquad Z\big(\mathrm{Hom}_A^{gr}(V, W)\big) = Z\big(\mathrm{Hom}_A(|V|, W)\big) \cap \mathrm{Hom}_A^{gr}(V, W),$$

because $\mathrm{Hom}_A^{gr}(V, W)$ is a submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k with differentiation.

We also have that

$$\text{(8.7.5)} \qquad B\big(\mathrm{Hom}_A^{gr}(V, W)\big) = B\big(\mathrm{Hom}_A(|V|, W)\big) \cap \mathrm{Hom}_A^{gr}(V, W).$$

More precisely, the fact that the left side is contained in the right side follows from $\mathrm{Hom}_A^{gr}(V, W)$ being a submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k with differentiation. The opposite inclusion can be verified using the definition of $\mathrm{Hom}_A^{gr}(V, W)$.

Using (8.7.5),

$$\text{(8.7.6)} \qquad \begin{array}{l} \text{we may consider } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \\ \text{as a submodule of } H\big(\mathrm{Hom}_A(|V|, W)\big), \end{array}$$

as a module over k. Similarly,

(8.7.7) we may consider $Z'\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ as a submodule of $Z'\big(\mathrm{Hom}_A(|V|,W)\big)$,

as a module over k.

There is a natural homomorphism

(8.7.8) from $Z\big(\mathrm{Hom}_A(|V|,W)\big)$ into $\mathrm{Hom}_A(H(|V|),W)$,

as modules over k, as in (8.6.4). One can check that the restriction of this homomorphism to $Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ defines a homomorphism

(8.7.9) from $Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ into $\mathrm{Hom}_A^{gr}(H(V),W)$,

as modules over k, of degree 0.

There is a natural homomorphism

(8.7.10) α' from $H\big(\mathrm{Hom}_A(|V|,W)\big)$ into $\mathrm{Hom}_A(H(|V|),W)$,

as modules over k, as in (8.6.7). One can check that

(8.7.11) α' maps $H\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ into $\mathrm{Hom}_A^{gr}(H(V),W)$,

and that

(8.7.12) the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V,W)\big)$ has degree 0.

There is an natural injective homomorphism

(8.7.13) from $\mathrm{Hom}_A(Z'(|V|),W)$ into $\mathrm{Hom}_A(|V|,W)$,

as modules over k, as in (8.6.8). More precisely, this homomorphism maps $\mathrm{Hom}_A(Z'(|V|),W)$ onto $Z\big(\mathrm{Hom}_A(|V|,W)\big)$, as in (8.6.10). One can check that this homomorphism

(8.7.14) maps $\mathrm{Hom}_A^{gr}(Z'(V),W)$ onto $Z\big(\mathrm{Hom}_A^{gr}(V,W)\big)$,

and that the restriction of this homomorphism to $\mathrm{Hom}_A^{gr}(Z'(V),W)$ has degree 0.

There is a natural homomorphism

(8.7.15) η from $\mathrm{Hom}_A(Z'(|V|),W)$ onto $H\big(\mathrm{Hom}_A(|V|,W)\big)$,

as modules over k, as in (8.6.11). One can verify that

(8.7.16) η maps $\mathrm{Hom}_A^{gr}(Z'(V),W)$ onto $H\big(\mathrm{Hom}_A^{gr}(V,W)\big)$,

and that

(8.7.17) the restriction of η to $\mathrm{Hom}_A^{gr}(Z'(V),W)$ has degree 0.

The restriction of η to $\mathrm{Hom}_A^{gr}(Z'(V), W)$ is the same as the composition of the restriction of the homomorphism as in (8.7.13) to $\mathrm{Hom}_A^{gr}(Z'(V), W)$, considered as a homomorphism onto $Z\big(\mathrm{Hom}_A^{gr}(V, W)\big)$, with the natural quotient mapping from $Z\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ onto $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$.

There is a natural homomorphism

$$\xi \text{ from } \mathrm{Hom}_A(Z'(|V|), W) \text{ into } \mathrm{Hom}_A(H(|V|), W), \tag{8.7.18}$$

as modules over k, as in (8.6.12). One can check that

$$\xi \text{ maps } \mathrm{Hom}_A^{gr}(Z'(V), W) \text{ into } \mathrm{Hom}_A^{gr}(H(V), W), \tag{8.7.19}$$

and that

$$\text{the restriction of } \xi \text{ to } \mathrm{Hom}_A^{gr}(Z'(V), W) \text{ has degree } 0. \tag{8.7.20}$$

Note that $\alpha' \circ \eta = \xi$ on $\mathrm{Hom}_A(Z'(|V|), W)$, as in (8.6.13). In particular, this holds on $\mathrm{Hom}_A^{gr}(Z'(V), W)$. The restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is uniquely determined by this equality of homomorphisms on $\mathrm{Hom}_A^{gr}(Z'(V), W)$, because of (8.7.16).

There is a natural homomorphism

$$\text{from } \mathrm{Hom}_A(|V|, W) \text{ into } \mathrm{Hom}_A(Z(|V|), W), \tag{8.7.21}$$

as modules over k, which sends an element of $\mathrm{Hom}_A(|V|, W)$ to its restriction to $Z(|V|)$, as in (8.6.14). The restriction of this homomorphism to $\mathrm{Hom}_A^{gr}(V, W)$ defines a natural homomorphism

$$\text{from } \mathrm{Hom}_A^{gr}(V, W) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), W), \tag{8.7.22}$$

as modules over k. More precisely, this homomorphism has degree 0.

There is a natural homomorphism

$$\text{from } Z(|V|) \text{ into } Z'(|V|), \tag{8.7.23}$$

as modules over A, as in (8.6.15). The restriction of this homomorphism to $Z(V)$ defines a homomorphism

$$\text{from } Z(V) \text{ into } Z'(V), \tag{8.7.24}$$

as modules over A, of degree 0. This is the same as the restriction to $Z(V)$ of the natural quotient mapping from V onto $Z'(V)$, and the composition of the natural quotient mapping from $Z(V)$ onto $H(V)$ with the natural inclusion from $H(V)$ into $Z'(V)$.

There is a natural homomorphism

$$\text{from } \mathrm{Hom}_A(Z'(|V|), W) \text{ into } \mathrm{Hom}_A(Z(|V|), W), \tag{8.7.25}$$

as modules over k, as in (8.6.16). The restriction of this homomorphism to $\mathrm{Hom}_A^{gr}(Z'(V), W)$ defines a homomorphism

$$\text{from } \mathrm{Hom}_A^{gr}(Z'(V), W) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), W), \tag{8.7.26}$$

as modules over k, of degree 0.

There is a natural homomorphism

$$\zeta \text{ from } H\big(\mathrm{Hom}_A(|V|, W)\big) \text{ into } \mathrm{Hom}_A(Z(|V|), W), \tag{8.7.27}$$

as modules over k, as in (8.6.17). One can check that

$$\zeta \text{ maps } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), W), \tag{8.7.28}$$

and that

$$\text{the restriction of } \zeta \text{ to } H\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ has degree } 0. \tag{8.7.29}$$

Remember that $\zeta \circ \eta$ is the same as the homomorphism as in (8.7.25).

There is a natural injective homomorphism

$$\tau \text{ from } \mathrm{Hom}_A(H(|V|), W) \text{ into } \mathrm{Hom}_A(Z(|V|), W), \tag{8.7.30}$$

as modules over k, as in (8.6.19). One can verify that

$$\tau \text{ maps } \mathrm{Hom}_A^{gr}(H(V), W) \text{ into } \mathrm{Hom}_A^{gr}(Z(V), W), \tag{8.7.31}$$

and that

$$\text{the restriction of } \tau \text{ to } \mathrm{Hom}_A^{gr}(H(V), W) \text{ has degree } 0. \tag{8.7.32}$$

We also have that $\tau \circ \xi$ is the same as the homomorphism as in (8.7.25), as before. This implies that $\tau \circ \xi = \zeta \circ \eta$, as in (8.6.22). In particular, this holds on $\mathrm{Hom}_A^{gr}(Z'(V), W)$.

Remember that $\tau \circ \alpha' = \zeta$ on $H\big(\mathrm{Hom}_A(|V|, W)\big)$, as in (8.6.23). In particular, this holds on $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$.

These homomorphisms and their properties correspond to the commutative diagram (1) on p64 of [3], and to parts of Proposition 6.1a on p65 of [3], under slightly different conditions.

We may consider W as a graded module over A with differentiation that is a complex, with grading as in (8.4.7), and $d_W = 0$, as in Section 8.4. In this case, $B(W) = \{0\}$, $Z(W) = H(W) = Z'(W) = W$, and the remarks in this and the previous section correspond to those in Sections 6.13 – 6.15.

8.8 Splitting $Z'(V)$ and α'

Let k be a commutative ring with a mutliplciative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that V and W are both left or both right modules over A, and that (V, d_V) is a module over A with differentiation. Thus $\mathrm{Hom}_A(V, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,W)}$ as in (8.4.1).

Remember that $Z'(V) = V/d_V(V)$, and suppose that

$$Z'(V) \text{ corresponds to the direct sum of } H(V) \text{ and another submodule of } Z'(V), \tag{8.8.1}$$

as a module over A. This means that

$$\text{there is a homomorphism from } Z'(V) \text{ onto } H(V), \text{ as modules over } A, \text{ that is equal to the identity mapping on } H(V). \tag{8.8.2}$$

We can compose the natural quotient mapping from V onto $Z'(V)$ with this homomorphism to get a homomorphism

$$\beta \text{ from } V \text{ onto } H(V), \tag{8.8.3}$$

as modules over A. Of course,

$$\beta \circ d_V = 0 \tag{8.8.4}$$

on V, by construction. Note that

$$\text{the restriction of } \beta \text{ to } Z(V) \text{ is the same as the natural quotient mapping from } Z(V) \text{ onto } H(V), \tag{8.8.5}$$

by (8.8.2).

We can use β to get a homomorphism

$$\text{from } \mathrm{Hom}_A(H(V), W) \text{ into } \mathrm{Hom}_A(V, W), \tag{8.8.6}$$

as modules over k. This homomorphism sends an element of $\mathrm{Hom}_A(H(V), W)$ to its composition with β, to get an element of $\mathrm{Hom}_A(V, W)$. The composition of this element of $\mathrm{Hom}_A(V, W)$ with d_V is equal to 0, because of (8.8.4). This means that we get an element of $Z\big(\mathrm{Hom}_A(V, W)\big)$, by the definition of d. Thus we get a homomorphism

$$\text{from } \mathrm{Hom}_A(H(V), W) \text{ into } Z\big(\mathrm{Hom}_A(V, W)\big) \tag{8.8.7}$$

in this way.

This leads to a homomorphism

$$\text{from } \mathrm{Hom}_A(H(V), W) \text{ into } H\big(\mathrm{Hom}_A(V, W)\big), \tag{8.8.8}$$

as modules over k, by composing the previous homomorphism as in (8.8.7) with the natural quotient mapping

$$\text{from } Z\big(\mathrm{Hom}_A(V, W)\big) \text{ onto } H\big(\mathrm{Hom}(V, W)\big). \tag{8.8.9}$$

Let α' be the homomorphism from $H\big(\mathrm{Hom}_A(V, W)\big)$ into $\mathrm{Hom}_A(H(V), W)$ defined in Section 8.6. One can check that

$$\text{the composition of the homomorphism as in (8.8.8) with } \alpha' \text{ is the identity mapping on } \mathrm{Hom}_A(H(V), W). \tag{8.8.10}$$

More precisely, the composition of the homomorphism as in (8.8.8) with α' is the same as the composition of the homomorphism as in (8.8.7) with the homomorphism as in (8.6.4), by construction. To get that this is the identity mapping on $\mathrm{Hom}_A(H(V), W)$, one can use (8.8.5).

Using this, we get that

$$\alpha'\big(H\big(\mathrm{Hom}_A(V, W)\big)\big) = \mathrm{Hom}_A(H(V), W). \tag{8.8.11}$$

We can also use (8.8.10) to get that

(8.8.12) $H\big(\mathrm{Hom}_A(V, W)\big)$ corresponds to the direct sum of $\ker \alpha'$ and another submodule of $H\big(\mathrm{Hom}_A(V, W)\big)$,

as a module over k.

Let us suppose now that (V, d_V) is a graded module over A with differentiation that is a complex, and that

(8.8.13) $Z'(V)$ corresponds to the direct sum of $H(V)$ and another homogeneous submodule of $Z'(V)$,

as a module over A. Equivalently, this means that $Z'(V)$ corresponds to the direct sum of $H(V)$ and another submodule, as a graded module over A. This implies that

(8.8.14) the homomorphism as in (8.8.2) has degree 0.

It follows that

(8.8.15) the homomorphism β as in (8.8.3) has degree 0.

We can define $\mathrm{Hom}^{gr}_A(V, W)$ as a graded module over k as in (8.4.4), and as a complex, with differentiation operator $d = d_{\mathrm{Hom}^{gr}_A(V,W)}$ as in (8.4.9). The underlying module $|V|$ over A corresponding to V, without a grading, may be considered as a module with differentiation with respect to d_V, so that $\mathrm{Hom}_A(|V|, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(|V|,W)}$ defined as in (8.4.1). Remember that $\mathrm{Hom}^{gr}_A(V, W)$ may be considered as a submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k with differentiation, as in Section 8.4.

We can use β to get a homomorphism

(8.8.16) from $\mathrm{Hom}_A(H(|V|), W)$ into $\mathrm{Hom}_A(|V|, W)$,

as modules over k, as in (8.8.6). This may be considered as a homomorphism

(8.8.17) from $\mathrm{Hom}_A(H(|V|), W)$ into $Z\big(\mathrm{Hom}_A(|V|, W)\big)$,

as modules over k, as in (8.8.7). One can check that

(8.8.18) the homomorphism as in (8.8.16) maps $\mathrm{Hom}^{gr}_A(H(V), W)$ into $\mathrm{Hom}^{gr}_A(V, W)$.

The restriction of this homomorphism to $\mathrm{Hom}_A^{gr}(H(V), W)$ may be considered as a homomorphism

(8.8.19) $$\text{from } \mathrm{Hom}_A^{gr}(H(V), W) \text{ into } Z\big(\mathrm{Hom}_A^{gr}(V, W)\big),$$

as modules over k. Observe that

(8.8.20) the homomorphism as in (8.8.19) has degree 0,

because of (8.8.15).

As before, we can compose the homomorphism as in (8.8.17) with the natural quotient mapping

(8.8.21) $$\text{from } Z\big(\mathrm{Hom}_A(|V|, W)\big) \text{ onto } H\big(\mathrm{Hom}_A(|V|, W)\big)$$

to get a homomorphism

(8.8.22) $$\text{from } \mathrm{Hom}_A(H(|V|), W) \text{ into } H\big(\mathrm{Hom}_A(|V|, W)\big),$$

as modules over k. Note that this homomorphism maps $\mathrm{Hom}_A^{gr}(H(V), W)$ into $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$. More precisely, the restriction of this homomorphism to $\mathrm{Hom}_A^{gr}(H(V), W)$ is the same as the homomorphism

(8.8.23) $$\text{from } \mathrm{Hom}_A^{gr}(H(V), W) \text{ into } H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$$

obtained by composing the homomorphism as in (8.8.19) with the natural quotient mapping

(8.8.24) $$\text{from } Z\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ onto } H\big(\mathrm{Hom}_A^{gr}(V, W)\big).$$

It is easy to see that

(8.8.25) the homomorphism as in (8.8.23) has degree 0,

because of (8.8.20).

Let α' be the homomorphism

(8.8.26) $$\text{from } H\big(\mathrm{Hom}_A(H(|V|), W)\big) \text{ into } \mathrm{Hom}_A(H(|V|), W)$$

discussed in the previous two sections. Thus

(8.8.27) the composition of the homomorphism as in (8.8.22) with α' is the identity mapping on $\mathrm{Hom}_A(H(|V|), W)$,

as in (8.8.10). This implies that

(8.8.28) $$\alpha'\big(H\big(\mathrm{Hom}_A(|V|, W)\big)\big) = \mathrm{Hom}_A(H(|V|), W))$$

and

(8.8.29) $H\big(\mathrm{Hom}_A(|V|, W)\big)$ corresponds to the direct sum of $\ker \alpha'$ and another submodule of $H\big(\mathrm{Hom}_A(|V|, W)\big)$, as a module over k, as in (8.8.11) and (8.8.12).

Remember that α' maps $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ into $\mathrm{Hom}_A^{gr}(H(V), W)$, and the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ has degree 0, as in (8.7.11) and (8.7.12). We also have that

(8.8.30) the composition of the homomorphism as in (8.8.23) with the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is the identity mapping on $\mathrm{Hom}_A^{gr}(H(V), W)$,

because of (8.8.27). This implies in particular that

$$\alpha'\Big(H\big(\mathrm{Hom}_A^{gr}(V, W)\big)\Big) = \mathrm{Hom}_A^{gr}(H(V), W). \tag{8.8.31}$$

Note that

(8.8.32) the kernel of the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is a homogeneous submodule of $H\big(\mathrm{Hom}_A(V, W)\big)$,

as a graded module over k, because the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ has degree 0. Similarly,

(8.8.33) the image of $\mathrm{Hom}_A^{gr}(H(V), W)$ under the homomorphism as in (8.8.23) is a homogeneous submodule of $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$,

(8.8.34)

because of (8.8.25). Using (8.8.30), we get that

(8.8.35) $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ corresponds to the direct sum of the homogeneous submodules in (8.8.32) and (8.8.33),

as a graded module over k. This corresponds to Proposition 6.2a on p66 of [3], under slightly different conditions.

We may also consider W as a complex, with the grading as in (8.4.7), and $d_W = 0$, as in Section 8.4. Under these conditions, the previous remarks correspond to those in Section 7.3.

8.9 Additional conditions and α'

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V and W be both left or both right modules over A again, and suppose that (V, d_V) is a module over A with differentiation. This means that $\mathrm{Hom}_A(V, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(V,W)}$ as in (8.4.1).

Remember that

$$\begin{aligned} (8.9.1)\quad Z\big(\mathrm{Hom}_A(V,W)\big) &= \{\phi \in \mathrm{Hom}_A(V,W) : d(\phi) = \phi \circ d_V = 0\} \\ &= \{\phi \in \mathrm{Hom}_A(V,W) : B(V) \subseteq \ker \phi\}, \end{aligned}$$

as in Section 8.6. We have also seen that

$$\begin{aligned} (8.9.2)\quad B\big(\mathrm{Hom}_A(V,W)\big) &= \{\phi \in \mathrm{Hom}_A(V,W) : \phi = d(\psi) = \psi \circ d_V \\ &\qquad\qquad \text{for some } \psi \in \mathrm{Hom}_A(V,W)\} \end{aligned}$$

is contained in

$$(8.9.3)\qquad \{\phi \in \mathrm{Hom}_A(V,W) : Z(V) \subseteq \ker \phi\}.$$

Note that (8.9.3) is contained in (8.9.1). We may be interested in situations in which

$$(8.9.4)\qquad B\big(\mathrm{Hom}_A(V,W)\big) \text{ is equal to (8.9.3).}$$

It is easy to see that (8.9.3) is equal to

$$(8.9.5)\quad \{\phi \in \mathrm{Hom}_A(V,W) : \phi = \psi_0 \circ d_V \text{ for some } \psi_0 \in \mathrm{Hom}_A(B(V),W)\}.$$

If W is injective as a module over A, then this is the same as (8.9.2), so that (8.9.4) holds. If

$$\begin{aligned} (8.9.6)\qquad & V \text{ corresponds to the direct sum of } B(V) \\ & \text{and another submodule of } V, \end{aligned}$$

as a module over A, then (8.9.5) is the same as (8.9.2), so that (8.9.4) holds again.

Let α' be the homomorphism from $H\big(\mathrm{Hom}_A(V,W)\big)$ into $\mathrm{Hom}_A(H(V),W)$, as modules over k, defined in Section 8.6. By construction, the composition of the natural quotient mapping

$$(8.9.7)\qquad \text{from } Z\big(\mathrm{Hom}_A(V,W)\big) \text{ onto } H\big(\mathrm{Hom}_A(V,W)\big)$$

with α' is the same as the homomorphism

$$(8.9.8)\qquad \text{from } Z\big(\mathrm{Hom}_A(V,W)\big) \text{ into } \mathrm{Hom}_A(H(V),W)$$

mentioned in (8.6.4). One can check that the kernel of the homomorphism as in (8.9.8) is equal to (8.9.3). This means that

$$\begin{aligned} (8.9.9)\qquad & \ker \alpha' \text{ is the same as the image of (8.9.3)} \\ & \text{under the natural quotient mapping as in (8.9.7).} \end{aligned}$$

In particular,

$$(8.9.10)\qquad \alpha' \text{ is injective if and only if (8.9.4) holds.}$$

Remember that ξ is the natural homomorphism from $\mathrm{Hom}_A(Z'(V),W)$ into $\mathrm{Hom}_A(H(V),W)$ that sends an element of $\mathrm{Hom}_A(Z'(V),W)$ to its restriction to $H(V)$, considered as a submodule of $Z'(V)$, as in Section 8.6. We have seen that ξ is the same as the composition of the natural homomorphism η from $\mathrm{Hom}_A(Z'(V), W)$ onto $H\big(\mathrm{Hom}_A(V, W)\big)$ defined in Section 8.6 with α'. This implies that

$$\alpha'\big(H\big(\mathrm{Hom}_A(V, W)\big)\big) = \xi\big(\mathrm{Hom}_A(Z'(V), W)\big). \tag{8.9.11}$$

Thus

$$\alpha'\big(H\big(\mathrm{Hom}_A(V, W)\big)\big) = \mathrm{Hom}_A(H(V), W) \tag{8.9.12}$$

if and only if

$$\xi\big(\mathrm{Hom}_A(Z'(V), W)\big) = \mathrm{Hom}_A(H(V), W). \tag{8.9.13}$$

If W is injective as a module over A, then (8.9.13) holds. This implies that (8.9.12) holds, as before.

If $Z'(V)$ corresponds to the direct sum of $H(V)$ and another submodule of $Z'(V)$, as a module over A, then it is easy to see that (8.9.13) holds. This implies that (8.9.12) holds, which could also be obtained as in (8.8.11).

8.10 Additional conditions and complexes

We continue with the same notation and hypotheses as in the previous section, and suppose in addition that (V, d_V) is a graded module over A that is a complex. Thus $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a graded module over k as in (8.4.4), and as a complex, with differentiation operator $d = d_{\mathrm{Hom}_A^{gr}(V,W)}$ as in (8.4.9). We may define $\mathrm{Hom}_A^{gr}(B(V), W)$, $\mathrm{Hom}_A^{gr}(H(V), W)$, and $\mathrm{Hom}_A^{gr}(Z'(V), W)$ as graded modules over k too, as in (8.4.4).

We may consider the underlying module $|V|$ over A corresponding to V as a module with differentiation operator d_V, so that $\mathrm{Hom}_A(|V|, W)$ is a module over k with differentiation operator $d = d_{\mathrm{Hom}_A(|V|,W)}$ defined as in (8.4.1). Remember that $\mathrm{Hom}_A^{gr}(V, W)$ may be considered as a submodule of $\mathrm{Hom}_A(|V|, W)$, as a module over k with differentiation, as in Section 8.4.

As in Section 8.7, $H(|V|)$ and $Z'(|V|)$ are the same as $H(V)$ and $Z'(V)$, respectively, without the gradings, so that they are the same as $|H(V)|$ and $|Z'(V)|$, respectively. This means that $\mathrm{Hom}_A^{gr}(H(V), W)$ and $\mathrm{Hom}_A^{gr}(Z'(V), W)$ may be considered as submodules of $\mathrm{Hom}_A(H(|V|), W)$ and $\mathrm{Hom}_A(Z'(|V|), W)$, respectively, as modules over k.

Remember that

$$Z\big(\mathrm{Hom}_A^{gr}(V, W)\big) = Z\big(\mathrm{Hom}_A(|V|, W)\big) \cap \mathrm{Hom}_A^{gr}(V, W) \tag{8.10.1}$$

and

$$B\big(\mathrm{Hom}_A^{gr}(V, W)\big) = B\big(\mathrm{Hom}_A(|V|, W)\big) \cap \mathrm{Hom}_A^{gr}(V, W), \tag{8.10.2}$$

as in Section 8.7. The latter implies that $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ may be considered as a submodule of $H\big(\mathrm{Hom}_A(|V|, W)\big)$, as a modules over k, as before.

Of course, $Z\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ consists of the $\phi \in \mathrm{Hom}_A^{gr}(V, W)$ such that

$$d(\phi)^{j+1} = \phi^j \circ d_V^{-j-1} = 0 \tag{8.10.3}$$

for every $j \in \mathbf{Z}$. Equivalently, this means that

$$B(V)^{-j} \subseteq \ker \phi^j \tag{8.10.4}$$

for every $j \in \mathbf{Z}$. Similarly, $B\big(\operatorname{Hom}^{gr}_A(V, W)\big)$ is the same as the set of ϕ in $\operatorname{Hom}^{gr}_A(V, W)$ for which there is a $\psi \in \operatorname{Hom}^{gr}_A(V, W)$ such that

$$\phi^{j+1} = d(\psi)^{j+1} = \psi^j \circ d_V^{-j-1} \tag{8.10.5}$$

for every $j \in \mathbf{Z}$. This implies that

$$Z(V)^{-j-1} \subseteq \ker \phi^{j+1} \tag{8.10.6}$$

for every $j \in \mathbf{Z}$.

Consider

$$\{\phi \in \operatorname{Hom}^{gr}_A(V, W) : Z(V)^{-j} \subseteq \ker \phi^j \text{ for every } j \in \mathbf{Z}\}, \tag{8.10.7}$$

which is a homogeneous submodule of $Z\big(\operatorname{Hom}^{gr}_A(V, W)\big)$, as a module over k. This is the same as the intersection of (8.9.3) with $\operatorname{Hom}^{gr}_A(V, W)$. Note that $B\big(\operatorname{Hom}^{gr}_A(V, W)\big)$ is contained in (8.10.7). As in the previous section, we may be interested in situations in which

$$B\big(\operatorname{Hom}^{gr}_A(V, W)\big) \text{ is equal to (8.10.7).} \tag{8.10.8}$$

Observe that $\phi \in \operatorname{Hom}^{gr}_A(V, W)$ is an element of (8.10.7) if and only if for each $j \in \mathbf{Z}$ there is a homomorphism $\psi_{0,j}$ from $B(V)^{-j} = d_V^{-j-1}(V^{-j-1})$ into W, as modules over A, such that

$$\phi^{j+1} = \psi_{0,j} \circ d_V^{-j-1}. \tag{8.10.9}$$

Of course, if $\phi^{j+1} = 0$, then one can simply take $\psi_{0,j} = 0$. It follows that $\phi \in \operatorname{Hom}^{gr}_A(V, W)$ is an element of (8.10.7) if and only if there is a ψ_0 in $\operatorname{Hom}^{gr}_A(B(V), W)$ such that

$$\phi^{j+1} = \psi_0^j \circ d_V^{-j-1} \tag{8.10.10}$$

for every $j \in \mathbf{Z}$.

If W is injective as a module over A, then one can take $\psi_{0,j}$ to be a homomorphism from V^{-j} into W, as modules over A, in (8.10.9). This permits one to take $\psi_0 \in \operatorname{Hom}^{gr}_A(V, W)$ in (8.10.10), so that (8.10.8) holds.

Suppose for the moment that

$$\begin{aligned} &V \text{ corresponds to the direct sum of } B(V) \\ &\text{and another homogeneous submodule of } V, \end{aligned} \tag{8.10.11}$$

as a module over A. This means that V^{-j} corresponds to the direct sum of $B(V)^{-j}$ and another submodule of V^{-j}, as a module over A, for each $j \in \mathbf{Z}$. In this case, one can take $\psi_{0,j}$ to be a homomorphism from V^{-j} into W, as modules over A, in (8.10.9) again. This permits one to take $\psi_0 \in \mathrm{Hom}_A^{gr}(V, W)$ in (8.10.10), so that (8.10.8) holds, as before.

The natural quotient mapping

(8.10.12) $$\text{from } Z\big(\mathrm{Hom}_A^{gr}(V, W)\big) \text{ onto } H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$$

may be identified with the restriction to (8.10.1) of the natural quotient mapping

(8.10.13) $$\text{from } Z\big(\mathrm{Hom}_A(|V|, W)\big) \text{ onto } H\big(\mathrm{Hom}_A(|V|, W)\big),$$

because of (8.10.2). Let α' be the homomorphism

(8.10.14) $$\text{from } H\big(\mathrm{Hom}_A(|V|, W)\big) \text{ into } \mathrm{Hom}_A(H(|V|), W)$$

defined in Sections 8.6 and 8.7. One can verify that

(8.10.15) the kernel of the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is the same as the image of (8.10.7) under the natural quotient mapping as in (8.10.12).

This is similar to (8.9.9), and can be obtained from it. It follows in particular that

(8.10.16) the restriction of α' to $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$ is injective if and only if (8.10.8) holds.

Let ξ be the natural homomorphism

(8.10.17) $$\text{from } \mathrm{Hom}_A(Z'(|V|), W) \text{ into } \mathrm{Hom}_A(H(|V|), W)$$

that sends an element of $\mathrm{Hom}_A(Z'(|V|), W)$ to its restriction to $H(|V|)$, considered as a submodule of $Z'(|V|)$, as in Sections 8.6 and 8.7. Note that

(8.10.18) $$\alpha'\big(H\big(\mathrm{Hom}_A(|V|, W)\big)\big) = \xi\big(\mathrm{Hom}_A(Z'(|V|), W)\big),$$

as in (8.9.11). The restriction of ξ to $\mathrm{Hom}_A^{gr}(V, W)$ is a homomorphism into $\mathrm{Hom}_A^{gr}(H(V), W)$ of degree 0, as in Section 8.7. One can check that

(8.10.19) $$\alpha'\big(H\big(\mathrm{Hom}_A^{gr}(V, W)\big)\big) = \xi\big(\mathrm{Hom}_A^{gr}(Z'(V), W)\big)$$

using the same type of argument as before. More precisely, let η be the natural homomorphism from $\mathrm{Hom}_A(Z'(|V|), W)$ onto $H\big(\mathrm{Hom}_A(|V|, W)\big)$ defined in Section 8.7. We have seen that $\alpha' \circ \eta = \xi$ on $\mathrm{Hom}_A(Z'(|V|), W)$, and that η maps $\mathrm{Hom}_A^{gr}(Z'(V), W)$ onto $H\big(\mathrm{Hom}_A^{gr}(V, W)\big)$, which implies (8.10.19). It follows that

(8.10.20) $$\alpha'\big(H\big(\mathrm{Hom}_A^{gr}(V, W)\big)\big) = \mathrm{Hom}_A^{gr}(H(V), W)$$

if and only if

(8.10.21) $$\xi\big(\mathrm{Hom}_A^{gr}(Z'(V), W)\big) = \mathrm{Hom}_A^{gr}(H(V), W).$$

One can check that (8.10.21) holds when W is injective as a module over A. Thus (8.10.20) holds in this case.

Suppose now that $Z'(V)$ corresponds to the direct sum of $H(V)$ and another homogeneous submodule of $Z'(V)$, as a module over A. This means that $Z'(V)^{-j}$ corresponds to the direct sum of $H(V)^{-j}$ and another submodule of $Z'(V)^{-j}$, as a module over A, for each $j \in \mathbf{Z}$. It is easy to see that (8.10.21) holds under these conditions. This implies that (8.10.20) holds, as before. This could also be obtained as in (8.8.31).

The properties of α' discussed in this section are related to Propositions 7.2 and 7.4 on p68, 70 of [3], and some remarks on p70 of [3].

8.11 Easy complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a graded left or right module over A, with

$$V^j = \{0\} \quad \text{when } j \neq 0, 1. \tag{8.11.1}$$

Suppose that d_V is a homomorphism from V into itself, as a module over A, with degree 1, so that

$$d_V^j = 0 \quad \text{when } j \neq 0. \tag{8.11.2}$$

Under these conditions, $d_V \circ d_V = 0$ automatically, so that (V, d_V) is a complex. The underlying module with differentiation is the same as in Section 5.5.

In this case,

$$\begin{aligned} Z(V)^j &= \ker d_V^0 && \text{when } j = 0 \\ &= V^1 && \text{when } j = 1 \\ &= \{0\} && \text{otherwise.} \end{aligned} \tag{8.11.3}$$

Similarly,

$$\begin{aligned} B(V)^j &= d_V^0(V^0) && \text{when } j = 1 \\ &= \{0\} && \text{otherwise.} \end{aligned} \tag{8.11.4}$$

Thus

$$\begin{aligned} H(V)^j &= \ker d_V^0 && \text{when } j = 0 \\ &= V^1/d_V^0(V^0) && \text{when } j = 1 \\ &= \{0\} && \text{otherwise.} \end{aligned} \tag{8.11.5}$$

We also have that

$$\begin{aligned} Z'(V)^j &= V^0 && \text{when } j = 0 \\ &= V^1/d_V^0(V^0) && \text{when } j = 1 \\ &= \{0\} && \text{otherwise.} \end{aligned} \tag{8.11.6}$$

Similarly,

$$B'(V)^j = V^0 / \ker d_V^0 \quad \text{when } j = 0 \tag{8.11.7}$$
$$= \{0\} \quad \text{otherwise.}$$

Using this, the remarks in Section 3.13 may be considered as a particular case of those in Section 5.12. This corresponds to the remark after Proposition 2.10 on p23 of [1].

Lemmas 3.2 and 3.3 on p40 of [3] deal with many of the same properties as in Section 3.13, under broader conditions appropriate to the particular parts. Some of the statements in Section 5.3 could be obtained using this, as on p55 of [3].

8.12 Sums, products, and differentiation

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let $\{V_\alpha\}_{\alpha \in I}$ be a nonempty family of all left or all right modules over A. Thus the direct sum $\bigoplus_{\alpha \in I} V_\alpha$ and direct product $\prod_{\alpha \in I} V_\alpha$ may be defined as left or right modules over A, as appropriate.

Suppose that (V_α, d_{V_α}) is a module with differentiation for each $\alpha \in I$. We can define differentiation operators on $\bigoplus_{\alpha \in I}$ and $\prod_{\alpha \in I} V_\alpha$ in the obvious way, using d_{V_α} in each coordinate. It is easy to see that $\bigoplus_{\alpha \in I} V_\alpha$ and $\prod_{\alpha \in I} V_\alpha$ become modules with differentiation in this way. More precisely, $\bigoplus_{\alpha \in I} V_\alpha$ is a submodule of $\prod_{\alpha \in I} V_\alpha$, as a module with differentiation.

Observe that

$$Z\Big(\bigoplus_{\alpha \in I} V_\alpha\Big) = \bigoplus_{\alpha \in I} Z(V_\alpha) \tag{8.12.1}$$

and

$$Z\Big(\prod_{\alpha \in I} V_\alpha\Big) = \prod_{\alpha \in I} Z(V_\alpha) \tag{8.12.2}$$

under these conditions. Similarly,

$$B\Big(\bigoplus_{\alpha \in I} V_\alpha\Big) = \bigoplus_{\alpha \in I} B(V_\alpha) \tag{8.12.3}$$

and

$$B\Big(\prod_{\alpha \in I} V_\alpha\Big) = \prod_{\alpha \in I} B(V_\alpha). \tag{8.12.4}$$

We also have that

$$H\Big(\bigoplus_{\alpha \in I} V_\alpha\Big) = \bigoplus_{\alpha \in I} H(V_\alpha) \tag{8.12.5}$$

and

$$H\Big(\prod_{\alpha \in I} V_\alpha\Big) = \prod_{\alpha \in I} H(V_\alpha). \tag{8.12.6}$$

Similarly, we get that

$$Z'\Big(\bigoplus_{\alpha \in I} V_\alpha\Big) = \bigoplus_{\alpha \in I} Z'(V_\alpha) \tag{8.12.7}$$

and

$$Z'\Big(\prod_{\alpha \in I} V_\alpha\Big) = \prod_{\alpha \in I} Z'(V_\alpha), \tag{8.12.8}$$

as well as

$$B'\Big(\bigoplus_{\alpha \in I} V_\alpha\Big) = \bigoplus_{\alpha \in I} B'(V_\alpha) \tag{8.12.9}$$

and

$$B'\Big(\prod_{\alpha \in I} V_\alpha\Big) = \prod_{\alpha \in I} B'(V_\alpha). \tag{8.12.10}$$

This corresponds to Proposition 9.3 on p98 of [3].

Suppose now that V_α is a graded module over A for each $\alpha \in I$. In this case, we can identify $\bigoplus_{\alpha \in I} V_\alpha$ with

$$\bigoplus_{j \in \mathbf{Z}} \Big(\bigoplus_{\alpha \in I} V_\alpha^j\Big) \tag{8.12.11}$$

in a straightforward way. This makes $\bigoplus_{\alpha \in I} V_\alpha$ a graded module too, with

$$\Big(\bigoplus_{\alpha \in I} V_\alpha\Big)^j = \bigoplus_{\alpha \in I} V_\alpha^j \tag{8.12.12}$$

for each $j \in \mathbf{Z}$.

Put

$$V^j = \prod_{\alpha \in I} V_\alpha^j \tag{8.12.13}$$

for each $j \in \mathbf{Z}$, which is a left or right module over A, as appropriate. Thus

$$V = \bigoplus_{j \in \mathbf{Z}} V^j \tag{8.12.14}$$

is a graded module over A. Of course, (8.12.12) is a submodule of (8.12.13), as a module over A, for each $j \in \mathbf{Z}$. This means that (8.12.11) may be considered as a homogeneous submodule of (8.12.14).

Note that (8.12.14) may be considered as a submodule of

$$\prod_{j \in \mathbf{Z}} V^j, \tag{8.12.15}$$

as a module over A. As before, we can identify

$$\prod_{\alpha \in I} \Big(\prod_{j \in \mathbf{Z}} V_\alpha^j\Big) \tag{8.12.16}$$

with (8.12.15), as modules over A, in a straightforward way. Of course,

$$\prod_{\alpha \in I} V_\alpha = \prod_{\alpha \in I} \Big(\bigoplus_{j \in \mathbf{Z}} V_\alpha^j \Big) \tag{8.12.17}$$

may be considered as a submodule of (8.12.16), as a module over A. It is easy to see that (8.12.14) corresponds to a submodule of (8.12.17), with respect to the identification between (8.12.15) and (8.12.16) just mentioned. This is compatible with considering $\bigoplus_{\alpha \in I} V_\alpha$ as a submodule of (8.12.16), as a module over A, by considering V_α as a submodule of $\prod_{j \in I} V_\alpha^j$ for each $\alpha \in I$.

In the next section, we shall consider the case where (V_α, d_{V_α}) is a complex for each $\alpha \in I$.

8.13 Sums, products, and complexes

Let k be a commutative ring with a multiplicative identity element again, and let A be an associative algebra over k with a multiplicative identity element e_A. Let (U, d_U) be a graded left or right module over A with differentiation that this a complex. Put

$$W = \prod_{j \in Z} U^j, \tag{8.13.1}$$

so that W is a left or right module over A, as appropriate, and U corresponds to a submodule of W, as a module over A.

If $w \in W$, then define $d_W(w) \in W$ by saying that for each $j \in \mathbf{Z}$, the jth coordinate of $d_W(w)$ is equal to d_U^{j-1} of the $(j-1)$th coordinate of w. More precisely, the $(j-1)$th coordinate of w is in U^{j-1}, so that d_U^{j-1} sends it into U^j. One can check that this makes (W, d_W) a module over A with differentiation. We also have that U corresponds to a submodule of W, as a module with differentiation.

Suppose now that $\{(V_\alpha, d_{V_\alpha})\}_{\alpha \in I}$ is a nonempty family of all left or all right graded modules over A with differentiation that are complexes. If $\alpha \in I$, then put

$$W_\alpha = \prod_{j \in \mathbf{Z}} V_\alpha^j, \tag{8.13.2}$$

which is a left or right module over A. We can define d_{W_α} on W_α as in the preceding paragraph, so that (W, d_{W_α}) is a module over A with differentiation. Note that V_α corresponds to a submodule of W_α, as a module with differentiation, as before.

Using this, we can define a differentation operator on

$$\prod_{\alpha \in I} W_\alpha, \tag{8.13.3}$$

to get a module over A with differentiation, as in the previous section. Of course, (8.13.3) is the same as (8.12.16), as a module over A. One can verify that (8.12.17) is a submodule of (8.13.3), as a module with differentiation. This corresponds to defining a differentiation operator on $\prod_{\alpha \in I} V_\alpha$, as discussed near the beginning of the previous section.

Similarly, one can check that the module V defined in (8.12.14) corresponds to a submodule of (8.13.3), as a module over A with differentiation, using the identification between (8.12.15) and (8.12.16) mentioned earlier. More precisely, the restriction of the differentiation operator to (8.12.14) has degree 1 with respect to the natural grading on V. This makes V a complex, as a graded module over A with differentiation.

It is easy to see that $\bigoplus_{\alpha \in I} V_\alpha$ is a submodule of (8.13.3), as a module with differentiation. As before, the restriction of the differentiation operator to $\bigoplus_{\alpha \in I} V_\alpha$ has degree 1 with respect to the grading as in (8.12.12), so that $\bigoplus_{\alpha \in I} V_\alpha$ becomes a complex.

If $j \in \mathbf{Z}$, then

$$Z\Big(\bigoplus_{\alpha \in I} V_\alpha\Big)^j = \bigoplus_{\alpha \in I} Z(V_\alpha)^j \tag{8.13.4}$$

and

$$Z(V)^j = \prod_{\alpha \in I} Z(V_\alpha)^j. \tag{8.13.5}$$

Similarly,

$$B\Big(\bigoplus_{\alpha \in I} V_\alpha\Big)^j = \bigoplus_{\alpha \in I} B(V_\alpha)^j \tag{8.13.6}$$

and

$$B(V)^j = \prod_{\alpha \in I} B(V_\alpha)^j. \tag{8.13.7}$$

Using this, we get that

$$H\Big(\bigoplus_{\alpha \in I} V_\alpha\Big)^j = \bigoplus_{\alpha \in I} H(V_\alpha)^j \tag{8.13.8}$$

and

$$H(V)^j = \prod_{\alpha \in I} H(V_\alpha)^j. \tag{8.13.9}$$

We also get that

$$Z'\Big(\bigoplus_{\alpha \in I} V_\alpha\Big)^j = \bigoplus_{\alpha \in I} Z'(V_\alpha)^j \tag{8.13.10}$$

and

$$Z'(V)^j = \prod_{\alpha \in I} Z'(V_\alpha)^j, \tag{8.13.11}$$

and similarly

$$B'\Big(\bigoplus_{\alpha \in I} V_\alpha\Big)^j = \bigoplus_{\alpha \in I} B'(V_\alpha)^j \tag{8.13.12}$$

and

$$B'(V)^j = \prod_{\alpha \in I} B'(V_\alpha)^j. \tag{8.13.13}$$

This corresponds to Proposition 9.3 on p98 of [3] again.

8.14 Direct limits and differentiation

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let $(I, \preceq)$ be a nonempty pre-directed set, let (V_j, d_j) be a module over A with differentiation for each $j \in I$. More precisely, we suppose that either V_j is a left module over A with differentiation for every $j \in I$, or that V_j is a right module over A with differentiation for every $j \in I$.

Suppose that for every $j, l \in I$ with $j \preceq l$, we have a homomorphism $\nu_{j,l}$ from V_j into V_l, as modules over A with differentiation. As usual, we ask that $\nu_{j,j}$ be the identity mapping on V_j for every $j \in I$. If $j, l, r \in I$ and $j \preceq l \preceq r$, then we also ask that

$$\nu_{l,r} \circ \nu_{j,l} = \nu_{j,r}, \tag{8.14.1}$$

as before. Under these conditions, we get a *direct* or *inductive system* of modules with differentiation over $(I, \preceq)$.

Let $\varinjlim V_j$ be the direct limit of the V_j's, as a direct system of modules over A, as in Section 3.2. This is a left or right module over A, as appropriate. If $l \in I$, then we also have a homomorphism ν_l from V_l into $\varinjlim V_j$, as modules over A, as before. Remember that

$$\nu_l = \nu_r \circ \nu_{l,r} \tag{8.14.2}$$

for every $l, r \in I$ with $l \preceq r$.

If $j, l \in I$ and $j \preceq l$, then

$$d_l \circ \nu_{j,l} = \nu_{j,l} \circ d_j, \tag{8.14.3}$$

because $\nu_{j,l}$ is supposed to be a homomorphism from V_j into V_l, as modules with differentiation. Thus the family of d_j's, $j \in I$, defines a homomorphism from the direct systems of V_j's, $j \in I$, into itself, as in Section 3.4. This implies that there is unique homomorphism $d = \varinjlim d_j$ from $\varinjlim V_j$ into itself, as a module over A, such that

$$d \circ \nu_l = \nu_l \circ d_l \tag{8.14.4}$$

for every $l \in I$, as before.

Of course, $d_j \circ d_j = 0$ on V_j for each $j \in I$, because (V_j, d_j) is a module with differentiation. This implies that

$$d \circ d = 0 \text{ on } \varinjlim V_j, \tag{8.14.5}$$

as in Section 3.4. This means that $\varinjlim V_j$ is a module with differentiation with respect to d.

If $j, l \in I$ and $j \preceq l$, then

$$\nu_{j,l}(Z(V_j)) \subseteq Z(V_l), \tag{8.14.6}$$

because of (8.14.3). Let $\nu^Z_{j,l}$ be the restriction of $\nu_{j,l}$ to $Z(V_j)$, considered as a mapping into $Z(V_l)$. Of course, $\nu^Z_{j,j}$ is the identity mapping on $Z(V_j)$ for each $j \in I$. If $j, l, r \in I$ and $j \preceq l \preceq r$, then

$$\nu^Z_{l,r} \circ \nu^Z_{j,l} = \nu^Z_{j,r}, \tag{8.14.7}$$

by (8.14.1).

Thus the family of modules $Z(V_j)$ and homomorphisms $\nu^Z_{j,l}$ forms a direct system over $(I, \preceq)$, so that the direct limit $\varinjlim Z(V_j)$ can be defined as a module over A in the usual way. If $l \in I$, then we get a homomorphism ν^Z_l from $Z(V_l)$ into $\varinjlim Z(V_j)$, as modules over A, as in Section 3.2. If $r \in I$ and $l \preceq r$, then

$$\nu^Z_l = \nu^Z_r \circ \nu^Z_{l,r}, \tag{8.14.8}$$

as before.

The family of obvious inclusion mappings from $Z(V_j)$ into V_j, $j \in I$, defines a homomorphism from the direct system of $Z(V_j)$'s into the direct system of V_j's, $j \in I$, as in Section 3.4. This leads to a natural homomorphism

$$\text{from } \varinjlim Z(V_j) \text{ into } \varinjlim V_j, \tag{8.14.9}$$

as modules over A, as before. This homomorphism is characterized by the fact that for each $l \in I$, the composition of ν^Z_l with the homomorphism as in (8.14.9) is the same as the composition of the natural inclusion mapping from $Z(V_l)$ into V_l with ν_l.

If $j \in I$, then the composition of the obvious inclusion mapping from $Z(V_j)$ into V_j with d_j is equal to 0, by construction. It follows that

(8.14.10) the composition of the homomorphism as in (8.14.9) with d is equal to 0,

as in Section 3.4. This means that

(8.14.11) the homomorphism as in (8.14.9) maps $\varinjlim Z(V_j)$ into $Z\big(\varinjlim V_j\big)$.

One can check that

(8.14.12) the homomorphism as in (8.14.9) is injective, using the remarks in Section 3.4, or more directly. Similarly, one can verify that

(8.14.13) the homomorphism as in (8.14.9) maps $\varinjlim Z(V_j)$ onto $Z\big(\varinjlim V_j\big)$,

using the remarks in Section 3.4. This corresponds to part of Proposition 9.3* on p100 of [3].

Similarly, if $j, l \in I$ and $j \preceq l$, then

$$\nu_{j,l}(B(V_j)) \subseteq B(V_l), \tag{8.14.14}$$

by (8.14.3). Let $\nu_{j,l}^B$ be the restriction of $\nu_{j,l}$ to $B(V_j)$, considered as a mapping into $B(V_l)$. Thus $\nu_{j,j}^B$ is the identity mapping on $B(V_j)$ for each $j \in I$, and

$$\nu_{l,r}^B \circ \nu_{j,l}^B = \nu_{j,r}^B \tag{8.14.15}$$

when $j, l, r \in I$ satisfy $j \preceq l \preceq r$, by (8.14.1). This means that the family of modules $B(V_j)$ and homomorphisms $\nu_{j,l}^B$ forms a direct system over $(I, \preceq)$, so that the direct limit $\varinjlim B(V_j)$ can be defined as a module over A. As before, we get a homomorphism ν_l^B from $B(V_l)$ into $\varinjlim B(V_j)$, as modules over A, for each $l \in I$, with

$$\nu_l^B = \nu_r^B \circ \nu_{l,r}^B \tag{8.14.16}$$

when $r \in I$ and $l \preceq r$.

The family of obvious inclusion mappings from $B(V_j)$ into $Z(V_j)$, $j \in I$, defines a homomorphism from the direct system of $B(V_j)$'s into the direct system of $Z(V_j)$'s, $j \in I$, as in Section 3.4. This leads to a natural homomorphism

(8.14.17) from $\varinjlim B(V_j)$ into $\varinjlim Z(V_j)$,

as modules over A, which is characterized by the property that for each $l \in I$, the composition of ν_l^B with this homomorphism is the same as the composition of the natural inclusion mapping from $B(V_l)$ into $Z(V_l)$ with ν_l^Z. One could also use the family of obvious inclusion mappings from $B(V_j)$ into V_j, $j \in I$, to get a homomorphism from the direct system of $B(V_j)$'s into the direct system of V_j's, $j \in I$, and thus a homomorphism

(8.14.18) from $\varinjlim B(V_j)$ into $\varinjlim V_j$,

as modules over A. This homomorphism is the same as the composition of the homomorphisms as in (8.14.17) and (8.14.9), as in Section 3.4.

If $j \in I$, then let d_j^B be d_j, considered as a mapping from V_j onto $B(V_j)$. Of course,

$$d_l^B \circ \nu_{j,l}^B = \nu_{j,l}^B \circ d_j^B \tag{8.14.19}$$

when $l \in I$ satisfies $j \preceq l$, as in (8.14.3). This means that the family of d_j^B's, $j \in I$, defines a homomorphism from the direct system of V_j's into the direct system of $B(V_j)$'s, $j \in I$, as in Section 3.4. Thus there is a unique homomorphism $d^B = \varinjlim d_j^B$ from $\varinjlim V_j$ into $\varinjlim B(V_j)$, as modules over A, such that

(8.14.20) $$d^B \circ \nu_l^B = \nu_l^B \circ d_l^B$$

for every $l \in I$. More precisely,

(8.14.21) $$d^B\big(\varinjlim V_j\big) = \varinjlim B(V_j),$$

because $d_j^B(V_j) = B(V_j)$ for every $j \in I$, as in Section 3.4.

The composition of d^B with the homomorphism as in (8.14.18) is the same as d, because d_j is the same as the composition of d_j^B with the obvious inclusion mapping from $B(V_j)$ into V_j for each $j \in I$. It follows that

(8.14.22) $B\big(\varinjlim V_j\big)$ is the same as the image of $\varinjlim B(V_j)$ under the homomorphism as in (8.14.18).

Note that

(8.14.23) the homomorphism as in (8.14.17) is injective,

as in Section 3.4. Similarly,

(8.14.24) the homomorphism as in (8.14.18) is injective,

which could also be obtained from (8.14.12) and (8.14.23).

If $j, l \in I$ and $j \preceq l$, then $\nu_{j,l}$ induces a homomorphism

(8.14.25) $$\nu_{j,l}^{Z'} \text{ from } Z'(V_j) \text{ into } Z'(V_l),$$

as modules over A, as in Section 5.2. Clearly $\nu_{j,j}^{Z'}$ is the identity mapping on $Z'(V_j) = V_j/d_j(V_j)$ for each $j \in I$, and

(8.14.26) $$\nu_{l,r}^{Z'} \circ \nu_{j,l}^{Z'} = \nu_{j,r}^{Z'}$$

for every $j, l, r \in I$ with $j \preceq l \preceq r$, because of (8.14.1). Thus the family of modules $Z'(V_j)$ and homomorphisms $\nu_{j,l}^{Z'}$ forms a direct system over $(I, \preceq)$, so that the direct limit $\varinjlim Z'(V_j)$ can be defined as a module over A. We also get a homomorphism $\nu_l^{Z'}$ from $Z'(V_l)$ into $\varinjlim Z'(V_j)$, as modules over A, for each $l \in I$, with

(8.14.27) $$\nu_l^{Z'} = \nu_r^{Z'} \circ \nu_{l,r}^{Z'}$$

when $r \in I$ satisfies $l \preceq r$, as usual.

Let $q_j^{Z'}$ be the natural quotient mapping from V_j onto $Z'(V_j)$ for each $j \in I$. If $j, l \in I$ and $j \preceq l$, then

(8.14.28) $$q_l^{Z'} \circ \nu_{j,l} = \nu_{j,l}^{Z'} \circ q_j^{Z'},$$

by construction. This means that the family of quotient mappings $q_j^{Z'}$, $j \in I$, defines a homomorphism from the direct system of V_j's into the direct system of $Z'(V_j)$'s, $j \in I$, as in Section 3.4. This leads to a natural homomorphism

$$q^{Z'} = \lim_{\longrightarrow} q_j^{Z'} \text{ from } \lim_{\longrightarrow} V_j \text{ into } \lim_{\longrightarrow} Z'(V_j), \tag{8.14.29}$$

as modules over A, which is characterized by the property that

$$q^{Z'} \circ \nu_l = \nu_l^{Z'} \circ q_l^{Z'} \tag{8.14.30}$$

for every $l \in I$. More precisely,

$$q^{Z'}\big(\lim_{\longrightarrow} V_j\big) = \lim_{\longrightarrow} Z'(V_j), \tag{8.14.31}$$

because $q_j^{Z'}(V_j) = Z'(V_j)$ for every $j \in I$, as in Section 3.4.

Of course, $\ker q_j^{Z'} = B(V_j)$ for each $j \in I$, by construction. This implies that the kernel of $q^{Z'}$ is the same as the image of $\lim\limits_{\longrightarrow} B(V_j)$ under the homomorphism as in (8.14.18), as in Section 3.4. It follows that

$$\ker q^{Z'} = B\big(\lim_{\longrightarrow} V_j\big), \tag{8.14.32}$$

because of (8.14.22). This means that

$$Z'\big(\lim_{\longrightarrow} V_j\big) \text{ is isomorphic to } \lim_{\longrightarrow} Z'(V_j), \tag{8.14.33}$$

as modules over A, in such a way that $q^{Z'}$ corresponds to the natural quotient mapping from $\lim\limits_{\longrightarrow} V_j$ onto $Z'\big(\lim\limits_{\longrightarrow} V_j\big)$. This corresponds to another part of Proposition 9.3* on p100 of [3].

If $j, l \in I$ and $j \preceq l$, then $\nu_{j,l}$ induces a homomorphism

$$\nu_{j,l}^H \text{ from } H(V_j) \text{ into } H(V_l), \tag{8.14.34}$$

as modules over A, as in Section 5.2. This is the same as the restriction of $\nu_{j,l}^{Z'}$ to $H(V_j) = Z(V_j)/d_j(V_j)$, considered as a mapping into $H(V_l)$. As before, $\nu_{j,j}^H$ is the identity mapping on $H(V_j)$ for each $j \in I$, and

$$\nu_{l,r}^H \circ \nu_{j,l}^H = \nu_{j,r}^H \tag{8.14.35}$$

for every $j, l, r \in I$ with $j \preceq l \preceq r$. This means that the family of modules $H(V_j)$ and homomorphisms $\nu_{j,l}^H$ forms a direct system over $(I, \preceq)$, so that the direct limit $\lim\limits_{\longrightarrow} H(V_j)$ can be defined as a module over A in the usual way. If $l \in I$, then we get a homomorphism ν_l^H from $H(V_l)$ into $\lim\limits_{\longrightarrow} H(V_j)$, as modules over A, with

$$\nu_l^H = \nu_r^H \circ \nu_{l,r}^H \tag{8.14.36}$$

for every $r \in I$ with $l \preceq r$, as before.

Let q_j^H be the natural quotient mapping from $Z(V_j)$ onto $H(V_j)$ for each $j \in I$, which is the same as the restriction of $q_j^{Z'}$ to $Z(V_j)$, considered as a mapping into $H(V_j)$. If $j, l \in I$ and $j \preceq l$, then

$$q_l^H \circ \nu_{j,l}^Z = \nu_{j,l}^H \circ q_j^H, \tag{8.14.37}$$

as mappings from $Z(V_j)$ into $H(V_l)$. Thus the family of quotient mappings q_j^H, $j \in I$, defines a homomorphism from the direct system of $Z(V_j)$'s into the direct system of $H(V_j)$'s, $j \in I$, as in Section 3.4. This leads to a natural homomorphism

$$q^H = \varinjlim q_j^H \text{ from } \varinjlim Z(V_j) \text{ into } \varinjlim H(V_j), \tag{8.14.38}$$

as modules over A, which is characterized by the property that

$$q^H \circ \nu_l^Z = \nu_l^H \circ q_l^H \tag{8.14.39}$$

for every $l \in I$. In fact,

$$q^H\big(\varinjlim Z(V_j)\big) = \varinjlim H(V_j), \tag{8.14.40}$$

because $q_j^H(Z(V_j)) = H(V_j)$ for every $j \in I$, as in Section 3.4.

Observe that

$$\begin{aligned} &\ker q^H \text{ is the same as the image of } \varinjlim B(V_j) \\ &\text{under the homomorphism as in (8.14.17),} \end{aligned} \tag{8.14.41}$$

because $\ker q_j^H = B(V_j)$ for each $j \in I$, as in Section 3.4. Remember that the homomorphism as in (8.14.9) defines an isomorphism from $\varinjlim Z(V_j)$ onto $Z\big(\varinjlim V_j\big)$, as modules over A, as in (8.14.12) and (8.14.13). We have also seen that the composition of the homomorphisms as in (8.14.17) and (8.14.9) is the same as the homomorphism as in (8.14.18). It follows that

$$\begin{aligned} &\text{the image of } \ker q^H \text{ under the homomorphism as in (8.14.9)} \\ &\text{is equal to } B\big(\varinjlim V_j\big), \end{aligned} \tag{8.14.42}$$

because of (8.14.22).

Thus q^H corresponds to a homomorphism

$$\text{from } Z\big(\varinjlim V_j\big) \text{ onto } \varinjlim H(V_j), \tag{8.14.43}$$

as modules over A, with kernel equal to $B\big(\varinjlim V_j\big)$. This shows that

$$H\big(\varinjlim V_j\big) \text{ is isomorphic to } \varinjlim H(V_j), \tag{8.14.44}$$

as modules over A, where the natural quotient mapping from $Z\big(\varinjlim V_j\big)$ onto $H\big(\varinjlim V_j\big)$ corresponds to the homomorphism as in (8.14.43). This corresponds to part of Proposition 9.3* on p100 of [3] again.

Let us mention the analogous statements for $B'(V_j) = V_j / \ker d_j$, for the sake of completeness. If $j, l \in I$ and $j \preceq l$, then $\nu_{j,l}$ induces a homomorphism

$$\nu_{j,l}^{B'} \text{ from } B'(V_j) \text{ into } B'(V_l), \tag{8.14.45}$$

as modules over A, as in Section 5.2. This is the identity mapping on $B'(V_j)$ when $j = l$, and

$$\nu_{l,r}^{B'} \circ \nu_{j,l}^{B'} = \nu_{j,r}^{B'} \tag{8.14.46}$$

for every $j, l, r \in I$ with $j \preceq l \preceq r$, by (8.14.1). Thus the family of modules $B'(V_j)$ and homomorphisms $\nu_{j,l}^{B'}$ forms a direct system over $(I, \preceq)$, so that the direct limit $\varinjlim B'(V_j)$ can be defined as a module over A. If $l \in I$, then we also get a homomorphism $\nu_l^{B'}$ from $B'(V_l)$ into $\varinjlim B'(V_j)$, as modules over A, with

$$\nu_l^{B'} = \nu_r^{B'} \circ \nu_{l,r}^{B'} \tag{8.14.47}$$

for every $r \in I$ with $l \preceq r$.

Let $q_j^{B'}$ be the natural quotient mapping from V_j onto $B'(V_j)$ for each $j \in I$. If $j, l \in I$ and $j \preceq l$, then

$$q_l^{B'} \circ \nu_{j,l} = \nu_{j,l}^{B'} \circ q_j^{B'}, \tag{8.14.48}$$

as mappings from V_j into $B'(V_l)$. It follows that the family of quotient mappings $q_j^{B'}$ defines a homomorphism from the direct system of V_j's into the direct system of $B'(V_j)$'s, $j \in I$, as in Section 3.4. This leads to a natural homomorphism

$$q^{B'} = \varinjlim q_j^{B'} \text{ from } \varinjlim V_j \text{ into } \varinjlim B'(V_j), \tag{8.14.49}$$

as modules over A, which is characterized by the property that

$$q^{B'} \circ \nu_l = \nu_l^{B'} \circ q_l^{B'} \tag{8.14.50}$$

for every $l \in I$. We also have that

$$q^{B'}\big(\varinjlim V_j\big) = \varinjlim B'(V_j), \tag{8.14.51}$$

because $q_j^{B'}(V_j) = B'(V_j)$ for every $j \in I$, as in Section 3.4.

The kernel of $q^{B'}$ is the same as the image of $\varinjlim Z(V_j)$ under the homomorphism as in (8.14.9), because $\ker q_j^{B'} = Z(V_j)$ for each $j \in I$, as in Section 3.4. This means that

$$\ker q^{B'} = Z\big(\varinjlim V_j\big), \tag{8.14.52}$$

because of (8.14.13). It follows that

$$B'\big(\varinjlim V_j\big) \text{ is isomorphic to } \varinjlim B'(V_j), \tag{8.14.53}$$

as modules over A, in such a way that $q^{B'}$ corresponds to the natural quotient mapping from $\varinjlim V_j$ onto $B'\big(\varinjlim V_j\big)$.

8.15 Direct sums and limits

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let $(I, \preceq)$ be a nonempty pre-directed set, and let B be a nonempty set. Suppose that V_j^β is a module over A for each $j \in I$ and $\beta \in B$. More precisely, these should be all left modules over A, or all right modules over A.

Suppose that for every $\beta \in B$ and $j, l \in I$ with $j \preceq l$ we have a homomorphism $\nu_{j,l}^\beta$ from V_j^β into V_l^β, as modules over A, with the usual two properties as in Section 3.2. Namely, $\nu_{j,j}^\beta$ should be the identity mapping on V_j^β for every $j \in I$ and $\beta \in B$, and we should have that

$$\nu_{l,r}^\beta \circ \nu_{j,l}^\beta = \nu_{j,r}^\beta \tag{8.15.1}$$

for every $\beta \in B$ and $j, l, r \in I$ with $j \preceq l \preceq r$. Let

$$V^\beta = \varinjlim V_j^\beta \tag{8.15.2}$$

be the corresponding direct limit of the V_j^β's, $j \in I$, for each $\beta \in B$, which is a module over A. If $\beta \in B$ and $l \in I$, then we get a homomorphism ν_l^β from V_l^β into V^β, as modules over A, as before. Remember that this homomorphism satisfies

$$\nu_l^\beta = \nu_r^\beta \circ \nu_{l,r}^\beta \tag{8.15.3}$$

when $r \in I$ and $l \preceq r$.

If $j \in I$, then put

$$V_j = \bigoplus_{\beta \in B} V_j^\beta, \tag{8.15.4}$$

which is a module over A. If $j, l \in I$ and $j \preceq l$, then let $\nu_{j,l}$ be the homomorphism from V_j into V_l, as modules over A, defined using $\nu_{j,l}^\beta$ on V_j^β for each $\beta \in B$. Thus $\nu_{j,j}$ is the identity mapping on V_j for each $j \in I$, and $\nu_{l,r} \circ \nu_{j,l} = \nu_{j,r}$ for every $j, l, r \in I$ with $j \preceq l \preceq r$, because of the analogous properties of the $\nu_{j,l}^\beta$'.

Let

$$V = \varinjlim V_j \tag{8.15.5}$$

be the direct limit of the V_j's, $j \in I$, which is a module over A. If $l \in I$, then we get a homomorphism ν_l from V_l into V, as modules over A, with $\nu_l = \nu_r \circ \nu_{l,r}$ when $r \in I$ and $l \preceq r$, as usual.

If $\alpha \in B$ and $j \in I$, then let i_j^α be the obvious mapping from V_j^α into V_j, which sends $v_j^\alpha \in V_j^\alpha$ to the element of V_j whose component in V_j^α is equal to v_j^α, and whose component in V_j^β is equal to 0 when $\alpha \neq \beta$. If $l \in I$ and $j \preceq l$, then it is easy to see that

$$i_l^\alpha \circ \nu_{j,l}^\alpha = \nu_{j,l} \circ i_j^\alpha. \tag{8.15.6}$$

Thus the family of i_j^α's, $j \in I$, defines a homomorphism from the direct system of V_j^α's into the direct system of V_j's, $j \in I$, as in Section 3.4. This means that there is a unique homomorphism $i^\alpha = \varinjlim i_j^\alpha$ from V^α into V, as modules over A, such that

$$i^\alpha \circ \nu_l^\alpha = \nu_l \circ i_l^\alpha \tag{8.15.7}$$

for every $l \in I$, as before.

Similarly, if $\alpha \in B$ and $j \in I$, then let π_j^α be the usual coordinate projection from V_j onto V_j^α. If $l \in I$ and $j \preceq l$, then

$$\pi_l^\alpha \circ \nu_{j,l} = \nu_{j,l}^\alpha \circ \pi_j^\alpha, \tag{8.15.8}$$

by construction. This means that the family of π_j^α's, $j \in I$, defines a homomorphism from the direct system of V_j's into the direct system of V_j^α's, $j \in I$, as in Section 3.4 again. It follows that there is a unique homomorphism $\pi^\alpha = \varinjlim \pi_j^\alpha$ from V into V^α, as modules over A, such that

$$\pi^\alpha \circ \nu_l = \nu_l^\alpha \circ \pi_l^\alpha \tag{8.15.9}$$

for every $l \in I$.

Of course,

$$\pi_j^\alpha \circ i_j^\alpha \text{ is the identity mapping on } V_j^\alpha \tag{8.15.10}$$

for every $\alpha \in B$ and $j \in I$. If $\alpha, \gamma \in B$ and $\alpha \neq \gamma$, then

$$\pi_j^\gamma \circ i_j^\alpha = 0 \tag{8.15.11}$$

for every $j \in I$. This implies that

$$\pi^\alpha \circ i^\alpha \text{ is the identity mapping on } V^\alpha \tag{8.15.12}$$

for every $\alpha \in B$, and that

$$\pi^\gamma \circ i^\alpha = 0 \tag{8.15.13}$$

for every $\alpha, \gamma \in B$ with $\alpha \neq \gamma$, as in Section 3.4.

Every $v \in V$ can be expressed as $v = \nu_l(v_l)$ for some $l \in I$ and $v_l \in V_l$, as in Section 3.2. In this case,

$$\pi^\alpha(v) = \pi^\alpha(\nu_l(v_l)) = \nu_l^\alpha(\pi_l^\alpha(v_l)) \tag{8.15.14}$$

for every $\alpha \in B$. In particular,

$$\pi^\alpha(v) = 0 \text{ for all but finitely many } \alpha \in B, \tag{8.15.15}$$

because $\pi_l^\alpha(v_l) = 0$ for all but finitely many $\alpha \in B$.

If $\nu_l^\alpha(\pi_l^\alpha(v_l)) = 0$ for some $\alpha \in B$, then there is an $r(\alpha) \in I$ such that $l \preceq r(\alpha)$ and

$$\nu_{l,r(\alpha)}^\alpha(\pi_l^\alpha(v_l)) = 0, \tag{8.15.16}$$

as in Section 3.2. If $\nu_l^\alpha(\pi_l^\alpha(v_l)) = 0$ for every $\alpha \in B$, then it follows that there is an $r \in I$ such that $l \preceq r$ and

$$\nu_{l,r}^\alpha(\pi_l^\alpha(v_l)) = 0 \tag{8.15.17}$$

for every $\alpha \in B$, because $(I, \preceq)$ is a pre-directed system, and this holds automatically when $\pi_l^\alpha(v_l) = 0$. Equivalently, this means that

$$\pi_r^\alpha(\nu_{l,r}(v_l)) = 0 \tag{8.15.18}$$

for every $\alpha \in B$, so that $\nu_{l,r}(v_l) = 0$. This implies that

$$v = \nu_l(v_l) = \nu_r(\nu_{l,r}(v_l)) = 0. \tag{8.15.19}$$

Using the i^α's, we get a homomorphism

$$\text{from } \bigoplus_{\beta \in B} V^\beta \text{ into } V, \tag{8.15.20}$$

as modules over A. Similarly, we get a homomorphism

$$\text{from } V \text{ into } \bigoplus_{\beta \in B} V^\beta, \tag{8.15.21}$$

as modules over A, using the π^α's and (8.15.15). The composiiton of these two homomorphisms is the identity mapping on $\bigoplus_{\beta \in B} V^\beta$, because of (8.15.12) and (8.15.13), as in Section 1.15. We also have that the homomorphism as in (8.15.21) is injective, by the remarks in the previous two paragraphs. One can use this to get that the homomorphism as in (8.15.20) is surjective, so that these two homomorphisms are inverses of each other.

Alternatively, one can check that $\bigoplus_{\beta \in B} V^\beta$ satisfies conditions that characterize the direct limit of the V_j's, $j \in I$, up to isomorphism, as in Section 3.2. More precisely, for each $l \in I$, we can get a homomorphism $\widetilde{\nu}_l$ from V_l into $\bigoplus_{\beta \in B} V^\beta$, as modules over A, using the homomorphisms ν_l^β from V_l^β into V^β for each $\beta \in B$. It is easy to see that

$$\widetilde{\nu}_l = \widetilde{\nu}_r \circ \nu_{l,r} \tag{8.15.22}$$

for every $l, r \in I$ with $l \preceq r$, using (8.15.3). The $\widetilde{\nu}_l$'s are the analogues of the ν_l's for $\bigoplus_{\beta \in B} V^\beta$, as a version of the direct limit of the V_l's. We would like to verify that $\bigoplus_{\beta \in B} V^\beta$ has the same property as the direct limit, in terms of homomorphisms from the V_l's into other modules over A that satisfy suitable compatibility conditions.

Let Z be another left or right module over A, as appropriate, and let ζ_l be a homomorphism from V_l into Z, as modules over A, for each $l \in I$. Suppose that

$$\zeta_l = \zeta_r \circ \nu_{l,r} \tag{8.15.23}$$

for every $l, r \in I$ with $l \preceq r$. If $l \in I$, then ζ_l determines a homomorphism ζ_l^β from V_l^β into Z, as modules over A, for each $\beta \in B$. We also have that

$$\zeta_l^\beta = \zeta_r^\beta \circ \nu_{l,r}^\beta \tag{8.15.24}$$

for every $l, r \in I$ with $l \preceq r$ and every $\beta \in B$, because of (8.15.23), and the way that $\nu_{l,r}$ was defined. This implies that for each $\beta \in B$, there is a unique homomorphism ζ^β from V^β into Z, as modules over A, such that

$$\zeta^\beta \circ \nu_l^\beta = \zeta_l^\beta \tag{8.15.25}$$

for every $l \in I$, as in Section 3.2.

Let $\widetilde{\zeta}$ be the homomorphism from $\bigoplus_{\beta \in B} V^\beta$ into Z obtained from the ζ^β's, $\beta \in B$, in the obvious way. Observe that

$$\widetilde{\zeta} \circ \widetilde{\nu}_l = \zeta_l \tag{8.15.26}$$

for every $l \in I$, by construction. It is easy to see that $\widetilde{\zeta}$ is uniquely determined by this property, because ζ^β is uniquely determined by (8.15.25) for each $\beta \in B$. This implies that $\bigoplus_{\beta \in B} V^\beta$ is isomorphic to V, as modules over A, as in Section 3.2. Of course, $\widetilde{\nu}_l$ corresponds to ν_l under this isomorphism, for each $l \in I$.

Let us now take $B = \mathbf{Z}$, so that V_j may be considered as a graded module over A for each $j \in I$. More precisely, $\nu_{j,l}$ has degree 0 as a homomorphism from V_j into V_l for every $j, l \in I$ with $j \preceq l$, by construction. We may consider V as a graded module over A too, as in the preceding paragraph. If $l \in I$, then ν_l has degree 0 as a homomorphism from V_l into V, with respect to this grading on V. It is easy to see that this grading on V is uniquely determined by this property.

Suppose that V_j is in fact a graded module over A with differentiation that is a complex for each $j \in I$, and that the family of V_j's is a direct system of modules with differentiation over $(I, \preceq)$, as in the previous section. This means that $\nu_{j,l}$ is a homomorphism from V_j into V_l as modules over A with differentiation for each $j, l \in I$ with $j \preceq l$, as before. The direct limit V may be considered as a module over A with differentiation, as in the previous section. One can check that V is a complex, using the grading defined here. This means that the differentiation operator d on V has degree 1, which can be verified using the fact that the differentiation operator d_j on V_j has degree 1 for each $j \in I$, by hypothesis.

Part III

More rings, modules, and complexes

Chapter 9

Some rings and modules

9.1 A reformulation of projectivity

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that a left module V over A has the following property. Let W_1 and Z_1 be left modules over A, let ψ_1 be a homomorphism from W_1 onto Z_1, and let ϕ_{Z_1} be a homomorphism from V into Z_1, as left modules over A. If W_1 is injective as a left module over A, then there is a homomorphism ϕ_{W_1} from V into W_1, as left modules over A, such that

(9.1.1) $$\phi_{Z_1} = \psi_1 \circ \phi_{W_1}.$$

This is the same as the definition of projectivity of V, as in Section 2.7, restricted to the case where W_1 is injective.

Proposition 5.1 on p12 of [3] states that this condition implies that V is projective. To see this, let W and Z be left modules over A, let ψ be a homomorphism from W onto Z, and let ϕ_Z be a homomorphism from V into Z, as left modules over A. Thus

(9.1.2) $$W_0 = \ker \psi$$

is a submodule of W, and we may as well take

(9.1.3) $$Z = W/W_0,$$

and ψ to be the natural quotient mapping from W onto W/W_0. We may suppose too that W is a submodule of an injective left module W_1 over A, as mentioned in Section 2.8. Put

(9.1.4) $$Z_1 = W_1/W_0,$$

and let ψ_1 be the natural quotient mapping from W_1 onto Z_1.

Of course, Z may be considered as a submodule of Z_1, and so we may take

(9.1.5) $$\phi_{Z_1} = \phi_Z,$$

considered as a homomorphism from V into Z_1. By hypothesis, there is a homomorphism ϕ_{W_1} from V into W_1 that satisfies (9.1.1). Note that

$$\psi_1(\phi_{W_1}(V)) = \phi_{Z_1}(V) = \phi_Z(V) \subseteq Z, \tag{9.1.6}$$

which implies that

$$\phi_{W_1}(V) \subseteq W. \tag{9.1.7}$$

Put

$$\phi_W = \phi_{W_1}, \tag{9.1.8}$$

considered as a mapping from V into W. Clearly $\phi_Z = \psi \circ \phi_W$, because of (9.1.1), as desired.

There are analogous statements for right modules over A, as usual.

9.2 A reformulation of injectivity

Let k be a commutative ring with a multiplicative identity element again, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that a left module W over A has the following property. Let V_1 be a left module over A, let $V_{1,0}$ be a submodule of V_1, and let $\phi_{1,0}$ be a homomorphism from $V_{1,0}$ into W, as modules over A. If V_1 is projective as a left module over A, then we ask that there be a homomorphism ϕ_1 from V_1 into W, as left modules over A, such that

$$\phi_1 = \phi_{1,0} \text{ on } V_{1,0}. \tag{9.2.1}$$

This is the same as the injectivity of W, as in Section 2.8, with the additional hypotheses that V_1 be projective.

Proposition 5.2 on p12 of [3] states that this condition implies that W is injective. To show this, let V be a left module over A, let V_0, be a submodule of V, and let ϕ_0 be a homomorphism from V_0 into W, as left modules over A. Remember that there is a projective left module V_1 over A and a homomorphism ξ from V_1 onto V, as left modules over A, as in Section 2.7. More precisely, one can take V_1 to be a free left module over A. Of course,

$$V_{1,0} = \xi^{-1}(V_0) \tag{9.2.2}$$

is a submodule of V_1.

Let ξ_0 be the restriction of ξ to $V_{1,0}$, and put

$$\phi_{1,0} = \phi_0 \circ \xi_0. \tag{9.2.3}$$

This is a homomorphism from $V_{1,0}$ into W, as left modules over A, which can be extended to a homomorphism ϕ_1 from V_1 into W, as left modules over A, by hypothesis. Observe that the kernel of ξ is contained in $V_{1,0}$, by construction, so that

$$\ker \xi = \ker \xi_0 \subseteq \ker \phi_{1,0} \subseteq \ker \phi_1. \tag{9.2.4}$$

This implies that there is a unique homomorphism ϕ from V into W, as left modules over A, such that

$$\phi_1 = \phi \circ \xi. \tag{9.2.5}$$

It is easy to see that $\phi = \phi_0$ on V_0, because of the analogous property of ϕ_1.

Of course, there are analogous statements for right modules over A.

9.3 Hereditary rings

Let A be a ring, with a nonzero multiplicative identity element e_A. We say that A is *left hereditary* if every left ideal in A is projective, as a left module over A, as on p13 of [3]. The definition of what it means for A to be *right hereditary* can be defined analogously, as mentioned on p15 of [3].

If $x \in A$, then

$$A\,x = \{a\,x : a \in A\} \tag{9.3.1}$$

is a left ideal in A. Of course,

$$a \mapsto a\,x \tag{9.3.2}$$

defines a homomorphism from A onto $A\,x$, as left modules over A. Suppose for the moment that A has no nontrivial zero divisors, so that the product of any two nonzero elements of A is nonzero as well. If $x \neq 0$, then (9.3.2) defines an isomorphism from A onto $A\,x$, as left modules over A. In particular, this means that $A\,x$ is free as a left ideal over A, and thus projective.

If A has no nontrivial zero divisors, and every left ideal in A is of the form $A\,x$ for some $x \in A$, then it follows that A is left hereditary. This corresponds to a remark at the bottom of p13 of [3].

Let V be a free left module over A, and let V_0 be a submodule of V. If A is left hereditary, then a theorem of Kaplansky states that

(9.3.3) V_0 is isomorphic to a direct sum of a family of left modules over A, each of which is isomorphic to a left ideal in A,

as a left module over A. This is Theorem 5.3 on p13 of [3], and we shall return to this in the next section.

Let V be a projective left module over A, so that V may be considered as a submodule of a free left module over A, as in Section 2.7. This means that every submodule of V may be considered as a submodule of a free left module over A. If A is left hereditary, then Kaplansky's theorem implies that every submodule of V is isomorphic to a direct sum of projective left modules over A. It follows that

(9.3.4) every submodule of V is projective as a left module over A

under these conditions, as in Section 2.7.

Conversely, suppose that

$$\text{every submodule of a projective left module over } A \text{ is projective as a left module over } A. \tag{9.3.5}$$

Of course, A is free as a left module over itself, and thus projective. It follows that every left ideal in A is projective as a left module over A, so that A is left hereditary. This corresponds to the equivalence of parts (a) and (b) of Theorem 5.4 on p14 of [3].

Suppose that (9.3.5) holds, and let us show that

$$\text{every quotient of an injective left module over } A \text{ is injective as a left module over } A. \tag{9.3.6}$$

Let W be an injective left module over A, and let W_0 be a submodule of W. To check that W/W_0 is injective, we let V_1 be a projective left module over A, $V_{1,0}$ be a submodule of V_1, and $\phi_{1,0}$ be a homomorphism from $V_{1,0}$ into W/W_0, as left modules over A. Using (9.3.5), we get that $V_{1,0}$ is projective as a left module over A too. This implies that there is a homomorphism ϕ_W from $V_{1,0}$ into W, as left modules over A, such that

$$\phi_{1,0} \text{ is the same as the composition of } \phi_W \text{ with the natural quotient mapping from } W \text{ onto } W/W_0. \tag{9.3.7}$$

Because W is injective, ϕ_W can be extended to a homomorphism ϕ from V_1 into W, as left modules over A. The composition of ϕ with the natural quotient mapping from W onto W_0 is a homomorphism from V into W/W_0, as left modules over A, that is equal to $\phi_{1,0}$ on $V_{1,0}$. It follows that W/W_0 is injective as a left module over A, as in the previous section.

Conversely, suppose that (9.3.6) holds, and let us show that (9.3.5) holds. Let V be a projective left module over A, and let V_0 be a submodule of V. To show that V_0 is projective, let W_1 and Z_1 be left modules over A, with W_1 injective, let ψ_1 be a homomorphism from W_1 onto Z_1, and let ϕ_{V_0,Z_1} be a homomorphism from V_0 into Z_1, as left modules over A. It follows that Z_1 is injective as a left module over A, because of (9.3.6). Thus ϕ_{V_0,Z_1} can be extended to a homomorphism ϕ_{V,Z_1} from V into Z_1, as left modules over A. Because V is projective, there is a homomorphism ϕ_{V,W_1} from V into W_1, as left modules over A, such that

$$\phi_{V,Z_1} = \psi_1 \circ \phi_{V,W_1}. \tag{9.3.8}$$

The restriction of ϕ_{V,W_1} to V_0 defines a homomorphism ϕ_{V_0,W_1} from V_0 into W_1, as left modules over A, such that

$$\phi_{V_0,Z_1} = \psi_1 \circ \phi_{V_0,W_1}. \tag{9.3.9}$$

This implies that V_0 is projective, as in Section 9.1. This corresponds to the equivalence of parts (b) and (c) in Theorem 5.4 on p14 of [3].

9.4 Kaplansky's theorem

Let A be a ring with a nonzero multiplicative identity element e_A again, and let V be a free left module over A. More precisely, suppose that $\{x_\alpha\}_{\alpha \in B}$ is a basis for V, as a free left module over A, and that B is well ordered by $\preceq$. If $\beta \in B$, then we let V_β be the submodule of V generated by x_α, with $\alpha \in B$, $\alpha \preceq \beta$, and $\alpha \neq \beta$. Similarly, let $\overline{V}_\beta$ be the submodule of V generated by x_α with $\alpha \in B$ and $\alpha \preceq \beta$.

Let U be a submodule of V. If $\beta \in B$ and $v \in U \cap \overline{V}_\beta$, then v can be expressed in a unique way as

$$v = w + a \cdot x_\beta, \tag{9.4.1}$$

where $w \in V_\beta$ and $a \in A$. Note that $v \mapsto a$ defines a homomorphism from $U \cap \overline{V}_\beta$ into A, as left modules over A. Let $\mathcal{I}_\beta$ be the image of $U \cap \overline{V}_\beta$ under this homomorphism, which is a left ideal in A. The kernel of this homomorphism is equal to

$$U \cap V_\beta. \tag{9.4.2}$$

Suppose that A is left hereditary, so that $\mathcal{I}_\beta$ is projective as a left module over A. This implies that $U \cap \overline{V}_\beta$ corresponds to the direct sum of $U \cap V_\beta$ and another submodule C_β, as left modules over A, as in Section 2.7. More precisely, C_β is isomorphic to $\mathcal{I}_\beta$, as a left module over A. We would like to show that U corresponds to the direct sum of the submodules C_β, $\beta \in B$, as a left module over A.

Let $\beta_1, \ldots, \beta_n$ be finitely many distinct elements of B, with

$$\beta_1 \preceq \beta_2 \preceq \cdots \preceq \beta_n. \tag{9.4.3}$$

Suppose that $c_j \in C_{\beta_j}$, $1 \leq j \leq n$, satisfy

$$\sum_{j=1}^{n} c_j = 0. \tag{9.4.4}$$

Note that $c_j \in U \cap \overline{V}_{\beta_j}$ for each $j = 1, \ldots, n$, so that $c_j \in U \cap V_{\beta_n}$ when $j < n$. Because $U \cap \overline{V}_{\beta_n}$ corresponds to the direct sum of $U \cap V_{\beta_n}$ and C_{β_n}, we get that

$$\sum_{j=1}^{n-1} c_j = c_n = 0. \tag{9.4.5}$$

One can repeat the process to get that $c_j = 0$ for each $j = 1, \ldots, n$.

Let $\sum_{\beta \in B} C_\beta$ be the submodule of U generated by the C_β's, $\beta \in B$. We would like to verify that

$$U = \sum_{\beta \in B} C_\beta. \tag{9.4.6}$$

Otherwise, there is an $\alpha \in B$ such that

$$U \cap \overline{V}_\alpha \not\subseteq \sum_{\beta \in B} C_\beta, \tag{9.4.7}$$

and we may as well take α to be the minimal element of B with this property. One can check that

$$U \cap V_\alpha \subseteq \sum_{\beta \in B} C_\beta, \tag{9.4.8}$$

using the minimality of α. This implies that

$$U \cap \overline{V}_\alpha = (U \cap V_\alpha) + C_\alpha \subseteq \sum_{\beta \in B} C_\beta, \tag{9.4.9}$$

which is a contradiction.

9.5 Semi-hereditary rings

Let A be a ring with a nonzero multiplicative identity element e_A. We say that A is *left semi-hereditary* if every left ideal in A that is finitely generated as a left module over A is projective, as a left module over A, as on p14 of [3]. The right-hereditary condition can be defined analogously, as mentioned on p15 of [3].

Let V be a free left module over A, and let V_0 be a finitely-generated submodule of V. If A is left semi-hereditary, then

(9.5.1) V_0 is isomorphic to a direct sum of finitely many left modules over A, each of which is isomorphic to a left ideal in A that is finitely generated as a left module over A.

More precisely, V_0 is isomorphic to such a direct sum, as a left module over A. This is Proposition 6.1 on p14 of [3].

To see this, one can first reduce to the case where V is freely generated by finitely many elements $x_1, \ldots, x_n$, as a left module over A. More precisely, it suffices to choose the x_j's so that V_0 is contained in the submodule of V generated by them.

The statement holds trivially if $n = 0$, with suitable interpretations. If $n = 1$, then it is easy to see that V_0 is isomorphic to a left ideal in A, as a left module over A. Otherwise, we can use induction, as follows.

Let W be the submodule of V generated by $x_1, \ldots, x_{n-1}$, which is interpreted as being $\{0\}$ when $n = 1$. Thus every $v \in V$ can be expressed in a unique way as

$$v = w + a \cdot x_n, \tag{9.5.2}$$

with $w \in W$ and $a \in A$. Of course, $v \mapsto a$ defines a homomorphism from V onto A, as left modules over A. This homomorphism maps V_0 onto a left ideal

$\mathcal{I}_0$ in A, and the kernel of the restriction of this homomorphism to V_0 is equal to

$$V_0 \cap W. \tag{9.5.3}$$

Note that $\mathcal{I}_0$ is finitely generated as a left module over A, because V_0 is finitely generated, by hypothesis.

If A is left semi-hereditary, then it follows that $\mathcal{I}_0$ is projective, as a left module over A. This implies that V_0 corresponds to the direct sum of $V_0 \cap W$ and another submodule C_0, as left modules over A, as in Section 2.7. We also have that C_0 is isomorphic to $\mathcal{I}_0$, as a left module over A.

Observe that $V_0 \cap W$ is finitely generated as a left module over A, because there is a homomorphism from V_0 onto $V_0 \cap W$. This permits us to use induction, to get that $V_0 \cap W$ is isomorphic to a direct sum of finitely many left modules over A, each of which is isomorphic to a left ideal in A that is finitely generated as a left module over A. It follows that V_0 has the same property, as desired.

Proposition 6.2 on p15 of [3] states that A is left semi-hereditary if and only if

(9.5.4) every finitely-generated submodule of a projective left module over A is projective as a left module over A.

Indeed, any projective left module over A may be considered as a submodule of a free left module over A, as in Section 2.7. Thus the "only if" part follows from (9.5.1), and the fact that direct sums of projective modules are projective. The "if" part follows from the fact that A is free as a left module over itself, and thus projective.

9.6 Hereditary rings and differentiation

Let A be a ring with a nonzero multiplicative identity element, and let (V, d_V) be a left or right module over A with differentiation. Suppose that A is left or right hereditary, as appropriate, and that V is projective as a module over A. This implies that

(9.6.1) $B(V) = d_V(V)$ is projective as a module over A,

as in Section 9.3, because $B(V)$ is a submodule of V. It follows that V corresponds to the direct sum of $Z(V) = \ker d_V$ and another submodule of V, as a module over A, as in Section 2.7.

Note that

(9.6.2) $B'(V) = V/Z(V) = V/\ker d_V$ is projective as a module over A,

because it is isomorphic to $B(V)$. Remember that there is a natural homomorphism from $Z'(V) = V/d_V(V)$ onto $B'(V)$, because $D_V(V) \subseteq \ker d_V$, as in Section 5.1. One can use this to get that $Z'(V)$ corresponds to the direct sum of $H(V) = Z(V)/B(V)$ and another submodule of $Z'(V)$, as a module over A, as in Section 2.7. Of course, this could also be obtained by expressing V as a direct sum of $Z(V)$ and another submodule, as in the preceding paragraph.

Suppose that (V, d_V) is a graded module over A with differentiation that is a complex. In this case, V is projective as a module over A if and only if V^j is projective as a module over A for every $j \in \mathbf{Z}$, as in Section 2.7. It is easy to see that V corresponds to the direct sum of $Z(V)$ and another homogeneous submodule of V, as a module over A, under these conditions. We also have that $Z'(V)$ corresponds to the direct sum of $H(V)$ and another homogeneous submodule of $Z'(V)$, as a module over A, as before. This is related to part of the proof of Theorem 3.2 on p113 of [3].

Let (V, d_V) be a left or right module over A with differentiation again, and suppose now that V is injective as a module over A. This implies that

(9.6.3) $B(V)$ is injective as a module over A,

as in Section 9.3. It follows that V corresponds to the direct sum of $B(V)$ and another submodule of V, as a module over A, as in Section 2.8. We also get that $Z(V)$ corresponds to the direct sum of $B(V)$ and another submodule of $Z(V)$, as a module over A.

Suppose that (V, d_V) is a graded module over A with differentiation that is a complex, and that V^j is injective as a module over A for every $j \in \mathbf{Z}$. This implies that

(9.6.4) $B(V)^j$ is injective as a module over A for every $j \in \mathbf{Z}$,

as before. It follows that V^j corresponds to the direct sum of $B(V)^j$ and another submodule of V^j, as a module over A, for every $j \in \mathbf{Z}$. We also get that $Z(V)^j$ corresponds to the direct sum of $B(V)^j$ and another submodule of $Z(V)^j$, as a module over A, for every $j \in \mathbf{Z}$. This is related to Theorem 3.2a on p114 of [3].

9.7 Noetherian conditions

Let A be a ring with a multiplicative identity element e_A, and let V be a left or right module over A. We say that V is *Noetherian* if

(9.7.1) every submodule of V is finitely generated, as a module over A,

as on p15 of [3]. The Noetherian condition is often defined in terms of the ascending chain condition for the collection of submodules of V, as a partially-ordered set with respect to inclusion, which says that any monotonically-increasing sequence of submodules of V with respect to inclusion is eventually constant. Another formulation asks that any nonempty collection of submodules of V have a maximal element. The equivalence of these conditions can be shown using standard arguments, as in Propositions 6.1 and 6.2 on p74f of [1].

Let V_0 be a submodule of V. If V_0 and the corresponding quotient module V/V_0 are finitely generated as modules over A, then one can check that V is finitely generated as a module over A, as in Exercise 9 on p32 of [1], and the first part of Exercise 2 on p16 of [3]. If V is finitely generated as a module over A, then it is easy to see that V/V_0 is finitely generated too. Of course, if V is Noetherian, then V_0 is finitely generated as well, as in the second part of Exercise 2 on p16 of [3].

One can check that V is Noetherian if and only if

$$V_0 \text{ and } V/V_0 \text{ are Noetherian,} \tag{9.7.2}$$

as in part (i) of Proposition 6.3 on p75 of [1]. The argument given there uses the ascending chain condition, and one could also verify this using the definition in terms of finitely-generated submodules.

Let $V_1, \ldots, V_n$ be finitely many Noetherian modules over A, which are all left modules over A, or all right modules over A. Under these conditions, their direct sum

$$\bigoplus_{j=1}^{n} V_j \text{ is Noetherian as a module over } A, \tag{9.7.3}$$

as in Corollary 6.4 on p76 of [1]. This can be seen using induction on n, and the statement in the preceding paragraph.

We say that A is *left or right Noetherian* as a ring if A is Noetherian as a left or right module over itself, as appropriate, as on p76 of [1], and p15 of [3]. This means that every left or right ideal in A is finitely generated as a module over A.

In this case, if V is a finitely-generated left or right module over A, as appropriate, then

$$V \text{ is Noetherian as a module over } A, \tag{9.7.4}$$

as in Proposition 6.5 on p76 of [1], and Proposition 7.1 on p15 of [3]. In the proof in [1], one expresses V as a quotient of A^n for some $n \geq 1$, and one uses the results mentioned in the previous two paragraphs. The proof in [3] uses induction on the number of generators of V, and basically involves analogous arguments.

An example is given on p16 of [3] of a ring that is left Noetherian and not right Noetherian, and which is due to J. Dieudonné. Let k be a commutative ring with a nonzero multiplicative identity element, which one can take to be $\mathbf{Z}$ for the example. The example may be considered as an associative algebra A over k with a nonzero multiplicative identity element e_A, which is generated by e_A and two additional nonzero elements x, y, as an algebra over k. More precisely, x and y are supposed to satisfy

$$y\,x = y\,y = 0, \tag{9.7.5}$$

and no other relations.

One might as well take x to be an indeterminate, so that the subalgebra of A generated by e_A and x may be identified with the usual algebra $k[x]$ of formal polynomials in x with coefficients in k. Every element of A can be expressed in a unique way as an element of $k[x]$ plus an element of $k[x]$ times y, so that A corresponds to the direct sum of $k[x]$ and $(k[x])\, y$, as a module over k. If k is a Noetherian ring, then $k[x]$ is a Noetherian ring, by Hilbert's Basis Theorem, which is Theorem 7.5 on p81 of [1]. In particular, this holds when $k = \mathbf{Z}$, or k is a field.

If we consider A as a module over $k[x]$ with respect to multiplication by elements of $k[x]$ on the left, then A is isomorphic to the direct sum of two copies of $k[x]$. If k is a Noetherian ring, so that $k[x]$ is a Noetherian ring, then it follows that A is Noetherian as a module over $k[x]$, as before. In this case, it is easy to see that A is Noetherian as a left module over itself, because every submodule of A as a left module over itself is also a submodule of A as a module over $k[x]$ with respect to multiplication by elements of $k[x]$ on the left. This means that A is left Noetherian as a ring under these conditions.

Observe that $(k[x])\, y$ is a right ideal in A, because of the relations (9.7.5). Using these relations, we also get that any set of generators for $(k[x])\, y$ as a right module over A generates $(k[x])\, y$ as a module over k. In particular, $(k[x])\, y$ is not finitely generated as a right module over A, because $(k[x])\, y$ is not finitely generated as a module over k. This means that A is not Noetherian as a right module over itself. Equivalently, A is not right Noetherian as a ring.

9.8 A criterion for finite generation

Let A be a ring with multiplicative identity element e_A, and let V be a left or right module over A. Also let V_1 and V_2 be submodules of V, so that $V_1 + V_2$ and $V_1 \cap V_2$ are submodules of V as well. If $V_1 + V_2$ and $V_1 \cap V_2$ are finitely generated as modules over A, then V_1 and V_2 are finitely generated as modules over A. This is Exercise 4 on p16 of [3].

This is a bit simpler when

$$V_1 \cap V_2 = \{0\}. \tag{9.8.1}$$

In this case, $V_1 + V_2$ corresponds to the direct sum of V_1 and V_2, as modules over A. In particular, this leads to homomorphisms from $V_1 + V_2$ onto V_1 and V_2, as modules over A. If $w_1, \ldots, w_n$ are finitely many elements of $V_1 + V_2$ that generate $V_1 + V_2$ as a module over A, then their images in V_1 and V_2 generate these modules.

Otherwise, consider the quotients

$$(V_1 + V_2)/V_1 \tag{9.8.2}$$

and

$$(V_1 + V_2)/V_2, \tag{9.8.3}$$

which are modules over A. More precisely, these are finitely-generated modules over A, because $V_1 + V_2$ is finitely generated. Let q_1, q_2 be the natural quotient mappings from $V_1 + V_2$ onto (9.8.2), (9.8.3), respectively. Observe that

$$q_1(V_2) = (V_1 + V_2)/V_1, \tag{9.8.4}$$

and that the kernel of the restriction of q_1 to V_2 is $V_1 \cap V_2$. It follows that V_2 is finitely generated as a module over A, because (9.8.2) and $V_1 \cap V_2$ are finitely generated, as in the previous section. Similarly, one can get that V_1 is finitely generated as a module over A, using the restriction of q_2 to V_1. This is related to the diagram in Exercise 1 on p16 of [3].

Alternatively, consider the quotient

$$(V_1 + V_2)/(V_1 \cap V_2). \tag{9.8.5}$$

As before, this is a finitely-generated module over A, because $V_1 + V_2$ is finitely generated. Let $q_{1,2}$ be the natural quotient mapping from $V_1 + V_2$ onto (9.8.5). Of course,

$$q_{1,2}(V_1) + q_{1,2}(V_2) = q_{1,2}(V_1 + V_2) = (V_1 + V_2)/(V_1 \cap V_2). \tag{9.8.6}$$

Let us check that

$$q_{1,2}(V_1) \cap q_{1,2}(V_2) = \{0\}. \tag{9.8.7}$$

If $v_1 \in V_1$ and $v_2 \in V_2$ satisfy $q_{1,2}(v_1) = q_{1,2}(v_2)$, then

$$v_1 - v_2 \in V_1 \cap V_2. \tag{9.8.8}$$

This implies that $v_1 \in V_2$ and $v_2 \in V_1$, so that $v_1, v_2 \in V_1 \cap V_2$, and thus $q_{1,2}(v_1) = q_{1,2}(v_2) = 0$, as desired.

This shows that (9.8.5) corresponds to the direct sum of $q_{1,2}(V_1)$ and $q_{1,2}(V_2)$, as a module over A. It follows that $q_{1,2}(V_1)$ and $q_{1,2}(V)$ are finitely generated as modules over A, because (9.8.5) is finitely generated as a module over A, as before. One can use this to get that V_1 and V_2 are finitely generated, because $V_1 \cap V_2$ is finitely generated, as in the previous section.

9.9 Direct sums and injective modules

Let A be a ring with multiplicative identity element e_A, let B be a nonempty set, and let V_β be a left module over A for each $\beta \in B$. If V_β is an injective module over A for each $\beta \in B$, and A is left Noetherian, then

$$\bigoplus_{\beta \in B} V_\beta \text{ is injective as a left module over } A. \tag{9.9.1}$$

Of course, there is an analogous statement for right modules. Remember that direct products of injective modules are injective, as in Section 2.8, without additional conditions on A. Of course, if B has only finitely many elements, then $\bigoplus_{\beta \in B} V^\beta$ is the same as the direct product.

To show (9.9.1), let $\mathcal{I}$ be a left ideal in A, and let ϕ be a homomorphism from $\mathcal{I}$ into $\bigoplus_{\beta \in B} V_\beta$, as left modules over A. It suffices to find

$$v \in \bigoplus_{\beta \in B} V_\beta \tag{9.9.2}$$

such that

$$\phi(a) = a \cdot v \tag{9.9.3}$$

for every $a \in \mathcal{I}$, as in Section 2.8. Let $\phi_\beta(a)$ be the component of $\phi(a)$ in V_β for each $a \in \mathcal{I}$ and $\beta \in B$. Thus ϕ_β defines a homomorphism from $\mathcal{I}$ into V_β, as left modules over A, for each $\beta \in B$. Note that (9.9.3) is the same as saying that

$$\phi_\beta(a) = a \cdot v_\beta \tag{9.9.4}$$

for every $a \in \mathcal{I}$ and $\beta \in B$, where v_β is the component of v in V_β.

If A is left Noetherian, then there are finitely many elements $x_1, \ldots, x_n$ of $\mathcal{I}$ such that $\mathcal{I}$ is generated by $x_1, \ldots, x_n$, as a left module over A. Put

$$B_j = \{\beta \in B : \phi_\beta(x_j) \neq 0\} \tag{9.9.5}$$

for each $j = 1, \ldots, n$, which is a finite subset of B, because $\phi(x_j) \in \bigoplus_{\beta \in B} V_\beta$. Thus

$$\bigcup_{j=1}^n B_j \tag{9.9.6}$$

is a finite subset of B too. If $a \in \mathcal{I}$, then it is easy to see that

$$\phi_\beta(a) = 0 \text{ when } \beta \in B \setminus \Big(\bigcup_{j=1}^n B_j\Big), \tag{9.9.7}$$

because $\mathcal{I}$ is generated by $x_1, \ldots, x_n$, as a left module over A. Equivalently, $\phi_\beta = 0$ when $\beta \in B \setminus \left(\bigcup_{j=1}^n B_j\right)$.

If $\beta \in \bigcup_{j=1}^n B_j$, then let v_β be an element of V_β such that (9.9.4) holds for every $a \in \mathcal{I}$. This uses the injectivity of V_β as a left module over A, as in Section 2.8. If $\beta \in B \setminus \left(\bigcup_{j=1}^n B_j\right)$, then we can take $v_\beta = 0$. This defines an element v of $\bigoplus_{\beta \in B} V_\beta$, whose component in V_β is equal to v_β for every $\beta \in B$. Under these conditions, (9.9.3) holds for every $a \in \mathcal{I}$, as desired.

Note that if (9.9.1) holds, then V_α is injective as a left module over A for every $\alpha \in B$. This can be seen using the same argument as for direct products in Section 2.8.

9.10 Some finitely-generated modules

Let k be a commutative ring with a multiplicative identity element, and let V, W be modules over k. If V and W are finitely generated as modules over k, then $V \bigotimes_k W$ is finitely generated as a module over k. This is the first part of Exercise 6 on p32 of [3]. More precisely, suppose that V, W are generated by subsets E_V, E_W, respectively, as modules over k. Under these conditions, $V \bigotimes_k W$ is generated by

$$\big\{v \otimes w : v \in E_V,\, w \in E_W\big\}, \tag{9.10.1}$$

as a module over k.

Let $v_1, \ldots, v_n$ be finitely many elements of V for some positive integer n, and let W^n be the usual space of n-tuples of elements of W. Thus W^n is a module over k, with respect to coordinatewise addition and scalar multiplication. If ϕ is a homomorphism from V into W, as modules over k, then put

$$L(\phi) = (\phi(v_1), \ldots, \phi(v_n)), \tag{9.10.2}$$

which is an element of W^n. This defines a homomorphism from $\mathrm{Hom}_k(V, W)$ into W^n, as modules over k. If V is freely generated by $v_1, \ldots, v_n$, as a module over k, then L is an isomorphism from $\mathrm{Hom}_k(V, W)$ onto W^n, as modules over k.

Suppose that W is finitely generated, as a module over k. This implies that W^n is finitely generated too, as a module over k. Suppose that k is also Noetherian as a ring. It follows that W^n is Noetherian as a module over k, as in Section 9.7. In particular, this means that

$$L\big(\mathrm{Hom}_k(V, W)\big) \text{ is finitely generated, as a module over } k. \tag{9.10.3}$$

If V is generated by $v_1, \ldots, v_n$, as a module over k, then it is easy to see that L is injective, as a mapping from $\mathrm{Hom}_k(V, W)$ into W^n. In this case, (9.10.3) implies that

$$\mathrm{Hom}_k(V, W) \text{ is finitely generated, as a module over } k. \tag{9.10.4}$$

Thus (9.10.4) holds when V and W are finitely generated as modules over k, and k is Noetherian. This is the second part of Exercise 6 on p32 of [3]. Of course, (9.10.4) also holds when V is freely generated by finitely many elements, and W is finitely generated, as modules over k.

If V is finitely generated as a module over k, then there is a homomorphism from a module X over k onto V, where X is freely generated by finitely many elements. If V is projective as a module over k, then X corresponds to the direct sum of two submodules, one of which is isomorphic to V, as in Section 2.7. This implies that $\mathrm{Hom}_k(X, W)$ is isomorphic to the direct sum of the corresponding spaces of homomorphisms from each of the two submodules just mentioned into W. In particular, $\mathrm{Hom}_k(X, W)$ is isomorphic to the direct sum of $\mathrm{Hom}_k(V, W)$ and another module over k. If W is finitely generated as a module over k, then $\mathrm{Hom}_k(X, W)$ is finitely generated as a module over k, as before. In this case, it follows that $\mathrm{Hom}_k(V, W)$ is finitely generated as a module over k. This means that (9.10.4) holds when V and W are finitely generated modules over k, and V is projective.

9.11 Some remarks about division rings

Let A be a ring, with a nonzero multiplicative identity element e_A. In this section, we suppose that A is a *division ring*, so that every nonzero element of A has a multiplicative inverse in A. This implies that A and $\{0\}$ are the only left or right ideals in A. In particular, A is left and right hereditary, and left and right Noetherian.

Of course, if A is commutative, then A is a field. Sometimes the term "field" is used for division rings that may not be commutative, but here we mean a commutative division ring. We may also refer to a *division algebra* over a field k, which is a division ring that is an algebra over k as well.

A left or right module over A is sometimes called a *left or right vector space* over A, as appropriate. Many of the usual properties of vector spaces over fields work for left and right vector spaces over division rings too, suitably interpreted. Let us consider a left module V over A, the corresponding statements for right modules being analogous.

A collection $v_1, \ldots, v_n$ of finitely many elements of V is said to be *linearly dependent* if there are elements $a_1, \ldots, a_n$ of A, not all equal to 0, such that

$$\sum_{j=1}^{n} a_j \cdot v_j = 0. \tag{9.11.1}$$

In this case, it is easy to see that a smaller collection of elements of V will generate the same submodule of V, as a left module over A.

A subset E of V is said to be *linearly independent* if there are no nonempty finite subsets of E that are linearly dependent. This means that the submodule of V generated by E is freely generated by E.

Let E_0 be a linearly independent subset of V. It is well known that there is a linearly independent subset E of V such that

$$E_0 \subseteq E, \tag{9.11.2}$$

and E is maximal with respect to inclusion. This can be shown using Zorn's lemma, or Hausdorff's maximality principle. One can use maximality of E to get that V is generated by E, as a left module over A. This implies that V is freely generated by E, as a left module over A, as before.

If V is generated by E_0 and finitely or countably many other elements of V, as a left module over A, then one can get a linearly independent set $E \subseteq V$ such that (9.11.2) holds and V is generated by E more directly. More precisely, one can go through the list of new elements and drop those that are in the submodule of V generated by E_0 and the previous elements in the list.

In particular, this implies that any left module over A is a free module over A. If V_1 is any submodule of V, as a left module, then one can use this to get a linearly independent set $E_1 \subseteq V_1$ such that V_1 is generated by E_1, as a left module over A. This can be extended to a linearly independent set $E \subseteq V$ that generates V as a left module over A, as before. Let V_2 be the submodule of V generated by $E_2 = E \setminus E_1$. It is easy to see that

$$V \text{ corresponds to the direct sum of } V_1 \text{ and } V_2, \tag{9.11.3}$$

as a left module over A, under these conditions.

Every left module over A is injective as a left module over A too. This follows easily from the characterization of injectivity in Section 2.8, because A and $\{0\}$ are the only left ideals in A. Alternatively, this can be obtained from the remarks in the preceding paragraph.

9.12 Polynomials and division rings

Let A be a division ring, let T be an indeterminate, and let $A[T]$ be the corresponding ring of formal polynomials in T with coefficients in A, as in Section 4.3. If $f(T) = \sum_{j=0}^{n} f_j\, T^j \in A[T]$ and $f_n \neq 0$, then the degree $\deg f(T)$ of $f(T)$ is defined to be n, as usual. If $f(T) = 0$, then the degree of $f(T)$ may be defined to be $-\infty$. It is easy to see that the degree of the product of two elements of $A[T]$ is equal to the sum of their degrees.

Let $a(T) \in A[T]$ be given, with $a(T) \neq 0$. If $f(T) \in A[T]$, then there are $b(T), r(T) \in A[T]$ such that

$$f(T) = b(T)\, a(T) + r(T) \tag{9.12.1}$$

and $\deg r(T) < \deg a(T)$. This can be seen in essentially the same way as when A is commutative. Of course, if $\deg f(T) < \deg a(T)$, then one can simply take $b(T) = 0$.

Otherwise, one can find $c \in A$ and a nonnegative integer j such that

$$\deg(f(T) - (c\, T^j)\, a(T)) < \deg f(T). \tag{9.12.2}$$

One can repeat the process until the degree of the remainder is strictly less than $\deg a(T)$. The analogous statement with $a(T)$ multiplied on the right by an element of $A[T]$ can be shown in the same way.

Let $\mathcal{I}$ be a left ideal in $A[T]$, and let $a(T)$ be a nonzero element of $\mathcal{I}$ of minimal degree. Under these conditions,

$$\mathcal{I} = \{b(T)\, a(T) : b(T) \in A[T]\}. \tag{9.12.3}$$

Clearly the right side of (9.12.3) is contained in $\mathcal{I}$, because $a(T) \in \mathcal{I}$. If $f(T) \in \mathcal{I}$, then $f(T)$ can be expressed as in (9.12.1). In this case, $r(T) \in \mathcal{I}$, because $\mathcal{I}$ is a left ideal in $A[T]$. The minimality of the degree of $a(T)$ implies that $r(T) = 0$, so that $f(T)$ is in the right side of (9.12.3). Of course, there is an analogous statement for nonzero right ideals in $A[T]$.

Note that

$$b(T) \mapsto b(T)\, a(T) \tag{9.12.4}$$

defines a homomorphism from $A[T]$ onto $\mathcal{I}$, as left modules over $A[T]$. This homomorphism is injective, because products of nonzero elements of $A[T]$ are nonzero, as before. Thus $\mathcal{I}$ is isomorphic to $A[T]$ as a left module over $A[T]$, and in particular $\mathcal{I}$ is projective as a left module over $A[T]$. This implies that $A[T]$ is left hereditary as a ring. Similarly, $A[T]$ is right hereditary as a ring.

Let $A[[T]]$ be the ring of formal power series in T with coefficients in A, as in Section 4.3. If j_0 is a nonnegative integer, then it is easy to see that

$$(A[[T]])\, T^{j_0} \tag{9.12.5}$$

is a left and right ideal in $A[[T]]$. Conversely, any nonzero left or right ideal $\mathcal{I}$ in $A[[T]]$ is of this form. To see this, let $f(T) = \sum_{j=j_0}^{\infty} f_j T^j$ be an element of $\mathcal{I}$ with $f_{j_0} \neq 0$ and j_0 as small as possible. Thus $\mathcal{I}$ is contained in (9.12.5), by construction. Note that an element of $A[[T]]$ is invertible when its coefficient of T^0 is nonzero, because A is a division ring, as in Section 4.5. One can use this to check that $\mathcal{I}$ is equal to (9.12.5).

It is easy to see that the product of two nonzero elements of $A[[T]]$ is nonzero as well, by considering the smallest powers of T for which the corresponding coefficients are not zero. It follows that any nonzero left or right ideal $\mathcal{I}$ in $A[[T]]$ is isomorphic to $A[[T]]$, as a left or right module over $A[[T]]$, as appropriate. In particular, this means that $\mathcal{I}$ is projective as a left or right module over $A[[T]]$, as appropriate. This shows that $A[[T]]$ is left and right hereditary, as a ring.

9.13 An interesting family of rings

Let A_1 be a commutative ring with a nonzero multiplicative identity element e. Also let ε be a ring homomorphism from A_1 into itself, with

$$\varepsilon(e) = e. \tag{9.13.1}$$

Put

$$A_2 = A_1 \times A_1, \tag{9.13.2}$$

considered initially as a commutative group with respect to coordinatewise addition. Of course, this is the same as the direct sum of two copies of A_1, as a commutative group with respect to addition.

Let us define multiplication on A_2 by putting

$$(a, b)\, (a', b') = (a\, a', a\, b' + \varepsilon(a')\, b) \tag{9.13.3}$$

for every $a, a', b, b' \in A_1$, as in Exercise 5 on p159 of [3]. This defines a mapping from $A_2 \times A_2$ into A_2 that is additive in each of (a, b) and (a', b'). One can check that this definition of multiplication on A_2 is associative, so that A_2 is a ring. If ε is the identity mapping on A_1, then multiplication on A_2 is commutative.

It is easy to see that $(e, 0)$ is the multiplicative identity element in A_2, because of (9.13.1). Observe that

$$a \mapsto (a, 0) \tag{9.13.4}$$

is an injective ring homomorphism from A_1 into A_2. Thus $A_1 \times \{0\}$ is a subring of A_2 that is isomorphic to A_1.

Suppose for the moment that k is a commutative ring with a nonzero multiplicative identity element, and that A_1 is the algebra $k[x]$ of formal polynomials in an indeterminate x with coefficients in k. Let ε be the mapping from $k[x]$ into itself that sends a formal polynomial in x to its constant term. In this case, A_2 corresponds to the algebra over k in Dieudonné's example, discussed in Section 9.7, as in Exercise 5 on p159 of [3].

Clearly

$$(a, b) \mapsto a \tag{9.13.5}$$

defines a ring homomorphism from A_2 onto A_1. If $(a, b) \in A_2$ has a left or right inverse in A_2, then it follows that A is invertible in A_1. If $b \in A_1$, then

$$(e, b)\,(e, -b) = (e, -b)\,(e, b) = (e, 0). \tag{9.13.6}$$

This means that (e, b) is invertible in A_2, with inverse $(e, -b)$. If $(a, b) \in A_2$ and a is invertible in A_1, then it follows that

$$(a, b) = (a, 0)\,(e, a^{-1}\,b) \tag{9.13.7}$$

is invertible in A_2, as in Exercise 5 on p159 of [3].

Suppose for the moment that A_1 is a local ring, and let $\mathcal{I}_1$ be the unique maximal ideal in A_1. Thus $\mathcal{I}_1$ consists exactly of the noninvertible elements of A_1, as in Section 4.13. Under these conditions,

$$\mathcal{I}_2 = \{(a, b) \in A_2 : a \in \mathcal{I}_1\} \tag{9.13.8}$$

is a two-sided ideal in A_2, which is the same as the collection of elements of A_2 without a left inverse, or without a right inverse. This means that A_2 is a local ring as well, as in Section 4.13. Note that $A_1/\mathcal{I}_1$ is isomorphic to $A_2/\mathcal{I}_2$, as in Exercise 5 on p159 of [3].

It is easy to see that $\{0\} \times A_1$ is a two-sided ideal in A_2, which is the same as the kernel of (9.13.5). We may consider A_2 as a module over A_1 with respect to both left and right multiplication, using (9.13.4). Clearly $A_1 \times \{0\}$ and $\{0\} \times A_1$ are both submodules of A_2, as a module over A_1 with respect to each of left and right multiplication. As modules over A_1 with respect to left multiplication, $A_1 \times \{0\}$ and $\{0\} \times A_1$ are both isomorphic to A_1.

If A_1 is Noetherian as a ring, then $A_1 \times \{0\}$ and $\{0\} \times A_1$ are Noetherian as modules over A_1 with respect to multiplication on the left. This implies that A_2 is Noetherian as a module over A_1 with respect to multiplication on the left, because A_2 corresponds to the direct sum of $A_1 \times \{0\}$ and $\{0\} \times A_1$. In this case, it follows that A_2 is left Noetherian, because left ideals in A_2 are submodules of A_2 as a module over A_1 with respect to multiplication on the left.

Let k be a commutative ring with a nonzero multiplicative identity element again, and suppose now that A_1 is the algebra $k[[x]]$ of formal power series in an indeterminate x with coefficients in k. Also let ε be the mapping from $k[[x]]$ to itself that sends a formal power series in x to its constant term, so that A_2 may be defined as before. If k is a field, then $k[[x]]$ is a local ring, so that A_2 is a local ring, as mentioned earlier.

If k is Noetherian, then it is well known that $k[[x]]$ is a Noetherian ring too, as on p81 of [1]. This implies that A_2 is left Noetherian, as before.

It is easy to see that any submodule W of

$$\{0\} \times A_1 = \{0\} \times k[[x]], \tag{9.13.9}$$

as a module over k, is a right ideal in A_2, because of (9.13.3) and the definition of ε. Similarly, any set of generators for W, as a right module over A_2, also generates W as a module over k. One can use this to get that A_2 is not right Noetherian, as in Exercise 5 on p159 of [3].

9.14 Graded rings

Let A be a ring, with a multiplicative identity element e_A. Suppose that A is graded as a commutative group with respect to addition, or equivalently as a module over $\mathbf{Z}$, as in Section 5.9. Suppose also that

$$A^j = \{0\} \quad \text{for every } j < 0, \tag{9.14.1}$$

and that

$$A^j \cdot A^l \subseteq A^{j+l} \tag{9.14.2}$$

for every $j, l \geq 0$. Under these conditions, A is said to be a *graded ring*, as on p146 of [3].

In this case, one can check that the component e_A^0 of e_A in A^0 is a multiplicative identity element of A. This means that $e_A^0 = e_A$, so that $e_A \in A^0$. Note that A^0 is a subring of A.

If $a \in A$, then let

$$\varepsilon(a) = a^0 \tag{9.14.3}$$

be the component of a in A^0. This defines a ring homomorphism from A onto A^0. Of course, the kernel of ε corresponds to the direct sum of A^j, with $j \geq 1$.

Let k be a commutative ring with a multiplicative identity element, and suppose that A is an associative algebra over k. If A is graded as a module over k, with respect to the same grading as before, then A is a *graded* as an associative algebra over k, as on p164 of [3].

Let B be a subring of A, and suppose that B is homogeneous as a subgroup of A, as a graded commutative group with respect to addition, as in Section 5.9. Thus B is a graded commutative group with respect to addition too, using the grading induced by the one on A. Clearly B is a graded ring with respect to this grading.

Let A be any ring with multiplicative identity element e_A, and let $T_1, \ldots, T_n$ be commuting determinates. Thus the corresponding ring $A[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in A may be defined as in Section 4.3. An element $f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, t^\alpha$ of $A[T_1, \ldots, T_n]$ is said to be *homogeneous of degree* j for some nonnegative integer j if

$$f_\alpha = 0 \text{ when } \alpha_1 + \cdots + \alpha_n \neq j. \tag{9.14.4}$$

If we take $(A[T_1, \ldots, T_n])^j$ to be the set of elements of $A[T_1, \ldots, T_n]$ that are homogeneous of degree j for each $j \geq 0$, and to be $\{0\}$ when $j < 0$, then

$$A[T_1, \ldots, T_n] \text{ is a graded ring,} \tag{9.14.5}$$

as on p146 of [3]. Of course, $(A[T_1, \ldots, T_n])^0$ can be identified with A in a natural way.

Let E be a nonempty set, and let $\Sigma(E)$ be the free semigroup generated by E, as in Section 4.10. One can use $\Sigma(E)$ to get the corresponding semigroup ring $A(\Sigma(E))$, as in Sections 4.9 and 4.10. This is the free ring over A with generating set E, as on p146, 148 of [3]. By construction, the elements of $A(\Sigma(E))$ correspond to finite sums of elements of A times formal products of finitely many elements of E. If j is a nonnegative integer, then let $(A(\Sigma(E)))^j$ be the set of elements of $A(\Sigma(E))$ that can be expressed as a finite sum of elements of A times formal products of elements of E of length j. If we take $(A(\Sigma(E)))^j = \{0\}$ when $j < 0$, then it is easy to see that

$$A(\Sigma(E)) \text{ is a graded ring,} \tag{9.14.6}$$

as on p146 of [3]. Note that $(A(\Sigma(E)))^0$ can be identified with A in a natural way.

Let V be a bimodule over A, so that V is both a left and right module over A, and the actions of A on V on the left and right commute with each other. The corresponding tensor ring $T(V)$ can be defined as in Section 4.11, and

$$T(V) \text{ is a graded ring,} \tag{9.14.7}$$

with $(T(V))^j = T^j(V)$ for each $j \geq 0$, and $(T(V))^j = \{0\}$ when $j < 0$. In particular, $(T(V))^0 = T^0(V) = A$, by construction.

Let n be a positive integer, and let $x_1, \ldots, x_n$ be indeterminates. Remember that the corresponding exterior ring $E_A(x_1, \ldots, x_n)$ in $x_1, \ldots, x_n$ with coefficients in A is defined as in Section 4.15. If $0 \leq j \leq n$, then let $(E_V(x_1, \ldots, x_n))^j$ be the set of elements of $E_V(x_1, \ldots, x_n)$ of the form

$$\sum_{\substack{I \subseteq \{1, \ldots, n\} \\ |I| = j}} a_I \, x_I, \tag{9.14.8}$$

Here x_I is as in (4.15.2) for each $I \subseteq \{1, \ldots, n\}$, $|I|$ is the number of elements of I, and $a_I \in A$. If $j < 0$ or $j > n$, then we take $(E_A(x_1, \ldots, x_n))^j = \{0\}$. It is easy to see that

$$E_A(x_1, \ldots, x_n) \text{ is a graded ring,} \tag{9.14.9}$$

as on p146 of [3]. Note that $(E_A(x_1, \ldots, x_n))^0$ can be identified with A, as before.

Remember that the ring $A[d]$ of dual numbers associated to A as in Section 5.4 consists of expressions of the form $a_1 + a_2 \, d$, where $a_1, a_2 \in A$, and d is a formal symbol that commutes with all elements of A and satisfies $d^2 = 0$. This corresponds to taking $n = 1$ in the preceding paragraph, as on p146 of [3]. In particular,

(9.14.10) $$A[d] \text{ is a graded ring,}$$

with $(A[d])^0 = A$, $(A[d])^1 = A\,d$, and $(A[d])^j = \{0\}$ otherwise.

9.15 Graded modules over graded rings

Let A be a ring, with a multiplicative identity element e_A, and suppose that A is graded, as in the previous section. Also let V be a right module over A. Suppose that V is graded as a commutative group with respect to addition, or equivalently as a module over $\mathbf{Z}$, as in Section 5.9. Let us say that this grading on V is *compatible* with the grading on A if

(9.15.1) $$V^j \cdot A^l \subseteq V^{j+l}$$

for every $j, l \in \mathbf{Z}$, where of course we may as well take $l \geq 0$. Equivalently, this means that

(9.15.2) $$v^j \cdot a^l \in V^{j+l}$$

for every $v^j \in V^j$ and $a^l \in A^l$.

Remember that A^0 is a subring of A, so that V may be considered as a right module over A^0. If the grading on V is compatible with the grading on A, then V is a graded module over A^0 in the usual sense, as in Section 5.9.

Suppose that V is positive with respect to its grading, so that $V^j = \{0\}$ when $j < 0$, as in Section 5.9. If the grading on V is compatible with the grading on A, then V is said to be *graded* as a module over A, as a graded ring, as on p154 of [3]. Of course, there are analogous notions for left modules. Note that A may be considered as a graded module over itself, as a graded ring.

Let U be a submodule of V, as a module over A. Suppose that U is homogeneous as a subgroup of V, as a graded commutative group with respect to addition, as in Section 5.9. If the grading on V is compatible with the grading on A, then the grading on U induced by the grading on V is compatible with the grading on A as well. In particular, if V is graded as a module over A, as a graded ring, then U is graded as a module over A, as a graded ring, with respect to the induced grading.

Let B be a subring of A that contains e_A, and suppose that B is homogeneous as a subgroup of A, as a commutative group with respect to addition, so that B may be considered as a graded ring with respect to the induced grading, as in the previous section. Of course, V may be considered as a module over B. If the grading on V is compatible with the grading on A, then the grading on V is compatible with the grading on B induced by the grading on A. If V is graded as a module over A, as a graded ring, then V is graded as a module over B, as a graded ring, with the induced grading.

Let us now take A to be any ring with a multiplicative identity element e_A, and let $T_1, \ldots, T_n$ be commuting indeterminates, so that the corresponding ring $A[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in A may be defined as in Section 4.3. This may be considered as a graded ring, as in the previous section. Let V be a left or right module over A, and let $V[T_1, \ldots, T_n]$ be the space of formal polynomials in $T_1, \ldots, T_n$ with coefficients in V, as in Section 4.3 again. This may be considered as a left or right module over $A[T_1, \ldots, T_n]$, as appropriate, as before.

Let us say that an element $f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha$ of $V[T_1, \ldots, T_n]$ is *homogeneous of degree* j for some nonnegative integer j if $f_\alpha = 0$ when

(9.15.3)
$$\alpha_1 + \cdots + \alpha_n \neq j,$$

as usual. Let $(V[T_1, \ldots, T_n])^j$ be the set of elements of $V[T_1, \ldots, T_n]$ that are homogeneous of degree j when $j \geq 0$, and be equal to $\{0\}$ when $j < 0$. Using this, it is easy to see that

(9.15.4) $V[T_1, \ldots, T_n]$ is a graded module over $A[T_1, \ldots, T_n]$, as a graded ring.

Let $x_1, \ldots, x_n$ be indeterminates, so that the exterior ring $E_A(x_1, \ldots, x_n)$ in $x_1, \ldots, x_n$ with coefficients in A is defined as in Section 4.15. This may be considered as a graded ring, as in the previous section. Let V be a left or right module over A again, and let $\mathcal{E}_V(x_1, \ldots, x_n)$ be as in Section 4.15. Remember that $\mathcal{E}_V(x_1, \ldots, x_n)$ may be considered as a left or right module over $E_A(x_1, \ldots, x_n)$, as appropriate.

Let $(\mathcal{E}_V(x_1, \ldots, x_n))^j$ be the set of elements of $\mathcal{E}_V(x_1, \ldots, x_n)$ of the form

(9.15.5)
$$\sum_{\substack{I \subseteq \{1, \ldots, n\} \\ |I| = j}} v_I \, x_I$$

when $0 \leq j \leq n$. As before, x_I is as in (4.15.2) for each $I \subseteq \{1, \ldots, n\}$, $|I|$ is the number of element of I, and $v_I \in V$. If we also take $(\mathcal{E}_V(x_1, \ldots, n))^j = \{0\}$ when $j < 0$ or $j > n$, then it is easy to see that

(9.15.6) $\mathcal{E}_V(x_1, \ldots, x_n)$ is a graded module over $E_A(x_1, \ldots, x_n)$, as a graded ring.

The ring $A[d]$ of dual numbers associated to A as in Section 5.4 may be considered as a graded ring, as in Section 9.14. Remember that a left or right module (V, d_V) over A with differentiation corresponds exactly to a left or right module over $A[d]$, as appropriate. Suppose that V is graded as a module over A, as in Section 5.9. Observe that this grading is compatible with the grading on $A[d]$ exactly when d_V has degree 1, which means that V is a complex, as in Section 5.10. If V is positive as a graded module, then it follows that

(9.15.7) V is a graded module over $A[d]$, as a graded ring.

Chapter 10

Left and right complexes

10.1 Left complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a left or right module over A. As on p75 of [3], and mentioned in Section 7.5, we may consider V as a graded module over A with differentiation that this a complex, by taking $V^0 = V$, $V^j = \{0\}$ when $j \neq 0$, and differentiation operator $d_V = 0$. In this case, $Z(V)$, $Z'(V)$, and $H(V)$ are the same as V, and $B(V), B'(V) = \{0\}$.

Let (X, d_X) be a graded left or right module over A, as appropriate, with differentiation that is a complex. Suppose that X is negative as a graded module, so that

$$X^j = \{0\} \quad \text{when } j > 0, \tag{10.1.1}$$

as in Section 5.9. Let ε be a map from X into V, as complexes. This means that ε is a homomorphism from X into V, as modules over A with differentiation, of degree 0, as in Section 5.11. The combination of X and ε is called a *left complex over* V, and ε is called the *augmentation* of the left complex, as on p75 of [3].

The condition that ε have degree 0 means that $\varepsilon(X^j) \subseteq V^j$ for each $j \in \mathbf{Z}$, as in Section 5.9. Remember that the restriction of ε to X^j is denoted ε^j for each j, as before. In this case,

$$\varepsilon^j = 0 \quad \text{when } j \neq 0, \tag{10.1.2}$$

because $V^j = \{0\}$ when $j \neq 0$. Thus ε is determined by ε^0, which is a homomorphism from X^0 into $V^0 = V$, as modules over A.

The condition that ε be a homomorphism from X into V, as modules with differentiation, means that

$$\varepsilon \circ d_X = d_V \circ \varepsilon = 0, \tag{10.1.3}$$

as in Section 5.2. This is the same as saying that

$$\varepsilon^j \circ d_X^{j-1} = d_V^{j-1} \circ \varepsilon^{j-1} = 0 \tag{10.1.4}$$

for each j, as in Section 5.11. This holds automatically when $j \neq 0$ here, so that we are reduced to the condition that

$$\varepsilon^0 \circ d_X^{-1} = 0, \tag{10.1.5}$$

as on p75 of [3].

The left complex X is said to be *projective* if X^j is projective as a module over A for each j, as on p75 of [3].

Remember that ε induces a homomorphism

$$\text{from } H(X) \text{ into } H(V) \tag{10.1.6}$$

of degree 0, as in Sections 5.2 and 5.11. Thus we get an induced homomorphism

$$\text{from } H(X)^j \text{ into } H(V)^j \tag{10.1.7}$$

for each $j \in \mathbf{Z}$. The left complex X over V is said to be *acyclic* if the homomorphism as in (10.1.6) is an isomorphism, as on p75 of [3]. Equivalently, this means that the homomorphism as in (10.1.7) is an isomorphism for each j.

If $j \neq 0$, then $H(V)^j = \{0\}$, and the condition that the homomorphism as in (10.1.7) be an isomorphism means that

$$H(X)^j = \{0\}. \tag{10.1.8}$$

This holds automatically when $j > 0$, because of (10.1.1).

Note that $d_X^0 = 0$, because $X^1 = \{0\}$, as in (10.1.1). Thus

$$Z(X)^0 = X^0. \tag{10.1.9}$$

The condition (10.1.5) is the same as saying that

$$B(X)^0 = d_X^{-1}(X^{-1}) \subseteq \ker \varepsilon^0. \tag{10.1.10}$$

The homomorphism as in (10.1.7) is injective when $j = 0$ if and only if

$$d_X^{-1}(X^{-1}) = \ker \varepsilon^0. \tag{10.1.11}$$

The homomorphism as in (10.1.7) is surjective when $j = 0$ if and only if

$$\varepsilon^0(X^0) = V^0 = V. \tag{10.1.12}$$

It is often convenient to put $X_j = X^{-j}$, as in Section 5.9. Similarly, we may put

$$d_j = d_{X,j} = d_X^{-j}, \tag{10.1.13}$$

which maps $X^{-j} = X_j$ into $X^{-j+1} = X_{j-1}$. We may also put

$$\varepsilon_j = \varepsilon^{-j}, \tag{10.1.14}$$

which maps $X^{-j} = X_j$ into $V^{-j} = V_j$. Of course, this means that $\varepsilon_j = 0$ when $j \neq 0$, and $\varepsilon_0 = \varepsilon^0$. Using this notation, the acyclicity of X as a left complex over V is equivalent to the exactness of the sequence

$$\cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon_0} V \longrightarrow 0, \tag{10.1.15}$$

as on p75 of [3].

10.2 Projective resolutions

Let us continue with the same notation and hypotheses as in the previous section. A left complex X over V is said to be a *projective resolution* of V if it is projective and acyclic, as on p75 of [3].

Every module V over A has a projective resolution, as in Proposition 1.2 on p77 of [3]. One can start with a projective module X_0 over A and a homomorphism ε_0 from X_0 onto V, as in Section 2.7. Put

$$Z_0 = \ker \varepsilon_0. \tag{10.2.1}$$

Similarly, let d_1 be a homomorphism from a projective module X_1 over A onto Z_0. More precisely, we shall consider d_1 as a homomorphism from X_1 into X_0, as modules over A.

Continuing in this way, suppose that d_j is a homomorphism from X_j into X_{j-1} for some $j \geq 1$, and put

$$Z_j = \ker d_j. \tag{10.2.2}$$

We take d_{j+1} to be a homomorphism from a projective module X_{j+1} over A onto Z_j, considered as a homomorphism into X_j, as before.

To get a left complex X over V, we take $X^j = \{0\}$ when $j > 0$, $X^{-j} = X_j$ when $j \geq 0$, $\varepsilon^0 = \varepsilon_0$, and $\varepsilon^j = 0$ when $j \neq 0$. We also take $d_X^j = 0$ when $j \geq 0$, and $d_X^{-j} = d_j$ when $j \geq 1$. This is a projective resolution of V, by construction.

Note that we can take X_j to be a free module over A for each $j \geq 0$, as in Section 2.7, and as mentioned on p77 of [3].

If V is projective as a module over A, then we can take $X_0 = V$, and ε_0 to be the identity mapping. In this case, $Z_0 = \{0\}$, and we can take $X_j = \{0\}$ and $d_j = 0$ when $j \geq 1$.

If $Z_j = \{0\}$ for any $j \geq 0$, then we can take

$$X_l = \{0\},\ d_l = 0 \quad \text{when } l > j. \tag{10.2.3}$$

If Z_j is a projective module over A for some $j \geq 0$, then we can take $X_{j+1} = Z_j$ and d_{j+1} to be the obvious inclusion mapping from Z_j into X_j. This means that

$$Z_{j+1} = \{0\}, \tag{10.2.4}$$

so that we can take

$$X_l = \{0\},\ d_l = 0 \quad \text{when } l > j + 1. \tag{10.2.5}$$

Suppose for the moment that A is left or right hereditary, as appropriate, as in Section 9.3. We can start with a projective module X_0 over A and a homomorphism ε_0 from X_0 onto V as before. In this case, Z_0 is projective as a module over A, because Z_0 is a submodule of X_0, as in Section 9.3. This permits us to take $X_1 = Z_1$, and d_1 to be the obvious inclusion mapping from X_1 into X_0, as in the preceding paragraph. This means that

$$Z_1 = \ker d_1 = \{0\}, \tag{10.2.6}$$

so that we can take $X_j = \{0\}$ and $d_j = 0$ when $j \geq 2$, as before.

If V is finitely generated as a module over A, then we can take X_0 to be a finitely-generated free module over A. If A is left or right Noetherian as a ring, as appropriate, then Z_0 is finitely generated as a module over A too. Under these conditions, we can repeat the process, and take X_j to be a finitely-generated free module over A for each $j \geq 0$. This corresponds to Proposition 1.3 on p78 of [3].

Suppose now that A has no nontrivial zero divisors, so that if $a, b \in A$ and $a, b \neq 0$, then $a\,b \neq 0$. Let x be a nonzero element of A, so that $A\,x$ is a left ideal in A, and $a \mapsto a\,x$ is an isomorphism from A onto $A\,x$, as left modules over A. This implies that $A\,x$ is free as a left module over A. Consider

$$V = A/A\,x, \tag{10.2.7}$$

which is a left module over A. We can take $X_0 = A$, considered as a free left module over A, and ε_0 to be the natural quotient mapping from A onto V. The kernel of ε_0 is $A\,x$, which is free as a left module over A, and thus projective. As before, we can take $X_1 = A\,x$, d_1 to be the obvious inclusion mapping from $A\,x$ into A, and $X_j = \{0\}$, $d_j = 0$ when $j \geq 2$. Of course, there are analogous statements for $x\,A$ and $A/x\,A$, as right modules over A.

10.3 Maps over homomorphisms

Let us continue with the same notation and hypotheses as in Section 10.1. In particular, we let V be a left or right module over A, and we suppose that X is a left complex over V with augmentation ε. Let $\widetilde{V}$ be another left or right module over A, as appropriate, and suppose that $\widetilde{X}$ is a left complex over $\widetilde{V}$, with augmentation $\widetilde{\varepsilon}$.

Let ϕ be a homomorphism from V into $\widetilde{V}$, as modules over A. Suppose that Φ is a map from X into $\widetilde{X}$, as complexes, as in Section 5.11. This means that Φ is a homomorphism from X into $\widetilde{X}$, as modules over A with differentiation, and that Φ has degree 0. If

$$\widetilde{\varepsilon} \circ \Phi = \phi \circ \varepsilon, \tag{10.3.1}$$

then Φ is said to be a *map over* ϕ, as on p76 of [3].

Suppose that

$$X \text{ is a projective left complex over } V, \tag{10.3.2}$$

and that

$$\widetilde{X} \text{ is acyclic as a left complex over } \widetilde{V}. \tag{10.3.3}$$

Under these conditions,

$$\text{there is a map } \Phi \text{ from } X \text{ into } \widetilde{X} \text{ over } \phi, \tag{10.3.4}$$

as in Proposition 1.1 on p76 of [3].

Let

$$d = d_X, \ \widetilde{d} = d_{\widetilde{X}} \tag{10.3.5}$$

be the differentiation operators on X, $\widetilde{X}$, respectively. As in Section 10.1, we put $X_j = X^{-j}$, $d_j = d^{-j}$, and $\varepsilon_j = \varepsilon^{-j}$, and similarly for $\widetilde{X}$. Thus $X_j = \widetilde{X}_j = \{0\}$ when $j < 0$, by hypothesis. The restriction of Φ to X_j is $\Phi_j = \Phi^{-j}$, which is 0 when $j < 0$. We want to define Φ_j recursively for $j \geq 0$.

Note that $\phi \circ \varepsilon_0$ is a homomorphism from X_0 into $\widetilde{V}$, as modules over A. We also have that

$$\widetilde{\epsilon}(\widetilde{X}_0) = \widetilde{V}, \tag{10.3.6}$$

because $\widetilde{X}$ is acyclic as a left complex over $\widetilde{V}$, as in (10.1.12). This implies that there is a homomorphism Φ_0 from X_0 into $\widetilde{X}_0$, as modules over A, such that

$$\widetilde{\varepsilon}_0 \circ \Phi_0 = \phi \circ \varepsilon_0, \tag{10.3.7}$$

because X_0 is projective as a module over A, by hypothesis.

Similarly, $\Phi_0 \circ d_1$ defines a homomorphism from X_1 into $\widetilde{X}_0$, as modules over A. Observe that

$$\widetilde{\varepsilon}_0 \circ \Phi_0 \circ d_1 = \phi \circ \varepsilon_0 \circ d_1 = 0, \tag{10.3.8}$$

using (10.3.7) in the first step, and (10.1.5) in the second step. This means that

$$(\Phi_0 \circ d_1)(X_1) \subseteq \ker \widetilde{\varepsilon}_0 = \widetilde{d}_1(\widetilde{X}_1), \tag{10.3.9}$$

where the second step is as in (10.1.11), because $\widetilde{X}$ is acyclic as a left complex over $\widetilde{V}$. It follows that there is a homomorphism Φ_1 from X_1 into $\widetilde{X}_1$, as modules over A, such that

$$\widetilde{d}_1 \circ \Phi_1 = \Phi_0 \circ d_1, \tag{10.3.10}$$

because X_1 is projective as a module over A.

Suppose now that $l \geq 2$ is an integer, and that Φ_j has been defined as a homomorphism from X_j into $\widetilde{X}_j$, as modules over A, for $j = 0, \ldots, l-1$. Suppose also that

$$\widetilde{d}_j \circ \Phi_j = \Phi_{j-1} \circ d_j \tag{10.3.11}$$

for $j = 1, \ldots, l-1$. Thus $\Phi_{l-1} \circ d_l$ defines a homomorphism from X_l into $\widetilde{X}_{l-1}$, as modules over A, and

$$\widetilde{d}_{l-1} \circ \Phi_{l-1} \circ d_l = \Phi_{l-2} \circ d_{l-1} \circ d_l = 0. \tag{10.3.12}$$

This implies that

$$(\Phi_{l-1} \circ d_l)(X_l) \subseteq \ker \widetilde{d}_{l-1} = \widetilde{d}_l(\widetilde{X}_l), \tag{10.3.13}$$

where the second step is as in (10.1.8), because $\widetilde{X}$ is acyclic as a left complex over $\widetilde{V}$. This means that there is a homomorphism Φ_l from X_l into $\widetilde{X}_l$, as modules over A, such that

$$\widetilde{d}_l \circ \Phi_l = \Phi_{l-1} \circ d_l, \tag{10.3.14}$$

because X_l is projective as a module over A.

Continuing in this way, we can define Φ_j for every j, to get a homomorphism Φ from X into $\widetilde{X}$ of degree 0. More precisely, Φ is a homomorphism from X into $\widetilde{X}$, as modules with differentiation, because (10.3.11) holds when $j \geq 1$, by construction, and this also holds automatically when $j \leq 0$. Thus Φ is a map from X into $\widetilde{X}$, as complexes. Note that (10.3.1) follows from (10.3.7), so that Φ is a map over ϕ, as desired.

10.4 Some related homotopies

We continue with the same notation and hypotheses as in the previous section. Let ϕ be a homomorphism from V into $\widetilde{V}$ again, as modules over A, and suppose now that Φ, Ψ are maps from X into $\widetilde{X}$ over ϕ. If X is a projective left complex over V, and $\widetilde{X}$ is acyclic as a left complex over V, then

(10.4.1) there is a homotopy Σ between Φ and Ψ,

as maps between complexes, as in Section 5.11. This is another part of Proposition 1.1 on p76 of [3].

In particular, Σ is supposed to be a homomorphism from X into $\widetilde{X}$ of degree -1, so that Σ^j maps X^j into $\widetilde{X}^{j-1}$ for every integer j. As before, we put

$$\Sigma_j = \Sigma^{-j}, \tag{10.4.2}$$

which should map X_j into $\widetilde{X}_{j+1}$ for every j. This is automatically equal to 0 when $j < 0$, and we want to define Σ_j recursively when $j \geq 0$.

Observe that

$$\widetilde{\varepsilon}_0 \circ (\Phi_0 - \Psi_0) = \widetilde{\varepsilon} \circ \Phi_0 - \widetilde{\varepsilon} \circ \Psi_0 = \phi \circ \epsilon_0 - \phi \circ \epsilon_0 = 0, \tag{10.4.3}$$

because Φ and Ψ are both maps over ϕ. Thus

$$(\Phi_0 - \Psi_0)(X_0) \subseteq \ker \widetilde{\varepsilon}_0 = \widetilde{d}_1(\widetilde{X}_1), \tag{10.4.4}$$

where the second step is as in (10.1.11), because $\widetilde{X}$ is acyclic as a left complex over $\widetilde{V}$. This implies that there is a homomorphism Σ_0 from X_0 into $\widetilde{X}_1$, as modules over A, such that

$$\widetilde{d}_1 \circ \Sigma_0 = \Phi_0 - \Psi_0, \tag{10.4.5}$$

because X_0 is projective as a module over A, by hypothesis.

Let l be a positive integer, and suppose that Σ_j has been defined as a homomorphism from X_j into $\widetilde{X}_{j+1}$, as modules over A, for $j = 0, \ldots, l-1$. Suppose also that

$$\widetilde{d}_{j+1} \circ \Sigma_j + \Sigma_{j-1} \circ d_j = \Phi_j - \Psi_j \tag{10.4.6}$$

when $0 < j \leq l-1$. Note that $\Sigma_{l-1} \circ d_l$ defines a homomorphism from X_l into $\widetilde{X}_l$, as modules over A.

Because Φ and Ψ are homomorphisms from X into $\widetilde{X}$, as modules with differentiation, we have that

$$\begin{aligned} \widetilde{d}_l \circ (\Phi_l - \Psi_l - \Sigma_{l-1} \circ d_l) &= \widetilde{d}_l \circ \Phi_l - \widetilde{d}_l \circ \Psi_l - \widetilde{d}_l \circ \Sigma_{l-1} \circ d_l \\ &= \Phi_{l-1} \circ d_l - \Psi_{l-1} \circ d_l - \widetilde{d}_l \circ \Sigma_{l-1} \circ d_l. \end{aligned} \tag{10.4.7}$$

If $l = 1$, then the right side is equal to 0, by (10.4.5). If $l \geq 2$, then we can take $j = l - 1$ in (10.4.6), to get that

$$\widetilde{d}_l \circ \Sigma_{l-1} \circ d_l = \Phi_{l-1} \circ d_l - \Psi_{l-1} \circ d_l. \tag{10.4.8}$$

This implies that the right side of (10.4.7) is equal to 0 when $l \geq 2$ as well.

It follows that

$$(\Phi_l - \Psi_l - \Sigma_{l-1} \circ d_l)(X_l) \subseteq \ker \widetilde{d}_l = \widetilde{d}_{l+1}(X_l), \tag{10.4.9}$$

where the second step is as in (10.1.8), because $\widetilde{X}$ is acyclic as a left complex over $\widetilde{V}$. Thus there is a homomorphism Σ_l from X_l into $\widetilde{X}_{l+1}$, as modules over A, such that

$$\widetilde{d}_{l+1} \circ \Sigma_l = \Phi_l - \Psi_l - \Sigma_{l-1} \circ d_l, \tag{10.4.10}$$

because X_l is projective as a module over A. This means that (10.4.6) holds, with $j = l$.

We can continue in this way to define Σ_j for every j, and get a homomorphism from X into $\widetilde{X}$ that is homogeneous of degree -1. By construction, (10.4.6) holds when $j > 0$. If $j = 0$, then (10.4.6) reduces to (10.4.5), and (10.4.6) holds automatically when $j < 0$. This implies that Σ is a homotopy between Φ and Ψ, as desired.

10.5 Some remarks about compositions

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, $\widetilde{V}$, and $\widehat{V}$ be all left or all right modules over A. Suppose that X, $\widetilde{X}$, and $\widehat{X}$ are left complexes over V, $\widetilde{V}$, and $\widehat{V}$, respectively, as in Section 10.1. Let ε, $\widetilde{\varepsilon}$, and $\widehat{\varepsilon}$ be the corresponding augmentations on X, $\widetilde{X}$, and $\widehat{X}$, respectively.

Let ϕ be a homomorphism from V into $\widetilde{V}$, and let ψ be a homomorphism from $\widetilde{V}$ into $\widehat{V}$, as modules over A. Suppose that Φ is a map from X into $\widetilde{X}$ over ϕ, and that Ψ is a map from $\widetilde{X}$ into $\widehat{X}$ over ψ, as in Section 10.3. Note that $\Psi \circ \Phi$ is a map from X into $\widehat{X}$, as complexes. Of course, $\psi \circ \phi$ is a homomorphism from V into $\widehat{V}$, as modules over A. We also have that

$$\widehat{\varepsilon} \circ \Psi \circ \Phi = \psi \circ \widetilde{\varepsilon} \circ \Phi = \psi \circ \phi \circ \varepsilon, \tag{10.5.1}$$

so that $\Psi \circ \Phi$ is a map over $\psi \circ \phi$.

Now let X and Y be projective resolutions of V, as in Section 10.2. There is a map Φ from X into Y over the identity map on V, as in Section 10.3.

Similarly, there is a map Ψ from Y into X over the identity mapping on V. It follows that $\Psi \circ \Phi$ is a map from X into itself over the identity mapping on V, as in the preceding paragraph. Similarly, $\Phi \circ \Psi$ is a map from Y into itself over the identity mapping on V.

It is easy to see that the identity mapping on X is a map over the identity mapping on V, as a map from X into itself. It follows that $\Psi \circ \Phi$ is homotopic to the identity on X, as maps from X into itself as a complex, as in the previous section. Similarly, $\Phi \circ \Psi$ is homotopic to the identity map on Y, as maps from Y into itself, as a complex. This correspoonds to a remark on p77 of [3], after Proposition 1.2. One may say that X and Y have the same homotopy type under these conditions, as in [3].

10.6 Right complexes

Let k be a commutative ring with a multiplicative identity element, let A be an associative algebra over k with a multiplicative identity element e_A, and let V be a left or right module over A again. As before, we may consider V to be a complex, with $V^0 = V$, $V^j = \{0\}$ when $j \neq 0$, and $d_V = 0$.

Let (X, d_X) be a graded left or right module over A, as appropriate, with differentiation that is a complex, and suppose that X is positive as a graded module. This means that

$$X^j = \{0\} \quad \text{when } j < 0, \tag{10.6.1}$$

as in Section 5.9. Let ε be a map from V into X as complexes, so that ε is a homomorphism from V into X, as modules over A with differentiation, of degree 0, as in Section 5.11. The combination of X and ε is called a *right complex over* V, and ε is called the *augmentation* of the right complex, as on p78 of [3].

The condition that ε have degree 0 means that $\varepsilon(V^j) \subseteq X^j$ for every $j \in \mathbf{Z}$, as in Section 5.9. Of course, the restriction ε^j of ε to V^j is equal to 0 when $j \neq 0$, so that ε is determined by ε^0.

The condition that ε be a homomorphism from V into X, as modules with differentiation, means that

$$d_X \circ \varepsilon = \varepsilon \circ d_V = 0, \tag{10.6.2}$$

as in Section 5.2. This is the same as saying that

$$d_X^j \circ \varepsilon^j = \varepsilon^{j+1} \circ d_V^j = 0 \tag{10.6.3}$$

for each j, as in Section 5.11. This reduces to the condition that

$$d_X^0 \circ \varepsilon^0 = 0, \tag{10.6.4}$$

because $\varepsilon^j = 0$ when $j \neq 0$.

The right complex X is said to be *injective* if X^j is injective as a module over A for each j, as on p78 of [3].

Remember that ε induces a homomorphism

$$\text{from } H(V) \text{ into } H(X) \tag{10.6.5}$$

of degree 0, as in Sections 5.2 and 5.11. This leads to an induced homomorphism

$$\text{from } H(V)^j \text{ into } H(X)^j \tag{10.6.6}$$

for each $j \in \mathbf{Z}$. The right complex X over V is said to be *acyclic* if the homomorphism as in (10.6.5) is an isomorphism, as on p78 of [3]. This is the same as saying that the homomorphism as in (10.6.6) is an isomorphism for every j.

If $j \neq 0$, then this means that

$$H(X)^j = \{0\}, \tag{10.6.7}$$

because $H(V)^j = \{0\}$. This holds automatically when $j < 0$, by (10.6.1).

Observe that

$$B(X)^0 = \{0\}, \tag{10.6.8}$$

because $X^{-1} = \{0\}$, as in (10.6.1). The condition (10.6.4) is the same as saying that

$$\varepsilon^0(V^0) \subseteq Z(X)^0. \tag{10.6.9}$$

The homomorphism as in (10.6.6) is injective when $j = 0$ if and only if

$$\ker \varepsilon^0 = \{0\}, \tag{10.6.10}$$

because of (10.6.8). The homomorphism as in (10.6.6) is surjective when $j = 0$ if and only if

$$\varepsilon^0(V^0) = Z(X)^0. \tag{10.6.11}$$

The acyclicity of X as a right complex is equivalent to the exactness of the sequence

$$0 \longrightarrow V \xrightarrow{\varepsilon} X^0 \xrightarrow{d_X^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots, \tag{10.6.12}$$

as on p78 of [3].

10.7 Injective resolutions

We continue with the same notation and hypotheses as in the previous section. A right complex X over V is said to be an *injective resolution* of V if it is injective and acyclic, as on p78 of [3].

Every module V over A has an injective resolution, as in Proposition 1.2a on p78 of [3]. We can begin here with an injective homomorphism ε from V into an injective module X^0 over A, as in Section 2.8. Put

$$Y^0 = X^0/\varepsilon(V), \tag{10.7.1}$$

which is another module over A.

As before, there is an injective homomorphism

$$\text{from } Y^0 \text{ into an injective module } X^1 \text{ over } A. \tag{10.7.2}$$

Let d^0 be the composition of the natural quotient mapping from X^0 onto Y^0 with this injection, so that d^0 is a homomorphism from X^0 into X^1, as modules over A. Note that

$$\ker d^0 = \varepsilon(V), \tag{10.7.3}$$

by construction.

Suppose that modules X^j, X^{j+1} over A have already been chosen for some $j \geq 0$, as well as a homomorphism d^j from X^j into X^{j+1}, as modules over A. Put

$$Y^{j+1} = X^{j+1}/d^j(X^j), \tag{10.7.4}$$

which is a module over A too. There is an injective homomorphism

$$\text{from } Y^{j+1} \text{ into an injective module } X^{j+2} \text{ over } A, \tag{10.7.5}$$

as usual. Let d^{j+1} be the composition of the natural quotient mapping from X^{j+1} onto Y^{j+1} with this injection, so that d^{j+1} is a homomorphism from X^{j+1} into X^{j+2}, as modules over A. Thus

$$\ker d^{j+1} = d^j(X^j), \tag{10.7.6}$$

by construction.

Continuing in this way, we get injective modules X^j for each $j \geq 0$. We take $X^j = \{0\}$ when $j < 0$, to get a positive graded module X over A. More precisely, X is a complex, with $d_X^j = d^j$ as before when $j \geq 0$, and of course $d_X^j = 0$ when $j < 0$. We also take $\varepsilon^0 = \varepsilon$, and $\varepsilon^j = 0$ when $j \neq 0$. It is easy to see that this makes X a right complex over V, which is an injective resolution of V.

If V is an injective module over A, then we can take $X^0 = V$, and ε to be the identity mapping. This means that $Y^0 = \{0\}$, and we can take $X^{j+1} = \{0\}$ and $d^j = 0$ for every $j \geq 0$.

Similarly, if $Y^j = \{0\}$ for any $j \geq 0$, then we can take

$$X^{l+1} = \{0\},\ d^l = 0 \quad \text{when } l \geq j. \tag{10.7.7}$$

If Y^j is an injective module over A for some $j \geq 0$, then we can take $X^{j+1} = Y^j$, and use the identity mapping on Y^j as the injective homomorphism from Y^j into X^{j+1}. In this case, d^j is the natural quotient mapping from X^j onto Y^j, so that $d^j(X^j) = Y^j = X^{j+1}$, and

$$Y^{j+1} = \{0\}. \tag{10.7.8}$$

Thus we can take

$$X^{l+1} = \{0\},\ d^l = 0 \quad \text{when } l \geq j+1, \tag{10.7.9}$$

as before.

Suppose that A is left or right hereditary, as appropriate, as in Section 9.3. Let ε be an injective homomorphism from V into an injective module X^0 over A, as before. If Y^0 is as in (10.7.1), then Y^0 is injective as a module over A, as in Section 9.3. This means that we can take $X^1 = Y^0$, and use the identity mapping on Y^0 as the injective homomorphism from Y^0 into X^1, as in the preceding paragraph. It follows that d^0 is the natural quotient mapping from X^0 onto Y^0, and

$$Y^1 = \{0\}, \tag{10.7.10}$$

so that we can take $X^{l+1} = \{0\}$ and $d^l = 0$ when $l \geq 1$, as before.

10.8 Maps and right complexes

Let us continue with the same notation and hypotheses as in Section 10.6. We let V be a left or right module over A again, and suppose that X is a right complex over V with augmentation ε. Let $\widetilde{V}$ be another left or right module over A, as appropriate, and suppose that $\widetilde{X}$ is a right complex over $\widetilde{V}$, with augmentation $\widetilde{\varepsilon}$.

Let ϕ be a homomorphism from V into $\widetilde{V}$, as modules over A. Suppose that Φ is a map from X into $\widetilde{X}$, as complexes, as in Section 5.11, so that Φ is a homomorphism from X into $\widetilde{X}$, as modules over A with differentiation, of degree 0. We say that Φ is a *map over* ϕ if

$$\Phi \circ \varepsilon = \widetilde{\varepsilon} \circ \phi, \tag{10.8.1}$$

as on p78 of [3].

Suppose that

$$X \text{ is acyclic as a right complex over } V, \tag{10.8.2}$$

and that

$$\widetilde{X} \text{ is an injective right complex over } \widetilde{V}. \tag{10.8.3}$$

In this case,

$$\text{there is a map } \Phi \text{ from } X \text{ into } \widetilde{X} \text{ over } \phi, \tag{10.8.4}$$

as in Proposition 1.1a on p78 of [3].

Of course, the restriction Φ^j of Φ to X^j is 0 when $j < 0$, because $X^j = \{0\}$. We would like to define Φ^j recursively when $j \geq 0$.

We would like to begin by choosing Φ^0 to be a homomorphism from X^0 into $\widetilde{X}^0$, as modules over A, such that

$$\Phi^0 \circ \varepsilon^0 = \widetilde{\varepsilon}^0 \circ \phi. \tag{10.8.5}$$

Note that ε^0 is injective, because X is acyclic as a right complex over V, as in (10.6.10). Thus (10.8.5) determines Φ^0 on $\varepsilon^0(V) \subseteq X^0$, as a homomorphism into $\widetilde{X}^1$, because $\widetilde{\varepsilon}^0 \circ \phi$ is a homomorphism from V into $\widetilde{X}^0$, as modules over

A. It follows that there is a homomorphism Φ^0 from X^0 into $\widetilde{X}^0$ as in (10.8.5), because $\widetilde{X}^0$ is injective as a module over A, by hypothesis.

Similarly, we would like to choose Φ^1 to be a homomorphism from X^1 into $\widetilde{X}^1$, as modules over A, such that

$$\Phi^1 \circ d_X^0 = d_{\widetilde{X}}^0 \circ \Phi^0. \tag{10.8.6}$$

Observe that

$$d_{\widetilde{X}}^0 \circ \Phi^0 \circ \varepsilon^0 = d_{\widetilde{X}}^0 \circ \widetilde{\varepsilon}^0 \circ \phi = 0, \tag{10.8.7}$$

using (10.8.5) in the first step, and the analogue of (10.6.4) for $\widetilde{\varepsilon}^0$ in the second step. This implies that the right side of (10.8.6) is equal to 0 on the kernel of d_X^0, because of (10.6.11). This means that Φ^1 is well-defined on $d_X^0(X^0)$ by (10.8.6). Thus there is a homomorphism Φ^1 from X^1 into $\widetilde{X}^1$ as in (10.8.6), because $\widetilde{X}^1$ is injective as a module over A.

Suppose that $l \geq 2$ is an integer, and that Φ^j has been defined as a homomorphism from X^j into $\widetilde{X}^j$, as modules over A, for $j = 0, \ldots, l-1$. Suppose also that

$$\Phi^j \circ d_X^{j-1} = d_{\widetilde{X}}^{j-1} \circ \Phi^{j-1} \tag{10.8.8}$$

for $j = 1, \ldots, l-1$. We would like to choose Φ^l to be a homomorphism from X^l into $\widetilde{X}^l$, as modules over A, such that

$$\Phi^l \circ d_X^{l-1} = d_{\widetilde{X}}^{l-1} \circ \Phi^{l-1}, \tag{10.8.9}$$

which is the same as (10.8.8) with $j = l$. Observe that

$$d_{\widetilde{X}}^{l-1} \circ \Phi^{l-1} \circ d_X^{l-2} = d_{\widetilde{X}}^{l-1} \circ d_{\widetilde{X}}^{l-2} \circ \Phi^{l-2} = 0, \tag{10.8.10}$$

using (10.8.8) with $j = l-1$ in the first step. This implies that

$$d_{\widetilde{X}}^{l-1} \circ \Phi^{l-1} = 0 \tag{10.8.11}$$

on $d_X^{l-2}(X^{l-2})$, which is the same as $\ker d_X^{l-1}$, because $H(X)^{l-1} = \{0\}$, as in (10.6.7). Thus Φ^l is well-defined on $d_X^{l-1}(X^{l-1})$ by (10.8.9). It follows that there is a homomorphism Φ^l from X^l into $\widetilde{X}^l$ that satisfies (10.8.9), because $\widetilde{X}^l$ is injective as a module over A.

We can continue in this way to define Φ^j for every j, to get a homomorphism from X into $\widetilde{X}$ of degree 0. We also have that Φ is a homomorphism from X into $\widetilde{X}$ as modules with differentiation, because (10.8.8) holds when $j \geq 1$ by construction, and automatically when $j \leq 0$. This means that Φ is a map from X into $\widetilde{X}$, as complexes. Of course, (10.8.1) follows from (10.8.5), so that Φ is a map over ϕ.

10.9 Homotopies and right complexes

Let us continue with the same notation and hypotheses as in the previous section, with ϕ a homomorphism from V into $\widetilde{V}$, as modules over A, in particular. Suppose that Φ and Ψ are maps from X into $\widetilde{X}$ over ϕ, in the sense defined in the previous section. If X is acyclic as a right complex over V, and $\widetilde{X}$ is an injective right complex over $\widetilde{V}$, then

(10.9.1) $$\text{there is a homotopy } \Sigma \text{ between } \Phi \text{ and } \Psi,$$

as maps between complexes, as in Section 5.11. This is the second part of Proposition 1.1a on p78 of [3].

By definition of a homotopy, Σ is supposed to be a homomorphism from X into $\widetilde{X}$ of degree -1, so that Σ^j maps X^j into $\widetilde{X}^{j-1}$ for every j. Of course, Σ^j is automatically 0 when $j \leq 0$, and we would like to define it recursively when $j > 0$.

Note that

(10.9.2) $$(\Phi^0 - \Psi^0) \circ \varepsilon^0 = \Phi^0 \circ \varepsilon^0 - \Psi^0 \circ \varepsilon^0 = \widetilde{\varepsilon}^0 \circ \phi - \widetilde{\varepsilon}^0 \circ \phi = 0,$$

because Φ and Ψ are maps over ϕ. This means that

(10.9.3) $$\ker(\Phi^0 - \Psi^0) \supseteq \varepsilon^0(X^0) = Z(X)^0,$$

using the acyclicity of X as a right complex over V in the second step, as in (10.6.11). We would like to choose Σ^1 to be a homomorphism from X^1 into $\widetilde{X}^0$, as modules over A, such that

(10.9.4) $$\Sigma^1 \circ d_X^0 = \Phi^0 - \Psi^0.$$

It follows from (10.9.3) that Σ^1 is well defined on $d_X^0(X^0)$ by (10.9.4). Thus there is a homomorphism Σ^1 from X^1 into $\widetilde{X}^0$ that satisfies (10.9.4), because $\widetilde{X}^0$ is injective as a module over A.

Let $l \geq 2$ be an integer, and suppose now that Σ^j has been defined as a homomorphism from X^j into $\widetilde{X}^{j-1}$, as modules over A, for $j = 1, \ldots, l-1$. Suppose also that

(10.9.5) $$\Sigma^j \circ d_X^{j-1} + d_{\widetilde{X}}^{j-2} \circ \Sigma^{j-1} = \Phi^{j-1} - \Psi^{j-1}$$

when $2 \leq j \leq l-1$. We would like to choose Σ^l to be a homomorphism from X^l into $\widetilde{X}^{l-1}$, as modules over A, such that

(10.9.6) $$\Sigma^l \circ d_X^{l-1} = -d_{\widetilde{X}}^{l-2} \circ \Sigma^{l-1} + \Phi^{l-1} - \Psi^{l-1},$$

which is equivalent to (10.9.5), with $j = l$.

Observe that

(10.9.7) $$\begin{aligned}
&(-d_{\widetilde{X}}^{l-2} \circ \Sigma^{l-1} + \Phi^{l-1} - \Psi^{l-1}) \circ d_X^{l-2} \\
&\quad = \quad -d_{\widetilde{X}}^{l-2} \circ \Sigma^{l-1} \circ d_X^{l-2} + \Phi^{l-1} \circ d_X^{l-2} - \Psi^{l-1} \circ d_X^{l-2} \\
&\quad = \quad -d_{\widetilde{X}}^{l-2} \circ \Sigma^{l-1} \circ d_X^{l-2} + d_{\widetilde{X}}^{l-2} \circ \Phi^{l-2} - d_{\widetilde{X}}^{l-2} \circ \Psi^{l-2},
\end{aligned}$$

because Φ, Ψ are homomorphisms from X into $\widetilde{X}$, as modules over A with differentiation. If $l = 2$, then the right side of (10.9.7) is equal to 0, by (10.9.4). If $l \geq 3$, then we can take $j = l - 1$ in (10.9.5), to get that

$$d_{\widetilde{X}}^{l-2} \circ \Sigma^{l-1} \circ d_X^{l-2} = d_{\widetilde{X}}^{l-2} \circ \Phi^{l-2} - d_{\widetilde{X}}^{l-2} \circ \Psi^{l-2}. \tag{10.9.8}$$

This implies that the right side of (10.9.7) is equal to 0 in this case too.

It follows that the right side of (10.9.6) is equal to 0 on $d_X^{l-2}(X^{l-2})$. This is the same as saying that the right side of (10.9.6) is equal to 0 on $\ker d_X^{l-1}$, because $H(X)^{l-1} = \{0\}$, as in (10.6.7). This means that Σ^l is well-defined on $d_X^{l-1}(X^{l-1})$ by (10.9.6). Thus we can get a homomorphism Σ^l from X^l into $\widetilde{X}^{l-1}$, as modules over A, that satisfies (10.9.6), because $\widetilde{X}^{l-1}$ is injective as a module over A.

Continuing in this way, we can define Σ^j for every j, to get a homomorphism Σ from X into $\widetilde{X}$ that is homogeneous of degree -1. We also have that (10.9.5) holds when $j \geq 2$, by construction. If $j = 1$, then (10.9.5) follows from (10.9.4). If $j \leq 0$, then (10.9.5) holds automatically, so that Σ is a homotopy between Φ and Ψ, as desired.

10.10 Compositions and right complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, $\widetilde{V}$, and $\widehat{V}$ be all left or all right modules over A, and suppose that X, $\widetilde{X}$, and $\widehat{X}$ are right complexes over V, $\widetilde{V}$, and $\widehat{V}$, respectively, as in Section 10.6. These include the augmentations ε, $\widetilde{\varepsilon}$, and $\widehat{\varepsilon}$, as before.

Let ϕ, ψ be homomorphisms from V, $\widetilde{V}$ into $\widetilde{V}$, $\widehat{V}$, respectively, as modules over A, and suppose that Φ, Ψ are maps fom X, $\widetilde{X}$ into $\widetilde{X}$, $\widehat{X}$, over ϕ, ψ, respectively, as in Section 10.8. Thus $\psi \circ \phi$ is a homomorphism from V into $\widehat{V}$, as modules over A, and $\Psi \circ \Phi$ is a map from X into $\widehat{X}$, as complexes. Observe that

$$\Psi \circ \Phi \circ \varepsilon = \Psi \circ \widetilde{\varepsilon} \circ \phi = \widehat{\varepsilon} \circ \psi \circ \phi, \tag{10.10.1}$$

so that $\Psi \circ \Phi$ is a map over $\psi \circ \phi$.

Suppose now that X, Y are injective resolutions of V, as in Section 10.7. There are maps Φ, Ψ from X, Y into Y, X, respectively, over the identity map on V, as in Section 10.8. It follows that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are maps from X and Y into themselves, respectively, over the identity mapping on V, as in the preceding paragraph.

Note that the identity mappings on X, Y are maps over the identity mapping on V, as maps from X, Y into themselves. Using the remarks in the previous section, we get that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are homotopic to the identity mappings on X and Y, respectively, as maps from X, Y into themselves, as complexes.

10.11 Tensor products and left complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a right module over A, let W be a left module over A, and let $V \bigotimes_A W$ be a tensor product of V and W, as modules over A. Suppose that X, Y are left complexes over V, W, as in Section 10.1, with augmentations ε_X, ε_Y, respectively.

Under these conditions, $X \bigotimes_A Y$ can be defined as a doubly-graded module over k, with

$$(10.11.1) \qquad \big(X \bigotimes_A Y\big)^{j,l} = X^j \bigotimes_A Y^l$$

for every $j, l \in \mathbf{Z}$, as in Section 6.1. Of course, this is equal to $\{0\}$ when $j > 0$ or $l > 0$, because X and Y are negative as graded modules, by hypothesis. One can get a single grading on $X \bigotimes_A Y$ from the double grading as in Section 5.13, as usual. It is easy to see that $X \bigotimes_A Y$ is negative with respect to this single grading, because of the analogous property of the double grading just mentioned.

More precisely, $X \bigotimes_A Y$ can be defined as a double complex over k, as in Section 6.2. The two differentiation operators on $X \bigotimes_A Y$, as a double complex, can be combined to get a single differentiation operator, so that $X \bigotimes_A Y$ becomes a single complex with respect to the single grading mentioned in the preceding paragraph, as in Section 5.14. Thus

$$(10.11.2) \qquad H\Big(X \bigotimes_A Y\Big)_n = H\Big(X \bigotimes_A Y\Big)^{-n}$$

can be defined as a module over k in the usual way for every integer n. This is equal to $\{0\}$ when $n < 0$, because $X \bigotimes_A Y$ is negative with respect to its single grading.

Remember that V, W may be considered as complexes, as in Section 10.1. Using this, one can consider $V \bigotimes_A W$ as a double complex, which leads to a single complex, as before. This is the same as the single complex obtained from $V \bigotimes_A W$ as a module over k, as in Section 10.1 again. One can use the augmentations ε_X, ε_Y to get a map

$$(10.11.3) \qquad \varepsilon \text{ from } X \bigotimes_A Y \text{ into } V \bigotimes_A W,$$

as complexes, as in Sections 5.15, 6.1, and 6.2. This means that

$$(10.11.4) \quad X \bigotimes_A Y \text{ is a left complex over } V \bigotimes_A W \text{ with respect to } \varepsilon,$$

as in Section 10.1.

Let $\widetilde{V}$ be another right module over A, let $\widetilde{W}$ be another left module over A, and let $\widetilde{V} \bigotimes_A \widetilde{W}$ be a tensor product of $\widetilde{V}$ and $\widetilde{W}$, as modules over A. Also let $\widetilde{X}$, $\widetilde{Y}$ be left complexes over $\widetilde{V}$, $\widetilde{Y}$, respectively, so that $\widetilde{X} \bigotimes_A \widetilde{Y}$ is a left complex over $\widetilde{V} \bigotimes_A \widetilde{W}$, as before. Suppose that ϕ, ψ are homomorphisms from

V, W into $\widetilde{V}$, $\widetilde{W}$, respectively, as modules over A, and that Φ, Ψ are maps from X, Y into $\widetilde{X}$, $\widetilde{Y}$ over ϕ, ψ, respectively, as in Section 10.3.

Using ϕ and ψ, we get a homomorphism

(10.11.5) $$\text{from } V \bigotimes_A W \text{ into } \widetilde{V} \bigotimes_A \widetilde{W},$$

as modules over k, as in Section 1.9. Similarly, we get a map

(10.11.6) $$\Phi \otimes \Psi \text{ from } X \bigotimes_A Y \text{ into } \widetilde{X} \bigotimes_A \widetilde{Y},$$

as double complexes, as in Section 6.2. One can check that

(10.11.7) $\Phi \otimes \Psi$ is a map over the homomorphism as in (10.11.5),

as in Section 10.3. Note that $\Phi \otimes \Psi$ is a map from $X \bigotimes_A Y$ into $\widetilde{X} \bigotimes_A \widetilde{Y}$, considered as single complexes, as in Section 5.15. Thus $\Phi \otimes \Psi$ induces a homomorphism from (10.11.2) into its analogue for $\widetilde{X} \bigotimes_A \widetilde{Y}$ for each n, as in Section 5.11.

Let Φ', Ψ' be another pair of maps from X, Y into $\widetilde{X}$, $\widetilde{Y}$ over ϕ, ψ, respectively, which leads to a map

(10.11.8) $$\Phi' \otimes \Psi' \text{ from } X \bigotimes_A Y \text{ into } \widetilde{X} \bigotimes_A \widetilde{Y},$$

as double complexes, as before. Suppose that Φ, Ψ are homotopic to Φ', Ψ', respectively, as maps between complexes, as in Section 5.11. This implies that

(10.11.9) $$\Phi \otimes \Psi \text{ is homotopic to } \Phi' \otimes \Psi',$$

as maps between double complexes, as in Section 6.2. It follows that $\Phi \otimes \Psi$ is homotopic to $\Phi' \otimes \Psi'$ as maps between single complexes, as in Section 5.15. This means that $\Phi \otimes \Psi$ and $\Phi' \otimes \Psi'$ induce the same mappings on (10.11.2) for each n, as in Section 5.11.

We may be particularly interested in (10.11.2) when X and Y are projective resolutions of V and W, respectively. In this case,

(10.11.10) (10.11.2) is uniquely determined by V and W

for each n, up to isomorphism. This follows from the fact that projective resolutions are unique up to homotopy equivalence, as in Section 10.5.

These remarks correspond to some of those on p82-4 of [3] in the left case, and for tensor products. This is also related to some remarks on p107 of [3].

10.12 Left complexes in one factor

Let us return to the same notation and hypotheses as at the beginning of the previous section. Remember that $X \bigotimes_A W$ and $V \bigotimes_A Y$ can be defined as graded modules over k, with

(10.12.1) $$\big(X \bigotimes_A W\big)^j = X^j \bigotimes_A W, \quad \big(V \bigotimes_A Y\big)^j = V \bigotimes_A Y^j$$

for every integer j, as in Section 7.5. More precisely, $X \bigotimes W$ and $V \bigotimes_A Y$ are negative as graded modules, because X and Y are negative, by hypothesis. In fact, $X \bigotimes_A W$ and $V \bigotimes_A Y$ can be defined as complexes over k, because X and Y are complexes, as in Section 7.5 again.

One can define suitable augmentations on $X \bigotimes_A W$ and $V \bigotimes_A Y$, so that

$$X \bigotimes_A W \text{ and } V \bigotimes_A Y \text{ are left complexes over } V \bigotimes_A W, \tag{10.12.2}$$

as in Section 10.1. If one considers V as a complex, as in Section 10.1, then one can consider $V \bigotimes_A W$ as a complex too, as before. In this case, one can get an augmentation map from $X \bigotimes_A W$ into $V \bigotimes_A W$ using the augmentation map ε_X on X and the identity mapping on W. This is a map from $X \bigotimes_A W$ into $V \bigotimes_A W$, as complexes, as in Section 7.6. The augmentation on $V \bigotimes_A Y$ is defined analogously, by considering W as a complex, and using the augmentation ε_Y on Y and the identity mapping on V.

Let $\widetilde{V}$ be another right module over A again, and let $\widetilde{V} \bigotimes_A W$ be a tensor product of $\widetilde{V}$ and W, as modules over A. Also let $\widetilde{X}$ be a left complex over $\widetilde{V}$, so that $\widetilde{X} \bigotimes_A W$ is a left complex over $\widetilde{V} \bigotimes_A W$, as before. Suppose that ϕ is a homomorphism from V into $\widetilde{V}$, as modules over A, and that Φ is a map from X into $\widetilde{X}$ over ϕ, as in Section 10.3. Using ϕ and the identity mapping on W, we get a homomorphism

$$\text{from } V \bigotimes_A W \text{ into } \widetilde{V} \bigotimes_A W, \tag{10.12.3}$$

as modules over k. Similarly, we can use Φ and the identity mapping on W to get a map

$$\text{from } X \bigotimes_A W \text{ into } \widetilde{X} \bigotimes_A W, \tag{10.12.4}$$

as complexes, as in Section 7.6. One can check that this is a map over the homomorphism as in (10.12.3), as in Section 10.3. Of course, this uses augmentations on $X \bigotimes_A W$ and $\widetilde{X} \bigotimes_A W$ as in the preceding paragraph.

Let Φ' be another map from X into $\widetilde{X}$ over ϕ, which leads to a map from $X \bigotimes_A W$ into $\widetilde{X} \bigotimes_A W$, as complexes, as before. Suppose that Φ is homotopic to Φ', as maps between complexes, as in Section 5.11. This implies that the corresponding maps from $X \bigotimes_A W$ into $\widetilde{X} \bigotimes_A W$ are homotopic, as maps between complexes, as in Section 7.6. Of course, there are analogous statements when $\widetilde{W}$ is another left module over A, $\widetilde{Y}$ is a left complex over $\widetilde{W}$, ψ is a homomorphism from W into $\widetilde{W}$, as modules over A, and Ψ is a map from Y into $\widetilde{Y}$ over ψ. In this case, we consider the corresponding mappings from $V \bigotimes_A W$ into $V \bigotimes_A \widetilde{W}$, and from $V \bigotimes_A Y$ into $V \bigotimes_A \widetilde{Y}$.

If n is an integer, then

$$H\Big(X \bigotimes_A W\Big)_n = H\Big(X \bigotimes_A W\Big)^{-n} \tag{10.12.5}$$

and

$$H\Big(V \bigotimes_A Y\Big)_n = H\Big(V \bigotimes_A Y\Big)^{-n} \tag{10.12.6}$$

can be defined as modules over k in the usual way. These are both equal to $\{0\}$ when $n < 0$, because $X \bigotimes_A W$ and $V \bigotimes_A Y$ are negative as graded modules. We may be particularly interested in (10.12.5) and (10.12.6) when X and Y are projective resolutions of V and W, respectively. In this case,

(10.12.7) (10.12.5) and (10.12.6) are uniquely determined, up to isomorphism, by V and W instead of X and Y, respectively,

for each n, because projective resolutions are unique up to homotopy equivalence, as in Section 10.5. These remarks correspond to some of those on p82-4 of [3] in the left case again, where modules are transformed by taking the tensor product with a fixed module.

If X and Y are projective resolutions of V and W, respectively, then (10.12.5) and (10.12.6) can also be described in terms of "satellites", which are discussed in Chapter III of [3]. This uses Theorem 6.1 on p90 of [3].

Remember V, W can be used to define complexes in a simple way, as in Section 10.1. Using these complexes, $X \bigotimes_A W$ and $V \bigotimes_A Y$ may be defined as double complexes, as in Sections 6.1 and 6.2. These double complexes lead to single complexes in the usual way, as in Sections 5.13 and 5.14. The single complexes that we get this way are the same as before, using the gradings in (10.12.1), as in Section 7.5.

The complexes associated to V, W as in the preceding paragraph may be considered as left complexes over V, W, respectively, where the augmentation mappings correspond to the identity mappings on these complexes. The augmentation maps ε_X, ε_Y may be considered as maps from X, Y into the complexes associated to V, W over the identity mappings on V, W, respectively, as in Section 10.3. We can use these maps and the identity mappings on X and Y to get maps

(10.12.8) from $X \bigotimes_A Y$ into $X \bigotimes_A W$

and

(10.12.9) from $X \bigotimes_A Y$ into $V \bigotimes_A Y$,

as double complexes, as in the previous section. These maps induce homomorphisms

(10.12.10) from (10.11.2) into (10.12.5)

and

(10.12.11) from (10.11.2) into (10.12.6)

for each n, as in Sections 5.11 and 5.15. This corresponds to the homomorphisms as in (1a) on p97 of [3], in the case of tensor products.

If X and Y are projective resolutions of V and W, respectively, then it is well known that

(10.12.12) the homomorphisms as in (10.12.10) and (10.12.11) are isomorphisms for each n. This uses the analogue of Theorem 8.1 on p95 of [3] in the left case. This also uses Proposition 1.1a on p106 of [3] to get the "left balanced" condition condition defined on p97 of [3], which was discussed near the end of Section 2.7. This fact is mentioned on p107 of [3] too, as well as the description in terms of satellites. Of course, (10.12.12) implies that

(10.12.13) (10.12.5) and (10.12.6) are isomorphic to each other

for each n.

10.13 Homomorphisms and resolutions

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V and W be both left or both right modules over A. Suppose that X is a left complex over V with augmentation ε_X, and that Y is a right complex over W with an augmentation ε_Y, as in Sections 10.1 and 10.6, respectively.

We can define $\mathrm{Hom}_A^{gr}(X, Y)$ as a doubly-graded module over k, with

$$(10.13.1) \qquad \big(\mathrm{Hom}_A^{gr}(X, Y)\big)^{j,l} = \mathrm{Hom}_A(X^{-j}, Y^l)$$

for every $j, l \in \mathbf{Z}$, as in Section 6.3. This is equal to $\{0\}$ when $j < 0$ or $l < 0$, because X is negative and Y is positive as graded modules, by hypothesis. Remember that there is a single grading on $\mathrm{Hom}_A^{gr}(X, Y)$ obtained from the double grading as in Section 5.13. Note that $\mathrm{Hom}_A^{gr}(X, Y)$ is positive with respect to this single grading, because of the analogous property of the double grading just mentioned.

In fact, $\mathrm{Hom}_A^{gr}(X, Y)$ can be defined as a double complex over k, as in Section 6.5. As usual, the two differentiation operators on $\mathrm{Hom}_A^{gr}(X, Y)$, as a double complex, can be combined to get a single differentiation operator, which makes $\mathrm{Hom}_A^{gr}(X, Y)$ a single complex with respect to the single grading mentioned in the previous paragraph, as in Section 5.14. This means that

$$(10.13.2) \qquad H\Big(\mathrm{Hom}_A^{gr}(X, Y)\Big)^n$$

can be defined as a module over k in the usual way for every $n \in \mathbf{Z}$. This is equal to $\{0\}$ when $n < 0$, because $\mathrm{Hom}_A^{gr}(X, Y)$ is positive with respect to its single grading.

We may consider V, W as complexes, as in Sections 10.1 and 10.6. We can use this to consider $\mathrm{Hom}_A^{gr}(V, W)$ as a double complex, which leads to a single complex, as before. This is the same as the single complex obtained from $\mathrm{Hom}_A(V, W)$, as a module over k, in the usual way. We can use the augmentations ε_X and ε_Y to get a map

(10.13.3) ε from $\mathrm{Hom}_A^{gr}(V, W)$ into $\mathrm{Hom}_A^{gr}(X, Y)$, as complexes, as in Sections 5.15, 6.4, and 6.5. It follows that

(10.13.4) $\quad \mathrm{Hom}^{gr}_A(X, Y)$ is a right complex over $\mathrm{Hom}_A(V, W)$ with respect to ε,

as in Section 10.6.

Let $\widetilde{V}$, $\widetilde{W}$ be additional left or right modules over A, as appropriate. Also let $\widetilde{X}$ be a left complex over $\widetilde{V}$, and let $\widetilde{Y}$ be a right complex over $\widetilde{W}$. Thus $\mathrm{Hom}^{gr}_A(\widetilde{X}, \widetilde{Y})$ is a right complex over $\mathrm{Hom}_A(V, W)$, as before. Suppose that ϕ is a homomorphism from $\widetilde{V}$ into V, and that ψ is a homomorphism from W into $\widetilde{W}$, as modules over A. Let Φ be a map from $\widetilde{X}$ into X over ϕ, and let Ψ be a map from Y into $\widetilde{Y}$ over ψ, as in Sections 10.3 and 10.8, respectively.

Using ϕ and ψ, we get a homomorphism

(10.13.5) $\quad \mathrm{Hom}(\phi, \psi)$ from $\mathrm{Hom}_A(V, W)$ into $\mathrm{Hom}_A(\widetilde{V}, \widetilde{W})$,

as modules over k, as in Section 6.3. Similarly, we can use Φ and Ψ to get a map

(10.13.6) $\quad \mathrm{Hom}^{gr}(\Phi, \Psi)$ from $\mathrm{Hom}^{gr}_A(X, Y)$ into $\mathrm{Hom}^{gr}_A(\widetilde{X}, \widetilde{Y})$,

as double complexes, as in Sections 6.4 and 6.5. One can verify that

(10.13.7) $\quad \mathrm{Hom}^{gr}(\Phi, \Psi)$ is a map over $\mathrm{Hom}(\phi, \psi)$,

as in Section 10.8. Remember that $\mathrm{Hom}^{gr}(\Phi, \Psi)$ may also be considered as a map from $\mathrm{Hom}^{gr}_A(X, Y)$ into $\mathrm{Hom}^{gr}_A(\widetilde{X}, \widetilde{Y})$ as single complexes, as in Section 5.15. This means that $\mathrm{Hom}^{gr}(\Phi, \Psi)$ induces a homomorphism from (10.13.2) into its analogue for $\mathrm{Hom}^{gr}_A(\widetilde{X}, \widetilde{Y})$ for each n, as in Section 5.11.

Let Φ' be another map from $\widetilde{X}$ into X over ϕ, and let Ψ' be another map from Y into $\widetilde{Y}$ ove ψ. This leads to a map

(10.13.8) $\quad \mathrm{Hom}^{gr}(\Phi', \Psi')$ from $\mathrm{Hom}^{gr}_A(X, Y)$ into $\mathrm{Hom}^{gr}_A(\widetilde{X}, \widetilde{Y})$,

as double complexes, as before. Suppose that Φ, Ψ are homotopic to Φ', Ψ', respectively, as maps between complexes, as in Section 5.11. This implies that

(10.13.9) $\quad \mathrm{Hom}^{gr}(\Phi, \Psi)$ is homotopic to $\mathrm{Hom}^{gr}(\Phi', \Psi')$,

as maps between double complexes, as in Section 6.5, and thus as maps between single complexes, as in Section 5.15. It follows that $\mathrm{Hom}^{gr}(\Phi, \Psi)$ and $\mathrm{Hom}^{gr}(\Phi', \Psi')$ induce the same mappings on (10.13.2) for each n, as in Section 5.11.

We may be particularly interested in (10.13.2) when X is a projective resolution of V, and Y is an injective resolution of W. Under these conditions,

(10.13.10) $\quad$ (10.13.2) is uniquely determined by V and W

for each n, up to isomorphism. This uses the uniqueness of projective and injective resolutions up to homotopy equivalence, as in Sections 10.5 and 10.10.

These remarks correspond to some of those on p82f of [3], for spaces of homomorphisms. This is related to some remarks on p107 of [3] as well.

10.14 Left complexes and $\mathrm{Hom}(\cdot, W)$

We return to the same notation and hypotheses as at the beginning of the previous section. Remember that $\mathrm{Hom}_A^{gr}(X, W)$ can be defined as a graded module over k, with

$$(\mathrm{Hom}_A^{gr}(X, W))^j = \mathrm{Hom}_A(X^{-j}, W) \tag{10.14.1}$$

for every integer j, as in Section 8.4. Note that $\mathrm{Hom}_A^{gr}(X, W)$ is positive as a graded module, because X is negative, by hypothesis. More precisely, $\mathrm{Hom}_A^{gr}(X, W)$ can be defined as a complex over k, because X is a complex, as in Section 8.4.

One can define an augmentation on $\mathrm{Hom}_A^{gr}(X, W)$, so that

(10.14.2) $\quad \mathrm{Hom}_A^{gr}(X, W)$ is a right complex over $\mathrm{Hom}_A(V, W)$,

as in Section 10.6. If V is considered to be a complex, as in Section 10.6, then $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a complex, as in the preceding paragraph. This is the same as the complex associated to $\mathrm{Hom}_A(V, W)$ in the usual way. One can get an augmentation map from $\mathrm{Hom}_A(V, W)$ into $\mathrm{Hom}_A^{gr}(X, W)$ using the augmentation map ε_X on X and the identity mapping on W. This corresponds to a map from $\mathrm{Hom}_A^{gr}(V, W)$ into $\mathrm{Hom}_A^{gr}(X, W)$, as complexes, as in Section 8.5.

Let $\widetilde{V}$ be another left or right module over A, as appropriate, and let $\widetilde{X}$ be a left complex over $\widetilde{V}$. Thus $\mathrm{Hom}_A^{gr}(\widetilde{X}, W)$ can be defined as a right complex over $\mathrm{Hom}_A(\widetilde{V}, W)$, as before. Suppose that ϕ is a homomorphism from $\widetilde{V}$ into V, as modules over A, and let Φ be a map from $\widetilde{X}$ into X over ϕ, as in Section 10.3. Using ϕ and the identity mapping on W, we get a homomorphism

(10.14.3) $\quad$ from $\mathrm{Hom}_A(V, W)$ into $\mathrm{Hom}_A(\widetilde{V}, W)$,

as modules over k, as in Section 6.3. Similarly, we can use Φ and the identity mapping on W to get a map

(10.14.4) $\quad$ from $\mathrm{Hom}_A^{gr}(X, W)$ into $\mathrm{Hom}_A^{gr}(\widetilde{X}, W)$,

as complexes, as in Section 8.5. One can check that this is a map over the homomorphism as in (10.14.3), as in Section 10.8. This uses augmentations on $\mathrm{Hom}_A^{gr}(X, W)$ and $\mathrm{Hom}_A^{gr}(\widetilde{X}, W)$ as in the previous paragraph.

Let Φ' be another map from $\widetilde{X}$ into X over ϕ, which leads to a map from $\mathrm{Hom}_A^{gr}(X, W)$ into $\mathrm{Hom}_A^{gr}(\widetilde{X}, W)$, as complexes, as before. If Φ is homotopic to Φ', as maps between complexes, as in Section 5.11, then the corresponding maps from $\mathrm{Hom}_A^{gr}(X, W)$ into $\mathrm{Hom}_A^{gr}(\widetilde{X}, W)$ are homotopic, as maps between complexes, as in Section 8.5.

Because $\mathrm{Hom}_A^{gr}(X, W)$ is a complex over k,

$$H\Big(\mathrm{Hom}_A^{gr}(X, W)\Big)^n \tag{10.14.5}$$

can be defined as a module over k for each integer n in the usual way. This is equal to $\{0\}$ when $n < 0$, because $\mathrm{Hom}_A^{gr}(X, W)$ is positive as a graded module. We may be particularly interested in (10.14.5) when X is a projective resolution of V, in which case

(10.14.6) (10.14.5) is uniquely determined by V

for each n, up to isomorphism, because projective resolutions are unique up to homotopy equivalence, as in Section 10.5. These remarks correspond to some of those on p82f of [3] again.

Note that (10.14.5) can also be described in terms of "satellites" under these conditions, as in Theorem 6.1 on p90 of [3].

If we consider W as a complex in the usual way, as in Section 10.6, then $\mathrm{Hom}_A^{gr}(X, W)$ can be defined as a double complex, as in Sections 6.3 and 6.5. This double complex leads to a single complex, as in Sections 5.13 and 5.14. This single complex is the same as before, with the grading as in (10.14.1), as in Section 8.4.

The complex associated to W as in the previous paragraph may be considered as a right complex over W, where the augmentation mapping corresponds to the identity mapping on this complex. The augmentation map ε_Y may be considered as a map from the complex associated to W into Y over the identity mapping on W, as in Section 10.8. We can use this map and the identity mapping on X to get a map

(10.14.7) from $\mathrm{Hom}_A^{gr}(X, W)$ into $\mathrm{Hom}_A^{gr}(X, Y)$,

as double complexes, as in the previous section. This map induces a homomorphism

(10.14.8) from (10.14.5) into (10.13.2)

for each n, as in Sections 5.11 and 5.15. This corresponds to a homomorphism as in (1) on p94 of [3], in the case of spaces of homomorphisms.

If X is a projective resolution of V, and Y is an injective resolution of W, then it is well known that

(10.14.9) the homomorphism as in (10.14.8) is an isomorphism

for each n, as in Theorem 8.1 on p95 of [3]. This is mentioned on p107 of [3] as well.

10.15 Right complexes and $\mathrm{Hom}(V, \cdot)$

Let us return to the same notation and hypotheses as at the beginning of Section 10.13. Note that $\mathrm{Hom}_A^{gr}(V, Y)$ can be defined as a graded module over k, with

(10.15.1) $$\big(\mathrm{Hom}_A^{gr}(V, Y)\big)^l = \mathrm{Hom}_A(V, Y^l)$$

for every integer l, as in Section 7.12. This is positive as a graded module, because Y is positive, by hypothesis. In fact, $\mathrm{Hom}_A^{gr}(V, Y)$ can be defined as a complex over k, because Y is a complex, as in Section 7.12.

One can define an augmentation on $\mathrm{Hom}_A^{gr}(V, Y)$, so that

$$\mathrm{Hom}_A^{gr}(V, Y) \text{ is a right complex over } \mathrm{Hom}_A(V, W), \tag{10.15.2}$$

as in Section 10.6. If W is considered as a complex, as in Section 10.6, then $\mathrm{Hom}_A^{gr}(V, W)$ may be defined as a complex, as in the previous paragraph. This is the same as the complex associated to $\mathrm{Hom}_A(V, W)$ in the analogous way. An augmentation map from $\mathrm{Hom}_A(V, W)$ into $\mathrm{Hom}_A^{gr}(V, Y)$ can be obtained from the identity mapping on V and the augmentation map ε_Y from W into Y. This corresponds to a map from $\mathrm{Hom}_A^{gr}(V, W)$ into $\mathrm{Hom}_A(V, Y)$, as complexes, as in Section 7.13.

Let $\widetilde{W}$ be another left or right module over A, as appropriate, and let $\widetilde{Y}$ be a right complex over $\widetilde{W}$. This means that $\mathrm{Hom}_A^{gr}(V, \widetilde{Y})$ can be defined as a right complex over $\mathrm{Hom}_A(V, \widetilde{W})$, as in the previous two paragraphs. Let ψ be a homomorphism from W into $\widetilde{W}$, as modules over A, and let Ψ be a map from Y into $\widetilde{Y}$ over ψ, as in Section 10.8. We can use ψ and the identity mapping on V to get a homomorphism

$$\text{from } \mathrm{Hom}_A(V, W) \text{ into } \mathrm{Hom}_A(V, \widetilde{W}), \tag{10.15.3}$$

as modules over k, as in Section 6.3. We can also use Ψ and the identity mapping on V to get a map

$$\text{from } \mathrm{Hom}_A^{gr}(V, Y) \text{ into } \mathrm{Hom}_A^{gr}(V, \widetilde{Y}), \tag{10.15.4}$$

as complexes, as in Section 7.13. One can verify that this is a map over the homomorphism as in (10.15.3), as in Section 10.8. This uses the augmentations on $\mathrm{Hom}_A^{gr}(V, Y)$ and $\mathrm{Hom}_A^{gr}(V, \widetilde{Y})$ that are defined as in the preceding paragraph.

Let Ψ' be another map from Y into $\widetilde{Y}$ over ψ, which leads to a map from $\mathrm{Hom}_A^{gr}(V, Y)$ into $\mathrm{Hom}_A^{gr}(V, \widetilde{Y})$, as complexes, as before. If Ψ is homotopic to Ψ', as maps between complexes, as in Section 5.11, then the corresponding maps from $\mathrm{Hom}_A^{gr}(V, Y)$ into $\mathrm{Hom}_A^{gr}(V, \widetilde{Y})$ are homotopic, as maps between complexes, as in Section 7.13.

We can define

$$H\Big(\mathrm{Hom}_A^{gr}(V, Y)\Big)^n \tag{10.15.5}$$

as a module over k for every integer n in the usual way, because $\mathrm{Hom}_A^{gr}(V, Y)$ is a complex over k. This module is $\{0\}$ when $n < 0$, because $\mathrm{Hom}_A^{gr}(V, Y)$ is positive, as a graded module. We may be particularly interested in this module when Y is an injective resolution of W, in which case

$$\text{(10.15.5) is uniquely determined by } W \tag{10.15.6}$$

for each n, up to isomorphism, because injective resolutions are unique up to homotopy equivalence, as in Section 10.10. These remarks correspond to some of those on p82f of [3].

Under these conditions, (10.15.5) can be described in terms of "satellites" too, as in Theorem 6.1 on p90 of [3].

We may consider V as a complex in the usual way, as in Section 10.6, so that $\mathrm{Hom}_A^{gr}(V, Y)$ may be defined as a double complex, as in Sections 6.3 and 6.5. Using this double complex, we get a single complex, as in Sections 5.13 and 5.14. This single complex is the same as at the beginning of the section, with the grading as in (10.15.1), as in Section 7.12.

The complex associated to V as in the preceding paragraph may be considered as a left complex over V, where the augmentation mapping corresponds to the identity mapping on this complex. The augmentation map ε_X may be considered as a map from X into the complex associated to V over the identity mapping on V, as in Section 10.3. We can use this map and the identity mapping on Y to get a map

(10.15.7) $$\text{from } \mathrm{Hom}_A^{gr}(V, Y) \text{ into } \mathrm{Hom}_A^{gr}(X, Y),$$

as double complexes, as in Section 10.13. This map induces a homomorphism

(10.15.8) $$\text{from (10.15.5) into (10.13.2)}$$

for each n, as in Sections 5.11 and 5.15. This corresponds to a homomorphism as in (1) on p94 of [3] again, in the case of spaces of homomorphisms.

If X is a projective resolution of V, and Y is an injective resolution of W, then

(10.15.9) the homomorphism as in (10.15.8) is an isomorphism

for each n, as in Theorem 8.1 on p95 of [3]. This is also mentioned on p107 of [3].

Chapter 11

Integral domains

11.1 Torsion and divisible elements

Let k be an *integral domain*, which is to say a commutative ring with a nonzero multiplicative identity element such that the product of any two nonzero elements of k is not zero too. Also let V be a module over k.

An element v of V is said to be a *torsion element* of V if there is a $t \in k$ such that $t \neq 0$ and

$$t \cdot v = 0. \tag{11.1.1}$$

The collection $\tau(V)$ of torsion elements of V is a submodule of V, as in Exercise 12 on p45 of [1], and as mentioned on p127 of [3].

If

$$\tau(V) = V, \tag{11.1.2}$$

then V is said to be a *torsion module*, as a module over k. It is easy to see that $\tau(V)$ is a torsion module, as on p127 of [3].

If

$$\tau(V) = \{0\}, \tag{11.1.3}$$

then V is said to be *torsion-free*, as a module over k. One can check that

$$V/\tau(V) \tag{11.1.4}$$

is torsion-free as a module over k, as in the first part of Exercise 12 on p45 of [1], and as mentioned on p127 of [3]. Note that submodules of torsion-free modules are torsion-free as well.

Let I be a nonempty set, and let V_j be a module over k for each $j \in I$. One can verify that

$$\tau\Big(\bigoplus_{j \in I} V_j\Big) = \bigoplus_{j \in I} \tau(V_j), \tag{11.1.5}$$

as mentioned on p127 of [3].

Of course, k is torsion-free as a module over itself, by hypothesis. Thus free modules over k are torsion-free.

If V is projective as a module over k, then V is isomorphic to a submodule of a free module over k, as in Section 2.7. This implies that V is torsion-free, as in Proposition 1.1 on p127 of [3].

An element v of V is said to be *divisible* if for every $t \in k$ with $t \neq 0$ there is a $w \in V$ such that

$$v = t \cdot w. \tag{11.1.6}$$

The collection $\delta(V)$ of divisible elements of V is a submodule of V, as on p127f of [3].

If

$$\delta(V) = V, \tag{11.1.7}$$

then V is said to be *divisible*, as a module over k. In this case, it is easy to see that every quotient of V is divisible too, as on p128 of [3].

One can check that

$$\delta(V/\delta(V)) = \{0\}, \tag{11.1.8}$$

as mentioned on p128 of [3]. More precisely, let q be the natural quotient mapping from V onto $V/\delta(V)$. If $v \in V$ and $q(v)$ is divisible in $V/\delta(V)$, then one can verify that v is divisible in V.

Let I be a nonempty set again, and let V_j be a module over k for each $j \in I$. One can check that

$$\delta\Big(\prod_{j \in I} V_j\Big) = \prod_{j \in I} \delta(V_j), \tag{11.1.9}$$

as on p128 of [3].

11.2 More on torsion and divisibility

Let k be an integral domain again, and let V be a module over k. If V is injective as a module over k, then V is divisible as a module over k, as in Proposition 1.2 on p128 of [3]. To see this, let $v \in V$ and $t_0 \in k$ with $t_0 \neq 0$ be given, and note that $t_0\, k = \{t_0\, x : x \in k\}$ is an ideal in k. If $t \in t_0\, k$, then we would like to put

$$\phi(t\, t_0) = t \cdot v. \tag{11.2.1}$$

It is easy to see that this is well defined, because k has no nontrivial zero divisors.

More precisely, ϕ defines a homomorphism from $t_0\, k$ into V, as modules over k. Because V is injective, there is a $w \in V$ such that

$$\phi(t\, t_0) = (t\, t_0) \cdot w \tag{11.2.2}$$

for every $t \in k$, as in Section 2.8. It follows that

$$v = t_0 \cdot w, \tag{11.2.3}$$

as desired.

If V is torsion-free and divisible, then V is injective as a module over k, as in Proposition 1.3 on p128 of [3]. Indeed, let $\mathcal{I}$ be an ideal in k, and let ϕ be a homomorphism from $\mathcal{I}$ into V, as modules over k. It suffices to show that there is a $w \in V$ such that

$$\phi(t) = t \cdot w \tag{11.2.4}$$

for every $t \in \mathcal{I}$, as in Section 2.8. Of course, if $\mathcal{I} = \{0\}$, then we can take $w = 0$, and so we may suppose that $\mathcal{I} \neq \{0\}$.

If $t \in \mathcal{I}$ and $t \neq 0$, then there is a unique $w_t \in V$ such that

$$\phi(t) = t \cdot w_t, \tag{11.2.5}$$

because V is torsion-free and divisible. If $t_1, t_2 \in \mathcal{I}$ and $t_1, t_2 \neq 0$, then

$$\begin{aligned} (t_1\, t_2) \cdot w_{t_1} &= t_2 \cdot (t_1 \cdot w_{t_1}) = t_2 \cdot \phi(t_1) = \phi(t_1\, t_2) \\ &= t_1 \cdot \phi(t_2) = t_1 \cdot (t_2 \cdot w_{t_2}) = (t_1\, t_2) \cdot w_{t_2}. \end{aligned} \tag{11.2.6}$$

This implies that $w_{t_1} = w_{t_2}$, because $t_1\, t_2 \neq 0$ in k, and V is torsion-free. If we let w be the common value of w_t, $t \in \mathcal{I}$, $t \neq 0$, then (11.2.4) holds for every $t \in \mathcal{I}$, as desired.

If $t \in k$, then

$$v \mapsto t \cdot v \tag{11.2.7}$$

defines a homomorphism from V into itself, as a module over k. The condition that V be torsion-free says exactly that this mapping is injective when $t \neq 0$. Similarly, V is divisible exactly when (11.2.7) maps V onto itself when $t \neq 0$.

Let W be another module over k, and let $V \bigotimes_k W$ be a tensor product of V and W, as modules over k. If $t \in k$, then the action of t on $V \bigotimes_k W$ is the unique homomorphism from $V \bigotimes_k W$ into itself, as a module over k, such that

$$v \otimes w \mapsto t \cdot (v \otimes w) \tag{11.2.8}$$

for every $v \in V$ and $w \in W$. This is the same as the homomorphism from $V \bigotimes_k W$ into itself, as a module over k, corresponding to (11.2.7) and the identity mapping on W.

If V is divisible, then (11.2.7) maps V onto itself when $t \neq 0$, as before. This implies that the action of t on $V \bigotimes_k W$ is surjective when $t \neq 0$, as in Section 1.3. It follows that

$$V \bigotimes_k W \text{ is divisible} \tag{11.2.9}$$

as a module over k in this case, as in Corollary 1.5 on p128 of [3]. Similarly, (11.2.9) holds when W is divisible, as a module over k.

If V is torsion-free and divisible, then (11.2.7) defines an isomorphism from V onto itself, as a module over k, when $t \neq 0$. This implies that the action of t on $V \bigotimes_k W$ is an isomorphism from $V \bigotimes_k W$ onto itself, as a module over k, when $t \neq 0$, as in Section 1.3. This means that

$$V \bigotimes_k W \text{ is torsion-free and divisible} \tag{11.2.10}$$

as a module over k, as in the last part of Proposition 1.4 on p128 of [3]. This also works when W is torsion-free and divisible.

If W is torsion-free as a module over k, then it is easy to see that

$$\mathrm{Hom}_k(V, W) \text{ is torsion-free,} \tag{11.2.11}$$

as a module over k. One can check that this also holds when V is divisible, using the surjectivity of (11.2.7) when $t \neq 0$. This is the second part of Corollary 1.5 on p128 of [3].

If V is torsion-free and divisible, then the action of t on $\mathrm{Hom}_k(V, W)$ is an isomorphism when $t \neq 0$, because (11.2.7) is an isomorphism from V onto itself. This implies that

$$\mathrm{Hom}_k(V, W) \text{ is torsion-free and divisible,} \tag{11.2.12}$$

as a module over k. Similarly, if W is torsion-free and divisible, then the action of t on W is an isomorphism from W onto itself, as a module over k, when $t \neq 0$. This implies that the action of t on $\mathrm{Hom}_k(V, W)$ is an isomorphism from $\mathrm{Hom}_k(V, W)$ onto itself, as a module over k, when $t \neq 0$, so that (11.2.12) holds. This corresponds to the last part of Proposition 1.4 on p128 of [3] again.

If V is a torsion module and W is divisible, then

$$V \bigotimes_k W = \{0\}, \tag{11.2.13}$$

as in Proposition 1.8 on p129 of [3]. To see this, let $v \in V$ and $w \in W$ be given, and let us check that

$$v \otimes w = 0 \tag{11.2.14}$$

in $V \bigotimes_k W$. Because V is a torsion module, there is a $t \in k$ such that $t \neq 0$ and $t \cdot v = 0$. There is also a $u \in W$ such that $w = t \cdot u$, because W is divisible. It follows that

$$v \otimes w = v \otimes (t \cdot u) = (t \cdot v) \otimes u = 0, \tag{11.2.15}$$

as desired.

If ϕ is a homomorphism from V into W, as modules over k, then it is easy to see that

$$\phi(\tau(V)) \subseteq \tau(W), \tag{11.2.16}$$

as in the second part of Exercise 12 on p45 of [1]. Similarly,

$$\phi(\delta(V)) \subseteq \delta(W). \tag{11.2.17}$$

If V is a torsion module and W is torsion-free, then (11.2.16) implies that

$$\phi = 0. \tag{11.2.18}$$

This means that

$$\mathrm{Hom}_k(V, W) = \{0\} \tag{11.2.19}$$

under these conditions, as in Proposition 1.8 on p129 of [3].

11.3 Some easy homomorphisms

Let k be an integral domain, and let V be a module over k. If $v_1, \ldots, v_n$ are finitely many torsion elements of V, then it is easy to see that there is a $t \in k$ such that $t \neq 0$ and

$$t \cdot v_j = 0 \tag{11.3.1}$$

for every $j = 1, \ldots, n$. Of course, this implies that $t \cdot v = 0$ for every v in the submodule of V generated by $v_1, \ldots, v_n$. If V is finitely generated and torsion, as a module over k, then there is a $t \in k$ such that $t \neq 0$ and

$$t \cdot v = 0 \text{ for every } v \in V. \tag{11.3.2}$$

This corresponds to part of the proof of Proposition 1.6 on p128 of [3].

If $t \in k$, then

$$t \cdot V = \{t \cdot v : v \in V\} \tag{11.3.3}$$

is a submodule of V. Similarly,

$${}_tV = \{v \in V : t \cdot v = 0\} \tag{11.3.4}$$

is a submodule of V. Put

$$V_t = V/(t \cdot V), \tag{11.3.5}$$

which is another module over k.

Note that

$$x \mapsto t\,x \tag{11.3.6}$$

defines a homomorphism from k into itself, as a module over itself. This homomorphism is injective when $t \neq 0$, because k is an integral domain.

Remember that V satisfies the requirements of a tensor product of k with V, as modules over k, as in Section 1.4. Using (11.3.6) and the identity mapping on V, we get a homomorphism

$$\text{from } k \bigotimes\nolimits_k V \text{ into itself,} \tag{11.3.7}$$

as a module over k, as in Section 1.3. This corresponds to $v \mapsto t \cdot v$, as a homomorphism from V into itself. Of course, (11.3.4) is the kernel of this homomorphism on V.

If $v \in V$, then

$$y \mapsto y \cdot v \tag{11.3.8}$$

defines a homomorphism from k into V, as modules over k. Every homomorphism from k into V, as modules over k, corresponds to a unique $v \in V$ in this way. The restriction of (11.3.8) to $y \in t\,k$ defines a homomorphism from $t\,k$ into V, as modules over k. This homomorphism is equal to 0 on $t\,k$ exactly when $v \in {}_tV$.

Suppose that $t \neq 0$, and let ϕ be a homomorphism from $t\,k$ into V, as modules over k. Thus

$$\phi(t\,x) = x \cdot \phi(t) \tag{11.3.9}$$

for every $x \in k$. Note that $\phi(t)$ can be any element of V. It is easy to see that (11.3.8) is equal to ϕ on $t\,k$ if and only if

$$\phi(t) = t \cdot v. \tag{11.3.10}$$

In particular, ϕ has an extension to a homomorphism from k into V, as modules over k, exactly when $\phi(t) \in t \cdot V$.

These remarks correspond to some of those on p129 of [3].

11.4 Fields of fractions

Let k be an integral domain, and let Q_k be the corresponding field of fractions, or quotients. We may identify k with a subring of Q_k, as usual. Note that Q_k is torsion-free and divisible, and thus injective, as a module over k. It follows that the quotient Q_k/k is divisible, as a module over k.

If $t \in k$ and $t \neq 0$, then

$$(1/t)\,k = \{x/t : x \in k\} \tag{11.4.1}$$

is a submodule of Q_k, as a module over k. We also have that

$$Q_k = \bigcup \{(1/t)\,k : t \in k,\, t \neq 0\}, \tag{11.4.2}$$

by the definition of Q_k.

Note that

$$(1/r)\,k,\ (1/t)\,k \subseteq (1/(r\,t))\,k \tag{11.4.3}$$

when $r, t \in k$ and $r, t \neq 0$. More precisely, if $t_1, t_2 \in k$ and $t_1, t_2 \neq 0$, then

$$(1/t_1)\,k \subseteq (1/t_2)\,k \tag{11.4.4}$$

if and only if $1/t_1 \in (1/t_2)\,k$, which means that

$$t_2 = r\,t_1 \tag{11.4.5}$$

for some $r \in k$.

Put

$$t_1 \preceq t_2 \tag{11.4.6}$$

when $t_1, t_2 \in k \setminus \{0\}$ satisfy (11.4.5) for some $r \in k$. This defines a pre-order on $k \setminus \{0\}$, which makes $k \setminus \{0\}$ a directed system, because

$$t_1,\, t_2 \preceq t_1\,t_2 \tag{11.4.7}$$

for every $t_1, t_2 \in k \setminus \{0\}$.

One may identify Q_k, as a module over k, with the direct limit of the submodules (11.4.1), as in Exercise 12 on p45 of [1], and mentioned on p130 of [3]. More precisely, one may consider these submodules to be indexed by $k \setminus \{0\}$, or simply ordered by inclusion, as in Section 3.3.

Let V be a module over k, and let $Q_k \bigotimes_k V$ be a tensor product of Q_k and V, as modules over k. There is a natural homomorphism

(11.4.8) $$\phi \text{ from } V \text{ into } Q_k \bigotimes_k V,$$

as modules over k, with

(11.4.9) $$\phi(v) = 1 \otimes v$$

for every $v \in V$. If we identify V with $k \bigotimes_k V$, as in Section 1.4, then ϕ corresponds to the homomorphism from $k \bigotimes_k V$ into $Q_k \bigotimes_k V$ associated to the natural inclusion mapping from k into Q_k and the identity mapping on V, as in Section 1.3. We would like to show that

(11.4.10) $$\ker \phi = \tau(V),$$

as in the fourth part of Exercise 12 on p45 of [1], and Proposition 2.1 on p130 of [3].

If $t \in k$ and $t \neq 0$, then let $((1/t)\, k) \bigotimes_k V$ be a tensor product of $(1/t)\, k$ and V, as modules over k. There is a natural homomorphism

(11.4.11) $$\text{from } ((1/t)\, k) \bigotimes_k V \text{ into } Q_k \bigotimes_k V,$$

as modules over k, associated to the obvious inclusion mapping from $(1/t)\, k$ into Q_k and the identity mapping on V, as in Section 1.3. There is also a natural homomorphism

(11.4.12) $$\phi_t \text{ from } V \text{ into } ((1/t)\, k) \bigotimes_k V,$$

as modules over k, with

(11.4.13) $$\phi_t(v) = 1 \otimes v$$

for every $v \in V$. As before, ϕ_t corresponds to the homomorphism from $k \bigotimes_k V$ into $((1/t)\, k) \bigotimes_k V$ associated to the natural inclusion mapping from k into $(1/t)\, k$ and the identity mapping on V. Observe that

(11.4.14) $$\begin{array}{l} \phi \text{ is the same as the composition of } \phi_t \\ \text{with the homomorphism as in (11.4.11).} \end{array}$$

In particular,

(11.4.15) $$\ker \phi_t \subseteq \ker \phi.$$

In fact,

(11.4.16) $$\ker \phi = \bigcup \{\ker \phi_t : t \in k,\, t \neq 0\}.$$

This can be seen by identifying $Q_k \bigotimes_k V$ with the direct limit of $((1/t)\, k) \bigotimes_k V$, as in Section 3.6.

Let $t \in k \setminus \{0\}$ be given, and note that

(11.4.17) $$(1/t)\, x \mapsto x$$

defines an isomorphism from $(1/t)\,k$ onto k, as modules over k. This leads to an isomorphism

$$\psi_t \text{ from } ((1/t)\,k) \bigotimes_k V \text{ onto } k \bigotimes_k V, \tag{11.4.18}$$

as modules over k, using the identity mapping on V, as in Section 1.3. This may be considered as an isomorphism from $((1/t)\,k) \bigotimes_k V$ onto V, by identifying $k \bigotimes_k V$ with V, as before. It follows that

$$\ker \phi_t = \ker(\psi_t \circ \phi_t). \tag{11.4.19}$$

One can check that $\psi_t \circ \phi_t$ corresponds to

$$v \mapsto t \cdot v, \tag{11.4.20}$$

as a homomorphism from V into itself. Thus

$$\ker(\psi_t \circ \phi_t) = \{v \in V : t \cdot v = 0\}. \tag{11.4.21}$$

This implies (11.4.10), as desired.

11.5 Some injective mappings

Let k be an integral domain, and let Q_k be the corresponding field of fractions. If V is a finitely-generated torsion-free module over k, then there is an injective homomorphism from V into a free module over k with a finite basis, as in Proposition 2.4 on p131 of [3]. To see this, let $Q_k \bigotimes_k V$ be a tensor product of Q_k and V, as modules over k, as in the previous section. Thus (11.4.9) defines an injective homomorphism ϕ from V into $Q_k \bigotimes_k V$, as modules over k, as before.

Note that $Q_k \bigotimes_k V$ may be considered as a module over Q_k, as in Section 1.10. This means that $Q_k \bigotimes_k V$ may be considered as a vector space over Q_k, because Q_k is a field. Let $v_1, \ldots, v_n$ be generators for V, as a module over k. It is easy to see that $Q_k \bigotimes_k V$ is spanned by $\phi(v_1), \ldots, \phi(v_n)$, as a vector space over Q_k. In particular, $Q_k \bigotimes_k V$ has finite dimension, as a vector space over Q_k.

Let $e_1, \ldots, e_m$ be a basis for $Q_k \bigotimes_k V$, as a vector space over Q_k. Thus, for each $j = 1, \ldots, n$, we can express $\phi(v_j)$ as

$$\phi(v_j) = \sum_{l=1}^{m} q_{j,l}\, e_l, \tag{11.5.1}$$

where $q_{j,l} \in Q_k$ for every $l = 1, \ldots, m$. It is easy to see that there is a $t \in k$ such that $t \neq 0$ and

$$t\, q_{j,l} \in k \tag{11.5.2}$$

for every $j = 1, \ldots, n$ and $l = 1, \ldots, m$. Of course, (11.5.1) is the same as saying that

$$\phi(v_j) = \sum_{l=1}^{n} t\, q_{j,l}\, (t^{-1}\, e_l) \tag{11.5.3}$$

for every $j = 1, \ldots, n$.

Let W be the submodule of $Q_k \bigotimes_k V$, as a module over k, generated by

$$t^{-1}\,e_1, \ldots, t^{-1}\,e_m. \tag{11.5.4}$$

Observe that W is freely generated by (11.5.4), because $e_1, \ldots, e_m$ is a basis for $Q_k \bigotimes_k V$, as a vector space over Q_k. We also have that

$$\phi(V) \subseteq W, \tag{11.5.5}$$

because $\phi(v_j) \in W$ for each $j = 1, \ldots, n$. This means that ϕ may be considered as a homomorphism from V into W, as modules over k, as desired.

Let U be a projective module over k, and let U_0 be a submodule of U that is also projective, as a module over k. Let V be another module over k, and let $U_0 \bigotimes_k V$, $U \bigotimes_k V$ be tensor products of U_0, U with V, respectively, as modules over k. This leads to a homomorphism

$$\text{from } U_0 \bigotimes_k V \text{ into } U \bigotimes_k V, \tag{11.5.6}$$

as modules over k, using the obvious inclusion mapping from U_0 into U, and the identity mapping on V. If V is torsion-free, then Proposition2.5 on p131 of [3] states that

$$\text{the homomorphism as in (11.5.6) is injective.} \tag{11.5.7}$$

To show this, we start with the case where V is finitely generated as a module over k, so that there is an injective homomorphism from V into a free module W over k, as before. Let $U_0 \bigotimes_k W$, $U \bigotimes_k W$ be tensor products of U_0, U with W, respectively, as modules over k. We get homomorphisms

$$\text{from } U_0 \bigotimes_k V \text{ into } U_0 \bigotimes_k W \tag{11.5.8}$$

and

$$\text{from } U \bigotimes_k V \text{ into } U \bigotimes_k W, \tag{11.5.9}$$

as modules over k, using the identity mappings on U_0, U, respectively, and the homomorphism from V into W mentioned before. We also get a homomorphism

$$\text{from } U_0 \bigotimes_k W \text{ into } U \bigotimes_k W, \tag{11.5.10}$$

as modules over k, using the obvious inclusion mapping from U_0 into U, and the identity mapping on W.

Under these conditions, each of the three homomorphisms mentioned in the preceding paragraph is injective, by the remarks near the end of Section 2.7. This uses the projectivity of U_0, U, and W, and the injectivity of the inclusion mapping from U_0 into U, and the homomorphism from V into W being considered.

It is easy to see that the composition of the homomorphism as in (11.5.6) with the homomorphism as in (11.5.9) is the same as the composition of the homomorphism as in (11.5.8) with the homomorphism as in (11.5.10). More precisely, both compositions are the same as the homomorphism

$$\text{from } U_0 \bigotimes_k V \text{ into } U \bigotimes_k W, \tag{11.5.11}$$

as modules over k, obtained from the obvious inclusion mapping from U_0 into U, and the homomorphism from V into W mentioned earlier. One can use this and the injectivity conditions mentioned in the previous paragraph to get (11.5.7) in this case.

If V is not necessarily finitely generated as a module over k, then one may identify it with a direct limit of finitely-generated submodules, as in Section 3.12. We may also identify $U_0 \bigotimes_k V$, $U \bigotimes_k V$ with direct limits of tensor products of U_0, U with these finitely-generated submodules of V, as in Section 3.6. Of course, these submodules of V are torsion-free too. One can use this to obtain (11.5.7) from the analogous statement for finitely-generated torsion-free modules.

11.6 Another injective mapping

Let k be an integral domain, and let Q_k be the corresponding field of fractions again. If Y is any module over k, then there is an injective homomorphism from Y into a divisible module over k, as in Proposition 2.6 on p132 of [3]. To show this, one can start with a homomorphism η from a torsion-free module V over k onto Y. More precisely, one can take V to be a free module over k, as in Section 2.7, for instance.

Let $Q_k \bigotimes_k V$ be a tensor product of Q_k and V, as modules over k, and let ϕ be the homomorphism from V into $Q_k \bigotimes_k V$ with $\phi(v) = 1 \otimes v$ for every $v \in V$, as in Section 11.4. Thus ϕ is injective, because V is torsion-free, as before. If V is a free module over k, then $Q_k \bigotimes_k V$ is a direct sum of a family of copies of Q_k, and the injectivity of ϕ can be seen more directly.

Using ϕ, we get a homomorphism

$$\text{from } V/\ker\eta \text{ into } \big(Q_k \bigotimes_k V\big)/\phi(\ker\eta), \tag{11.6.1}$$

as modules over k. It is easy to see that this homomorphism is injective, because ϕ is injective. Of course,

$$\phi(\ker\eta) = \ker(\phi\circ\eta), \tag{11.6.2}$$

because ϕ is injective. Note that $Q_k \bigotimes_k V$ is divisible as a module over k, because Q_k is divisible as a module over k, as in Section 11.2. This implies that

$$\big(Q_k \bigotimes_k V\big)/\phi(\ker\eta) \text{ is divisible} \tag{11.6.3}$$

as a module over k, as in Section 11.1.

Using η, we get an isomorphism

$$\text{from } V/\ker\eta \text{ onto } Y, \tag{11.6.4}$$

as modules over k. Thus the homomorphism as in (11.6.1) leads to an injective homomorphism

$$\text{from } Y \text{ into } \big(Q_k \bigotimes_k V\big)/\phi(\ker\eta), \tag{11.6.5}$$

as modules over k, as desired.

Now let k be any commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a left module over A. It is well known that V is projective as a module over A if and only if there is a family $\{v_j\}_{j\in I}$ of elements of V and a family $\{\phi_j\}_{j\in I}$ of homomorphisms from V into A, as left modules over A, with the following properties: for every $v \in V$,

$$\phi_j(v) = 0 \text{ for all but finitely many } j \in I, \tag{11.6.6}$$

and

$$v = \sum_{j\in I} \phi_j(v)\cdot v_j. \tag{11.6.7}$$

This is Proposition 3.1 on p132 of [3]. Of course, there is an analogous statement for right modules over A.

If V is projective, then we can start with a free left module U over A and a homomorphism ψ from U onto V, as modules over A, as in Section 2.7. Projectivity of V implies that there is a homomorphism ϕ from V into U, as modules over A, such that $\psi\circ\phi$ is the identity mapping on V. Let $\{e_j\}_{j\in I}$ be a basis for U as a free left module over A, and put

$$v_j = \psi(e_j) \tag{11.6.8}$$

for each $j \in I$. Every $u \in U$ can be expressed in a unique way as

$$u = \sum_{j\in I} u_j \cdot e_j, \tag{11.6.9}$$

where $u_j \in A$ for every $j \in I$, and $u_j = 0$ for all but finitely many $j \in I$. If $v \in V$, then we can take $u = \psi(v)$, to get that

$$v = \phi(\psi(v)) = \sum_{j\in I} (\psi(v))_j\, v_j, \tag{11.6.10}$$

with $(\psi(v))_j = 0$ for all but finitely many $j \in I$.

More precisely, in this argument, we can start with any family $\{v_j\}_{j\in I}$ of elements of V such that V is generated by the v_j's, $j \in I$, as a left module over A. We can take U to be a free left module over A with a basis $\{e_j\}_{j\in I}$, using the same set I. In this case, there is a unique homomorphism ψ from U into V, as left modules over A, that satisfies (11.6.8), and which maps U onto V.

Conversely, suppose that there are families $\{v_j\}_{j \in I}$ of elements of V and $\{\phi_j\}_{j \in I}$ of homomorphisms from V into A that satisfy (11.6.6) and (11.6.7). Let U be a free left module over A with basis $\{e_j\}_{j \in I}$, for some family of elements indexed by the same set I. Thus there is a unique homomorphism ψ from U onto V, as left modules over A, as in (11.6.8). Note that

$$\phi(v) = \sum_{j \in I} \phi_j(v) \cdot e_j \tag{11.6.11}$$

defines a homomorphism from V into U, as left modules over A. If $v \in V$, then

$$\psi(\phi(v)) = \sum_{j \in I} \phi_j(v)\, v_j = v, \tag{11.6.12}$$

by (11.6.7). This implies that ϕ is injective, and that U corresponds to the direct sum of $\phi(V)$ and $\ker \psi$, as a left module over A. It follows that V is projective, as in Section 2.7.

11.7 Fractional and invertible ideals

Let k be an integral domain, and let Q_k be the corresponding field of fractions. A submodule M of Q_k, as a module over k, is called a *fractional ideal* of k if there is an $x \in k$ such that $x \neq 0$ and

$$x\, M \subseteq k, \tag{11.7.1}$$

as on p96 of [1]. The ordinary ideals in k satisfy this condition with $x = 1$, and may be called *integral ideals* of k.

If (11.7.1) holds for some $x \in Q_k$ with $x \neq 0$, then it is easy to see that M is a fractional ideal of k. More precisely, one can multiply x by a suitable nonzero element of k, to get another nonzero element of k.

If M is a fractional ideal of k, then we put

$$(k : M) = \{x \in Q_k : x\, M \subseteq k\}, \tag{11.7.2}$$

as on p96 of [1]. It is easy to see that this is a submodule of Q_k, as a module over k. Note that $(k : M) \neq \{0\}$, by the definition of a fractional ideal.

If $y \in M$, then

$$y\,(k : M) \subseteq k, \tag{11.7.3}$$

by the definition of $(k : M)$. This implies that $(k : M)$ is a fractional ideal of k when $M \neq \{0\}$.

Let M be a submodule of Q_k, as a module over k. If M is finitely generated as a module over k, then M is a fractional ideal of k, as on p96 of [1]. Indeed, suppose that M is generated by $y_1, \ldots, y_n \in Q_k$ as a module over k. It is easy to see that there are $x, z_1, \ldots, z_n \in k$ such that

$$y_j = z_j / x \tag{11.7.4}$$

for each $j = 1, \ldots, n$. This implies that (11.7.1) holds, as desired.

Of course, if k is Noetherian as a ring, then every integral ideal of k is finitely generated as a module over k. This implies that every fractional ideal of k is finitely generated as a module over k, as on p96 of [1].

Let M, N be submodules of Q_k, as a module over k. The product $M\,N$ is defined to be the subset of Q_k consisting of finite sums of products of elements of M and N. This is a submodule of Q_k, as a module over k, as well. Note that multiplication in this sense is commutative and associative, with k as the identity element.

We say that M is an *invertible ideal* of k if there is a submodule N of Q_k, as a module over k, such that

$$M\,N = k, \tag{11.7.5}$$

as on p96 of [1]. In particular, this implies that $M, N \neq \{0\}$, so that M, N are fractional ideals of k. We also get that

$$N \subseteq (k : M), \tag{11.7.6}$$

and that $k = M\,N \subseteq M\,(k : M) \subseteq k$. In fact, we have that

$$(k : M) \subseteq (k : M)\,M\,N \subseteq k\,N = N, \tag{11.7.7}$$

so that

$$N = (k : M), \tag{11.7.8}$$

as in [1].

If M is an invertible ideal, then there are finitely many elements $x_1, \ldots, x_n$ of M and $y_1, \ldots, y_n$ of $(k : M)$ such that

$$\sum_{j=1}^{n} x_j\,y_j = 1. \tag{11.7.9}$$

This implies that

$$x = \sum_{j=1}^{n} (x\,y_j)\,x_j \tag{11.7.10}$$

for every $x \in M$, with $x\,y_j \in k$ for each $j = 1, \ldots, n$, because $y_j \in (k : M)$. It follows that M is generated by $x_1, \ldots, x_n$ as a module over k, and in particular that M is finitely generated as a module over k, as on p96 of [1].

Conversely, if M is a fractional ideal of k, and there are $x_1, \ldots, x_n \in M$ and $y_1, \ldots, y_n \in (k : M)$ such that (11.7.9) holds, then it is easy to see that $M\,(k : M) = k$, so that M is invertible.

11.8 Inversible ideals

Let k be an integral domain, and let Q_k be the corresponding field of fractions. An ideal $\mathcal{I}$ of k is said to be *inversible* if there are finitely many elements $a_1, \ldots, a_n$ of $\mathcal{I}$ and $q_1, \ldots, q_n$ of Q_k such that

$$q_j\,\mathcal{I} \subseteq k \tag{11.8.1}$$

for every $j = 1, \ldots, n$, and

$$\sum_{j=1}^{n} q_j\, a_j = 1, \tag{11.8.2}$$

as on p132 of [3]. This is the same as the characterization of the invertibility of $\mathcal{I}$ as a fractional ideal of k mentioned in the previous section. Although these conditions are practically the same, it is convenient to use the slightly different terms to reflect the slightly different situations in which they are used. Note that $\mathcal{I} \neq \{0\}$ in this case.

Proposition 3.2 on p132 of [3] states that a nonzero ideal $\mathcal{I}$ in k is inversible if and only if it is projective as a module over k. Suppose that $\mathcal{I}$ is inversible, and let $a_1, \ldots, a_n$ and $q_1, \ldots, q_n$ be as in the preceding paragraph. Put

$$\phi_j(x) = q_j\, x \tag{11.8.3}$$

for each $j = 1, \ldots, n$ and $x \in \mathcal{I}$, which defines a homomorphism from $\mathcal{I}$ into k, as modules over k. If $x \in \mathcal{I}$, then

$$x = \sum_{j=1}^{n} q_j\, x\, a_j = \sum_{j=1}^{n} \phi_j(x)\, a_j, \tag{11.8.4}$$

using (11.8.2) in the first step. This implies that $\mathcal{I}$ is projective as a module over k, as in Section 11.6.

Before showing the converse, let us mention a helpful fact. Let $\mathcal{I}$ be a nonzero ideal in k, and let ϕ be a homomorphism from $\mathcal{I}$ into k. If $x, y \in \mathcal{I}$, then

$$x\, \phi(y) = \phi(x\, y) = y\, \phi(x). \tag{11.8.5}$$

It follows that there is a $q \in Q_k$ such that

$$q = \phi(x)/x \tag{11.8.6}$$

for every $x \in \mathcal{I}$ with $x \neq 0$. This means that

$$\phi(x) = q\, x \tag{11.8.7}$$

for every $x \in \mathcal{I}$.

Suppose now that $\mathcal{I}$ is a nonzero ideal in k that is projective as a module over k. This implies that there is a nonempty family $\{a_j\}_{j \in L}$ of elements of $\mathcal{I}$ and a family $\{\phi_j\}_{j \in L}$ of homomorphisms from $\mathcal{I}$ into k, as modules over k, such that for every $x \in \mathcal{I}$, $\phi_j(x) = 0$ for all but finitely many $j \in L$, and

$$x = \sum_{j \in L} \phi_j(x)\, a_j, \tag{11.8.8}$$

as in Section 11.6. Using the remarks in the preceding paragraph, we get that for each $j \in L$, there is a unique $q_j \in Q_k$ such that (11.8.3) holds for every $x \in \mathcal{I}$. Note that $q_j = 0$ for all but finitely many $j \in L$, because (11.8.3) is supposed to be equal to 0 for all but finitely many $j \in L$, and we can take $x \neq 0$. We may as well replace L with the finite subset of $j \in L$ such that $q_j \neq 0$.

It follows from (11.8.8) that

$$x = \sum_{j \in L} x\, q_j\, a_j = x \sum_{j \in L} q_j\, a_j \tag{11.8.9}$$

for every $x \in \mathcal{I}$. Taking $x \neq 0$, we obtain that

$$\sum_{j \in L} q_j\, a_j = 1. \tag{11.8.10}$$

Note that (11.8.1) holds for every $j \in L$, because $\phi_j(\mathcal{I}) \subseteq k$.

Proposition 3.3 on p132 of [3] states that inversible ideals in k are finitely generated as modules over k. This was mentioned already in the previous section, for invertible fractional ideals of k. One could also have used this in the previous argument, to get L to be finite at the beginning.

Let $\mathcal{I}$ be an ideal in k, and let V be a module over k. There is a natural homomorphism

$$\text{from } \mathrm{Hom}_k(k, V) \text{ into } \mathrm{Hom}_k(\mathcal{I}, V), \tag{11.8.11}$$

as modules over k, which sends a homomorphism from k into V, as modules over k, to its restriction to $\mathcal{I}$. If $\mathcal{I}$ is inversible and V is divisible, then Proposition 3.4 on p133 of [3] states that

(11.8.12) the homomorphism as in (11.8.11) is surjective.

Equivalently, let ϕ be a homomorphism from $\mathcal{I}$ into V, as modules over k. We would like to show that there is a $v \in V$ such that

$$\phi(x) = x \cdot v \tag{11.8.13}$$

for every $x \in \mathcal{I}$.

Because $\mathcal{I}$ is inversible, there are $a_1, \ldots, a_n \in \mathcal{I}$ and $q_1, \ldots, q_n \in Q_k$ satisfying (11.8.1) and (11.8.2). Let us choose, for each $j = 1, \ldots, n$, an element v_j of V such that

$$\phi(a_j) = a_j \cdot v_j. \tag{11.8.14}$$

More precisely, we can take $v_j = 0$ when $a_j = 0$, and otherwise use the divisibility of V. We may also require that $a_j \neq 0$ for each j, by dropping and a_j's equal to 0.

If $x \in \mathcal{I}$, then

$$\phi(x) = \phi\Big(\sum_{j=1}^{n} q_j\, x\, a_j\Big) = \sum_{j=1}^{n} (q_j\, x) \cdot \phi(a_j), \tag{11.8.15}$$

using (11.8.2) in the first step, and the fact that $q_j\, x \in k$ for each $j = 1, \ldots, n$ in the second step. This implies that

$$\phi(x) = \sum_{j=1}^{n} (q_j\, x) \cdot (a_j \cdot v_j) = \sum_{j=1}^{n} (q_j\, x\, a_j) \cdot v_j, \tag{11.8.16}$$

using (11.8.14) in the first step. It follows that

$$\phi(x) = x \cdot \Big(\sum_{j=1}^{n} (q_j\, a_j) \cdot v_j \Big), \tag{11.8.17}$$

because $q_j\, a_j \in k$ for every $j = 1, \ldots, n$. Thus (11.8.13) holds, with $v = \sum_{j=1}^{n} (q_j\, a_j) \cdot v_j$.

11.9 Prüfer rings

Let k be a commutative ring with a multiplicative identity element. As in Section 9.5, we say that k is *semi-hereditary* if every ideal in k that is finitely generated as a module over k is projective as a module over k.

Suppose now that k is an integral domain. In this case, k is semi-hereditary if and only if every nonzero ideal in k that is finitely generated as a module over k is inversible, because inversibility is equivalent to projectivity, as in the previous section. Under these conditions, k is said to be a *Prüfer ring*, as on p133 of [3].

Proposition 4.1 on p133 of [3] states that k is a Prüfer ring if and only if

(11.9.1) every finitely-generated torsion-free module over k is projective.

Note that ideals in k are torsion-free as modules over k, because k is an integral domain. Thus (11.9.1) implies that k is semi-hereditary.

Conversely, let V be a finitely-generated torsion-free module over k. This implies that there is an injective homomorphism from V into a free module over k with a finite basis, as in Section 11.5. Thus we may identify V with a submodule of a free module over k. If k is semi-hereditary, then every finitely-generated submodule of a free module over k is projective, as in Section 9.5. This means that V is projective as a module over k, as desired.

Suppose that k is a Prüfer ring, and let V be a torsion-free module over k. Also let U_1, U_2, U_3 be modules over k, and let $\theta\ \theta'$ be homomorphisms from U_1, U_2 into U_2, U_3, respectively, as modules over k. If V_0 is a submodule of V, then we let $U_1 \bigotimes_k V_0$, $U_2 \bigotimes_k V_0$, $U_3 \bigotimes_k V_0$ be tensor products of U_1, U_2, U_3 with V_0, as modules over k, respectively. This leads to homomorphisms

$$\Theta_{V_0} \text{ from } U_1 \bigotimes_k V_0 \text{ into } U_2 \bigotimes_k V_0 \tag{11.9.2}$$

and

$$\Theta'_{V_0} \text{ from } U_2 \bigotimes_A V_0 \text{ into } U_3 \bigotimes_k V_0, \tag{11.9.3}$$

as modules over k, using θ, θ', respectively, and the identity mapping on V. Suppose that

$$U_1 \xrightarrow{\theta} U_2 \xrightarrow{\theta'} U_3 \tag{11.9.4}$$

is exact, so that $\theta(U_1) = \ker \theta'$.

If V_0 is a finitely-generated submodule of V, then V_0 is projective as a module over k, because V_0 is torsion-free. In this case,

$$U_1 \bigotimes_A V_0 \xrightarrow{\Theta_{V_0}} U_2 \bigotimes_A V_0 \xrightarrow{\Theta'_{V_0}} U_3 \bigotimes_A V_0 \tag{11.9.5}$$

is exact, as in Section 2.7. We can use this to get that

$$U_1 \bigotimes_k V \xrightarrow{\Theta_V} U_2 \bigotimes_k V \xrightarrow{\Theta'_V} U_3 \bigotimes_k V \tag{11.9.6}$$

is exact, as in Corollary 4.3 on p134 of [3].

More precisely, V corresponds to the direct limit of its finitely-generated submodules, as in Section 3.12. Similarly, for each $j = 1, 2, 3$, $U_j \bigotimes_k V$ may be considered as a direct limit of $U_j \bigotimes_k V_0$, where V_0 is a finitely-generated submodule of V, as in Section 3.6. We may consider Θ_V, Θ'_V as direct limits of Θ_{V_0}, Θ'_{V_0}, respectively, where V_0 is a finitely-generated submodule of V, as in Section 3.4. This permits us to obtain the exactness of (11.9.6) from the exactness of (11.9.5), as before. Of course, there are analogous statements for $V \bigotimes_k U_j$ instead of $U_j \bigotimes_k V$.

Suppose that k is a Prüfer ring again, let V, W be torsion-free modules over k, and let $V \bigotimes_k W$ be a tensor product of V and W over k. Under these conditions,

$$V \bigotimes_k W \text{ is torsion-free} \tag{11.9.7}$$

as a module over k, as in Proposition 4.5 on p134 of [3]. To see this, let Q_k be the field of fractions corresponding to k, and let $Q_k \bigotimes_k V$ be a tensor product of Q_k and V, as modules over k. The natural homomorphism from V into $Q_k \bigotimes_k V$ discussed in Section 11.4 is injective, because V is torsion-free.

Let $\big(Q_k \bigotimes_k V\big) \bigotimes_k W$ be a tensor product of $Q_k \bigotimes V$ and W, as modules over k. Consider the homomorphism

$$\text{from } V \bigotimes_k W \text{ into } \big(Q_k \bigotimes_k V\big) \bigotimes_k W, \tag{11.9.8}$$

as modules over k, obtained from the mapping from V into $Q_k \bigotimes_k V$ mentioned in the preceding paragraph, and the identity mapping on W. This homomorphism is injective, because the mapping from V into $Q_k \bigotimes_k V$ is injective, and W is torsion-free, by the previous statement about exactness.

Of course, we may identify $\big(Q_k \bigotimes_k V\big) \bigotimes_k W$ with $Q_k \bigotimes_k \big(V \bigotimes_k W\big)$. Using this identification, one can check that the homomorphism as in (11.9.8) corresponds to the natural mapping from $V \bigotimes_k W$ into $Q_k \bigotimes_k \big(V \bigotimes_k W\big)$, as in Section 11.4. The injectivity of this mapping implies (11.9.7), as before.

11.10 Dedekind rings

Let k be a commutative ring with a multiplicative identity element. We say that k is *hereditary* if every ideal in k is projective as a module over k, as in Section

9.3. Similarly, k is *Noetherian* as a ring if k is Noetherian as a module over itself, as in Section 9.7. This means that every ideal in k is finitely generated as a module over k, as before.

Suppose that k is an integral domain. Observe that

$$k \text{ is hereditary if and only if every nonzero ideal in } k \text{ is inversible,} \tag{11.10.1}$$

because inversibility is equivalent to projectivity, as in Section 11.8. In this case, k is said to be a *Dedekind ring*, as on p134 of [3].

Equivalently, k is a Dedekind ring if and only if

$$\text{every nonzero fractional ideal of } k \text{ is invertible.} \tag{11.10.2}$$

Remember that an ideal in k is inversible if and only if it is invertible as a fractional ideal, as in Section 11.8. If M is a fractional ideal of k, $x \in k$, $x \neq 0$, and $x\,M \subseteq k$, then it is easy to see that M is invertible if and only if $x\,M$ is invertible.

Dedekind domains are defined another way on p95 of [1]. This formulation includes the condition that k have dimension one in a standard sense involving prime ideals that will be discussed in Section 14.8, and which implies that k is not a field. The definition of a Dedekind domain on p10 of [15] is equivalent to the one in [1], except that k is only asked to have dimension less than or equal to one in this sense, so that k may be a field.

If k is not a field, then the equivalence of these definitions of Dedekind rings and domains is given in Theorem 9.8 on p97 of [1]. The fact that Dedekind domain satisfy (11.10.2) is also mentioned in Proposition 5 on p11 of [15].

Note that

$$\text{Dedekind rings are Noetherian,} \tag{11.10.3}$$

because inversible ideals in k are finitely generated as modules over k, as in Section 11.8. More precisely, Dedekind rings are the same as Prüfer rings that are Noetherian, as on p134 of [3].

If k is a Dedekind ring and V is a divisible module over k, then Proposition 5.1 on p134 of [3] states that

$$V \text{ is injective as a module over } k. \tag{11.10.4}$$

To see this, it suffices to show that if $\mathcal{I}$ is an ideal in k and ϕ is a homomorphism from $\mathcal{I}$ into V, as modules over k, then there is a $v \in V$ such that $\phi(x) = x \cdot v$ for every $x \in \mathcal{I}$, as in Section 2.8. Of course, this holds trivially when $\mathcal{I} = \{0\}$. Otherwise, if $\mathcal{I} \neq \{0\}$, then $\mathcal{I}$ is inversible as an ideal in k, by hypothesis. In this case, the condition was shown in Section 11.8.

Conversely, if k is an integral domain, and every divisible module over k is injective, then k is a Dedekind ring. This is another part of Proposition 5.1 on p134 of [3]. Remember that injective modules over k are divisible, as in Section 11.2. We also have that quotients of divisible modules are divisible, as in Section 11.1. Thus our hypothesis implies that quotients of injective modules over k are injective too. This implies that k is hereditary, as in Section 9.3. This means that k is a Dedekind ring, as desired.

11.11 Some injective modules

If k is an integral domain and Y is a module over k, then we have seen that there is an injective homomorphism from Y into a divisible module W over k, as in Section 11.6. If k is a Dedekind ring, then W is injective as a module over k, as in the previous section. This gives another approach to the analogous result for modules over arbitrary rings, as in Section 2.8. This corresponds to the remark on p135 of [3].

Let A be a ring with a nonzero multiplicative identity element e_A. There is a natural ring homomorphism ϕ from $\mathbf{Z}$ into A with $\phi(1) = e_A$. Let V be a left module over A, which may be considered as a module over $\mathbf{Z}$. It is easy to see that $\mathbf{Z}$ is a Dedekind ring. Thus one can use the argument described in the preceding paragraph to get an injective module W over $\mathbf{Z}$ and an injective homomorphism from V into W, as modules over $\mathbf{Z}$.

Remember that the contravariant ϕ-extension of W is the space

$$ {}^{(\phi)}W = \mathrm{Hom}_{\mathbf{Z}}(A, W) \tag{11.11.1}$$

of homomorphisms from A into W, as modules over $\mathbf{Z}$, as in Section 2.11, which is a left module over A. Let $V(\mathbf{Z})$ be V considered as a module over $\mathbf{Z}$, so that ${}^{(\phi)}Z(V)$ may be defined as a left module over A in the same way. There is a natural injective homomorphism

$$ \text{from } V \text{ into } {}^{(\phi)}V(\mathbf{Z}), \tag{11.11.2}$$

as left modules over A, because V is a left module over A, as in Section 2.13. The injective homomorphism from V into W, as modules over $\mathbf{Z}$, leads to an injective homomorphism

$$ \text{from } {}^{(\phi)}V(\mathbf{Z}) \text{ into } {}^{(\phi)}W, \tag{11.11.3}$$

as left modules over A, using composition of homomorphisms. The composition of these two homomorphisms defines an injective homomorphism

$$ \text{from } V \text{ into } {}^{(\phi)}W, \tag{11.11.4}$$

as left modules over A.

We also have that ${}^{(\phi)}W$ is injective as a left module over A, because W is injective as a module over $\mathbf{Z}$, as in Section 2.11. This is another approach to getting an injective homomorphism from a left module over A into an injective left module over A, as in Section 2.8. Of course, there is an analogous argument for right modules. This corresponds to the remarks after Proposition 6.3 on p31 of [3]. As mentioned in [3], this proof was communicated to the authors by B. Echmann, and a similar proof was also found by H. A. Forrester.

11.12 Some divisible modules

Let k be an integral domain, and let Q_k be the corresponding field of fractions. We may consider k and Q_k as modules over k, so that the quotient Q_k/k is a module over k too. In fact, Q_k/k is divisible, as mentioned in Section 11.4. It is easy to see that Q_k/k is a torsion module, as a module over k. If k is a Dedekind ring, then Q_k/k is injective as a module over k, as in Section 11.10.

Let Y be a module over k, and let η be a homomorphism from a torsion-free module V over k onto Y, as in Section 11.6. Also let $Q_k \bigotimes_k V$ be a tensor product of Q_k and V, as modules over k, and let ϕ be the homomorphism from V into $Q_k \bigotimes_k V$ with $\phi(v) = 1 \otimes v$ for every $v \in V$, as before. Remember that ϕ is injective, so that ϕ leads to an injective homomorphism from $V/\ker\eta$ into

$$\big(Q_k \bigotimes_k V\big)/\phi(\ker\eta), \tag{11.12.1}$$

as modules over k. We have seen that $Q_k \bigotimes_k V$ is divisible as a module over k, so that (11.12.1) is divisible too. Of course, $V/\ker\eta$ is isomorphic to Y, as a module over k, so that we get an injective homomorphism from Y into (11.12.1), as modules over k, as before.

Suppose now that Y is a torsion module, as a module over k. If $v \in V$, then it follows that there is a $t \in k$ such that $t \neq 0$ and

$$\eta(t \cdot v) = t \cdot \eta(v) = 0, \tag{11.12.2}$$

so that $t \cdot v \in \ker\eta$. In this case,

$$t \cdot (1 \otimes v) = 1 \otimes (t \cdot v) = \phi(t \cdot v) \in \phi(\ker\eta). \tag{11.12.3}$$

Using this, one can check that

$$\text{(11.12.1) is a torsion module,} \tag{11.12.4}$$

as a module over k.

Suppose that k is a Dedekind ring as well. Thus (11.12.1) is injective as a module over k, because it is divisible, as in Section 11.10. This shows that if Y is a torsion module over k, then there is an injective homomorphism from Y into an injective torsion module over k.

One can use this to get an injective resolution of Y consisting of torsion modules, as in Section 10.7. This corresponds to the first part of Exercise 2 on p139 of [3]. Of course, injective resolutions are simpler for hereditary rings, as before.

Chapter 12

Commutative rings and fractions

12.1 Rings of fractions

Let k be a commutative ring with a multiplicative identity element. A subset S of k is said to be *multiplicatively closed* in k if

(12.1.1) $$1 \in S$$

and S is closed under multiplication in the usual sense that

(12.1.2) $$x\,y \in S \text{ for every } x, y \in S.$$

Thus S is a commutative semigroup with multiplicative identity element 1, which is a sub-semigroup of k, as a semigroup with respect to multiplication.

Consider the binary relation $\simeq$ defined on $k \times S$ by putting

(12.1.3) $$(x, r) \simeq (y, t)$$

when

(12.1.4) $$x\,t\,v - y\,r\,v = 0$$

for some $v \in S$. This relation is clearly reflexive and symmetric on $k \times S$. To show that $\simeq$ is transitive on $k \times S$, suppose that (12.1.3) and (12.1.4) hold, and that $(y, t) \simeq (z, u)$ for some $(z, u) \in k \times S$, so that

(12.1.5) $$y\,u\,w - z\,t\,w = 0$$

for some $w \in S$. It follows that

(12.1.6) $$(x\,t\,v - y\,r\,v)\,u\,w = (y\,u\,w - z\,t\,w)\,r\,v = 0,$$

so that

(12.1.7) $$x\,u\,t\,v\,w - z\,r\,t\,v\,w = 0.$$

This means that $(x,r) \simeq (z,u)$, because $t\,v\,w \in S$, as desired.

Thus $\simeq$ defines an equivalence relation on $k \times S$. The corresponding set of equivalence classes may be denoted $S^{-1}\,k$, as on p36 of [1]. This construction is also mentioned in Exercise 9 on p141 of [3], with the additional condition that

$$0 \notin S. \tag{12.1.8}$$

Note that the multiplicative identity element in a ring is normally supposed to be nonzero in [3], as on p3 of [3], whereas $1 = 0$ in k is permitted in [1], as on p1 of [1]. The set of equivalence classes is denoted k_S in [3], but we shall normally use the notation $S^{-1}\,k$ here, to avoid confusion with some other notation in [1].

If $x \in k$ and $r \in S$, then we let x/r denote the equivalence class of (x,r) with respect to $\simeq$, as on p36 of [1]. If $y \in k$ and $t \in S$ as well, then we would like to define the sum and product of x/r and y/t in $S^{-1}\,k$ by

$$(x/r) + (y/t) = (x\,r + y\,t)/(r\,t) \tag{12.1.9}$$

and

$$(x/r)\,(y/t) = (x\,y)/(r\,t). \tag{12.1.10}$$

One can check that these determine well-defined operations of addition and multiplication on $S^{-1}\,k$, which make $S^{-1}\,k$ a commutative ring, as in [1, 3]. This is called the *ring of fractions of k with respect to S*, as on p37 of [1]. Clearly $1/1$ is the multiplicative identity element in $S^{-1}\,k$.

We also have that

$$x \mapsto x/1 \tag{12.1.11}$$

is a ring homomorphism from k into $S^{-1}\,k$, as on p37 of [1]. If $r \in S$, then $r/1$ has multplicative inverse $1/r$ in $S^{-1}\,k$. If $x \in k$, then it is easy to see that

$$x/1 = 0 \tag{12.1.12}$$

in $S^{-1}\,k$ if and only if

$$x\,t = 0 \tag{12.1.13}$$

in k for some $t \in S$.

If $0 \in S$, then (12.1.4) holds automatically with $v = 0$, so that $S^{-1}\,k = \{0\}$. Conversely, if $S^{-1}\,k = \{0\}$, then one can check that $0 \in S$, as in Example 2 on p38 of [1].

Observe that k is an integral domain exactly when

$$S = k \setminus \{0\} \tag{12.1.14}$$

is multiplicatively closed in k. In this case, $S^{-1}\,k$ is the same as the usual field Q_k of fractions of k, as in the remark on p37 of [1].

12.2 Modules of fractions

Let k be a commutative ring with a multiplicative identity element, let S be a multiplicatively closed subset of k, and let V be a module over k. Consider the binary relation $\cong$ defined on $V \times S$ by putting

$$(v, r) \cong (v', r') \tag{12.2.1}$$

when

$$t\, r'\, v - t\, r\, v' = 0 \tag{12.2.2}$$

for some $t \in S$. One can check that this defines an equivalence relation on $V \times S$, as in the previous section. The corresponding set of equivalence classes may be denoted $S^{-1}\, V$, as on p38 of [1]. If $v \in V$ and $r \in S$, then we let v/r denote the equivalence class of (v, r) with respect to $\cong$, as in [1].

One can define addition and scalar multiplication on $S^{-1}\, V$ in natural ways, so that $S^{-1} V$ becomes a module over $S^{-1}\, k$, as in [1]. In Exercise 9 on p141 of [3], $S^{-1}\, V$ is defined initially as a module over k, before $S^{-1}\, k$ is defined as a ring. This module is denoted V_S in [3], but we shall not use this notation here, to avoid confusion with some other notation from [1], as before. In particular, one can use this to define $S^{-1}\, k$ initially as a module over k, by considering k as a module over itself. One can define multiplication on $S^{-1}\, k$ as in (12.1.10) again to get a commutative ring with a multiplicative identity element, and define scalar multiplication on $S^{-1}\, V$ so that it becomes a module over $S^{-1}\, k$, as before.

If $S^{-1}\, V$ is defined as a module over $S^{-1}\, k$ in this way, then $S^{-1}\, V$ may also be considered as a module over k, using the ring homomorphism (12.1.11) from k into $S^{-1}\, k$. This is the same as defining $S^{-1}\, V$ initially as a module over k, as in [3]. It is easy to see that

$$v \mapsto v/1 \tag{12.2.3}$$

defines a homomorphism from V into $S^{-1}\, V$, as modules over k, as on p142 of [3]. Note that

$$v/1 = 0 \tag{12.2.4}$$

in $S^{-1}\, V$ if and only if

$$t\, v = 0 \tag{12.2.5}$$

for some $t \in S$, as in [3].

Let W be another module over k, and let f be a homomorphism from V into W, as modules over k. This leads to a homomorphism $S^{-1}\, f$ from $S^{-1}\, V$ into $S^{-1}\, W$, as modules over $S^{-1}\, k$, with

$$(S^{-1}\, f)(v/r) = f(v)/r \tag{12.2.6}$$

for every $v \in V$ and $r \in S$, as on p38 of [1].

Let Z be a third module over k, and let g be a homomorphism from W into Z, as modules over k. Thus $g \circ f$ is a homomorphism from V into Z, as modules over k, and

$$S^{-1}\, (g \circ f) = (S^{-1}\, g) \circ (S^{-1}\, f), \tag{12.2.7}$$

as on p38 of [1]. If $g \circ f = 0$, then it follows that

$$(S^{-1}\, g) \circ (S^{-1}\, f) = 0. \tag{12.2.8}$$

Suppose that

$$f(V) = \ker g, \tag{12.2.9}$$

and let us check that

$$(S^{-1}\, f)(S^{-1}\, V) = \ker(S^{-1}\, g). \tag{12.2.10}$$

This corresponds to Proposition 3.3 on p39 of [3], and the first part of Exercise 10 on p142 of [3]. Of course, (12.2.9) implies that $g \circ f = 0$, so that

$$(S^{-1}\, f)(S^{-1}\, V) \subseteq \ker(S^{-1} g), \tag{12.2.11}$$

by (12.2.8). To get the opposite inclusion, let $w \in W$ and $r \in S$ be given, with

$$g(w)/r = (S^{-1}\, g)(w/r) = 0 \tag{12.2.12}$$

in $S^{-1}\, Z$. This implies that there is a $t \in S$ such that

$$g(t\, w) = t\, g(w) = 0 \tag{12.2.13}$$

in Z. It follows that there is a $v \in V$ such that

$$f(v) = t\, w. \tag{12.2.14}$$

This means that

$$(S^{-1}\, f)(v/(r\, t)) = f(v)/(r\, t) = (t\, w)/(r\, t) = w/r \tag{12.2.15}$$

in $S^{-1}\, W$, as desired.

In particular, if g is injective, then $S^{-1}\, g$ is injective, which can also be seen a bit more directly. If W is a submodule of Z, then we can apply this to the obvious inclusion mapping from W into Z. This permits us to consider $S^{-1}\, W$ as a submodule of $S^{-1}\, Z$ in this case, as on p39 of [1].

Let V_1, V_2 be submodules of V, so that $S^{-1}\, V_1$ and $S^{-1}\, V_2$ may be considered as submodules of $S^{-1}\, V$, as in the preceding paragraph. It is easy to see that

$$S^{-1}\,(V_1 + V_2) = (S^{-1}\, V_1) + (S^{-1}\, V_2), \tag{12.2.16}$$

as submodules of $S^{-1}\, V$, as in part (i) of Corollary 3.4 on p39 of [1].

Similarly, let us check that

$$S^{-1}\,(V_1 \cap V_2) = (S^{-1}\, V_1) \cap (S^{-1}\, V_2), \tag{12.2.17}$$

as in part (ii) of Corollary 3.4 on p39 of [1]. The left side is clearly contained in the right side, and so it suffices to verify the opposite inclusion. Let $v_1 \in V_1$, $v_2 \in V_2$, and $r_1, r_2 \in S$ be given, with

$$v_1/r_1 = v_2/r_2. \tag{12.2.18}$$

This means that

$$t\, r_2\, v_1 = t\, r_1\, v_2 \tag{12.2.19}$$

for some $t \in S$. If u denotes the common value in (12.2.19), then $u \in V_1 \cap V_2$. The common value in (12.2.18) is equal to

$$u/(t\, r_1\, r_2). \tag{12.2.20}$$

This is an element of $S^{-1}\,(V_1 \cap V_2)$, as desired.

Let q_1 be the natural quotient mapping from V onto the quotient module V/V_1. This leads to a homomorphism

$$S^{-1}\, q_1 \text{ from } S^{-1}\, V \text{ onto } S^{-1}\,(V/V_1), \tag{12.2.21}$$

as modules over $S^{-1}\, k$, as before. We also have that

$$\ker(S^{-1}\, q_1) = S^{-1}\, V_1, \tag{12.2.22}$$

as in (12.2.10). This means that $S^{-1}\, q_1$ induces an isomorphism

$$\text{from } (S^{-1}\, V)/(S^{-1}\, V_1) \text{ onto } S^{-1}\,(V/V_1), \tag{12.2.23}$$

as modules over $S^{-1}\, k$, as in part (iii) of Corollary 3.4 on p39 of [1].

12.3 Fractions and tensor products

Let k be a commutative ring with a multiplicative identity element, let S be a multiplicatively closed subset of k, and let V be a module over k. Also let $(S^{-1}\, k) \bigotimes_k V$ be a tensor product of $S^{-1}\, k$ and V, as modules over k. An arbitrary element of this tensor product may be expressed as a finite sum

$$\sum_{j=1}^{n} (x_j/r_j) \otimes v_j, \tag{12.3.1}$$

where $x_j \in k$, $r_j \in S$, and $v_j \in V$ for each $j = 1, \ldots, n$. Put

$$r = \prod_{j=1}^{n} r_j \tag{12.3.2}$$

and

$$t_l = \prod_{j \ne l} r_j \tag{12.3.3}$$

for each $l = 1, \ldots, n$. These are elements of S, with $r = r_l\, t_l$ for each l.

Thus (12.3.1) is equal to

$$\begin{aligned} \sum_{j=1}^{n} (t_j\, x_j/r) \otimes v_j &= \sum_{j=1}^{n} (1/r) \otimes ((t_j\, x_j) \cdot v_j) \\ &= (1/r) \otimes \Big(\sum_{j=1}^{n} (t_j\, x_j) \cdot v_j \Big). \end{aligned} \tag{12.3.4}$$

This shows that every element of $(S^{-1}\,k) \bigotimes_k V$ may be expressed as $(1/r) \otimes v$ for some $r \in S$ and $v \in V$. This corresponds to part of the proof of Proposition 3.5 on p39 of [1].

One can check that

$$(x/r, v) \mapsto (x \cdot v)/r \tag{12.3.5}$$

defines a mapping from $(S^{-1}\,k) \times V$ into $S^{-1}\,V$ that is bilinear over k. This leads to a unique homomorphism

$$\text{from } (S^{-1}\,k) \bigotimes_k V \text{ into } S^{-1}\,V, \tag{12.3.6}$$

as modules over k, with

$$(x/r) \otimes v \mapsto (x \cdot v)/r \tag{12.3.7}$$

for every $x \in k$, $r \in S$, and $v \in V$. It is easy to see that this homomorphism is surjective, and we would like to check that it is injective as well.

Let $r \in S$ and $v \in V$ be given, and suppose that

$$v/r = 0 \tag{12.3.8}$$

in $S^{-1}\,V$. This implies that $t \cdot v = 0$ for some $t \in S$. It follows that

$$(1/r) \otimes v = (t/(r\,t)) \otimes v = (1/(r\,t)) \otimes (t \cdot v) = 0 \tag{12.3.9}$$

in $(S^{-1}\,k) \bigotimes_k V$. This shows that the kernel of the homomorphism mentioned in the preceding paragraph is trivial.

Note that $(S^{-1}\,k) \bigotimes_k V$ may be considered as a module over $S^{-1}\,k$, because $S^{-1}\,k$ may be considered as a module over itself, as in Section 1.10. This may also be considered as an example of extension of scalars, as in Section 2.9. It is easy to see that the homomorphism as in (12.3.6) is linear over $S^{-1}\,k$. Thus we get an isomorphism as in (12.3.6), as modules over $S^{-1}\,k$. This corresponds to Proposition 3.5 on p39 of [1], and to part of Exercise 9 on p141 of [3].

Let W be another module over k, and let $V \bigotimes_k W$ be a tensor product of V and W. This is a module over k, so that $S^{-1}\,(V \bigotimes_k W)$ may be defined as a module over $S^{-1}\,k$, as before. Let $(S^{-1}\,V) \bigotimes_{S^{-1}\,k} (S^{-1}\,W)$ be a tensor product of $S^{-1}\,V$ and $S^{-1}\,W$, as modules over $S^{-1}\,k$. One can check that

$$((v/r), (w/t)) \mapsto (v \otimes w)/(r\,t) \tag{12.3.10}$$

defines a mapping from $(S^{-1}\,V) \times (S^{-1}\,W)$ into $S^{-1}\,(V \bigotimes_k W)$ that is bilinear over $S^{-1}\,k$. This leads to a unique homomorphism

$$\text{from } (S^{-1}\,V) \bigotimes_{S^{-1}\,k} (S^{-1}\,W) \text{ into } S^{-1}\,(V \bigotimes_k W), \tag{12.3.11}$$

as modules over $S^{-1}\,k$, with

$$(v/r) \otimes (w/t) \mapsto (v \otimes w)/(r\,t) \tag{12.3.12}$$

for every $v \in V$, $w \in W$, and $r, t \in k$.

It is easy to see that this homomorphism is surjective. Proposition 3.7 on p40 of [1] states that this homomorphism is an isomorphism. This can be obtained from the previous result and isomorphisms between double tensor products, as in Section 1.12.

If Z is a module over $S^{-1}\,k$, then Z may be considered as a module over k, using the natural ring homomorphism from k into $S^{-1}\,k$. If Z is considered as a module over k, then $S^{-1}\,Z$ may be defined as a module over $S^{-1}\,k$, as before. Under these conditions, one can check that the natural homomorphism from Z into $S^{-1}\,Z$, as modules over k, is an isomorphism between modules over $S^{-1}\,k$. This is another part of Exercise 9 on p141 of [3].

12.4 Fractions and prime ideals

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}$ be a proper ideal in k. Note that $\mathcal{I}$ is a prime ideal in k if and only if

$$S = k \setminus \mathcal{I} \tag{12.4.1}$$

is multiplicatively closed in k, as in Example 1 on p38 of [1]. Supposing that this is the case, we may put

$$k_{\mathcal{I}} = S^{-1}\,k, \tag{12.4.2}$$

as in [1].

It is easy to see that

$$\mathcal{I}_1 = \{x/r : x \in \mathcal{I},\, r \in k \setminus \mathcal{I}\} \tag{12.4.3}$$

is an ideal in $k_{\mathcal{I}}$. One can check that $1/1 \notin \mathcal{I}_1$, so that $\mathcal{I}_1 \neq k_{\mathcal{I}}$. If $y/t \in k_{\mathcal{I}} \setminus \mathcal{I}_1$, then $y \in k \setminus \mathcal{I}$, and y/t is invertible in $k_{\mathcal{I}}$. This implies that $\mathcal{I}_1$ is a maximal ideal in $k_{\mathcal{I}}$, and in fact $k_{\mathcal{I}}$ is a local ring, as in Section 4.13. Equivalently, $\mathcal{I}_1$ is the unique maximal proper ideal in $k_{\mathcal{I}}$, as on p38 of [1].

If V is a module over k, then we may put

$$V_{\mathcal{I}} = S^{-1}\,V, \tag{12.4.4}$$

with S as in (12.4.1), as on p38 of [1]. Of course, if $V = \{0\}$, then $V_{\mathcal{I}} = \{0\}$.

Suppose that $V \neq \{0\}$, and let v be a nonzero element of V. Thus

$$\{t \in k : t \cdot v = 0\} \tag{12.4.5}$$

is a proper ideal in k. Let $\mathcal{I}_0$ be a proper maximal ideal in k that contains (12.4.5), which can be obtained using Zorn's lemma or Hausdorff's maximality principle.

Of course, maximal ideals are prime ideals, by a standard argument, so that $V_{\mathcal{I}_0}$ may be defined as before. If

$$v/1 = 0 \text{ in } V_{\mathcal{I}_0}, \tag{12.4.6}$$

then

$$r \cdot v = 0 \tag{12.4.7}$$

for some $r \in k \setminus \mathcal{I}_0$. However, this would contradict the fact that (12.4.5) is contained in $\mathcal{I}_0$. This shows that

$$V = \{0\} \tag{12.4.8}$$

when

$$V_{\mathcal{I}} = \{0\} \tag{12.4.9}$$

for all maximal proper ideals $\mathcal{I}$ in k. This corresponds to Proposition 3.8 on p40 of [1], and the first part of Exercise 11 on p142 of [3].

Let W be another module over k, and let ϕ be a homomorphism from V into W, as modules over k. If $\mathcal{I}$ is a proper prime ideal in k and S is as in (12.4.1), then we get an induced homomorphism $\phi_{\mathcal{I}} = S^{-1}\,\phi$ from $V_{\mathcal{I}}$ into $W_{\mathcal{I}}$, as in Section 12.2. If ϕ is surjective, then it is easy to see that $\phi_{\mathcal{I}}$ is surjective. If ϕ is injective, then $\phi_{\mathcal{I}}$ is injective too, as before. This corresponds to part of Proposition 3.9 on p40 of [1].

More precisely,

$$\ker \phi_{\mathcal{I}} = (\ker \phi)_{\mathcal{I}}. \tag{12.4.10}$$

This follows from (12.2.10), with f equal to the obvious inclusion mapping from $\ker g$ into V, and $g = \phi$. If (12.4.10) is equal to $\{0\}$ for all maximal proper ideals $\mathcal{I}$ in k, then it follows that $\ker g = \{0\}$, as in (12.4.8). Equivalently, if $\phi_{\mathcal{I}}$ is injective for all maximal proper ideals $\mathcal{I}$ in k, then ϕ is injective. This is another part of Proposition 3.9 on p40 of [1].

Similarly, if $\phi_{\mathcal{I}}$ is surjective for all maximal proper ideals $\mathcal{I}$ in k, then ϕ is surjective, as in Proposition 3.9 on p40 of [1].

12.5 The nilradical

Let k be a commutative ring with a multiplicative identity element. An element x if k is said to be *nilpotent* if

$$x^n = 0 \tag{12.5.11}$$

in k for some positive integer n, as on p2 of [1]. The set $\mathcal{N}$ of all nilpotent elements of k is called the *nilradical* of k, as on p5 of [1].

Suppose that $x, y \in \mathcal{N}$, so that $x^m = y^n = 0$ for some positive integers m, n. Under these conditions, $m + n - 1$ is a positive integer, and one can check that

$$(x + y)^{m+n-1} = 0. \tag{12.5.12}$$

Indeed, the left side can be expanded into a sum of terms of the form

$$x^j\, y^l, \tag{12.5.13}$$

where j and l are nonnegative integers with $j + l = m + n - 1$. It is easy to see that $j \geq m$ or $l \geq n$, so that (12.5.13) is equal to 0. This implies (12.5.12).

One can use this to verify that $\mathcal{N}$ is an ideal in k. One can also check that every nilpotent element of the quotient ring $k/\mathcal{N}$ is equal to 0. This is Proposition 1.7 on p5 of [1].

Propostion 1.8 on p5 of [1] states that $\mathcal{N}$ is equal to the intersection of all the prime ideals in k. It is easy to see that every nilpotent element of k is contained in every prime ideal of k.

Suppose that $x \in k$ is not nilpotent, so that for each positive integer n, $x^n \neq 0$. One can use Zorn's lemma or Hausdorff's maximality principle to get an ideal $\mathcal{I}$ in k such that for each positive integer n,

$$x^n \notin \mathcal{I}, \tag{12.5.14}$$

and $\mathcal{I}$ is maximal with respect to inclusion. We would like to check that $\mathcal{I}$ is a prime ideal in k.

Let $y, z \in k \setminus \mathcal{I}$ be given, and let $\mathcal{I}_y$, $\mathcal{I}_z$ be the ideals in k generated by $\mathcal{I}$ and y, z, respectively. There are positive integers m, n such that

$$x^m \in \mathcal{I}_y, \ x^n \in \mathcal{I}_z, \tag{12.5.15}$$

by the maximality of $\mathcal{I}$.

Let $\mathcal{I}_{yz}$ be the ideal in k generated by $\mathcal{I}$ and $y\,z$. One can verify that

$$x^{m+n} \in \mathcal{I}_{yz}, \tag{12.5.16}$$

using (12.5.15).

This implies that $y\,z \notin \mathcal{I}$, so that $\mathcal{I}$ is prime. Another version of this using a suitable ring of fractions of k is mentioned in Remark 2 on p42 of [1], and this will be discussed in the next section.

12.6 Using powers to get fractions

Let k be a commutative ring with a multiplicative identity element, and let $x \in k$ be given. Observe that

$$S = \{x^n : n \in (\mathbf{Z}_+ \cup \{0\})\} \tag{12.6.1}$$

is a multiplicatively closed subset of k. This is Example 3 on p38 of [1]. We also have that $0 \in S$ if and only if x is nilpotent in k.

Suppose now that x is not nilpotent in k, so that $0 \notin S$. This implies that $S^{-1}\,k \neq \{0\}$, as in Section 12.1. It follows that there is a maximal proper ideal $\mathcal{I}_0$ in $S^{-1}\,k$. In particular, $\mathcal{I}_0$ is a prime ideal in $S^{-1}\,k$.

Put

$$\mathcal{I}_1 = \{w \in k : w/1 \in \mathcal{I}_0\}. \tag{12.6.2}$$

It is easy to see that this is an ideal in k. Note that $1 \notin \mathcal{I}_1$, because $1/1 \notin \mathcal{I}_0$. One can check that $\mathcal{I}_1$ is a prime ideal in k, because $\mathcal{I}_0$ is a prime ideal in $S^{-1}\,k$.

If n is a positive integer, then $x^n/1$ is invertible in $S^{-1}\,k$, because $x^n \in S$, as in Section 12.1. This means that

$$x^n/1 \notin \mathcal{I}_0, \tag{12.6.3}$$

because $\mathcal{I}_0$ is a proper ideal in $S^{-1}\,k$. It follows that

$$x^n \notin \mathcal{I}_1. \tag{12.6.4}$$

This is the argument from Remark 2 on p42 of [1] that was mentioned in the previous section.

12.7 Extensions and contractions of ideals

Let k_1 and k_2 be commutative rings with multiplicative identity elements 1_{k_1}, 1_{k_2}, respectively, and let f be a ring homomorphism from k_1 into k_2, with $f(1_{k_1}) = 1_{k_2}$. Also let $\mathcal{I}_1$ be an ideal in k_1, and note that $f(\mathcal{I}_1)$ is not necessarily an ideal in k_2. Under these conditions, the *extension* of $\mathcal{I}_1$ with respect to f is the ideal $\mathcal{I}_1^e$ in k_2 generated by $f(\mathcal{I}_1)$. Equivalently, $\mathcal{I}_1^e$ consists of finite sums of products of elements of $f(\mathcal{I}_1)$ and k_2, as on p9 of [1].

If $\mathcal{I}_2$ is an ideal in k_2, then $f^{-1}(\mathcal{I}_2)$ is an ideal in k_1. This is the *contraction* of $\mathcal{I}_2$ with respect to f, which may be denoted $\mathcal{I}_2^c$, as on p9 of [1]. If $\mathcal{I}_2 \ne k_2$, then $1_{k_2} \notin \mathcal{I}_2$, which implies that $1_{k_1} \notin \mathcal{I}_2^c$. If $\mathcal{I}_2$ is a prime ideal in k_2, then it is easy to see that $\mathcal{I}_2^c$ is a prime ideal in k_1.

It is easy to see that

$$\mathcal{I}_1 \subseteq \mathcal{I}_1^{ec}, \tag{12.7.1}$$

where $\mathcal{I}_1^{ec}$ is the contraction of $\mathcal{I}_1^e$. Similarly,

$$\mathcal{I}_2^{ce} \subseteq \mathcal{I}_2, \tag{12.7.2}$$

where $\mathcal{I}_2^{ce}$ is the extension of $\mathcal{I}_2^c$. This is part (i) of Proposition 1.17 on p10 of [1]. Part (ii) of this proposition states that

$$\mathcal{I}_2^{cec} = \mathcal{I}_2^c \tag{12.7.3}$$

and

$$\mathcal{I}_1^{ece} = \mathcal{I}_1^e, \tag{12.7.4}$$

where the contraction and extension operations are used again, as indicated. This follows from the previous part, as in [1].

Let $\mathcal{C}$ be the collection of all ideals in k_1 that are contractions of ideals in k_2, and let $\mathcal{E}$ be the collection of all ideals in k_2 that are extensions of ideals in k_1. One can check that $\mathcal{C}$ is the same as the collection of all ideals $\mathcal{I}_1$ in k_1 such that

$$\mathcal{I}_1^{ec} = \mathcal{I}_1. \tag{12.7.5}$$

Similarly, one can verify that $\mathcal{E}$ is the same as the collection of all ideals $\mathcal{I}_2$ in k_2 such that

$$\mathcal{I}_2^{ce} = \mathcal{I}_2. \tag{12.7.6}$$

This is part of part (iii) of Proposition 1.17 in [1].

In fact,

$$\mathcal{I}_1 \mapsto \mathcal{I}_1^e \tag{12.7.7}$$

defines a one-to-one mapping from $\mathcal{C}$ onto $\mathcal{E}$. Similarly,

$$\mathcal{I}_2 \mapsto \mathcal{I}_2^c \tag{12.7.8}$$

is a one-to-one mapping from $\mathcal{E}$ onto $\mathcal{C}$, which is the inverse of the previous mapping. This is the other part of part (iii) of Proposition 1.17 in [1].

12.8 Products and ideal quotients

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}_1$, $\mathcal{I}_2$ be ideals in k. The *product* $\mathcal{I}_1 \mathcal{I}_2$ is the ideal in k generated by products of elements of $\mathcal{I}_1$ and $\mathcal{I}_2$, as on p6 of [1]. Equivalently, $\mathcal{I}_1 \mathcal{I}_2$ consists of finite sums of products of elements of $\mathcal{I}_1$ and $\mathcal{I}_2$. Of course,

$$\mathcal{I}_1 \mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2. \tag{12.8.1}$$

Let V be a module over k, and let V_1, V_2 be submodules of V. Put

$$(V_1 : V_2) = \{t \in k : t \cdot V_2 \subseteq V_1\}, \tag{12.8.2}$$

where $t \cdot V_2 = \{t \cdot v_2 : v_2 \in V_2\}$. It is easy to see that (12.8.2) is an ideal in k, as on p19 of [1]. Note that this is not the same as in Section 11.7, although the notation is similar.

If we take $V_1 = \{0\}$ in (12.8.2), then we get the *annihilator*

$$\operatorname{Ann}(V_2) = \{t \in k : t \cdot V_2 = \{0\}\} \tag{12.8.3}$$

of V_2 in k. Observe that

$$\operatorname{Ann}(V_1 + V_2) = \operatorname{Ann}(V_1) \cap \operatorname{Ann}(V_2). \tag{12.8.4}$$

One can also verify that

$$(V_1 : V_2) = \operatorname{Ann}((V_1 + V_2)/V_1). \tag{12.8.5}$$

These two statements correspond to parts (i) and (ii) of Exercise 2.2 on p20 of [1], respectively.

The *product* $\mathcal{I}_1 \cdot V_2$ of $\mathcal{I}_1$ and V_2 may be defined as the subset of V_2 consisting of finite sums of elements of the form $t_1 \cdot v_2$, where $t_1 \in \mathcal{I}_1$ and $v_2 \in V_2$. This is submodule of V_2, as on p19 of [1]. Note that

$$(V_1 : V_2) \cdot V_2 \subseteq V_1, \tag{12.8.6}$$

as in part (ii) of Exercise 1.12 on p8 of [1].

Of course, $\mathcal{I}_1$, $\mathcal{I}_2$ may be considered as submodules of k, as a module over itself. Their *ideal quotient* is defined to be $(\mathcal{I}_1 : \mathcal{I}_2)$, as in (12.8.2), with $V = k$, as on p8 of [1]. Clearly

$$\mathcal{I}_1 \subseteq (\mathcal{I}_1 : \mathcal{I}_2), \tag{12.8.7}$$

as in part (i) of Exercise 1.12 on p8 of [1]. If $x \in k$, then we put

$$(\mathcal{I}_1 : x) = \{t \in k : t\,x \in \mathcal{I}_1\}. \tag{12.8.8}$$

This is the same as $(\mathcal{I}_1 : \mathcal{I}_2)$, with $\mathcal{I}_2$ taken to be the ideal in k generated by x, as in [1].

12.9 More on extensions and contractions

Let k_1, k_2 be commutative rings with multiplicative identity elements 1_{k_1}, 1_{k_2}, respectively, and let f be a ring homomorphism from k_1 into k_2 with $f(1_{k_1}) = 1_{k_2}$, as in Section 12.7. Also let $\mathcal{I}_1$, $\widetilde{\mathcal{I}}_1$ be ideals in k_1, and let $\mathcal{I}_2$, $\widetilde{\mathcal{I}}_2$ be ideals in k_2. We would like to mention some properties of extensions and contractions with respect to f, as in Exercise 1.18 on p10 of [1].

Note that $\mathcal{I}_1 + \widetilde{\mathcal{I}}_1$ and $\mathcal{I}_2 + \widetilde{\mathcal{I}}_2$ are ideals in k_1 and k_2, respectively. It is easy to see that

$$(\mathcal{I}_1 + \widetilde{\mathcal{I}}_1)^e = \mathcal{I}_1^e + \widetilde{\mathcal{I}}_1^e. \tag{12.9.1}$$

We also have that

$$\mathcal{I}_2^c + \widetilde{\mathcal{I}}_2^c \subseteq (\mathcal{I}_2 + \widetilde{\mathcal{I}}_2)^c. \tag{12.9.2}$$

Of course, $\mathcal{I}_1 \cap \widetilde{\mathcal{I}}_1$ and $\mathcal{I}_2 \cap \widetilde{\mathcal{I}}_2$ are ideals in k_1 and k_2, respectively. One can check that

$$(\mathcal{I}_1 \cap \widetilde{\mathcal{I}}_1)^e \subseteq \mathcal{I}_1^e \cap \widetilde{\mathcal{I}}_1^e. \tag{12.9.3}$$

It is easy to see that

$$(\mathcal{I}_2 \cap \widetilde{\mathcal{I}}_2)^c = \mathcal{I}_2^c \cap \widetilde{\mathcal{I}}_2^c. \tag{12.9.4}$$

The products $\mathcal{I}_1\,\widetilde{\mathcal{I}}_1$ and $\mathcal{I}_2\,\widetilde{\mathcal{I}}_2$ are defined as ideals in k_1 and k_2, respectively, as in the previous section. One can verify that

$$(\mathcal{I}_1\,\widetilde{\mathcal{I}}_1)^e = \mathcal{I}_1^e\,\widetilde{\mathcal{I}}_1^e. \tag{12.9.5}$$

Clearly

$$\mathcal{I}_2^c\,\widetilde{\mathcal{I}}_2^c \subseteq (\mathcal{I}_2\,\widetilde{\mathcal{I}}_2)^c. \tag{12.9.6}$$

The quotient ideals $(\mathcal{I}_1 : \widetilde{\mathcal{I}}_1)$ and $(\mathcal{I}_2 : \widetilde{\mathcal{I}}_2)$ in k_1 and k_2, respectively, may be defined as in the previous section. Similarly, $(\mathcal{I}_1^e : \widetilde{\mathcal{I}}_1^e)$ is an ideal in k_2, and $(\mathcal{I}_2^c : \widetilde{\mathcal{I}}_2^c)$ is an ideal in k_1. One can check that

$$(\mathcal{I}_1 : \widetilde{\mathcal{I}}_1)^e \subseteq (\mathcal{I}_1^e : \widetilde{\mathcal{I}}_1^e). \tag{12.9.7}$$

Similarly,

$$(\mathcal{I}_2 : \widetilde{\mathcal{I}}_2)^c \subseteq (\mathcal{I}_2^c : \widetilde{\mathcal{I}}_2^c). \tag{12.9.8}$$

12.10 Radicals of ideals

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}$ be an ideal in k. The *radical* $r(\mathcal{I})$ of $\mathcal{I}$ in k is defined by

$$r(\mathcal{I}) = \{x \in k : x^n \in \mathcal{I} \text{ for some } n \in \mathbf{Z}_+\}, \tag{12.10.1}$$

as on p8 of [1].

Let q be the natural quotient homomorphism from k onto $k/\mathcal{I}$, and let $\mathcal{N}(k/\mathcal{I})$ be the nilradical of $k/\mathcal{I}$, as in Section 12.5. It is easy to see that

$$r(\mathcal{I}) = q^{-1}(\mathcal{N}(k/\mathcal{I})), \tag{12.10.2}$$

as on p8 of [1]. This implies that $r(\mathcal{I})$ is an ideal in k as well, because $\mathcal{N}(k/\mathcal{I})$ is an ideal in $k/\mathcal{I}$, as before. In particular, $r(\{0\})$ is the same as the nilradical of k.

Let us mention some basic properties of the radical, as in Exercise 1.13 on p9 of [1]. Clearly

$$\mathcal{I} \subseteq r(\mathcal{I}). \tag{12.10.3}$$

It is easy to see that

$$r(r(\mathcal{I})) = r(\mathcal{I}). \tag{12.10.4}$$

Let $\widetilde{\mathcal{I}}$ be another ideal in k. One can check that

$$r(\mathcal{I}\,\widetilde{\mathcal{I}}) = r(\mathcal{I} \cap \widetilde{\mathcal{I}}). \tag{12.10.5}$$

We also have that

$$r(\mathcal{I} \cap \widetilde{\mathcal{I}}) = r(\mathcal{I}) \cap r(\widetilde{\mathcal{I}}). \tag{12.10.6}$$

It is easy to see that $r(\mathcal{I}) = k$ if and only if $\mathcal{I} = k$. Indeed, if $1 \in r(\mathcal{I})$, then $1 \in \mathcal{I}$. One can verify that

$$r(\mathcal{I} + \widetilde{\mathcal{I}}) = r(r(\mathcal{I}) + r(\widetilde{\mathcal{I}})). \tag{12.10.7}$$

If n is a positive integer, then let $\mathcal{I}^n$ be the nth power of $\mathcal{I}$ with respect to multiplication of ideals as in Section 12.8. Equivalently, $\mathcal{I}^n$ consists of all finite sums of products of n elements of $\mathcal{I}$, as on p6 of [1]. This is interpreted as being equal to k when $n = 0$, as in [1].

It is easy to see that

$$r(\mathcal{I}^n) = r(\mathcal{I}) \tag{12.10.8}$$

for every $n \geq 1$. If $\mathcal{I}$ is a prime ideal in k, then one can check that

$$r(\mathcal{I}) = \mathcal{I}. \tag{12.10.9}$$

This implies that

$$r(\mathcal{I}^n) = \mathcal{I} \tag{12.10.10}$$

for each $n \geq 1$.

Proposition 1.14 on p9 of [1] states that $r(\mathcal{I})$ is the same as the intersection of all of the prime ideals in k that contain $\mathcal{I}$. This can be obtained from the analogous statement for the nilradical in Section 12.5 and (12.10.2).

Let k_1, k_2 be commutative rings with multiplicative identity elements 1_{k_1}, 1_{k_2}, respectively, and let f be a ring homomorphism from k_1 into k_2 with $f(1_{k_1}) = 1_{k_2}$. If $\mathcal{I}_1$ is an ideal in k_1, then one can check that

$$r(\mathcal{I}_1)^e \subseteq r(\mathcal{I}_1^e), \tag{12.10.11}$$

where the radicals are taken in k_1 and k_2, respectively. If $\mathcal{I}_2$ is an ideal in k_2, then

$$r(\mathcal{I}_2)^c = r(\mathcal{I}_2^c), \tag{12.10.12}$$

where the radicals are taken in k_2 and k_1, respectively. This is another part of Exercise 1.18 on p10 of [1].

12.11 Ring homomorphisms and fractions

Let k_1, k_2 be commutative rings with multiplicative identity elements 1_{k_1}, 1_{k_2}, respectively, and let ϕ be a ring homomorphism from k_1 into k_2 with $\phi(1_{k_1}) = 1_{k_2}$. Also let S_1 be a multiplicatively closed subset of k_1, and observe that

$$S_2 = \phi(S_1) \tag{12.11.1}$$

is a multiplicatively closed subset of k_2. Thus $S_1^{-1}\, k_1$ and $S_2^{-1}\, k_2$ may be defined as commutative rings with multiplicative identity elements as in Section 12.1. We also get the corresponding ring homomorphisms

$$x_1 \mapsto x_1/1_{k_1} \tag{12.11.2}$$

and

$$x_2 \mapsto x_2/1_{k_2} \tag{12.11.3}$$

from k_1, k_2 into $S_1^{-1}\, k_1$, $S_2^{-1}\, k_2$, respectively, as before.

We would like to define a mapping Φ from $S_1^{-1}\, k_1$ into $S_2^{-1}\, k_2$ by putting

$$\Phi(x_1/r_1) = \phi(x_1)/\phi(r_1) \tag{12.11.4}$$

for every $x_1 \in k_1$ and $r_1 \in S_1$. It is easy to see that this is a well-defined mapping from $S_1^{-1}\, k_1$ into $S_2^{-1}\, k_2$, and a ring homomorphism. Note that the composition of (12.11.2) with Φ is the same as the composition of ϕ with (12.11.3). If

$$\phi(k_1) = k_2, \tag{12.11.5}$$

then

$$\Phi(S_1^{-1}\, k_1) = S_2^{-1}\, k_2. \tag{12.11.6}$$

Of course, the kernel of ϕ is an ideal in k_1, which may be considered as a submodule of k_1, as a module over itself. Thus $S_1^{-1}(\ker \phi)$ may be considered as a submodule of $S_1^{-1} k_1$, as a module over itself, as in Section 12.2. This means that $S_1^{-1}(\ker \phi)$ may be considered as an ideal in $S_1^{-1} k_1$.

It is easy to see that (12.11.4) is equal to 0 in $S_2^{-1} k_2$ if and only if there is a $t_1 \in S_1$ such that

$$\phi(t_1 x_1) = \phi(t_1)\, \phi(x_1) = 0 \tag{12.11.7}$$

in k_2. One can use this to get that

$$\ker \Phi = S_1^{-1}(\ker \phi). \tag{12.11.8}$$

Let V be k_2, considered as a module over k_1, using ϕ. Thus we can define $S_1^{-1} V$ as a module over $S_1^{-1} k_1$, as in Section 12.2. This corresponds to considering $S_2^{-1} k_2$ as a module over $S_1^{-1} k_1$ using Φ, as in Exercise 4 on p44 of [1].

12.12 More on rings of fractions

Let k_1, k be commutative rings with multiplicative identity elements 1_{k_1}, 1_k, respectively, and let g be a ring homomorphism from k_1 into k with $g(1_{k_1}) = 1_k$. Also let S_1 be a multiplicatively closed subset of k_1, and suppose that for each $r_1 \in S_1$,

$$g(r_1) \text{ has a multiplicative inverse in } k. \tag{12.12.1}$$

Under these conditions, Proposition 3.1 on p37 of [1] states that there is a unique ring homomorphism h from $S_1^{-1} k_1$ into k such that

$$h(x_1/1_{k_1}) = g(x_1) \tag{12.12.2}$$

for every $x_1 \in k_1$.

To get uniqueness, observe that for each $r_1 \in S_1$, we have that

$$h(1_{k_1}/r_1) = h(r_1/1_{k_1})^{-1} = g(r_1)^{-1}. \tag{12.12.3}$$

This implies that

$$h(x_1/r_1) = h(x_1/1_{k_1})\, h(1_{k_1}/r_1) = g(x_1)\, g(r_1)^{-1} \tag{12.12.4}$$

for every $x_1 \in k_1$ and $r_1 \in S_1$.

To get existence, we would like to define h as in (12.12.4), and we need to check that h is well-defined on $S_1^{-1} k_1$. Let $x_1, x_1' \in k_1$ and $r_1, r_1' \in S_1$ be given, with

$$x_1/r_1 = x_1'/r_1'. \tag{12.12.5}$$

This means that there is a $t_1 \in S_1$ such that

$$x_1\, r_1'\, t_1 = x_1'\, r_1\, t_1, \tag{12.12.6}$$

as in Section 12.1. Thus

$$g(x_1)\, g(r_1')\, g(t_1) = g(x_1')\, g(r_1)\, g(t_1), \tag{12.12.7}$$

so that

$$g(x_1)\,g(r_1)^{-1} = g(x_1')\,g(r_1')^{-1}, \tag{12.12.8}$$

because $g(t_1)$ is invertible in k. This implies that h is well-defined on $S_1^{-1}\,k_1$, and it is easy to see that h is a ring homomorphism.

Suppose now that g satisfies the following two additional conditions. First, if $x_1 \in k_1$ and $g(x_1) = 0$, then

$$x_1\,t_1 = 0 \tag{12.12.9}$$

in k_1 for some $t_1 \in S_1$. Second, every element of k is of the form

$$g(x_1)\,g(r_1)^{-1} \tag{12.12.10}$$

for some $x_1 \in k_1$ and $r_1 \in S_1$. In this case, Corollary 3.2 on p37 of [1] states that h is an isomorphism from $S_1^{-1}\,k_1$ onto k. Indeed, the first condition implies that the kernel of h is trivial, and the second condition implies that h is surjective.

Let k_2 be another commutative ring with multiplicative identity element 1_{k_2}, and let ϕ be a ring homomorphism from k_1 into k_2 with $\phi(1_{k_1}) = 1_{k_2}$. Remember that $S_2 = \phi(S_1)$ is a multiplicatively closed subset of k_2, as in Section 12.11. Let us take

$$k = S_2^{-1}\,k_2, \tag{12.12.11}$$

and let g be the ring homomorphism from k_1 into k defined by

$$g(x_1) = \phi(x_1)/1_{k_2} \tag{12.12.12}$$

for every $x_1 \in k_1$. This is the same as the composition of ϕ with the usual homomorphism $x_2 \mapsto x_2/1_{k_2}$ from k_2 into $S_2^{-1}\,k_2$. This homomorphism satisfies (12.12.1) by construction, and the corresponding homomorphism h is the same as the homomorphism Φ discussed in the previous section.

12.13 Extensions, contractions, and fractions

Let k be a commutative ring with a multiplicative identity element, and let S be a multiplicatively closed subset of k. Put

$$f(x) = x/1 \tag{12.13.1}$$

for every $x \in k$, which defines a ring homomorphism from k into $S^{-1}\,k$, as in Section 12.1. If $\mathcal{I}_1$ is an ideal in k, then we let $\mathcal{I}_1^e$ be its extension in $S^{-1}\,k$ with respect to f, as in Section 12.7. Similarly, if $\mathcal{I}_2$ is an ideal in $S^{-1}\,k$, then we let $\mathcal{I}_2^c$ be its contraction in k with respect to f. Let $\mathcal{C}$ be the collection of ideals in k that are contractions of ideals in $S^{-1}\,k$, and let $\mathcal{E}$ be the collection of ideals in $S^{-1}\,k$ that are extensions of ideals in k, as before.

If $\mathcal{I}_1$ is any ideal in k, then we would like to check that

$$\mathcal{I}_1^e = S^{-1}\,\mathcal{I}_1, \tag{12.13.2}$$

as on p41 of [1]. More precisely, $\mathcal{I}_1$ may be considered as a submodule of k, as a module over itself, so that $S^{-1}\,\mathcal{I}_1$ is defined as a module over $S^{-1}\,k$. This may be identified with a submodule of $S^{-1}\,k$, as a module over itself, as in Section 12.2. Thus $S^{-1}\,\mathcal{I}_1$ is an ideal in $S^{-1}\,k$, which contains $f(\mathcal{I}_1)$, and thus $\mathcal{I}_1^e$. It is clear that $S^{-1}\,\mathcal{I}_1$ is contained in $\mathcal{I}_1^e$, so that they are equal to each other.

Let $\mathcal{I}_2$ be an ideal in $S^{-1}\,k$, and let us check that $\mathcal{I}_2^{ce} = \mathcal{I}_2$. This means that $\mathcal{I}_2$ is the extension of an ideal in k, as in part (i) of Proposition 3.11 on p41 of [1]. It suffices to show that

$$\mathcal{I}_2 \subseteq \mathcal{I}_2^{ce}, \tag{12.13.3}$$

because of (12.7.2). Let x/r be an element of $\mathcal{I}_2$, where $x \in k$ and $r \in S$, as usual. Observe that $x/1 \in \mathcal{I}_2$, so that $x \in \mathcal{I}_2^c$, and $x/r \in \mathcal{I}_2^{ce}$, as desired.

Let $\mathcal{I}_1$ be an ideal in k again, and let us verify that

$$\mathcal{I}_1^{ec} = \bigcup_{r \in S} (\mathcal{I}_1 : r), \tag{12.13.4}$$

where $(\mathcal{I}_1 : r)$ is as in (12.8.8). Of course,

$$\mathcal{I}_1^{ec} = (S^{-1}\,\mathcal{I}_1)^c, \tag{12.13.5}$$

by (12.13.2). Thus $x \in k$ is an element of $\mathcal{I}_1^{ec}$ if and only if

$$x/1 = y/t \tag{12.13.6}$$

in $S^{-1}\,k$ for some $y \in \mathcal{I}_1$ and $t \in S$. Remember that (12.13.6) means that $x\,t\,v - y\,v = 0$ for some $v \in S$, as in Section 12.1. One can use this to get (12.13.4).

This is the first part of part (ii) of Proposition 3.11 on p41 of [1], and the second part says that

$$\mathcal{I}_1^e = S^{-1}\,k \quad \text{if and only if} \quad \mathcal{I}_1 \cap S \neq \emptyset. \tag{12.13.7}$$

More precisely, if $\mathcal{I}_1 \cap S \neq \emptyset$, then $1/1 \in \mathcal{I}_1^e$. Conversely, if $\mathcal{I}_1^e = S^{-1}\,k$, then $\mathcal{I}_1^{ec} = k$. This means that $1 \in \mathcal{I}_1^{ec}$, which implies that $\mathcal{I}_1 \cap S \neq \emptyset$, by (12.13.4).

Note that

$$\mathcal{I}_1 \in \mathcal{C} \quad \text{if and only if} \quad \mathcal{I}_1^{ec} \subseteq \mathcal{I}_1, \tag{12.13.8}$$

as in Section 12.7. Let q_1 be the natural quotient mapping from k onto $k/\mathcal{I}_1$. Part (iii) of Proposition 3.11 on p41 of [1] says that

$$\mathcal{I}_1 \in \mathcal{C} \quad \text{if and only if } q_1(S) \text{ has no zero-divisors in } k/\mathcal{I}_1. \tag{12.13.9}$$

Indeed, $\mathcal{I}_1 \in \mathcal{C}$ if and only if

$$\bigcup_{r \in S} (\mathcal{I}_1 : r) \subseteq \mathcal{I}_1, \tag{12.13.10}$$

by (12.13.4) and (12.13.8). This condition says exactly that for every $r \in S$, if $x \in k$ and $r\,x \in \mathcal{I}_1$, then $x \in \mathcal{I}_1$, as desired.

Consider the mapping

$$\mathcal{I}_1 \mapsto \mathcal{I}_1^e = S^{-1}\,\mathcal{I}_1 \tag{12.13.11}$$

from ideals in k to ideals in $S^{-1}\,k$. Part (iv) of Proposition 3.11 on p41 of [1] states that this defines a one-to-one mapping from the set of prime ideals in k that are disjoint from S onto the set of proper prime ideals in $S^{-1}\,k$. Of course, if $0 \in S$, then there are no ideals in k that are disjoint from S, and there are no proper ideals in $S^{-1}\,k = \{0\}$. Thus we may as well suppose that $0 \not\in S$.

Remember that (12.13.11) is one-to-one on $\mathcal{C}$, as in Section 12.7. If $\mathcal{I}_1$ is a prime ideal in k, then $k/\mathcal{I}_1$ has no nonzero zero-divisors. If $\mathcal{I}_1$ is also disjoint from S, then it follows that $\mathcal{I}_1 \in \mathcal{C}$, by (12.13.9). Thus (12.13.11) is one-to-one on the set of prime ideals in k that are disjoint from S.

If $\mathcal{I}_2$ is any ideal in $S^{-1}\,k$, then we have already seen that $\mathcal{I}_2^{ce} = \mathcal{I}_2$. If $\mathcal{I}_2 \neq S^{-1}\,k$, then it follows that

$$\mathcal{I}_2^c \cap S = \emptyset. \tag{12.13.12}$$

If $\mathcal{I}_2$ is a prime ideal in $S^{-1}\,k$, then $\mathcal{I}_2^c$ is a prime ideal in k, as in Section 12.7. This shows that every proper prime ideal in $S^{-1}\,k$ corresponds to a prime ideal in k that is disjoint from S as in (12.13.11).

Let $\mathcal{I}_1$ be a proper prime ideal in k, and let q_1 be the natural quotient mapping from k onto $k/\mathcal{I}_1$, which is an integral domain. Note that $q_1(S)$ is a multiplicatively closed subset of $k/\mathcal{I}_1$, as in Section 12.11. Suppose that

$$\mathcal{I}_1 \cap S = \emptyset, \tag{12.13.13}$$

so that

$$0 \not\in q_1(S). \tag{12.13.14}$$

This means that

$$q_1(S)^{-1}(k/\mathcal{I}_1) \neq \{0\}, \tag{12.13.15}$$

as in Section 12.1. In fact, one can check that

$$q_1(S)^{-1}(k/\mathcal{I}_1) \text{ is an integral domain,} \tag{12.13.16}$$

because $k/\mathcal{I}_1$ is an integral domain.

Using q_1, we get a ring homomorphism

$$Q_1 \text{ from } S^{-1}\,k \text{ onto } q_1(S)^{-1}(k/\mathcal{I}_1), \tag{12.13.17}$$

with kernel $S^{-1}\,\mathcal{I}_1$, as in Section 12.11. This leads to a ring isomorphism

$$\text{from } (S^{-1}\,k)/(S^{-1}\,\mathcal{I}_1) \text{ onto } q_1(S)^{-1}(k/\mathcal{I}_1). \tag{12.13.18}$$

It follows that $(S^{-1}\,k)/(S^{-1}\,\mathcal{I}_1)$ is an integral domain, which could also be verified more directly from the definitions. This implies that

$$S^{-1}\,\mathcal{I}_1 \text{ is a proper prime ideal in } k, \tag{12.13.19}$$

so that (12.13.11) maps the set of prime ideals in k that are disjoint from S into the set of proper prime ideals in $S^{-1}\,k$.

Part (v) of Proposition 3.11 on p42 of [1] states that (12.13.11) commutes with taking finite sums, products, intersections, and radicals of ideals in k. This was already mentioned in Section 12.9 for finite sums and products, and it follows from analogous statements in Section 12.2 for finite sums and intersections. If $\mathcal{I}_1$ is an ideal in k, then

$$S^{-1}(r(\mathcal{I}_1)) = r(\mathcal{I}_1)^e \subseteq r(\mathcal{I}_1^e) = r(S^{-1}\,\mathcal{I}_1), \tag{12.13.20}$$

as in Section 12.10. The opposite inclusion can be verified directly in this case.

If $\mathcal{N}(k)$ is the nilradical of k, then Corollary 3.12 on p42 of [1] states that

(12.13.21) the nilradical $\mathcal{N}(S^{-1}\,k)$ of $S^{-1}\,k$ is the same as $S^{-1}\,\mathcal{N}(k)$.

This follows from the previous statement about radicals, because the nilradical is the same as the radical of $\{0\}$.

Let $\mathcal{I}_0$ be a proper prime ideal in k, and let us take $S = k \setminus \mathcal{I}_0$, which is a multiplicatively closed subset of k, as in Section 12.4. In this case, (12.13.11) defines a one-to-one mapping from the set of prime ideals in k that are contained in $\mathcal{I}_0$ onto the set of proper prime ideals in $S^{-1}\,k = k_{\mathcal{I}_0}$, as before. This is Corollary 3.13 on p42 of [1].

12.14 More on modules of fractions

Let k be a commutative ring with a multiplicative identity element, and let S be a multiplicatively closed subset of k. If V is a finitely-generated module over k, then Proposition 3.14 on p 43 of [1] states that

$$S^{-1}\,\mathrm{Ann}(V) = \mathrm{Ann}(S^{-1}\,V). \tag{12.14.1}$$

Let W and Z be submodules of V, and suppose that they satisfy the corresponding analogues of (12.14.1). Let us first check that $W + Z$ has the same property, as in [1]. Observe that

$$\begin{aligned} S^{-1}\,\mathrm{Ann}(V+W) &= S^{-1}(\mathrm{Ann}(W) \cap \mathrm{Ann}(Z)) \\ &= (S^{-1}\,\mathrm{Ann}(W)) \cap (S^{-1}\,\mathrm{Ann}(Z)) \\ &= (\mathrm{Ann}(S^{-1}\,W)) \cap (\mathrm{Ann}(S^{-1}\,Z)), \end{aligned} \tag{12.14.2}$$

where the first step is as in Section 12.8, the second step is as in Section 12.2, and the third step is by hypothesis. The right side is equal to

$$\mathrm{Ann}((S^{-1}\,W) + (S^{-1}\,Z)) = \mathrm{Ann}(S^{-1}(W+Z)), \tag{12.14.3}$$

as in Sections 12.8 and 12.2 again.

This permits one to reduce to the case where V is generated by a single element, as a module over k. If $\mathcal{I} = \mathrm{Ann}(V)$, then it follows that V is isomorphic to $k/\mathcal{I}$, as a module over k. In this case, $S^{-1}\,V$ is isomorphic to $(S^{-1}\,k)/(S^{-1}\,\mathcal{I})$, as a module over $S^{-1}\,k$, as in Section 12.2. This implies that

$$\mathrm{Ann}(S^{-1}\,V) = S^{-1}\,\mathcal{I} = S^{-1}\ \mathrm{Ann}(V), \tag{12.14.4}$$

as desired.

Now let V be any module over k, and let W, Z be submodules of V. If Z is finitely generated as a module over k, then

$$S^{-1}\,(W : Z) = (S^{-1}\,W : S^{-1}\,Z), \tag{12.14.5}$$

as in Corollary 3.15 on p43 of [1]. Remember that $(W : Z)$ is an ideal in k, as in Section 12.8, so that $S^{-1}\,(W : Z)$ is an ideal in $S^{-1}\,k$. Similarly, $S^{-1}\,W$ and $S^{-1}\,Z$ are submodules of $S^{-1}\,V$, as a module over $S^{-1}\,k$, so that $(S^{-1}\,W : S^{-1}\,Z)$ is an ideal in $S^{-1}k$. To show (12.14.5), we use the fact that

$$(W : Z) = \mathrm{Ann}((W + Z)/W), \tag{12.14.6}$$

as in Section 12.8.

It is easy to see that $(W + Z)/W$ is finitely generated as a module over k, because Z is finitely generated, by hypothesis. It follows that

$$S^{-1}\,(W : Z) = \mathrm{Ann}(S^{-1}\,((W + Z)/W)), \tag{12.14.7}$$

by (12.14.1). We also have that $S^{-1}\,((W + Z)/W)$ is isomorphic to

$$((S^{-1}\,W) + (S^{-1}\,Z))/(S^{-1}\,W), \tag{12.14.8}$$

as modules over $S^{-1}\,k$, as in Section 12.2. This means that

$$S^{-1}\,(W : Z) = \mathrm{Ann}(((S^{-1}\,W) + (S^{-1}\,Z))/(S^{-1}\,W)), \tag{12.14.9}$$

by (12.14.7). The right side of (12.14.9) is the same as the right side of (12.14.5), as in Section 12.8 again.

12.15 Contractions of prime ideals

Let k_1 and k_2 be commutative rings with multiplicative identity elements 1_{k_1} and 1_{k_2}, respectively, and let f be a ring homomorphism from k_1 into k_2, with $f(1_{k_1}) = 1_{k_2}$. Also let $\mathcal{I}_1$ be an ideal in k_1, and remember that extensions of ideals in k_1 and contractions of ideals in k_2 with respect to f are defined as in Section 12.7. In particular, if $\mathcal{I}_1$ is the contraction of an ideal in k_2 with respect to f, then

$$\mathcal{I}_1 = \mathcal{I}_1^{ec}, \tag{12.15.1}$$

as before.

Suppose now that $\mathcal{I}_1$ is a proper prime ideal in k_1 that satisfies (12.15.1). We would like to show that $\mathcal{I}_1$ is the contraction of a prime ideal in k_2 with respect to f, as in Proposition 3.16 on p43 of [1]. Remember that

$$S_1 = k_1 \setminus \mathcal{I}_1 \tag{12.15.2}$$

is multiplicatively closed in k_1, because $\mathcal{I}_1$ is a prime ideal, as in Section 12.4. This implies that

$$S_2 = f(S_1) = f(k_1 \setminus \mathcal{I}_1) \tag{12.15.3}$$

is multiplicatively closed in k_2, as in Section 12.11.

Clearly

$$\mathcal{I}_1^{ec} \cap S_1 = \emptyset, \tag{12.15.4}$$

by (12.15.1). Using this, one can check that

$$\mathcal{I}_1^{e} \cap S_2 = \emptyset. \tag{12.15.5}$$

Of course,

$$x_2 \mapsto x_2/1_{k_2} \tag{12.15.6}$$

defines a ring homomorphism from k_2 into $S_2^{-1}\, k_2$, as in Section 12.1. Let $(\mathcal{I}_1^e)^e$ be the ideal in $S_2^{-1}\, k_2$ which is the extension of $\mathcal{I}_1^e$ with respect to (12.15.6). Observe that

$$(\mathcal{I}_1^e)^e \ne S_2^{-1}\, k_2, \tag{12.15.7}$$

by (12.13.7) and (12.15.5).

Let $\mathcal{I}_3$ be a maximal proper ideal in $S_2^{-1}\, k_2$ with

$$(\mathcal{I}_1^e)^e \subseteq \mathcal{I}_3. \tag{12.15.8}$$

In particular, $\mathcal{I}_3$ is a prime ideal in $S_2^{-1}\, k_2$, because it is maximal. Consider

$$\mathcal{I}_2 = (\mathcal{I}_3)^c = \{x_2 \in k_2 : x_2/1_{k_2} \in \mathcal{I}_3\}, \tag{12.15.9}$$

which is the contraction of $\mathcal{I}_3$ in k_2 with respect to (12.15.6). Thus $\mathcal{I}_2$ is a prime ideal in k_2, because $\mathcal{I}_3$ is a prime ideal in $S_2^{-1}\, k_2$, as in Section 12.7. Note that

$$\mathcal{I}_1^e \subseteq \mathcal{I}_2, \tag{12.15.10}$$

because of (12.15.8). We also have that

$$\mathcal{I}_2 \cap S_2 = \emptyset, \tag{12.15.11}$$

because $\mathcal{I}_3$ is a proper ideal in $S_2^{-1}\, k_2$. This means that $\mathcal{I}_2^c \subseteq \mathcal{I}_1$, by (12.15.3). It follows that

$$\mathcal{I}_2^c = \mathcal{I}_1, \tag{12.15.12}$$

by (12.15.10).

Part IV

More on rings and modules

Chapter 13

Nakayama's lemma and other matters

13.1 The Jacobson radical

Let k be a commutative ring with a multiplicative identity element. The *Jacobson radical* $\mathcal{R}$ in k is defined to be the intersection of all of the maximal proper ideals in k, as on p5 of [1]. Of course, this is an ideal in k. Note that k has a proper maximal ideal when $k \neq \{0\}$. If $k = \{0\}$, then we interpret $\mathcal{R}$ to be $\{0\}$.

If $x \in \mathcal{R}$ and $y \in k$, then

$$1 - x\,y \text{ has a multiplicative inverse in } k, \tag{13.1.1}$$

as in Proposition 1.9 on p6 of [1]. Otherwise, the ideal generated by $1 - x\,y$ would be a proper subset of k, which is contained in a maximal proper ideal $\mathcal{I}$ in k. We also have that $x \in \mathcal{I}$, because $\mathcal{R} \subseteq \mathcal{I}$, by construction, so that $x\,y \in \mathcal{I}$. This means that $1 \in \mathcal{I}$, which is a contradiction, because $\mathcal{I}$ is supposed to be a proper subset of k.

Conversely, suppose that $x \in k$ has the property that (13.1.1) holds for every $y \in k$. We would like to check that $x \in \mathcal{R}$, which is the other part of Proposition 1.9 on p6 of [1]. Let $\mathcal{I}$ be a maximal proper ideal in k, and suppose that $x \in k \setminus \mathcal{I}$. This means that the ideal in k generated by $\mathcal{I}$ and x is equal to k. It follows that there is a $w \in \mathcal{I}$ and $k \in k$ such that

$$w + x\,y = 1. \tag{13.1.2}$$

This implies that $w = 1 - x\,y$ is not invertible in k, because $\mathcal{I} \neq k$. Of course, $x \in k \setminus \mathcal{R}$ exactly when $x \in k \setminus \mathcal{I}$ for some maximal proper ideal $\mathcal{I}$ in k.

Remember that $\mathcal{N}$ denotes the nilradical of k, consisting of all nilpotent elements of k, as in Section 12.5. Equivalently, $\mathcal{N}$ is equal to the intersection of all prime ideals in k, as before. This implies that

$$\mathcal{N} \subseteq \mathcal{R}, \tag{13.1.3}$$

because maximal ideals in k are prime ideals.

If $z \in \mathcal{N}$, then

$$1 - z \text{ has a multiplicative inverse in } k, \tag{13.1.4}$$

by (13.1.1) and (13.1.3). This corresponds to the first part of Exercise 1 on p10 of [1]. Alternatively,

$$\sum_{j=0}^{\infty} z^j \tag{13.1.5}$$

reduces to a finite sum in k, because z is nilpotent. This sum is the multiplicative inverse of $1 - z$ in k, by a standard argument. If $x \in \mathcal{N}$ and $y \in k$, then $x\,y \in \mathcal{N}$, so that one can take $z = x\,y$. It follows that (13.1.1) holds under these conditions. This is another way to get (13.1.3), using the earlier characterization of $\mathcal{R}$.

Let $\mathcal{I}$ be an ideal in k. Of course, $\mathcal{I} \subseteq \mathcal{R}$ exactly when $\mathcal{I}$ is contained in every maximal proper ideal in k.

If $\mathcal{I} \subseteq \mathcal{R}$ and $x \in \mathcal{I}$, then $1 - x$ has a multiplicative inverse in k, as in (13.1.1). Conversely, if for every $x \in \mathcal{I}$, $1 - x$ has a multiplicative inverse in k, then one can check that $\mathcal{I} \subseteq \mathcal{R}$, using the characterization of $\mathcal{R}$ in terms of (13.1.1).

13.2 Ideals, modules, and homomorphisms

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}$ be an ideal in k. Also let V be a module over k, and remember that $\mathcal{I} \cdot V$ is the submodule of V consisting of finite sums of elements of the form $t \cdot v$, where $t \in \mathcal{I}$ and $v \in V$, as in Section 12.8.

Suppose that ϕ is a homomorphism from V into itself, as a module over k, such that

$$\phi(V) \subseteq \mathcal{I} \cdot V. \tag{13.2.1}$$

Suppose that V is finitely generated as a module over k as well. Under these condition, there are a positive integer n and elements $a_1, \ldots, a_n$ of $\mathcal{I}$ such that

$$\phi^n + a_1\,\phi^{n-1} + \cdots + a_{n-1}\,\phi + a_n\,I_V = 0, \tag{13.2.2}$$

where I_V is the identity mapping on V. This is Proposition 2.4 on p21 of [1].

Remember that the space $\mathrm{Hom}_k(V, V)$ of homomorphisms from V into itself, as a module over k, is an associative algebra over k with respect to composition of mappings. Let $\mathcal{A}_\phi$ be the subalgebra of $\mathrm{Hom}_k(V, V)$ generated by ϕ and I_V. This is a commutative subalgebra of $\mathrm{Hom}_k(V, V)$, consisting of finite linear combinations of I_V and powers of ϕ, with coefficients in k.

Let $v_1, \ldots, v_n$ be generators of V, as a module over k. Thus every element of $\mathcal{I} \cdot V$ can be expressed as

$$\sum_{l=1}^{n} t_l \cdot v_l, \tag{13.2.3}$$

where $t_1, \ldots, t_n \in \mathcal{I}$. In particular, for each $j = 1, \ldots, n$, $\phi(v_j)$, $\phi(v_j)$ can be expressed as

$$\phi(v_j) = \sum_{l=1}^{n} a_{j,l} \cdot v_l, \tag{13.2.4}$$

where $a_{j,1}, \ldots, a_{j,n} \in \mathcal{I}$.

Let $\delta_{j,l}$ be the usual $n \times n$ Kronecker delta matrix with entries in k, which is equal to 1 when $j = l$ and to 0 when $j \neq l$. Thus

$$\delta_{j,l}\, \phi - a_{j,l}\, I_V \tag{13.2.5}$$

defines an $n \times n$ matrix with entries in $\mathcal{A}_\phi$. This means that the determinant

$$\psi = \det(\delta_{j,l}\, \phi - a_{j,l}\, I_V) \tag{13.2.6}$$

of this matrix can be defined as an element of $\mathcal{A}_\phi$ in the usual way. In fact, ψ can be expressed as in the left side of (13.2.2). We would like to show that

$$\psi = 0. \tag{13.2.7}$$

Note that

$$\delta_{j,l}\, I_V \tag{13.2.8}$$

is the identity matrix, as an $n \times n$ matrix with entries in $\mathcal{A}_\phi$. It is well known that the product of the cofactor transpose of (13.2.5) with (13.2.5), as $n \times n$ matrices with entries in $\mathcal{A}_\phi$, is equal to the determinant times the identity matrix, which is to say

$$\delta_{j,l}\, \psi, \tag{13.2.9}$$

as in Cramer's rule. The cofactor transpose of a matrix is sometimes called the adjoint or classical adjoint of the matrix, which is different from other uses of the term adjoint for linear mappings.

Let us reexpress (13.2.4) as saying that

$$\sum_{l=1}^{n} (\delta_{j,l}\, \phi - a_{j,l}\, I_V)(v_l) = 0 \tag{13.2.10}$$

for each $j = 1, \ldots, n$. This implies that

$$\sum_{l=1}^{n} \delta_{r,l} \cdot \psi(v_l) = 0 \tag{13.2.11}$$

for every $r = 1, \ldots, n$, by the remarks in the preceding paragraph. This means that $\psi(v_r) = 0$ for every $r = 1, \ldots, n$. It follows that (13.2.7) holds, because V is generated by $v_1, \ldots, v_n$ as a module over k, by hypothesis.

13.3 Nakayama's lemma

Let k be a commutative ring with a multiplicative identity element, and let V be a finitely-generated module over k. Also let $\mathcal{I}$ be an ideal in k, and suppose that

$$\mathcal{I} \cdot V = V. \tag{13.3.1}$$

Under these conditions, there is an $x \in k$ such that

$$x - 1 \in \mathcal{I} \tag{13.3.2}$$

and

$$x \cdot V = 0. \tag{13.3.3}$$

This is Corollary 2.5 on p21 of [1]. To see this, one can take ϕ to be the identity mapping on V in (13.2.1). Let $a_1, \ldots, a_n \in \mathcal{I}$ be as in (13.2.2), and put

$$x = 1 + a_1 + \cdots + a_n. \tag{13.3.4}$$

Thus (13.3.2) holds by construction, and (13.3.3) follows from (13.2.2).

Suppose now that we also have that

$$\mathcal{I} \subseteq \mathcal{R}, \tag{13.3.5}$$

where $\mathcal{R}$ is the Jacobson radical of k, as in Section 13.1. In this case, *Nakayama's lemma* states that

$$V = \{0\}, \tag{13.3.6}$$

as in Proposition 2.6 on p21 of [1]. Indeed, (13.3.2) and (13.3.5) imply that x has a multiplicative inverse in k, as in Section 13.1. This means that (13.3.6) follows from (13.3.3). This is the first proof on p21 of [1].

Alternatively, suppose that V is generated by $v_1, \ldots, v_n$, as a module over k, for some positive integer n. One can check that every element of V can be expressed as a linear combination of $v_1, \ldots, v_n$, with coefficients in $\mathcal{I}$, because of (13.3.1). In particular,

$$v_n = a_1 \cdot v_1 + \cdots + a_n \cdot v_n \tag{13.3.7}$$

for some $a_1, \ldots, a_n \in \mathcal{I}$. Equivalently,

$$(1 - a_n) \cdot v_n = a_1 \cdot v_1 + \cdots a_{n-1} \cdot v_{n-1}. \tag{13.3.8}$$

Note that $a_n \in \mathcal{R}$, by (13.3.5), so that $1 - a_n$ has a multiplicative inverse in k, as in Section 13.1. Using this and (13.3.8), we get that V is generated by $v_1, \ldots, v_{n-1}$, as a module over k. We can repeat the process as needed to get that (13.3.6) holds. This corresponds to the second proof on p22 of [1].

Let W be a finitely-generated module over k, and let Z be a submodule of W. Also let $\mathcal{I}$ be an ideal in k again, and suppose that

$$\mathcal{I} \cdot W + Z = W. \tag{13.3.9}$$

If (13.3.5) holds, then

$$Z = W, \tag{13.3.10}$$

as in Corollary 2.7 on p22 of [1]. This can be obtained from Nakayama's lemma, with

$$V = W/Z. \tag{13.3.11}$$

More precisely, this uses the fact that

$$\mathcal{I} \cdot (W/Z) = (\mathcal{I} \cdot W + Z)/Z. \tag{13.3.12}$$

13.4 A noncommutative version

Let A be a ring with a nonzero multiplicative identity element e, and let $\mathcal{I}$ be a left ideal in A. Suppose that for every $x \in \mathcal{I}$,

$$e + x \text{ has a right inverse in } A. \tag{13.4.1}$$

If A is a commutative ring, then this is the same as saying that $\mathcal{I}$ is contained in the Jacobson radical of A, as in Section 13.1.

Suppose for the moment that A is a local ring, as in Section 4.13. Thus the collection $\mathcal{I}_0$ of $a \in A$ such that a does not have a left inverse in A is a left ideal in A. This implies that $\mathcal{I}_0$ is a two-sided ideal in A, and that every element of $A \setminus \mathcal{I}_0$ has a two-sided inverse in A, as before. If $a \in \mathcal{I}_0$, then $e + a \not\in \mathcal{I}_0$, because $e \not\in \mathcal{I}_0$, so that $e + a$ is invertible in A. In particular, $\mathcal{I}_0$ satisfies (13.4.1).

Let V be a right module over A, and let $V \cdot \mathcal{I}$ be the subset of V consisting of finite sums of elements of the form $v \cdot a$, with $v \in V$ and $a \in \mathcal{I}$. This is a subgroup of V, as a commutative group with respect to addition. Suppose that

$$V \cdot \mathcal{I} = V. \tag{13.4.2}$$

If V is finitely generated as a right module over A, then it follows that $V = \{0\}$. This is the same as Nakayama's lemma when A is commutative. If A is a local ring, and $\mathcal{I} = \mathcal{I}_0$ is as before, then this is the first part of Proposition 5.1' on p155 of [3]. The formulation using (13.4.1) is part of Exercise 9 on p160 of [3].

The proof in [3] is analogous to the second proof of Nakayama's lemma in the previous section. Suppose that V is generated by $v_1, \ldots, v_n$, as a right module over A, for some positive integer n. Using this and (13.4.2), one can check that every element of V can be expressed as

$$\sum_{j=1}^{n} v_j \cdot a_j \tag{13.4.3}$$

for some $a_1, \ldots, a_n \in \mathcal{I}$. This also uses the hypothesis that $\mathcal{I}$ be a left ideal in A.

In particular, v_n can be expressed as (13.4.3) for some $a_1, \ldots, a_n \in \mathcal{I}$. This means that

$$v_n \cdot (e - a_n) = \sum_{j=1}^{n-1} v_j \cdot a_j. \tag{13.4.4}$$

Let b be a right inverse of $e - a_n$ in A, as in (13.4.1). Observe that

$$v_n = (v_n \cdot (e - a_n)) \cdot b = \sum_{j=1}^{n-1} (v_j \cdot a_j) \cdot b = \sum_{j=1}^{n-1} v_j \cdot (a_j\, b). \tag{13.4.5}$$

This implies that V is generated by $v_1, \ldots, v_{n-1}$, as a right module over A, and one can repeat the process to get that $V = \{0\}$. Of course, there are analogous statements for left modules.

13.5 Ideals and fractions

Let k be a commutative ring with a multiplicative identity element, and let V be a module over k. Also let S be a multiplicatively closed subset of k, so that $S^{-1}\,k$ and $S^{-1}\,V$ may be defined as in Sections 12.1 and 12.2. If

$$r \cdot V = \{0\} \tag{13.5.1}$$

for some $r \in S$, then it is easy to see that

$$S^{-1}\,V = \{0\}. \tag{13.5.2}$$

If (13.5.2) holds, and if V is finitely generated as a module over k, then one can check that there is an $r \in S$ such that (13.5.1) holds. This corresponds to Exercise 1 on p43 of [1].

Let $\mathcal{I}$ be an ideal in k, so that $\mathcal{I} \cdot V$ may be defined as a submodule of V as in Section 12.8. If $\mathcal{I}$ is considered as a submodule of k, as a module over itself, then $S^{-1}\,\mathcal{I}$ may be considered as a submodule of $S^{-1}\,k$, as a module over itself, as in Section 12.2. Equivalently, $S^{-1}\,\mathcal{I}$ may be considered as an ideal in $S^{-1}\,k$. Thus $(S^{-1}\,\mathcal{I}) \cdot (S^{-1}\,V)$ may be defined as a submodule of $S^{-1}\,V$, as a module over $S^{-1}\,k$, in the usual way. One can check that

$$S^{-1}\,(\mathcal{I} \cdot V) = (S^{-1}\,\mathcal{I}) \cdot (S^{-1}\,V). \tag{13.5.3}$$

Let us now take

$$S = 1 + \mathcal{I} = \{1 + a : a \in \mathcal{I}\}. \tag{13.5.4}$$

It is easy to see that this is a multiplicatively closed subset of k, as in Example 4 on p38 of [1]. In this case, every element of $S^{-1}\,\mathcal{I}$ can be expressed as

$$a/(1-b), \tag{13.5.5}$$

where $a, b \in \mathcal{I}$. Note that

$$\begin{aligned} (1/1) - (a/(1-b)) &= ((1-b)/(1-b)) - (a/(1-b)) \\ &= (1-a-b)/(1-b). \end{aligned} \tag{13.5.6}$$

This is invertible as an element of $S^{-1}\,k$, because $1-b, 1-a-b \in S$. This implies that $S^{-1}\,\mathcal{I}$ is contained in the Jacobson radical of $S^{-1}\,k$, as in Section 13.1. This is the first part of Execise 2 on p43 of [1].

Let V be a module over k again, and suppose that

$$V = \mathcal{I} \cdot V. \tag{13.5.7}$$

This implies that

$$S^{-1} V = (S^{-1} \mathcal{I}) \cdot (S^{-1} V), \tag{13.5.8}$$

by (13.5.3). Suppose that V is finitely generated as a module over k, which implies that $S^{-1} V$ is finitely generated as a module over $S^{-1} k$. Under these conditions, we get that (13.5.2) holds, by Nakayama's lemma, because $S^{-1} \mathcal{I}$ is contained in the Jacobson radical of $S^{-1} k$, as in the preceding paragraph. This implies that (13.5.1) holds for some $r \in S$, as before.

This is another way to get (13.3.3), as in the second part of Exercise 2 on p43 of [1]. Of course, one should use the second proof of Nakayama's lemma in Section 13.3 here, since the first proof uses (13.3.3).

13.6 Modules over quotient rings

Let k be a commutative ring with a multiplicative identity element, let $\mathcal{I}$ be an ideal in k, and let V be a module over k. Suppose for the moment that

$$\mathcal{I} \subseteq \mathrm{Ann}(V), \tag{13.6.1}$$

where the annihilator $\mathrm{Ann}(V)$ of V in k is as in Section 12.8. This means that the action of each element of $\mathcal{I}$ on V is equal to 0. It follows that the action of any $t \in k$ on V only depends on the image of t in the quotient ring $k/\mathcal{I}$. This permits one to consider V as a module over $k/\mathcal{I}$, as on p19 of [1].

Let W be a module over k, and remember that $\mathcal{I} \cdot W$ is a submodule of W, as in Section 12.8. Thus the quotient $W/(\mathcal{I} \cdot W)$ is defined as a module over k. Of course,

$$\mathcal{I} \subseteq \mathrm{Ann}(W/(\mathcal{I} \cdot W)), \tag{13.6.2}$$

by construction. This means that $W/(\mathcal{I} \cdot W)$ may be considered as a module over $k/\mathcal{I}$, as in the preceding paragraph.

Let q_W be the natural quotient mapping from W onto $W/(\mathcal{I} \cdot W)$, as modules over k. Suppose that $w_1, \ldots, w_n$ are finitely many elements of W such that

(13.6.3) $W/(\mathcal{I} \cdot W)$ is generated by $q_W(w_1), \ldots, q_W(w_n)$, as a module over k.

Equivalently, this means that

(13.6.4) $W/(\mathcal{I} \cdot W)$ is generated by $q_W(w_1), \ldots, q_W(w_n)$, as a module over $k/\mathcal{I}$.

Let Z be the submodule of W, as a module over k, generated by $w_1, \ldots, w_n$. Thus

$$q_W(Z) = W/(\mathcal{I} \cdot W). \tag{13.6.5}$$

This is the same as saying that

$$Z + (\mathcal{I} \cdot W) = W. \tag{13.6.6}$$

Suppose now that W is also finitely generated, as a module over k. If $\mathcal{I}$ is contained in the Jacobson radical of k, then it follows that $W = Z$, as in Section 13.3. This means that

$$W \text{ is generated by } w_1, \ldots, w_n, \text{ as a module over } k. \tag{13.6.7}$$

In particular, if k is a local ring, and $\mathcal{I}$ is the unique maximal proper ideal in k, then $\mathcal{I}$ is the same as the Jacobson radical of k. This corresponds to Proposition 2.8 on p22 of [1].

13.7 Another noncommutative version

Let A be a ring with a nonzero multiplicative identity element e_A, and let $\mathcal{I}$ be a proper two-sided ideal in A. Thus the quotient $A/\mathcal{I}$ is a ring with a nonzero multiplicative identity element $e_{A/\mathcal{I}}$.

Let W be a right module over A, and let $W \cdot \mathcal{I}$ be as in Section 13.4. It is easy to see that

$$W \cdot \mathcal{I} \text{ is a submodule of } W, \text{ as a right module over } A, \tag{13.7.1}$$

because $\mathcal{I}$ is a right ideal in A.

This means that the quotient $W/(W \cdot \mathcal{I})$ is defined as a right module over A. The action of any element of $\mathcal{I}$ on $W/(W \cdot \mathcal{I})$ on the right is equal to 0, by construction. Equivalently, the action of $a \in A$ on $W/(W \cdot \mathcal{I})$ on the right only depends on the image of a in $A/\mathcal{I}$, under the natural quotient homomorphism from A onto $A/\mathcal{I}$. It follows that $W/(W \cdot \mathcal{I})$ may be considered as a right module over $A/\mathcal{I}$ as well, with respect to this action.

Let q_W be the natural quotient homomorphism from W onto $W/(W \cdot \mathcal{I})$, as right modules over A. Suppose that $w_1, \ldots, w_n$ are finitely many elements of W such that

$$\begin{aligned} &W/(W \cdot \mathcal{I}) \text{ is generated by } q_W(w_1), \ldots, q_W(w_n), \\ &\text{as a right module over } A. \end{aligned} \tag{13.7.2}$$

This is the same as saying that

$$\begin{aligned} &W/(W \cdot \mathcal{I}) \text{ is generated by } q_W(w_1), \ldots, q_W(w_n), \\ &\text{as a right module over } A/\mathcal{I}. \end{aligned} \tag{13.7.3}$$

Let Z be the submodule of W, as a right module over A, generated by $w_1, \ldots, w_n$. Note that

$$q_W(Z) = W/(W \cdot \mathcal{I}), \tag{13.7.4}$$

so that

$$Z + (W \cdot \mathcal{I}) = W. \tag{13.7.5}$$

Consider the quotient $V = W/Z$, which is another right module over A. Thus $V \cdot \mathcal{I}$ may be defined in the usual way, and is a submodule of V, as a right module over A. It is easy to see that

$$V \cdot \mathcal{I} = (W/Z) \cdot \mathcal{I} = (W \cdot \mathcal{I} + Z)/Z = W/Z = V, \tag{13.7.6}$$

because of (13.7.5).

Suppose that for every $x \in \mathcal{I}$,

$$e_A + x \text{ has a right inverse in } A. \tag{13.7.7}$$

In particular, this holds when A is a local ring, and $\mathcal{I}$ is the corresponding two-sided ideal in A, as in Sections 4.13 and 13.4.

Suppose also that W is finitely generated as a right module over A. This implies that V is finitely generated as a right module over A too. Under these conditions, we get that $V = \{0\}$, because of (13.7.6), as in Section 13.4. This means that $W = Z$, so that

$$W \text{ is generated by } w_1, \ldots, w_n, \text{ as a right module over } A. \tag{13.7.8}$$

This basically corresponds to a simplification of the first part of Proposition 5.2 on p155 of [3]. This also uses the first part of Proposition 5.1' on p155 of [3], and the corresponding part of Exercise 9 on p160 of [3], as in Section 13.4. There are analogous statements for left modules, as usual.

13.8 Some quotients by ideals

Let A be a ring with a nonzero multiplicative identity element e_A again, and let $\mathcal{I}_R$ be a right ideal in A, so that

$$W = A/\mathcal{I}_R \tag{13.8.1}$$

is a right module over A. Also let $\mathcal{I}$ be a left ideal in A, and let $\mathcal{I}_R \mathcal{I}$ be the subset of A consisting of finite sums of products of elements of $\mathcal{I}_R$ and $\mathcal{I}$. Note that

$$\mathcal{I}_R \mathcal{I} \subseteq \mathcal{I}_R \cap \mathcal{I}. \tag{13.8.2}$$

If $\mathcal{I}_R$ is considered as a right module over A, then $\mathcal{I}_R \mathcal{I}$ is the same as the subset $\mathcal{I}_R \cdot \mathcal{I}$ of $\mathcal{I}_R$ defined in Section 13.4. Similarly, if $\mathcal{I}$ is considered as a left module over A, then $\mathcal{I}_R \mathcal{I}$ is the same as the subset $\mathcal{I}_R \cdot \mathcal{I}$ of $\mathcal{I}$ defined analogously.

Let

$$W \cdot \mathcal{I} = (A/\mathcal{I}_R) \cdot \mathcal{I} \tag{13.8.3}$$

be as in Section 13.4 again. This is a subgroup of W, as a commutative group with respect to addition. Of course, $\mathcal{I}_R + \mathcal{I}$ is a subgroup of A, as a commutative group with respect to addition. It is easy to see that

$$W \cdot \mathcal{I} = (\mathcal{I}_R + \mathcal{I})/\mathcal{I}_R. \tag{13.8.4}$$

The quotients

$$W/(W \cdot \mathcal{I}) \tag{13.8.5}$$

and

$$A/(\mathcal{I}_R + \mathcal{I}) \tag{13.8.6}$$

are defined as commutative groups with respect to addition in the usual way. These quotients are isomorphic to each other in a natural way, because of (13.8.4), by standard arguments.

Suppose now that $\mathcal{I}$ is a proper two-sided ideal in A, so that the quotient $A/\mathcal{I}$ is a ring with a nonzero multiplicative identity element. Observe that $\mathcal{I}_R + \mathcal{I}$, $\mathcal{I}_R \cap \mathcal{I}$, and $\mathcal{I}_R \mathcal{I}$ are right ideals in A, because $\mathcal{I}$ is a right ideal. We also have that $W \cdot \mathcal{I}$ is a submodule of W, as a right module over A, as in the previous section. Thus the quotients (13.8.5) and (13.8.6) are defined as right modules over A in this case. These quotients are isomorphic as right modules over A in a natural way, as before.

In fact, (13.8.5) and (13.8.6) may be considered as right modules over $A/\mathcal{I}$, as in the previous section. They are isomorphic as right modules over $A/\mathcal{I}$ in a natural way, as usual. Let q_W be the natural quotient mapping from W onto (13.8.5), and let q_0 be the natural quotient mapping from A onto W. Put

$$q_1 = q_W \circ q_0, \tag{13.8.7}$$

which corresponds to the natural quotient mapping from A onto (13.8.6).

If $x \in \mathcal{I}$, then

$$q_1(e_A + x) = q_1(e_A). \tag{13.8.8}$$

This implies that (13.8.5) is generated by $q_1(e_A + x)$, as a right module over A.

If $e_A + x$ has a right inverse in A, then A is generated by $e_A + x$, as a right module over itself. This implies that W is generated by $q_0(e_A + x)$, as a right module over A. Of course, one could consider left modules over A obtained by taking quotients of A by left ideals as well.

13.9 Some freely-generated modules

Let A be a ring, with a nonzero multiplicative identity element e_A. Suppose that A is a local ring, as in Sections 4.13 and 13.4, with corresponding two-sided ideal $\mathcal{I}_0$. Let $W \neq \{0\}$ be a finitely-generated right module over A, and let $W \cdot \mathcal{I}_0$ be as in Section 13.4. This is a submodule of W, as a right module over A, as in Section 13.7. Let q_W be the natural quotient homomorphism from W onto $W/(W \cdot \mathcal{I}_0)$, as right modules over A.

Remember that $W/(W \cdot \mathcal{I}_0)$ may be considered as a right module over $A/\mathcal{I}_0$ in a natural way, as in Section 13.7. Let $w_1, \ldots, w_n$ be finitely many elements of W such that

(13.9.1) $W/(W \cdot \mathcal{I}_0)$ is generated by $q_W(w_1), \ldots, q_W(w_n)$, as a right module over $A/\mathcal{I}_0$.

This implies that W is generated by $w_1, \ldots, w_n$, as a right module over A, as in Section 13.7.

We may as well ask that $q_W(w_1), \ldots, q_W(w_n)$ be a minimal set of generators of $W/(W \cdot \mathcal{I}_0)$, as a right module over $A/\mathcal{I}_0$. This implies that

(13.9.2) $W/(W \cdot \mathcal{I}_0)$ is freely generated by $q_W(w_1), \ldots, q_W(w_n)$, as a right module over $A/\mathcal{I}_0$,

because $A/\mathcal{I}_0$ is a division ring. This can be verified in basically the same way as for vector spaces over (commutative) fields.

Let A^n be the usual space of n-tuples of elements of A, considered as a right module over A. If $a \in A^n$, then put

$$\phi(a) = \sum_{j=1}^{n} w_j \cdot a_j. \tag{13.9.3}$$

This defines a homomorphism from A^n onto W, as right modules over A.

By considering A^n as a right module over A, $A^n \cdot \mathcal{I}_0$ is defined as a submodule of A^n as in Sections 13.4 and 13.7. This is the same as the set of n-tuples of elements of $\mathcal{I}_0$, but it is better not to use the notation $\mathcal{I}_0^n$ for this set, because that may also be used for the nth power of $\mathcal{I}_0$ in A. Observe that

$$\ker \phi \subseteq A^n \cdot \mathcal{I}_0. \tag{13.9.4}$$

More precisely,

$$\ker \phi \subseteq \ker(q_W \circ \phi) = A^n \cdot \mathcal{I}_0, \tag{13.9.5}$$

using (13.9.2) in the second step.

Suppose now in addition that

(13.9.6) W is projective as a right module over A.

This implies that A^n corresponds to the direct sum of $\ker \phi$ and another submodule of A^n, as a right module over A, as in Section 2.7. This means that there is a homomorphism ψ from A^n onto $\ker \phi$, as right modules over A, that is equal to the identity mapping on $\ker \phi$. In particular, it follows that

(13.9.7) $\ker \phi$ is finitely generated, as a right module over A.

As usual, $(\ker \phi) \cdot \mathcal{I}_0$ is defined as a submodule of $\ker \phi$, as a right module over A, as in Sections 13.7 and 13.7. One can check that

$$\ker \phi = (\ker \phi) \cdot \mathcal{I}_0, \tag{13.9.8}$$

using (13.9.4) and the remarks in the preceding paragraph. More precisely,

$$(\ker \phi) \cdot \mathcal{I}_0 = \psi(A^n) \cdot \mathcal{I}_0 = \psi(A^n \cdot \mathcal{I}_0) = \ker \phi, \tag{13.9.9}$$

using (13.9.4) and the fact that ψ is the identity mapping on $\ker \phi$ in the last step.

Under these conditions, we get that

(13.9.10) $$\ker \phi = \{0\},$$

as in Section 13.4. This means that

(13.9.11) $$\phi \text{ is an isomorphism from } A^n \text{ onto } W,$$

as right modules over A. This is the same as saying that

(13.9.12) $$W \text{ is freely generated by } w_1, \ldots, w_n, \text{ as a right module over } A.$$

One could just as well consider left modules over A, as usual.

The remarks in this section basically correspond to parts of Propositions 5.1' and 5.2 and Theorem 5.3 on p155f of [3], and their proofs. This may be considered as a simplification of Theorem 6.1' on p157 of [3]. More precisely, (13.9.6) was not part of the hypotheses in [3], and was used here to get (13.9.7) and (13.9.8). In [3], (13.9.7) was obtained by asking A to be right Noetherian as a ring. Another condition was used to get (13.9.8) in [3], which is implied by (13.9.6).

The condition used in [3] to get (13.9.8) may be described in terms of "augmented rings", which are discussed in Chapter 24. Here A may be considered as an augmented ring, with augmentation ideal $\mathcal{I}_0$. The condition used in [3] may be described as saying that the first homology group of A, with respect to this augmentation, and with coefficients in W, is equal to $\{0\}$. This uses the definition of the homology groups on p143 of [3].

13.10 Complementary ideals

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a nonzero multiplicative identity element e_A. Let us say that left ideals $\mathcal{I}_1$, $\mathcal{I}_2$ in A are *complementary* in A if A corresponds to the direct sum of $\mathcal{I}_1$ and $\mathcal{I}_2$, as a left module over itself. It is easy to see that this happens if and only if A corresponds to the direct sum of $\mathcal{I}_1$ and $\mathcal{I}_2$, as a module over k. Of course, there are analogous notions and statements for right ideals and two-sided ideals in A. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are complementary two-sided ideals in A, then A corresponds to the direct sum of $\mathcal{I}_1$ and $\mathcal{I}_2$, as an algebra over k.

If $\mathcal{I}_1, \mathcal{I}_2$ are left ideals in A, then the quotients $A/\mathcal{I}_1$, $A/\mathcal{I}_2$ may be considered as left modules over A. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are complementary left ideals in A, then $A/\mathcal{I}_1$, $A/\mathcal{I}_2$ are isomorphic to $\mathcal{I}_2$, $\mathcal{I}_1$, respectively, as left ideals over A. In this case, $\mathcal{I}_1$ and $\mathcal{I}_2$ are projective as left modules over A, so that $A/\mathcal{I}_1$ and $A/\mathcal{I}_2$ are projective as left modules over A.

Let $\mathcal{I}_1$ be a left ideal in A, and suppose that $A/\mathcal{I}_1$ is projective as a left module over A. This implies that there is a left ideal $\mathcal{I}_2$ in A that is complementary to $\mathcal{I}_1$, as in Section 2.7.

Let θ_1 be an *idempotent* element of A, so that

$$\theta_1^2 = \theta_1. \tag{13.10.1}$$

Note that

$$\theta_2 = e_A - \theta_1 \tag{13.10.2}$$

is idempotent in A as well. We also have that

$$\theta_1\,\theta_2 = \theta_2\,\theta_1 = 0. \tag{13.10.3}$$

Put

$$\mathcal{I}_j = \{a\,\theta_j : a \in A\} \tag{13.10.4}$$

for $j = 1, 2$, which are left ideals in A. One can check that $\mathcal{I}_1$ and $\mathcal{I}_2$ are complementary left ideals in A under these conditions.

Conversely, suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ are complementary left ideals in A. This implies that there are unique elements θ_1, θ_2 of $\mathcal{I}_1$, $\mathcal{I}_2$ such that

$$\theta_1 + \theta_2 = e_A. \tag{13.10.5}$$

If $a \in A$, then $a\,\theta_1 \in \mathcal{I}_1$, $a\,\theta_2 \in I_2$, and

$$a = a\,\theta_1 + a\,\theta_2. \tag{13.10.6}$$

In fact, $a\,\theta_1$ and $a\,\theta_2$ are uniquely determined by these properties, by hypothesis. In particular, if $a \in \mathcal{I}_1$, then

$$a\,\theta_1 = a \text{ and } a\,\theta_2 = 0, \tag{13.10.7}$$

and similarly for elements of $\mathcal{I}_2$. It follows that θ_1, θ_2 are idempotent elements of A that satisfy (13.10.3), and that $\mathcal{I}_1, \mathcal{I}_2$ are as in (13.10.4). This corresponds to a remark on p164 of [14].

13.11 A class of modules

Let A be a ring with a nonzero multiplicative identity element e_A, and let $\mathcal{I}$ be a proper two-sided ideal in A. If W is a right module over A, then $W \cdot \mathcal{I}$ may be defined as in Section 13.4, and is a submodule of W, as in Section 13.7. Let us say that W is *proper with respect to* $\mathcal{I}$ if either $W = \{0\}$ or

$$W \neq W \cdot \mathcal{I}, \tag{13.11.1}$$

as on p154 of [3]. Thus Nakayama's lemma and its analogue for noncommutative rings give criteria for modules to be proper in this sense. Note that free modules over A are proper with respect to $\mathcal{I}$, as in [3].

Suppose now that A is a graded ring, as in Section 9.14. Remember that A^0 is a subring of A that contains e_A, and that $\varepsilon(a) = a^0$ defines a ring homomorphism from A onto A^0. In this case, we take

$$\mathcal{I} = \ker \varepsilon. \tag{13.11.2}$$

This is a proper two-sided ideal in A, which corresponds to the direct sum of A^j, $j \geq 1$, as a subgroup of A, as a commutative group with respect to addition.

Let W be a graded right module over A, as a graded ring, as in Section 9.15. Under these conditions,

$$W \text{ is proper with respect to } \mathcal{I}, \tag{13.11.3}$$

as in Proposition 5.1 on p154 of [3]. To see this, suppose that $W \neq \{0\}$, and let m be the smallest nonnegative integer such that

$$W^m \neq \{0\}. \tag{13.11.4}$$

It is easy to see that $W \cdot \mathcal{I}$ is contained in the subgroup of W generated by W^j, $j \geq m = 1$, as a commutative group with respect to addition. This implies (13.11.1), because of (13.11.4).

Of course,

$$W^j \cap (W \cdot \mathcal{I}) \tag{13.11.5}$$

is a subgroup of W, as a commutative group with respect to addition, for every $j \geq 0$. This subgroup consists of finite sums of elements of the form $w \cdot a$, where $w \in W^l$ and $a \in A^r$ for some $l \geq 0$ and $r \geq 1$ with $j = l + r$. In particular, $W^0 = \{0\}$. Observe that $W \cdot \mathcal{I}$ corresponds to the direct sum of these subgroups, with $j \geq 1$. Thus $W \cdot \mathcal{I}$ is homogeneous as a subgroup of W, as a graded commutative group with respect to addition, as in Section 5.9.

This defines a grading on $W \cdot \mathcal{I}$, as a commutative group with respect to addition. This grading on $W \cdot \mathcal{I}$ is compatible with the grading on A, as in Section 9.15. Thus $W \cdot \mathcal{I}$ may be considered as a graded module over A, as a graded ring, as before.

13.12 Quotients and gradings

Let us continue with the same notation and hypotheses as in the previous section, so that A is a graded ring, W is a graded right module over A, as a graded ring, and so on.

Let U be a submodule of W, as a right module over A, that is homogeneous as a subgroup of W, as a graded commutative group with respect to addition, as in Section 5.9. The grading on U induced by the grading on W is compatible with the grading on A, so that

$$U \text{ is graded as a module over } A, \text{ as a graded ring}, \tag{13.12.1}$$

as in Section 9.15. The quotient W/U is graded as a commutative group with respect to addition in a natural way, as in Section 5.9 again. Of course, W/U is also a right module over A, and the grading on W/U is compatible with the grading on A. This means that

$$W/U \text{ is graded as a module over } A, \text{ as a graded ring}. \tag{13.12.2}$$

If we take $U = W \cdot \mathcal{I}$, then we get that $W/(W \cdot \mathcal{I})$ is a graded module over A, as a graded ring. If $a \in \mathcal{I}$, then the action of a on the right on $W/(W \cdot \mathcal{I})$ is equal to 0, by construction. The action of A on $W/(W \cdot \mathcal{I})$ on the right leads to an action of $A/\mathcal{I}$ on $W/(W \cdot \mathcal{I})$ on the right, as in Section 13.7. This corresponds to the action of A^0 on $W/(W \cdot \mathcal{I})$ on the right, using ε. Note that $W/(W \cdot \mathcal{I})$ is a graded right module over A^0 in the usual sense, as in Sections 5.9 and 9.15.

If W is any right module over A^0, then W may be considered as a right module over A, using ϵ. In this case, if W is considered as a right module over A, then

$$W \cdot \mathcal{I} = \{0\}, \tag{13.12.3}$$

by construction. If W is a graded right module over A^0 in the usual sense, then W may be considered as a graded module over A, as a graded ring, in this way.

Let W be any graded right module over A, as a graded ring, again, and let U be a submodule of W, as a right module over A, that is homogeneous as a subgroup of W, as a graded commutative group with respect to addition. Thus W/U is a graded right module over A, as a graded ring, as before, so that $(W/U) \cdot \mathcal{I}$ is a submodule of W/U, as a graded right module over U, that is homogeneous as a subgoup of W/U, as a graded commutative group with respect to addition, as in the previous section. If $W/U \neq \{0\}$, then

$$W/U \neq (W/U) \cdot \mathcal{I}, \tag{13.12.4}$$

as in (13.11.1). Equivalently, this means that if $U \neq W$, then

$$U + (W \cdot \mathcal{I}) \neq W. \tag{13.12.5}$$

Let E be a nonempty subset of W consisting of homogeneous elements of W, as in Section 5.9. Thus each $x \in E$ is an element of W^j for some nonnegative integer j. Let U_E be the submodule of W, as a right module over A, generated by E. It is easy to see that U_E is homogeneous as a subgroup of W, as a graded commutative group with respect to addition, under these conditions.

Let q_W be the natural quotient mapping from W onto $W/(W \cdot \mathcal{I})$. Note that $W/(W \cdot \mathcal{I})$ is generated by $q_W(E)$, as a right module over A, if and only if $W/(W \cdot \mathcal{I})$ is generated by $q_W(E)$, as a right module over $A/\mathcal{I}$. This is also the same as saying that

$$U_E + (W \cdot \mathcal{I}) = W. \tag{13.12.6}$$

This implies that

$$U_E = W, \tag{13.12.7}$$

as before. This corresponds to the first part of Proposition 5.2 on p155 of [3], in the case of graded modules over graded rings.

13.13 Some graded free modules

Let A be a ring with a nonzero multiplicative identity element e_A. Also let E be a nonempty set, and let F_E be the free right module over A generated by E.

This may be defined precisely as the space $c_{00}(E, A)$ of A-valued functions on E with finite support, on which A acts by multiplication on the right. Let us identify each element of E with the A-valued function on E equal to e_A at that element, and to 0 otherwise. Thus F_E is freely generated by the elements of E, as a right module over A.

Suppose now that A is a graded ring, as in Section 9.14. Let λ be a function on E with values in the set $\mathbf{Z}_+ \cup \{0\}$ of nonnegative integers. If j is a nonnegative integer, then let $(F_E)^j$ be the subset of F_E consisting of finite sums of elements of the form $x \cdot a$, where $x \in E$, $a \in A^r$ for some nonnegative integer r, and

$$j = \lambda(x) + r. \tag{13.13.1}$$

It is easy to see that this defines a grading on F_E, as a commutative group with respect to addition, as in Section 5.9. This grading is compatible with the grading on A, by construction. Thus

$$\begin{aligned} &F_E \text{ may be considered as a graded right module over } A, \\ &\text{as a graded ring,} \end{aligned} \tag{13.13.2}$$

as in Section 9.15. Of course, one can always take $\lambda(x) = 0$ for every $x \in E$, in which case F_E corresponds to a direct sum of copies of A, indexed by E, as graded right modules over itself.

Let W be a right module over A. Any mapping ϕ from E into W has a unique extension to a homomorphism Φ from F_E into W, as right modules over A. Suppose now that W is a graded right module over A, as a graded ring, and that

$$\phi(x) \in W^{\lambda(x)} \tag{13.13.3}$$

for every $x \in E$. This implies that

$$\Phi \text{ has degree 0 as a homomorphism from } F_E \text{ into } W, \tag{13.13.4}$$

as graded commutative groups with respect to addition, as in Section 5.9. This corresponds to a remark on p155 of [3], in the proof of Proposition 5.1.

It follows that

$$\begin{aligned} &\ker \Phi \text{ and } \Phi(F_E) \text{ are homogeneous} \\ &\text{as subgroups of } F_E \text{ and } W, \text{ respectively,} \end{aligned} \tag{13.13.5}$$

as graded commutative groups with respect to addition. This means that

$$\ker \Phi \text{ and } \Phi(F_E) \text{ are graded modules over } A, \text{as a graded ring,} \tag{13.13.6}$$

with respect to the gradings induced by those on F_E and W, respectively, as in Section 9.15. This corresponds to another remark on p155 of [3].

Suppose that E is a nonempty subset of W consisting of homogeneous elements, with

$$x \in W^{\lambda(x)} \tag{13.13.7}$$

for every $x \in E$. This is the same as (13.13.3), with ϕ equal to the obvious inclusion mapping from E into W. In this case, $\Phi(F_E)$ is the same as the submodule U_E of W, as a right module over A, generated by E.

13.14 Some more freely-generated modules

Let A be a ring with a nonzero multiplicative identity element e_A, and suppose that A is a graded ring, as in Section 9.14. Thus A^0 is a subring of A that contains e_A, $\varepsilon(a) = a^0$ defines a ring homomorphism from A onto A^0, and $\mathcal{I} = \ker \varepsilon$ is a proper two-sided ideal in A. In this section, we also ask that

$$A^0 \text{ be a division ring.} \tag{13.14.1}$$

Let $W \neq \{0\}$ be a graded right module over A, as a graded ring, as in Section 9.15. Remember that $W \cdot \mathcal{I}$ is as in Section 13.4, and is a submodule of W, as a right module over A, as in Section 13.7. Let q_W be the natural quotient homomorphism from W onto $W/(W \cdot \mathcal{I})$, as right modules over A. We have seen that $W/(W \cdot \mathcal{I})$ may be considered as a right module over $A/\mathcal{I}$, which corresponds to considering $W/(W \cdot \mathcal{I})$ as a right module over A^0, using ε, as in Section 13.12.

Let E be a nonempty subset of W consisting of homogeneous elements. Suppose that $W/(W \cdot \mathcal{I})$ is generated by $q_W(E)$, as a right module over $A/\mathcal{I}$. This is the same as saying that $W/(W \cdot \mathcal{I})$ is generated by $q_W(E)$, as a right module over A, as before.

One can show that $W/(W \cdot \mathcal{I})$ is freely generated by a subset of $q_W(E)$, as a right module over $A/\mathcal{I}$, because $A/\mathcal{I}$ is a division ring. This is analogous to standard arguments for vector spaces over (commutative) fields.

This implies that there is a nonempty subset $\widetilde{E}$ of E such that

$$\begin{aligned} &W/(W \cdot \mathcal{I}) \text{ is freely generated by } q_W(\widetilde{E}), \\ &\text{as a right module over } A/\mathcal{I}, \end{aligned} \tag{13.14.2}$$

and

$$q_W \text{ is one-to-one on } \widetilde{E}. \tag{13.14.3}$$

It follows that

$$W \text{ is generated by } \widetilde{E}, \text{ as a right module over } A, \tag{13.14.4}$$

as in Section 13.12.

If $x \in \widetilde{E}$, then let $\lambda(x)$ be a nonnegative integer such that $x \in W^{\lambda(x)}$. Let $F_{\widetilde{E}}$ be the free right module over A generated by $\widetilde{E}$, with the grading defined using λ, as in the previous section. The obvious inclusion mapping $\widetilde{\phi}$ from $\widetilde{E}$ into W has a unique extension to a homomorphism $\widetilde{\Phi}$ from $F_{\widetilde{E}}$ into W, as right modules over A. We also have that

$$\widetilde{\Phi} \text{ has degree 0 as a homomorphism from } F_{\widetilde{E}} \text{ into } W, \tag{13.14.5}$$

as graded commutative groups with respect to addition, as in the previous section.

Note that

$$\widetilde{\Phi}(F_{\widetilde{E}}) = W, \tag{13.14.6}$$

by (13.14.4). As in the previous section, $\ker \widetilde{\Phi}$ is homogeneous as a subgroup of $F_{\widetilde{E}}$, as a graded commutative group with respect to addition. This implies that

$$\ker \widetilde{\Phi} \text{ is a graded module over } A, \text{ as a graded ring,} \tag{13.14.7}$$

with respect to the grading induced by the one on $F_{\widetilde{E}}$, as before.

Of course, $F_{\widetilde{E}} \cdot \mathcal{I}$ can be defined in the usual way, because $F_{\widetilde{E}}$ is a right module over A. In this case, $F_{\widetilde{E}} \cdot \mathcal{I}$ consists of finite sums of elements of the form $x \cdot a$, where $x \in \widetilde{E}$ and $a \in \mathcal{I}$. Observe that

$$\ker(q_W \circ \widetilde{\Phi}) = F_{\widetilde{E}} \cdot \mathcal{I}, \tag{13.14.8}$$

because of (13.14.2) and (13.14.3). This means that

$$\ker \widetilde{\Phi} \subseteq F_{\widetilde{E}} \cdot \mathcal{I}. \tag{13.14.9}$$

Suppose that

$$W \text{ is projective as a right module over } A. \tag{13.14.10}$$

It follows that $F_{\widetilde{E}}$ corresponds to the direct sum of $\ker \widetilde{\Phi}$ and another submodule of $F_{\widetilde{E}}$, as a right module over A, as in Section 2.7. This implies that there is a homomorphism Ψ from $F_{\widetilde{E}}$ onto $\ker \widetilde{\Phi}$, as right modules over A, that is equal to the identity mapping on $\ker \widetilde{\Phi}$.

Note that $(\ker \widetilde{\Phi}) \cdot \mathcal{I}$ may be defined in the usual way, and is a submodule of $\ker \widetilde{\Phi}$, as a right module over A. Under these conditions, we have that

$$(\ker \widetilde{\Phi}) \cdot \mathcal{I} = \Psi(F_{\widetilde{E}}) \cdot \mathcal{I} = \Psi(F_{\widetilde{E}} \cdot \mathcal{I}) = \ker \widetilde{\Phi}, \tag{13.14.11}$$

using (13.14.9) and the fact that Ψ is the identity mapping on $\ker \widetilde{\Phi}$ in the last step.

This implies that

$$\ker \widetilde{\Phi} = \{0\}, \tag{13.14.12}$$

because of (13.14.7), as in Section 13.11. It follows that $\widetilde{\Phi}$ is an isomorphism from $F_{\widetilde{E}}$ onto W, as right modules over A.

This basically corresponds to parts of Propositions 5.1 and 5.2 and Theorem 5.3 on p154ff of [3], and their proofs. This may also be considered as a simplification of Theorem 6.1 on p156 of [3]. More precisely, in [3], the first homology group of A, as an augmented ring with augmentation ideal $\mathcal{I}$, and with coefficients in W, was asked to be $\{0\}$, instead of (13.14.10). This condition is implied by (13.14.10), and was used to get the same conclusion as in (13.14.11). Of course, there are analogous statements for left moduules.

One might note that some of the arguments used here and in Section 13.12 are similar to arguments in Sections 13.7 and 13.9. In [3], some conditions are formulated on p154 that include both cases, and Proposition 5.2 and Theorem 5.3 on p155f are stated in terms of these conditions.

13.15 Some quotients as tensor products

Let A be a ring with a multiplicative identity element e_A, and let $\mathcal{I}$ be a left ideal in A. If W is a right module over A, then $W \cdot \mathcal{I}$ is defined as in Section 13.4, and is a subgroup of W, as a commutative group with respect to addition. This means that the quotient

$$W/(W \cdot \mathcal{I}) \tag{13.15.1}$$

is defined as a commutative group as well. If $\mathcal{I}$ is a two-sided ideal in A, then $W \cdot \mathcal{I}$ is a submodule of W, as a right module over A, as in Section 13.7. In this case, (13.15.1) may be considered as a right module over A too.

Put

$$Q = A/\mathcal{I}, \tag{13.15.2}$$

which may be considered as a left module over A. If $\mathcal{I}$ is a two-sided ideal in A, then Q may be considered as a ring, and both a left and right module over A.

Let $W \bigotimes_A \mathcal{I}$ and $W \bigotimes_A Q$ be tensor products of W with $\mathcal{I}$ and Q, respectively, which are commutative groups with respect to addition. If $\mathcal{I}$ is a two-sided ideal in A, then $W \bigotimes_A \mathcal{I}$ and $W \bigotimes_A Q$ may be considered as right modules over A, as in Section 1.10.

Using the identity mapping on W and the obvious inclusion mapping from $\mathcal{I}$ into A, we get a homomorphism

$$\text{from } W \bigotimes_A \mathcal{I} \text{ into } W \bigotimes_A A, \tag{13.15.3}$$

as commutative groups with respect to addition. This may be identified with a homomorphism

$$\text{from } W \bigotimes_A \mathcal{I} \text{ into } W, \tag{13.15.4}$$

as commutative groups with respect to addition, because of the usual identification of $W \bigotimes_A A$ with W. The image of this homomorphism is equal to $W \cdot \mathcal{I}$. If $\mathcal{I}$ is a two-sided ideal in A, then we get module homomorphisms in (13.15.3) and (13.15.4).

Similarly, we can use the identity mapping on W and the natural quotient mapping from A onto Q to get a homomorphism

$$\text{from } W \bigotimes_A A \text{ onto } W \bigotimes_A Q, \tag{13.15.5}$$

as commutative groups with respect to addition. This may be identified with a homomorphism

$$\text{from } W \text{ onto } W \bigotimes_A Q, \tag{13.15.6}$$

as commutative groups with respect to addition, by identifying $W \bigotimes_A A$ with W again. If $\mathcal{I}$ is a two-sided ideal in A, then we get module homomorphisms in (13.15.5) and (13.15.6), as before.

The image of the homomorphism as in (13.15.3) is equal to the kernel of the homomorphism as in (13.15.5), as in Section 2.5. Equivalently, the image of the homomorphism as in (13.15.4) is equal to the kernel of the homomorphism as in (13.15.6). This leads to an isomorphism

$$\text{from } W/(W \cdot \mathcal{I}) \text{ onto } W \bigotimes\nolimits_A Q, \tag{13.15.7}$$

as commutative groups with respect to addition. If $\mathcal{I}$ is a two-sided ideal in A, then this is an isomorphism between right modules over A.

This corresponds to Exercise 2 on p31 of [1] when A is commutative. This also corresponds to (2) on p144 of [3], and to (1) on p154 of [3]. Of course, there are analogous statements for left modules over A and right ideals in A.

Chapter 14

Fractions, dimension, and valuations

14.1 Pairs of multiplicatively closed sets

Let k_1, k_2 be commutative rings with multiplicative identity elements 1_{k_1}, 1_{k_2}, respectively, and let ϕ be a ring homomorphism from k_1 into k_2 with $\phi(1_{k_1}) = 1_{k_2}$. Also let S_1, S_2 be multiplicatively closed subsets of k_1, k_2, respectively, and suppose that

(14.1.1) $$\phi(S_1) \subseteq S_2.$$

Thus $S_1^{-1}\, k_1$, $S_2^{-1}\, k_2$ may be defined as in Section 12.1, with the corresponding ring homomorphisms

(14.1.2) $$x_1 \mapsto x_1/1_{k_1}$$

and

(14.1.3) $$x_2 \mapsto x_2/1_{k_2}$$

from k_1, k_2 into $S_1^{-1}\, k_1$, $S_2^{-1}\, k_2$, respectively. As in Section 12.11,

(14.1.4) $$\Phi(x_1/r_1) = \phi(x_1)/\phi(r_1)$$

defines a ring homomorphism from $S_1^{-1}\, k_1$ into $S_2^{-1}\, k_2$. Note that

(14.1.5) $$S_1^{-1}(\ker \phi) \subseteq \ker \Phi.$$

Alternatively,

(14.1.6) $$x_1 \mapsto \phi(x_1)/1_{k_2}$$

defines a ring homomorphism from k_1 into $S_2^{-1}\, k_2$, which is the same as the composition of ϕ with (14.1.3). This homomorphism sends elements of S_1 to invertible elements of $S_2^{-1}\, k_2$, because of (14.1.1). This leads to a unique ring homomorphism Φ from $S_1^{-1}\, k_1$ into $S_2^{-1}\, k_2$ such that

(14.1.7) $$\Phi(x_1/1_{k_1}) = \phi(x_1)/1_{k_2}$$

for every $x_1 \in k_1$, as in Section 12.12. Of course, this is the same as the homomorphism Φ mentioned in the preceding paragraph.

Let k be a commutative ring with a multiplicative identity element 1_k, and let S, T be multiplicatively closed subsets of k. This implies that

$$S\,T = \{r\,t : r \in S,\, t \in T\} \tag{14.1.8}$$

is a multiplicatively closed set in k. Let U be the image of T under the natural homomorphism from k into $S^{-1}\,k$, which is a multiplicatively closed set in $S^{-1}\,k$. Exercise 3 on p43 of [1] asks one to show that

(14.1.9) $(S\,T)^{-1}\,k$ and $U^{-1}\,(S^{-1}\,k)$ are isomorphic as rings.

There is a natural ring homomorphism

(14.1.10) from k into $U^{-1}\,(S^{-1}\,k)$,

obtained by composing the natural homomorphism from k into $S^{-1}\,k$ with the natural homomorphism from $S^{-1}\,k$ into $U^{-1}\,(S^{-1}\,k)$. It is easy to see that this homomorphism sends elements of $S\,T$ to invertible elements of $U^{-1}\,(S^{-1}\,k)$. This leads to a natural ring homomorphism

(14.1.11) from $(S\,T)^{-1}\,k$ into $U^{-1}\,(S^{-1}\,k)$,

as in Section 12.12. More precisely, the natural homomorphism as in (14.1.10) is equal to the composition of the natural homomorphism from k into $(S\,T)^{-1}\,k$ with the homomorphism as in (14.1.11).

Note that $S \subseteq S\,T$. This leads to a natural ring homomorphism

(14.1.12) from $S^{-1}\,k$ into $(S\,T)^{-1}\,k$,

as discussed at the beginning of the section. More precisely, the composition of the natural homomorphism from k into $S^{-1}\,k$ with the homomorphism as in (14.1.12) is the same as the natural homomorphism from k into $(S\,T)^{-1}\,k$. In particular, the image of U under the homomorphism as in (14.1.12) is the same as the image of T under the natural homomorphism from k into $(S\,T)^{-1}\,k$. Of course, the natural homomorphism from k into $(S\,T)^{-1}\,k$ sends elements of T to invertible elements of $(S\,T)^{-1}\,k$. This means that the homomorphism as in (14.1.12) sends elements of U to invertible elements of $(S\,T)^{-1}\,k$. Thus the homomorphism as in (14.1.12) leads to a natural ring homomorphism

(14.1.13) from $U^{-1}\,(S^{-1}\,k)$ into $(S\,T)^{-1}\,k$,

as in Section 12.12. One can check that this is the inverse of the homomorphism as in (14.1.11).

14.2 Fractions and integral domains

Let k be an integral domain, so that k is a commutative ring with a nonzero multiplicative identity element and no nonzero zero divisors. Thus

$$S_0 = k \setminus \{0\} \tag{14.2.1}$$

is a multiplicatively closed set in k, so that the corresponding ring of fractions $S_0^{-1}\,k$ may be defined as in Section 12.1. In this case, the natural homomorphism from k into $S_0^{-1}\,k$ is injective, so that k may be identified with a subring of $S_0^{-1}\,k$. In fact, $S_0^{-1}\,k$ is the same as the usual field Q_k of fractions associated to k.

Let S be another multiplicatively closed set in k, with $0 \notin S$. Note that the natural homomorphism from k into $S^{-1}\,k$ is injective, so that k may be identified with a subring of $S^{-1}\,k$, as before. There is also a natural ring homomorphism from $S^{-1}\,k$ into Q_k, as in the previous section. One can check that this homomorphism is injective too, so that $S^{-1}\,k$ may be identified with a subring of Q_k as well. More precisely, $S^{-1}\,k$ corresponds to the set of elements of Q_k of the form x/r, with $x \in k$ and $r \in S$.

Of course, $S^{-1}\,k$ is an integral domain, and one can define its field of fractions in the usual way. In fact, the field of fractions of $S^{-1}\,k$ is isomorphic to Q_k in a natural way.

We may consider Q_k as a module over k, and as a module over $S^{-1}\,k$. If we consider Q_k as a module over k, then $S^{-1}\,Q_k$ may be defined as a module over $S^{-1}\,k$, as in Section 12.2. One can check that $S^{-1}\,Q_k$ is isomorphic to Q_k, considered as a module over $S^{-1}\,k$, in a natural way.

Let M be a submodule of Q_k, as a module over k. Remember that $S^{-1}\,M$ may be considered as a submodule of $S^{-1}\,Q_k$, as in Section 12.2. In this case, $S^{-1}\,M$ may be identified with a submodule of Q_k, as a module over $S^{-1}\,k$. Using this identification, $S^{-1}\,M$ consists of elements of Q_k of the form y/r, with $y \in M$ and $r \in S$.

Remember that M is called a fractional ideal of k if there is an $x \in k$ such that $x \neq 0$ and $x\,M \subseteq k$, as in Section 11.7, and on p96 of [1]. Similarly, a submodule N of Q_k, as a module over $S^{-1}\,k$, is a fractional ideal of $S^{-1}\,k$ if there are $x \in k$ and $r \in S$ such that $x \neq 0$ and

$$(x/r)\,N \subseteq S^{-1}\,k. \tag{14.2.2}$$

This implies that

$$x\,N \subseteq S^{-1}\,k, \tag{14.2.3}$$

so that we might as well take $r = 1$.

If (14.2.3) holds for some $x \in Q_k$ with $x \neq 0$, then N is a fractional ideal of $S^{-1}\,k$, as in Section 11.7. More precisely, we can multiply x by a nonzero element of k to get another nonzero element of k in this case, as before.

If M is a fractional ideal of k, then it is easy to see that $S^{-1}\,M$ is a fractional ideal of $S^{-1}\,k$.

14.3 Fractions and invertible ideals

Let us continue with the same notation and hypotheses as in the previous section. If M, N are submodules of Q_k, as a module over k, then $M\,N$ is defined to be the set of finite sums of products of elements of M and N, as in Section 11.7. This is a submodule of Q_k, as a module over k, as before. Similarly, if M and N are submodules of Q_k, as a module over $S^{-1}\,k$, then $M\,N$ can be defined in the same way, and is a submodule of Q_k, as a module over $S^{-1}\,k$.

Remember that a submodule M of Q_k, as a module over k, is said to be an invertible ideal of k if there is a submodule N of Q_k, as a module over k, such that $M\,N = k$, as in Section 11.7, and on p96 of [1]. This implies that M, N are fractional ideals of k, as before. Similarly, a submodule M of Q_k, as a module over $S^{-1}\,k$, is said to be an invertible ideal of $S^{-1}\,k$ if there is a submodule N of Q_k, as a module over $S^{-1}\,k$, such that

$$M\,N = S^{-1}\,k. \tag{14.3.1}$$

This implies that M and N are fractional ideals of $S^{-1}\,k$, for the same reasons as before.

If M, N are submodules of Q_k, as a module over k, then $S^{-1}\,M$, $S^{-1}\,N$ are submodules of Q_k, as a module over $S^{-1}\,k$, as in the previous section. It is easy to see that

$$S^{-1}(M\,N) = (S^{-1}\,M)\,(S^{-1}\,N). \tag{14.3.2}$$

If M is an invertible ideal of k, then there is a submodule N of Q_k, as a module over k, such that $M\,N = k$, and we get that

$$(S^{-1}\,M)\,(S^{-1}\,N) = S^{-1}\,k. \tag{14.3.3}$$

This implies that $S^{-1}\,M$ is an invertible ideal of $S^{-1}\,k$, as in Proposition 9.6 on p97 of [1].

If M is a submodule of Q_k, as a module over k, then put

$$(k : M)_{Q_k} = \{x \in Q_k : x\,M \subseteq k\}. \tag{14.3.4}$$

This was denoted $(k : M)$ in Section 11.7, and we include the subscript Q_k on the left here, because similar notation was used in Section 12.8 for something else. Note that $(k : M)_{Q_k}$ is a submodule of Q_k, as a module over k, that is nonzero exactly when M is a fractional ideal of k, as before. If $M \neq \{0\}$, then $(k : M)_{Q_k}$ is a fractional ideal of k, as before.

Similarly, if M is a submodule of Q_k, as a module over $S^{-1}\,k$, then put

$$(S^{-1}\,k : M)_{Q_k} = \{x \in Q_k : x\,M \subseteq S^{-1}\,k\}. \tag{14.3.5}$$

This is a submodule of Q_k, as a module over $S^{-1}\,k$, that is nonzero exactly when M is a fractional ideal of $S^{-1}\,k$. If $M \neq \{0\}$, then $(S^{-1}\,k : M)_{Q_k}$ is a fractional ideal of $S^{-1}\,k$.

Suppose that M is a submodule of Q_k, as a module over k, that is an invertible ideal of k, so that there is a submodule N of Q_k, as a module over k, such that $M\,N = k$. Under these conditions,

$$N = (k : M)_{Q_k}, \tag{14.3.6}$$

as in Section 11.7. Similarly, let M be a submodule of Q_k, as a module over $S^{-1}\,k$, that is an invertible ideal of $S^{-1}\,k$, so that there is a submodule N of Q_k, as a module over $S^{-1}\,k$, that satisfies (14.3.1). In this case, we have that

$$N = (S^{-1}\,k : M)_{Q_k}, \tag{14.3.7}$$

for the same reasons as before.

If M_1, M_2 are submodules of Q_k, as a module over k, then it is easy to see that

$$(k : M_1 + M_2)_{Q_k} = (k : M_1)_{Q_k} \cap (k : M_2)_{Q_k}. \tag{14.3.8}$$

Similarly, if M_1, M_2 are submodules of Q_k, as a module over ${}^{-1}\,k$, then

$$(S^{-1}\,k : M_1 + M_2)_{Q_k} = (S^{-1}\,k : M_1)_{Q_k} \cap (S^{-1}\,k : M_2)_{Q_k}. \tag{14.3.9}$$

14.4 Fractions and $(k : M)_{Q_k}$

We continue with the same notation and hypotheses as in the previous two sections. Let M be a submodule of Q_k, as a module over k, and suppose that M is finitely generated, as a module over k. We would like to check that

$$(S^{-1}\,k : S^{-1}\,M)_{Q_k} = S^{-1}(k : M)_{Q_k}. \tag{14.4.1}$$

This is analogous to Proposition 3.15 on p43 of [1], which was discussed in Section 12.14.

Let M_1, M_2 be submodules of Q_k, as a module over k, and suppose that they satisfy the corresponding analogues of (14.4.1). We would like to verify that $M_1 + M_2$ has the same property. Using (14.3.8), we get that

$$S^{-1}(k : M_1 + M_2)_{Q_k} = S^{-1}((k : M_1)_{Q_k} \cap (k : M_2)_{Q_k}). \tag{14.4.2}$$

This is equal to

$$(S^{-1}(k : M_1)_{Q_k}) \cap (S^{-1}(k : M_2)_{Q_k}), \tag{14.4.3}$$

as in Section 12.2. This reduces to

$$(S^{-1}\,k : S^{-1}\,M_1)_{Q_k} \cap (S^{-1}\,k : S^{-1}\,M_2)_{Q_k}, \tag{14.4.4}$$

by hypothesis. This is the same as

$$(S^{-1}\,k : (S^{-1}\,M_1) + (S^{-1}\,M_2))_{Q_k}, \tag{14.4.5}$$

because of (14.3.9). This is equal to

$$(S^{-1}\,k : S^{-1}(M_1 + M_2))_{Q_k}, \tag{14.4.6}$$

as in Section 12.2 again.

The remarks in the preceding paragraph permit one to reduce to the case where M is generated by a single element, as a module over k. In this case, (14.4.1) can be verified directly.

Alternatively, it is easy to see that

$$S^{-1}(k : M)_{Q_k} \subseteq (S^{-1}\, k : S^{-1}\, M)_{Q_k} \tag{14.4.7}$$

for any submodule of Q_k, as a module over k. Thus it suffices to show that

$$(S^{-1}\, k : S^{-1}\, M)_{Q_k} \subseteq S^{-1}(k : M)_{Q_k} \tag{14.4.8}$$

when M is finitely generated as a module over k.

Suppose that M is generated by $y_1, \ldots, y_n \in M$, as a module over k, for some positive integer n. If $x \in Q_k$ is in the left side of (14.4.8), then

$$x\, y_j \in S^{-1}\, k \tag{14.4.9}$$

for each $j = 1, \ldots, n$. This means that for each $j = 1, \ldots, n$ there is an $r_j \in S$ such that

$$r_j\, x\, y_j \in k. \tag{14.4.10}$$

Put $r = \prod_{j=1}^{n} r_j$, so that $r \in S$, and

$$r\, x\, y_j \in k \tag{14.4.11}$$

for each $j = 1, \ldots, n$. This implies that

$$r\, x \in (k : M)_{Q_k}, \tag{14.4.12}$$

so that $x \in S^{-1}(k : M)_{Q_k}$.

14.5 Primary ideals

Let k be a commutative ring with a multiplicative identity element $1_k = 1$. An ideal $\mathcal{I} \neq k$ in k is said to be *primary* if for every $x, y \in k$ with $x\, y \in \mathcal{I}$, we have that

$$x \in \mathcal{I} \text{ or } y^n \in \mathcal{I} \text{ for some } n \in \mathbf{Z}_+, \tag{14.5.1}$$

as on p50 of [1]. This is the same as saying that $k/\mathcal{I} \neq \{0\}$, and that

$$\text{every zero-divisor in } k/\mathcal{I} \text{ is nilpotent,} \tag{14.5.2}$$

as in [1]. Note that proper prime ideals are primary.

Let k_2 be another commutative ring with multiplicative identity element 1_{k_2}, and let f be a ring homomorphism from k into k_2 with $f(1_k) = 1_{k_2}$. Let $\mathcal{I}_2$ be an ideal in k_2, and remember that $f^{-1}(\mathcal{I}_2)$ is an ideal in k, which is the contraction $\mathcal{I}_2^c$ of $\mathcal{I}_2$ with respect to f, as in Section 12.7. If $\mathcal{I}_2 \neq k_2$, then $1_{k_2} \notin \mathcal{I}_2$, which implies that $1_k \notin f^{-1}(\mathcal{I}_2)$, so that $f^{-1}(\mathcal{I}_2) \neq k$. If

$$\mathcal{I}_2 \text{ is a primary ideal in } k_2, \tag{14.5.3}$$

then

$$f^{-1}(\mathcal{I}_2) \text{ is a primary ideal in } k, \tag{14.5.4}$$

as mentioned on p50 of [1]. Although this can be verified directly, it can also be obtained by observing that f induces a ring isomorphism from $k/f^{-1}(\mathcal{I}_2)$ onto a subring of $k_2/\mathcal{I}_2$, as in [1].

Let $\mathcal{I}$ be a primary ideal in k, and remember that $r(\mathcal{I})$ is the radical of $\mathcal{I}$ in k, as in Section 12.10. Let us check that

$$r(\mathcal{I}) \text{ is a prime ideal in } k, \tag{14.5.5}$$

as in Proposition 4.1 on p50 of [1]. If $x, y \in k$ and $x\,y \in r(\mathcal{I})$, then

$$x^m\,y^m = (x\,y)^m \in \mathcal{I} \tag{14.5.6}$$

for some positive integer m. This implies that either $x^m \in \mathcal{I}$, or that $y^{m\,n} = (y^m)^n \in \mathcal{I}$ for some positive integer n, because $\mathcal{I}$ is primary. This means that either x or y is an element of $r(\mathcal{I})$, as desired.

Remember that $r(\mathcal{I})$ is the same as the intersection of all of the prime ideals in k that contain $\mathcal{I}$, as in Section 12.10. It follows that $r(\mathcal{I})$ is the smallest prime ideal in k that contains $\mathcal{I}$ in this case, as in Proposition 4.1 on p50 of [1]. If

$$\mathcal{I}_0 = r(\mathcal{I}), \tag{14.5.7}$$

then $\mathcal{I}$ is said to be $\mathcal{I}_0$-*primary*, as on p51 of [1].

Let $\mathcal{I}$ be a proper ideal in k, and suppose that

$$r(\mathcal{I}) \text{ is a maximal proper ideal in } k. \tag{14.5.8}$$

Remember that $r(\mathcal{I}) \neq k$ when $\mathcal{I} \neq k$, as in Section 12.10. Under these conditions,

$$\mathcal{I} \text{ is a primary ideal in } k, \tag{14.5.9}$$

as in Proposition 4.2 on p51 of [1]. To see this, note that the image of $r(\mathcal{I})$ in $k/\mathcal{I}$ under the natural quotient mapping is the nilradical $\mathcal{N}(k/\mathcal{I})$ of $k/\mathcal{I}$, as in Section 12.10. Of course, the image of $r(\mathcal{I})$ in $k/\mathcal{I}$ is also a maximal proper ideal in $k/\mathcal{I}$, so that

$$\mathcal{N}(k/\mathcal{I}) \text{ is a maximal proper ideal in } k/\mathcal{I}. \tag{14.5.10}$$

Remember that $\mathcal{N}(k/\mathcal{I})$ is the same as the intersection of all prime ideals in $k/\mathcal{I}$, as in Section 12.5. In particular, every prime ideal in $k/\mathcal{I}$ contains $\mathcal{N}(k/\mathcal{I})$. It follows that

$$\mathcal{N}(k/\mathcal{I}) \text{ is the only proper prime ideal in } k/\mathcal{I}, \tag{14.5.11}$$

because it is maximal.

This means that $\mathcal{N}(k/\mathcal{I})$ is the only proper maximal ideal in $k/\mathcal{I}$, because maximal ideals are prime. Thus every element of $k/\mathcal{I}$ that is not in $\mathcal{N}(k/\mathcal{I})$ has a multiplicative inverse in $k/\mathcal{I}$. This implies that every zero-divisor in $k/\mathcal{I}$ is an element of $\mathcal{N}(k/\mathcal{I})$. It follows that $\mathcal{I}$ is a primary ideal in k, as before.

If $\mathcal{I}_0$ is a maximal proper ideal in k, then $\mathcal{I}_0$ is a prime ideal in k, and

$$r(\mathcal{I}_0^n) = \mathcal{I}_0 \tag{14.5.12}$$

for every positive integer n, as in Section 12.10. This implies that $\mathcal{I}_0^n$ is a primary ideal in k, which is another part of Proposition 4.2 on p51 of [1].

14.6 More on $S^{-1}\,k$

Let k be a commutative ring with a multiplicative identity element. Also let S be a multiplicatively closed subset of k, so that $S^{-1}\,k$ is the corresponding ring of fractions, as in Section 12.1.

Suppose that $\mathcal{I}_0$ is a prime ideal in k, and that $\mathcal{I}$ is an $\mathcal{I}_0$-primary ideal in k. If

$$S \cap \mathcal{I}_0 \neq \emptyset, \tag{14.6.1}$$

then

$$S^{-1}\,\mathcal{I} = S^{-1}\,k, \tag{14.6.2}$$

as in part (i) of Proposition 4.8 on p53 of [1]. Indeed, if $r \in S \cap \mathcal{I}_0$, then $r^n \in S \cap \mathcal{I}$ for some positive integer n. This means that $r^n/1$ is an element of $S^{-1}\,\mathcal{I}$ that is invertible in $S^{-1}\,k$, so that (14.6.2) holds.

Suppose now that

$$S \cap \mathcal{I}_0 = \emptyset, \tag{14.6.3}$$

so that $S^{-1}\,\mathcal{I}_0$ is a proper prime ideal in $S^{-1}\,k$, as in Section 12.13. Part (ii) of Proposition 4.8 on p53 of [1] states that

$$S^{-1}\,\mathcal{I} \text{ is } (S^{-1}\,\mathcal{I}_0)\text{-primary in } S^{-1}\,k, \tag{14.6.4}$$

and that

$$\text{the contraction of } S^{-1}\,\mathcal{I} \text{ is equal to } \mathcal{I}. \tag{14.6.5}$$

More precisely, this means the contraction of $S^{-1}\,\mathcal{I}$ with respect to the natural homomorphism from k into $S^{-1}\,k$, as in Section 12.7. Remember that $S^{-1}\,\mathcal{I}$ is the same as the extension $\mathcal{I}^e$ of $\mathcal{I}$ with respect to the natural homomorphism from k into $S^{-1}\,k$, as in Section 12.13.

Thus (14.6.5) is the same as saying that the contraction $\mathcal{I}^{ec}$ of $\mathcal{I}^e$ is the same as $\mathcal{I}$. Of course, $\mathcal{I} \subseteq \mathcal{I}^{ec}$, and so we only need to check the opposite inclusion. To do this, we use the characterization of $\mathcal{I}^{ec}$ in Section 12.13. Let $t \in S$ and $x \in k$ be given, with $t\,x \in \mathcal{I}$. It suffices to show that

$$x \in \mathcal{I}. \tag{14.6.6}$$

If $x \notin \mathcal{I}$, then $t^n \in \mathcal{I}$ for some positive integer n, because $\mathcal{I}$ is primary. This would imply that $t \in r(\mathcal{I}) = \mathcal{I}_0$, which would contradict (14.6.3).

Observe that

$$r(S^{-1}\,\mathcal{I}) = S^{-1}\,r(\mathcal{I}) = S^{-1}\,\mathcal{I}_0, \tag{14.6.7}$$

where the first step is as in Section 12.13, and the second step follows from the fact that $r(\mathcal{I}) = \mathcal{I}_0$, by hypothesis. One can check directly that $S^{-1}\,\mathcal{I}$ is primary in $S^{-1}\,k$, to get (14.6.4).

If $\mathcal{I}_2$ is a primary ideal in $S^{-1}\,k$, then its contraction $\mathcal{I}_2^c$ with respect to the natural homomorphism from k into $S^{-1}\,k$ is a primary ideal in k, as in the previous section. Remember that there is a natural one-to-one correspondence between contracted ideals in k and extended ideals in $S^{-1}\,k$, as in Section 12.7. In this case, all ideals in $S^{-1}\,k$ are extended, as in Section 12.13. Using the previous statements, we get a one-to-one correspondence between the primary ideals in $S^{-1}\,k$ and the primary ideals in k that are disjoint from S. This is the last part of Proposition 4.8 on p53 of [1].

Remember that k is Noetherian if every ideal in k is finitely generated as a module over k, as in Section 9.7. This is equivalent to the ascending chain condition for the collection of ideals in k, as before. This is also equivalent to asking that any nonempty collection of ideals in k have a maximal element.

If k is Noetherian, then

$$S^{-1}\,k \text{ is Noetherian,} \tag{14.6.8}$$

as in Proposition 7.3 on p80 of [1]. Remember that every ideal in $S^{-1}\,k$ is the extension of its contraction with respect to the natural homomorphism $x \mapsto x/1$ from k into $S^{-1}\,k$, as in Section 12.13. There is also a one-to-one correspondence between contractions of ideals in $S^{-1}\,k$ and extensions of ideals in k, as in Section 12.7. One can use this to get the ascending chain condition for ideals in $S^{-1}\,k$, or the existence of maximal elements in nonempty collections of ideals in $S^{-1}\,k$, from the corresponding property of k.

Alternatively, every ideal in $S^{-1}\,k$ is of the form $S^{-1}\,\mathcal{I}_1$ for some ideal $\mathcal{I}_1$ in k, as in Section 12.13. More precisely, any ideal in $S^{-1}\,k$ is the extension of an ideal $\mathcal{I}_1$ in k with respect to the natural homomorphism from k into $S^{-1}\,k$, and the extension of $\mathcal{I}_1$ is equal to $S^{-1}\,\mathcal{I}_1$, as before. If k is Noetherian, then $\mathcal{I}_1$ is generated by finitely many elements $x_1, \ldots, x_n$, as an ideal in k. In this case, it is easy to see that $S^{-1}\,\mathcal{I}_1$ is generated by $x_1/1, \ldots, x_n/1$, as an ideal in $S^{-1}\,k$, as in [1].

14.7 Powers of radicals

Let k be a commutative ring with a multiplicative identity element, let $\mathcal{I}$ be an ideal in k, and let $r(\mathcal{I})$ be the radical of $\mathcal{I}$ in k, as in Section 12.10. Suppose that $\mathcal{I}_1$ is an ideal in k such that

$$\mathcal{I}_1 \text{ is finitely generated, as an ideal in } k \tag{14.7.1}$$

and

$$\mathcal{I}_1 \subseteq r(\mathcal{I}). \tag{14.7.2}$$

Under these conditions, there is a positive integer m such that

$$\mathcal{I}_1^m \subseteq \mathcal{I}, \tag{14.7.3}$$

as in Proposition 7.14 on p83 of [1]. Remember that powers of ideals in k are defined using products of ideals as in Section 12.8. Of course, one might as well take $\mathcal{I}_1 = r(\mathcal{I})$ when $r(\mathcal{I})$ is finitely generated as an ideal in k, which holds automatically when k is Noetherian.

Let $x_1, \ldots, x_l$ be finitely many elements of $\mathcal{I}_1$ that generate $\mathcal{I}_1$, as an ideal in k. Thus, for each $j = 1, \ldots, l$, there is a positive integer n_j such that

$$x_j^{n_j} \in \mathcal{I}, \tag{14.7.4}$$

because $x_j \in r(\mathcal{I})$. If m is any positive integer, then $\mathcal{I}_1^m$ is generated as an ideal in k by elements of the form

$$x_1^{r_1}\, x_2^{r_2} \cdots x_l^{r_l}, \tag{14.7.5}$$

where $r_1, \ldots, r_l$ are nonnegative integers such that

$$\sum_{j=1}^{l} r_j = m. \tag{14.7.6}$$

If

$$m = \sum_{j=1}^{l} (n_j - 1) + 1, \tag{14.7.7}$$

then (14.7.6) implies that $r_j \geq n_j$ for some j. This means that (14.7.5) is an element of $\mathcal{I}$, as desired.

If $\mathcal{I} = \{0\}$, then $r(\mathcal{I})$ is the same as the nilradical $\mathcal{N}$ of k, as in Section 12.5. If

$$\mathcal{N} \text{ is finitely generated, as an ideal in } k, \tag{14.7.8}$$

then it follows that

$$\mathcal{N}^m = \{0\} \tag{14.7.9}$$

for some positive integer m, as before. In particular, this holds when k is Noetherian, as in Corollary 7.15 on p83 of [1].

Let $\mathcal{I}_0$ be an ideal in k. If

$$r(\mathcal{I}) = \mathcal{I}_0, \tag{14.7.10}$$

and if $\mathcal{I}_0$ is finitely generated as an ideal in k, then

$$\mathcal{I}_0^m \subseteq \mathcal{I} \subseteq \mathcal{I}_0 \tag{14.7.11}$$

for some positive integer m, as in Proposition 7.16 on p83 of [1]. In particular, (14.7.10) implies that (14.7.11) holds for some $m \geq 1$ when k is Noetherian.

Of course, the second inclusion in (14.7.11) follows directly from (14.7.10). The first inclusion follows from (14.7.3) under these conditions.

If $\mathcal{I}_0$ is any ideal in k, and (14.7.11) holds for some $m \geq 1$, then we get that

$$r(\mathcal{I}) = r(\mathcal{I}_0), \tag{14.7.12}$$

because $r(\mathcal{I}_0^m) = r(\mathcal{I}_0)$, as in Section 12.10. If $\mathcal{I}_0$ is a prime ideal in k, then $r(\mathcal{I}_0) = \mathcal{I}_0$, as before, so that (14.7.10) holds. Of course, if $\mathcal{I}_0$ is a maximal proper ideal in k, then $\mathcal{I}_0$ is prime. This corresponds to another part of Proposition 7.16 on p83 of [1].

14.8 Dimension via prime ideals

Let k be a commutative ring with a multiplicative identity element. Suppose that

$$\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \cdots \subseteq \mathcal{I}_n \tag{14.8.1}$$

is a strictly increasing sequence of proper prime ideals in k for some nonnegative integer n, so that $\mathcal{I}_j \neq \mathcal{I}_{j+1}$ for each $j = 0, 1, \ldots, n-1$. Such a sequence is said to be a *chain* of prime ideals in k of *length* n. Note that if $k \neq \{0\}$, then k has a proper maximal ideal, which is prime.

If $k \neq \{0\}$, then the *dimension* of k is defined to be the supremum of the lengths of chains of prime ideals in k, as on p90 of [1]. This is either a nonnegative integer, or $+\infty$ if the lengths of the chains of prime ideals in k are unbounded. The dimension of a field in this sense is equal to 0.

If $k \neq \{0\}$, then

$$k \text{ has dimension } 0 \tag{14.8.2}$$

if and only if

$$\text{every proper prime ideal in } k \text{ is maximal.} \tag{14.8.3}$$

This uses the facts that every proper ideal in k is contained in a proper maximal ideal, and that proper maximal ideals are prime.

Suppose for the moment that k is an integral domain, so that $\{0\}$ is a proper prime ideal in k. If the dimension of k is 0, then $\{0\}$ is the only proper prime ideal in k. Remember that every non-invertible element of k is contained in a proper maximal ideal in k, which is prime. It follows that k is a field in this case.

Suppose that $k \neq \{0\}$, and remember that $\mathcal{N}$ denotes the nilradical of k, as in Section 12.5. If

$$\mathcal{N} \text{ is a maximal proper ideal in } k, \tag{14.8.4}$$

then

$$\mathcal{N} \text{ is the only proper prime ideal in } k. \tag{14.8.5}$$

This follows from the earlier remarks about (14.5.10), with $\mathcal{I} = \{0\}$, because $\mathcal{N}$ is the same as the radical of $\{0\}$ in k. This also corresponds to part of Exercise 10 on p11 of [1]. Of course, this implies that the dimension of k is 0.

Remember that $\mathcal{N}$ is equal to the intersection of all of the prime ideals in k, as in Section 12.5. More precisely, $\mathcal{N}$ is the intersection of all of the proper prime ideals in k, because $k \neq \{0\}$. If

$$k \text{ has only one proper prime ideal,} \tag{14.8.6}$$

then that ideal is equal to $\mathcal{N}$. Of course, k has dimension 0 in this case, and (14.8.4) holds. This corresponds to another part of Exercise 10 on p11 of [1].

Suppose now that k is a local ring, and let $\mathcal{I}$ be the unique maximal proper ideal in k. If k has dimension 0, then $\mathcal{I}$ is the unique proper prime ideal in k. This implies that

$$\mathcal{I} = \mathcal{N}, \tag{14.8.7}$$

as in the preceding paragraph. This is related to some remarks on p90 of [1].

14.9 Noetherian local rings

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}$ be an ideal in k. Remember that the nth power $\mathcal{I}^n$ of $\mathcal{I}$ is defined for each positive integer n using multiplication of ideals, as in Section 12.10. Note that

$$\mathcal{I}^{n+1} \subseteq \mathcal{I}^n \tag{14.9.1}$$

for each $n \geq 1$. Of course, we either have that

$$\mathcal{I}^n \neq \mathcal{I}^{n+1} \text{ for each } n \geq 1, \tag{14.9.2}$$

or

$$\mathcal{I}^n = \mathcal{I}^{n+1} \text{ for some } n \geq 1, \tag{14.9.3}$$

as on p90 of [1].

If $\mathcal{I}$ is finitely generated as an ideal in k, then it is easy to see that $\mathcal{I}^m$ is finitely generated for every $m \geq 1$. Of course, this holds when k is Noetherian, as in Section 9.7.

Suppose that k is a local ring, with unique maximal proper ideal $\mathcal{I}$. Suppose also that (14.9.3) holds, which may be reexpressed as

$$\mathcal{I}\,\mathcal{I}^n = \mathcal{I}^n. \tag{14.9.4}$$

Note that $\mathcal{I}$ is the same as the Jacobson radical of k in this case. If $\mathcal{I}^n$ is finitely generated as an ideal in k, then we get that

$$\mathcal{I}^n = \{0\}, \tag{14.9.5}$$

by Nakayama's lemma, as in Section 13.3. This corresponds to part of Proposition 8.6 on p90 of [1].

In particular, (14.9.5) implies that every element of $\mathcal{I}$ is nilpotent, so that $\mathcal{I}$ is contained in the nilradical $\mathcal{N}$ of k. This implies that $\mathcal{I} = \mathcal{N}$, because the opposite inclusion follows from the fact that $\mathcal{I}$ is a prime ideal. It follows that $\mathcal{I}$ is the only prime ideal in k, so that k has dimension 0, as in the previous section. This corresponds to another part of Proposition 8.6 on p90 of [1]. Alternatively, one can use the fact that any prime ideal in k contains $\mathcal{I}^n$ to get that it contains $\mathcal{I}$, and thus is equal to $\mathcal{I}$, as in [1].

Suppose for the moment that $\mathcal{I}$ is generated, as an ideal in k, by a single element x of k. This means that $\mathcal{I}^n$ is generated by x^n, as an ideal in k, for every positive integer n. Thus (14.9.5) is the same as saying that

$$x^n = 0. \tag{14.9.6}$$

If (14.9.4) holds, then

$$x^n = a\, x^{n+1} \tag{14.9.7}$$

for some $a \in k$. This means that

$$(1 - a\,x)\, x^n = 0, \tag{14.9.8}$$

which implies (14.9.6), because $1 - a\,x$ is invertible in k under these conditions.

Suppose now for the moment that k is an integral domain. In this case, (14.9.5) implies that $\mathcal{I} = \{0\}$. This means that k is a field, which shows more directly that k has dimension 0.

If x is any element of k, and the ideal in k generated by x^n is equal to the ideal in k generated by x^{n+1} for some positive integer n, then (14.9.7) holds for some $a \in k$, as before. If k is an integral domain, then (14.9.8) implies that either $x = 0$, or x is invertible in k.

14.10 Local rings of dimension 0

Let k be a commutative ring with a multiplicative identity element, and suppose that k is a local ring, with unique maximal proper ideal $\mathcal{I}_0$. Thus

$$k_0 = k/\mathcal{I}_0 \tag{14.10.1}$$

is a field. Note that $\mathcal{I}_0/\mathcal{I}_0^2$ may be considered as a vector space over k_0, as in Section 13.6. Let us suppose throughout this section that

$$\mathcal{I}_0 \text{ is finitely generated as an ideal in } k, \tag{14.10.2}$$

which happens in particular when k is Noetherian. This implies that $\mathcal{I}_0/\mathcal{I}_0^2$ has finite dimension, as a vector space over k_0, as on p91 of [1].

Suppose for the moment that

$$\mathcal{I}_0/\mathcal{I}_0^2 = \{0\}, \tag{14.10.3}$$

which is to say that

$$\mathcal{I}_0^2 = \mathcal{I}_0. \tag{14.10.4}$$

Nakayama's lemma implies that

$$\mathcal{I}_0 = \{0\}, \tag{14.10.5}$$

as in Section 13.3, because $\mathcal{I}$ is the same as the Jacobson radical of k. This means that k is a field in this case. This corresponds to part of Proposition 8.8 on p91 of [1], and its proof.

Suppose now that

$$\dim_{k_0}(\mathcal{I}_0/\mathcal{I}_0^2) = 1, \tag{14.10.6}$$

which is to say that the dimension of $\mathcal{I}_0/\mathcal{I}_0^2$, as a vector space over k_0, is equal to one. Let q_0 be the natural quotient mapping from $\mathcal{I}_0$ onto $\mathcal{I}_0/\mathcal{I}_0^2$, and let x be an element of $\mathcal{I}_0 \setminus \mathcal{I}_0^2$, so that $q_0(x) \neq 0$. This means that $\mathcal{I}_0/\mathcal{I}_0^2$ is spanned by $q_0(x)$, as a vector space over k_0, by (14.10.6). It follows that

$$\mathcal{I}_0 \text{ is generated by } x, \text{ as an ideal in } k, \tag{14.10.7}$$

as in Section 13.6. This corresponds to another part of Proposition 8.8 on p91 of [1].

Let us suppose in addition that

$$k \text{ has dimension } 0, \tag{14.10.8}$$

in the sense of Section 14.8. This implies that $\mathcal{I}_0$ is the unique proper prime ideal in k, and that $\mathcal{I}_0$ is the same as the nilradical $\mathcal{N}$ in k, as before. It follows that

$$\mathcal{I}_0^m = \{0\} \tag{14.10.9}$$

for some positive integer m, as in Section 14.7. Note that for each positive integer n,

$$\mathcal{I}_0^n \text{ is generated by } x^n, \text{ as an ideal in } k, \tag{14.10.10}$$

by (14.10.7).

Let $\mathcal{I}$ be a proper ideal in k, and observe that

$$\mathcal{I} \subseteq \mathcal{I}_0, \tag{14.10.11}$$

because $\mathcal{I}$ is contained in a maximal proper ideal in k. If $\mathcal{I} \neq \{0\}$, then there is a positive integer r such that

$$\mathcal{I} \subseteq \mathcal{I}_0^r \text{ and } \mathcal{I} \not\subseteq \mathcal{I}_0^{r+1}. \tag{14.10.12}$$

Let y be an element of $\mathcal{I} \setminus \mathcal{I}_0^{r+1}$. Thus y can be expressed as

$$y = a\, x^r \tag{14.10.13}$$

for some $a \in k$, by (14.10.10). We also have that $a \notin \mathcal{I}_0$, because $y \notin \mathcal{I}_0^{r+1}$.

This means that a is invertible in k, because $\mathcal{I}_0$ is the unique maximal proper ideal in k. It follows that $x^r \in \mathcal{I}$, which implies that $\mathcal{I}_0^r \subseteq \mathcal{I}$, by (14.10.10). This shows that

$$\mathcal{I} = \mathcal{I}_0^r, \tag{14.10.14}$$

as in Proposition 8.8 on p91 of [1], and its proof.

14.11 One-dimensional integral domains

Let k be a commutative ring with a multiplicative identity element that is an integral domain, so that $1 \neq 0$ in k, and k has no nonzero zero-divisors. Equivalently, this means that $\{0\}$ is a proper prime ideal in k. Remember that k has dimension 0, in the sense of Section 14.8, if and only if k is a field. In this section, we ask that

$$k \text{ have dimension } 1, \tag{14.11.1}$$

in the sense of Section 14.8. This is the same as saying that k is not a field, and that

$$\text{every proper prime ideal } \neq \{0\} \text{ in } k \text{ is maximal,} \tag{14.11.2}$$

as on p93 of [1].

Let $\mathcal{I}_1$ be a proper ideal in k with $\mathcal{I}_1 \neq \{0\}$. It is easy to see that

$$k/\mathcal{I}_1 \text{ has dimension } 0, \tag{14.11.3}$$

in the sense of Section 14.8. More precisely, proper prime ideals in $k/\mathcal{I}_1$ correspond to proper prime ideals in k that contain $\mathcal{I}_1$, and are not equal to $\{0\}$ in particular. Thus proper prime ideals in $k/\mathcal{I}_1$ are maximal, because of (14.11.2).

Suppose now that k is also a local ring, with unique maximal proper ideal $\mathcal{I}_0$. The condition that k not be a field means that

$$\mathcal{I}_0 \neq \{0\}. \tag{14.11.4}$$

Note that

$$\mathcal{I}_0 \text{ is the only proper prime ideal } \neq \{0\} \text{ in } k, \tag{14.11.5}$$

because of (14.11.2). We ask that $\mathcal{I}_0$ be finitely generated as an ideal in k, which happens in particular when k is Noetherian.

Put $k_0 = k/\mathcal{I}_0$ again, which is a field. Remember that $\mathcal{I}_0/\mathcal{I}_0^2$ may be considered as a vector space over k_0, as in Section 13.6. This vector space has finite dimension, because $\mathcal{I}_0$ is finitely generated.

Let $\mathcal{I} \neq \{0\}$ be a proper ideal in k. Of course, $\mathcal{I} \subseteq \mathcal{I}_0$, because $\mathcal{I}$ is contained in a maximal proper ideal in k. Remember that the radical of $\mathcal{I}$ in k is equal to the intersection of all of the prime ideals in k that contain $\mathcal{I}$, as in Section 12.10. It follows that

$$r(\mathcal{I}) = \mathcal{I}_0 \tag{14.11.6}$$

in this case, because of (14.11.5). This implies that $\mathcal{I}$ is a primary ideal in k, as in Section 14.5. We also get that

$$\mathcal{I}_0^m \subseteq \mathcal{I} \subseteq \mathcal{I}_0 \tag{14.11.7}$$

for some positive integer m, as in Section 14.7. This corresponds to Remark (A) on p95 of [1].

Remark (B) on p95 of [1] states that

$$\mathcal{I}_0^n \neq \mathcal{I}_0^{n+1} \tag{14.11.8}$$

for each positive integer n. Otherwise, k would have dimension 0, as in Section 14.9. More precisely, because k is an integral domain, one could simply say that k would have to be a field otherwise.

Suppose for the moment that

$$\mathcal{I}_0 \text{ is generated by a single element of } k, \text{ as an ideal in } k. \tag{14.11.9}$$

This implies that the dimension of $\mathcal{I}_0/\mathcal{I}_0^2$, as a vector space over k_0, is less than or equal to one. In fact, we get that

$$\dim_{k_0}(\mathcal{I}_0/\mathcal{I}_0^2) = 1, \tag{14.11.10}$$

because of (14.11.8). This is part of Proposition 9.2 on p94 of [1].

14.12 One-dimensional local domains

Let us continue with the same notation and hypotheses as in the previous section. We would like to show that if (14.11.10) holds, then

$$\text{every ideal } \mathcal{I} \neq \{0\} \text{ in } k \text{ is of the form } \mathcal{I}_0^r \tag{14.12.1}$$

for some nonnegative integer r. This is another part of Proposition 9.2 on p94 of [1]. Remember that $\mathcal{I}_0^r$ is interpreted as being equal to k when $r = 0$.

Let $\mathcal{I} \neq \{0\}$ be a proper ideal in k, so that (14.11.7) holds for some positive integer m. Observe that $k/\mathcal{I}_0^m$ is a commutative ring with a multiplicative identity element that is a local ring, with unique maximal proper ideal $\mathcal{I}_0/\mathcal{I}_6^m$. Of course,

$$\mathcal{I}_0/\mathcal{I}_0^m \text{ is finitely generated as an ideal in } k/\mathcal{I}_0^m, \tag{14.12.2}$$

because $\mathcal{I}_0$ is finitely generated as an ideal in k, by hypothesis. Clearly

$$(k/\mathcal{I}_0^m)/(\mathcal{I}_0/\mathcal{I}_0^m) \tag{14.12.3}$$

is isomorphic to k_0 as a field.

We may as well suppose that $m \geq 2$, because otherwise $\mathcal{I} = \mathcal{I}_0$. It is easy to see that

$$(\mathcal{I}_0/\mathcal{I}_0^m)^2 = \mathcal{I}_0^2/\mathcal{I}_0^m, \tag{14.12.4}$$

where the left side is defined as an ideal in $k/\mathcal{I}_0^m$ in the usual way. Note that

$$(\mathcal{I}_0/\mathcal{I}_0^m)/(\mathcal{I}_0/\mathcal{I}_0^m)^2 \tag{14.12.5}$$

may be considered as a vector space over (14.12.3), as before. This is the same as

$$(\mathcal{I}_0/\mathcal{I}_0^m)/(\mathcal{I}_0^2/\mathcal{I}_0^m), \tag{14.12.6}$$

because of (14.12.4). This may be identified with $\mathcal{I}/\mathcal{I}_0^2$, as a vector space over k_0.

Thus (14.11.10) implies the analogous condition for (14.12.5). Note that $\mathcal{I}_0^m \neq \{0\}$, because $\mathcal{I}_0 \neq \{0\}$, and k is an integral domain. This implies that $k/\mathcal{I}_0^m$ has dimension 0, as in (14.11.3). We may as well suppose that $\mathcal{I} \neq \mathcal{I}_0^m$, which means that $\mathcal{I}/\mathcal{I}_0^m \neq \{0\}$. It follows that

$$\mathcal{I}/\mathcal{I}_0^m = (\mathcal{I}_0/\mathcal{I}_0^m)^r \tag{14.12.7}$$

in $k/\mathcal{I}_0^m$ for some positive integer r, as in Section 14.10.

We also have $r < m$ in this case, because otherwise the right side of (14.12.7) would be $\{0\}$. Thus the right side of (14.12.7) is equal to $\mathcal{I}_0^r/\mathcal{I}_0^m$. This implies that $\mathcal{I} = \mathcal{I}_0^r$, as desired.

Suppose now that (14.12.1) holds, and let us show that

(14.12.8) there is an $x \in k$ such that every nonzero ideal in k is generated by x^l for some nonnegative integer l.

This is another part of Proposition 9.2 on p94 of [1]. Note that (14.12.8) implies (14.11.9).

If we take $n = 1$ in (14.11.8), then we get that $\mathcal{I}_0 \neq \mathcal{I}_0^2$. Thus there is an $x \in \mathcal{I}_0$ such that $x \notin \mathcal{I}_0^2$. In particular, $x \neq 0$, and (14.12.1) implies that there is a nonnegative integer r such that $\mathcal{I}_0^r$ is generated by x. It is clear in this case that $r = 1$. This shows directly that (14.12.1) implies (14.11.9).

If l is a nonnegative integer, then it follows that $\mathcal{I}_0^l$ is generated by x^l. This and (14.12.1) imply (14.12.8), as desired.

14.13 Valuations

Let k_1 be a field. A real-valued function v on $k_1 \setminus \{0\}$ is said to be a *valuation* if it satisfies the following two conditions. First,

$$v(x\,y) = v(x) + v(y) \tag{14.13.1}$$

for every $x, y \in k_1 \setminus \{0\}$. Of course, this is the same as saying that v is a homomorphism from $k_1 \setminus \{0\}$, as a commutative group with respect to multiplication, into $\mathbf{R}$, as a commutative group with respect to addition. In particular, this implies that $v(1) = 0$. The second condition is that

$$v(x + y) \geq \min(v(x), v(y)) \tag{14.13.2}$$

for every $x, y \in k_1 \setminus \{0\}$ such that $x + y \neq 0$. It is customary to put

$$v(0) = +\infty, \tag{14.13.3}$$

so that v is a function on k_1 with values in $\mathbf{R} \cup \{+\infty\}$. Note that (14.13.1) and (14.13.2) hold, with suitable interpretations, when x, y, or $x + y$ is 0.

This definition is mentioned on p24 of [9]. This corresponds to a valuation of k_1 with values in $\mathbf{R}$, as in Exercise 31 on p72 of [1]. This may also be called a real valuation of k_1, as on p26 of [15].

This is closely related to ultrametric or non-archimedean absolute value functions on fields, as in Definition 2.1.1 on p21f of [9], and Problems 28 and 63 on p24, 39 of [9], respectively. This is mentioned on p26 of [15] as well. The term "valuation" is used a bit differently in Definition 1.1 on p12 of [4], which basically corresponds to a quasimetric version of an absolute value function. A non-archimedean valuation in the sense of Definition 1.3 on p15 of [4] is the same as an ultrametric or non-archimedean absolute value function, which corresponds to a valuation in the sense used here as before.

Note that we can get a valuation on k_1 by putting

$$v(x) = 0 \tag{14.13.4}$$

for every $x \in k_1 \setminus \{0\}$, and using (14.13.3), as before. This may be described as the *trivial valuation* on k_1. This corresponds to the trivial absolute value function in Example 2 on p22 of [9], or the trivial valuation on p12 of [4].

If v is a valuation on k_1 and a is a positive real number, then it is easy to see that

$$a\, v(x) \tag{14.13.5}$$

defines a valuation on k_1 as well. This valuation is said to be *equivalent* to v on k_1. This corresponds to equivalence of absolute value functions, as in Lemma 3.1.2 on p42 of [9]. This also corresponds to equivalence of valuations in the sense of [4], as in Definition 1.2 on p13 of [4].

Let v be a valuation on k_1, and note that

$$v(k_1 \setminus \{0\}) \text{ is a subgroup of } \mathbf{R}, \tag{14.13.6}$$

as a commutative group with respect to addition. This is called the *value group* of v, as in Exercise 31 on p72 of [1], and Problem 29 on p24 of [9]. This corresponds to the valuation group of a valuation in the sense of [4], as on p42 of [4]. More precisely, the valuation group in this sense is a subgroup of the multiplicative group of positive real numbers.

If A is any subgroup of $\mathbf{R}$, as a commutative group with respect to addition, then it is well known and not difficult to show that

$$0 \text{ is not a limit point of } A, \tag{14.13.7}$$

with respect to the standard topology on $\mathbf{R}$, if and only if

$$A = \{0\} \tag{14.13.8}$$

or

$$A = a\,\mathbf{Z} \quad \text{for some positive real number } a. \tag{14.13.9}$$

Otherwise, if 0 is a limit point of A, then one can check that

$$A \text{ is dense in } \mathbf{R}, \tag{14.13.10}$$

with respect to the standard topology.

If $A = v(k_1 \setminus \{0\})$ satisfies (14.13.7), then v corresponds to a "discrete valuation" in the sense defined on p42 of [4]. However, we shall use this term a bit differently here, as on p94 of [1], and p5, 26 of [15].

14.14 Some related subgroups and ideals

Let us continue with the same notation and hypotheses as in the previous section. Note that

$$v(1) = 0. \tag{14.14.1}$$

If $x \in k_1$ and $x^n = 1$ for some positive integer n, then

$$v(x) = 0, \tag{14.14.2}$$

because $n\, v(x) = v(x^n) = 0$. In particular, $v(-1) = 0$. This implies that

$$v(-y) = v(y) \tag{14.14.3}$$

for every $y \in k_1$.

If $t \in \mathbf{R}$, then it is easy to see that

$$\{x \in k_1 : v(x) > t\} \tag{14.14.4}$$

and

$$\{x \in k_1 : v(x) \ge t\} \tag{14.14.5}$$

are subgroups of k_1, as a commutative group with respect to addition. In fact,

$$\{x \in k_1 : v(x) \ge 0\} \tag{14.14.6}$$

is a subring of k_1. This is called the *valuation ring* of v, as in Exercise 31 on p72 of [1], and on p94 of [1]. This corresponds to the valuation ring of an ultrametric absolute value function, as in Definition 2.4.2 on p38 of [9]. This is the same as the ring of (valuation-)integers with respect to a non-archimedean valuation on p41 of [4].

If $x \in k_1 \setminus \{0\}$, then

$$v(x^{-1}) = -v(x). \tag{14.14.7}$$

In particular, if $v(x) = 0$, then

$$v(x^{-1}) = 0. \tag{14.14.8}$$

This implies that x is invertible as an element of the valuation ring of v. Conversely, if x is an invertible element of the valuation ring of v, then it is easy to see that $v(x) = 0$, using (14.14.7).

If $t \ge 0$, then (14.14.4) and (14.14.5) are ideals in the valuation ring of v. One can check that

$$\{x \in k_1 : v(x) > 0\} \tag{14.14.9}$$

is a maximal proper ideal in the valuation ring (14.14.6). More precisely, if x is an element of the valuation ring of v that is not in (14.14.9), then $v(x) = 0$, so that x is invertible in the valuation ring. This means that the valuation ring of v is a local ring, with (14.14.9) as its unique maximal proper ideal. This corresponds to the valuation ideal of an ultrametric absolute value function, as in Definition 2.4.2 on p38 of [9].

If v is not the trivial valuation on k_1, then there is an $x \in k_1$ such that $x \neq 0$ and $v(x) \neq 0$. This implies that there are $y, z \in k_1 \setminus \{0\}$ such that

$$v(y) > 0 \text{ and } v(z) < 0, \tag{14.14.10}$$

using x and $1/x$. This means that (14.14.9) is not equal to $\{0\}$, and in particular that the valuation ring (14.14.6) is not a field. Of course, if v is the trivial valuation on k_1, then the valuation ring (14.14.6) is equal to k_1, and (14.14.9) is equal to $\{0\}$.

Suppose for the moment that $A = v(k_1 \setminus \{0\})$ satisfies (14.13.7), and thus either (14.13.8) or (14.13.9). Of course, (14.13.8) says exactly that v is the trivial valuation on k_1. If (14.13.9) holds, then it is easy to see that every nonzero ideal in the valuation ring (14.14.6) is of the form (14.14.5), where t is a nonnegative integer multiple of a. In particular, the valuation ring (14.14.6) is Noetherian in these two cases. Otherwise, if (14.13.10) holds, then one can check that the valuation ring (14.14.6) is not Noetherian.

If

$$v(k_1 \setminus \{0\}) = \mathbf{Z}, \tag{14.14.11}$$

then v is called a *discrete valuation*, as on p94 of [1]. This implies that v is not the trivial valuation on k_1, and that $A = v(k_1 \setminus \{0\})$ satisfies (14.13.7). If v is a nontrivial valuation on k_1 such that $A = v(k_1 \setminus \{0\})$ satisfies (14.13.7), then v is equivalent to a discrete valuation in this sense.

14.15 Valuations on integral domains

Let k be an integral domain. A real-valued function v on $k \setminus \{0\}$ is said to be a *valuation* if it satisfies the same two conditions as before, as in Problem 28 on p24 of [9]. Namely, if $x, y \in k \setminus \{0\}$, then we ask that

$$v(x\,y) = v(x) + v(y), \tag{14.15.1}$$

and that

$$v(x + y) \geq \min(v(x), v(y)) \tag{14.15.2}$$

when $x + y \neq 0$. Observe that $v(1) = 0$, by taking $x = y = 1$ in (14.15.1). It is customary to put $v(0) = +\infty$, as before.

If $x \in k$ and $x^n = 1$ for some positive integer n, then $v(x) = 0$, as in the previous section. It follows that $v(-1) = 0$, so that $v(-y) = v(y)$ for every $y \in k$, as before. We also have that

$$\{x \in k : v(x) > t\} \tag{14.15.3}$$

and

$$\{x \in k : v(x) \geq t\} \tag{14.15.4}$$

are subgroups of k, as a commutative group with respect to addition, for every $t \in \mathbf{R}$. Similarly,

$$\{x \in k : v(x) \geq 0\} \tag{14.15.5}$$

is a subring of k. If $t \geq 0$, then (14.15.3) and (14.15.4) are ideals in (14.15.5), as before.

If x is an invertible element of k, then

$$v(x^{-1}) = -v(x). \tag{14.15.6}$$

If we also have that $v(x) = 0$, then $v(x^{-1}) = 0$, so that x is an invertible element of (14.15.5). Conversely, if x is an invertible element of (14.15.5), then x is an invertible element of k, and $v(x) = 0$. Suppose for the moment that

$$v(y) \geq 0 \text{ for every } y \in k, \tag{14.15.7}$$

so that (14.15.5) is the same as k. If x is an invertible element of k, then we get that $v(x) = 0$.

Let Q_k be the field of fractions corresponding to k. One can extend v to $Q_k \setminus \{0\}$ in such a way that

$$v(y/z) = v(y) - v(z) \tag{14.15.8}$$

for every $y, z \in k$ with $y, z \neq 0$. One can check that this defines a valuation on Q_k, as in Problem 28 on p24 of [9]. More precisely, any pair of nonzero elements of Q_k may be expressed as x/z, y/z for some $x, y, z \in k \setminus \{0\}$. If $x + y \neq 0$, then it is easy to see that

$$v((x/z) + (y/z)) \geq \min(v(x/z), v(y/z)), \tag{14.15.9}$$

using (14.15.2) and (14.15.8).

Suppose for the moment that if $y, z \in k \setminus \{0\}$ satisfy

$$v(y) \geq v(z), \tag{14.15.10}$$

then there is a $w \in k$ such that

$$y = w\, z. \tag{14.15.11}$$

This is the same as saying that if

$$v(y/z) \geq 0, \tag{14.15.12}$$

then

$$y/z \in k. \tag{14.15.13}$$

This means that the valuation ring of v in Q_k is contained in k.

Chapter 15

Discrete valuations and Dedekind domains

15.1 Discrete valuation rings

Let k be an integral domain, and let Q_k be the corresponding field of fractions. Suppose that v is a discrete valuation on Q_k, as in Section 14.14. Suppose also that k is the valuation ring of v, so that

(15.1.1) $$k = \{x \in Q_k : v(x) \geq 0\}.$$

Under these conditions, k is said to be a *discrete valuation ring*, as on p94 of [1]. An equivalent formulation is given on p5 of [15], and we shall say more about that in Section 15.4.

Of course,

(15.1.2) $$v(Q_k \setminus \{0\}) = \mathbf{Z},$$

by the definition of a discrete valuation. This implies that

(15.1.3) $$v(k \setminus \{0\}) = \mathbf{Z}_+ \cup \{0\}.$$

As in Section 14.14, k is a local ring, with

(15.1.4) $$\{x \in k : v(x) > 0\}$$

as its unique maximal proper ideal. If l is a nonnegative integer, then

(15.1.5) $$\{x \in k : v(x) \geq l\}$$

is an ideal in k, as before. Note that these ideals are all nonzero in this case, and that (15.1.5) is equal to (15.1.4) when $l = 1$. We also have that l is uniquely determined by (15.1.5), because of (15.1.3).

If $x, y \in k$, $x, y \neq 0$, and

(15.1.6) $$v(x) = v(y),$$

then $v(x\,y^{-1}) = 0$, so that $x\,y^{-1}$ is an invertible element of k. This implies that x and y generate the same ideal in k. The ideal that they generate is the same as (15.1.5), with $l = v(x) = v(y)$.

Let $\mathcal{I}$ be a nonzero ideal in k, and let l be the smallest nonnegative integer for which there is a $y \in \mathcal{I}$ such that $v(y) = l$. One can check that $\mathcal{I}$ is equal to (15.1.5), as on p94 of [1]. It follows that k is Noetherian, as in [1]. More precisely, $\mathcal{I}$ is equal to (15.1.5) for only one nonnegative integer l.

Suppose that $x \in k$ satisfies

$$v(x) = 1. \tag{15.1.7}$$

This implies that

$$v(x^l) = l \tag{15.1.8}$$

for every integer l. It follows that (15.1.5) is the same as the ideal in k generated by x^l when $l \geq 0$, as before. This corresponds to a remark on p94 of [1].

It follows from the remarks in the previous paragraphs that (15.1.4) is the only nonzero proper prime ideal in k. In particular, k has dimension one in the sense of Section 14.8, as on p94 of [1].

If k is a discrete valuation ring, then it is easy to see that the corresponding valuation v is unique. More precisely, one can check that v is uniquely determined on k, and thus on Q_k.

15.2 Getting discrete valuation rings

Let k_1 be a field, and let v_1 be a valuation on k_1. Remember that

$$\{w \in k_1 : v_1(w) \geq 0\} \tag{15.2.1}$$

is a subring of k_1, which is the valuation ring of v_1. It is easy to see that k_1 may be identified with the field of fractions of this ring. If v_1 is a discrete valuation on k_1, then it follows that

(15.2.2) (15.2.1) is a discrete valuation ring.

This is related to Proposition 1 on p6 of [15], although simpler, with the formulation of the definition of a discrete valuation ring in the previous section.

Let k be an integral domain with Q_k as its field of fractions again. Suppose that

(15.2.3) there is an $x \in k$ such that every nonzero ideal in k is generated by x^l for some nonnegative integer l,

as in (14.12.8). Suppose also that k is not a field, so that $x \neq 0$, and x is not invertible in k. This implies that the ideal in k generated by x^{l+1} is a proper subset of the ideal generated by x^l for each $l \geq 0$, because k is an integral domain, as mentioned at the end of Section 14.9.

If $y \in k$ and $y \neq 0$, then there is a unique nonnegative integer l such that the ideal in k generated by y is the same as the ideal generated by x^l, and we put

$$v(y) = l. \tag{15.2.4}$$

This happens exactly when y is equal to the product of x^l and an invertible element of k, as in the next section. If z is another nonzero element of k, then one can check that

$$v(y\,z) = v(y) + v(z). \tag{15.2.5}$$

One can also verify that

$$v(y + z) \geq \min(v(y), v(z)) \tag{15.2.6}$$

when $y + z \neq 0$. Thus v defines a valuation on k, as in Section 14.15.

We can extend v to a valuation on Q_k, as before. Note that every nonzero element of Q_k can be expressed as $u\,x^l$ for some invertible element u of k and integer l, in which case we have that

$$v(u\,x^l) = l. \tag{15.2.7}$$

It follows that v is a discrete valuation on Q_k, and that (15.1.1) holds. This means that k is a discrete valuation ring, as in Proposition 9.2 on p94 of [1].

Remember that some conditions under which (15.2.3) holds were discussed in Sections 14.11 and 14.12. Thus these are conditions under which k is a discrete valuation ring, as in [1].

15.3 Some remarks about integral domains

Let k be a commutative ring with a multiplicative identity element. One may say that $x, y \in k$ are *associates* in k if

$$x = a\,y, \;\; y = b\,x \tag{15.3.1}$$

for some $a, b \in k$, as on p111 of [12]. Of course, if $y = x\,u$ for some invertible element u of k, then $x = y\,u^{-1}$, and x, y are associates in k.

Equivalently, $x, y \in k$ are associates when x is an element of the ideal in k generated by y, and y is an element of the ideal generated by x. This is the same as saying that the ideals in k generated by x and y are the same.

Suppose that k is an integral domain. If $x, y \in k$ are associates in k, then it is well known that $y = x\,u$ for some invertible element u of k, as on p111 of [12]. More precisely, if $a, b \in k$ are as in (15.3.1), then $x = a\,b\,x$, so that

$$(a\,b - 1)\,x = 0. \tag{15.3.2}$$

If $x \neq 0$, then it follows that $a\,b = 1$, so that a and b are inverses of each other. Otherwise, if $x = 0$, then $y = 0$, and the statement is trivial.

An element $x \neq 0$ of k is said to be *reducible* if it can be expressed as the product of two non-invertible elements of k. Otherwise, x is said to be *irreducible*. Note that irreducible elements of k are sometimes said to be prime, as on p111 of [12].

If the ideal in k generated by x is prime, then x is irreducible. Indeed, if

$$x = y\,z \tag{15.3.3}$$

for some $y, z \in k$, and the ideal generated by x is prime, then at least one of y and z is in that ideal. This implies that

$$y = b\,x \text{ or } z = c\,x \tag{15.3.4}$$

for some b or c in k. This means that at least one of y and z is an associate of x in k, because of (15.3.3). It follows that the other one is invertible in k, as before.

Suppose now that k is a *principal ideal domain*, so that every ideal in k is generated by a single element. If x is an irreducible element of k, then it is well known that the ideal in k generated by x is prime, as on p125 of [12].

It is well known that every nonzero element of k is either invertible or a product of irreducible elements of k, as in Theorem 24 on p118 of [12]. In the second case, the factorization is unique up to permutations and using other associates of the irreducible factors, as in [12].

15.4 Principal ideal domains

Let k be a principal ideal domain. It is well known that

$$k \text{ has dimension less than or equal to one,} \tag{15.4.1}$$

in the sense of Section 14.8. This is the same as saying that every nonzero proper prime ideal in k is maximal. This corresponds to Example (3) on p5 of [1], as mentioned on p96 of [1]. This also follows from a remark on p125 of [12].

To see this, let $\mathcal{I}_1$ be a nonzero proper prime ideal in k, and let $\mathcal{I}_2$ be an ideal in k that contains $\mathcal{I}_1$. By hypothesis, $\mathcal{I}_1$ and $\mathcal{I}_2$ are generated by nonzero elements x and y of k, respectively. Because $\mathcal{I}_1 \subseteq \mathcal{I}_2$, we have that $x \in \mathcal{I}_2$, so that $x = y\,z$ for some $z \in k$. This implies that y or z is invertible in k, because x is irreducible, as in the previous section. This means that $\mathcal{I}_2$ is equal to $\mathcal{I}_1$ or k, as desired.

Of course, k is a field if and only if $\{0\}$ is the only proper ideal in k. Suppose that k is a local ring that is not a field, so that k has a unique nonzero maximal proper ideal $\mathcal{I}_0$. Of course, $\mathcal{I}_0$ is a prime ideal in k, and in fact

$$\mathcal{I}_0 \text{ is the unique nonzero proper prime ideal in } k, \tag{15.4.2}$$

by the previous remarks. Conversely, if k has a unique proper prime ideal, then that is the unique maximal proper ideal in k.

This can be used as another way to define discrete valuation rings, as on p5 of [15]. Note that discrete valuation rings have these properties, as in Section 15.1. This corresponds to Proposition 1 on p6 of [15].

Conversely, suppose that k satisfies the conditions mentioned before, and that

$$\mathcal{I}_0 \text{ is generated by } x \in k, \text{ as an ideal in } k. \tag{15.4.3}$$

Note that $x \neq 0$, because $\mathcal{I}_0 \neq \{0\}$, and that x is irreducible, because $\mathcal{I}_0$ is prime. The condition that $\mathcal{I}_0$ be the unique nonzero prime ideal in k implies that every irreducible element of k is an associate of x.

If $y \in k$ and $y \neq 0$, then y is invertible, or y can be expressed as a product of irreducible elements of k, as in the previous section. This implies that y is an associate of x^l for some nonnegative integer l in this case. It follows that every nonzero ideal in k is generated by x^l for some nonnegative integer l, because k is a principal ideal domain. This implies that k is a discrete valuation ring, as in Section 15.2.

Alternatively, if k satisfies the conditions mentioned before, then there is an $x \in k$ such that every nonzero ideal in k is generated by x^l for some nonnegative integer l, as in Sections 14.11 and 14.12. More precisely, this uses the fact that $\mathcal{I}_0$ is generated by a single element of k, as an ideal in k, because k is a principal ideal domain. This means that k is a discrete valuation ring, as in Section 15.2 again.

15.5 Local rings and valuations

Let k be a commutative ring with a nonzero multiplicative identity element, and suppose that k is a local ring that is not a field. Thus k has a unique maximal proper ideal $\mathcal{I}_0$, with $\mathcal{I}_0 \neq \{0\}$. Let us suppose for the rest of the section that

$$\mathcal{I}_0 \text{ is generated by } w_0 \in k, \text{ as an ideal in } k. \tag{15.5.1}$$

Note that $w_0 \neq 0$, because $\mathcal{I}_0 \neq \{0\}$.

Suppose in addition that

$$\bigcap_{j=1}^{\infty} \mathcal{I}_0^j = \{0\}. \tag{15.5.2}$$

This holds automatically when k is Noetherian, as in the proof of Proposition 2 on p7 of [15]. This will be discussed further in the next section. More precisely, (15.5.2) holds when k is Noetherian, without asking that $\mathcal{I}_0$ be generated by a single element, as mentioned in [15].

If k is an integral domain, then

$$k \text{ is a discrete valuation ring,} \tag{15.5.3}$$

as in Exercise 4 on p99 of [1]. This is related to part of the proof of Proposition 2 on p7 of [15]. In [15], one asks that w_0 not be nilpotent, instead of asking that k be an integral domain. Of course, if k is an integral domain, then w_0 is not nilpotent, because $w_0 \neq 0$. Note that discrete valuation rings satisfy all of these conditions.

Let $y \in k \setminus \{0\}$ be given. Observe that there is a largest nonnegative integer l such that

$$y \in \mathcal{I}_0^l, \tag{15.5.4}$$

because of (15.5.2). Thus

$$y = a\, w_0^l \tag{15.5.5}$$

for some $a \in k$ with $a \notin \mathcal{I}_0$. This means that a is invertible in k.

If k is an integral domain, then we get (15.5.3), as in Section 15.2. If we only ask that w_0 not be nilpotent, then one can use (15.5.5) to get that k is an integral domain, as in [15]. Note that there is no condition here on the dimension of k, in the sense of Section 14.8, which is a bit different from the discussion in Sections 14.11 and 14.12.

15.6 More on the additional condition

Let k be as at the beginning of the prevous section again, and suppose now that k is Noetherian as well. Consider the set $\mathcal{I}$ of $x \in k$ such that

$$x\, w_0^m = 0 \tag{15.6.1}$$

for some nonnegative integer m, and thus all sufficiently large m. It is easy to see that this is an ideal in k. It follows that $\mathcal{I}$ is finitely generated as an ideal in k, because k is Noetherian. This implies that there is a nonnegative integer m_0 such that

$$x\, w_0^{m_0} = 0 \tag{15.6.2}$$

for every $x \in \mathcal{I}$, as in the proof of Proposition 2 on p7 of [15].

To show (15.5.2), let $y \in \bigcap_{j=1}^{\infty} \mathcal{I}_0^j$ be given. Thus, for each positive integer j, there is an $x_j \in k$ such that

$$y = x_j\, w_0^j. \tag{15.6.3}$$

Observe that

$$(x_j - x_{j+1}\, w_0)\, w_0^j = 0 \tag{15.6.4}$$

for each j, which implies that

$$x_j - x_{j+1}\, w_0 \in \mathcal{I}. \tag{15.6.5}$$

Let $\mathcal{I}(j)$ be the ideal in k generated by $\mathcal{I}$ and x_j for each j. It follows that

$$\mathcal{I}(j) \subseteq \mathcal{I}(j+1) \tag{15.6.6}$$

for each j, because of (15.6.5). The sequence of $\mathcal{I}(j)$'s is eventually constant, because k is Noetherian. This means that

$$x_{j+1} \in \mathcal{I}(j) \tag{15.6.7}$$

when j is sufficiently large.

If (15.6.7) holds for some j, then

(15.6.8) $$x_{j+1} - t_j\, x_j \in \mathcal{I}$$

for some $t_j \in k$, by the definition of $\mathcal{I}(j)$. Note that $t_j\, x_j - t_j\, x_{j+1}\, w_0 \in \mathcal{I}$, by (15.6.5). It follows that

(15.6.9) $$x_{j+1}\,(1 - t_j\, w_0) \in \mathcal{I}.$$

We also have that $1 - t_j\, w_0$ is invertible in k, because $t_j\, w_0 \in \mathcal{I}_0$. This implies that

(15.6.10) $$x_{j+1} \in \mathcal{I}$$

for all sufficiently large j. Thus

(15.6.11) $$x_{j+1}\, w_0^{m_0} = 0$$

for all sufficiently large j, because of (15.6.2). This means that $y = 0$, as desired, because of (15.6.3).

If k is an integral domain, then the previous argument can be simplified, as mentioned on p7 of [15]. In this case, $\mathcal{I} = \{0\}$, and $x_j = x_{j+1}\, w_0$ for each j. Similarly, $\mathcal{I}(j)$ reduces to the ideal in k generated by x_j for each j.

15.7 Discrete valuation rings and invertibility

Let k be an integral domain, and let Q_k be the corresponding field of fractions. Remember that a submodule M of Q_k, as a module over k, is called a fractional ideal of k if there is a $y \in k$ such that $y \neq 0$ and $y\, M \subseteq k$, as in Section 11.7. On p8 of [15], a submodule M of Q_k, as a module over k, is called a fractional ideal of Q_k with respect to k if M is finitely generated as a module over k. This implies that M is a fractional ideal of k in the sense used here, and the converse holds when k is Noetherian, as in Section 11.7.

If there is a submodule N of Q_k, as a module over k, such that $M\, N = k$, then M is said to be an invertible ideal of k, as before. In this case, N is equal to the set $(k : M)_{Q_k}$ of $x \in Q_k$ such that $x\, M \subseteq k$, as in Section 11.7. Remember that this set was initially denoted $(k : M)$, and that we used the notation $(k : M)_{Q_k}$ in Section 14.3, because similar notation was used in Section 12.8 for something else.

If $M \neq \{0\}$, then $(k : M)_{Q_k}$ is a fractional ideal of k, as in Section 11.7. If k is Noetherian, then $(k : M)_{Q_k}$ is a fractional ideal of Q_k with respect to k, as before. A nonzero ideal $\mathcal{I}$ of k is said to be invertible in the remark on p9 of [15] when

(15.7.1) $$\mathcal{I}\,(k : \mathcal{I})_{Q_k} = k.$$

This is equivalent to the invertibility of $\mathcal{I}$ as a fractional ideal of k.

Suppose for the moment that k is a discrete valuation ring, and let us show that

(15.7.2) every nonzero fractional ideal of k is invertible.

This corresponds to part of Proposition 9.7 on p97 of [1], and it is also mentioned in the proof of Proposition 5 on p11 of [15]. Remember that there is an $x \in k$ such that every nonzero ideal in k is generated by x^l for some nonnegative integer l, as in Section 15.1. Let M be a nonzero fractional ideal in k, so that $y\,M \subseteq k$ for some $y \in k$ with $y \neq 0$. This means that $y\,M$ is a nonzero ordinary or integral ideal in k, which is generated by x^l for some $l \geq 0$, as before.

Similarly, the ideal in k generated by y is the same as the ideal generated by x^n for some nonnegative integer n. It follows that M is the submodule of Q_k, as a module over k, generated by x^{l-n}. If N is the submodule of Q_k, as a module over k, generated by x^{n-l}, then we get that $M\,N = k$, as desired.

15.8 Local domains and fractional ideals

Let k be an integral domain that is not a field, with the corresponding field Q_k of fractions. Suppose that k is also a local ring, with unique maximal proper ideal $\mathcal{I}_0$, that satisfies (15.7.2). We would like to show that k is a discrete valuation ring, which is the other part of Proposition 9.7 on p97 of [1]. Another approach, based on some arguments on p8f of [15], will be discussed in the next section.

Remember that invertible ideals of k are finitely generated as modules over k, as in Section 11.7. It follows that any ideal in k is finitely generated as a module over k, so that k is Noetherian.

We would like to show that

(15.8.1) $$\text{every nonzero ideal in } k \text{ is af the form } \mathcal{I}_0^r$$

for some nonnegative integer r, as in (14.12.1). Let Σ be the collection of nonzero ideals in k that cannot be expressed as a nonnegative power of $\mathcal{I}_0$, and suppose for the sake of a contradiction that $\Sigma \neq \emptyset$. This implies that Σ has a maximal element $\mathcal{I}_1$, because k is Noetherian, as in Section 9.7. Of course, $\mathcal{I}_1 \neq k$, so that $\mathcal{I}_1$ is contained in a maximal proper ideal in k. This means that $\mathcal{I}_1 \subseteq \mathcal{I}_0$, and we also have that $\mathcal{I}_1 \neq \mathcal{I}_0$.

Note that $\mathcal{I}_0 \neq \{0\}$, because k is not a field, so that $\mathcal{I}_0$ is invertible as a fractional ideal in k, by hypothesis. This means that there is a submodule N_0 of Q_k, as a module over k, such that

(15.8.2) $$\mathcal{I}_0\,N_0 = k.$$

In particular,

(15.8.3) $$\mathcal{I}_1\,N_0 \subseteq k,$$

so that $\mathcal{I}_1\,N_0$ is an ideal in k. We also have that

(15.8.4) $$\mathcal{I}_1\,N_0 \neq k,$$

because $\mathcal{I}_0\,N_0\,\mathcal{I}_1 = \mathcal{I}_1$.

Remember that $N_0 = (k : \mathcal{I}_0)_{Q_k}$, so that $1 \in N_0$. This implies that

(15.8.5) $$\mathcal{I}_1 \subseteq \mathcal{I}_1\,N_0,$$

which could also be obtained from the fact that $\mathcal{I}_1\,\mathcal{I}_0 \subseteq \mathcal{I}_1$. If $\mathcal{I}_1 = \mathcal{I}_1\,N_0$, then $\mathcal{I}_1\,\mathcal{I}_0 = \mathcal{I}_1$. This would imply that $\mathcal{I}_1 = \{0\}$, by Nakayama's lemma, as in Section 13.3. This also uses the fact that k is Noetherian, to get that $\mathcal{I}_1$ is finitely generated as an ideal in k. However, $\mathcal{I}_1 \neq \{0\}$, by construction. Thus we get that $\mathcal{I}_1 \neq \mathcal{I}_1\,N_0$.

It follows that $\mathcal{I}_1\,N_0$ is not an element of Σ, by the maximality of $\mathcal{I}_1$. This means that $\mathcal{I}_1\,N_0$ is a nonnegative power of $\mathcal{I}_0$. In fact, $\mathcal{I}_1\,N_0$ is a positive power of $\mathcal{I}_0$, because of (15.8.4). This implies that $\mathcal{I}_1$ is a nonnegative power of $\mathcal{I}_0$, which is a contradiction.

Thus (15.8.1) holds under these conditions. One can use this to get that there is an $x \in k$ such that every nonzero ideal in k is generated by x^l for some nonnegative integer l, as in (14.12.8). More precisely, k was asked to have dimension one, in the sense of Section 14.8, in Sections 14.11 and 14.12. This was not needed for the argument just mentioned, and one could also obtain it from (15.8.1) here. Indeed, if $\mathcal{I}$ is a nonzero proper prime ideal in k and $\mathcal{I} = \mathcal{I}_0^j$ for some positive integer j, then one can get $\mathcal{I} = \mathcal{I}_0$, by taking radicals, as in Section 12.10.

This implies that k is a discrete valuation ring, as in Section 15.2.

15.9 Local domains and invertible ideals

Let k be an integral domain, with the corresponding field of fractions Q_k, and suppose that k is a local ring, with unique maximal proper ideal $\mathcal{I}_0$. Also let M be a nonzero fractional ideal of k that is invertible, so that there is a submodule N of Q_k, as a module over k, such that $M\,N = k$. We would like to show that

(15.9.1) $\quad M$ is generated by a single element, as a module over k.

This is mentioned in the remark on p9 of [15].

There are finitely many elements $x_1, \ldots, x_l$ of M and $y_1, \ldots, y_l$ of N such that

$$\sum_{j=1}^{l} x_j\,y_j = 1, \tag{15.9.2}$$

because $1 \in M\,N$. Note that $x_j\,y_j \in k$ for each $j = 1, \ldots, l$, because $M\,N \subseteq k$. It follows that $x_{j_0}\,y_{j_0}$ is not in $\mathcal{I}_0$ for some j_0. This means that $x_{j_0}\,y_{j_0}$ is invertible in k in this case.

Using this, we get $x \in M$ and $y \in N$ such that $x\,y = 1$. If $z \in M$, then

$$z = x\,(y\,z), \tag{15.9.3}$$

and $y\,z \in M\,N = k$. This implies that M is generated by x, as a module over k. This corresponds to the proof of I on p8 of [15], as mentioned in the remark on p9 of [15].

Suppose now that k is not a field, and that every nonzero fractional ideal of k is invertible. In particular, this means that $\mathcal{I}_0$ is invertible, because $\mathcal{I}_0 \neq \{0\}$.

The previous argument implies that $\mathcal{I}_0$ is generated by a single element, as an ideal in k.

Remember that k is Noetherian under these conditions, as in the previous section. It follows that k is a discrete valuation ring, as in Sections 15.5 and 15.6. This corresponds to a statement on p8 of [15], shortly before the proof of the statement I.

Alternatively, if every nonzero ideal in k is invertible, then the previous argument shows that k is a principal ideal domain. If k is not a field, then it follows that k is a discrete valuation ring, as in Section 15.4.

If k is a local domain of dimension one, in the sense of Section 14.8, then one could get that k is a discrete valuation ring from the condition that $\mathcal{I}_0$ be generated by a single element, as an ideal in k, as in Sections 14.11 and 14.12. More precisely, in Sections 14.11 and 14.11, one only ever seems to use a Noetherian condition on k to get that $\mathcal{I}_0$ is finitely generated, as an ideal in k.

15.10 A criterion for invertibility

Let k be an integral domain, and let Q_k be the corresponding field of fractions. Remember that a submodule M of Q_k, as a module over k, is said to be a fractional ideal of k if there is an $x \in k$ such that $x \neq 0$ and $x\,M \subseteq k$, as in Section 11.7. If there is another submodule N of Q_k, as a module over k, such that $M\,N = k$, then M is said to be invertible, as before. In this case, we have seen that N is equal to the set $(k : M)_{Q_k}$ of $x \in Q_k$ such that $x\,M \subseteq k$.

Of course,

$$M\,(k : M)_{Q_k} \subseteq k \tag{15.10.1}$$

holds automatically, by definition of $(k : M)_{Q_k}$. This means that $M\,(k : M)_{Q_k}$ is an ideal in k, because it is a submodule of Q_k, as a module over k. Invertibility of M is equivalent to

$$M\,(k : M)_{Q_k} = k. \tag{15.10.2}$$

Let $\mathcal{I}$ be a maximal proper ideal in k, which is prime in particular. Thus

$$S_{\mathcal{I}} = k \setminus \mathcal{I} \tag{15.10.3}$$

is multiplicatively closed in k, as in Section 12.4. The corresponding ring of fractions $S_{\mathcal{I}}^{-1}\,k$, as in Section 12.1, is an integral domain whose field of fractions may be identified with Q_k, as in Section 14.2. The module of fractions $S_{\mathcal{I}}^{-1}\,M$ of M with respect to $S_{\mathcal{I}}$, as in Section 12.2, may be identified with a submodule of Q_k, as a module over $S_{\mathcal{I}}^{-1}\,k$, as before. This is a fractional ideal of $S_{\mathcal{I}}^{-1}\,k$.

Suppose that

$$S_{\mathcal{I}}^{-1}\,M \text{ is invertible as a fractional ideal of } S_{\mathcal{I}}^{-1}\,k. \tag{15.10.4}$$

Remember that $(S_{\mathcal{I}}^{-1}\,k : S_{\mathcal{I}}^{-1}\,M)_{Q_k}$ is the set of $x \in Q_k$ such that

$$x\,(S_{\mathcal{I}}^{-1}\,M) \subseteq S_{\mathcal{I}}^{-1}\,k, \tag{15.10.5}$$

as in Section 14.3. This is a submodule of Q_k, as a module over $S_{\mathcal{I}}^{-1}\,k$, and (15.10.4) is the same as saying that

$$(S_{\mathcal{I}}^{-1}\,M)\,(S_{\mathcal{I}}^{-1}\,k : S_{\mathcal{I}}^{-1}\,M)_{Q_k} = S_{\mathcal{I}}^{-1}\,k. \tag{15.10.6}$$

Suppose that

$$M \text{ is finitely generated, as a module over } k. \tag{15.10.7}$$

This implies that $(S_{\mathcal{I}}^{-1}\,k : S_{\mathcal{I}}^{-1}\,M)_{Q_k} = S_{\mathcal{I}}^{-1}\,(k : M)_{Q_k}$, as in Section 14.4. It follows that

$$(S_{\mathcal{I}}^{-1}\,M)\,(S_{\mathcal{I}}^{-1}\,(k : M)_{Q_k}) = S_{\mathcal{I}}^{-1}\,k, \tag{15.10.8}$$

by (15.10.6). This means that

$$S_{\mathcal{I}}^{-1}(M\,(k : M)_{Q_k}) = S_{\mathcal{I}}^{-1}\,k, \tag{15.10.9}$$

as in Section 14.3.

In particular, $1/1$ is an element of $S_{\mathcal{I}}^{-1}(M\,(k : M)_{Q_k})$. This implies that there are $v \in M\,(k : M)_{Q_k}$ and $r, t \in S_{\mathcal{I}}$ such that $t\,v = t\,r \in S_{\mathcal{I}}$, as in Section 12.2. Note that $v \in k$, by (15.10.1). It follows that $v \notin \mathcal{I}$, by the definition of $S_{\mathcal{I}}$, and because $\mathcal{I}$ is an ideal in k. Thus

$$M\,(k : M)_{Q_k} \not\subseteq \mathcal{I}. \tag{15.10.10}$$

If this holds for every maximal proper ideal $\mathcal{I}$ in k, then it follows that (15.10.2) holds, because $M\,(k : M)_{Q_k}$ is an ideal in k. This corresponds to part of Proposition 9.6 on p97 of [1].

15.11 Dedekind domains

Let k be a Noetherian integral domain. Remember that if $\mathcal{I}$ is a proper prime ideal in k, then $S_{\mathcal{I}} = k \setminus \mathcal{I}$ is multiplicatively closed in k, as in Section 12.4. In this case,

$$k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\,k \tag{15.11.1}$$

may be defined as in Section 12.1.

Suppose that for every proper prime ideal $\mathcal{I} \neq \{0\}$ in k,

$$k_{\mathcal{I}} \text{ is a discrete valuation ring,} \tag{15.11.2}$$

as in Section 15.1. Under these conditions, k is said to be a *Dedekind domain*, as in p95 of [1], and p10 of [15]. More precisely, the formulation in [1] also asks that k not be a field. If we do not include this condition, then this definition is equivalent to the definition of a Dedekind ring in Section 11.10, as before.

The formulation in [1] also includes the condition that k have dimension one, in the sense of Section 14.8. Of course, this implies that k not be a field, which would have dimension 0. It is easy to see that the definition in the preceding paragraph implies that

(15.11.3) k have dimension less than or equal to one,

which is also mentioned in [15].

Indeed, let $\mathcal{I}$ be a nonzero maximal proper ideal in k. Note that $k_{\mathcal{I}}$ has dimension one, by hypothesis, as in Section 15.1. There is also a one-to-one correspondence between the prime ideals in k that are contained in $\mathcal{I}$ and the proper prime ideals in $k_{\mathcal{I}}$, as mentioned at the end of Section 12.13. It follows that $\mathcal{I}$ is the only nonzero prime ideal in k contained in $\mathcal{I}$, as desired.

Remember that an integral domain has dimension less than or equal to one if and only if every proper nonzero prime ideal is maximal, as in Section 14.11. If (15.11.2) holds for every nonzero maximal proper ideal $\mathcal{I}$ in k, then every nonzero proper prime ideal in k is maximal, as in the preceding paragraph. This means that (15.11.2) holds for every nonzero proper prime ideal in k.

Suppose now that k is a principal ideal domain. It is easy to see that k is a hereditary ring, as in Section 9.3, and thus a Dedekind ring, as in Section 11.10.

Let us verify that k is a Dedekind domain, without using the equivalence with Dedekind rings mentioned earlier. This is Example (1) on p96 of [1], which is also mentioned on p10 of [15]. Note that k is Noetherian, because every ideal in k is finitely generated.

Let S be a multiplicatively closed set in k, with $0 \notin k$, and remember that $S^{-1}\,k$ is an integral domain, as in Section 14.2. We have also seen that every ideal in $S^{-1}\,k$ is the extension of an ideal in k, as in Section 12.13. Using this, one can check that every ideal in $S^{-1}\,k$ is generated by a single element, because of the analogous property of k, by hypothesis. This means that $S^{-1}\,k$ is a principal ideal domain, and in particular that $S^{-1}\,k$ has dimension less than or equal to one, as before.

Let $\mathcal{I}$ be a nonzero proper prime ideal in k, and let $S_{\mathcal{I}}$ and $k_{\mathcal{I}}$ be as before. Remember that $k_{\mathcal{I}}$ is a local ring, as in Section 12.4. Note that $k_{\mathcal{I}}$ is not a field under these conditions. More precisely, $S_{\mathcal{I}}^{-1}\,\mathcal{I}$ is a nonzero proper ideal in $k_{\mathcal{I}}$. We also have that $k_{\mathcal{I}}$ is a principal ideal domain, as in the preceding paragraph. It follows that $k_{\mathcal{I}}$ is a discrete valuation ring, as in Sections 14.11, 14.12, and 15.2, or Section 15.4. Of course, if k is a field, then there are no nonzero proper ideals in k, and the previous statement holds vacuously.

15.12 Dedekind domains and invertibility

Let k be an integral domain, and let Q_k be the corresponding field of fractions. Suppose that k is a Dedekind domain, and let us show that

(15.12.1) every nonzero fractional ideal of k is invertible.

This corresponds to part of Theorem 9.8 on p97 of [1], and to Proposition 5 on p11 of [15]. Remember that "fractional ideals" are defined a bit differently in

[15], as in Section 15.7. However, the difference does not matter here, because k is Noetherian, by hypothesis.

Let M be a nonzero fractional ideal of k, so that M is a submodule of Q_k, as a module over k, such that $x\,M \subseteq k$ for some $x \in k$ with $x \neq 0$, as in Section 11.7. Remember that k is supposed to be Noetherian, as in Section 15.11. This implies that M is finitely generated as a module over k, as in Section 11.7.

Let $\mathcal{I}$ be a maximal proper ideal in k. Remember that $S_{\mathcal{I}} = k \setminus \mathcal{I}$ is multiplicatively closed in k, because $\mathcal{I}$ is a prime ideal, as in Section 12.4. The ring of fractions $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\,k$, as in Section 12.1, is an integral domain whose field of fractions may be identified with Q_k, as in Section 14.2.

The module of fractions

$$M_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\,M, \tag{15.12.2}$$

as in Section 12.2, may be identified with a submodule of Q_k, as a module over $k_{\mathcal{I}}$, as before. This is a fractional ideal of $k_{\mathcal{I}}$, as in Section 15.10.

Let us check that

$$M_{\mathcal{I}} \text{ is invertible as a fractional ideal of } k_{\mathcal{I}}. \tag{15.12.3}$$

Of course, if k is a field, then $\mathcal{I} = \{0\}$, and there is nothing to do.

If k is not a field, then $\mathcal{I} \neq \{0\}$, and $k_{\mathcal{I}}$ is a discrete valuation ring, as in Section 15.11. It is easy to see that

$$M_{\mathcal{I}} \neq \{0\}, \tag{15.12.4}$$

because $M \neq \{0\}$, by hypothesis, and k is an integral domain. It follows that (15.12.3) holds under these conditions, as in Section 15.7.

This implies that M is invertible as a fractional ideal of k, as in Section 15.10. This uses the fact that M is finitely generated as a module over k, as before.

15.13 Dedekind domains from invertibility

Let k be an integral domain, and let Q_k be the corresponding field of fractions, as usual. In this section, we suppose that every nonzero fractional ideal of k is invertible, and we would like to show that k is a Dedekind domain. This is the other part of Theorem 9.8 on p97 of [1].

Remember that invertible ideals of k are finitely generated as modules over k, as in Section 11.7. This implies that k is Noetherian, as in Section 15.8.

Let $\mathcal{I}$ be a nonzero proper prime ideal in k, and remember that $S_{\mathcal{I}} = k \setminus \mathcal{I}$ is multiplicatively closed in k, as in Section 12.4. Thus $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\,k$ may be defined as in Section 12.1, and we would like to show that $k_{\mathcal{I}}$ is a discrete valuation ring.

Remember that $k_{\mathcal{I}}$ is a local ring, as in Section 12.4. More precisely, the unique maximal proper ideal in $k_{\mathcal{I}}$ is $S_{\mathcal{I}}^{-1}\,\mathcal{I}$, using the notation mentioned near the beginning of Section 12.13. We also have that $k_{\mathcal{I}}$ is an integral domain, as mentioned in Section 14.2, and that $k_{\mathcal{I}}$ is Noetherian, as in Section 14.6. Note that $k_{\mathcal{I}}$ is not a field, because $S_{\mathcal{I}}^{-1}\,\mathcal{I}$ is a nonzero proper ideal in $k_{\mathcal{I}}$, as in Section 15.11.

It suffices to show that every nonzero fractional ideal of $k_{\mathcal{I}}$ is invertible, as in Sections 15.8 and 15.9. It is easy to reduce to the case of a nonzero ordinary (integral) ideal in $k_{\mathcal{I}}$.

Let $\mathcal{I}_2$ be a nonzero ideal in $k_{\mathcal{I}}$, and remember that $\mathcal{I}_2^{ce} = \mathcal{I}_2$, as in Section 12.13. Here $\mathcal{I}_2^c$ is the contraction of $\mathcal{I}_2$ with respect to the natural homomorphism from k into $k_{\mathcal{I}}$, and $\mathcal{I}_2^{ce}$ is the extension of $\mathcal{I}_2^c$ in $k_{\mathcal{I}}$. In this case, $k_{\mathcal{I}}$ may be identified with a subring of Q_k, as in Section 14.2, so that

$$\mathcal{I}_2^c = \mathcal{I}_2 \cap k. \tag{15.13.1}$$

It is easy to see that $\mathcal{I}_2^c \neq \{0\}$, because $\mathcal{I}_2 \neq \{0\}$.

It follows that $\mathcal{I}_2^c$ is an invertible ideal of k, by hypothesis. This implies that $S_{\mathcal{I}}^{-1}\,\mathcal{I}_2^c$ is an invertible ideal of $k_{\mathcal{I}}$, as in Section 14.3. Note that

$$S_{\mathcal{I}}^{-1}\,\mathcal{I}_2^c = \mathcal{I}_2^{ce} = \mathcal{I}_2, \tag{15.13.2}$$

as in Section 12.13. This means that $\mathcal{I}_2$ is an invertible ideal of $k_{\mathcal{I}}$, as desired.

15.14 Primary ideals in Dedekind domains

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}_1 \neq \{0\}$ be a primary ideal in k, as in Section 14.5. Remember that $r(\mathcal{I}_1)$ is the radical of $\mathcal{I}_1$, as in Section 12.10, and note that $r(\mathcal{I}_1) \neq \{0\}$, because $\mathcal{I}_1 \neq \{0\}$. We also have that $r(\mathcal{I}_1) \neq k$, because $\mathcal{I}_1 \neq k$, by the definition of a primary ideal. In fact, $r(\mathcal{I}_1)$ is a prime ideal in k, as in Section 14.5. Thus

$$S = k \setminus r(\mathcal{I}_1) \tag{15.14.1}$$

is a multiplicatively closed set in k, as in Section 12.4.

Let $S^{-1}\,k$ be as in Section 12.1, and remember that $x \mapsto x/1$ defines a ring homomorphism from k into $S^{-1}\,k$. The extensions $\mathcal{I}_1^e$ and $r(\mathcal{I}_1)^e$ of $\mathcal{I}_1$ and $r(\mathcal{I}_1)$, respectively, with respect to this homomorphism are ideals in $S^{-1}\,k$, as in Section 12.7. These are the same as the ideals $S^{-1}\,\mathcal{I}_1$ and $S^{-1}\,r(\mathcal{I}_1)$ in $S^{-1}\,k$, as in Section 12.13. In this case, $S^{-1}\,k$ is a local ring, with $S^{-1}\,r(\mathcal{I}_1)$ as its unique maximal proper ideal, as in Section 12.4.

The contractions of $S^{-1}\,\mathcal{I}_1$ and $S^{-1}\,r(\mathcal{I}_1)$ in k with respect to the natural homomorphism from k into $S^{-1}\,k$ are equal to $\mathcal{I}_1$ and $r(\mathcal{I}_1)$, respectively, as in Section 14.6. This uses the fact that $\mathcal{I}_1$ and $r(\mathcal{I}_1)$ are disjoint from S, by construction.

If k is a Dedekind domain, then $S^{-1}\,k$ is a discrete valuation ring, as in Section 15.11. This implies that

$$S^{-1}\,\mathcal{I}_1 = (S^{-1}\,r(\mathcal{I}_1))^l \tag{15.14.2}$$

for a unique positive integer l, as in Section 15.1.

Note that

$$(S^{-1}\, r(\mathcal{I}_1))^l = S^{-1}\, r(\mathcal{I}_1)^l, \tag{15.14.3}$$

as in Sections 12.9 and 12.13, and which can be verified directly anyway. Thus (15.14.2) is the same as saying that

$$S^{-1}\, \mathcal{I}_1 = S^{-1}\, r(\mathcal{I}_1)^l. \tag{15.14.4}$$

This holds for a unique positive integer l, as before.

If k is a Dedekind domain, then $r(\mathcal{I}_1)$ is a maximal proper ideal in k, because $r(\mathcal{I}_1)$ is a nonzero proper prime ideal in k. This implies that $r(\mathcal{I}_1)^l$ is a primary ideal in k, as in Section 14.5. It follows that the contraction of $S^{-1}\, r(\mathcal{I}_1)^l$ in k with respect to the natural homomorphism from k into $S^{-1}\, k$ is equal to $r(\mathcal{I}_1)^l$, as in Section 14.6 again. Using this, we get that

$$\mathcal{I}_1 = r(\mathcal{I}_1)^l. \tag{15.14.5}$$

This corresponds to part of Theorem 9.3 on p95 of [1].

Of course, (15.14.5) implies (15.14.4). This means that l is uniquely determined by (15.14.5).

15.15 One-dimensional Noetherian domains

Let k be an integral domain of dimension one in the sense of Section 14.8. Thus k is not a field, and every nonzero proper prime ideal in k is maximal. If k is Noetherian, and if every primary ideal in k is a power of a prime ideal, then k is a Dedekind domain, as in Theorem 9.3 on p95 of [1].

More precisely, if $\mathcal{I}_1$ is a primary ideal in k, then the radical $r(\mathcal{I}_1)$ is a proper prime ideal in k, as in Section 14.5. The hypothesis mentioned in the preceding paragraph means that

$$\mathcal{I}_1 = r(\mathcal{I}_1)^l \tag{15.15.1}$$

for some positive integer l.

Let $\mathcal{I} \neq \{0\}$ be a proper prime ideal in k, so that $S_{\mathcal{I}} = k \setminus \mathcal{I}$ is multiplicatively closed in k, as in Section 12.4. We would like to show that $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\, k$ is a discrete valuation ring, as in Section 15.11. The hypothesis that k have dimension one implies that $\mathcal{I}$ is a maximal proper ideal in k, and it is sufficient to consider only that case anyway, as before.

Remember that $k_{\mathcal{I}}$ is a local ring, with unique maximal proper ideal $S_{\mathcal{I}}^{-1}\, \mathcal{I}$, as in Section 12.4. We have seen that $k_{\mathcal{I}}$ is an integral domain, because k is an integral domain, as in Section 14.2. We also have that $k_{\mathcal{I}}$ is Noetherian, because k is Noetherian, as in Section 14.6. Note that $S_{\mathcal{I}}^{-1}\, \mathcal{I} \neq \{0\}$, because $\mathcal{I} \neq 0$, and k is an integral domain.

There is a one-to-one correspondence between the set of prime ideals in k that are contained in $\mathcal{I}$ and the set of proper prime ideals in $k_{\mathcal{I}}$, as mentioned at the end of Section 12.13. This implies that

$$k_{\mathcal{I}} \text{ has dimension one,} \tag{15.15.2}$$

because k has dimension one. In fact, $S_{\mathcal{I}}^{-1}\,\mathcal{I}$ is the only nonzero proper prime ideal in $k_{\mathcal{I}}$.

We would like to show that if $\mathcal{I}_2 \ne \{0\}$ is an ideal in $k_{\mathcal{I}}$, then

$$\mathcal{I}_2 = (S_{\mathcal{I}}^{-1}\,\mathcal{I})^l \tag{15.15.3}$$

for some nonnegative integer l. This will imply that $k_{\mathcal{I}}$ is a discrete valuation ring, as in Sections 14.12 and 15.2. We may as well suppose that $\mathcal{I}_2 \ne k_{\mathcal{I}}$, since otherwise we could take $l = 0$ in (15.15.3). This means that

$$\mathcal{I}_2 \subseteq S_{\mathcal{I}}^{-1}\,\mathcal{I}, \tag{15.15.4}$$

because $S_{\mathcal{I}}^{-1}\,\mathcal{I}$ is the unique maximal proper ideal in $k_{\mathcal{I}}$.

Remember that the radical $r(\mathcal{I}_2)$ of $\mathcal{I}_2$ is the same as the intersection of all of the prime ideals in $k_{\mathcal{I}}$ that contain $\mathcal{I}_2$, as in Section 12.10. This implies that

$$r(\mathcal{I}_2) = S_{\mathcal{I}}^{-1}\,\mathcal{I}, \tag{15.15.5}$$

because $\mathcal{I}_2 \ne \{0\}$, and $S_{\mathcal{I}}^{-1}\,\mathcal{I}$ is the only nonzero proper prime ideal in $k_{\mathcal{I}}$. It follows that $\mathcal{I}_2$ is a primary ideal in $k_{\mathcal{I}}$, because $S_{\mathcal{I}}^{-1}\,\mathcal{I}$ is a maximal proper ideal in $k_{\mathcal{I}}$, as in Section 14.5.

Let $\mathcal{I}_2^c$ be the contraction of $\mathcal{I}_2$ in k, with respect to the natural homomorphism $x \mapsto x/1$ from k into $k_{\mathcal{I}}$, as in Section 12.7. Note that $\mathcal{I}_2^c$ is a primary ideal in k, because $\mathcal{I}_2$ is a primary ideal in $k_{\mathcal{I}}$, as in Section 14.5. It is easy to see that $\mathcal{I}_2^c \ne \{0\}$, because $\mathcal{I}_2 \ne \{0\}$. We also have that that $\mathcal{I}_2^c \ne k$, because $\mathcal{I}_2 \ne k_{\mathcal{I}}$.

Remember that a prime ideal in k that is disjoint from $S_{\mathcal{I}}$ is the contraction of an ideal in $k_{\mathcal{I}}$, as in Section 12.13. This implies that a prime ideal in k that is disjoint from $S_{\mathcal{I}}$ is equal to the contraction of its extension, as in Section 12.7. In particular, $\mathcal{I}$ is a prime ideal in k that is disjoint from $S_{\mathcal{I}}$, so that

$$\mathcal{I} = \mathcal{I}^{ec} = (S_{\mathcal{I}}^{-1}\,\mathcal{I})^c, \tag{15.15.6}$$

where the second step is as in Section 12.13. It follows that

$$\mathcal{I}_2^c \subseteq (S_{\mathcal{I}}^{-1}\,\mathcal{I})^c = \mathcal{I}, \tag{15.15.7}$$

using (15.15.4) in the first step.

The radical $r(\mathcal{I}_2^c)$ of $\mathcal{I}_2^c$ is the smallest prime ideal in k that contains $\mathcal{I}_2^c$, because $\mathcal{I}_2^c$ is a primary ideal in k, as in Section 14.5. Note that

$$r(\mathcal{I}_2^c) \subseteq \mathcal{I}, \tag{15.15.8}$$

by (15.15.7), and that $r(\mathcal{I}_2^c) \ne \{0\}$, because $\mathcal{I}_2^c \ne \{0\}$. This means that

$$r(\mathcal{I}_2^c) = \mathcal{I}, \tag{15.15.9}$$

because k has dimension one, by hypothesis.

Thus

$$\mathcal{I}_2^c = \mathcal{I}^l \tag{15.15.10}$$

for some positive integer l, as in (15.15.1). Remember that $\mathcal{I}_2 = \mathcal{I}_2^{ce}$, as in Section 12.13. Using this, we get that

$$\mathcal{I}_2 = (\mathcal{I}^l)^e = (\mathcal{I}^e)^l = (S_{\mathcal{I}}^{-1}\,\mathcal{I})^l, \tag{15.15.11}$$

where the second step is as in Section 12.9, and the third step is as in Section 12.13. This shows that (15.15.3) holds, as desired.

Chapter 16

Valuation rings and integral elements

16.1 Valuation rings

An integral domain k is said to be a *valuation ring* of its field Q_k of fractions if for every $x \in Q_k$ with $x \neq 0$, we have that $x \in k$, or $x^{-1} \in k$, or both, as on p65 of [1].

Let k_1 be a field, and let k be a subring of k_1 that contains the multiplicative identity element $1 = 1_{k_1}$ of k_1. This implies that k is an integral domain, for which the corresponding field of fractions may be identified with the subfield of k_1 generated by k. One might say that k is a valuation ring of k_1 if for every $x \in k_1$ with $x \neq 0$, we have that $x \in k$, $x^{-1} \in k$, or both. This would imply in particular that k_1 is generated as a field by k, so that k_1 may be identified with the field Q_k of fractions of k. This means that k would be a valuation ring of Q_k, as in the preceding paragraph.

Let v be a valuation on k_1, as in Section 14.13. Remember that

$$\{x \in k_1 : v(x) \geq 0\} \tag{16.1.1}$$

is a subring of k_1 that contains 1_{k_1}, called the valuation ring of v, as in Section 14.14. If $x \in k_1$ and $x \neq 0$, then x or x^{-1} is an element of (16.1.1), depending on whether $v(x) \geq 0$ or $v(x) \leq 0$. This implies that k_1 may be identified with the field of fractions of (16.1.1), and that (16.1.1) is a valuation ring in its field of fractions, as in the previous paragraph. This corresponds to a remark on p94 of [1].

Let k be a commutative ring with a multiplicative identity element. If x, y are elements of k and $x\,y$ is invertible in k, then it is easy to see that

$$x \text{ and } y \text{ are invertible in } k. \tag{16.1.2}$$

Put

$$\mathcal{I}_0 = \{x \in k : x \text{ is not invertible in } k\}. \tag{16.1.3}$$

If $x \in \mathcal{I}_0$ and $y \in k$, then

(16.1.4) $$x\,y \in \mathcal{I}_0,$$

by (16.1.2).

Now let k be an integral domain, with field of fractions Q_k. In this case,

(16.1.5) $$\mathcal{I}_0 = \{x \in k : x = 0 \text{ or } x^{-1} \in Q_k \setminus k\}.$$

Suppose that k is a valuation ring of Q_k. If $x, y \in k$ and $x, y \neq 0$, then

(16.1.6) $$x\,y^{-1} \in k \text{ or } x^{-1}\,y \in k.$$

Let $x, y \in \mathcal{I}_0$ be given, and let us check that

(16.1.7) $$x + y \in \mathcal{I}_0.$$

This follows automatically when $x = 0$ or $y = 0$, and so we may suppose that $x, y \neq 0$. If $x\,y^{-1} \in k$, then

(16.1.8) $$x + y = (x\,y^{-1} + 1)\,y \in \mathcal{I}_0,$$

as in (16.1.4). Similarly, (16.1.7) holds when $x^{-1}\,y \in k$.

This shows that $\mathcal{I}_0$ is an ideal in k, as in the proof of the first part of Proposition 5.18 on p65 of [1]. This means that $\mathcal{I}_0$ is the unique maximal proper ideal in k, so that k is a local ring, as in [1].

16.2 Noetherian valuation rings

Let k be an integral domain that is a valuation ring of its field Q_k of fractions. If $x, y \in k$ and $x, y \neq 0$, then either $x = t\,y$ for some $t \in k$, or $y = t'\,x$ for some $t' \in k$, by (16.1.6). This implies that the ideals in k that are generated by a single element are linearly ordered by inclusion.

Let $\mathcal{I}$ be an ideal in k generated by finitely many elements $x_1, \ldots, x_n$. It is easy to see that

(16.2.1) $$\mathcal{I} \text{ is generated by } x_j$$

for some j, using the remark in the preceding paragraph.

Suppose now that k is also Noetherian, so that every ideal in k is finitely generated. This implies that every ideal in k is generated by a single element, as in the previous paragraph, so that k is a principal ideal domain.

If k is not a field, then k is a discrete valuation ring under these conditions, as in Exercise 3 on p99 of [1]. Remember that principal ideal domains have dimension less than or equal to one, in the sense of Section 14.8, as in Section 15.4. This means that k has dimension one, because k is not a field.

Let $\mathcal{I}_0$ be the unique maximal proper ideal in k, as in the previous section. Note that $\mathcal{I}_0$ is generated as an ideal in k by a single element, as before. One can get that k is a discrete valuation ring using the arguments in Sections 14.11, 14.12, and 15.2.

Alternatively, one could use the characterization of discrete valuation rings in Section 15.4. One could also use the criterion for a commutative ring to be a discrete valuation ring discussed in Sections 15.5 and 15.6.

Remember that discrete valuation rings are Noetherian, as in Section 15.1.

16.3 Integral elements of larger rings

Let k_1 be a commutative ring with a multiplicative identity element $1 = 1_{k_1}$, and let k be a subring of k_1 that contains 1_{k_1}. An element x of k_1 is said to be *integral over* k if it satisfies a monic polynomial equation with coefficients in k. This means that there are a positive integer n and n elements $a_1, \ldots, a_n$ of k such that

$$x^n + a_1\, x^{n-1} + \cdots + a_{n-1}\, x + a_n = 0, \tag{16.3.1}$$

as on p59 of [1], and p8 of [15]. Note that every element of k is integral over k in this sense.

If every element of k_1 that is integral over k is an element of k, then k is said to be *integrally closed* in k_1, as on p60 of [1], and p8 of [15].

Suppose now that k is an integral domain, and let Q_k be its field of fractions. If k is integrally closed in Q_k, then k is simply said to be *integrally closed*, as on p62 of [1], and p8 of [15].

It is well known that the ring $\mathbf{Z}$ of integers is integrally closed in the field $\mathbf{Q}$ of rational numbers, as in Example 5.0 on p59 of [1]. The same argument can be used to show that any unique factorization domain is integrally closed, as mentioned on p63 of [1].

Suppose for the moment that k is a valuation ring of Q_k, as in Section 16.1. In this case, k is integrally closed in Q_k, as in Proposition 5.18 on p65 of [1]. Indeed, suppose that $x \in Q_k$ is integral over k, so that x satisfies a monic polynomial equation with coefficients in k as in (16.3.1). If $x \not\in k$, then $x \neq 0$ and $x^{-1} \in k$, and we can multiply both sides of (16.3.1) by x^{1-n} to get that

$$x = -a_1 - \cdots - a_{n-1}\, x^{2-n} - a_n\, x^{1-n}. \tag{16.3.2}$$

This implies that $x \in k$.

Let k_1 be a field, and let v be a valuation on k_1, as in Section 14.13. If $x, y \in k_1$ and $v(x) \le v(y)$, then

$$v(x+y) \ge v(x), \tag{16.3.3}$$

by definition of a valuation. We also have that

$$v(x) \ge \min(v(x+y), v(-y)) = \min(v(x+y), v(y)). \tag{16.3.4}$$

If $v(x) > v(y)$, then it follows that

$$v(x) \ge v(x+y). \tag{16.3.5}$$

This means that

$$v(x + y) = v(x) \tag{16.3.6}$$

when $v(x) > v(y)$.

Remember that the valuation ring of v,

$$\{x \in k_1 : v(x) \geq 0\}, \tag{16.3.7}$$

is a valuation ring of k_1, as in Section 16.1. Thus (16.3.7) is integrally closed in k_1, as before. Another way to see this is mentioned on p8 of [15]. We can do this using the remarks in the preceding paragraph, as follows.

Suppose that $x \in k_1$ is integral over (16.3.7), so that x satisfies a monic polynomial equation with coefficients in (16.3.7), as in (16.3.1). If $v(x) < 0$ and $1 \leq l \leq n$, then

$$v(x^n) = n\, v(x) > (n - l)\, v(x) \geq v(a_l\, x^{n-l}). \tag{16.3.8}$$

This implies that

$$v(x^n + a_1\, x^{n-1} + \cdots + a_{n-1}\, x + a_n) = v(x^n), \tag{16.3.9}$$

as in the preceding paragraph. This contradicts (16.3.1), so that $v(x) \geq 0$, as desired.

16.4 Some properties of integral elements

Let k_0 be a commutative ring with a multiplicative identity element, and let V be a module over k_0. Remember that the annihilator $\mathrm{Ann}(V)$ of V in k_0 consists of $t \in k_0$ such that $t \cdot V = \{0\}$, as in Section 12.8. If

$$\mathrm{Ann}(V) = \{0\}, \tag{16.4.1}$$

then V is said to be *faithful* as a module over k_0, as on p20 of [1].

Let k_1 be a commutative ring with a multiplicative identity element $1 = 1_{k_1}$, and let k be a subring of k_1 that contains 1_{k_1}. If $x \in k_1$, then let $k[x]$ be the subring of k_1 generated by k and x. This is the same as the submodule of k_1, as a module over k, generated by the nonnegative integer powers of x in k_1. Proposition 5.1 on p59f of [1] gives some properties of $k[x]$ that are equivalent to saying that

$$x \text{ is integral over } k. \tag{16.4.2}$$

If (16.4.2) holds, then it is easy to see that

$$k[x] \text{ is finitely generated as a module over } k. \tag{16.4.3}$$

More precisely, suppose that x satisfies a monic polynomial equation of degree n with coefficients in k, as in (16.3.1). If r is a nonnegative integer, then we get that

$$x^{n+r} = -a_1\, x^{n+r-1} - \cdots - a_{n-1}\, x^{r+1} - a_n\, x^r. \tag{16.4.4}$$

One can use this to check that $k[x]$ is generated by $1, x, \ldots, x^{n-1}$, as a module over k, as on p60 of [1].

Clearly (16.4.3) implies that

$$k[x] \text{ is contained in a subring } C \text{ of } k_1 \text{ such that } C \text{ is finitely generated as a module over } k, \tag{16.4.5}$$

by taking $C = k[x]$. Let us check that (16.4.5) implies that

$$\text{there is a faithful module } V \text{ over } k[x] \text{ that is finitely generated as a module over } k. \tag{16.4.6}$$

Of course, if (16.4.5) holds, then C is a module over $k[x]$ that is finitely generated as a module over k. It is easy to see that C is also faithful as a module over $k[x]$, because $1_{k_1} \in C$, as on p60 of [1].

We would like to show that (16.4.6) implies (16.4.2), to get that these four conditions are equivalent, as in [1]. To do this, we use the remarks in Section 13.2, with V considered as a module over k. The ideal $\mathcal{I}$ of k mentioned in Section 13.2 is taken to be k here, so that $\mathcal{I} \cdot V = V$.

If $y \in k[x]$ and $v \in V$, then put

$$\phi_y(v) = y \cdot v, \tag{16.4.7}$$

where the right side is defined as an element of V, because V is a module over $k[x]$. Note that ϕ_y is a homomorphism from V into itself, as a module over k. Thus

$$y \mapsto \phi_y \tag{16.4.8}$$

defines a mapping from $k[x]$ into the space $\mathrm{Hom}_k(V, V)$ of homomorphisms from V into itself, as a module over k. More precisely, (16.4.8) is a homomorphism from $k[x]$ into $\mathrm{Hom}_k(V, V)$, as associative algebras over k. This homomorphism is injective, because V is faithful as a module over $k[x]$, by hypothesis.

If we take $\phi = \phi_x$, then the remarks in Section 13.2 imply that ϕ satisfies a monic polynomial equation with coefficients in k. This uses the hypothesis that V be finitely generated as a module over k. It follows that x satisfies the analogous polynomial equation, with the same coefficients, because of the remarks in the preceding paragraph, as on p60 of [1].

16.5 Some corollaries

Let us continue with the same notation and hypotheses as in the previous section. Let $x_1, \ldots, x_n$ be finitely many elements of k_1, and let $k[x_1, \ldots, x_n]$ be the subring of k_1 generated by k and $x_1, \ldots, x_n$. If

$$x_j \text{ is integral over } k \tag{16.5.1}$$

for each $j = 1, \ldots, n$, then Corollary 5.2 on p60 of [1] states that

$$k[x_1, \ldots, x_n] \text{ is finitely generated as a module over } k. \tag{16.5.2}$$

Note that this reduces to (16.4.3) when $n = 1$.

Put $A_r = k[x_1, \ldots, x_r]$ for each $r = 1, \ldots, n$. Suppose that $n > 1$, and observe that A_n is the same as the subring $A_{n-1}[x_n]$ of k_1 generated by A_{n-1} and x_n. Of course, x_n is integral over A_{n-1}, because x_n is integral over k, by hypothesis. This implies that $A_{n-1}[x_n]$ is finitely generated as a module over A_{n-1}, as in (16.4.3).

If we use induction, then we may suppose that A_{n-1} is finitely generated as a module over k. One can use this to get that $A_n = A_{n-1}[x_n]$ is finitely generated as a module over k, as on p60 of [1].

If $x, y \in k_1$ are integral over k, then $k[x, y]$ is finitely generated as a module over k, as in (16.5.2). This implies that $x \pm y$ and $x\,y$ are integral over k, because (16.4.5) implies (16.4.2). It follows that

$$C = \{x \in k_1 : x \text{ is integral over } k\} \tag{16.5.3}$$

is a subring of k_1, as in Corollary 5.3 on p60 of [1].

The subring C of k_1 is called the *integral closure* of k in k_1, as on p60 of [1]. Note that k is integrally closed in k_1 exactly when $C = k$. If $C = k_1$, then k_1 is said to be *integral over* k, as in [1]. Of course, C is integral over k.

Let k_2 be a commutative ring with a multiplicative identity element $1 = 1_{k_2}$, and suppose that k_1 is a subring of k_2 that contains 1_{k_2}. If k_1 is integral over k and k_2 is integral over k_1, then

$$k_2 \text{ is integral over } k, \tag{16.5.4}$$

as in Corollary 5.4 on p60 of [1]. To see this, let $x \in k_2$ be given, so that x satisfies a monic polynomial equation

$$x^n + b_1\,x^{n-1} + \cdots + b_{n-1}\,x + b_n = 0 \tag{16.5.5}$$

with coefficients in k_1. These coefficients are integral over k, by hypothesis, so that

$$k[b_1, \ldots, b_n] \text{ is finitely generated as a module over } k, \tag{16.5.6}$$

as in (16.5.2). Note that

$$x \text{ is integral over } k[b_1, \ldots, b_n]. \tag{16.5.7}$$

It follows that

$$(k[b_1, \ldots, b_n])[x] \text{ is finitely generated as a module over } k[b_1, \ldots, b_n], \tag{16.5.8}$$

as in the previous section. This implies that

$$(k[b_1, \ldots, b_n])[x] \text{ is finitely generated as a module over } k, \tag{16.5.9}$$

because of (16.5.6). Using this, we get that x is integral over k, as in the previous section again.

Let C be as in (16.5.3), and let $\widetilde{C}$ be the integral closure of C in k_1. We would like to check that

$$\widetilde{C} = C, \tag{16.5.10}$$

as in Corollary 5.5 on p61 of [1]. Of course, $C \subseteq \widetilde{C}$ automatically. We also have that C is integral over k, as before, and similarly that $\widetilde{C}$ is integral over C. This implies that $\widetilde{C}$ is integral over k, as in (16.5.4), so that $\widetilde{C} \subseteq C$.

16.6 Getting a discrete valuation

Let k be a Noetherian integral domain. Suppose that k is a local ring, and let $\mathcal{I}_0$ be the unique maximal proper ideal in k. Suppose also that k has dimension one in the sense of Section 14.8. Equivalently, this means that $\mathcal{I}_0 \neq \{0\}$, and that $\mathcal{I}_0$ is the only nonzero proper prime ideal in k, as in Section 14.11.

Suppose in addition that k is integrally closed. We would like to show that

$$\mathcal{I}_0 \text{ is a principal ideal in } k. \tag{16.6.1}$$

This will imply that k is a discrete valuation ring, using the results discussed in Sections 14.11, 14.12, and 15.2, or in Sections 15.5 and 15.6. This corresponds to parts of Proposition 9.2 on p94 of [1], and Proposition 3 on p7 of [15].

Let us begin with the approach in [1]. Let $a \in \mathcal{I}_0$ be given, with $a \neq 0$, and let $\mathcal{I}(a)$ be the ideal in k generated by a. Under these conditions, there is a positive integer m such that

$$\mathcal{I}_0^m \subseteq \mathcal{I}(a) \subseteq \mathcal{I}_0, \tag{16.6.2}$$

as in Section 14.11. We may as well take m to be as small as possible, so that

$$\mathcal{I}_0^{m-1} \not\subseteq \mathcal{I}(a). \tag{16.6.3}$$

Remember that $\mathcal{I}_0^0$ is interpreted as being equal to k, as in Section 12.10.

Let b be an element of $\mathcal{I}_0^{m-1}$ that is not in $\mathcal{I}(a)$. In particular, $b \neq 0$, so that a/b is defined as an element of the field Q_k of fractions of k. Note that

$$b/a \notin k, \tag{16.6.4}$$

because $b \notin \mathcal{I}(a)$. This means that

$$b/a \text{ is not integral over } k \tag{16.6.5}$$

in Q_k, because k is integrally closed in Q_k, by hypothesis.

Suppose for the sake of a contradiction that

$$(b/a)\,\mathcal{I}_0 \subseteq \mathcal{I}_0. \tag{16.6.6}$$

Let $k[b/a]$ be the subring of Q_k generated by k and b/a, as in Section 16.4. If (16.6.6) holds, then we may consider $\mathcal{I}_0$ as a module over $k[b/a]$, because $\mathcal{I}_0$ is an ideal in k. In fact, $\mathcal{I}_0$ would be faithful as a module over $k[b/a]$, because $\mathcal{I}_0 \neq \{0\}$, and Q_k is a field. We also have that $\mathcal{I}_0$ is finitely generated as a module over k, because k is Noetherian, by hypothesis. Under these conditions, we would get that b/a is integral over k, as in Section 16.4 again. This is a contradiction, which means that

$$(b/a)\,\mathcal{I}_0 \not\subseteq \mathcal{I}_0. \tag{16.6.7}$$

Remember that $b \in \mathcal{I}_0^{m-1}$, so that

$$b\,\mathcal{I}_0 \subseteq \mathcal{I}_0^m \subseteq \mathcal{I}(a). \tag{16.6.8}$$

This implies that

$$(b/a)\,\mathcal{I}_0 \subseteq k. \tag{16.6.9}$$

More precisely, $(b/a)\,\mathcal{I}_0$ is an ideal in k. It follows that

$$(b/a)\,\mathcal{I}_0 = k, \tag{16.6.10}$$

because of (16.6.7), and the hypothesis that $\mathcal{I}_0$ be the unique maximal proper ideal in k.

This means that $\mathcal{I}_0$ is the same as the ideal in k generated by a/b, as desired.

16.7 Another argument using fractional ideals

Let k be an integral domain that is a local ring, with unique maximal proper ideal $\mathcal{I}_0$. Suppose also that k is not a field, so that $\mathcal{I}_0 \neq \{0\}$. We would like to consider another proof of (16.6.1) when k is Noetherian, of dimension one, and integrally closed, as on p8f of [15].

Remember that $(k : \mathcal{I}_0)_{Q_k}$ is the set of x in the field Q_k of fractions of k such that $x\,\mathcal{I}_0 \subseteq k$, as in Section 15.7. This is a fractional ideal of k, because $\mathcal{I}_0 \neq \{0\}$, as before. More precisely, if k is Noetherian, then $(k : \mathcal{I}_0)_{Q_k}$ is finitely generated as a module over k, as in Section 11.7. This means that $(k : \mathcal{I}_0)_{Q_k}$ is a fractional ideal of Q_k with respect to k, as in [15], in this case.

Of course,

$$\mathcal{I}_0\,(k : \mathcal{I}_0)_{Q_k} \subseteq k, \tag{16.7.1}$$

by construction. In fact, $\mathcal{I}_0\,(k : \mathcal{I}_0)_{Q_k}$ is an ideal in k. We also have that

$$k \subseteq (k : \mathcal{I}_0)_{Q_k}, \tag{16.7.2}$$

because $\mathcal{I}_0$ is an ideal in l. This implies that

$$\mathcal{I}_0 \subseteq \mathcal{I}_0\,(k : \mathcal{I}_0)_{Q_k}. \tag{16.7.3}$$

It follows that

$$\mathcal{I}_0\,(k : \mathcal{I}_0)_{Q_k} = \mathcal{I}_0 \tag{16.7.4}$$

or

$$\mathcal{I}_0\,(k : \mathcal{I}_0)_{Q_k} = k, \tag{16.7.5}$$

because $\mathcal{I}_0$ is a maximal proper ideal in k. If (16.7.5) holds, then $\mathcal{I}_0$ is invertible as a fractional ideal of k, or equivalently in the sense mentioned in the remark on p9 of [15], as in Section 15.7. In this case, (16.6.1) holds, as in Section 15.9. This corresponds to the statement I on p8 of [15].

If (16.7.4) holds and k is Noetherian and integrally closed, then Statement II on p8 of [15] says that

(16.7.6) $$(k : \mathcal{I}_0)_{Q_k} = k.$$

Of course, it suffices to show that

(16.7.7) $$(k : \mathcal{I}_0)_{Q_k} \subseteq k,$$

because of (16.7.2). To see this, let $x \in (k : \mathcal{I}_0)_{Q_k}$ be given, so that

(16.7.8) $$x\,\mathcal{I}_0 \subseteq \mathcal{I}_0,$$

by (16.7.4). This implies that

(16.7.9) $$x^l\,\mathcal{I}_0 \subseteq \mathcal{I}_0$$

for every positive integer l.

Let V_n be the submodule of Q_k, as a module over k, generated by $1, x, \ldots, x^n$ for each positive integer n. Thus

(16.7.10) $$V_n \subseteq V_{n+1}$$

for every $n \geq 1$, by construction. We also have that

(16.7.11) $$V_n \subseteq (k : \mathcal{I}_0)_{Q_k}$$

for each n, by (16.7.9) and the definition of $(k : \mathcal{I}_0)_{Q_k}$. Note that $(k : \mathcal{I}_0)_{Q_k}$ is Noetherian as a module over k, because k is Noetherian, and $(k : \mathcal{I}_0)_{Q_k}$ is finitely generated as a module over k, as in Section 9.7. This implies that

(16.7.12) $$V_{n+1} = V_n$$

when n is sufficiently large.

If (16.7.12) holds, then $x^{n+1} \in V_n$, and x is integral over k. This implies that $x \in k$, because k is integrally closed in Q_k, by hypothesis. This means that (16.7.7) holds, as desired. Alternatively, we could get that x is integral over k using (16.7.8) and a criterion for this in Section 16.4, as in Section 16.6. The approach discussed here, from p9 of [15], seems to be more direct in this case.

If k is Noetherian and has dimension one, then statement III on p8 of [15] says that (16.7.6) does not hold. This will be discussed in Section 16.9, after a preliminary result in the next section.

16.8 Powers in the denominator

Let k be an integral domain that is a local ring and not a field again, with unique maximal proper ideal $\mathcal{I}_0 \neq \{0\}$. Also let $x \in \mathcal{I}_0$ with $x \neq 0$ be given. Note that

(16.8.1) $$S = \{x^n : n \in (\mathbf{Z}_+ \cup \{0\})\}$$

is a multiplicatively closed set in k, as in Section 12.6. Of course, $0 \notin S$, because $x \neq 0$ and k is an integral domain.

The corresponding ring of fractions $S^{-1}\,k$ may be identified with a subring of the field Q_k of fractions of k, as in Section 14.2. This subring consists of elements of Q_k of the form y/x^n, where $y \in k$, and n is a nonnegative integer.

Suppose that k has dimension one, in the sense of Section 14.8. This means that $\mathcal{I}_0$ is the unique nonzero proper prime ideal in k, as in Section 14.11. Let us check that

$$S^{-1}\,k = Q_k, \tag{16.8.2}$$

as in the first part of the proof of III on p9 of [15].

If (16.8.2) does not hold, then $S^{-1}\,k$ is not a field. This would imply that there is a nonzero maximal proper ideal $\mathcal{I}_1$ in $S^{-1}\,k$. Observe that

$$x \notin \mathcal{I}_1, \tag{16.8.3}$$

because x is invertible in $S^{-1}\,k$. It follows that

$$\mathcal{I}_0 \not\subseteq \mathcal{I}_1 \cap k. \tag{16.8.4}$$

Note that $\mathcal{I}_1$ is a prime ideal in $S^{-1}\,k$, which implies that $\mathcal{I}_1 \cap k$ is a prime ideal in k. Because $\mathcal{I}_1 \neq \{0\}$, there is an element of $\mathcal{I}_1$ of the form y/x^n, where $y \in k$, $y \neq 0$, and n is a nonnegative integer. It follows that $y \in \mathcal{I}_1 \cap k$, so that $\mathcal{I}_1 \cap k \neq \{0\}$. This contradicts the fact that $\mathcal{I}_0$ is the unique nonzero proper prime ideal in k.

Alternatively, if $z \in k$ and $z \neq 0$, then we would like to show that there is a $y \in k$ such that

$$y\,z = x^n \tag{16.8.5}$$

for some nonnegative integer n. This is clear when z is invertible in k, and so we may suppose that z is not invertible in k, so that the ideal $\mathcal{I}(z)$ in k generated by z is a proper ideal in k. Remember that the radical $r(\mathcal{I}(z))$ of $\mathcal{I}(z)$ in k is equal to the intersection of all of the prime ideals in k that contain $\mathcal{I}(z)$, as in Section 12.10. This means that

$$r(\mathcal{I}(z)) = \mathcal{I}_0, \tag{16.8.6}$$

because $\mathcal{I}(z)$ is a nonzero proper ideal in k, and $\mathcal{I}_0$ is the unique nonzero proper prime ideal in k. It follows that

$$x \in r(\mathcal{I}(z)), \tag{16.8.7}$$

so that (16.8.5) holds for some $y \in k$ and $n \geq 0$.

Equivalently,

$$1/z = y/x^n \in S^{-1}\,k. \tag{16.8.8}$$

This implies (16.8.2). Note that the previous argument for (16.8.2) implies that $1/z$ can be expresed as in (16.8.8), so that (16.8.5) holds for some $y \in k$ and $n \geq 0$.

16.9 Using powers in the denominator

Let us return to the same notation and hypotheses as in Section 16.7. Thus k is an integral domain that is a local ring and not a field, with unique maximal proper ideal $\mathcal{I}_0 \neq \{0\}$, as in the previous section as well. We would like to show that if k is Noetherian and has dimension one, then

$$(k : \mathcal{I}_0)_{Q_k} \neq k, \tag{16.9.1}$$

as in statement III on p8 of [15]. More precisely, the hypothesis that k be Noetherian is only used here to get that $\mathcal{I}_0$ be finitely generated as an ideal in k. Remember that k is automatically contained in $(k : \mathcal{I}_0)_{Q_k}$, so that (16.9.1) is the same as saying that $(k : \mathcal{I}_0)_{Q_k}$ is not contained in k.

Let $z \in k$ with $z \neq 0$ be given, and let $\mathcal{I}(z)$ be the ideal in k generated by z. If $x \in \mathcal{I}_0$, then

$$x^n \in \mathcal{I}(z) \tag{16.9.2}$$

for some positive integer n, as in the previous section. Of course, this uses the hypothesis that k have dimension one. Equivalently, $\mathcal{I}_0$ is contained in the radical $r(\mathcal{I}(z))$ of $\mathcal{I}(z)$. If $\mathcal{I}_0$ is finitely generated as an ideal in k, then it follows that

$$\mathcal{I}_0^m \subseteq \mathcal{I}(z) \tag{16.9.3}$$

for some positive integer m, as in Section 14.7.

Let us now take $z \in \mathcal{I}_0$, with $z \neq 0$. Remember that $\mathcal{I}_0 \neq \{0\}$, by hypothesis. Let m be the smallest positive integer such that (16.9.3) holds. This implies that

$$\mathcal{I}_0^{m-1} \not\subseteq \mathcal{I}(z), \tag{16.9.4}$$

where $\mathcal{I}_0^0$ is interpreted as being equal to k, as usual. Note that $\mathcal{I}(z) \subseteq \mathcal{I}_0$, so that (16.9.4) holds automatically when $m = 1$.

Let y be an element of $\mathcal{I}_0^{m-1}$ that is not in $\mathcal{I}(z)$. Observe that

$$y\,\mathcal{I}_0 \subseteq \mathcal{I}_0^m \subseteq \mathcal{I}(z), \tag{16.9.5}$$

by (16.9.3). This means that

$$(y/z)\,\mathcal{I}_0 \subseteq k, \tag{16.9.6}$$

so that

$$y/z \in (k : \mathcal{I}_0)_{Q_k}. \tag{16.9.7}$$

However, $y/z \notin k$, because $y \notin \mathcal{I}(z)$. This implies (16.9.1), as desired.

16.10 Fractions and integral elements

Let k_1 be a commutative ring with a multiplicative identity element $1 = 1_{k_1}$, and let k be a subring of k_1 that contains 1_{k_1}. Also let S be a multiplicatively closed set in k, as in Section 12.1. Note that S may be considered as a multiplicatively closed set in k_1 as well. Thus $S^{-1}\,k$ and $S^{-1}\,k_1$ may be defined as in Section 12.1.

The obvious inclusion mapping from k into k_1 leads to a ring homomorphism from $S^{-1}\,k$ into $S^{-1}\,k_1$ in a natural way, as in Section 12.11. The induced homomorphism from $S^{-1}\,k$ into $S^{-1}\,k_1$ is injective, because the inclusion mapping from k into k_1 is injective, as before. Equivalently, if $x \in k$ and $r \in S$, then $x/r = 0$ in $S^{-1}\,k$ if and only if this holds in $S^{-1}\,k_1$, which can also be verified directly. Thus we may identify $S^{-1}\,k$ with a subring of $S^{-1}\,k_1$.

Suppose that $x \in k_1$ is integral over k, so that

(16.10.1) $$x^n + a_1\,x^{n-1} + \cdots + a_{n-1}\,x + a_n = 0$$

in k_1 for some positive integer n and $a_1, \ldots, a_n \in k$. If $r \in S$, then we get that

(16.10.2) $$(x/r)^n + (a_1/r)\,(x/r)^{n-1} + \cdots + (a_{n-1}/r^{n-1})\,(x/r) + (a_n/r^n) = 0$$

in $S^{-1}\,k$. This implies that

(16.10.3) $$x/r \in S^{-1}\,k_1 \text{ is integral over } S^{-1}\,k.$$

In particular, if k_1 is integral over k, then

(16.10.4) $$S^{-1}\,k_1 \text{ is integral over } S^{-1}\,k,$$

as in the second part of Proposition 5.6 on p61 of [1]. This also corresponds to part of the remark on p10 of [15].

Conversely, suppose that (16.10.3) holds for some $x \in k_1$ and $r \in S$. This means that

(16.10.5) $$(x/r)^n + (a_1/r_1)\,(x/r)^{n-1} + \cdots + (a_{n-1}/r_{n-1})\,(x/r) + (a_n/r_n) = 0$$

for some positiove integer n, $a_1, \ldots, a_n \in k$, and $r_1, \ldots, r_n \in S$. Put

(16.10.6) $$t = r_1\,r_2 \cdots r_n \in S,$$

and observe that

(16.10.7) $$y = x\,t \in k_1 \text{ is integral over } k.$$

Thus

(16.10.8) $$x/r = y/(r\,t),$$

with $r\,t \in S$.

Let C be the integral closure of k in k_1, as in Section 16.5, so that C is a subring of k_1 that contains k. We may identify $S^{-1}\,C$ with a subring of $S^{-1}\,k_1$ that contains $S^{-1}\,k$, as before. Using (16.10.8), we get that

(16.10.9) $$x/r \in S^{-1}\,C.$$

Note that $S^{-1}\,C$ is integral over $S^{-1}\,k$, as in (16.10.4). It follows that

(16.10.10) $$S^{-1}\,C \text{ is the integral closure of } S^{-1}\,k \text{ in } S^{-1}\,k_1.$$

This is Proposition 5.12 on p62 of [1]. This corresponds to the remark on p10 of [15] as well.

16.11 More on fractions, integral elements

Let k be an integral domain, let Q_k be the corresponding field of fractions, and let C be the integral closure of k in Q_k. Also let S be a multiplicatively closed set in k, with $0 \notin S$, so that $S^{-1}\,k$ may be identified with a subring of Q_k, as in Section 14.2. Similarly, $S^{-1}\,C$ may be identified with a subring of Q_k, which is the integral closure of $S^{-1}\,k$ in Q_k, as in (16.10.10). Of course, $S^{-1}\,Q_k$ may be identified with Q_k, and Q_k may be identified with the field of fractions of $S^{-1}\,k$.

In particular,

(16.11.1) if k is integrally closed, then $S^{-1}\,k$ is integrally closed.

This reduces to the fact that $S^{-1}\,C = S^{-1}\,k$ when $C = k$. This corresponds to part of the proof of Proposition 5.13 on p63 of [1].

Let ϕ be the obvious inclusion mapping from k into C, and let Φ be the obvious inclusion mapping from $S^{-1}\,k$ into $S^{-1}\,C$. These are ring homomorphisms, and one may consider Φ as corresponding to ϕ as in Section 12.11.

Let W be C, considered as a module over k. We can define $S^{-1}\,W$ as a module over $S^{-1}\,k$, as in Section 12.2. We may also consider $S^{-1}\,C$ as a module over $S^{-1}\,k$, because $S^{-1}\,k$ is a subring of $S^{-1}\,C$. This corresponds exactly to $S^{-1}\,W$, as a module over $S^{-1}\,k$, as in Section 12.11.

Similarly, let V be k, considered as a module over itself. We can define $S^{-1}\,V$ as a module over $S^{-1}\,k$ as in Section 12.2 again. This corresponds exactly to $S^{-1}\,k$ as a module over itself, as before.

Let f be the obvious inclusion mapping from V into W, as modules over k. This leads to a homomorphism $S^{-1}\,f$ from $S^{-1}\,V$ into $S^{-1}\,W$, as modules over $S^{-1}\,k$, as in Section 12.2. It is easy to see that $S^{-1}\,f$ corresponds exactly to Φ, with respect to the identifications of $S^{-1}\,V$ and $S^{-1}\,W$ with $S^{-1}\,k$ and $S^{-1}\,C$, respectively, mentioned in the previous two paragraphs.

If $\mathcal{I}$ is a proper prime ideal in k, then $S_{\mathcal{I}} = k \setminus \mathcal{I}$ is a multiplicatively closed subset of k that does not contain 0, as in Section 12.4. In particular, this holds when $\mathcal{I}$ is a proper maximal ideal in $\mathcal{I}$.

Suppose that

(16.11.2) $$S_{\mathcal{I}}^{-1}\,k \text{ is integrally closed}$$

for every maximal proper ideal $\mathcal{I}$ in k. This is the same as saying that

(16.11.3) $$S_{\mathcal{I}}^{-1}\,k = S_{\mathcal{I}}^{-1}\,C$$

for every maximal proper ideal $\mathcal{I}$ in k, because $S_{\mathcal{I}}^{-1}\,C$ corresponds to the integral closure of $S_{\mathcal{I}}^{-1}\,k$ in Q_k, as before. Equivalently, this means that $S_{\mathcal{I}}^{-1}\,f$ maps $S_{\mathcal{I}}^{-1}\,V$ onto $S_{\mathcal{I}}^{-1}\,W$ for every maximal proper ideal $\mathcal{I}$ in k, by the earlier remarks.

Under these conditions, f maps V onto W, as in Section 12.4. This means that $k = C$, so that k is integrally closed. This is part of Proposition 5.13 on p63 of [1].

Essentially the same point is addressed in the proof that (i) implies (ii) in Proposition 4 on p10 of [15]. Suppose that (16.11.2) holds for every maximal proper ideal $\mathcal{I}$ in k again, and let $w \in C$ be given. This implies that

$$w \in S_{\mathcal{I}}^{-1} C = S_{\mathcal{I}}^{-1} k \tag{16.11.4}$$

for every maximal proper ideal $\mathcal{I}$ in k.

Observe that

$$\{x \in k : x\, w \in k\} \tag{16.11.5}$$

is an ideal in k. If $\mathcal{I}$ is a maximal proper ideal in k, then (16.11.4) implies that there is an element of $k \setminus \mathcal{I}$ in (16.11.5). This means that (16.11.5) is not contained in $\mathcal{I}$. It follows that (16.11.5) is equal to k, so that $w \in k$.

16.12 Dedekind domains and integral elements

Let k be a Noetherian integral domain. If $\mathcal{I}$ is a proper prime ideal in k, then $S_{\mathcal{I}} = k \setminus \{0\}$ is a multiplicatively closed set in k that does not contain 0, as before. Remember that k is a Dedekind domain if for every maximal proper ideal $\mathcal{I} \neq \{0\}$ in k,

$$k_{\mathcal{I}} = S_{\mathcal{I}}^{-1} k \text{ is a discrete valuation ring,} \tag{16.12.1}$$

as in Section 15.11. This implies that k has dimension less than or equal to one, in the sense of Section 14.8, as before. More precisely, the formulation on p95 of [1] also asks that k not be a field, which would imply that k have dimension equal to 0.

Note that (16.12.1) implies that

$$k_{\mathcal{I}} \text{ is integrally closed,} \tag{16.12.2}$$

as in Section 16.3. Of course, $\mathcal{I} = \{0\}$ is a maximal proper ideal in k if and only if k is a field, in which case k is obviously integrally closed. If k is a Dedekind domain, then it follows that k is integrally closed, as in the previous section. This corresponds to parts of Theorem 9.3 on p95 of [1], and Proposition 4 on p10 of [15].

Conversely, suppose that k has dimension less than or equal to one, and that k is integrally closed. Let $\mathcal{I} \neq \{0\}$ be a maximal proper ideal in k, and observe that (16.12.2) holds, as in (16.11.1). We also have that $k_{\mathcal{I}}$ is a Noetherian integral domain, as in Sections 14.2 and 14.6. Remember that $k_{\mathcal{I}}$ is a local ring too, as in Section 12.4. It is easy to see that the unique maximal proper ideal in $k_{\mathcal{I}}$ is nonzero, because $\mathcal{I} \neq \{0\}$.

Under these conditions, k is not a field, because $\mathcal{I} \neq \{0\}$, so that k has dimension one. Remember that there is a one-to-one correspondence between the set of prime ideals in k that are contained in $\mathcal{I}$ and the set of proper prime ideals in $k_{\mathcal{I}}$, as mentioned at the end of Section 12.13. It follows that $k_{\mathcal{I}}$ has dimension one as well.

Observe that (16.12.2) holds, because k is integrally closed, by hypothesis, as in (16.11.1). It follows that (16.12.1) holds, as in Section 16.6. This means that k is a Dedekind domain, as in Section 15.11. This corresponds to other parts of Theorem 9.3 on p95 of [1], and Proposition 4 on p10 of [15].

16.13 Fields, quotients, and integral elements

Let k_1, k_2 be commutative rings with multiplicative identity elements 1_{k_1}, 1_{k_2}, respectively, and let k be a subring of k_1 that contains 1_{k_1}. Also let ϕ be a ring homomorphism from k_1 onto k_2, so that $\phi(1_{k_1}) = 1_{k_2}$, and $\phi(k)$ is a subring of k_2 that contains 1_{k_2}. If $x \in k_1$ is integral over k, then it is easy to see that

(16.13.1) $$\phi(x) \text{ is integral over } \phi(k) \text{ in } k_2.$$

If k_1 is integral over k, then it follows that

(16.13.2) $$k_2 \text{ is integral over } \phi(k).$$

This corresponds to the first part of Proposition 5.6 on p61 of [1].

Suppose now that k_1 is an integral domain, and that k is a subring of k that contains 1_{k_1} again. If k_1 is integral over k, then

(16.13.3) $$k \text{ is a field if and only if } k_1 \text{ is a field.}$$

This is Proposition 5.7 on p61 of [1].

Suppose first that k is a field, and let $y \in k_1$ be given, with $y \neq 0$. Because y is integral over k, by hypothesis, we have that

(16.13.4) $$y^n + a_1\, y^{n-1} + \cdots + a_{n-1}\, y + a_n = 0$$

for some positive integer n and $a_1, \ldots, a_n \in k$. This implies that

(16.13.5) $$y\,(y^{n-1} + a_1\, y^{n-2} + \cdots + a_{n-1}) = -a_n.$$

We may as well take n to be as small as possible, so that

(16.13.6) $$y^{n-1} + a_1\, y^{n-2} + \cdots + a_{n-1} \neq 0.$$

This means that $a_n \neq 0$, because k_1 is an integral domain, by hypothesis. It follows that a_n is invertible in k, and thus in k_1, because k is a field. This implies that y is invertible in k_1, with

(16.13.7) $$y^{-1} = -a_n^{-1}\,(y^{n-1} + a_1\, y^{n-2} + \cdots + a_{n-1}),$$

as desired.

Conversely, suppose that k_1 is a field, and let $x \in k$ be given, with $x \neq 0$. Under these conditions, x is invertible in k_1, and x^{-1} is integral over k. This means that

(16.13.8) $$x^{-m} + c_1\, x^{1-m} + \cdots + c_{m-1}\, x^{-1} + c_m = 0$$

for some positive integer m and $c_1, \ldots, c_m \in k$. We can multiply both side by x^{m-1}, to get that

$$x^{-1} = -c_1 - \cdots - c_{m-1}\, x^{m-2} - c_m\, x^{m-1}. \tag{16.13.9}$$

This implies that $x^{-1} \in k$, as desired.

16.14 Two corollaries, and another argument

Let k_1 be a commutative ring with a multiplicative identity element $1 = 1_{k_1}$, let k be a subring of k that contains 1_{k_1}, and suppose that k_1 is integral over k. Suppose for the moment that $\mathcal{I}_1$ is a proper prime ideal in k_1, and note that

$$\mathcal{I} = \mathcal{I}_1 \cap k \tag{16.14.1}$$

is a proper prime ideal in k. Under these conditions,

$$\begin{aligned} &\mathcal{I}_1 \text{ is a maximal proper ideal in } k_1 \text{ if and only if} \\ &\mathcal{I} \text{ is a maximal proper ideal in } k, \end{aligned} \tag{16.14.2}$$

as in Corollary 5.8 on p61 of [1].

To see this, put $k_2 = k_1/\mathcal{I}_1$, and let ϕ be the natural quotient homomorphism from k_1 onto k_2. Of course, $\phi(k)$ may be identified with $k/\mathcal{I}$, and k_2 is an integral domain. We also have that k_2 is integral over $\phi(k)$, because k_1 is integral over k, as in (16.13.2). It follows that k_2 is a field if and only if $\phi(k)$ is a field, as in (16.13.3). This is equivalent to (16.14.2), as desired.

Suppose now that $\mathcal{I}_1$, $\mathcal{I}_1'$ are proper prime ideals in k_1 such that

$$\mathcal{I}_1 \subseteq \mathcal{I}_1'. \tag{16.14.3}$$

If

$$\mathcal{I}_1 \cap k = \mathcal{I}_1' \cap k, \tag{16.14.4}$$

then

$$\mathcal{I}_1 = \mathcal{I}_1'. \tag{16.14.5}$$

This is Corollary 5.9 on p61 of [1], which corresponds to Lemma 2 on p14 of [15].

Let us begin with the proof in [1]. Let $\mathcal{I}$ be as in (16.14.4), which is a proper prime ideal in k. Thus $S = k \setminus \mathcal{I}$ is a multiplicatively closed set in k, as in Section 12.4. We may also consider S as a multiplicatively closed set in k_1, so that $S^{-1}\, k$ and $S^{-1}\, k_1$ may be defined as commutative rings as in Section 12.1. We may identify $S^{-1}\, k$ with a subring of $S^{-1}\, k_1$, as in Section 16.10.

Remember that $S^{-1}\, \mathcal{I}$ may be considered as an ideal in $S^{-1}\, k$, and similarly $S^{-1}\, \mathcal{I}_1$, $S^{-1}\, \mathcal{I}_1'$ may be considered as ideals in $S^{-1}\, k_1$, as in Section 12.13. We have seen that $S^{-1}\, \mathcal{I}$ is the same as the extension of $\mathcal{I}$ with respect to the natural ring homomorphism $x \mapsto x/1$ from k into $S^{-1}\, k$, and similarly $S^{-1}\, \mathcal{I}_1$, $S^{-1}\, \mathcal{I}_1'$ are the same as the extensions of $\mathcal{I}_1$, $\mathcal{I}_1'$ with respect to the natural ring homomorphism from k_1 in $S^{-1}\,k_1$, respectively. We also have that $S^{-1}\,k$ is a local ring, with $S^{-1}\,\mathcal{I}$ as its unique maximal proper ideal, as in Section 12.4.

Of course,

$$S^{-1}\,\mathcal{I}_1 \subseteq S^{-1}\,\mathcal{I}_1', \tag{16.14.6}$$

because of (16.14.3). One can check that

$$(S^{-1}\,\mathcal{I}_1) \cap (S^{-1}\,k) = S^{-1}\,\mathcal{I}, \tag{16.14.7}$$

because $\mathcal{I} = \mathcal{I}_1 \cap k$, by construction. More precisely, it is easy to see that

$$S^{-1}\,\mathcal{I} \subseteq (S^{-1}\,\mathcal{I}_1) \cap (S^{-1}\,k), \tag{16.14.8}$$

and one can verify that

$$(S^{-1}\,\mathcal{I}_1) \cap (S^{-1}\,k) \subseteq S^{-1}\,\mathcal{I} \tag{16.14.9}$$

too. Similarly,

$$(S^{-1}\,\mathcal{I}_1') \cap (S^{-1}\,k) = S^{-1}\,\mathcal{I}. \tag{16.14.10}$$

Note that $\mathcal{I}_1$ and $\mathcal{I}_1'$ are disjoint from S, because $\mathcal{I}$ is as in (16.14.4). This implies that $S^{-1}\,\mathcal{I}_1, S^{-1}\,\mathcal{I}_1' \neq S^{-1}\,k_1$, as in Section 12.13. In fact, $S^{-1}\,\mathcal{I}_1$ and $S^{-1}\,\mathcal{I}_1'$ are proper prime ideals in $S^{-1}\,k_1$, as before.

We also have that $S^{-1}\,k_1$ is integral over $S^{-1}\,k$, because k_1 is integral over k, by hypothesis, as in Section 16.10. It follows that $S^{-1}\,\mathcal{I}_1$, $S^{-1}\,\mathcal{I}_1'$ are maximal proper ideals in $S^{-1}\,k_1$, because $S^{-1}\,\mathcal{I}$ is a maximal proper ideal in $S^{-1}\,k$, as in (16.14.2). This implies that

$$S^{-1}\,\mathcal{I}_1 = S^{-1}\,\mathcal{I}_1', \tag{16.14.11}$$

by (16.14.6). We can get (16.14.5) from (16.14.11), because $\mathcal{I}_1$ and $\mathcal{I}_1'$ are prime ideals in k_1 that are disjoint from S, as in Section 12.13.

Alternatively, let ϕ be the natural quotient homomorphism from k_1 onto $k_2 = k_1/\mathcal{I}_1$ again. Thus k_2 is an integral domain that it integral over $\phi(k)$, as before. Suppose for the sake of a contradiction that (16.14.5) does not hold, and let x be an element of $\mathcal{I}_1'$ that is not in $\mathcal{I}_1$, so that $\phi(x) \neq 0$. Note that $\phi(x)$ is integral over $\phi(x)$, which means that

$$\phi(x)^n + a_1\,\phi(x)^{n-1} + \cdots + a_{n-1}\,\phi(x) + a_n = 0 \tag{16.14.12}$$

for some positive integer n and $a_1, \ldots, a_n \in \phi(k)$. If we take n to be as small as possible, then we get that $a_n \neq 0$, because k_2 is an integral domain, as in the previous section.

Of course, $\phi(\mathcal{I}_1')$ is an ideal in k_2. It is easy to see that $a_n \in \phi(\mathcal{I}_1')$, because of (16.14.12). Thus there are $y \in k$ and $z \in \mathcal{I}_1'$ such that

$$a_n = \phi(y) = \phi(z). \tag{16.14.13}$$

In particular, $\phi(y - z) = 0$, so that $y - z \in \mathcal{I}_1$. It follows that $y \in \mathcal{I}_1'$, because of (16.14.3). This means that $y \in \mathcal{I}_1' \cap k$, so that $y \in \mathcal{I}_1 \cap k$, by (16.14.4). This implies that $\phi(y) = 0$, contradicting the fact that $a_n \neq 0$. This corresponds to the proof of Lemma 2 on p14 of [15].

16.15 Getting prime ideals in k_1

Let k_1 be a commutative ring with a multiplicative identity element $1 = 1_{k_1}$ again, let k be a subring of k that contains 1_{k_1}, and suppose that k_1 is integral over k. If $\mathcal{I}$ is a proper prime ideal in k, then there is a proper prime ideal $\mathcal{I}_1$ in k_1 such that

$$\mathcal{I}_1 \cap k = \mathcal{I}. \tag{16.15.1}$$

This is Theorem 5.10 on p62 of [1].

As in the previous section, $S = k \setminus \mathcal{I}$ is a multiplicatively closed set in k, which may be considered as a multiplicatively closed set in k_1 as well. Thus $S^{-1}\,k$ and $S^{-1}\,k_1$ may be defined as commutative rings, and we may identify $S^{-1}k$ with a subring of $S^{-1}\,k_1$, as before. We also have that $S^{-1}\,k_1$ is integral over $S^{-1}\,k$, because k_1 is integral over k, as in Section 16.10.

Let α, α_1 be the natural ring homomorphisms from k, k_1 into $S^{-1}\,k$, $S^{-1}\,k_1$, respectively. The composition of α with the inclusion mapping from $S^{-1}\,k$ into $S^{-1}\,k_1$ is equal to the composition of the obvious inclusion mapping from k into k_1 with α_1, as on p62 of [1]. This corresponds to some remarks in Sections 12.11 and 16.10.

Let $\mathcal{I}_2$ be a maximal proper ideal in $S^{-1}\,k_1$. Observe that

$$\mathcal{I}_2 \cap (S^{-1}\,k) \tag{16.15.2}$$

is a maximal proper ideal in $S^{-1}\,k$, as in (16.14.2). Remember that $S^{-1}\,k$ is a local ring, with $S^{-1}\,\mathcal{I}$ as its unique maximal proper ideal, as in Section 12.4. Thus

$$\mathcal{I}_2 \cap (S^{-1}\,k) = S^{-1}\,\mathcal{I}. \tag{16.15.3}$$

We also have that

$$\mathcal{I}_1 = \alpha_1^{-1}(\mathcal{I}_2) \tag{16.15.4}$$

is a proper prime ideal in k_1 because $\mathcal{I}_2$ is a proper prime ideal in $S^{-1}\,k_1$, as in Section 12.13. More precisely,

$$\mathcal{I}_1 \cap S = \emptyset, \tag{16.15.5}$$

as before. This means that $\mathcal{I}_1 \cap k$ is disjoint from S, so that

$$\mathcal{I}_1 \cap k \subseteq \mathcal{I}. \tag{16.15.6}$$

In fact,

$$\mathcal{I}_1 \cap k = \alpha_1^{-1}(\mathcal{I}_2) \cap k = \alpha^{-1}(\mathcal{I}_2 \cap (S^{-1}\,k)) = \alpha^{-1}(S^{-1}\,\mathcal{I}), \tag{16.15.7}$$

using the earlier remarks about α and α_1 in the second step, and (16.15.3) in the third step. This implies that $\mathcal{I} \subseteq \mathcal{I}_1 \cap k$, so that (16.15.1) holds.

Suppose now that

$$\mathcal{I}_{k,1} \subseteq \mathcal{I}_{k,2} \subseteq \cdots \subseteq \mathcal{I}_{k,n} \tag{16.15.8}$$

is a chain of proper prime ideals in k for some positive integer n,

$$\mathcal{I}_{k_1,1} \subseteq \mathcal{I}_{k_1,2} \subseteq \cdots \subseteq \mathcal{I}_{k_1,m} \tag{16.15.9}$$

is a chain of proper prime ideals in k_1 for some positive integer $m < n$, and

$$\mathcal{I}_{k_1,j} \cap k = \mathcal{I}_{k,j} \tag{16.15.10}$$

for each $j = 1, \ldots, m$. Under these conditions, (16.15.9) can be extended to a chain

$$\mathcal{I}_{k_1,1} \subseteq \mathcal{I}_{k_1,2} \subseteq \cdots \subseteq \mathcal{I}_{k_1,n} \tag{16.15.11}$$

of proper prime ideals in k_1 that satisfies (16.15.10) for each $j = 1, \ldots, n$. This is Theorem 5.11 on p62 of [1], which is called the "going-up theorem".

To see this, we can use induction to reduce to the case where $m = 1$ and $n = 2$. Put

$$k_2 = k_1/\mathcal{I}_{k_1,1}, \tag{16.15.12}$$

and let ϕ be the natural quotient homomorphism from k_1 onto k_2. Note that $\phi(k)$ is isomorphic to $k/\mathcal{I}_{k,1}$ as a ring, and that k_2 is integral over $\phi(k)$, because k_1 is integral over k, as in Section 16.13. It is easy to see that $\phi(\mathcal{I}_{k,2})$ is a proper prime ideal in $\phi(k)$, because $\mathcal{I}_{k,2}$ is a proper prime ideal in k, and $\mathcal{I}_{k,1}$ is contained in $\mathcal{I}_{k,2}$.

It follows that there is a proper prime ideal $\mathcal{I}_{k_2,2}$ in k_2 such that

$$\mathcal{I}_{k_2,2} \cap \phi(k) = \phi(\mathcal{I}_{k,2}), \tag{16.15.13}$$

as in (16.15.1). Put

$$\mathcal{I}_{k_1,2} = \phi^{-1}(\mathcal{I}_{k_2,2}), \tag{16.15.14}$$

which is a proper prime ideal in k_1 that contains $\mathcal{I}_{k_1,1}$. We also have that

$$\begin{aligned} \mathcal{I}_{k_1,2} \cap k = \phi^{-1}(\mathcal{I}_{k_2,2}) \cap k &= \phi^{-1}(\mathcal{I}_{k_2,2} \cap \phi(k)) \cap k \\ &= \phi^{-1}(\phi(\mathcal{I}_{k,2})) \cap k = \mathcal{I}_{k,2}, \end{aligned} \tag{16.15.15}$$

using the fact that $\mathcal{I}_{k,1}$ is contained in $\mathcal{I}_{k,2}$ in the last step.

Part V

Rings, modules, and field extensions

Chapter 17

Some properties of ideals

17.1 Sums, products, and intersections

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ be ideals in k. Of course,

$$\mathcal{I}_1 + \mathcal{I}_2 = \{x_1 + x_2 : x_1 \in \mathcal{I}_1,\, x_2 \in \mathcal{I}_2\} \tag{17.1.1}$$

is an ideal in k as well.

Remember that the product $\mathcal{I}_1\,\mathcal{I}_2$ is the ideal in k generated by products of elements of $\mathcal{I}_1$ and $\mathcal{I}_2$, as in Section 12.8. It is easy to see that the distributive law

$$\mathcal{I}_1\,(\mathcal{I}_2 + \mathcal{I}_3) = \mathcal{I}_1\,\mathcal{I}_2 + \mathcal{I}_1\,\mathcal{I}_3 \tag{17.1.2}$$

holds, as on p6 of [1].

Note that

$$\mathcal{I}_1 \cap \mathcal{I}_2 + \mathcal{I}_1 \cap \mathcal{I}_3 \subseteq \mathcal{I}_1 \cap (\mathcal{I}_2 + \mathcal{I}_3). \tag{17.1.3}$$

If

$$\mathcal{I}_2 \subseteq \mathcal{I}_1 \text{ or } \mathcal{I}_3 \subseteq \mathcal{I}_1, \tag{17.1.4}$$

then one can check that

$$\mathcal{I}_1 \cap (\mathcal{I}_2 + \mathcal{I}_3) = \mathcal{I}_1 \cap \mathcal{I}_2 + \mathcal{I}_1 \cap \mathcal{I}_3. \tag{17.1.5}$$

This is the *modular law* mentioned on p6 of [1]. This corresponds to the modular identity in a lattice, as in Exercise 10 on p376 of [2], and L5 on p479 of [12].

We also have that

$$(\mathcal{I}_1 + \mathcal{I}_2)\,(\mathcal{I}_1 \cap \mathcal{I}_2) = \mathcal{I}_1\,(\mathcal{I}_1 \cap \mathcal{I}_2) + \mathcal{I}_2\,(\mathcal{I}_1 \cap \mathcal{I}_2) \subseteq \mathcal{I}_1\,\mathcal{I}_2, \tag{17.1.6}$$

as on p6 of [1]. Remember that $\mathcal{I}_1\,\mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2$, as in Section 12.8. If

$$\mathcal{I}_1 + \mathcal{I}_2 = k, \tag{17.1.7}$$

then we get that

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I}_1 \, \mathcal{I}_2, \tag{17.1.8}$$

as on p6 of [1].

If (17.1.7) holds, then $\mathcal{I}_1$ and $\mathcal{I}_2$ are said to be *coprime* or *comaximal* as ideals in k, as on p7 of [1]. Equivalently, this means that there are $x_1 \in \mathcal{I}_1$ and $x_2 \in \mathcal{I}_2$ such that

$$x_1 + x_2 = 1. \tag{17.1.9}$$

Remember that the radical $r(\mathcal{I})$ of an ideal $\mathcal{I}$ in k is defined in Section 12.10, and that $\mathcal{I} \subseteq r(\mathcal{I})$ automatically. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are coprime in k, then it follows that their radicals $r(\mathcal{I}_1)$ and $r(\mathcal{I}_2)$ are coprime in k as well.

Conversely, if $r(\mathcal{I}_1)$ and $r(\mathcal{I}_2)$ are coprime in k, then $\mathcal{I}_1$ and $\mathcal{I}_2$ are coprime in k, as in Proposition 1.16 on p9 of [1]. To see this, remember that

$$r(\mathcal{I}_1 + \mathcal{I}_2) = r(r(\mathcal{I}_1) + r(\mathcal{I}_2)), \tag{17.1.10}$$

as in Section 12.10. If $r(\mathcal{I}_1)$ and $r(\mathcal{I}_2)$ are coprime in k, then we get that

$$r(\mathcal{I}_1 + \mathcal{I}_2) = k. \tag{17.1.11}$$

This implies (17.1.7), as in Section 12.10.

17.2 Finitely many ideals

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be finitely many ideals in k. If

$$\mathcal{I}_j \text{ and } \mathcal{I}_l \text{ are coprime when } j \neq l, \tag{17.2.1}$$

then

$$\prod_{j=1}^{n} \mathcal{I}_j = \bigcap_{j=1}^{n} \mathcal{I}_j, \tag{17.2.2}$$

as in the first part of Proposition 1.10 on p7 of [1]. Of course, then $n = 2$ case corresponds to (17.1.8).

Suppose that $n > 2$, and that the analogous statement holds for $n - 1$. Put

$$\mathcal{I} = \prod_{j=1}^{n-1} \mathcal{I}_j = \bigcap_{j=1}^{n-1} \mathcal{I}_j. \tag{17.2.3}$$

Because

$$\mathcal{I}_j + \mathcal{I}_n = k \tag{17.2.4}$$

for each $j = 1, \ldots, n - 1$, by (17.2.1), there are $x_j \in \mathcal{I}_j$ and $y_j \in \mathcal{I}_n$ for each $j = 1, \ldots, n - 1$ such that

$$x_j + y_j = 1. \tag{17.2.5}$$

This implies that

$$\prod_{j=1}^{n-1} x_j = \prod_{j=1}^{n-1} (1 - y_j) = 1 + \text{ an element of } \mathcal{I}_n. \tag{17.2.6}$$

Note that the left side is an element of $\mathcal{I}$.

It follows that

$$\mathcal{I} + \mathcal{I}_n = k, \tag{17.2.7}$$

as in the previous section. Thus

$$\prod_{j=1}^{n} \mathcal{I}_j = \mathcal{I}\,\mathcal{I}_n = \mathcal{I} \cap \mathcal{I}_n = \bigcap_{j=1}^{n} \mathcal{I}_j, \tag{17.2.8}$$

using the $n = 2$ case in the second step.

Let q_j be the natural homomorphism from k onto the quotient ring $k/\mathcal{I}_j$ for each $j = 1, \ldots, n$. Using these homomorphisms, we get a ring homomorphism ϕ from k into the Cartesian product

$$\prod_{j=1}^{n} (k/\mathcal{I}_j). \tag{17.2.9}$$

The second part of Proposition 1.10 on p7 of [1] says that

$$\phi \text{ is surjective} \tag{17.2.10}$$

if and only if (17.2.1) holds.

If (17.2.10) holds, then for each $j = 1, \ldots, n$ there is an $x_j \in k$ such that $q_j(x_j) = 1$ in $k/\mathcal{I}_j$, and $q_l(x_j) = 0$ in $k/\mathcal{I}_l$ when $j \neq l$. This means that $1 - x_j \in \mathcal{I}_j$, and $x_l \in \mathcal{I}_l$ when $j \neq l$. It follows that

$$1 \in \mathcal{I}_j + \mathcal{I}_l \tag{17.2.11}$$

when $j \neq l$, so that (17.2.1) holds.

Conversely, suppose that (17.2.1) holds. If $j \neq l$, then there are $u_{j,l} \in \mathcal{I}_j$ and $v_{j,l} \in \mathcal{I}_l$ such that

$$u_{j,l} + v_{j,l} = 1, \tag{17.2.12}$$

because $\mathcal{I}_j + \mathcal{I}_l = k$. Let x_j be the product of $v_{j,l}$ over $l = 1, \ldots, n$ with $j \neq l$. This is the same as the product of $(1 - u_{j,l})$ over $l = 1, \ldots, n$ with $j \neq l$. It follows that

$$x_j = 1 + \text{ an element of } \mathcal{I}_j \tag{17.2.13}$$

for each $j = 1, \ldots, n$, so that $q_j(x_j) = 1$ in $k/\mathcal{I}_j$. We also have that $x_j \in \mathcal{I}_l$ when $j \neq l$, so that $q_l(x_j) = 0$. One can use this to get that (17.2.10) holds, as desired.

Of course,

$$\ker \phi = \bigcap_{j=1}^{n} \mathcal{I}_n, \tag{17.2.14}$$

by construction. Thus ϕ is injective if and only if $\bigcap_{j=1}^{n} \mathcal{I}_j = \{0\}$, as in the third part of Proposition 1.10 on p7 of [1].

17.3 Finite unions and intersections

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be finitely many prime ideals in k. If $\mathcal{I}$ is an ideal in k such that

$$\mathcal{I} \subseteq \bigcup_{j=1}^{n} \mathcal{I}_j, \tag{17.3.1}$$

then

$$\mathcal{I} \subseteq \mathcal{I}_j \tag{17.3.2}$$

for some j. This is the first part of Proposition 1.11 on p8 of [1].

Equivalently, suppose that

$$\mathcal{I} \not\subseteq \mathcal{I}_j \tag{17.3.3}$$

for each $j = 1, \ldots, n$. Under these conditions, we would like to show that

$$\mathcal{I} \not\subseteq \bigcup_{j=1}^{n} \mathcal{I}_j. \tag{17.3.4}$$

Of course, this is clear when $n = 1$. Suppose now that $n > 1$, and that the analogous statement holds for $n - 1$.

Using the induction hypothesis, we get that for each $l = 1, \ldots, n$ there is an $x_l \in \mathcal{I}$ such that

$$x_l \not\in \mathcal{I}_j \tag{17.3.5}$$

when $j \neq l$. If $x_l \not\in \mathcal{I}_l$ for some l, then (17.3.3) holds, as desired. Otherwise, we have that

$$x_l \in \mathcal{I}_l \tag{17.3.6}$$

for each $l = 1, \ldots, n$.

Put

$$y_l = x_1\, x_2\, \cdots\, x_{l-2}\, x_{l-1}\, x_{l+1}\, x_{l+2} \cdots x_n, \tag{17.3.7}$$

for each $l = 1, \ldots, n$, which is to say the product of the x_j's, $1 \leq j \leq n$, except for $j = l$. Observe that $y_l \in \mathcal{I}$, and that $y_l \in \mathcal{I}_j$ when $j \neq l$, because of (17.3.6). We also have that $y_l \not\in \mathcal{I}_l$, because of (17.3.5), and because $\mathcal{I}_l$ is a prime ideal.

It follows that

$$y = \sum_{l=1}^{n} y_l \tag{17.3.8}$$

is an element of $\mathcal{I}$ that is not in $\mathcal{I}_r$ for any $r = 1, \ldots, n$. This implies (17.3.4), as desired.

Now let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be finitely many ideals in k, and let $\mathcal{I}$ be a prime ideal in k. If

$$\bigcap_{j=1}^{n} \mathcal{I}_j \subseteq \mathcal{I}, \tag{17.3.9}$$

then

$$\mathcal{I}_l \subseteq \mathcal{I} \tag{17.3.10}$$

for some l. In particular, if

$$\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j, \tag{17.3.11}$$

then

$$\mathcal{I} = \mathcal{I}_l \tag{17.3.12}$$

for some l. This is the second part of Proposition 1.11 on p8 of [1].

To see this, suppose that $\mathcal{I}_l \not\subseteq \mathcal{I}$ for any l. Thus, for each $l = 1, \ldots, n$, there is an $x_l \in \mathcal{I}_l$ such that $x_l \notin \mathcal{I}$. This implies that

$$\prod_{l=1}^{n} x_l \in \prod_{l=1}^{n} \mathcal{I}_l \subseteq \bigcap_{j=1}^{n} \mathcal{I}_j, \tag{17.3.13}$$

and that

$$\prod_{l=1}^{n} x_l \notin \mathcal{I}, \tag{17.3.14}$$

because $\mathcal{I}$ is a prime ideal. It follows that (17.3.9) does not hold.

If (17.3.11) holds, then (17.3.10) implies (17.3.12), because $\mathcal{I} \subseteq \mathcal{I}_l$ automatically.

17.4 Irreducible ideals

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}$ be an ideal in k. Suppose that if $\mathcal{I}_1$, $\mathcal{I}_2$ are ideals in k such that

$$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2, \tag{17.4.1}$$

then

$$\mathcal{I} = \mathcal{I}_1 \text{ or } \mathcal{I}_2. \tag{17.4.2}$$

Under these conditions, $\mathcal{I}$ is said to be *irreducible* as an ideal in k. Note that prime ideals in k are irreducible, as in the previous section.

If k is Noetherian, then

(17.4.3) every ideal in k may be expressed as the intersection of finitely many irreducible ideals in k,

as in Lemma 7.11 on p83 of [1]. Indeed, suppose for the sake of a contradiction that there are ideals in k that cannot be expressed as the intersection of finitely many irreducible ideals. Because k is Noetherian, there is an ideal $\mathcal{I}$ in k that cannot be expressed as the intersection of finitely many irreducible ideals in k, and which is maximal with respect to this property. In particular, $\mathcal{I}$ is reducible in k, in the sense that there are ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ in k such that (17.4.1) holds, and (17.4.2) does not hold. It follows that $\mathcal{I}$ is a proper subset of each of $\mathcal{I}_1$ and $\mathcal{I}_2$.

The maximality of $\mathcal{I}$ implies that each of $\mathcal{I}_1$ and $\mathcal{I}_2$ may be expressed as the intersection of finitely many irreducible ideals in k. This means that $\mathcal{I}$ may be expressed as the intersection of finitely many irreducible ideals in k, because of (17.4.1). This contradicts the hypothesis that $\mathcal{I}$ cannot be expressed in this way, as desired.

If $\mathcal{I}$ is an irreducible ideal in k, $\mathcal{I} \neq k$, and k is Noetherian, then

$$\mathcal{I} \text{ is a primary ideal in } k, \tag{17.4.4}$$

as in Lemma 7.12 on p83 of [1]. Remember that $\mathcal{I}$ is a primary ideal in k if and only if $k/\mathcal{I} \neq \{0\}$, and every zero-divisor in $k/\mathcal{I}$ is nilpotent, as in Section 14.5. It is easy to see that $\mathcal{I}$ is irreducible in k if and only if $\{0\}$ is irreducible in $k/\mathcal{I}$.

If k is Noetherian, then $k/\mathcal{I}$ is Noetherian too, as in Proposition 6.6 on p76 of [1]. More precisely, if k is Noetherian as a commutative ring, then $k/\mathcal{I}$ is Noetherian as a module over k, as in Section 9.7. It is easy to see that this implies that $k/\mathcal{I}$ is Noetherian as a commutative ring, because any ideal in $k/\mathcal{I}$ may be considered as a submodule of $k/\mathcal{I}$, as a module over k.

This permits us to reduce the earlier statement to the case where $\mathcal{I} = \{0\}$, as in [1]. Thus we would like to show that if $\{0\}$ is irreducible in k, $k \neq \{0\}$, and k is Noetherian, then every zero-divisor in k is nilpotent.

Let $x \in k$ be a zero-divisor, so that $x\,y = 0$ for some $y \in k$ with $y \neq 0$. If $w \in k$, then let $\mathcal{I}(w)$ be the ideal in k generated by w. Remember that the annihilator $\mathrm{Ann}(\mathcal{I}(w))$ of $\mathcal{I}(w)$ in k as defined in Section 12.8, and is an ideal in k. Note that

$$\mathrm{Ann}(\mathcal{I}(w)) = \{t \in k : t\,w = 0\} \tag{17.4.5}$$

for each $\in k$. It is easy to see that

$$\mathrm{Ann}(\mathcal{I}(x^n)) \subseteq \mathrm{Ann}(\mathcal{I}(x^{n+1})) \tag{17.4.6}$$

for every $n \geq 1$.

If k is Noetherian, then there is a positive integer n_0 such that

$$\mathrm{Ann}(\mathcal{I}(x^n)) = \mathrm{Ann}(\mathcal{I}(x^{n_0})) \tag{17.4.7}$$

for every $n \geq n_0$. Let us check that

$$\mathcal{I}(x^{n_0}) \cap \mathcal{I}(y) = \{0\} \tag{17.4.8}$$

in this case. If $z \in \mathcal{I}(y)$, then $x\,z = 0$, because $x\,y = 0$, by hypothesis. Suppose that $z \in \mathcal{I}(x^{n_0})$ too, so that $z = t\,x^{n_0}$ for some $t \in k$. Under these conditions, we get that

$$t\,x^{n_0+1} = x\,z = 0, \tag{17.4.9}$$

so that $t \in \mathrm{Ann}(\mathcal{I}(x^{n_0+1}))$. This implies that $t \in \mathrm{Ann}(\mathcal{I}(x^{n_0}))$, by (17.4.7). It follows that $z = t\,x^{n_0} = 0$, so that (17.4.8) holds.

If $\{0\}$ is irreducible as an ideal in k, then (17.4.8) implies that

$$\mathcal{I}(x^{n_0}) = \{0\}, \tag{17.4.10}$$

because $y \neq 0$, by hypothesis. This means that $x^{n_0} = 0$, so that x is nilpotent, as desired.

17.5 More on primary ideals

Let k be a commutative ring with a multiplicative identity element. Remember that an ideal $\mathcal{I} \neq k$ in k is said to be primary if for every $x, y \in k$ with $x\,y \in \mathcal{I}$, we have that $x \in \mathcal{I}$, or $y^n \in \mathcal{I}$ for some positive integer n, as in Section 14.5. The radical $r(\mathcal{I})$ of $\mathcal{I}$ in k was defined in Section 12.10, and we have seen that $r(\mathcal{I}) = k$ if and only if $\mathcal{I} = k$. If $\mathcal{I}$ is a primary ideal in k, then $r(\mathcal{I})$ is a proper prime ideal in k, as in Section 14.5. In this case, if $\mathcal{I}_0 = r(\mathcal{I})$, then $\mathcal{I}$ is said to be $\mathcal{I}_0$-primary, as before.

Let $\mathcal{I}_0$ be a proper prime ideal in k, and let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be finitely many $\mathcal{I}_0$-primary ideals in k. Under these conditions,

$$\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j \text{ is } \mathcal{I}_0\text{-primary in } k, \tag{17.5.1}$$

as in Lemma 4.3 on p51 of [1]. To see this, observe first that

$$r(\mathcal{I}) = \bigcap_{j=1}^{n} r(\mathcal{I}_j) = \mathcal{I}_0, \tag{17.5.2}$$

where the first step is as in Section 12.10.

Suppose that $x, y \in k$, $x\,y \in \mathcal{I}$, and $y \notin \mathcal{I}$. This implies that $y \notin \mathcal{I}_j$ for some j, while $x\,y \in \mathcal{I}_j$. It follows that $x^n \in \mathcal{I}_j$ for some positive integer n, because $\mathcal{I}_j$ is primary. This means that $x \in r(\mathcal{I}_j) = \mathcal{I}_0$. It follows that $x^m \in \mathcal{I}$ for some positive integer m, because of (17.5.2).

Let $\mathcal{I}_0$ be a proper prime ideal in k again, let $\mathcal{I}$ be a $\mathcal{I}_0$-primary ideal in k, and let $x \in k$ be given. Remember that

$$(\mathcal{I} : x) = \{t \in k : t\,x \in \mathcal{I}\} \tag{17.5.3}$$

is an ideal in k, as in Section 12.8. Note that

$$\mathcal{I} \subseteq (\mathcal{I} : x), \tag{17.5.4}$$

as before. Some properties of $(\mathcal{I} : x)$ are mentioned in Lemma 4.4 on p51 of [1], as follows.

Of course, if $x \in \mathcal{I}$, then

$$(\mathcal{I} : x) = k. \tag{17.5.5}$$

This means that

$$r((\mathcal{I} : x)) = k. \tag{17.5.6}$$

More precisely, this works for any ideal $\mathcal{I}$ in k.

If $x \notin \mathcal{I}$, then

$$(\mathcal{I} : x) \text{ is } \mathcal{I}_0\text{-primary in } k. \tag{17.5.7}$$

Let us first check that

$$(\mathcal{I} : x) \subseteq \mathcal{I}_0 \tag{17.5.8}$$

in this case. If $y \in (\mathcal{I} : x)$, so that $x\,y \in \mathcal{I}$, then $y^n \in \mathcal{I}$ for some positive integer n, because $x \notin \mathcal{I}$. This means that $y \in r(\mathcal{I}) = \mathcal{I}_0$, as desired. It follows from this and (17.5.4) that

$$r((\mathcal{I} : x)) = r(\mathcal{I}) = \mathcal{I}_0, \tag{17.5.9}$$

because $r(\mathcal{I}_0) = \mathcal{I}_0$, as in Section 12.10.

Suppose that $y, z \in k$ and $y\,z \in (\mathcal{I} : x)$, and let us show that $z \in (\mathcal{I} : x)$ or $y^n \in (\mathcal{I} : x)$ for some positive integer n. The second possibility is the same as saying that y is an element of (17.5.9). Suppose that $y \notin \mathcal{I}_0$, and let us verify that $z \in (\mathcal{I} : x)$. The hypothesis that $y\,z \in (\mathcal{I} : x)$ means that $x\,z\,y \in \mathcal{I}$. This implies that $x\,z \in \mathcal{I}$, because $\mathcal{I}$ is $\mathcal{I}_0$-primary, and $y \notin \mathcal{I}_0$. Thus $z \in (\mathcal{I} : x)$. It follows that $(\mathcal{I} : x)$ is a primary ideal in k.

If $x \notin \mathcal{I}_0$, then

$$(\mathcal{I} : x) = \mathcal{I}. \tag{17.5.10}$$

More precisely, one can check that $(\mathcal{I} : x) \subseteq \mathcal{I}$ in this case, because $\mathcal{I}$ is $\mathcal{I}_0$-primary. Note that (17.5.10) implies (17.5.9).

17.6 Primary decompositions

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}$ be a proper ideal in k. A *primary decomposition* of $\mathcal{I}$ is an expression of $\mathcal{I}$ as the intersection of finitely many primary ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ in k,

$$\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j, \tag{17.6.1}$$

as on p51 of [1]. If $\mathcal{I}$ has a primary decomposition, then one may say that $\mathcal{I}$ is *decomposable*, as on p52 of [1].

If k is Noetherian, then every proper ideal in k is decomposable, as in Section 17.4. This corresponds to Theorem 7.13 on p83 of [1]. One might consider $\mathcal{I} = k$ as having a primary decomposition, with $n = 0$ in (17.6.1).

Let $\mathcal{I}$ be a proper ideal in k with a primary decomposition as in (17.6.1). One can reduce to the case where

$$r(\mathcal{I}_1), \ldots, r(\mathcal{I}_n) \text{ are distinct}, \tag{17.6.2}$$

as mentioned on p52 of [1]. More precisely, if some of the $\mathcal{I}_j$'s have the same radical, then their intersection is a primary ideal in k with the same radical, as in the previous section.

One can also reduce to the case where for each $l = 1, \ldots, n$,

$$\bigcap_{j \neq l} \mathcal{I}_j \not\subseteq \mathcal{I}_l, \tag{17.6.3}$$

by simply dropping any of the primary ideals that are not needed to get (17.6.1), as on p52 of [1]. A primary decomposition (17.6.1) that satisfies (17.6.2) and (17.6.3) is said to be *minimal*, *irredundant*, *reduced*, or *normal*, etc., as in [1].

Let $\mathcal{I}$ be a proper ideal in k with primary decomposition as in (17.6.1) again. Also let $x \in k$ be given, so that $(\mathcal{I} : x)$ may be defined as in (17.5.3), and similarly for $(\mathcal{I}_j : x)$. Observe that

$$(\mathcal{I} : x) = \bigcap_{j=1}^{n} (\mathcal{I}_j : x). \tag{17.6.4}$$

This implies that

$$r((\mathcal{I} : x)) = \bigcap_{j=1}^{n} r((\mathcal{I}_j : x)), \tag{17.6.5}$$

as in Section 12.10.

If $x \in \mathcal{I}_j$ for some j, then $(\mathcal{I}_j : x) = k$, so that $r((\mathcal{I}_j : x)) = k$, as in the previous section. Similarly, if $x \in \mathcal{I}$, then $(\mathcal{I} : x) = k$, and $r((\mathcal{I} : x)) = k$. If $x \notin \mathcal{I}$, then

$$\begin{aligned} r((\mathcal{I} : x)) &= \bigcap \{r((\mathcal{I}_j : x)) : 1 \leq j \leq n, \, x \notin \mathcal{I}_j\} \\ &= \bigcap \{r(\mathcal{I}_j) : 1 \leq j \leq n, \, x \notin \mathcal{I}_j\}, \end{aligned} \tag{17.6.6}$$

using (17.6.5) in the first step, and (17.5.9) in the second step.

If $r((\mathcal{I} : x))$ is a proper prime ideal in k, then it follows that $x \notin \mathcal{I}$, and that

$$r((\mathcal{I} : x)) = r(\mathcal{I}_l) \text{ for some } l, \tag{17.6.7}$$

$1 \leq l \leq n$, as in Section 17.3. Conversely, if $1 \leq l \leq n$ and (17.6.3) holds, then there is an element x_l of

$$\Big(\bigcap_{j \neq l} \mathcal{I}_j\Big) \setminus \mathcal{I}_l. \tag{17.6.8}$$

Under these conditions, we get that

$$r((\mathcal{I} : x_l)) = \bigcap_{j=1}^{n} r((\mathcal{I}_j : x)) = r(\mathcal{I}_l), \tag{17.6.9}$$

by (17.6.6).

If (17.6.1) is a minimal primary decomposition of $\mathcal{I}$, then we get that

$$r(\mathcal{I}_1), \ldots, r(\mathcal{I}_n) \tag{17.6.10}$$

are the same as the proper prime ideals in k of the form $r((\mathcal{I} : x))$ for some $x \in k$. In particular, the collection of these proper prime ideals in k does not depend on the minimal primary decomposition (17.6.1) of $\mathcal{I}$. This is Theorem 4.5 on p52 of [1], which is the *first uniqueness theorem* for minimal primary decompositions.

If x_l is an element of (17.6.8), then

$$(\mathcal{I} : x_l) = (\mathcal{I}_l : x_l), \tag{17.6.11}$$

by (17.6.4), and because $(\mathcal{I}_j : x_l) = k$ when $j \neq l$, as before. Note that (17.6.11) is $r(\mathcal{I}_l)$-primary in k, as in (17.5.7). This corresponds to the first remark on p52 of [1].

17.7 More on primary decompositions

Let k be a commutative ring with a multiplicative identity element, let $\mathcal{I}$ be a proper ideal in k, and suppose that (17.6.1) is a minimal primary decomposition of $\mathcal{I}$ in k. The prime ideals $r(\mathcal{I}_j)$, $1 \le j \le n$, are said to *belong to* $\mathcal{I}$ or be *associated with* $\mathcal{I}$, as on p52 of [1]. Note that

$$\mathcal{I} \subseteq \mathcal{I}_l \subseteq r(\mathcal{I}_l) \tag{17.7.1}$$

for each $l = 1, \ldots, n$.

The prime ideals among those in (17.6.10) that are minimal with respect to inclusion are called the *minimal* or *isolated* prime ideals belonging to $\mathcal{I}$, as on p52 of [1]. The other prime ideals among those in (17.6.10) are called the *embedded* prime ideals belonging to $\mathcal{I}$.

Suppose that $\mathcal{I}_0$ is a proper prime ideal in k such that

$$\mathcal{I} \subseteq \mathcal{I}_0. \tag{17.7.2}$$

This means that

$$\bigcap_{j=1}^{n} \mathcal{I}_j \subseteq \mathcal{I}_0, \tag{17.7.3}$$

so that

$$\bigcap_{j=1}^{n} r(\mathcal{I}_j) = r\Big(\bigcap_{j=1}^{n} \mathcal{I}_j\Big) \subseteq r(\mathcal{I}_0) = \mathcal{I}_0, \tag{17.7.4}$$

using some basic properties of radicals of ideals, as in Section 12.10. This implies that

$$r(\mathcal{I}_l) \subseteq \mathcal{I}_0 \tag{17.7.5}$$

for some l, as in Section 17.3. It follows that

$$\mathcal{I}_0 \text{ contains a minimal prime ideal belonging to } \mathcal{I}, \tag{17.7.6}$$

because $r(\mathcal{I}_l)$ contains a minimal prime ideal belonging to $\mathcal{I}$. This is the first part of Proposition 4.6 on p52 of [1].

Consider

(17.7.7) the set of all proper prime ideals in k that contain $\mathcal{I}$.

Using (17.7.6), we get that every minimal element of (17.7.7) is a minimal prime ideal belonging to $\mathcal{I}$. Conversely, we also get that every minimal prime ideal belonging to $\mathcal{I}$ is a minimal element of (17.7.7). Thus the minimal prime ideals belonging to $\mathcal{I}$ are the same as the proper prime ideals in k that contain $\mathcal{I}$ and are minimal with respect to inclusion. This is the second part of Proposition 4.6 on p52 of [1].

The first part of Proposition 4.7 on p53 of [1] states that

$$\bigcup_{l=1}^{n} r(\mathcal{I}_l) = \{x \in k : (\mathcal{I} : x) \neq \mathcal{I}\}. \tag{17.7.8}$$

If $x \in k$ is not an element of $r(\mathcal{I}_l)$ for some l, then

$$(\mathcal{I}_l : x) = \mathcal{I}_l, \tag{17.7.9}$$

as in (17.5.10). If x is not an element of $r(\mathcal{I}_l)$ for any l, then it follows that

$$(\mathcal{I} : x) = \bigcap_{l=1}^{n} (\mathcal{I}_l : x) = \bigcap_{l=1}^{n} \mathcal{I}_l = \mathcal{I}, \tag{17.7.10}$$

using (17.6.4) in the first step. This implies that the right side of (17.7.8) is contained in the left side. To get equality, we shall reduce to the case where $\mathcal{I} = \{0\}$, as in [1], after some preliminary remarks in the next section.

17.8 Surjective ring homomorphisms

Let k and $\widetilde{k}$ be commutative rings with multiplicative identity elements 1_k, $1_{\widetilde{k}}$, respectively. Also let ϕ be a ring homomorphism from k onto $\widetilde{k}$. which implies that $\phi(1_k) = 1_{\widetilde{k}}$. If $\mathcal{I}$ is any ideal in k, then it is easy to see that $\phi(\mathcal{I})$ is an ideal in $\widetilde{k}$. This is the same as the extension of $\mathcal{I}$ with respect to ϕ, as in Section 12.7. If $\widetilde{\mathcal{I}}$ is an ideal in $\widetilde{k}$, then $\phi^{-1}(\widetilde{\mathcal{I}})$ is an ideal in k, which is the contraction of $\widetilde{\mathcal{I}}$ with respect to ϕ, as before.

If $\mathcal{I}$ is an ideal in k, then

$$\phi^{-1}(\phi(\mathcal{I})) = \mathcal{I} + \ker\phi. \tag{17.8.1}$$

In particular, if

$$\ker\phi \subseteq \mathcal{I}, \tag{17.8.2}$$

then

$$\mathcal{I} = \phi^{-1}(\phi(\mathcal{I})). \tag{17.8.3}$$

If $\widetilde{\mathcal{I}}$ is an ideal in $\widetilde{k}$, then

$$\ker \phi \subseteq \phi^{-1}(\widetilde{\mathcal{I}}), \tag{17.8.4}$$

and

$$\phi(\phi^{-1}(\widetilde{\mathcal{I}})) = \widetilde{\mathcal{I}}. \tag{17.8.5}$$

It follows that

$$\mathcal{I} \mapsto \phi(\mathcal{I}) \tag{17.8.6}$$

defines a one-to-one mapping fom the set of ideals in k that contain $\ker \phi$ onto the set of ideals in $\widetilde{k}$, with inverse given by

$$\widetilde{\mathcal{I}} \mapsto \phi^{-1}(\widetilde{\mathcal{I}}). \tag{17.8.7}$$

This corresponds to Proposition 1.1 on p2 of [1], which is also mentioned on p9 of [1].

Note that $\phi^{-1}(\widetilde{\mathcal{I}})$ is a proper ideal in k when $\widetilde{\mathcal{I}}$ is a proper ideal in $\widetilde{k}$. If $\mathcal{I}$ is a proper ideal in k that contains $\ker \phi$, then $\phi(\mathcal{I})$ is a proper ideal in $\widetilde{k}$. If $\widetilde{\mathcal{I}}$ is a prime ideal in $\widetilde{k}$, then $\phi^{-1}(\widetilde{\mathcal{I}})$ is a prime ideal in k, as in Section 12.7. If $\mathcal{I}$ is a prime ideal in k that contains $\ker \phi$, then one can check that

$$\phi(\mathcal{I}) \text{ is a prime ideal in } \widetilde{k}, \tag{17.8.8}$$

using (17.8.3). This corresponds to a remark on p9 of [1].

If $\widetilde{\mathcal{I}}$ is a primary ideal in $\widetilde{k}$, then $\phi^{-1}(\widetilde{\mathcal{I}})$ is a primary ideal in k, as in Section 14.5. If $\mathcal{I}$ is a primary ideal in k that contains $\ker \phi$, then one can verify that

$$\phi(\mathcal{I}) \text{ is a primary ideal in } \widetilde{k}, \tag{17.8.9}$$

using (17.8.3) again. This corresponds to part of a remark near the beginning of the proof of Proposition 4.7 on p53 of [1].

If $\widetilde{\mathcal{I}}$ is an ideal in $\widetilde{k}$, then

$$\phi^{-1}(r(\widetilde{\mathcal{I}})) = r(\phi^{-1}(\widetilde{\mathcal{I}})), \tag{17.8.10}$$

as in Section 12.10. Here $r(\cdot)$ is the radical of an ideal in k or $\widetilde{k}$, as appropriate. If $\mathcal{I}$ is an ideal in k, then it is easy to see that

$$\phi(r(\mathcal{I})) \subseteq r(\phi(\mathcal{I})), \tag{17.8.11}$$

as before. If $\mathcal{I}$ contains $\ker \phi$, then one can check that

$$\phi(r(\mathcal{I})) = r(\phi(\mathcal{I})), \tag{17.8.12}$$

using (17.8.3).

Let $\mathcal{I}$ be an ideal in k again, and let $x \in k$ be given. It is easy to see that

$$\phi((\mathcal{I} : x)) \subseteq (\phi(\mathcal{I}) : \phi(x)), \tag{17.8.13}$$

where $(\mathcal{I} : x)$ and $(\phi(\mathcal{I}) : \phi(x))$ are the ideals in k and $\widetilde{k}$, respectively, defined as in Section 17.5. If $\mathcal{I}$ contains $\ker \phi$, then one can verify that

$$\phi((\mathcal{I} : x)) = (\phi(\mathcal{I}) : \phi(x)), \tag{17.8.14}$$

using (17.8.3). Note that

$$\ker \phi \subseteq \mathcal{I} \subseteq (\mathcal{I} : x) \tag{17.8.15}$$

when $\mathcal{I}$ contains $\ker \phi$. In this case, if $x \in k$, then we get that

$$(\mathcal{I} : x) = \mathcal{I} \text{ if and only if } (\phi(\mathcal{I}) : \phi(x)) = \phi(\mathcal{I}). \tag{17.8.16}$$

If A_1, A_2 are subsets of k, then

$$\phi(A_1 \cap A_2) \subseteq \phi(A_1) \cap \phi(A_2). \tag{17.8.17}$$

If A_1, A_2 are subgroups of k, as a commutative group with respect to addition, with

$$\ker \phi \subseteq A_j \tag{17.8.18}$$

for $j = 1, 2$, then one can check that

$$\phi(A_1 \cap A_2) = \phi(A_1) \cap \phi(A_2). \tag{17.8.19}$$

Similarly, if $A_1, \ldots, A_n$ are finitely many subgroups of k, as a commutative group with respect to addition, and if (17.8.18) holds for each $j = 1, \ldots, n$, then

$$\phi\Big(\bigcap_{j=1}^{n} A_j\Big) = \bigcap_{j=1}^{n} \phi(A_j). \tag{17.8.20}$$

This could also be obtained from the fact that $\phi^{-1}(\phi(A)) = A$ when A is a subgroup of k, as a commutative group with respect to addition, that contains $\ker \phi$.

Let $\mathcal{I}$ be a proper ideal in k, and let $\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j$ be a primary decomposition of $\mathcal{I}$ in k. Suppose that $\mathcal{I}$ contains $\ker \phi$, so that

$$\ker \phi \subseteq \mathcal{I}_j \tag{17.8.21}$$

for each $j = 1, \ldots, n$. This implies that $\phi(\mathcal{I})$ is a proper ideal in $\widetilde{k}$, and that

$$\phi(\mathcal{I}) = \bigcap_{j=1}^{n} \phi(\mathcal{I}_j), \tag{17.8.22}$$

as in (17.8.20). This is a primary decomposition of $\phi(\mathcal{I})$ in $\widetilde{k}$, because $\phi(\mathcal{I}_j)$ is a primary ideal in $\widetilde{k}$ for each $j = 1, \ldots, n$, as in (17.8.9). This corresponds to another part of a remark near the beginning of the proof of Proposition 4.7 on p53 of [1].

Note that $\ker \phi$ is contained in $r(\mathcal{I}_j)$ for each $j = 1, \ldots, n$, by (17.8.21). Suppose now that $\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j$ is a minimal primary decomposition of $\mathcal{I}$. One can check that (17.8.22) is a minimal primary decomposition of $\phi(\mathcal{I})$ under these conditions.

The analogue of (17.7.8) for $\phi(\mathcal{I})$ in $\widetilde{k}$ is that

$$\bigcup_{l=1}^{n} r(\phi(\mathcal{I}_l)) = \{y \in \widetilde{k} : (\phi(\mathcal{I}) : y) \neq \phi(\mathcal{I})\}. \tag{17.8.23}$$

One can verify that this holds if and only if (17.7.8) holds.

In particular, we can take ϕ to be the natural quotient homomorphism from k onto $\widetilde{k} = k/\mathcal{I}$. This permits us to reduce (17.7.8) to the case where $\mathcal{I} = \{0\}$, as in the proof of Proposition 4.7 on p53 of [1]. This case will be discussed in Section 17.10.

17.9 Some remarks about zero-divisors

Let k be a commutative ring with a multiplicative identity element, and let D be the set of zero-divisors in k, which is to say the set of $x \in k$ such that $x\,y = 0$ for some $y \in k$ with $y \neq 0$. If $x \in k$ and $\mathcal{I}$ is an ideal in k, then $(\mathcal{I} : x)$ is the set of $t \in k$ such that $t\,x \in \mathcal{I}$, which is an ideal in k, as before. In particular,

$$(\{0\} : x) = \{t \in k : t\,x = 0\}, \tag{17.9.1}$$

so that

$$x \in D \text{ if and only if } (\{0\} : x) \neq \{0\}. \tag{17.9.2}$$

Alternatively, $(\{0\} : x)$ is the same as the annihilator of the ideal in k generated by x, as in Section 12.8. We also have that

$$D = \bigcup_{x \neq 0} (\{0\} : x), \tag{17.9.3}$$

as on p8 of [1]. More precisely, the union on the right is taken over all $x \in k$ with $x \neq 0$.

If E is any subset of k, then the *radical* $r(E)$ of E in k may be defined by

$$r(E) = \{x \in k : x^n \in E \text{ for some } n \in \mathbf{Z}_+\}, \tag{17.9.4}$$

as on p9 of [1]. This is the same as in the definition of the radical of an ideal in k, as in Section 12.10, although this is not necessarily an ideal in k. It is easy to see that the radical of a union of a family of subsets of k is the same as the union of the radicals of the sets, as in [1]. Note that

$$E \subseteq r(E) \tag{17.9.5}$$

for any $E \subseteq k$. We also have that

$$r(r(E)) = r(E) \tag{17.9.6}$$

for every $E \subseteq k$, as before.

If $x \in k$, then $x \in r(D)$ if and only if

(17.9.7) $$x^n\, y = 0$$

for some positive integer n and $y \in k$ with $y \neq 0$. We can take n as small as possible, to get that $x \in D$. This means that

(17.9.8) $$r(D) = D,$$

as in the proof of Proposition 1.15 on p9 of [1].

Using (17.9.3) and (17.9.8), we get that

(17.9.9) $$D = r\Big(\bigcup_{x \neq 0} (\{0\} : x)\Big) = \bigcup_{x \neq 0} r((\{0\} : x)).$$

This is Proposition 1.15 on p9 of [1].

17.10 Primary decompositions of $\{0\}$

Let k be a commutative ring with a nonzero multiplicative identity element, and suppose that

(17.10.1) $$\bigcap_{j=1}^{n} \mathcal{I}_j = \{0\}$$

is a minimal primary decomposition of $\{0\}$ in k, as in Section 17.6. We would like to show that

(17.10.2) $$\bigcup_{l=1}^{n} r(\mathcal{I}_l) = \{x \in K : (\{0\} : x) \neq \{0\}\},$$

which is the same as (17.7.8), with $\mathcal{I} = \{0\}$. This will imply (17.7.8) for arbitrary $\mathcal{I}$, as mentioned near the end of Section 17.8.

Let D be the set of zero-divisors in k, as in the previous section. Observe that (17.10.2) is equivalent to saying that

(17.10.3) $$\bigcup_{l=1}^{n} r(\mathcal{I}_l) = D,$$

because of (17.9.2). This corresponds to the second part of Proposition 4.7 on p53 of [1].

If $x \in k$ and $x \neq 0$, then

(17.10.4) $$r((\{0\} : x)) = \bigcap \{r(\mathcal{I}_j) : 1 \leq j \leq n,\, x \notin \mathcal{I}_j\},$$

as in (17.6.6). In particular, this implies that

(17.10.5) $$r((\{0\} : x)) \subseteq \mathcal{I}_j$$

for some j. It follows that

$$D \subseteq \bigcup_{l=1}^{n} r(\mathcal{I}_l), \tag{17.10.6}$$

because of (17.9.9).

We also have that for each $l = 1, \ldots, n$, there is an $x_l \in k$ such that

$$r(\mathcal{I}_l) = r((\{0\} : x_l)), \tag{17.10.7}$$

as in Section 17.6. More precisely, $x_l \neq 0$, by the previous argument, or because $r(\mathcal{I}_l) \neq k$. This implies that

$$\bigcup_{l=1}^{n} r(\mathcal{I}_l) \subseteq D, \tag{17.10.8}$$

because of (17.9.9). Thus (17.10.3) holds, by (17.10.6) and (17.10.8).

Remember that the nilradical $\mathcal{N}$ of k is the same as the intersection of all the prime ideals in k, as in Section 12.5. Because of (17.10.1), we can use (17.7.6) with $\mathcal{I} = \{0\}$ to get that $\mathcal{N}$ is equal to the intersection of the minimal prime ideals belonging to $\{0\}$. This corresponds to a remark on p53 of [1].

17.11 Zero-divisors and prime ideals

Let k be a commutative ring with a nonzero multiplicative identity element. Also let Σ be the collection of all ideals $\mathcal{I}$ in k such that every element of $\mathcal{I}$ is a zero-divisor in k, as in Exercise 14 on p12 of [1]. One can check that every element of Σ is contained in a maximal element of Σ, with respect to inclusion, as in [1], using Zorn's lemma or Hausdorff's maximality principle. More precisely, one can verify that the union of any nonempty chain of elements of Σ, with respect to inclusion, is an element of Σ.

If $\mathcal{I}_0$ is a maximal element of Σ, then $\mathcal{I}_0$ is a prime ideal in k, as in [1]. To see this, suppose that $x, y \in k$, $x\,y \in \mathcal{I}_0$, and $y \notin \mathcal{I}_0$, and let us show that $x \in \mathcal{I}_0$. Remember that the set $(\mathcal{I}_0 : x)$ of $t \in k$ such that $t\,x \in \mathcal{I}_0$ is an ideal in k that contains $\mathcal{I}_0$. We also have that $y \in (\mathcal{I}_0 : x)$, by hypothesis, so that

$$\mathcal{I}_0 \neq (\mathcal{I}_0 : x). \tag{17.11.1}$$

It follows that there is a $z \in (\mathcal{I}_0 : x)$ such that z is not a zero-divisor in k, by the maximality of $\mathcal{I}_0$.

Let $\mathcal{I}_1$ be the ideal in k generated by $\mathcal{I}_0$ and x, so that every element of $\mathcal{I}_1$ is of the form $a + b\,x$ for some $a \in \mathcal{I}_0$ and $b \in k$. If $w \in \mathcal{I}_1$, then

$$w\,z \in \mathcal{I}_0, \tag{17.11.2}$$

because $x\,z \in \mathcal{I}_0$, by construction. This implies that there is a $t \in k$ such that $t \neq 0$ and

$$t\,w\,z = 0, \tag{17.11.3}$$

because every element of $\mathcal{I}_0$ is a zero-divisor in k. Note that $t\,z \neq 0$, because z is not a zero-divisor in k. It follows that w is a zero-divisor in k.

Thus $\mathcal{I}_1 \in \Sigma$, which means that $\mathcal{I}_1 = \mathcal{I}_0$, because $\mathcal{I}_0 \subseteq \mathcal{I}_1$ by construction, and $\mathcal{I}_0$ is a maximal element of Σ. This implies that $x \in \mathcal{I}_0$, as desired.

If $x \in k$ is a zero-divisor, then every element of the ideal in k generated by x is a zero-divisor, so that the ideal in k generated by x is an element of Σ. This implies that the ideal in k generated by x is contained in a maximal element of Σ, as before. It follows that the set D of zero-divisors in k is equal to the union of the maximal elements of Σ. In particular, this shows that D can be expressed as the union of a family of proper prime ideals in k, as in [1].

17.12 Saturated multiplicatively closed sets

Let k be a commutative ring with a nonzero multiplicative identity element $1_k = 1$. Note that the set of all non-invertible elements of k is the same as the usion of all of the proper maximal ideals in k. In particular, this may be considered as the union of a family of proper prime ideals in k.

Let k_1 be another commutative ring with a nonzero multiplicative identity element 1_{k_1}, and let ϕ be a ring homomorphism from k into k_1 such that $\phi(1_k) = 1_{k_1}$. Remember that the inverse image of a proper prime ideal in k_1 under ϕ is a proper prime ideal in k, as in Section 12.7. This implies that

$$\{x \in k : \phi(x) \text{ is not invertible in } k_1\} \tag{17.12.1}$$

can be expressed as the union of a family of proper prime ideals in k. More precisely, this is the inverse image under ϕ of the union of a family of proper prime ideals in k_1, as in the preceding paragraph.

Let S be a multiplicatively closed subset of k, as in Section 12.1. We say that S is *saturated* if for every $x, y \in k$, we have that

$$x\,y \in S \text{ if and only if } x, y \in S, \tag{17.12.2}$$

as in Exercise 7 on p44 of [1]. Note that k may be considered as a saturated multiplicatively closed subset of itself. If S is a saturated multiplicatively closed subset of k and $0 \in S$, then $S = k$.

Observe that the group of invertible elements of k is a saturated multiplicativaly closed subset of k. If S is any saturated multiplicatively closed subset of k, then one can verify that S contains the group of invertible elements in k.

It is easy to see that the intersection of any nonempty family of multiplicatively closed subsets of k is a multiplicatively closed subset of k. Similarly, the intersection of any nonempty family of saturated multiplicatively closed subsets of k is a saturated multiplicatively closed subset of k.

If $\mathcal{I}$ is a proper prime ideal in k, then $k \setminus \mathcal{I}$ is a multiplicatively closed subset of k, as in Section 12.4. In fact, one can check that $k \setminus \mathcal{I}$ is saturated. It follows that the complement in k of the union of a nonempty family of proper prime ideals is a saturated multiplicatively closed set. This is part of part (i) of Exercise 7 on p44 of [1].

Suppose that S is a saturated multiplicatively closed subset of k, and consider the corresponding ring of fractions $S^{-1}\,k$, as in Section 12.1. Let $x \in k$ be given, and note that $x/1$ has a multiplicative inverse in $S^{-1}\,k$ if and only if there are $y \in k$ and $t \in S$ such that

$$(x/1)\,(y/t) = (x\,y)/t \tag{17.12.3}$$

is equal to $1/1$ in $S^{-1}\,k$. This happens if and only if there is a $v \in S$ such that

$$t\,v = x\,y\,v, \tag{17.12.4}$$

as before. Under these conditions, we get that $t\,v \in S$, and thus that $x, y \in S$, because S is saturated.

This shows that

$$S = \{x \in k : x/1 \text{ is invertible in } S^{-1}\,k\} \tag{17.12.5}$$

in this case. This means that $k \setminus S$ is as in (17.12.1), where ϕ is the natural ring homomorphism from k into $k_1 = S^{-1}\,k$. It follows that $k \setminus S$ can be expressed as the union of a family of proper prime ideals in k, as before. This is the other part of part (i) of Exercise 7 on p44 of [1].

Let D be the set of zero-divisors in k, as before. It is easy to see that the set

$$S_0 = k \setminus D \tag{17.12.6}$$

of all non-zero-divisors in k is a multiplicatively closed subset of k. In fact, one can check that S_0 is saturated as a multiplicatively closed subset of k, as in Exercise 9 on p44 of [1]. Using the remarks in the preceding paragraph, we get another way to see that

$$D = k \setminus S_0 \tag{17.12.7}$$

can be expressed as the union of a family of proper prime ideals in k, as in the previous section. This is another part of Exercise 9 on p44 of [1].

Let S be any multiplicatively closed subset of k, so that the ring $S^{-1}\,k$ of fractions of k with respect to S may be defined as in Section 12.1. Remember that for each $x \in k$, $x/1 = 0$ in $S^{-1}\,k$ if and only if $x\,t = 0$ in k for some $t \in S$. This implies that the natural ring homomorphism from k into $S^{-1}\,k$ is injective if and only if

$$S \subseteq S_0. \tag{17.12.8}$$

This corresponds to part (i) of Exercise 9 on p44 of [1]. The ring $S_0^{-1}\,k$ is called the *total ring of fractions* of k, as in [1].

17.13 Saturations of ideals

Let k be a commutative ring with a multiplicative identity element, and let S be a multiplicatively closed subset of k, so that the ring $S^{-1}\,k$ of fractions of k with respect to S may be defined as in Section 12.1. If $\mathcal{I}$ is an ideal in k, then $S^{-1}\,\mathcal{I}$

is an ideal in $S^{-1}\,k$, as in Section 12.13. This is the same as the extension of $\mathcal{I}$ with respect to the natural ring homomorphism from k into $S^{-1}\,k$, as before.

Let

$$S(\mathcal{I}) = (S^{-1}\,\mathcal{I})^c \tag{17.13.1}$$

be the contraction of $S^{-1}\,\mathcal{I}$ with respect to the natural ring homomorphism from k into $S^{-1}\,k$, as on p53 of [1]. This is the ideal in k obtained by taking the inverse image of $S^{-1}\,\mathcal{I}$ under the natural ring homomorphism from k into $S^{-1}\,k$, as in Section 12.7. This is the *saturation* of $\mathcal{I}$ in k with respect to S, as in Exercise 12 on p56 of [1].

Equivalently,

$$S(\mathcal{I}) = \{x \in k : x/1 \in S^{-1}\,\mathcal{I}\}. \tag{17.13.2}$$

Thus $x \in k$ is an element of $S(\mathcal{I})$ if and only if

$$x/1 = y/t \tag{17.13.3}$$

for some $y \in \mathcal{I}$ and $t \in S$. This is the same as saying that

$$x\,t\,v = y\,v \tag{17.13.4}$$

for some $v \in S$, as in Section 12.1. It follows that

$$S(\mathcal{I}) = \{x \in k : x\,r \in \mathcal{I} \text{ for some } r \in S\}. \tag{17.13.5}$$

If $\widetilde{\mathcal{I}}$ is another ideal in k, then part (i) of Exercise 12 on p56 of [1] states that

$$S(\mathcal{I}) \cap S(\widetilde{\mathcal{I}}) = S(\mathcal{I} \cap \widetilde{\mathcal{I}}). \tag{17.13.6}$$

This can be verified using (17.13.5), and the fact that S is multiplicatively closed. Alternatively,

$$(S^{-1}\mathcal{I}) \cap (S^{-1}\widetilde{\mathcal{I}}) = S^{-1}\,(\mathcal{I} \cap \widetilde{\mathcal{I}}), \tag{17.13.7}$$

as in Section 12.2. One can get (17.13.6) from (17.13.7) and the analogous property for contractions of intersections, as in Section 12.9.

Part (ii) of Exercise 12 on p56 of [1] states that

$$S(r(\mathcal{I})) = r(S(\mathcal{I})). \tag{17.13.8}$$

This can also be obtained using (17.13.5), or from the analogous statements for contractions and $S^{-1}\,\mathcal{I}$, as in Sections 12.10 and 12.13.

Part (iii) of Exercise 12 on p56 of [1] states that

$$S(\mathcal{I}) = k \text{ if and only if } \mathcal{I} \cap S \neq \emptyset. \tag{17.13.9}$$

One can check this using (17.13.5) again, or the fact that $S^{-1}\,\mathcal{I} = S^{-1}\,k$ if and only if $\mathcal{I}$ intersects S, as in Section 12.13.

Let S_1, S_2 be multiplicatively closed subsets of k, so that the set $S_1\,S_2$ of products of elements of S_1 and S_2 is also a multiplicatively closed set in k. Under these conditions, part (iv) of Exercsie 12 on p56 of [1] states that

$$S_1(S_2(\mathcal{I})) = (S_1\,S_2)(\mathcal{I}). \tag{17.13.10}$$

This can be verified using the analogue of (17.13.5) in the various relevant cases. Alternatively, one can use the ring isomorphism between $(S_1\,S_2)^{-1}\,k$ and an appropriate ring of fractions of $S_1^{-1}\,k$ mentioned in Section 14.1.

17.14 Primary decompositions and fractions

Let k be a commutative ring with a multiplicative identity element, let S be a multiplicatively closed subset of k, and let $\mathcal{I}$ be a proper ideal in k. Suppose that $\mathcal{I} = \bigcap_{j=1}^n \mathcal{I}_j$ is a minimal primary decomposition of $\mathcal{I}$, as in Section 17.6. We may reorder the $\mathcal{I}_j$'s, if necessary, to get a nonnegative integer $m \le n$ such that

$$r(\mathcal{I}_j) \cap S = \emptyset \text{ when } j \le m \tag{17.14.1}$$

and

$$r(\mathcal{I}_j) \cap S \ne \emptyset \text{ when } j > m. \tag{17.14.2}$$

If $m \ge 1$, then Proposition 4.9 on p54 of [1] states that

$$S^{-1}\,\mathcal{I} = \bigcap_{j=1}^{m} (S^{-1}\,\mathcal{I}_j) \tag{17.14.3}$$

and

$$S(\mathcal{I}) = \bigcap_{j=1}^{m} \mathcal{I}_j, \tag{17.14.4}$$

and that these are minimal primary decompositions.

To see this, observe first that

$$S^{-1}\,\mathcal{I} = \bigcap_{j=1}^{n} (S^{-1}\,\mathcal{I}_j), \tag{17.14.5}$$

as in Sections 12.2 and 12.13. If $j > m$, then

$$S^{-1}\,\mathcal{I}_j = S^{-1}\,k, \tag{17.14.6}$$

because of (17.14.2), as in Section 14.6. Thus (17.14.5) reduces to (17.14.3) when $m \ge 1$, and otherwise $S^{-1}\,\mathcal{I} = S^{-1}\,k$.

Remember that $r(\mathcal{I}_j)$ is a prime ideal in k for each j, as in Section 14.5. We also have that $r(\mathcal{I}_j) \ne k$, because $\mathcal{I}_j \ne k$, by hypothesis, as in Section 12.10. Note that

$$r(S^{-1}\,\mathcal{I}_j) = S^{-1}\,r(\mathcal{I}_j) \tag{17.14.7}$$

for each j, as in Section 12.13.

If $j \le m$, then

$$S^{-1}\,\mathcal{I}_j \text{ is a primary ideal in } S^{-1}\,k, \tag{17.14.8}$$

because of (17.14.1), as in Section 14.6. This implies that (17.14.3) is a primary decomposition of $S^{-1}\,\mathcal{I}$ in $S^{-1}\,k$ when $m \ge 1$.

If $j \le m$, then $S^{-1}\,r(\mathcal{I}_j)$ is a proper prime ideal in $S^{-1}\,k$, because of (17.14.1), as in Section 12.13. The $S^{-1}\,r(\mathcal{I}_j)$'s, $j \le m$, are distinct ideals in $S^{-1}\,k$ too, as in Section 12.13, because the $r(\mathcal{I}_j)$'s are distinct ideals in k, by the minimality of the primary decomposition of $\mathcal{I}$, as in Section 17.6. This is the same as saying that the $r(S^{-1}\,\mathcal{I}_j)$'s, $j \le m$, are disctinct ideals in $S^{-1}\,k$, by (17.14.7). This is the first of the two conditions required for minimality of the primary decomposition (17.14.3).

If $j \le m$, then

$$S(\mathcal{I}_j) = (S^{-1}\,\mathcal{I}_j)^c = \mathcal{I}_j, \tag{17.14.9}$$

because of (17.14.1), as in Section 14.6. One can use this to get the second condition required for the minimality of the primary decomposition (17.14.3), because of the analogous property of the primary decomposition of $\mathcal{I}$.

Note that

$$S(\mathcal{I}) = \bigcap_{j=1}^{n} S(\mathcal{I}_j), \tag{17.14.10}$$

because of (17.13.6). If $j > m$, then

$$S(\mathcal{I}_j) = (S^{-1}\,\mathcal{I}_j)^c = (S^{-1}\,k)^c = k, \tag{17.14.11}$$

using (17.14.6) in the second step. If $m \ge 1$, then we get that

$$S(\mathcal{I}) = \bigcap_{j=1}^{m} S(\mathcal{I}_j) = \bigcap_{j=1}^{m} \mathcal{I}_j, \tag{17.14.12}$$

using (17.14.9) in the second step. Of course, this is a primary decomposition of $S(\mathcal{I})$ in k, and the minimality of this primary decomposition follows from the minimality of the primary decomposition of $\mathcal{I}$ in k. If $m = 0$, then $S(\mathcal{I}) = k$.

17.15 The second uniqueness theorem

Let k be a commutative ring with a multiplicative identity element, let $\mathcal{I}$ be a proper ideal in k, and suppose that $\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j$ is a minimal primary decomposition of $\mathcal{I}$. Remember that the collection of proper prime ideals $r(\mathcal{I}_1), \ldots, r(\mathcal{I}_n)$ in k does not depend on the particular minimal primary decomposition of $\mathcal{I}$, as in Section 17.6. These are the prime ideals in k that are said to belong to $\mathcal{I}$, as in Section 17.7.

Let Σ be a collection of the prime ideals in k that belong to $\mathcal{I}$. Suppose that if $\mathcal{P}'$ is a prime ideal in k that belongs to $\mathcal{I}$, $\mathcal{P} \in \Sigma$, and $\mathcal{P}' \subseteq \mathcal{P}$, then

$$\mathcal{P}' \in \Sigma. \tag{17.15.1}$$

Under these conditions, Σ is said to be *isolated*, as on p54 of [1].

Put

$$S = k \setminus \Big(\bigcup_{\mathcal{P} \in \Sigma} \mathcal{P} \Big) = \bigcap_{\mathcal{P} \in \Sigma} (k \setminus \mathcal{P}). \tag{17.15.2}$$

This is a multiplicatively closed subset of k, because $k \setminus \mathcal{P}$ is a multiplicatively closed subset of k for every $\mathcal{P} \in \Sigma$, as in Section 12.4.

Let $\mathcal{P}'$ be a prime ideal in k that belongs to $\mathcal{I}$ and satisfies

$$\mathcal{P}' \not\in \Sigma. \tag{17.15.3}$$

We would like to check that

$$\mathcal{P}' \cap S \neq \emptyset, \tag{17.15.4}$$

as on p54 of [1]. Equivalently, this means that

$$\mathcal{P}' \not\subseteq \bigcup_{\mathcal{P} \in \Sigma} \mathcal{P}. \tag{17.15.5}$$

Otherwise, if

$$\mathcal{P}' \subseteq \bigcup_{\mathcal{P} \in \Sigma} \mathcal{P}, \tag{17.15.6}$$

then

$$\mathcal{P}' \subseteq \mathcal{P}_0 \tag{17.15.7}$$

for some $\mathcal{P}_0 \in \Sigma$, as in Section 17.3. However, this would contradict (17.15.3), because Σ is supposed to isolated.

Suppose that

$$\Sigma = \{r(\mathcal{I}_{l_1}), \ldots, r(\mathcal{I}_{l_m})\} \tag{17.15.8}$$

for some positive integer m, where $l_1 < \cdots < l_m$ are positive integers less than or equal to n. In this case, we get that

$$S(\mathcal{I}) = \mathcal{I}_{l_1} \cap \cdots \cap \mathcal{I}_{l_m}, \tag{17.15.9}$$

as in (17.14.12). Although we have not reordered the $\mathcal{I}_j$'s here as in the previous section, they have essentially the same properties with respect to S as before, because of (17.15.4) and the definition (17.15.2) of S.

In particular, (17.15.9) does not depend on the particular minimal primary decomposition of $\mathcal{I}$. This is Theorem 4.10 on p54 of [1], which is the *second uniqueness theorem.* Of course, (17.15.9) does depend on the choice of Σ.

Suppose that $r(\mathcal{I}_l)$ is minimal among the prime ideals belonging to $\mathcal{I}$ for some $l \leq n$. In this case,

$$\Sigma = \{r(\mathcal{I}_l)\} \tag{17.15.10}$$

is isolated in the sense defined earlier. If $\Sigma = k \setminus r(\mathcal{I}_l)$, as in (17.15.2), then we get that

$$S(\mathcal{I}) = \mathcal{I}_l, \tag{17.15.11}$$

as in (17.15.9). Thus $\mathcal{I}_l$ does not depend on the particular minimal primary decomposition of $\mathcal{I}$, as in Corollary 4.11 on p54 of [1]. More precisely, $\mathcal{I}_l$ is uniquely determined by $r(\mathcal{I}_l)$.

The primary components $\mathcal{I}_l$ corresponding to minimal prime ideals $r(\mathcal{I}_l)$ belonging to $\mathcal{I}$ may be called *isolated primary components*, as in [1]. The collection of these isolated primary components does not depend on the particular minimal primary decomposition of $\mathcal{I}$, as in the preceding paragraph.

Chapter 18

Some properties of ideals, 2

18.1 Intersections of prime ideals

Let k be a commutative ring with a multiplicative identity element. Also let A be a nonempty set, and let $\mathcal{I}_\alpha$ be an ideal in k for each $\alpha \in A$. It is easy to see that

$$r\Big(\bigcap_{\alpha \in A} \mathcal{I}_\alpha\Big) \subseteq \bigcap_{\alpha \in A} r(\mathcal{I}_\alpha). \tag{18.1.1}$$

If A has only finitely many elements, then

$$r\Big(\bigcap_{\alpha \in A} \mathcal{I}_\alpha\Big) = \bigcap_{\alpha \in A} r(\mathcal{I}_\alpha). \tag{18.1.2}$$

This is mentioned in Section 12.10 when A has only two elements, and otherwise one can use a similar argument, or use the previous statement repeatedly.

If $\mathcal{I}_\alpha$ is a prime ideal in k for every $\alpha \in A$, then $r(\mathcal{I}_\alpha) = \mathcal{I}_\alpha$ for each α, as before. This implies that

$$\bigcap_{\alpha \in A} r(\mathcal{I}_\alpha) = \bigcap_{\alpha \in A} \mathcal{I}_\alpha \subseteq r\Big(\bigcap_{\alpha \in A} \mathcal{I}_\alpha\Big), \tag{18.1.3}$$

so that (18.1.2) holds in this case too. More precisely, we get that

$$r\Big(\bigcap_{\alpha \in A} \mathcal{I}_\alpha\Big) = \bigcap_{\alpha \in A} r(\mathcal{I}_\alpha) = \bigcap_{\alpha \in A} \mathcal{I}_\alpha \tag{18.1.4}$$

under these conditions.

If $\mathcal{I}$ is any ideal in k, then its radical $r(\mathcal{I})$ is the same as the intersection of all of the prime ideals in k that contain $\mathcal{I}$, as in Section 12.10. In particular, if $r(\mathcal{I}) = \mathcal{I}$, then $\mathcal{I}$ can be expressed as the intersection of a family of prime ideals in k. Conversely, if $\mathcal{I}$ can be expressed as the intersection of a family of prime ideals in k, then $r(\mathcal{I}) = \mathcal{I}$, as in (18.1.4). Of course, if $\mathcal{I} \neq k$, then we may as well consider only proper prime ideals in k. This corresponds to Exercise 9 on p11 of [1].

Suppose now that $\mathcal{I}$ is a proper ideal in k, and that $\mathcal{I} = \bigcap_{j=1}^n \mathcal{I}_j$ is a primary decomposition of $\mathcal{I}$, as in Section 17.6. Note that

$$r(\mathcal{I}) = \bigcap_{j=1}^{n} r(\mathcal{I}_j), \tag{18.1.5}$$

as in (18.1.2). If $r(\mathcal{I}) = \mathcal{I}$, then it follows that

$$\mathcal{I} = \bigcap_{j=1}^{n} r(\mathcal{I}_j). \tag{18.1.6}$$

Remember that $r(\mathcal{I}_j)$ is a proper prime ideal in k for each j, because $\mathcal{I}_j$ is a primary ideal, as in Sections 14.5 and 17.5. Suppose that we have a minimal primary decomposition of $\mathcal{I}$, so that the $r(\mathcal{I}_j)$'s are distinct, as in Section 17.6.

Let

$$r(\mathcal{I}_{l_1}), \ldots, r(\mathcal{I}_{l_m}) \tag{18.1.7}$$

be the minimal elements with respect to inclusion among the ideals $r(\mathcal{I}_j)$, with $l_1 < \cdots < l_m$. Observe that

$$\mathcal{I} = r(\mathcal{I}_{l_1}) \cap \cdots \cap r(\mathcal{I}_{l_m}), \tag{18.1.8}$$

because of (18.1.6). More precisely, this uses the fact that each of the $r(\mathcal{I}_j)$'s contains a minimal element. Thus (18.1.8) may be considered as another primary decomposition of $\mathcal{I}$.

If any of the minimal elements in (18.1.7) contains the intersection of the others, then it contains one of the other minimal elements, as in Section 17.3. This is not possible, because of minimality. It follows that (18.1.8) is another minimal primary decomposition of $\mathcal{I}$, as in Section 17.6.

Remember that the prime ideals corresponding to a minimal primary decomposition of $\mathcal{I}$ do not depend on the particular minimal primary decomposition of $\mathcal{I}$, by the first uniqueness theorem, which was discussed in Section 17.6. Of course, the prime ideals (18.1.7) are the same as those associated to the minimal primary decomposition (18.1.8) of $\mathcal{I}$, because the radical of the radical of an ideal is equal to the radical of the ideal, as in Section 12.10. This means that the minimal elements (18.1.7) include all of the $r(\mathcal{I}_j)$'s, as in Exercise 2 on p55 of [1].

Remember that the isolated primary components of a minimal primary decomposition of $\mathcal{I}$ do not depend on the particular minimal primary decomposition of $\mathcal{I}$, as in Section 17.15. This implies that

$$\mathcal{I}_j = r(\mathcal{I}_j) \tag{18.1.9}$$

for each $j = 1, \ldots, n$.

18.2 Primary decompositions and $(\mathcal{I} : x)$

Let k be a commutative Noetherian ring with a multiplicative identity element, and let $\mathcal{I}$ be a proper ideal in k. Remember that $\mathcal{I}$ has a primary decomposition, which one may take to be minimal, as in Section 17.6.

If $x \in k$, then $(\mathcal{I} : x) = \{t \in k : t\,x \in \mathcal{I}\}$ is an ideal in k that contains $\mathcal{I}$, as in Section 12.8. Under these conditions, the prime ideals in k that belong to $\mathcal{I}$, as in Section 17.7, are the same as the proper prime ideals in k of the form $(\mathcal{I} : x)$ for some $x \in k$. This is Proposition 7.17 on p83 of [1].

Remember that the first uniquenss theorem, discussed in Section 17.6, is the analogous statement using the radicals of the ideals $(\mathcal{I} : x)$. However, the Noetherian condition was not needed for that result.

If $(\mathcal{I} : x)$ is a proper prime ideal in k for some $x \in k$, then

$$r((\mathcal{I} : x)) = (\mathcal{I} : x), \tag{18.2.1}$$

as in Section 12.10. This implies that $(\mathcal{I} : x)$ is one of the prime ideals in k that belong to $\mathcal{I}$, as in Section 17.6.

Conversely, we would like to show that each of the prime ideals in k that belong to $\mathcal{I}$ is of this form. To do this, the first step is to reduce to the case where $\mathcal{I} = \{0\}$, by replacing k with $k/\mathcal{I}$. This also uses the remarks in Section 17.8.

Thus we suppose now that $\bigcap_{j=1}^n \mathcal{I}_j = \{0\}$ is a minimal primary decomposition of $\{0\}$ in k. Put

$$\mathcal{A}_l = \bigcap_{j \ne l} \mathcal{I}_j \tag{18.2.2}$$

for each $l = 1, \ldots, n$, which is an ideal in k. The minimality of the primary decomposition implies that

$$\mathcal{A}_l \not\subseteq \mathcal{I}_l \tag{18.2.3}$$

for each l, as in Section 17.6. In particular, this means that $\mathcal{A}_l \ne \{0\}$ for each l.

Note that

$$\mathcal{A}_l \cap \mathcal{I}_l = \{0\} \tag{18.2.4}$$

for each l, by construction. Thus

$$\mathcal{A}_l \setminus \mathcal{I}_l = \mathcal{A}_l \setminus (\mathcal{A}_l \cap \mathcal{I}_l) = \mathcal{A}_l \setminus \{0\} \tag{18.2.5}$$

for each l. If x is an element of (18.2.5) for some l, then

$$r((\{0\} : x)) = r(\mathcal{I}_l). \tag{18.2.6}$$

as in (17.6.9). In particular, this means that

$$(\{0\} : x) \subseteq r(\mathcal{I}_l). \tag{18.2.7}$$

Let $1 \leq l \leq n$ be given, and note that $r(\mathcal{I}_l)$ is finitely-generated as an ideal in k, because k is Noetherian. It follows that

$$r(\mathcal{I}_l)^m \subseteq \mathcal{I}_l \tag{18.2.8}$$

for some positive integer m, as in Section 14.7. This implies that

$$\mathcal{A}_l\, r(\mathcal{I}_l)^m \subseteq \mathcal{A}_l \cap r(\mathcal{I}_l)^m \subseteq \mathcal{A}_l \cap \mathcal{I}_l = \{0\}, \tag{18.2.9}$$

where the first step is as in Section 12.8.

Let m_l be the smallest positive integer such that

$$\mathcal{A}_l\, r(\mathcal{I}_l)^{m_l} = \{0\}. \tag{18.2.10}$$

Thus $\mathcal{A}_l\, r(\mathcal{I}_l)^{m_l-1} \neq \{0\}$. More precisely, if $m_l = 1$, then this reduces to the fact that $\mathcal{A}_l \neq \{0\}$, as before.

Let x be a nonzero element of $\mathcal{A}_l\, r(\mathcal{I}_l)^{m_l-1}$. Observe that

$$r(\mathcal{I}_l) \subseteq (\{0\} : x), \tag{18.2.11}$$

by (18.2.10). We also have that (18.2.7) holds, because x is a nonzero element of $\mathcal{A}_l$. It follows that

$$r(\mathcal{I}_l) = (\{0\} : x) \tag{18.2.12}$$

under these conditions.

18.3 Primary decompositions and products

Let k be an integral domain of dimension one, in the sense of Section 14.8. Also let $\mathcal{I} \neq \{0\}$ be a proper ideal in k with a minimal primary decomposition $\mathcal{I} = \bigcap_{j=1}^n \mathcal{I}_j$. Note that

$$\{0\} \neq \mathcal{I} \subseteq \mathcal{I}_j \subseteq r(\mathcal{I}_j) \tag{18.3.1}$$

for each $j = 1, \ldots, n$. This implies that

$$r(\mathcal{I}_j) \text{ is a maximal proper ideal in } k \tag{18.3.2}$$

for each j, because $r(\mathcal{I}_j)$ is a proper prime ideal in k, and k is a one-dimensional integral domain. Remember that the $r(\mathcal{I}_j)$'s are distinct ideals in k, because of the minimality of the primary decomposition, as in Section 17.6.

It is easy to see that

$$\text{the } r(\mathcal{I}_j)\text{'s are pairwise coprime in } k, \tag{18.3.3}$$

as in Section 17.1, because they are distinct maximal proper ideals in k. This implies that

$$\text{the } \mathcal{I}_j\text{'s are pairwise coprime in } k \tag{18.3.4}$$

too, as in Section 17.1 again. It follows that the product of the $\mathcal{I}_j$'s is the same as their intersection, as in Section 17.2. This means that

$$\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j = \prod_{j=1}^{n} \mathcal{I}_j \tag{18.3.5}$$

in this case. This corresponds to the existence part of Theorem 9.1 on p93 of [1].

Suppose now that

$$\mathcal{I} = \prod_{j=1}^{n} \mathcal{I}_j, \tag{18.3.6}$$

where $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are finitely many primary ideals in k with distinct radicals. Under these conditions, (18.3.1) and thus (18.3.2) hold for each j, as before. This implies that (18.3.3) and (18.3.4) hold as well, as before. This means that the product of the $\mathcal{I}_j$'s is the same as their intersection, as in Section 17.2 again. Thus (18.3.5) holds under these conditions as well.

Let $1 \le l \le n$ be given, and let us check that

$$\mathcal{I}_l + \prod_{j \ne l} \mathcal{I}_j = k. \tag{18.3.7}$$

This is the same as (17.2.7) when $l = n$, and the same argument can be used otherwise. This implies that

$$\mathcal{I}_l + \bigcap_{j \ne l} \mathcal{I}_j = k, \tag{18.3.8}$$

because a product of ideals is contained in its intersection. It follows that $\bigcap_{j \ne l} \mathcal{I}_j$ is not contained in $\mathcal{I}_l$, because $\mathcal{I}_l$ is a primary ideal in k, and thus proper. This means that

(18.3.9) the $\mathcal{I}_j$'s form a minimal primary decomposition of $\mathcal{I}$,

as in Section 17.6.

Using this, we get that the collection of the radicals of the $\mathcal{I}_j$'s is uniquely determined, as in the first uniqueness theorem in Section 17.6. It is easy to see that $r(\mathcal{I}_l)$ is minimal among the $r(\mathcal{I}_j)$'s for each l, because k has dimension one, by hypothesis. It follows that

(18.3.10) the collection of $\mathcal{I}_j$'s is uniquely determined

under these conditions, as in Section 17.15. Of course, the $\mathcal{I}_j$'s can be reordered, without affecting their product. This corresponds to the uniqueness part of Theorem 9.1 on p93 of [1].

18.4 Products and Dedekind domains

Let k be an integral domain of dimension one, in the sense of Section 14.8, and suppose that k is also Noetherian. Any proper ideal $\mathcal{I}$ in k has a minimal primary decomposition $\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j$, as in Section 17.6. Suppose that $\mathcal{I} \neq \{0\}$, so that $\mathcal{I}_j \neq \{0\}$ for each j, and $\mathcal{I}$ can be expressed as the product of the $\mathcal{I}_j$'s, as in (18.3.5). Remember that the radical $r(\mathcal{I}_j)$ of $\mathcal{I}_j$ is a proper prime ideal in k for each j, as in Sections 14.5 and 17.5.

Suppose from now on in this section that k is a Dedekind domain. In this case, for each j there is a unique positive integer m_j such that

$$\mathcal{I}_j = r(\mathcal{I}_j)^{m_j}, \tag{18.4.1}$$

as in Section 15.14. It follows that

$$\mathcal{I} = \bigcap_{j=1}^{n} \mathcal{I}_j = \prod_{j=1}^{n} \mathcal{I}_j = \prod_{j=1}^{n} r(\mathcal{I}_j)^{m_j}. \tag{18.4.2}$$

One could also allow $\mathcal{I} = k$ here, with $n = 0$, or $m_j = 0$ for each j. This corresponds to the existence part of Corollary 9.4 on p95 of [1].

Concerning uniqueness, suppose that $\mathcal{I}$ can be expressed as the product of finitely many proper prime ideals in k. This can be arranged into a product of positive powers of distinct proper prime ideals in k. Thus $\mathcal{I}$ can be expressed as in (18.3.6), where the $\mathcal{I}_j$'s are positive powers of distinct proper prime ideals in k.

Remember that the radical of a positive power of a prime ideal in k is that prime ideal, as in Section 12.10. Thus the proper prime ideal in k corresponding to $\mathcal{I}_j$ is $r(\mathcal{I}_j)$ for each j. This means that $\mathcal{I}_j$ may be expressed as in (18.4.1) for some positive integer m_j for each j, so that $\mathcal{I}$ can be expressed as

$$\mathcal{I} = \prod_{j=1}^{n} \mathcal{I}_j = \prod_{j=1}^{n} r(\mathcal{I}_j)^{m_j}, \tag{18.4.3}$$

where the $r(\mathcal{I}_j)$'s are distinct proper prime ideals in k.

More precisely, (18.3.1) and (18.3.2) hold for each j in this case, as before. It follows that $\mathcal{I}_j$ is a primary ideal in k for each j, as in Section 14.5. Using this, we get that the collection of $r(\mathcal{I}_j)$'s is uniquely determined, and in fact the collection of $\mathcal{I}_j$'s is uniquely determined, as in the previous section. The m_j's are also uniquely determined, as in Section 15.14 again. This corresponds to the uniqueness part of Corollary 9.4 on p95 of [1].

18.5 Artinian modules

Let k be a commutative ring with a multiplicative identity element, and let V be a module over k. A sequence $V_1, V_2, V_3, \ldots$ of submodules of V is said to be monotonically decreasing with respect to inclusion if

$$V_{j+1} \subseteq V_j \tag{18.5.1}$$

for each j, as usual. If every such sequence is eventually constant, then V is said to be *Artinian* as a module over k, as on p74 of [1]. This is equivalent to the condition that every nonempty collection of submodules of V have a minimal element, as in Proposition 6.1 on p74 of [1].

A number of basic properties of Artinian modules over k are analogous to properties of Noetherian modules over k, as mentioned on p75 of [1]. In particular, if V_0 is a submodule of V, then one can check that V is Artinian if and only if

(18.5.2) $$V_0 \text{ and } V/V_0 \text{ are Artinian modules over } k,$$

as in part (ii) of Proposition 6.3 on p75 of [1]. One can use this to get that

(18.5.3) the direct sum of finitely many Artinian modules over k is Artinian as well,

as in Corollary 6.4 on p76 of [1].

We say that k is *Artinian* as a commutative ring if it is Artinian as a module over itself, as on p76 of [1]. In this case, if V is a finitely-generated module over k, then

(18.5.4) $$V \text{ is Artinian as a module over } k,$$

as in Proposition 6.5 on p76 of [1]. This follows from the statements mentioned in the preceding paragraph, by expressing V as the quotient of the direct sum of finitely many copies of k, as a module over itself.

Let $\mathcal{I}$ be an ideal in k, and suppose for the moment that $t \cdot v = 0$ for every $t \in \mathcal{I}$ and $v \in V$. This implies that V may be considered as a module over the quotient ring $k/\mathcal{I}$. In this case, the submodules of V as a module over k are the same as the submodules of V as a module over $k/\mathcal{I}$. It follows that V is Noetherian or Artinian as a module over k if and only if V has the same property as a module over $k/\mathcal{I}$.

If k is Noetherian or Artinian as a ring, then $k/\mathcal{I}$ has the same property as a ring, as in Proposition 6.6 on p76 of [1]. More precisely, k is Noetherian or Artinian as a module over itself, by hypothesis. This implies that $k/\mathcal{I}$ is Noetherian or Artinian as a module over k, as appropriate, as before. Equivalently, this means that $k/\mathcal{I}$ has the same property as a module over itself, and thus as a ring, as in the preceding paragraph.

Suppose for the moment that k is a field, so that V is a vector space over k. If V has finite dimension as a vector space over k, then it is easy to see that

(18.5.5) $$V \text{ is Noetherian and Artinian as a module over } k.$$

Otherwise, if V does not have finite dimension as a vector space over k, then there is an infinite sequence $v_1, v_2, v_3, \ldots$ of vectors in V that are linearly independent. One can use this to get that V is neither Noetherian nor Artinian as a module over k. This corresponds to part of Proposition 6.10 on p78 of [1].

18.6 Artinian and Noetherian rings

Let k be a commutative ring with a multiplicative identity element again, and let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be finitely many maximal proper ideals in k, which need not be distinct. Suppose that

$$\prod_{j=1}^{n} \mathcal{I}_j = \{0\}. \tag{18.6.1}$$

Under these conditions, Corollary 6.11 on p78 of [1] says that k is Noetherian if and only if k is Artinian.

To see this, note that

$$\prod_{j=1}^{l} \mathcal{I}_j \tag{18.6.2}$$

is an ideal in k for each $l = 1, \ldots, n$, which may be interpreted as being equal to k when $l = 0$. Similarly,

$$\prod_{j=1}^{l+1} \mathcal{I}_j \tag{18.6.3}$$

is an ideal in k for each $l = 0, 1, \ldots, n-1$, which is contained in (18.6.2). These ideals may be considered as modules over k, so that their quotient

$$\Big(\prod_{j=1}^{l} \mathcal{I}_j\Big) \Big/ \Big(\prod_{j=1}^{l+1} \mathcal{I}_j\Big) \tag{18.6.4}$$

is defined as a module over k. In fact, the quotient may be considered as a module over $k/\mathcal{I}_{l+1}$, because the action of any element of $\mathcal{I}_{l+1}$ on the quotient is equal to 0, by construction.

Of course, $k/\mathcal{I}_{l+1}$ is a field, because $\mathcal{I}_{l+1}$ is a maximal proper ideal in k, by hypothesis. Thus the Netherian and Artinian conditions for (18.6.4), as a module over $k/\mathcal{I}_{l+1}$, are equivalent to the quotient having finite dimension as a vector space over $k/\mathcal{I}_{l+1}$, as in the previous section. It follows that the Noetherian and Artinian conditions for (18.6.4), as a module over k, are equivalent as well.

Suppose that k is Noetherian or Artinian as a ring, and thus as a module over itself. This implies that (18.6.2) has the same property, as a module over k, for each l. It follows that the quotient (18.6.4) has the same property, as a module over k, for each l. This means that (18.6.4) is Noetherian and Artinian, as a module over k, for each l, as in the previous paragraph.

Observe that (18.6.4) is the same as (18.6.2) when $l = n - 1$, because of (18.6.1). If (18.6.4) is Artinian, as a module over k, for each l, then one can check that k is Artinian. More precisely, one can check that (18.6.2) is Artinian for each l, using the characterization in (18.5.2). Similarly, if (18.6.4) is Noetherian, as a module over k, for each l, then k is Noetherian. It follows that k is Noetherian if and only if k is Artinian in this case, using the remarks in the preceding paragraph.

18.7 Commutative Artin rings

Let k be a commutative ring with a multiplicative identity element. If k is Artinian, as in Section 18.5, then one may also call k an *Artin ring*, as on p89 of [1]. Remember that every quotient of k is an Artin ring too in this case.

If k is an Artin ring and $\mathcal{I}$ is a proper prime ideal in k, then Proposition 8.1 on p89 of [1] states that $\mathcal{I}$ is a maximal proper ideal in k. In this case, $k_1 = k/\mathcal{I}$ is an integral domain that is an Artin ring as well, and it suffices to show that k_1 is a field. To see this, let $x \in k_1$ with $x \neq 0$ be given, and observe that the sequence of ideals in k_1 generated by the powers of x decreases monotonically. Thus the sequence is eventually constant, because k_1 is an Artin ring. This implies that x^n is an element of the ideal generated by x^{n+1} for some positive integer n, so that

$$x^n = x^{n+1}\, y \tag{18.7.1}$$

for some $y \in k_1$. This means that

$$x^n\,(x\,y - 1) = 0 \tag{18.7.2}$$

in k_1, which implies that $x\,y = 1$, because k_1 is an integral domain, and $x \neq 0$. It follows that x is invertible in k_1, so that k_1 is a field, as desired.

If k is an Artin ring, then Proposition 8.3 on p89 of [1] states that k has only finitely many maximal proper ideals. Of course, this is trivial when $k = \{0\}$, and so we may as well suppose that $1 \neq 0$ in k. Because k is Artinian, the collection $\mathcal{C}$ of all ideals in k that can be expressed as the intersection of finitely many maximal proper ideals in k has a minimal element. Suppose that

$$\bigcap_{j=1}^{n} \mathcal{I}_j \tag{18.7.3}$$

is such a minimal element, where $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are maximal proper ideals in k. If $\mathcal{I}$ is any maximal proper ideal in k, then

$$\mathcal{I} \cap \Big(\bigcap_{j=1}^{n} \mathcal{I}_j\Big) \tag{18.7.4}$$

is another element of $\mathcal{C}$. Note that (18.7.4) is contained in (18.7.3), which implies that they are equal, because of the minimality of (18.7.3). This means that

$$\bigcap_{j=1}^{n} \mathcal{I}_j \subseteq \mathcal{I}. \tag{18.7.5}$$

It follows that $\mathcal{I}_l \subseteq \mathcal{I}$ for some l, because $\mathcal{I}$ is a prime ideal in k, as in Section 17.3. More precisely, $\mathcal{I} = \mathcal{I}_l$, because $\mathcal{I}_l$ is a maximal proper ideal in k. Thus $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are all of the maximal proper ideals in k, as desired.

A similar argument is used in the proof of Proposition 6 on p11 of [15], and we shall return to this in Section 18.10.

18.8 Nilpotency of the nilradical

Let k be a commutative ring with a multiplicative identity element. Remember that $\mathcal{N}$ denotes the nilradical of k, as in Section 12.5. If k is an Artin ring, then

$$\mathcal{N}^m = \{0\} \tag{18.8.1}$$

for some positive integer m, as in Proposition 8.4 on p89 of [1]. Another criterion for this was discussed in Section 14.7, which includes the case when k is Noetherian.

Of course, the sequence of powers of $\mathcal{N}$ decreases monotonically with respect to inclusion. If k is Artinian, then it follows that there is an ideal $\mathcal{I}_1$ in k such that

$$\mathcal{I}_1 = \mathcal{N}^m \tag{18.8.2}$$

for all sufficiently large m.

Suppose for the sake of a contradiction that $\mathcal{I}_1 \neq \{0\}$. Let Σ be the collection of all ideals $\widetilde{\mathcal{I}}$ in k such that

$$\mathcal{I}_1 \, \widetilde{\mathcal{I}} \neq \{0\}. \tag{18.8.3}$$

If l is a nonnegative integer, then $\mathcal{I}_1 \, \mathcal{N}^l = \mathcal{I}_1$, so that $\mathcal{N}^l \in \Sigma$. In particular, $\Sigma \neq \emptyset$. If k is Artinian, then Σ has a minimal element $\widetilde{\mathcal{I}}_0$.

Of course, $\mathcal{I}_1 \, \widetilde{\mathcal{I}}_0 \neq \{0\}$, because $\widetilde{\mathcal{I}}_0 \in \Sigma$. Thus there is an $x \in \widetilde{\mathcal{I}}_0$ whose product with an element of $\mathcal{I}_1$ is not zero. Let $\mathcal{I}(x)$ be the ideal in k generated by x, so that $\mathcal{I}(x) \in \Sigma$. Clearly $\mathcal{I}(x) \subseteq \widetilde{\mathcal{I}}_0$, because $x \in \widetilde{\mathcal{I}}_0$. This means that

$$\mathcal{I}(x) = \widetilde{\mathcal{I}}_0, \tag{18.8.4}$$

because $\widetilde{\mathcal{I}}_0$ is minimal in Σ.

Observe that

$$(\mathcal{I}(x)\, \mathcal{N})\, \mathcal{I}_1 = \mathcal{I}(x)\, (\mathcal{N}\, \mathcal{I}_1) = \mathcal{I}(x)\, \mathcal{I}_1 \neq 0, \tag{18.8.5}$$

so that $\mathcal{I}(x)\, \mathcal{N} \in \Sigma$. We also have that $\mathcal{I}(x)\, \mathcal{N} \subseteq \mathcal{I}(x)$, which implies that

$$\mathcal{I}(x)\, \mathcal{N} = \mathcal{I}(x), \tag{18.8.6}$$

because $\mathcal{I}(x)$ is minimal in Σ. It follows that

$$x = x\, y \tag{18.8.7}$$

for some $y \in \mathcal{N}$. This means that

$$x = x\, y^n \tag{18.8.8}$$

for every positive integer n. However, $y^n = 0$ for some n, because $y \in \mathcal{N}$, so that $x = 0$, which is a contradiction.

18.9 More on commutative Artin Rings

Let k be a commutative ring with a multiplicative identity element. Theorem 8.5 on p90 of [1] states that k is an Artin ring if and only if

$$k \text{ is Noetherian and the dimension of } k \text{ is equal to } 0, \tag{18.9.1}$$

in the sense of Section 14.8.

Suppose that k is an Artin ring, and note that the dimension of k is zero, because proper prime ideals in k are maximal, as in Section 18.7. We have also seen that there are only finitely many maximal proper ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ in k, which are all of the proper prime ideals in k in this case. Remember that the nilradical $\mathcal{N}$ of k is equal to the intersection of all of the prime ideals in k, as in Section 12.5. This means that

$$\mathcal{N} = \bigcap_{j=1}^{n} \mathcal{I}_j. \tag{18.9.2}$$

In particular, this implies that

$$\prod_{j=1}^{n} \mathcal{I}_j \subseteq \mathcal{N}. \tag{18.9.3}$$

Remember that $\mathcal{N}^m = \{0\}$ for some positive integer m, because k is Artinian, as in (18.8.1). It follows that

$$\prod_{j=1}^{n} \mathcal{I}_j^m = \{0\}, \tag{18.9.4}$$

because of (18.9.3). Under these conditions, the hypothesis that k be Artinian implies that k is Noetherian, as in Section 18.6.

Suppose now that (18.9.1) holds, and let us show that k is Artinian. Of course, this is trivial when $k = \{0\}$, and so we may suppose that $k \neq \{0\}$, so that $\{0\}$ is a proper ideal in k. Because k is Noetherian, $\{0\}$ has a primary decomposition in k, as in Section 17.6. This means that $\{0\}$ has a minimal primary decomposition in k, as before.

Let $\mathcal{I}_1', \ldots, \mathcal{I}_n'$ be the minimal prime ideals in k belonging to $\{0\}$ with respect to this minimal promary decomposition of $\{0\}$ in k, as in Section 17.7. These are the same as the proper prime ideals in k that are minimal with respect to inclusion, as before, and in particular this does not depend on the minimal primary decomposition of $\{0\}$. More precisely, normally one would consider the minimal proper prime ideals in k that contain the given ideal, and of course all ideals in k contain $\{0\}$.

If $\mathcal{I}_0$ is any proper prime ideal in k, then

$$\mathcal{I}_j' \subseteq \mathcal{I}_0 \tag{18.9.5}$$

for some j, as in Section 17.7. As before, this was stated previously for proper prime ideals in k that contain the given ideal, which is $\{0\}$ in this case. This implies that

$$\mathcal{N} = \bigcap_{j=1}^{n} \mathcal{I}_j'. \tag{18.9.6}$$

In particular,

$$\prod_{j=1}^{n} \mathcal{I}_j' \subseteq \mathcal{N}, \tag{18.9.7}$$

as before.

We also have that $\mathcal{N}^m = \{0\}$ for some positive integer m, because k is Noetherian, as in Section 14.7. Using this, we get that

$$\prod_{j=1}^{n} (\mathcal{I}_j')^m = \{0\}, \tag{18.9.8}$$

as before. Note that $\mathcal{I}_1', \ldots, \mathcal{I}_n'$ are maximal proper ideals in k, because they are proper prime ideals and k has dimension 0, by hypothesis. This means that the hypothesis that k be Noetherian implies that k is Artinian, as in Section 18.6 again.

18.10 Finite collections of prime ideals

Let k be a commutative ring with a nonzero multiplicative identity element, and suppose that $\bigcap_{j=1}^{n} \mathcal{I}_j = \{0\}$ is a minimal primary decomposition of $\{0\}$ in k, as in Section 17.6. If $\mathcal{I}_0$ is any prime ideal in k, then

$$r(\mathcal{I}_l) \subseteq \mathcal{I}_0 \tag{18.10.1}$$

for some l, as in Section 17.7. Remember that $r(\mathcal{I}_j)$ is a proper prime ideal in k for each j, as in Sections 14.5 and 17.5.

Suppose for the moment that k has dimension zero, in the sense of Section 14.8. If $\mathcal{I}_0$ is a proper prime ideal in k, then (18.10.1) implies that

$$r(\mathcal{I}_l) = \mathcal{I}_0. \tag{18.10.2}$$

In particular, this means that

(18.10.3) there are only finitely many proper prime ideals in k

under these conditions.

If k is Noetherian, then $\{0\}$ has a minimal primary decomposition in k, as in Section 17.6. Thus (18.10.3) holds when k is Noetherian and has dimension zero. Alternatively, k is Noetherian of dimension 0 if and only if k is an Artin ring, as in the previous section. If k is an Artin ring, then every proper prime ideal in k is a maximal proper ideal in k, and there are only finitely many maximal proper ideals in k, as in Section 18.7.

If k is Noetherian and $\mathcal{I}$ is an ideal in k, then

$$k/\mathcal{I} \text{ is Noetherian as a ring} \tag{18.10.4}$$

too. More precisely, if k is Noetherian as a ring, then k is Noetherian as a module over itself, and

$$k/\mathcal{I} \text{ is Noetherian as a module over } k \tag{18.10.5}$$

as well, as in Section 9.7. It is easy to see that

$$\text{a submodule of } k/\mathcal{I} \text{ as a module over } k \tag{18.10.6}$$

is the same as

$$\text{a submodule of } k/\mathcal{I} \text{ as a module over itself.} \tag{18.10.7}$$

One can use this to get that if $k/\mathcal{I}$ is Noetherian as a module over k, then $k/\mathcal{I}$ is Noetherian as a module over itself, and thus as a ring.

Suppose now that k is an integral domain of dimension one, in the sense of Section 14.8. Let x be a nonzero element of k, and let $\mathcal{I}(x)$ be the ideal in k generated by x. There is a natural one-to-one correspondence between the proper prime ideals in the quotient ring $k/\mathcal{I}(x)$ and the proper prime ideals in k that contain $\mathcal{I}(x)$, as in Section 17.8. Of course, an ideal in k contains $\mathcal{I}(x)$ if and only if it contains x. Using this correspondence, it is easy to see that

$$k/\mathcal{I}(x) \text{ has dimension zero,} \tag{18.10.8}$$

because $x \neq 0$, by hypothesis.

If k is Noetherian, then $k/\mathcal{I}(x)$ is Noetherian, as in (18.10.4), and it follows that

$$\text{there are only finitely many proper prime ideals in } k/\mathcal{I}(x), \tag{18.10.9}$$

as before. This implies that

$$\text{there are only finitely many proper prime ideals in } k \text{ that contain } x, \tag{18.10.10}$$

as in the preceding paragraph.

Proposition 6 on p11 of [15] states that (18.10.10) holds in particular when k is a Dedekind domain, with a more direct proof in that case. This will be discussed further in the next section.

18.11 Some arguments for Dedekind domains

Let k be a Dedekind domain, as in Section 15.11, and let Q_k be the corresponding field of fractions. Also let $x \in k$ with $x \neq 0$ be given, and let $\mathcal{I}$, $\mathcal{I}'$ be ideals in k with

$$x \in \mathcal{I} \subseteq \mathcal{I}'. \tag{18.11.1}$$

Of course, $\mathcal{I}$, $\mathcal{I}'$ are integral ideals of k, and fractional ideals of k in particular, as in Sections 11.7 and 15.7. Note that $\mathcal{I}, \mathcal{I}' \neq \{0\}$, because of (18.11.1).

It follows that $\mathcal{I}$, $\mathcal{I}'$ are invertible as fractional ideals of k, as in Section 15.12. Remember that the inverse $\mathcal{I}^{-1}$ of $\mathcal{I}$ is equal to

$$(k : \mathcal{I}) = (k : \mathcal{I})_{Q_k} = \{y \in Q_k : y\,\mathcal{I} \subseteq k\}, \tag{18.11.2}$$

as in Section 11.7, and similarly for the inverse $(\mathcal{I}')^{-1}$ of $\mathcal{I}'$. It follows that

$$k \subseteq (\mathcal{I}')^{-1} \subseteq \mathcal{I}^{-1} \subseteq x^{-1}\,k, \tag{18.11.3}$$

because of (18.11.1) and the fact that $\mathcal{I}' \subseteq k$. Note that $\mathcal{I}$ is the inverse of $\mathcal{I}^{-1}$, by the definition of the inverse in Section 11.7.

Remember that k is Noetherian as a ring, because it is a Dedekind domain, by hypothesis, as in Section 15.11. This means that k is Noetherian as a module over itself, which implies that $x^{-1}\,k$ is Notherian as a module over k. The ascending chain condition for submodules of $x^{-1}\,k$ implies the ascending chain condition for ideals in k that contain x, because of the remarks in the preceding paragraph. This corresponds to the first part of the proof of Proposition 6 on p11 of [15]. Equivalently, this means that

$$k/\mathcal{I}(x) \text{ is an Artin ring,} \tag{18.11.4}$$

as in Section 18.5, where $\mathcal{I}(x)$ is the ideal in k generated by x.

Suppose that $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots$ is an infinite sequence of proper prime ideals in k that each contain x. This implies that

$$\bigcap_{j=1}^{n} \mathcal{I}_j \tag{18.11.5}$$

is an ideal in k that contains x for each positive integer n, which defines a decreasing sequence of ideals in k, with respect to inclusion. It follows that there is a positive integer n_0 such that

$$\bigcap_{j=1}^{n} \mathcal{I}_j = \bigcap_{j=1}^{n_0} \mathcal{I}_j \tag{18.11.6}$$

when $n \geq n_0$, as in the previous paragraph. This means that

$$\bigcap_{j=1}^{n_0} \mathcal{I}_j \subseteq \mathcal{I}_l \tag{18.11.7}$$

for all positive integers l. If $l > n_0$, then we get that

$$\mathcal{I}_j \subseteq \mathcal{I}_l \tag{18.11.8}$$

for some $j \leq n_0$, as in Section 17.3.

Remember that k has dimension less than or equal to one, because k is a Dedekind domain, as in Section 15.11. This implies that $\mathcal{I}_j$ is a maximal proper ideal in k for each j, because $x \neq 0$, by hypothesis. Using this and (18.11.8), we get that each $\mathcal{I}_l$ is the same as $\mathcal{I}_j$ for some $j \leq n_0$. This means that there are only finitely many proper prime ideals in k that contain x, because otherwise we could consider an infinite sequence of distinct proper prime ideals in k that contain x. This corresponds to the second part of the proof of Proposition 6 on p11 of [15].

18.12 Valuations on Dedekind domains

Let k be an integral domain, and let v be a valuation on k, as in Section 14.15. Remember that

$$\{x \in k : v(x) \geq 0\} \tag{18.12.1}$$

is a subring of k, and that

$$\{x \in k : v(x) > 0\} \tag{18.12.2}$$

is an ideal in (18.12.1). Of course, (18.12.2) is a proper subset of (18.12.1), because $v(1) = 0$, as before.

It is easy to see that (18.12.2) is in fact a prime ideal in (18.12.1). If k is a field, then (18.12.2) is a maximal proper ideal in (18.12.1), as in Section 14.14.

Now let k be a Dedekind domain, as in Section 15.11. In particular, k is an integral domain, and we let Q_k be the corresponding field of fractions. If $\mathcal{I}$ is a proper prime ideal in k, then $S_{\mathcal{I}} = k \setminus \mathcal{I}$ is multiplicatively closed in k, as in Section 12.4, so that $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1} k$ may be defined as in Section 12.1. Suppose that $\mathcal{I} \neq \{0\}$, so that

$$k_{\mathcal{I}} \text{ is a discrete valuation ring,} \tag{18.12.3}$$

as in Section 15.1, by definition of a Dedekind domain.

We also have that $k_{\mathcal{I}}$ is an integral domain, whose field of fractions may be identified with Q_k, as in Section 14.2. We may identify $k_{\mathcal{I}}$ with a subring of Q_k, as before. As in Section 15.1, the condition that $k_{\mathcal{I}}$ be a discrete valuation ring means that there is a discrete valuation $v_{\mathcal{I}}$ on Q_k, as in Section 14.14, such that

$$k_{\mathcal{I}} = \{y \in Q_k : v_{\mathcal{I}}(y) \geq 0\}. \tag{18.12.4}$$

Remember that $S_{\mathcal{I}}^{-1} \mathcal{I}$ is the unique maximal proper ideal in $k_{\mathcal{I}}$, as in Section 12.4, and using notation mentioned near the beginning of Section 12.13. This means that

$$S_{\mathcal{I}}^{-1} \mathcal{I} = \{y \in Q_k : v_{\mathcal{I}}(y) > 0\} = \{y \in Q_k : v_{\mathcal{I}}(y) \geq 1\}, \tag{18.12.5}$$

as in Section 15.1.

It is easy to see that

$$k \cap (S_{\mathcal{I}}^{-1} \mathcal{I}) = \mathcal{I}, \tag{18.12.6}$$

because $\mathcal{I}$ is a prime ideal in k. This corresponds to a broader statement in Section 12.13. Another related statement was mentioned in Section 14.6. It follows that

$$\mathcal{I} = \{y \in k : v_{\mathcal{I}}(y) > 0\} = \{y \in k : v_{\mathcal{I}}(y) \geq 1\}. \tag{18.12.7}$$

If $x \in k$, then

$$v_{\mathcal{I}}(x) \geq 0 \tag{18.12.8}$$

for every proper prime ideal $\mathcal{I} \neq \{0\}$ in k, because of (18.12.4) and the fact that $x \in k_{\mathcal{I}}$. If we also have that $x \neq 0$, then

$$x \in \mathcal{I} \tag{18.12.9}$$

for only finitely many nonzero proper prime ideals $\mathcal{I}$ in k, as in the previous two sections. This means that

$$v_{\mathcal{I}}(x) > 0 \tag{18.12.10}$$

for only finitely many nonzero proper prime ideals $\mathcal{I}$ in k, because of (18.12.7). If $x \in Q_k$ and $x \neq 0$, then one can use the previous statement to get that

$$v_{\mathcal{I}}(x) \neq 0 \tag{18.12.11}$$

for only finitely many nonzero proper prime ideals $\mathcal{I}$ in k. This corresponds to the corollary to Proposition 6 on p11 of [15].

If x is an invertible element of k, then

$$v_{\mathcal{I}}(x) = 0 \tag{18.12.12}$$

for every nonzero proper prime ideal $\mathcal{I}$, because of (18.12.8) and the analogous statement for $1/x$. If $x \in k$, $x \neq 0$, and x is not invertible in k, then x is contained in a nonzero maximal proper ideal $\mathcal{I}$ in k, and $\mathcal{I}$ is a prime ideal in particular. This implies that (18.12.10) holds, as before.

18.13 Some remarks about ideals

Let k be a commutative ring with a multiplicative identity element, and let $\mathcal{I}$ be a proper prime ideal in k. Thus $S_{\mathcal{I}} = k \setminus \mathcal{I}$ is a multiplicatively closed set in k, as in Section 12.4, so that $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\, k$ may be defined as in Section 12.1. If $\mathcal{L}$ is an ideal in k, then $\mathcal{L}$ may be considered as a module over k, so that $S_{\mathcal{I}}^{-1}\, \mathcal{L}$ may be defined as a module over $S_{\mathcal{I}}^{-1}\, k$, as in Section 12.2. This may be identified with a submodule of $k_{\mathcal{I}}$, considered as a module over itself, as before. This means that $S_{\mathcal{I}}^{-1}\, \mathcal{L}$ is an ideal in $k_{\mathcal{I}}$, which is the same as the extension of $\mathcal{L}$ in $k_{\mathcal{I}}$ with respect to the natural ring homomorphism from k into $k_{\mathcal{I}}$, as in Section 12.13.

Let $\mathcal{L}'$ be another ideal in k with

$$\mathcal{L} \subseteq \mathcal{L}', \tag{18.13.1}$$

so that

$$S_{\mathcal{I}}^{-1}\,\mathcal{L} \subseteq S_{\mathcal{I}}^{-1}\,\mathcal{L}'. \tag{18.13.2}$$

If

$$S_{\mathcal{I}}^{-1}\,\mathcal{L} = S_{\mathcal{I}}^{-1}\,\mathcal{L}' \tag{18.13.3}$$

for every maximal proper ideal $\mathcal{I}$ in k, then

$$\mathcal{L} = \mathcal{L}'. \tag{18.13.4}$$

Indeed, the quotient $\mathcal{L}'/\mathcal{L}$ may be considered as a module over k, so that

$$S_{\mathcal{I}}^{-1}\,(\mathcal{L}'/\mathcal{L}) \tag{18.13.5}$$

may be defined as a module over $k_{\mathcal{I}}$, as in Section 12.2. This is isomorphic to

$$(S_{\mathcal{I}}^{-1}\,\mathcal{L}')/(S_{\mathcal{I}}^{-1}\,\mathcal{L}), \tag{18.13.6}$$

as modules over $k_{\mathcal{I}}$, as before. The hypothesis (18.13.3) is the same as saying that (18.13.6) is $\{0\}$, so that (18.13.5) is $\{0\}$ too. If this holds for every maximal proper ideal $\mathcal{I}$ in k, then

$$\mathcal{L}'/\mathcal{L} = \{0\}, \tag{18.13.7}$$

as in Section 12.4. Of course, this is the same as saying that (18.13.4) holds.

18.14 Ideals in Dedekind domains

Let k be a Dedekind domain, as in Section 15.11, and suppose that k is not a field. This means that the maximal proper ideals in k are the same as the nonzero prime ideals in k, because k is an integral domain of dimension one. If $\mathcal{I}$ is a nonzero proper prime ideal in k, then we take $S_{\mathcal{I}} = k \setminus \mathcal{I}$ and $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\,k$ again, so that $k_{\mathcal{I}}$ is a discrete valuation ring, as before. This leads to a discrete valuation $v_{\mathcal{I}}$ on the field Q_k of fractions of k, as in Section 18.12.

If $\mathcal{L}$ is an ideal in k, then $S_{\mathcal{I}}^{-1}\,\mathcal{L}$ is an ideal in $k_{\mathcal{I}}$, which is the same as the extension of $\mathcal{L}$ with respect to the natural ring homomorphism from k into $k_{\mathcal{I}}$, as in the previous section. Remember that this homomorphism is injective, because k is an integral domain, as in Section 14.2. Thus

$$S_{\mathcal{I}}^{-1}\,\mathcal{L} \neq \{0\} \tag{18.14.1}$$

when $\mathcal{L} \neq \{0\}$. In this case, there is a unique nonnegative integer $v_{\mathcal{I}}(\mathcal{L})$ such that

$$S_{\mathcal{I}}^{-1}\,\mathcal{L} = \{y \in k_{\mathcal{I}} : v_{\mathcal{I}}(y) \geq v_{\mathcal{I}}(\mathcal{L})\}, \tag{18.14.2}$$

as in Section 15.1. Equivalently,

$$S_{\mathcal{I}}^{-1}\mathcal{L} = \{y \in Q_k : v_{\mathcal{I}}(y) \geq v_{\mathcal{I}}(\mathcal{L})\}, \tag{18.14.3}$$

because of (18.12.4).

Observe that (18.14.2) is the same as saying that

$$S_{\mathcal{I}}^{-1}\,\mathcal{L} = (S_{\mathcal{I}}^{-1}\,\mathcal{I})^{v_{\mathcal{I}}(\mathcal{L})}, \tag{18.14.4}$$

because of (18.12.5), and where the right side is as in Section 12.10. This corresponds to the definition of $v_{\mathcal{I}}(\mathcal{L})$ on the bottom of p11 of [15]. In fact, $v_{\mathcal{I}}(\mathcal{L})$ is defined more broadly in [15], and we shall return to this in Section 19.3. One can check that

$$v_{\mathcal{I}}(\mathcal{L}) = \min\{v_{\mathcal{I}}(y) : y \in S_{\mathcal{I}}^{-1}\,\mathcal{L}\} = \min\{v_{\mathcal{I}}(y) : y \in \mathcal{L}\}. \tag{18.14.5}$$

If $y \in \mathcal{L}$, then

$$v_{\mathcal{I}}(\mathcal{L}) \le v_{\mathcal{I}}(y). \tag{18.14.6}$$

If $y \neq 0$, then

$$v_{\mathcal{I}}(y) = 0 \tag{18.14.7}$$

for all but finitely many nonzero proper prime ideals $\mathcal{I}$ in k, as in Section 18.12. This implies that

$$v_{\mathcal{I}}(\mathcal{L}) = 0 \tag{18.14.8}$$

for all but finitely many nonzero proper prime ideals $\mathcal{I}$ in k when $\mathcal{L} \neq \{0\}$, as mentioned at the bottom of p11 in [15].

One can show that $\mathcal{L}$ is uniquely determined by the $v_{\mathcal{I}}(\mathcal{L})$'s, as $\mathcal{I}$ runs through all nonzero proper prime ideals in k. This will be discussed in the next section.

If $x \in k$, then let $\mathcal{I}(x)$ be the ideal in k generated by x. If $x \neq 0$, then $\mathcal{I}(x) \neq \{0\}$, and it is easy to see that

$$v_{\mathcal{I}}(\mathcal{I}(x)) = v_{\mathcal{I}}(x) \tag{18.14.9}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k. An extension of this is mentioned on p12 of [15], and this will be discussed in Section 19.4.

Thus $\mathcal{I}(x)$ is uniquely determined, as an ideal in k, by the $v_{\mathcal{I}}(x)$'s, as $\mathcal{I}$ runs though all of the nonzero proper prime ideals in k. This means that x is determined up to multiplication by an invertible element of k by the $v_{\mathcal{I}}(x)$'s.

Let $\mathcal{L}'$ be another ideal in k, and suppose that

$$\mathcal{L} \subseteq \mathcal{L}'. \tag{18.14.10}$$

This implies that

$$S_{\mathcal{I}}^{-1}\,\mathcal{L} \subseteq S_{\mathcal{I}}^{-1}\,\mathcal{L}' \tag{18.14.11}$$

for every nonzero proper prime ideals $\mathcal{I}$ in k. Using this, it is easy to see that

$$v_{\mathcal{I}}(\mathcal{L}') \le v_{\mathcal{I}}(\mathcal{L}) \tag{18.14.12}$$

for all such $\mathcal{I}$. The converse to this will also be discussed in the next section.

18.15 Another ideal in k

Let us continue with the same notation and hypotheses as in the previous section. Let $\mathcal{L}$ be a nonzero ideal in k again, and put

$$\text{(18.15.1)} \quad \mathcal{L}_2 = \{y \in k : v_{\mathcal{I}}(y) \geq v_{\mathcal{I}}(\mathcal{L}) \text{ for every nonzero proper prime ideal } \mathcal{I} \text{ in } k\}.$$

One can check that this is an ideal in k, and that

$$\text{(18.15.2)} \qquad \mathcal{L} \subseteq \mathcal{L}_2.$$

We would like to show that

$$\text{(18.15.3)} \qquad \mathcal{L} = \mathcal{L}_2,$$

as mentioned near the top of p12 of [15]. In particular, this implies that

$$\text{(18.15.4)} \qquad \mathcal{L} \text{ is uniquely determined by the } v_{\mathcal{I}}(\mathcal{L})\text{'s},$$

where $\mathcal{I}$ runs through all nonzero proper prime ideals in k, as mentioned in the previous section.

In order to get (18.15.3), it suffices to verify that

$$\text{(18.15.5)} \qquad S_{\mathcal{I}}^{-1}\,\mathcal{L} = S_{\mathcal{I}}^{-1}\,\mathcal{L}_2$$

for every maximal proper ideal $\mathcal{I}$ in k, as in Section 18.13. These are the same as the nonzero proper prime ideals in k in this case, as mentioned at the beginning of the previous section. Of course,

$$\text{(18.15.6)} \qquad S_{\mathcal{I}}^{-1}\,\mathcal{L} \subseteq S_{\mathcal{I}}^{-1}\,\mathcal{L}_2$$

for all such $\mathcal{I}$, because of (18.15.2). Thus it is enough to check that

$$\text{(18.15.7)} \qquad S_{\mathcal{I}}^{-1}\,\mathcal{L}_2 \subseteq S_{\mathcal{I}}^{-1}\,\mathcal{L}$$

for all such $\mathcal{I}$.

This is the same as saying that

$$\text{(18.15.8)} \qquad v_{\mathcal{I}}(\mathcal{L}_2) \geq v_{\mathcal{I}}(\mathcal{L})$$

for all nonzero proper prime ideals $\mathcal{I}$ in k, because of (18.14.2) and its analogue for $\mathcal{L}_2$. Equivalently, this means that

$$\text{(18.15.9)} \qquad \min\{v_{\mathcal{I}}(y) : y \in \mathcal{L}_2\} \geq v_{\mathcal{I}}(\mathcal{L})$$

for every nonzero proper prime ideals $\mathcal{I}$ in k, because of the analogue of (18.14.5) for $\mathcal{L}_2$. This follows directly from the definition of $\mathcal{L}_2$.

Let $\mathcal{L}'$ be another nonzero ideal in k, and consider

$$\text{(18.15.10)} \quad \mathcal{L}_2' = \{y \in k : v_{\mathcal{I}}(y) \geq v_{\mathcal{I}}(\mathcal{L}') \text{ for every nonzero proper prime ideal } \mathcal{I} \text{ in } k\},$$

as before. If (18.14.12) holds for every nonzero proper prime ideal $\mathcal{I}$ in k, then

$$\mathcal{L}_2 \subseteq \mathcal{L}_2'. \tag{18.15.11}$$

This implies that (18.14.10) holds, because of (18.15.3), and its analogue for $\mathcal{L}'$.

There is another way to characterize $\mathcal{L}$ in terms of the $v_{\mathcal{I}}(\mathcal{L})$'s, as in Proposition 7 on p12 of [15]. This will be discussed in Section 19.2. In particular, the remarks in the preceding paragraph follow easily from this other characterization.

Let $x, y \in k$ be given, with $x, y \neq 0$, so that the ideals $\mathcal{I}(x)$ and $\mathcal{I}(y)$ in k that they generate are nonzero as well. Suppose that

$$v_{\mathcal{I}}(y) \leq v_{\mathcal{I}}(x) \tag{18.15.12}$$

for all nonzero proper prime ideal $\mathcal{I}$ in k, so that

$$v_{\mathcal{I}}(\mathcal{I}(y)) \leq v_{\mathcal{I}}(\mathcal{I}(y)) \tag{18.15.13}$$

for all such $\mathcal{I}$, as in (18.14.9). This implies that

$$\mathcal{I}(x) \subseteq \mathcal{I}(y), \tag{18.15.14}$$

as before. If

$$v_{\mathcal{I}}(x) = v_{\mathcal{I}}(y) \tag{18.15.15}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k, then it follows that

$$\mathcal{I}(x) = \mathcal{I}(y). \tag{18.15.16}$$

Note that $x/y \in Q_k$, and that (18.15.12) is the same as saying that

$$v_{\mathcal{I}}(x/y) = v_{\mathcal{I}}(x) - v_{\mathcal{I}}(y) \geq 0 \tag{18.15.17}$$

for all nonzero proper prime ideals $\mathcal{I}$ in k. Under these conditions, we have that

$$x/y \in k, \tag{18.15.18}$$

because $x \in \mathcal{I}(y)$, as in (18.15.14). This implies that

$$k = \{w \in Q_k : v_{\mathcal{I}}(w) \geq 0 \text{ for every nonzero proper prime ideal } \mathcal{I} \text{ in } k\}, \tag{18.15.19}$$

because $v_{\mathcal{I}}(w) \geq 0$ for every such $\mathcal{I}$ when $w \in k$, as in (18.12.8).

Similarly, if

$$v_{\mathcal{I}}(x/y) = v_{\mathcal{I}}(x) - v_{\mathcal{I}}(y) = 0 \tag{18.15.20}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k, then

$$x/y,\ y/x \in k. \tag{18.15.21}$$

This means that

$$\begin{aligned} &\{w \in k : w \text{ is invertible in } k\} \\ &\quad = \{w \in Q_k : v_{\mathcal{I}}(w) = 0 \text{ for every nonzero proper prime ideal } \mathcal{I} \text{ in } k\}, \end{aligned} \tag{18.15.22}$$

because $v_{\mathcal{I}}(w) = 0$ for every such $\mathcal{I}$ when $w \in k$ is invertible in k, as in (18.12.12).

Chapter 19

Ideals and field extensions

19.1 Products of ideals in k

Let us continue as in Sections 18.14 and 18.15, so that k is a Dedekind domain that is not a field, and for each nonzero proper prime ideal $\mathcal{I}$ in k, we take $S_{\mathcal{I}} = k \setminus \mathcal{I}$ and $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1} k$. In this case, $k_{\mathcal{I}}$ is discrete valuation ring, which leads to a discrete valuation $v_{\mathcal{I}}$ on the field Q_k of fractions of k, as in Section 18.12. If $\mathcal{L} \neq \{0\}$ is an ideal in k, then $v_{\mathcal{I}}(\mathcal{L}) \geq 0$ may be defined as in Section 18.14.

Let $\mathcal{L}' \neq \{0\}$ be another ideal in k, and consider the product $\mathcal{L}\,\mathcal{L}'$ of $\mathcal{L}$ and $\mathcal{L}'$, which is another ideal in k, as in Section 12.8. Note that

$$\mathcal{L}\,\mathcal{L}' \neq \{0\}, \tag{19.1.1}$$

because k is an integral domain. We also have that

$$S_{\mathcal{I}}^{-1}(\mathcal{L}\,\mathcal{L}') = (S_{\mathcal{I}}^{-1}\,\mathcal{L})\,(S_{\mathcal{I}}^{-1}\,\mathcal{L}') \tag{19.1.2}$$

for all nonzero proper prime ideals $\mathcal{I}$ in k, as ideals in $k_{\mathcal{I}}$, as in Section 12.13. More precisely, this was obtained from the analogous statement for extensions of products of ideals, as in Section 12.9.

One can check that

$$v_{\mathcal{I}}(\mathcal{L}\,\mathcal{L}') = v_{\mathcal{I}}(\mathcal{L}) + v_{\mathcal{I}}(\mathcal{L}') \tag{19.1.3}$$

for all nonzero proper prime ideals $\mathcal{I}$ in k, using (19.1.2) and the remarks in Section 15.1. This is stated more broadly on p12 of [15], and we shall return to that in Section 19.4.

If $\mathcal{I}$ is any nonzero proper prime ideal in k, then

$$v_{\mathcal{I}}(\mathcal{I}) = 1, \tag{19.1.4}$$

as in (18.12.5). Note that

$$v_{\mathcal{I}}(k) = 0. \tag{19.1.5}$$

Let

$$\mathcal{I}' \neq \mathcal{I} \tag{19.1.6}$$

be another nonzero proper prime ideal in k. Remember that $\mathcal{I}'$ is a maximal proper ideal in k, because k is an integral domain of dimension one, as in Section 15.11. This means that

$$\mathcal{I}' \not\subseteq \mathcal{I}. \tag{19.1.7}$$

Thus

$$S_{\mathcal{I}}^{-1}\,\mathcal{I}' = k_{\mathcal{I}}, \tag{19.1.8}$$

as in Section 12.13. It follows that

$$v_{\mathcal{I}}(\mathcal{I}') = 0 \tag{19.1.9}$$

under these conditions.

Let $\mathcal{I}_1, \ldots, \mathcal{I}_r$ be finitely many distinct nonzero proper prime ideals in k, and let $n_1, \ldots, n_r$ be nonnegative integers. Remember that $\mathcal{I}_j^{n_j}$ may be defined as an ideal in k as in Section 12.10, so that

$$\prod_{j=1}^{r} \mathcal{I}_j^{n_j} = \mathcal{I}_1^{n_1} \cdots \mathcal{I}_r^{n_r} \tag{19.1.10}$$

may be defined as an ideal in k as in Section 12.8.

If $\mathcal{I}$ is a nonzero proper prime ideal in k, then

$$\begin{aligned} v_{\mathcal{I}}\Big(\prod_{j=1}^{r} \mathcal{I}_j^{n_j}\Big) &= n_l \quad \text{when } \mathcal{I} = \mathcal{I}_l \text{ for some } l \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{19.1.11}$$

This follows from (19.1.3), (19.1.4), (19.1.5), and (19.1.9).

19.2 More on ideals in k

Let us continue with the same notation and hypotheses as in the previous section, and let $\mathcal{L}$ be a nonzero ideal in k again. Remember that $v_{\mathcal{I}}(\mathcal{L}) = 0$ for all but finitely many nonzero proper prime ideals $\mathcal{I}$ in k, as in Section 18.14. Put

$$\mathcal{L}_1 = \prod_{\mathcal{I}} \mathcal{I}^{v_{\mathcal{I}}(\mathcal{L})}, \tag{19.2.1}$$

as mentioned at the top of p12 of [15]. Although the product is formally taken over all nonzero proper prime ideals $\mathcal{I}$ in k, this is interpreted as the product over any finite set of such ideals that includes all of those with

$$v_{\mathcal{I}}(\mathcal{L}) > 0. \tag{19.2.2}$$

If $\mathcal{I}'$ is any nonzero proper prime ideal in k, then

$$v_{\mathcal{I}'}(\mathcal{L}_1) = v_{\mathcal{I}'}(\mathcal{L}), \tag{19.2.3}$$

as in (19.1.11). This implies that

$$\mathcal{L} = \mathcal{L}_1, \tag{19.2.4}$$

as in Section 18.15.

This corresponds to the existence part of Proposition 7 on p12 of [15], and the uniqueness part follows from (19.1.11). This is stated more broadly in [15], and we shall return to this in Section 19.5. Remember that the existence and uniqueness of this type of expression for $\mathcal{L}$ was also discussed in Section 18.4, using another approach.

Let $\mathcal{L}' \neq \{0\}$ be another ideal in k, and suppose that

$$v_{\mathcal{I}}(\mathcal{L}') \leq v_{\mathcal{I}}(\mathcal{L}) \tag{19.2.5}$$

for all nonzero proper prime ideals $\mathcal{I}$ in k. This implies that

$$\mathcal{L} \subseteq \mathcal{L}', \tag{19.2.6}$$

as in Sections 18.14 and 18.15. This could also be obtained from the expression for $\mathcal{L}$ as in (19.2.1), and its analogue for $\mathcal{L}'$.

If $x \in k$ and $x \neq 0$, then the ideal $\mathcal{I}(x)$ in k generated by x is nonzero too. Remember that $v_{\mathcal{I}}(x) = 0$ for all but finitely many nonzero proper prime ideals $\mathcal{I}$ in k, as in Section 18.12. Observe that

$$\mathcal{I}(x) = \prod_{\mathcal{I}} \mathcal{I}^{v_{\mathcal{I}}(x)}, \tag{19.2.7}$$

as in (19.2.1), because $v_{\mathcal{I}}(\mathcal{I}(x)) = v_{\mathcal{I}}(x)$ for every nonzero proper prime ideal $\mathcal{I}$ in k, as in (18.14.9). This gives another way to look at the statements about $\mathcal{I}(x)$ mentioned in Section 18.15.

19.3 Fractional ideals of k

Let k be a Dedekind domain that is not a field again, and for each nonzero proper prime ideal $\mathcal{I}$ in k, take $S_{\mathcal{I}} = k \setminus \mathcal{I}$ and $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\, k$, as before. Remember that $k_{\mathcal{I}}$ is a discrete valuation ring, which leads to a discrete valuation $v_{\mathcal{I}}$ on the field Q_k of fractions of k, as in Section 18.12.

As in Section 11.7, a fractional ideal of k is a submodule M of Q_k, as a module over k, such that

$$x\, M \subseteq k \tag{19.3.1}$$

for some $x \in k$ with $x \neq 0$. If $\mathcal{I}$ is a nonzero proper prime ideal in k, then one can defined fractional ideals of $k_{\mathcal{I}}$ in the same way. Remember that the field of fractions of $k_{\mathcal{I}}$ may be identified with Q_k, as in Section 14.2, so that a fractional ideal of $k_{\mathcal{I}}$ may be considered as a submodule of Q_k, as a module over $k_{\mathcal{I}}$.

If l is an integer, then

$$\{y \in Q_k : v_{\mathcal{I}}(y) \geq l\} \tag{19.3.2}$$

is a fractional ideal of $k_{\mathcal{I}}$. One can check that every nonzero fractional ideal of $k_{\mathcal{I}}$ is of this form, as in Section 15.7. Note that l is uniquely determined by (19.3.2), because $v_{\mathcal{I}}$ is a discrete valuation on Q_k, as in Sections 14.14 and 15.1. Remember that (19.3.2) is equal to $k_{\mathcal{I}}$ when $l = 0$, as in Sections 15.1 and 18.12.

Let M be a fractional ideal of k, so that $S_{\mathcal{I}}^{-1} M$ may be defined as a module over $k_{\mathcal{I}}$, as in Section 12.2. We may consider $S_{\mathcal{I}}^{-1} M$ as a submodule of $S_{\mathcal{I}}^{-1} Q_k$, as a module over $k_{\mathcal{I}}$, and $S_{\mathcal{I}}^{-1} Q_k$ may be identified with Q_k, as a module over $k_{\mathcal{I}}$, as in Section 14.2. Thus $S_{\mathcal{I}}^{-1} M$ may be considered as a submodule of Q_k, as a module over $k_{\mathcal{I}}$, as before.

It is easy to see that $S_{\mathcal{I}}^{-1} M$ is a fractional ideal of $k_{\mathcal{I}}$, as in Section 14.2. Suppose that $M \neq \{0\}$, which implies that

$$S_{\mathcal{I}}^{-1} M \neq \{0\}. \tag{19.3.3}$$

This implies that there is a unique integer $v_{\mathcal{I}}(M)$ such that

$$S_{\mathcal{I}}^{-1} M = \{y \in Q_k : v_{\mathcal{I}}(y) \geq v_{\mathcal{I}}(M)\}, \tag{19.3.4}$$

as before. If M is an ordinary ideal in k, then this is the same as the definition of $v_{\mathcal{I}}(M)$ in Section 18.14.

Remember that $S_{\mathcal{I}}^{-1} \mathcal{I}$ is the same as (19.3.2), with $l = 1$, as in Section 18.12. If $l \geq 0$, then

$$(S_{\mathcal{I}}^{-1} \mathcal{I})^l \tag{19.3.5}$$

may be defined as an ideal in $k_{\mathcal{I}}$, as in Section 12.10, and it is easy to see that this is the same as (19.3.2). We also have that (19.3.2) is invertible as a fractional ideal of $k_{\mathcal{I}}$ for every integer l, with inverse

$$\{y \in Q_k : v_{\mathcal{I}}(y) \geq -l\}, \tag{19.3.6}$$

as in Section 15.7. Thus in fact (19.3.2) is the same as (19.3.5) for every integer l, where (19.3.5) is defined in terms of inverses when $l < 0$. This means that (19.3.4) is the same as saying that

$$S_{\mathcal{I}}^{-1} M = (S_{\mathcal{I}}^{-1} \mathcal{I})^{v_{\mathcal{I}}(M)}, \tag{19.3.7}$$

which corresponds to the definition of $v_{\mathcal{I}}(M)$ on the bottom of p11 of [15].

If $x \in k$, $x \neq 0$, is as in (19.3.1), then

$$v_{\mathcal{I}}(y) \geq -v_{\mathcal{I}}(x) \tag{19.3.8}$$

for every $y \in M$. One can check that

$$v_{\mathcal{I}}(M) = \min\{v_{\mathcal{I}}(y) : y \in S_{\mathcal{I}}^{-1} M\} = \min\{v_{\mathcal{I}}(y) : y \in M\}, \tag{19.3.9}$$

as in Section 18.14. In particular,

$$v_{\mathcal{I}}(M) \geq -v_{\mathcal{I}}(x). \tag{19.3.10}$$

Of course,

$$v_{\mathcal{I}}(M) \leq v_{\mathcal{I}}(y) \tag{19.3.11}$$

for every $y \in M$.

If $w \in Q_k$ and $w \neq 0$, then $v_{\mathcal{I}}(w) = 0$ for all but finitely many nonzero proper prime ideals $\mathcal{I}$ in k, as in Section 18.12. This implies that

$$v_{\mathcal{I}}(M) = 0 \tag{19.3.12}$$

for all but finitely many nonzero proper prime ideals $\mathcal{I}$ in k, because of (19.3.10) and (19.3.11). This is mentioned at the bottom of p11 of [15].

19.4 More on $v_{\mathcal{I}}(M)$

Let us continue with the same notation and hypotheses as in the previous section. If $w \in Q_k$ and $w \neq 0$, then it is easy to see that

$$w\,M \text{ is another nonzero fractional ideal of } k. \tag{19.4.1}$$

We also have that

$$v_{\mathcal{I}}(w\,M) = v_{\mathcal{I}}(w) + v_{\mathcal{I}}(M). \tag{19.4.2}$$

In particular,

$$v_{\mathcal{I}}(w\,k) = v_{\mathcal{I}}(w), \tag{19.4.3}$$

because $v_{\mathcal{I}}(k) = 0$, as in Section 19.1. This is mentioned on p12 of [15].

Let M' be another nonzero fractional ideal of k. Remember that there is an $x \in k$ such that $x \neq 0$ and (19.3.1) holds, and similarly there is an $x' \in k$ such that $x' \neq 0$ and

$$x'\,M' \subseteq k. \tag{19.4.4}$$

It follows that $x\,x' \neq 0$, because k is an integral domain, and

$$x\,x'\,M,\ x\,x'\,M' \subseteq k, \tag{19.4.5}$$

because M, M' are submodules of Q_k, as a module over k.

If

$$v_{\mathcal{I}}(M) = v_{\mathcal{I}}(M') \tag{19.4.6}$$

for all nonzero proper prime ideals $\mathcal{I}$ in k, then

$$M = M'. \tag{19.4.7}$$

This follows from the analogous statement for nonzero ideals in k, as in Section 18.15, and (19.4.3), (19.4.5). Similarly, if

$$v_{\mathcal{I}}(M') \leq v_{\mathcal{I}}(M) \tag{19.4.8}$$

for all nonzero proper prime ideals $\mathcal{I}$ in k, then

(19.4.9) $$M \subseteq M',$$

as before.

Alternatively, if M is a nonzero fractional ideal of k, then it is easy to see that

(19.4.10) $$M_2 = \{y \in Q_k : v_{\mathcal{I}}(y) \geq v_{\mathcal{I}}(M) \text{ for every nonzero proper prime ideal } \mathcal{I} \text{ in } k\}$$

is a submodule of Q_k, as a module over k, with

(19.4.11) $$M \subseteq M_2.$$

One can show that

(19.4.12) $$M = M_2,$$

using the same type of argument as in Section 18.15, as indicated near the top of p12 of [15]. This uses some remarks in Sections 12.2 and 12.4, as in Section 18.13.

If M, N are submodules of Q_k, as a module over k, then their product $M\,N$ may be defined as a submodule of Q_k, as a module over k, as in Section 11.7. This includes products of ideals in k, as in Section 12.8. If M and N are fractional ideals of k, then it is easy to see that

(19.4.13) $$M\,N \text{ is a fractional ideal of } k$$

too. If $M, N \neq \{0\}$, then

(19.4.14) $$M\,N \neq \{0\}$$

as well.

If $\mathcal{I}$ is a nonzero proper prime ideal in k again, then

(19.4.15) $$S_{\mathcal{I}}^{-1}(M\,N) = (S_{\mathcal{I}}^{-1}\,M)\,(S_{\mathcal{I}}^{-1}\,N),$$

as in Section 14.3. One can use this to get that

(19.4.16) $$v_{\mathcal{I}}(M\,N) = v_{\mathcal{I}}(M) + v_{\mathcal{I}}(N)$$

for all nonzero fractional ideals M, N of k, as on p12 of [15].

Remember that every nonzero fractional ideal M of k is invertible, because k is a Dedekind domain, as in Section 15.12. The inverse M^{-1} satisfies

(19.4.17) $$v_{\mathcal{I}}(M^{-1}) = -v_{\mathcal{I}}(M),$$

because of (19.4.16).

Let $\mathcal{I}_1, \ldots, \mathcal{I}_r$ be finitely many distinct nonzero proper prime ideals in k, and let $n_1, \ldots, n_r$ be integers. We can define

(19.4.18) $$\mathcal{I}_j^{n_j}$$

as a nonzero fractional ideal of k, using inverses when n_j is negative. Thus

$$\prod_{j=1}^{r} \mathcal{I}_j^{n_j} = \mathcal{I}_1^{n_1} \cdots \mathcal{I}_r^{n_r} \tag{19.4.19}$$

may be defined as a nonzero fractional ideal of k as well. If $\mathcal{I}$ is a nonzero proper prime ideal in k, then

$$\begin{aligned} v_{\mathcal{I}}\Big(\prod_{j=1}^{r} \mathcal{I}_j^{n_j}\Big) &= n_l \quad \text{when } \mathcal{I} = \mathcal{I}_l \text{ for some } l \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{19.4.20}$$

as in Section 19.1.

19.5 More on fractional ideals

Let us continue with the same notation and hypotheses as in the previous two sections. Let M be a nonzero fractional ideal of k again, and remember that $v_{\mathcal{I}}(M) = 0$ for all but finitely many nonzero proper prim ideals $\mathcal{I}$ in k, as in (19.3.12). Consider

$$M_1 = \prod_{\mathcal{I}} \mathcal{I}^{v_{\mathcal{I}}(M)}, \tag{19.5.1}$$

as mentioned at the top og p12 of [15], which may be interpreted as the product over any finite set of nonzero proper prime ideals $\mathcal{I}$ in k that includes all of those with

$$v_{\mathcal{I}}(M) \neq 0. \tag{19.5.2}$$

If $\mathcal{I}'$ is any nonzero proper prime ideal in k, then

$$v_{\mathcal{I}'}(M_1) = v_{\mathcal{I}'}(M), \tag{19.5.3}$$

as in (19.4.20), so that

$$M = M_1, \tag{19.5.4}$$

as in the previous section. This corresponds to Proposition 7 on p12 of [15], as in Section 19.2.

If M and N are submodules of Q_k, as a module over k, then put

$$(M : N)_{Q_k} = \{y \in Q_k : y\, N \subseteq M\}, \tag{19.5.5}$$

as on p11 of [15]. This is the same as in Section 14.3 when $M = k$, which corresponds to $(k : N)$ in the notation of Section 11.7. As in Section 14.3, the subscript Q_k is included because similar notation was used in Section 12.8 for something else. It is easy to see that (19.5.5) is a submodule of Q_k, as a module over k, with

$$(M : N)_{Q_k}\, N \subseteq M. \tag{19.5.6}$$

If N is an invertible ideal of k, as in Section 11.7, then it follows that

$$(M : N)_{Q_k} = (M : N)_{Q_k} \, N \, N^{-1} \subseteq M \, N^{-1}. \tag{19.5.7}$$

In this case, we also have that

$$M \, N^{-1} \, N = M, \tag{19.5.8}$$

which implies that

$$M \, N^{-1} \subseteq (M : N)_{Q_k}. \tag{19.5.9}$$

This means that

$$(M : N)_{Q_k} = M \, N^{-1} \tag{19.5.10}$$

under these conditions. This was also mentioned in Section 11.7 when $M = k$, and it is implicit in a remark on p12 of [15]. Note that the remarks in this and the preceding paragraph can be used when k is any integral domain.

If M and N are nonzero fractional ideals of k, then they are invertible, because k is a Dedekind domain, as in Section 15.12. If $\mathcal{I}$ is a nonzero proper prime ideal in k, then we get that

$$v_{\mathcal{I}}((M : N)_{Q_k}) = v_{\mathcal{I}}(M \, N^{-1}) = v_{\mathcal{I}}(M) - v_{\mathcal{I}}(N), \tag{19.5.11}$$

as on p12 of [15]. This uses (19.4.16), (19.4.17) in the second step.

If M, M' are nonzero fractional ideals of k with

$$M \subseteq M', \tag{19.5.12}$$

then

$$S_{\mathcal{I}}^{-1} \, M \subseteq S_{\mathcal{I}}^{-1} \, M' \tag{19.5.13}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k. This implies that

$$v_{\mathcal{I}}(M') \leq v_{\mathcal{I}}(M), \tag{19.5.14}$$

by the definition of $v_{\mathcal{I}}(M)$.

If M and N are fractional ideals of k, then there is a $y \in k$ such that $y \neq 0$ and

$$y \, M, \, y \, N \subseteq k, \tag{19.5.15}$$

as in (19.4.5). Using this, it is easy to see that

$$M + N \text{ is a fractional ideal of } k, \tag{19.5.16}$$

which is nonzero if M or N is nonzero. Of course,

$$M, \, N \subseteq M + N, \tag{19.5.17}$$

so that

$$v_{\mathcal{I}}(M + N) \leq v_{\mathcal{I}}(M), \, v_{\mathcal{I}}(N) \tag{19.5.18}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k, as in (19.5.14), when $M, N \neq \{0\}$. In fact, one can check that

$$v_{\mathcal{I}}(M + N) = \min(v_{\mathcal{I}}(M), v_{\mathcal{I}}(N)), \tag{19.5.19}$$

as mentioned on p12 of [15].

19.6 Using some ideals in k

Let k be a Dedekind domain that is not a field, and take $S_{\mathcal{I}} = k \setminus \mathcal{I}$ and $k_{\mathcal{I}} = S_{\mathcal{I}}^{-1}\, k$ for each nonzero proper prime ideal $\mathcal{I}$ in k, as before. This leads to a discrete valuation $v_{\mathcal{I}}$ on the field Q_k of fractions of k, as in Section 18.12.

Let

$$\mathcal{I}_1, \ldots, \mathcal{I}_l \tag{19.6.1}$$

be finitely many distinct nonzero proper prime ideals in k, and let

$$x_1, \ldots, x_l \in Q_k \tag{19.6.2}$$

and integers $n_1, \ldots, n_l$ be given. Under these conditions, there is an $x \in Q_k$ such that

$$v_{\mathcal{I}_j}(x - x_j) \ge n_j \tag{19.6.3}$$

for each $j = 1, \ldots, l$, and

$$v_{\mathcal{I}}(x) \ge 0 \tag{19.6.4}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k that is not one of the ideals $\mathcal{I}_1, \ldots, \mathcal{I}_l$. This is the approximation lemma on p12 of [15].

Suppose first that

$$x_1, \ldots, x_l \in k. \tag{19.6.5}$$

If $l = 1$, then we can take $x_1 = x$, and so we may as well suppose that $l \ge 2$. It is easy to reduce to the case where all but one of $x_1, \ldots, x_l$ is equal to 0. We may as well suppose that

$$x_j = 0 \text{ for } j \ge 2, \tag{19.6.6}$$

by rearranging the $\mathcal{I}_j$'s if necessary. We may as well suppose also that $n_j \ge 0$ for each j.

Consider

$$\mathcal{L} = \mathcal{I}_1^{n_1} + \prod_{j=2}^{l} \mathcal{I}_j^{n_j}, \tag{19.6.7}$$

which is an ideal in k. We would like to check that

$$v_{\mathcal{I}}(\mathcal{L}) = 0 \tag{19.6.8}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k. Observe that

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{L}) &= \min\Big(v_{\mathcal{I}}(\mathcal{I}_1^{n_1}), v_{\mathcal{I}}\Big(\prod_{j=2}^{l} \mathcal{I}_j^{n_j}\Big)\Big) \\ &= \min\Big(n_1\, v_{\mathcal{I}}(\mathcal{I}_1), \sum_{j=2}^{l} n_j\, v_{\mathcal{I}}(\mathcal{I}_j)\Big), \end{aligned} \tag{19.6.9}$$

where the first step is as in (19.5.19), and the second step is as in Section 19.1.

Remember that $v_{\mathcal{I}}(\mathcal{I}_j) \geq 0$ for each j, as in Section 18.14. If $\mathcal{I} = \mathcal{I}_1$, then (19.6.8) follows from the fact that

$$v_{\mathcal{I}_1}(\mathcal{I}_j) = 0 \tag{19.6.10}$$

when $j \geq 2$, because $\mathcal{I}_1 \neq \mathcal{I}_j$, as in Section 19.1. Similarly, if $\mathcal{I} \neq \mathcal{I}_1$, then (19.6.8) follows from the fact that

$$v_{\mathcal{I}}(\mathcal{I}_1) = 0. \tag{19.6.11}$$

Using (19.6.8), we get that

$$\mathcal{L} = k, \tag{19.6.12}$$

as in Section 18.15. This means that x_1 may be expressed as

$$x_1 = x + y, \tag{19.6.13}$$

where $y \in \mathcal{I}_1^{n_1}$, and $x \in \prod_{j=2}^{l} \mathcal{I}_j^{n_j}$. It is easy to see that x has the desired properties in this case.

We shall consider arbitrary elements $x_1, \ldots, x_l$ of Q_k in the next section. In fact, we shall see that the problem can be reduced to the one considered in this section. A corollary of this will also be discussed.

19.7 Reducing to the previous case

Let us continue with the same notation and hypotheses as in the previous section. If $x_1, \ldots, x_l \in Q_k$, then each x_j may be expressed as

$$x_j = w_j/r, \tag{19.7.1}$$

where $w_j \in k$, $r \in k$ does not depend on j, and $r \neq 0$. In this case, we would like to find $x \in Q_k$ of the form

$$x = w/r, \tag{19.7.2}$$

$w \in k$, with the desired properties. These properties may be restated as saying that

$$v_{\mathcal{I}_j}(w - w_j) \geq n_j + v_{\mathcal{I}_j}(r) \tag{19.7.3}$$

for each $j = 1, \ldots, l$, and

$$v_{\mathcal{I}}(w) \geq v_{\mathcal{I}}(r) \tag{19.7.4}$$

for every nonzero proper prime ideal $\mathcal{I}$ in k that is not one of the ideals $\mathcal{I}_1, \ldots, \mathcal{I}_l$.

Remember that $v_{\mathcal{I}}(r) = 0$ for all but finitely many nonzero proper prime ideals $\mathcal{I}$ in k, because $r \neq 0$, as in Section 18.12. Of course, (19.7.4) is the same as saying that

$$v_{\mathcal{I}}(w) \geq 0 \tag{19.7.5}$$

when $v_{\mathcal{I}}(r) = 0$, and this condition holds automatically when $w \in k$. This means that there are only finitely many nonzero proper prime ideals $\mathcal{I}$ that are not one of the ideals $\mathcal{I}_1, \ldots, \mathcal{I}_l$, and for which (19.7.4) is more restrictive than (19.7.5). These ideals may be added to the collection $\mathcal{I}_1, \ldots, \mathcal{I}_l$, to get conditions like those considered in the first case. This shows that this second case may be obtained from the first case, using this larger collection of prime ideals in k.

Suppose now that

$$k \text{ has only finitely many prime ideals.} \tag{19.7.6}$$

Under these conditions, the corollary on p12 of [15] states that

$$k \text{ is a principal ideal domain.} \tag{19.7.7}$$

Remember that principal ideal domains are Dedekind domains, as in Section 15.11.

Remember also that ideals in k may be expressed as products of prime ideals, as in Section 19.2. In order to show that every ideal in k is principal, it suffices to show that every nonzero proper prime ideal $\mathcal{I}'$ in k is principal. Let us take

$$\mathcal{I}_1 = \mathcal{I}', \tag{19.7.8}$$

and let $\mathcal{I}_2, \ldots, \mathcal{I}_l$ be a list of all other nonzero proper prime ideals in k, without any repetitions.

Let x_1 be an element of $\mathcal{I}_1$ such that

$$v_{\mathcal{I}_1}(x_1) = 1. \tag{19.7.9}$$

More precisely, there is an element of $k_{\mathcal{I}_1} = S_{\mathcal{I}_1}^{-1} k$ with this property, because $k_{\mathcal{I}_1}$ is a discrete valuation ring. Any element of $k_{\mathcal{I}_1}$ with this property is an element of $S_{\mathcal{I}_1}^{-1} \mathcal{I}_1$, as in Section 18.12. Of course, the elements of $S_{\mathcal{I}_1} = k \setminus \mathcal{I}_1$ are invertible in $k_{\mathcal{I}_1}$, by construction, so that $v_{\mathcal{I}_1} = 0$ on $S_{\mathcal{I}_1}$. This leads to an element of $\mathcal{I}_1$ as in (19.7.9).

Let us take

$$x_j = 1 \text{ for } j \ge 2, \tag{19.7.10}$$

and

$$n_1 = 2, \; n_j = 1 \text{ for } j \ge 2. \tag{19.7.11}$$

One can use the argument in the previous section to get $x \in k$ such that

$$v_{\mathcal{I}_1}(x) = 1, \; v_{\mathcal{I}_j}(x) = 0 \text{ for } j \ge 2. \tag{19.7.12}$$

This means that

$$v_{\mathcal{I}_1}(x\,k) = 1, \; v_{\mathcal{I}_j}(x\,k) = 0 \text{ for } j \ge 2. \tag{19.7.13}$$

These are the same as the values of $v_{\mathcal{I}_j}(\mathcal{I}_1)$, as in Section 19.1. This implies that

$$\mathcal{I}_1 = x\,k, \tag{19.7.14}$$

as in Section 18.15.

19.8 A result about linear independence

Let k be a field, and let Σ be a nonempty semigroup, with the semigroup operation expressed multiplicatively. A homomorphism from Σ into the multiplicative group $k \setminus \{0\}$ may be called a *character* of Σ, as on p208 of [11].

Theorem 7 on p209 of [11], due to Artin, says that

(19.8.1) the characters on Σ of this type are linearly independent

in the space of all functions on Σ with values in k, as a vector space over k. This means that if $\chi_1, \ldots, \chi_n$ are finitely many distinct characters on Σ of this type, and if

$$a_1\,\chi_1 + \cdots + a_n\,\chi_n = 0 \tag{19.8.2}$$

for some $a_1, \ldots, a_n \in k$, then $a_1 = \cdots = a_n = 0$. Of course, this is clear when $n = 1$.

Suppose for the sake of a contradiction that (19.8.2) holds for some $n \geq 2$, and where $a_1, \ldots, a_n$ are not all 0. We may as well suppose also that n is as small as possible, which implies that

$$a_j \neq 0 \tag{19.8.3}$$

for each j.

Because $\chi_1 \neq \chi_2$, there is a $z \in \Sigma$ such that

$$\chi_1(z) \neq \chi_2(z). \tag{19.8.4}$$

If $x \in \Sigma$, then

$$a_1\,\chi_1(x\,z) + \cdots + a_n\,\chi_n(x\,z) = 0, \tag{19.8.5}$$

by hypothesis. This means that

$$a_1\,\chi_1(z)\,\chi_1 + \cdots + a_n\,\chi_n(z)\,\chi_n = 0 \tag{19.8.6}$$

on Σ, because $\chi_1, \ldots, \chi_n$ are characters.

It follows that

$$a_2\,(\chi_2(z)\,\chi_1(z)^{-1} - 1)\,\chi_2 + \cdots + a_n\,(\chi_n(z)\,\chi_1(z)^{-1} - 1)\,\chi_n = 0 \tag{19.8.7}$$

on Σ, by multiplying the left side of (19.8.6) by $\chi_1(z)^{-1}$, and subtracting the left side of (19.8.2). The coefficient of χ_2 is not zero, because of (19.8.3) and (19.8.4). Thus we can reduce the size of the relation, so that n is not minimal.

19.9 Algebraic elements and extensions

Let k be a field, and let X be an indeterminate. A *formal polynomial* in X with coefficients in k may be expressed as

$$f(X) = \sum_{j=0}^{n} f_j\,X^j \tag{19.9.1}$$

for some nonnegative integer n, with $f_j \in k$ for each j, as in Section 4.3. If

$$f_n \neq 0, \tag{19.9.2}$$

then the *degree* of $f(X)$ is equal to n. If

$$f_n = 1, \tag{19.9.3}$$

then $f(X)$ is a *monic* polynomial in X of degree n.

Let k_1 be a field that contains k as a subfield. Note that k_1 may be considered as a commutative associative algebra over k. If $x \in k_1$, then

$$f(x) = \sum_{j=0}^{n} f_j\, x^j \tag{19.9.4}$$

defines an element of k_1, as in Section 4.4. Remember that

$$f(X) \mapsto f(x) \tag{19.9.5}$$

defines a homomorphism from $k[X]$ into k_1, as algebras over k, under these conditions. If

$$f(x) = 0, \tag{19.9.6}$$

then x is said to be a *root* of $f(X)$ in k_1.

Of course, k_1 may be considered as a vector space over k in particular. If k_1 has finite dimension as a vector space over k, then k_1 may be called a *finite extension* of k. The dimension of k_1, as a vector space over k, may be denoted

$$[k_1 : k], \tag{19.9.7}$$

and may be called the *degree* of k_1 over k.

Let k_2 be a field that contains k_1 as a subfield, so that k_2 contains k as a subfield as well. If k_1 is a finite extension of k and k_2 is a finite extension of k_1, then it is well known that k_2 is a finite extension of k, with

$$[k_2 : k] = [k_2 : k_1]\,[k_1 : k]. \tag{19.9.8}$$

This corresponds to Corollary 1 on p433 of [2], Proposition 2 on p162 of [11], and Theorem 3 on p443 of [12].

An element of k_1 is said to be *algebraic* over k if it is the root of a polynomial whose coefficients are in k and not all equal to 0, as on p421 of [2], p161 of [11], and p439 of [12]. If

(19.9.9) every element of k_1 is algebraic over k,

then k_1 is said to be *algebraic* as an extension of k, as on p161 of [11], and p456 of [12]. If k_1 is a finite extension of k, then it is well known that k_1 is algebraic over k, as in Theorem 8 on p430 of [2], Proposition 1 on p161 of [11], and mentioned on p443 of [12].

It is also well known that an algebraic element x of k_1 over k is the root of a unique monic polynomial of minimal degree with coefficients in k, and that this polynomial is irreducible. This may be called the *minimal polynomial* of x over k, as on p423 of [2], and p440 of [12]. This is denoted $\mathrm{Irr}(x, k, X)$ on p161 of [11].

If x is any element of k_1, then let $k(x)$ be the subfield of k_1 generated by k and x. If x is algebraic over k, then

(19.9.10) $$k(x) \text{ is a finite extension of } k,$$

with degree equal to the degree of the minimal polynomial of x over k. This corresponds to Theorem 7 on p429 of [2], Proposition 3 on p163 of [11], and a remark on p443 of [12].

Similarly, if $x_1, \ldots, x_n$ are finitely many elements of k_1, then $k(x_1, \ldots, x_n)$ denotes the subfield of k_1 generated by k and $x_1, \ldots, x_n$. If $x_1, \ldots, x_n$ are algebraic over k, then

(19.9.11) $$k(x_1, \ldots, x_n) \text{ is a finite extension of } k,$$

as in Proposition 5 on p165 of [11]. This also works when x_1 is algebraic over k and x_l is algebraic over $k(x_1, \ldots, x_{l-1})$ when $l \geq 2$, as in Corollary 4 on p433 of [2].

If E is a finite extension of k, then E is generated by k and finitely many elements of E, as in Proposition 4 on p164 of [11].

19.10 Distinguished classes and the compositum

Let L be a field, and let k, E, and F be subfields of L, with

(19.10.1) $$k \subseteq E, F.$$

As on p163 of [11],

(19.10.2) $$\text{the } \textit{compositum} \text{ of } E \text{ and } F$$

is defined to be the smallest subfield of L that contains E and F. This may be denoted $E\,F$, as in [11], and of course

(19.10.3) $$k \subseteq E\,F.$$

One may consider $E\,F$ as an extension of F, which may be called the *translation* or *lifting* of E, as an extension of k, to F, as on p164 of [11].

Similarly, the *compositum* of any nonempty family of subfields of L may be defined as the smallest subfield of L that contains all of the subfields in the family, as on p164 of [11].

Let $x_1, \ldots, x_n$ be finitely many elements of L, and suppose for the moment that

(19.10.4) $$E = k(x_1, \ldots, x_n)$$

is the subfield of L generated by k and $x_1, \ldots, x_n$, as in the previous section. One may say that E is *finitely generated* over k in this case, as on p164 of [11]. One can check that

$$E\,F = F(x_1, \ldots, x_n) \tag{19.10.5}$$

under these conditions, as in [11]. Equivalently, the compositum of E and F is the subfield of L generated by F and $x_1, \ldots, x_n$. Indeed, $k(x_1, \ldots, x_n)$ consists of quotients of polynomials in $x_1, \ldots, x_n$ with coefficients in k, where the denominator is nonzero, as in [11], and similarly for $F(x_1, \ldots, x_n)$.

Let $\mathcal{C}$ be a class of extensions of fields. We say that

$$\mathcal{C} \text{ is } \textit{distinguished} \tag{19.10.6}$$

if it satisfies the following three properties, as on p165 of [11].

Let k, E, F be fields, with

$$k \subseteq F \subseteq E. \tag{19.10.7}$$

The first condition is that

$$E \text{ is in } \mathcal{C} \text{ as an extension of } k \tag{19.10.8}$$

if and only if

$$F \text{ is in } \mathcal{C} \text{ as an extension of } k \tag{19.10.9}$$

and

$$E \text{ is in } \mathcal{C} \text{ as an extension of } F. \tag{19.10.10}$$

Let k, E, F be subfields of a field L that satisfy (19.10.1). If (19.10.8) holds, then the second condition asks that

$$E\,F \text{ is in } \mathcal{C}, \text{ as an extension of } F. \tag{19.10.11}$$

Let k, E, F be subfields of a field L that satisfy (19.10.1) again. If (19.10.8) and (19.10.9) hold, then the third condition asks that

$$E\,F \text{ is in } \mathcal{C}, \text{ as an extension of } k. \tag{19.10.12}$$

This follows from the previous two conditions, as mentioned in [11].

Some examples of distinguished classes will be discussed in the next section, and some other examples will be considered later.

19.11 A couple of distinguished classes

Let us check that

$$\text{the class of finite extensions is distinguished,} \tag{19.11.1}$$

as mentioned on p166 of [11]. Indeed, the first condition for a distinguished class follows from the remarks in Section 19.9. Concerning the second condition, let

k, E, F be subfields of a field L with $k \subseteq E, F$, and suppose that E is a finite extension of k. This implies that E is generated by k and finitely many elements of E, so that the compositum $E\,F$ of E and F is generated by F and the same finitely many elements of E, as in the previous section. It follows that

$$E\,F \text{ is a finite extension of } F, \tag{19.11.2}$$

as in Section 19.9, because the elements of E are algebraic over k, and thus over F.

We also have that

$$\text{the class of algebraic extensions is distinguished,} \tag{19.11.3}$$

as in [11]. To see this, let k, E, F be fields, with $k \subseteq F \subseteq E$. If E is algebraic over k, then it is easy to see that F is algebraic over k, and that E is algebraic over F.

Conversely, suppose that F is algebraic over k, and that E is algebraic over F. Let $\alpha \in E$ be given, so that

$$a_n\,\alpha^n + a_{n-1}\,\alpha^{n-1} + \cdots + a_1\,\alpha + a_0 = 0 \tag{19.11.4}$$

for some $a_0, a_1, \ldots, a_n \in F$ that are not all equal to 0. Consider the subfield

$$F_0 = k(a_0, a_1, \ldots, a_n) \tag{19.11.5}$$

of F generated by k and $a_0, a_1, \ldots, a_n$. Note that

$$F_0 \text{ is a finite extension of } k, \tag{19.11.6}$$

as in Section 19.9, because $a_0, a_1, \ldots, a_n$ are algebraic over k, by hypothesis. Of course, α is algebraic over F_0, by construction.

Consider the subfield $F_0(\alpha)$ of E generated by F_0 and α. This is a finite extension of F_0, because α is algebraic over F_0. It follows that

$$F_0(\alpha) \text{ is a finite extension of } k, \tag{19.11.7}$$

because F_0 is a finite extension of k. This implies that α is algebraic over k. Thus E is algebraic over k, so that the class of algebraic extensions satisfies the first condition in the definition of a distinguished class.

Concerning the second condition in the definition of a distinguished class, let k, E, and F be subfields of a field L with $k \subseteq E, F$, and suppose that E is algebraic over k. In particular, this means that every element of E is algebraic over F. It follows that the subfield of $E\,F$ generated by F and finitely many elements of E is finite over F, as in Section 19.9. Thus the elements of such a subfield of $E\,F$ are algebraic over F. One can use this to get that

$$E\,F \text{ is algebraic over } F. \tag{19.11.8}$$

More precisely, $E\,F$ is equal to the union of all of the subfields generated by F and finitely many elements of E. This is because this union is a subfield of $E\,F$, and it contains E and F by construction.

19.12 Some remarks about embeddings

Let k and L be fields, and let σ be an *embedding* of k into L. This means that σ is an injective homomorphism from k into L, as on p167 of [11]. Thus $\sigma(k)$ is a subfield of L, and σ determines a field isomorphism from k onto $\sigma(k)$.

Let E be an extension of k. We shall sometimes be interested embeddings τ of E into L that are extensions of σ, i.e., which are equal to σ on k. One may say that

(19.12.1) τ is an embedding of E into L *over* σ

under these conditions, as on p167 of [11].

If k is a subfield of L, then we can take σ to be the identity mapping on k. In this case, one may say that

(19.12.2) τ is an embedding of E into L *over* k

when τ is equal to the identity mapping on k, as on p167 of [11].

If σ is an embedding of k into L and $t \in k$, then the notation

$$t^\sigma = \sigma(t) \tag{19.12.3}$$

is sometimes used, as on p167 of [11]. Let X be an indeterminate, and let $f(X)$ be a formal polynomial in X with coefficients in k, as in (19.9.1). Thus

$$f^\sigma(X) = \sum_{j=0}^{n} \sigma(f_j)\, X^j \tag{19.12.4}$$

is a formal polynomial in X with coefficients in $\sigma(k) \subseteq L$, as on p115 of [11].

Let τ be an extension of σ to an embedding of E into L, as before. If $x \in E$, then $f(x)$ is defined as an element of E as in Sections 4.4 and 19.9, and similarly $f^\sigma(\tau(x))$ may be defined as an element of L. It is easy to see that

$$\tau(f(x)) = f^\sigma(\tau(x)) \tag{19.12.5}$$

under these conditions. In particular, if x is a root of $f(X)$ in E, then

(19.12.6) $\tau(x)$ is a root of $f^\sigma(X)$ in L,

as on p167 of [11].

Let ρ be an embedding of E into itself over k, so that ρ is an injective homomorphism of E into itself that is equal to the identity mapping on k. In particular,

(19.12.7) ρ may be considered as a linear mapping from E into itself,

as a vector space over k, as on p167 of [11]. If E is a finite extension of k, then it follows that

$$\rho(E) = E, \tag{19.12.8}$$

by a well-known result in linear algebra.

Let E' be a subfield of E that contains k, and suppose that

$$\rho(E') \subseteq E'. \tag{19.12.9}$$

If E' is a finite extension of k, then

$$\rho(E') = E', \tag{19.12.10}$$

as in (19.12.8).

Let $f(X)$ be a formal polynomial in X with coefficients in k. Note that ρ maps the set of roots of $f(X)$ in E into itself, as before. If E' is the subfield of E generated by k and the roots of $f(X)$ in E, then (19.12.9) holds. If the coefficients of $f(X)$ are not all 0, then there are only finitely many roots of $f(X)$ in E, each of which is algebraic over k. This implies that E' is a finite extension of k, as in (19.9.11), so that (19.12.10) holds under these conditions.

Lemma 1 on p167 of [11] states that (19.12.8) also holds when E is an algebraic extension of k. This is because every element of E is contained in a subfield E' of E that is a finite extension of k that satisfies (19.12.9), as in the preceding paragraph.

19.13 Algebraically closed fields

Let k be a field again, and let X be an indeterminate. If every formal polynomial in X with coefficients in k of degree at least one has a root in k, then k is said to be *algebraically closed*, as on p169 of [11]. The term *algebraically complete* is also used for this, as on p437 of [2].

An algebraically closed field $\overline{k}$ that contains k as a subfield is said to be an *algebraic closure* of k if

$$\overline{k} \text{ is also algebraic over } k. \tag{19.13.1}$$

Every field k has an algebraic closure, as in the corollary on p170 of [11].

Let L be an algebraically closed field, and let σ be an embedding of k into L. If E is an algebraic extension of k, then one may be interested in extensions of σ to embeddings of E into L, as on p170 of [11].

Suppose for the moment that

$$E = k(x) \tag{19.13.2}$$

for some $x \in E$, which is to say that E is generated by k and x. Note that any extension of σ to an embedding τ from E into L is uniquely determined by $\tau(x)$.

Of course, if E is an algebraic extension of k, then x should be algebraic over k. Let $f(X)$ be the minimal polynomial of x over k, as in Section 19.9. This leads to a polynomial $f^\sigma(X)$ in X with coefficients in $\sigma(k)$, as in (19.12.4). If

τ is an embedding of E into L that is an extension of σ, then $\tau(x)$ is a root of $f^\sigma(X)$ in L, as in (19.12.6).

In fact, every root of $f^\sigma(X)$ in L occurs in this way, as on p171 of [11]. This implies that

(19.13.3) the number of extensions of σ to embeddings of E into L

is equal to

(19.13.4) the number of distinct roots of $f^\sigma(X)$ in L,

as in Proposition 8 on p171 of [11]. In particular, the number of these extensions is less than or equal to the degree of $f^\sigma(X)$, which is the same as the degree of $f(X)$. Of course, this also shows that there is such an extension, because L is algebraically closed.

If E is any algebraic extension of k, then Theorem 2 on p171 of [11] says that

(19.13.5) there is an extension of σ to an embedding of E into L.

If E is also algebraically closed, and L is algebraic over $\sigma(k)$, then any such extension of σ maps E onto L, as in [11].

If $\overline{k}$ and $\overline{k}'$ are algebraic closures of k, then there is an isomorphism from $\overline{k}$ onto $\overline{k}'$ that is equal to the identity mapping on k, as in the corollary on p172 of [11].

Let k_2 be a field that contains k as a subfield, and let k_1 be a subfield of k_2 that contains k. If k_1 is algebraic over k, and k_2 is algebraic over k_1, then

(19.13.6) k_2 is algebraic over k,

as in Corollary 4 on p444 of [12]. This is also part of Proposition 6 on p166 of [11].

If k_1 is algebraic over k, and $\overline{k_1}$ is an algebraic closure of k_1, then

(19.13.7) $\overline{k_1}$ is an algebraic closure of k.

This follows from the fact that $\overline{k_1}$ is algebraic over k, as in the preceding paragraph. This corresponds to a remark at the top of p174 of [11].

19.14 Splitting fields

Let k, k_1 be fields, with k a subfield of k_1, and let X be an indeterminate. Also let $f(X)$ be a formal polynomial in X with coefficients in k, as in Section 19.9, with degree at least one. Suppose that

(19.14.1) $f(X)$ can be expressed as a product of linear factors as a formal polynomial in X with coefficients in k_1,

and that

(19.14.2) k_1 is generated as a field containing k by the roots of $f(X)$ in k_1.

Under these conditions, k_1 may be called a *root field* of $f(X)$, as on p452 of [2], or a *splitting field* of $f(X)$, as on p173 of [11], and p447 of [12]. It is well known that $f(X)$ has a splitting field, as in Theorem 1 on p453 of [2], and the first part of Theorem 4 on p447 of [12].

Suppose that k_2 is a field that contains k, and that $f(X)$ can be expressed as a product of linear factors as a formal polynomial in X with coefficients in k_2. In this case,

(19.14.3) the subfield of k_2 generated by k and the roots of $f(X)$ in k_2 is a splitting field of $f(X)$.

In particular, one can take k_2 to an algebraic closure of k, as mentioned on p174 of [11].

If E and K are splitting fields of $f(X)$, then there is an isomorphism from E onto K that is the identity mapping on k, as in Theorem 2 on p454 of [2], the first part of Theorem 3 on p173 of [11], and the second part of Theorem 4 on p447 of [12]. If $\overline{k}$ is an algebraic closure of k and

$$K \subseteq \overline{k}, \tag{19.14.4}$$

then the second part of Theorem 3 on p173 of [11] says that any embedding of E into $\overline{k}$ that is the identity mapping on k maps E onto K.

Let I be a nonempty set, and suppose that $f_j(X)$ is a formal polynomial in X with coefficients in k and degree at least one for each $j \in I$. An extension k_1 of k is said to be a *splitting field* of $\{f_j(X)\}_{j\in I}$ if

(19.14.5) $f_j(X)$ splits into a product of linear factors with coefficients in k_1 for each $j \in I$,

and

(19.14.6) k_1 is generated as a field containing k by the set of the roots of all of the $f_j(X)$'s, $j \in I$,

as on p174 of [11]. If I has only finitely many elements, then this is the same as a splitting field of the product of the $f_j(X)$'s, $j \in I$, as in the remark on p175 of [11].

Let $\overline{k}$ be an algebraic closure of k. If $j \in I$, then let k_j be a splitting field of $f_j(X)$ in $\overline{k}$. The subfield of $\overline{k}$ generated by the subfields k_j, $j \in I$, is a splitting field of $\{f_j(X)\}_{j\in I}$, as mentioned on p174 of [11]. This is the smallest subfield of $\overline{k}$ that contains k_j for each $j \in I$, which may be called the compositum of these subfields, as in Section 19.10.

Suppose that E and K are splitting fields of $\{f_j(X)\}_{j\in I}$, and let $\overline{K}$ be an algebraic closure of K. Under these conditions, the corollary on p174 of [11] says that if σ is an embedding of E into $\overline{K}$ that is equal to the identity mapping on k, then

$$\sigma(E) = K. \tag{19.14.7}$$

More precisely, for each $j \in I$, $f_j(X)$ has a unique splitting field E_j in E, and a unique splitting field K_j in K. We also have that

$$\sigma(E_j) = K_j \tag{19.14.8}$$

for each $j \in I$, as before. One can use this to get (19.14.7), because E and K are generated by the E_j's and K_j's, $j \in I$, as mentioned earlier.

19.15 Galois groups and normal extensions

Let k and K be fields, with $k \subseteq K$. An automorphism of K as a field is said to be an automorphism *over* k if it is equal to the identity mapping on k. Of course, the collection of automorphisms of K as a field is a group with respect to composition of mappings. The collection $\mathrm{Gal}(K/k)$ of automorphisms of K over k is a subgroup of this group, which is the *Galois group* of K over k.

This corresponds to the definition on p460 of [2]. The term Galois group is used the definition on p461 of [2] in the case where K is the root or splitting field of a polynomial with coefficients in k. In this case, one may also refer to this as the Galois group of the polynomial.

Galois groups of field extensions are mentioned on p190 of [11], in connection with Proposition 12 on p189 of [11], and again on p192 of [11]. The notation $G = G(K/k)$ for the Galois group is used on p192 of [11]. The Galois group of a polynomial, as the Galois group of a splitting field of the polynomial, is mentioned on p199 of [11].

The Galois group of a field extension is defined on p458 of [12], using the notation $\Gamma = \Gamma(K/k)$. The Galois group of a polynomial is defined on p450 of [12], as the Galois group of a splitting field of the polynomial.

Let k, K be fields, with $k \subseteq K$. There seem to be a few related ways in which the *normality* of K as an extension of k is sometimes defined.

Normality of K over k is defined on p459 of [12] to mean that for each $u \in K$ with $u \notin k$, there is an automorphism θ of K over k such that

$$\theta(u) \neq u. \tag{19.15.1}$$

Equivalently, this means that

(19.15.2) k is exactly the set of points in K that are fixed by every element of the Galois group of K over k.

Let X be an indeterminate, and suppose now that K is an algebraic extension of k. Let us say that K is *normal* as an extension of k if

(19.15.3) for every irreducible formal polynomial $f(X)$ in X with coefficients in k that has a root in K,

we have that

(19.15.4) $f(X)$ can be expressed as a product of linear factors with coefficients in K.

This corresponds to the condition NOR 3 in Theorem 4 on p175 of [11]. Normality is defined in the same way on p468 of [2] when K is a finite extension of k.

If K is a finite extension of k, then Theorem 12 on p459 of [12] says that certain conditions on K are equivalent. Condition (ii) in that theorem is that K be normal over k in the sense of [12], as in (19.15.2). The definition of normality in the preceding paragraph corresponds to part of condition (iii) in that theorem. The other part of condition (iii) in that theorem is that K be separable over k, as in Section 20.4.

Part of Theorem 4 on p175 of [11] is that normality as in the condition NOR 3 is equivalent to the condition NOR 2, which says that

(19.15.5) K is the splitting field of a family of polynomials in $k[X]$.

Theorem 14 on p468 of [2] is the analogous statement for finite extensions, i.e., if K is a finite extension of k, then K is normal over k if and only if

(19.15.6) K is the root field, or equivalently splitting field, of some polynomial with coefficients in k.

Condition (i) in Theorem 12 on p459 of [12] is that K be the splitting field of a polynomial with coefficients in k that is separable, as in the next section.

Let $\overline{k}$ be an algebraic closure of k, and suppose that K is a subfield of $\overline{k}$. We can reduce to that case, using an embedding of K into $\overline{k}$ over k, as in Section 19.13. Theorem 4 on p175 of [11] states that the conditions NOR2 and NOR 3 are equivalent to another condition NOR 1. This condition says that

(19.15.7) every embedding σ of K into $\overline{k}$ over k is an automorphism of K.

Remember that σ is an embedding of K into $\overline{k}$ over k when σ is the identity mapping on k, as in Section 19.12.

Chapter 20

More on field extensions

20.1 Some properties of normal extensions

One can check that field extensions of degree 2 are normal, as mentioned on p175 of [11]. One can use this to get an example of fields k, E, and F with $k \subseteq F \subseteq E$ such that F is normal over k, E is normal over F, and E is not normal over k, as in [11]. This means that the class of normal algebraic field extensions does not satisfy the "if" part of the first condition in the definition of a distinguished class, in Section 19.10.

Let k, E, and K be fields, with

$$k \subseteq E \subseteq K. \tag{20.1.1}$$

Suppose that K is algebraic over k, which implies that E is algebraic over k, and that K is algebraic over E. If K is normal over k, then the second part of Theorem 5 on p176 of [11] states that

$$K \text{ is normal over } E. \tag{20.1.2}$$

This is the second half of the "only if" part of the first condition in the definition of a distinguished class, for the class of normal algebraic field extensions. Indeed, if X is an indeterminate, and if K is the splitting field of a family of polynomials in $k[X]$, then these polynomials may be considered as elements of $E[X]$. This means that K may be considered as the splitting field of a family of polynomials in $E[X]$. This corresponds to Exercise 5 on p470 of [2] in the case of finite extensions.

Alternatively, let $\overline{E}$ be an algebraic closure of E, and remember that $\overline{E}$ may be considered as an algebraic closure of k, as mentioned in Section 19.13. Suppose that K is a subfield of $\overline{E}$, and remember that we can reduce to this case using a suitable embedding, as in Section 19.13. If σ is any embedding of K into $\overline{E}$ over E, then σ may be considered as an embedding of K into $\overline{E}$ over k. The normality of K over k implies that

$$\sigma(K) = K, \tag{20.1.3}$$

and this implies that K is normal over E, as in the previous section. This argument is mentioned on p176 of [11].

Let k and L be fields, with $k \subseteq L$, and suppose that K, F are subfields of L that contain k. Thus the compositum $K\,F$ of K and F may be defined as a subfield of L that contains k, as in Section 19.10. Suppose that K is algebraic over k, so that

$$K\,F \text{ is algebraic over } F, \tag{20.1.4}$$

as in Section 19.11. If K is normal over k, then the first part of Theorem 5 on p176 of [11] says that

$$K\,F \text{ is normal over } F. \tag{20.1.5}$$

This is the second condition in the definition of a distinguished class, for the class of normal algebraic field extensions.

As before, if K is the splitting field of a family of polynomials in $k[X]$, then these polynomials may be considered as elements of $F[X]$, and $K\,F$ may be considered as the splitting field of this family of polynomials, as an extension of F. Alternatively, let $\overline{F}$ be an algebraic closure of F. There is an embedding of $K\,F$ into $\overline{F}$ over F, because of (20.1.4), as in Section 19.13. Using this, we can reduce to the case where

$$K\,F \subseteq \overline{F}. \tag{20.1.6}$$

In particular, this means that $K \subseteq \overline{F}$.

One can get an algebraic closure $\overline{k}$ of k that is a subfield of $\overline{F}$ by taking the union of the subfields of $\overline{F}$ that contain k and are algebraic over k, as in the proof of the corollary on p170 of [11]. Note that

$$K \subseteq \overline{k}, \tag{20.1.7}$$

because $k \subseteq K \subseteq \overline{F}$ and K is algebraic over k. Let σ be an embedding of $K\,F$ into $\overline{F}$ over F. Thus σ is equal to the identity mapping on F, and on k in particular, because $k \subseteq F$. This implies that

$$k = \sigma(k) \subseteq \sigma(K), \tag{20.1.8}$$

and that $\sigma(K)$ is algebraic over k.

It follows that

$$\sigma(K) \subseteq \overline{k}. \tag{20.1.9}$$

If K is normal over k, then we get that $\sigma(K) = K$, as in the previous section. This implies that

$$\sigma(K\,F) = \sigma(K)\,\sigma(F) = K\,F, \tag{20.1.10}$$

using the definition of the compositum in the first step. This means that $K\,F$ is normal over F, as in the previous section again. This argument is also mentioned on p176 of [11].

20.2 More on normal extensions

Let k and L be fields with $k \subseteq L$, and let K_1, K_2 be subfields of L that contain k, so that the compositum $K_1 K_2$ of K_1 and K_2 is a subfield of L that contains k. Suppose that K_1 and K_2 are algebraic over k, so that

(20.2.1) $$K_1 K_2 \text{ is algebraic over } k,$$

as in Section 19.11. If K_1 and K_2 are normal over k, then

(20.2.2) $$K_1 K_2 \text{ is normal over } k,$$

as in Theorem 5 on p176 of [11]. This is the third condition in the definition of a distinguished class, for the class of normal algebraic extensions. One way to see this is to use the fact that K_1 and K_2 are the splitting fields of families of polynomials with coefficients in k, as in Section 19.15, so that $K_1 K_2$ is the splitting field of the union of two such families.

Alternatively, let $\overline{k}$ be an algebraic closure of k. There is an embedding of $K_1 K_2$ into $\overline{k}$ over k, because of (20.2.1), as in Section 19.13. We can use this to reduce to the case where

(20.2.3) $$K_1 K_2 \subseteq \overline{k},$$

so that $K_1, K_2 \subseteq \overline{k}$. Let σ be any embedding of $K_1 K_2$ into $\overline{k}$ over k. Note that $K_1, K_2 \subseteq K_1 K_2$, and that

(20.2.4) $$\sigma(K_1) = K_1,\ \sigma(K_2) = K_2,$$

because K_1 and K_2 are normal over k, as in Section 19.15. This implies that

(20.2.5) $$\sigma(K_1 K_2) = \sigma(K_1)\,\sigma(K_2) = K_1 K_2,$$

so that (20.2.2) holds, as before. This argument is mentioned on p176 of [11].

Let K_1 and K_2 be subfields of L that contain k again, so that $K_1 \cap K_2$ is a subfield of L that contains k as well. Clearly

(20.2.6) $$K_1 \cap K_2 \text{ is algebraic over } k$$

when K_1 or K_2 is algebraic over k. If K_1 and K_2 are algebraic and normal over k, then

(20.2.7) $$K_1 \cap K_2 \text{ is normal over } k,$$

as in Theorem 5 on p176 of [11]. Indeed, let $f(X)$ be an irreducible formal polynomial in an indeterminate X with coefficients in k that has a root in $K_1 \cap K_2$. If K_1 and K_2 are normal over k, then $f(X)$ can be expressed as a product of linear factors with coefficients in K_1, and as a product of linear factors with coefficients in K_2, as in Section 19.15. One can check that these two expressions are the same, so that $f(X)$ can be expressed as a product of linear factors with coefficients in $K_1 \cap K_2$. This implies (20.2.7), as before.

Alternatively, if $\overline{k}$ is an algebraic closure of k again, then we can reduce to the case where (20.2.3) holds, as before. Let σ be any embedding of $K_1 \cap K_2$

into $\overline{k}$ over k, and let τ be an extension of σ to an embedding of $K_1\,K_2$ into $\overline{k}$, as in Section 19.13. Observe that

$$\sigma(K_1 \cap K_2) = \tau(K_1 \cap K_2) = \tau(K_1) \cap \tau(K_2). \tag{20.2.8}$$

If K_1 and K_2 are normal over k, then $\tau(K_1) = K_1$ and $\tau(K_2) = K_2$, as in Section 19.15, so that

$$\sigma(K_1 \cap K_2) = K_1 \cap K_2. \tag{20.2.9}$$

This implies (20.2.7), as on p176 of [11].

20.3 Separable polynomials

Let k be a field, and let X be an indeterminate. Also let $f(X)$ be a formal polynomial in X with coefficients in k, as in Section 19.9. Suppose that $f(X)$ has *degree* n for some positive integer n, so that the coefficient of X^j in $f(X)$ is equal to 0 when $j > n$, and is not zero when $j = n$. We say that $f(X)$ is *separable* over k when

(20.3.1) $f(X)$ has n distinct roots in some splitting field,

and thus in every splitting field, as on p465 of [2], p178 of [11], and p448 of [12]. Of course, $f(X)$ has at most n roots in any extension of k, by a standard argument.

Equivalently, this means that

(20.3.2) $f(X)$ has no multiple roots in the splitting field.

Otherwise, one may say that $f(X)$ is *inseparable*. These terms are used a bit differently on p121 of [4], and we shall say more about that in a moment.

Let $f'(X)$ be the formal derivative of $f(X)$, as in Section 4.6. It is well known and not difficult to show that

(20.3.3) a root of $f(X)$ is a multiple root
if and only if it is also a root of $f'(X)$,

as on p465 of [2], Proposition 1 on p131 of [11], and p454 of [12]. This implies that $f(X)$ is separable if and only if

(20.3.4) $f(X)$ and $f'(X)$ have no common factors,

as in Theorem 11 on p466 of [2], and Proposition 8 on p454 of [12].

Equivalently, $f(X)$ is separable if and only if

(20.3.5) the greatest common divisor of $f(X)$ and $f'(X)$ is equal to 1.

It is well known that the greatest common divisor of $f(X)$ and $f'(X)$ can be obtained using the Euclidean algorithm in $k[X]$, as in [2, 12]. More precisely, the condition (20.3.5) holds when $f(X)$ and $f'(X)$ are considered as formal polynomials in X with coefficients in k if and only if it holds when they are considered as formal polynomials with coefficients in a larger field, as in [2, 12].

Suppose that $f(X)$ is irreducible as a formal polynomial in X with coefficients in k. In this case, (20.3.5) holds if and only if

$$f'(X) \neq 0, \tag{20.3.6}$$

as in Corollary 1 on p466 of [2].

If k has characteristic 0, then (20.3.6) holds automatically, so that $f(X)$ is separable. This corresponds to Corollary 2 on p466 of [2], the first part of Proposition 9 on 178 of [11], and Proposition 9 on p454 of [12]. Separability of a formal polynomial $f(X)$ is defined on p121 of [4] to mean (20.3.6), but it seems to be used only for irreducible polynomials.

Let $f(X)$ and $g(X)$ be formal polynomials in X with coefficients in k, and suppose that

$$f(X)\ g(X) \text{ is separable.} \tag{20.3.7}$$

Under these conditions, one can check that

$$f(X) \text{ and } g(X) \text{ are separable.} \tag{20.3.8}$$

20.4 Separable elements and extensions

Let k_1 be a field, and let k be a subfield of k_1. An algebraic element x of k_1 over k is said to be *separable* over k if

(20.4.1) the minimal polynomial of x over k is separable,

as on p456 of [12]. This means that

(20.4.2) the coefficients of the derivative of the minimal polynomial of x are not all 0,

as in the previous section. This holds automatically when k has characteristic 0, as before.

If k_1 is an algebraic extension of k, then separability of $x \in k_1$ is defined in an equivalent way on p121 of [4]. Observe that

(20.4.3) the derivative of the minimal polynomial of x over k is not equal to 0 at x

in this case, because x is not a multiple root of its minimal polynomial over k, as in [4].

If an element x of an extension of k is a root of a separable polynomial with coefficients in k, then

(20.4.4) x is separable over k,

as mentioned on p178 of [11]. In this case, this polynomial can be expressed as the product of the minimal polynomial of x over k and another polynomial with coefficients in k, so that the minimal polynomial of x over k is separable, as in the previous section.

A finite extension k_1 of k is said to be *separable* over k on p465 of [2] if

(20.4.5) every $x \in k_1$ is the root of a separable polynomial with coefficients in k.

This is the same as saying that

(20.4.6) every element of k_1 is separable over k

in the previous sense, as in the preceding paragraph. This is how separability of k_1 over k is defined on p121 of [4]. Separability of k_1 over k is also defined in this way on p456 of [12] when k_1 is any algebraic extension of k.

Separability of finite extensions is defined another way on p178 of [11], and we shall say more about that in the next section. Separability of an algebraic element x of an extension of k is defined in [11] to mean that the subfield $k(x)$ of the extension generated by k and x is separable over k in this sense. It is mentioned afterwards that this is equivalent to the separability of the minimal polynomial of x over k. Theorem 7 on p180 of [11] states that a finite extension of k is separable if and only if every element of the extension is separable over k.

Let E be a field that contains k as a subfield, and let F be a subfield of E that contains k. If $x \in E$ is separable over k, then

(20.4.7) x is separable over F

as well, as mentioned on p180 of [11]. Indeed, if x is the root of a separable polynomial with coefficients in k, then this polynomial may also be considered as having coefficients in F, and it is separable as a polynomial with coefficients in F.

Suppose that k_1 is a finite extension of k, and that

(20.4.8) k_1 is generated by k and finitely many elements of k_1 that are separable over k.

Under these conditions, it is remarked on p181 of [11] that an argument in the proof of Theorem 7 on the previous page shows that k_1 is separable.

If k_1 is an algebraic extension of k, then separability of k_1 is defined on p181 of [11] to mean that

(20.4.9) every finitely generated subextension of k_1 is separable over k.

Note that this implies that every element of k_1 is separable over k, in the sense defined on p178 of [11].

Suppose that k_1 is an algebraic extension of k that is generated by a family of elements $\{x_j\}_{j \in I}$. If x_j is separable over k for each $j \in I$, then Theorem 8 on p181 of [11] states that

(20.4.10) $$k_1 \text{ is separable over } k.$$

20.5 Separable extensions and embeddings

Let k be a field, and let E be an algebraic extension of k. Also let L be an algebraically closed field, and let σ be an embedding of k into L. We would like to consider extensions of σ to embeddings of E into L, as on p176 of [11]. The existence of such an extension follows from Theorem 2 on p171 of [11], as mentioned in Section 19.13.

Note that any extension of σ to an embedding of E into L maps E onto a subfield of L that is algebraic over $\sigma(k)$. Because of this, we shall suppose that

(20.5.1) $$L \text{ is algebraic over } \sigma(k),$$

so that

(20.5.2) $$L \text{ is an algebraic closure of } \sigma(k),$$

as in [11].

Let L' be another algebraically closed field, and let τ be an embedding of k into L'. Suppose that

(20.5.3) $$L' \text{ is an algebraic closure of } \tau(k),$$

as before. Under these conditions, there is a field isomorphism λ from L onto L' such that

(20.5.4) $$\lambda = \tau \circ \sigma^{-1} \text{ on } \sigma(k),$$

as in [11]. This uses the second part of Theorem 2 on p171 of [11], which was also mentioned in Section 19.13.

Let S_σ, S_τ be the sets of all extensions of σ, τ to embeddings of E into L, L', respectively, as in [11]. One can use λ to get a one-to-one mapping from S_σ onto S_τ, as on p177 of [11]. In particular, this means that the cardinalities of S_σ and S_τ are the same. Thus this cardinality depends only on k and E, and it is denoted

(20.5.5) $$[E : k]_s,$$

and called the *separable degree* of E over k, as in [11].

If F is a subfield of E that contains k, then it is easy to see that F is algebraic over k, and that E is algebraic over F, because E is algebraic over k. This means that

(20.5.6) $$[E : F]_s \text{ and } [F : k]_s$$

may be defined in the same way as before. Under these conditions, the first part of Theorem 6 on p177 of [11] says that

(20.5.7) $$[E : k]_s = [E : F]_s \, [F : k]_s.$$

More precisely, if σ is an embedding of k into L, then σ has $[F : k]_s$ distinct extensions to embeddings of F into L, and each of these has $[E : F]_s$ distinct extensions to embeddings of E into L. Of course, any extension of σ to an embedding of E into L may be considered as an extension to E of an extension of σ to an embedding of F into L.

If E is a finite extension of k, then the second part of Theorem 6 on p177 of [11] states that the separable degree of E over k is finite, with

(20.5.8) $$[E : k]_s \le [E : k].$$

This follows from Proposition 8 on p171 of [11] when E is generated by a single element, as in Section 19.13.

Suppose that E is a finite extension of k, and that F is a subfield of E that contains k again. The corollary on p178 of [11] states that

(20.5.9) $$[E : k]_s = [E : k]$$

if and only if

(20.5.10) $$[E : F]_s = [E : F]$$

and

(20.5.11) $$[F : k]_s = [F : k].$$

If E is a finite extension of k, then *separability* of E over k is defined on p178 of [11] to mean that (20.5.9) holds. This is equivalent to the other formulations of separability of finite extensions discussed in the previous section, as mentioned earlier.

20.6 Separable extensions are distinguished

Remember that the notion of a distinguished class of field extensions is defined on p165 of [11], as mentioned in Section 19.10. Theorem 9 on p181 of [11] states that

(20.6.1) the class of separable algebraic extensions is distinguished.

To see this, let k, E, F be fields with $k \subseteq F \subseteq E$. Suppose for the moment that E is a separable algebraic extension of k, which implies that E is algebraic over F and that F is algebraic over k, as mentioned in Section 19.11. Note that every element of F is separable over k, so that F is separable over k. We also have that every element of E is separable over F, as in Section 20.4, so that E is separable over F.

Conversely, suppose that F is a separable algebraic extension of k, and that E is a separable algebraic extension of F. This implies that E is an algebraic extension of k, as in Section 19.11. If E is a finite extension of k, then F is a finite extension of k and E is a finite extension of F, so that

(20.6.2) $$[F : k]_s = [F : k] \text{ and } [E : F]_s = [E : F],$$

as in the previous section. One can use this to get (20.5.9) from (20.5.7). This means that E is a separable extension of k, as before.

Otherwise, let $\alpha \in E$ be given. Note that α is separable over F, because E is separable over F, by hypothesis. This means that α is the root of a separable polynomial with coefficients in F. Let $a_0, a_1, \ldots, a_n \in F$ be the coefficients of this polynomial, and consider the subfield

$$F_0 = k(a_0, a_1, \ldots, a_n) \tag{20.6.3}$$

of F generated by k and these coefficients. This is a finite extension of k, as in Section 19.9, because $a_0, a_1, \ldots, a_n$ are algebraic over k, by hypothesis.

We also have that F_0 is separable over k, because $a_0, a_1, \ldots, a_n$ are separable over k, by hypothesis. Of course, α is separable over F_0, by construction. This means that the subfield $F_0(\alpha)$ of E generated by F_0 and α is separable over F_0. It follows that

$$F_0(\alpha) \text{ is separable over } k, \tag{20.6.4}$$

because of the earlier argument for finite separable extensions. This implies that α is separable over k, so that E is separable over k.

This shows that separable extensions satisfy the first condition in the definition of a distinguished class. To get the second condition, let k, E, and F be subfields of a field L with $k \subseteq E, F$, and suppose that E is separable algebraic over k. Remember that the compositum $E\,F$ of E and F is algebraic over k, as in Section 19.11.

Every element of E is separable over k, by hypothesis, and thus over F. This implies that $E\,F$ is separable over F, as in Section 20.4.

20.7 Minimal normal extensions

Let k be a field, and let E be a finite extension of k. Also let $\overline{E}$ be an algebraic closure of E, and remember that $\overline{E}$ may be considered as an algebraic closure of k, as in Section 19.13. Note that E is finitely generated as a field extension of k, as mentioned in Section 19.9. One can get a finite normal extension of k in $\overline{E}$ that contains E using the splitting field of finitely many polynomials over k whose zeros include a set of generators of E as an extension of k.

If E is also separable over k, then one can use finitely many separable polynomials over k, to get a finite normal extension of k in $\overline{E}$ that contains E and is separable too. This uses the fact that an extension of k is separable if it is generated by elements that are separable over k, as in Section 20.4. This is related to the first part of Exercise 1 on p461 of [12].

The intersection of all of the normal algebraic extensions of k in $\overline{E}$ that contain E is a normal algebraic extension of k, as in Section 20.2. This is the smallest normal algebraic extension of k in $\overline{E}$ that contains E, as on p182 of [11]. This is a finite extension of k, as before.

Let $\sigma_1, \ldots, \sigma_n$ be a list of the distinct embeddings of E into $\overline{E}$ over k, so that $n = [E : k]_s$, as in Section 20.5. Consider the compositum

$$K = \sigma_1(E)\, \sigma_2(E) \cdots \sigma_n(E) \tag{20.7.1}$$

of the subfields $\sigma_1(E), \sigma_2(E), \ldots, \sigma_n(E)$ of $\overline{E}$, as in Section 19.10. This is a finite extension of k, because the class of finite extensions is distinguished, as in Section 19.11.

In fact,

$$K \text{ is a normal extension of } k, \tag{20.7.2}$$

as on p182 of [11]. To see this, let τ be any embedding of K into $\overline{E}$ over k. Observe that

$$\tau \circ \sigma_1, \tau \circ \sigma_2, \ldots, \tau \circ \sigma_n \tag{20.7.3}$$

are distinct embeddings of E into $\overline{E}$ over k. It follows that these embeddings are the same as $\sigma_1, \sigma_2, \ldots, \sigma_n$, possibly rearranged, so that

$$\tau(K) = K. \tag{20.7.4}$$

This implies (20.7.2), as in Section 19.15.

Remember that $\sigma_1, \ldots, \sigma_n$ may be extended to embeddings of any algebraic extension of E, as in Section 19.13. One can use this to get that any normal algebraic extension of k in $\overline{E}$ that contains E also contains $\sigma_1(E), \ldots, \sigma_n(E)$. This means that such an extension contains K. Thus K is the smallest normal algebraic extension of k in $\overline{E}$ that contains E, as on p182 of [11].

If E is separable over k, then

$$K \text{ is separable over } k, \tag{20.7.5}$$

as on p182, because separable algebraic extensions form a distinguished class. This is related to the second part of Exercise 1 on p461 of [12].

20.8 Galois extensions

Let k be a field, and let K be an algebraic extension of k. Remember that normality of K as an extension of k is defined as in Section 19.15. If K is normal and separable over k, then K is said to be a *Galois extension* of k, as on p192 of [11].

If K is any extension of k, then normality of K over k is defined on p459 of [12] to mean that

$$k = \{u \in K : \theta(u) = u \text{ for every } \theta \in \mathrm{Gal}(K/k)\}, \tag{20.8.1}$$

as mentioned in Section 19.15. Here $\mathrm{Gal}(K/k)$ is the Galois group of K over k, as before. If K is a finite extension of k, then condition (iii) in Theorem 12 on p459 of [12] is the same as saying that K is normal as an extension of k as in Section 19.15, and separable, so that K is a Galois extension. Condition (ii) in the theorem is that K be normal as an extension of k as defined in [12], and part of the theorem is that these two conditions are equivalent in this case.

Let K be an algebraic extension of k again. If K is a Galois extension of k, then the first part of Theorem 1 on p192 of [11] states that (20.8.1) holds. The proof of this will be mentioned in Section 21.5, as well as a related statement for normal extensions that may not be separable.

The proof in [12] that (20.8.1) implies that K is a Galois extension of k when K is a finite extension of k seems to work as well when K is an algebraic extension of k. This is related to Exercise 3 on p461 of [12].

Let $\overline{K}$ be an algebraic closure of K. This may be considered as an algebraic closure of k, as mentioned in Section 19.13. Every element of the Galois group of K over k may be considered as an embedding of K into $\overline{K}$ over k. As in Section 19.15, normality of K over k in the sense that we are using here is equivalent to the condition that every embedding of K into $\overline{K}$ over k correspond to an element of the Galois group of K over k in this way. This corresponds to a remark on p192 of [11].

More precisely, let S_0 be the set of embeddings of K into $\overline{K}$ over k, which is to say the set of embeddings of K into $\overline{K}$ that are equal to the identity mapping on k, as in Section 19.12. This corresponds to S_σ in Section 20.5, with $E = K$, $L = \overline{K}$, and σ equal to the identity mapping on k, considered as an embedding of k into $\overline{K}$. Thus

$$\mathrm{Gal}(K/k) \subseteq S_0, \tag{20.8.2}$$

and normality of K over k is equivalent to

$$\mathrm{Gal}(K/k) = S_0, \tag{20.8.3}$$

as in the preceding paragraph.

Suppose from now on in this section that K is a finite extension of k, and let $\# A$ denote the cardinality of a set A. In this case, we get that $\mathrm{Gal}(K/k)$ is finite, with

$$\#\,\mathrm{Gal}(K/k) \le \# S_0 = [K:k]_s \le [K:k], \tag{20.8.4}$$

where the second and third steps are as in Section 20.5. We also get that

$$\#\,\mathrm{Gal}(K/k) = \# S_0 = [K:k]_s \tag{20.8.5}$$

if and only if (20.8.3) holds, which means that K is normal over k, as before. Observe that

$$\#\,\mathrm{Gal}(K/k) = [K:k] \tag{20.8.6}$$

if and only if (20.8.5) holds and

$$[K:k]_s = [K:k]. \tag{20.8.7}$$

This means that (20.8.6) holds if and only if K is normal and separable over k, or equivalently Galois over k, as in Theorem 12 on p459 of [12].

Remember that K is normal over k if and only if k is the root field, or equivalently splitting field, of a polynomial with coefficients in k, as in Section

19.15. Another part of Theorem 12 on p459 of [12] states that K is Galois over k if and only if K is the splitting field of a separable polynomial with coefficients in k. The fact that the splitting field of a separable polynomial with coefficients in k is a separable extension of k also follows from a statement mentioned in Section 20.4. The corollary on p469 of [2] says that K is the root field of a separable polynomial with coefficients in k when K is normal and separable. Theorems 12 and 13 on p469 of [2] state that root fields of separable polynomials with coefficients in k have some of the properties of finite Galois extensions mentioned earlier.

20.9 Separable extensions and the trace

Let k be a field, and let E be a finite separable extension of k. Also let $\overline{E}$ be an algebraic closure of E, which may be considered as an algebraic closure of k, as in Section 19.13. One could start with an algebraic closure $\overline{k}$ of k, as on p210 of [11]. In this case, one could reduce to the case where E is a subfield of $\overline{k}$ that contains k, using an embedding of E into $\overline{k}$, as in Section 19.13, and as mentioned on p211 of [11]. Of course, this means that $\overline{k}$ may be considered as an algebraic closure of E.

Put

(20.9.1) $$r = [E : k]_s = [E : k],$$

and let $\sigma_1, \ldots, \sigma_r$ be the distinct embeddings of E into $\overline{E}$ over k. If $x \in E$, then the *trace* of x from E to k is defined by

(20.9.2) $$\mathrm{Tr}_k^E(x) = \sum_{j=1}^{r} \sigma_j(x),$$

as on p210 of [11]. More precisely, the trace is initially defined in a slightly different way for any finite extension of k in [11], and then it is mentioned that it is equal to 0 when the extension is not separable.

Although (20.9.2) may be defined initially as an element of $\overline{E}$, it is in fact an element of the smallest normal algebraic extension K of k in $\overline{E}$ that contains E, as in Section 20.7. It is easy to see that (20.9.2) is invariant under the Galois group of K over k, because any element of the Galois group of K over k simply permutes the terms in the sum on the right side of (20.9.2). This implies that

(20.9.3) $$\mathrm{Tr}_k^E(x) \in k,$$

as in the previous section, because K is Galois over k, as in Section 20.7. This corresponds to part of the first part of Theorem 8 on p210 of [11].

Clearly Tr_k^E is a homomorphism from E into k, as commutative groups with respect to addition, as in Theorem 8 on 210 of [11]. In fact, Tr_k^E is a linear mapping from E into k, as vector spaces over k, as mentioned on p211 of [11].

Let F be a subfield of E that contains k. Note that F is a finite separable extension of k, and that E is a finite separable extension of F, because separable algebraic extensions form a distinguished class, as in Section 20.6. Of course, $\overline{E}$ may be considered as an algebraic closure of F as well. Thus Tr_F^E may be defined as a mapping from E into F, and Tr_k^F may be defined as a mapping from F into k, as before. It is not too difficult to show that

$$\mathrm{Tr}_k^E = \mathrm{Tr}_k^F \circ \mathrm{Tr}_F^E, \tag{20.9.4}$$

as in the second part of Theorem 8 on p210 of [11].

Suppose for the moment that

$$E = k(\alpha) \tag{20.9.5}$$

for some $\alpha \in E$, so that E is generated by k and α. Let X be an indeterminate, and let

$$f(X) = X^n + a_{n-1}\, X^{n-1} + \cdots + a_0 \tag{20.9.6}$$

be the minimal polynomial of α over k, as in Section 19.9. Under these conditions, we have that

$$\mathrm{Tr}_k^{k(\alpha)}(\alpha) = -a_{n-1}, \tag{20.9.7}$$

as in the third part of Theorem 8 on p210 of [11].

Of course, $-a_{n-1}$ is the same as the sum of the roots of $f(X)$, with their appropriate multiplicaities. In this case, $f(X)$ has no multiple roots, because of separability, as in Section 20.4. The embeddings of $k(\alpha)$ over k into an algebraic closure correspond exactly to the roots of $f(X)$, as in Section 19.13. This means that $\mathrm{Tr}_k^{k(\alpha)}(\alpha)$ is the same as the sum of the roots of $f(X)$, as in (20.9.7).

Let E be any finite separable extension of k again. The first part of Theorem 9 on p211 of [11] states that

$$\mathrm{Tr}_k^E(x) \neq 0 \tag{20.9.8}$$

for some $x \in E$. To see this, we consider $E \setminus \{0\}$ as a commutative group with respect to multiplication, and $\sigma_1, \ldots, \sigma_r$ as group homomorphisms from $E \setminus \{0\}$ into $\overline{E} \setminus \{0\}$. These homomorphisms are linearly independent as functions on $E \setminus \{0\}$ with values in $\overline{E}$, as in Section 19.8, because they are distinct on $E \setminus \{0\}$, by hypothesis. This implies in particular that their sum is nonzero on $E \setminus \{0\}$.

If $x \in k$, then

$$\mathrm{Tr}_k^E(x) = r \cdot x, \tag{20.9.9}$$

which is the sum of r x's in k, as mentioned in the remark on p212 of [11]. This implies (20.9.8) when $x \neq 0$ and k has characteristic 0, and when r is not a multiple of the characteristic of k, if it is positive.

If $x \in E$, then

$$y \mapsto x\, y \tag{20.9.10}$$

defines a linear mapping from E into itself, as a vector space over k. The trace of this linear mapping may be defined as an element of k in the usual way, because E has finite dimension as a vector space over k, by hypothesis. It is well known that the trace of this linear mapping is equal to $\mathrm{Tr}_k^E(x)$, as in Exercise 4 on p351 of [11].

20.10 Traces and integral elements

Let k_0 be an integral domain, and let $k = Q_{k_0}$ be its field of fractions. Also let E be a finite extension of k, and let $\overline{E}$ be an algebraic closure of E. Suppose that

(20.10.1) $$x \in E \text{ is integral over } k_0,$$

so that it satisfies a monic polynomial equation with coefficients in k_0, as in Section 16.3. This polynomial may be expressed as the product of the minimal polynomial of x over k and another polynomial with coefficients in k. It follows that the roots of the minimal polynomial of x over k satisfy the same monic polynomial equation with coefficients in k_0 as x does, so that they are integral over k_0 too.

It is well known and easy to see that the coefficients of the minimal polynomial of x over k may be expressed as sums of products of its roots. This implies that

(20.10.2) the coefficients of the minimal polynomial of x over k are integral over k_0,

as in Section 16.5. This corresponds to part of the corollary on p240 of [11]. Note that all rings in Chapter IX of [11] are supposed to be commutative, as mentioned at the beginning of that chapter, and that the term *entire ring* is used for integral domains on p61 of [11].

Suppose now that E is a finite separable extension of k. Under these conditions,

(20.10.3) $$\mathrm{Tr}_k^E(x) \text{ is integral over } k_0$$

as well. This corresponds to another part of the corollary on p240 of [11]. This corresponds to remarks in the proofs of Proposition 5.17 on p64 of [1] and Proposition 8 on p13 of [15] too.

Although (20.10.3) can be obtained from (20.10.2), it can be verified more directly from the definition (20.9.2) of the trace, as follows. If $\sigma_1, \ldots, \sigma_r$ are the distinct embeddings of E into $\overline{E}$ over k, as in the previous section, then $\sigma_j(x)$ satisfies the same polynomial equations as x for each j. In particular, this means that

(20.10.4) $$\sigma_j(x) \text{ is integral over } k_0$$

for each j. It follows that their sum is integral over k_0, as in Section 16.5 again.

20.11 Traces and integral closures

Let k_0 be an integral domain, and let $k = Q_{k_0}$ be its field of fractions again. Also let E be a finite separable extension of k, and let $\overline{E}$ be an algebraic closure of E. Suppose that

(20.11.1) $$k_0 \text{ is integrally closed,}$$

so that k_0 is integrally closed in k, as in Section 16.3. Let

$$k_1 = \{x \in E : x \text{ is integral over } k_0\} \tag{20.11.2}$$

be the integral closure of k_0 in E, as in Section 16.5.

Under these conditions, Proposition 5.17 on p64 of [1] says that there is a basis $v_1, \ldots, v_r$ for E as a vector space over k such that

$$k_1 \subseteq k_0\, v_1 + \cdots + k_0\, v_r. \tag{20.11.3}$$

This is related to Exercise 5 on p252 of [11], and to Proposition 8 on p13 of [15], and we shall say more about that n a moment.

Of course, every element of E is algebraic over k, as mentioned in Section 19.9. This implies that every $x \in E$ satisfies a polynomial equation

$$a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_0 = 0 \tag{20.11.4}$$

with coefficients in k_0 and $a_n \neq 0$, because k is the field of fractions of k_0, by a standard argument. One can multiply both sides of the equation by a_n^{n-1} to get that $a_n\, x$ is integral over k_0, so that

$$a_n\, x \in k_1. \tag{20.11.5}$$

We can use this to get a basis $u_1, \ldots, u_r$ for E, as a vector space over k, with

$$u_1, \ldots, u_r \in k_1, \tag{20.11.6}$$

as in [1].

Let Tr_k^E be as in Section 20.9, and note that

$$\mathrm{Tr}_k^E(x\, y) \tag{20.11.7}$$

defines a symmetric bilinear form on E, as a vector space over k. This bilinear form is also nondegenerate on E, because Tr_k^E is nonzero on some elements of E, as before.

One can use this to get a basis $v_1, \ldots, v_r$ of E as a vector space over k such that

$$\begin{aligned} \mathrm{Tr}_k^E(u_j\, v_l) &= 1 \quad \text{when } j = l \\ &= 0 \quad \text{when } j \neq l. \end{aligned} \tag{20.11.8}$$

as in Corollary 1 on p212 of [11]. More precisely, one can use (20.11.7) to identify E with its dual space, as a vector space over k, in which case $v_1, \ldots, v_r$ corresponds to the dual basis associated to $u_1, \ldots, u_r$.

If $x \in E$, then x can be expressed in a unique way as a linear combination of $v_1, \ldots, v_r$ with coefficients in k, and in fact

$$x = \sum_{j=1}^{r} \mathrm{Tr}_k^E(x\, u_j)\, v_j, \tag{20.11.9}$$

because of (20.11.8). Remember that k_1 is a subring of E, as in Section 16.5. If $x \in k_1$, then $x\,u_j \in k_1$ for each j, and

$$\mathrm{Tr}_k^E(x\,u_j) \in k_1 \tag{20.11.10}$$

for each j, as in (20.10.3). This implies (20.11.3).

Suppose now that we also have that

$$k_0 \text{ is Noetherian} \tag{20.11.11}$$

as a commutative ring, and thus as a module over itself. The right side of (20.11.3) may be considered as a module over k_0, which is freely generated by $v_1, \ldots, v_r$. This module is isomorphic to the direct sum of r copies of k_0, and is thus Noetherian as a module over k_0, as in Section 9.7.

Note that k_1 may be considered as a module over k_0, because k_0 is a subring of k_1. It follows that

$$k_1 \text{ is finitely generated as a module over } k_0, \tag{20.11.12}$$

because of (20.11.3). This corresponds to Exercise 5 on p252 of [11], and to Proposition 8 on p13 of [15].

20.12 Dedekind domains and integral closures

Let us continue with the same notation and hypotheses as in the previous section, including (20.11.11). Observe that

$$k_1 \text{ is Noetherian, as a module over } k_0, \tag{20.12.1}$$

as in Section 9.7. This implies that

$$k_1 \text{ is Noetherian as a commutative ring,} \tag{20.12.2}$$

which is to say that k_1 is Noetherian as a module over itself.

Suppose now that k_0 is a Dedekind domain, which includes the conditions that k_0 be a Noetherian integral domain, as in Section 15.11. This implies that k be integrally closed as well, as in Section 16.12.

Of course, k_1 is an integral domain too. Note that k_1 is integrally closed in E, as in Section 16.5. It is easy to see that E is the field of fractions of k_1, using (20.11.5). This means that

$$k_1 \text{ is integrally closed,} \tag{20.12.3}$$

as in Section 16.3.

Remember that k_0 has dimension less than or equal to one, in the sense of Section 14.8, because k_0 is a Dedekind domain, as in Section 15.11. One can check that

$$k_1 \text{ has dimension less than or equal to one,} \tag{20.12.4}$$

because k_1 is integral over k_0. This uses the results mentioned in Section 16.14.

It follows that

$$k_1 \text{ is a Dedekind domain,} \tag{20.12.5}$$

as in Section 16.12. This corresponds to Proposition 9 on p13 of [15], and it is related to Theorem 9.5 on p96 of [1]. This also works if one requires Dedekind domains to not be fields, or equivalently to have dimension equal to one, as mentioned in Section 15.11. More precisely, if k_0 is not a field, then k_1 is not a field, because $k_0 = k \cap k_1$.

20.13 Some remarks about multiple roots

Let k be a field, and let X be an indeterminate. Also let $g(X)$ be a polynomial in X with coefficients in k, and suppose that

$$f(X) = (X - \alpha)^m \, g(X) \tag{20.13.1}$$

for some $\alpha \in k$ and $m \in \mathbf{Z}_+$. If $g(\alpha) \neq 0$, then m is said to be the *multiplicity* of α as a root of $f(X)$, as on p131, 178 of [11]. We say that α is a *multiple root* of $f(X)$ when $m > 1$, and otherwise α is a *simple root* of $f(X)$ when $m = 1$, as in [11].

Suppose that k has positive characteristic p. It is well known and not difficult to see that the binomial coefficient $\binom{p}{l}$ is a multiple of p when $1 \le l \le p-1$. This implies that

$$(x+y)^p = x^p + y^p \tag{20.13.2}$$

for every $x, y \in k$, as on p131 of [11]. Of course, $(x\,y)^p = x^p\,y^p$ for all $x, y \in k$, so that

$$x \mapsto x^p \tag{20.13.3}$$

is a homomorphism from k into itself, as a field. The kernel of this homomorphism is trivial, so that (20.13.3) is injective.

If r is a positive integer, then it follows that

$$x \mapsto x^{p^r} \tag{20.13.4}$$

is an injective homomorphism from k into itself, as a field, as on p132 of [11]. Let $c \in k$ be given, and consider

$$X^{p^r} - c. \tag{20.13.5}$$

If this polynomial has a root $a \in k$, so that

$$a^{p^r} = c, \tag{20.13.6}$$

then

$$X^{p^r} - c = X^{p^r} - a^{p^r} = (X-a)^{p^r}, \tag{20.13.7}$$

as a formal polynomial in X with coefficients in k. This means that a is the only root of (20.13.5), as on p132 of [11].

If

$$(20.13.3) \text{ maps } k \text{ onto itself,} \tag{20.13.8}$$

then k is said to be *perfect*, as on p190 of [11]. Any field of characteristic 0 is said to be perfect too, as in [11]. An equivalent formulation is mentioned on p121 of [4]. Note that (20.13.8) holds when k has only finitely many elements, because (20.13.3) is injective.

If $f(X)$ is a formal polynomial in X with coefficients in k and $f'(X) = 0$, then $f(X)$ can be expressed as a linear combination of powers of X^p with coefficients in k, as in Proposition 2 on p131 of [11]. If k is perfect, then it follows that $f(X)$ can be expressed as the pth power of a formal polynomial in X with coefficients in k, as mentioned on p121 of [4]. In particular, this implies that $f(X)$ is reducible when the degree of $f(X)$ is at least one, as in [4]. If k is perfect and $f(X)$ is ireducible, then it follows that $f'(X) \neq 0$, so that $f(X)$ is separable, as in Section 20.3. This means that algebraic elements of extensions of k are separable, so that algebraic extensions of k are separable.

20.14 More on separable elements

Let k be a field, let $\overline{k}$ be an algebraic closure of k, let α be an element of $\overline{k}$, and let $f(X)$ be the minimal polynomial of α over k. If k has characteristic 0, then all of the roots of $f(X)$ are simple, so that $f(X)$ is separable as a polynomial over k, as in Section 20.3. If k has positive characteristic p, then there is a nonnegative integer μ such that

$$\text{every root of } f(X) \text{ has multiplicity } p^{\mu}, \tag{20.14.1}$$

as in Proposition 9 on p178 of [11]. In this case, we have that

$$[k(\alpha) : k] = p^{\mu}\,[k(\alpha) : k]_s, \tag{20.14.2}$$

and that

$$\alpha^{p^{\mu}} \text{ is separable over } k \tag{20.14.3}$$

as in [11].

To see this, let $\alpha_1, \ldots, \alpha_r$ be a list of the distinct roots of $f(X)$ in $\overline{k}$, and let m_l be the multiplicity of α_l in $f(X)$ for each $l = 1, \ldots, r$. Note that

$$f(X) = \prod_{l=1}^{r} (X - \alpha_l)^{m_l}, \tag{20.14.4}$$

because $f(X)$ is a monic polynomial by hypothesis, as in Section 19.9.

If $1 \leq j \leq r$, then there is a unique isomorphism σ_j from $k(\alpha)$ onto $k(\alpha_j)$ over k such that

$$\sigma_j(\alpha) = \alpha_j, \tag{20.14.5}$$

as on p179 of [11]. This can be extended to an automorphism of $\overline{k}$ over k, as in Section 19.13, and we let σ_j denote this extension as well.

If $f^{\sigma_j}(X)$ is the formal polynomial associated to $f(X)$ and σ_j as in Section 19.12, then

(20.14.6) $$f^{\sigma_j}(X) = f(X),$$

because σ_j is the identity mapping on k, and $f(X)$ has coefficients in k. One can use this to get that

(20.14.7) $$f(X) = \prod_{l=1}^{r} (X - \sigma_j(\alpha_l))^{m_l},$$

as on p179 of [11]. If α has multiplicity m in $f(X)$, then it follows that

(20.14.8) $$m_j = m,$$

as in [11].

Let $f'(X)$ be the formal derivative of $f(X)$, as in Section 4.6. Remember that $f(X)$ has no multiple roots when $f'(X) \neq 0$, because $f(X)$ is irreducible, as in Section 20.3. In particular, this holds when k has characteristic 0, as before.

Suppose that k has positive characteristic p. If

(20.14.9) $$f'(X) = 0,$$

then the coefficient of X^l in $f(X)$ is 0 unless l is an integer multiple of p. This means that

(20.14.10) $$f(X) = g(X^p)$$

for some formal monic polynomial $g(X)$ with coefficients in k, as on p179 of [11]. It is easy to see that $g(X)$ is irreducible as a formal polynomial with coefficients in k, because $f(X)$ is irreducible. If $g'(X) \neq 0$, then we can stop.

Remember that $[k(\alpha) : k]$ is equal to the degree of f, as mentioned in Section 19.9. Note that

(20.14.11) $$g(\alpha^p) = f(\alpha) = 0,$$

so that $[k(\alpha^p) : k]$ is equal to the degree of g. Of course,

(20.14.12) $$k(\alpha^p) \subseteq k(\alpha).$$

It follows that

(20.14.13) $$[k(\alpha) : k(\alpha^p)] = p,$$

as mentioned on p179 of [11], using a standard properties of degrees of extensions mentioned in Section 19.9.

We can repeat the process as needed to get a nonnegative integer μ and an irreducible formal monic polynomial $h(X)$ with coefficients in k such that

(20.14.14) $$f(X) = h(X^{p^\mu})$$

and

(20.14.15) $$h'(X) \neq 0,$$

as on p179 in [11]. Thus $h(X)$ has no multiple roots, as before. Clearly

$$h(\alpha^{p^\mu}) = f(\alpha) = 0, \tag{20.14.16}$$

and $[k(\alpha^{p^\mu}) : k]$ is equal to the degree of h. Note that (20.14.3) follows from (20.14.16). We also have that

$$k(\alpha^{p^\mu}) \subseteq k(\alpha), \tag{20.14.17}$$

and

$$[k(\alpha) : k(\alpha^{p^\mu})] = p^\mu, \tag{20.14.18}$$

as on p179 of [11].

If $1 \le l \le r$, then

$$h(\alpha_l^{p^\mu}) = f(\alpha_l) = 0. \tag{20.14.19}$$

It follows that $h(X)$ can be expressed as the product of $X - \alpha_l^{p_\mu}$ and a formal polynomial in X with coefficients in $\overline{k}$ that is not equal to 0 at $\alpha_l^{p^\mu}$, because $h(X)$ does not have multiple roots. This means that $f(X)$ can be expressed as the product of

$$X^{p^\mu} - \alpha_l^{p^\mu} = (X - \alpha_l)^{p^\mu} \tag{20.14.20}$$

and a formal polynomial that is a linear combination of powers of X^{p^μ}, and which is not equal to 0 at α_l. Thus the multiplicity of α_l as a root of $f(X)$ is equal to p^μ, as in (20.14.1).

The degree of $f(X)$ is equal to $p^\mu\, r$, so that the degree of $h(X)$ is equal to r. Note that

$$\alpha_1^{p^\mu}, \ldots, \alpha_r^{p^\mu} \tag{20.14.21}$$

are distinct roots of $h(X)$, and that these are all of the roots of $h(X)$, because the degree of $h(X)$ is equal to r, as in [11].

Remember that $[k(\alpha) : k]_s$ is the number of embeddings of $k(\alpha)$ into $\overline{k}$ over k, as in Section 20.5. This is equal to the number of distinct roots of $f(X)$, by Proposition 8 on p171 of [11], which was mentioned in Section 19.13. This implies (20.14.2), because $[k(\alpha) : k]$ is equal to the degree of $f(X)$, as mentioned in Section 19.9. Note that

$$[k(\alpha^{p^\mu}) : k]_s = [k(\alpha^{p^\mu}) : k], \tag{20.14.22}$$

which is the degree of $h(X)$, and that

$$[k(\alpha) : k]_s = [k(\alpha^{p^\mu}) : k]_s, \tag{20.14.23}$$

as on p179 of [11].

20.15 More on algebraic extensions

Let k be a field of positive characteristic p, and let α be an algebraic element of an extension of k. If there is a nonnegative integer n such that

$$\alpha^{p^n} \in k, \tag{20.15.1}$$

then α is said to be *purely inseparable* over k, as on p186 of [11].

Let E be an algebraic extension of k, and consider the following four conditions on E. The first condition is that

$$[E : k]_s = 1, \tag{20.15.2}$$

where the left side is as in Section 20.5. The second condition is that

$$\text{every } \alpha \in E \text{ is purely inseparable over } k. \tag{20.15.3}$$

The third condition is that for every $\alpha \in E$, the minimal polynomial of α over k is of the form

$$X^{p^n} - a \tag{20.15.4}$$

for some nonnegative integer n and $a \in k$. The fourth condition is that there be a family $\{\alpha_j\}_{j \in I}$ of generators of E as an extension of k such that

$$\alpha_j \text{ is purely inseparable over } k \tag{20.15.5}$$

for each $j \in I$.

These four conditions are equivalent, as in [11]. To see this, suppose for the moment that (20.15.2) holds, and let $\alpha \in E$ be given. Remember that $k(\alpha)$ is the subfield of E generated by k and α, and observe that

$$[k(\alpha) : k]_s = 1, \tag{20.15.6}$$

as in Section 20.5.

Let $\overline{E}$ be an algebraic closure of E, which may be considered as an algebraic closure of k, as in Section 19.13. Also let X be an indeterminate, and let $f(X)$ be the minimal polynomial of α over k. Remember that $[k(\alpha) : k]_s$ is the same as the number of embeddings of $k(\alpha)$ into $\overline{E}$ over k, as in Section 20.5, and that this is the same as the number of distinct roots of $f(X)$ in $\overline{E}$, as in Section 19.13. Thus (20.15.6) is the same as saying that $f(X)$ has only one root in $\overline{E}$, which is α.

Put

$$m = [k(\alpha) : k], \tag{20.15.7}$$

which is the same as the degree of $f(X)$. The remarks in the preceding paragraph imply that

$$f(X) = (X - \alpha)^m, \tag{20.15.8}$$

as a formal polynomial in X with coefficients in $k(\alpha)$. Let us express m as

$$m = p^n\, r, \tag{20.15.9}$$

where n is a nonnegative integer, and r is a positive integer that is not a multiple of p. Observe that

$$f(X) = (X - \alpha)^{p^n r} = (X^{p^n} - \alpha^{p^n})^r. \tag{20.15.10}$$

It follows that

$$f(X) = X^{p^n r} - r \cdot \alpha^{p^n} X^{p^n (r-1)} + \cdots. \tag{20.15.11}$$

This implies that

$$r \, \alpha^{p^n} \in k, \tag{20.15.12}$$

because the coefficients of $f(X)$ are in k. This means that (20.15.1) holds, because r is not a multiple of p.

Suppose now that $\alpha \in E$ is purely inseparable over k, so that

$$\alpha^{p^{n_0}} \in k \tag{20.15.13}$$

for some nonnegative integer n_0. Let $f(X)$ be the minimal polynomial of α over k again, and let m be the degree of $f(X)$. Of course, α is a root of

$$X^{p^{n_0}} - \alpha^{p^{n_0}} = (X - \alpha)^{p^{n_0}}, \tag{20.15.14}$$

so that $X^{p^{n_0}} - \alpha^{p^{n_0}}$ is equal to the product of $f(X)$ and another monic formal polynomial in X with coefficients in k. It follows that α is the only root of $f(X)$.

If m is as in (20.15.9), then we get that (20.15.1) holds, as before. The irreducibility of $f(X)$ implies that $r = 1$, and that

$$f(X) = X^{p^n} - \alpha^{p^n}. \tag{20.15.15}$$

This shows that (20.15.3) implies the third condition mentioned earlier. The third condition clearly implies (20.15.3), which implies the fourth condition.

Suppose that E satisfies the fourth condition mentioned earlier, and let $f_j(X)$ be the minimal polynomial of α_j over k for each $j \in I$. If $j \in I$, then there is a nonnegative integer n_j such that

$$\alpha_j^{p^{n_j}} \in k, \tag{20.15.16}$$

as in (20.15.5). This implies that

$$X^{p^{n_j}} - \alpha_j^{p^{n_j}} = (X - \alpha_j)^{p^{n_j}} \tag{20.15.17}$$

is equal to the product of $f_j(X)$ and another monic formal polynomial in X with coefficients in k, as before. In fact, if we take n_j to be as small as possible, then we get that $f_j(X)$ is equal to (20.15.17), as before. However, for the moment we only need the fact that α_j is the only root of $f_j(X)$ in $\overline{E}$.

Any embedding of E into $\overline{E}$ over k sends α_j to a root of $f_j(X)$. This means that such an embedding sends α_j to itself for each j, because it is the only root of $f_j(X)$. It follows that such an embedding is equal to the identity mapping on E. This implies (20.15.2).

If E satisfies these conditions, then E is said to be *purely inseparable* as an extension of k, as on p187 of [11].

Part VI

Rings, fields, modules, and satellites

Chapter 21

More on field extensions, 2

21.1 Maximal separable subextensions

Let k be a field of positive characteristic p. Proposition 10 on p187 of [11] states that

(21.1.1) the class of purely inseparable extensions of k is distinguished,

in the sense described in Section 19.10. The first condition in the definition of a distinguished class can be obtainsd using (20.15.2) and a property of the separable degree on Section 20.5. The second condition in the definition of a distinguished class can be obtained from the fourth condition in Section 20.15.

Let E be an algebraic extension of k, and let E_0 be the compositum of all of the subfields F of E that contain k and that are separable over k, as in Section 19.10. Proposition 11 on p187 of [11] states that

(21.1.2) $$E_0 \text{ is separable over } k,$$

and that

(21.1.3) $$E \text{ is purely inseparable over } E_0.$$

More precisely, (21.1.2) holds because E_0 is generated by elements that are separable over k, as in Section 20.4. In fact, E_0 consists of all of the elements of E that are separable over k, as in [11]. If $\alpha \in E$, then

(21.1.4) $$\alpha^{p^n} \in E_0$$

for some nonnegative integer n, as in (20.14.3), which implies (21.1.3).

If E is both separable and purely inseparable over k, then Corollary 1 on p188 of [11] states that

(21.1.5) $$E = k.$$

Indeed, if $\alpha \in E$, then the minimal polynomial of α over k is of the form (20.15.4) for some $n \geq 0$, as before. However, if E is separable over k, then α

is separable over k, so that the formal derivative of this polynomial should not be 0, as in Sections 20.3 and 20.4. This means that $n = 0$, so that $\alpha \in k$.

If r is a nonnegative integer, then put

$$E^{p^r} = \{x^{p^r} : x \in E\}. \tag{21.1.6}$$

This is a subfield of E, because $x \mapsto x^{p^r}$ is a homomorphism from E into itself as a field, as in Section 20.13.

Consider the condition that

$$\text{the compositum of } E^p \text{ and } k \text{ is equal to } E, \tag{21.1.7}$$

as in Corollary 4 on p188 of [11]. If r is a positive integer, the one can use (21.1.7) to get that

$$\text{the compositum of } E^{p^r} \text{ and } k \text{ is equal to } E, \tag{21.1.8}$$

as mentioned in [11].

Suppose that E is a finite extension of k, and let E_0 be the maximal subfield of E that contains k and is separable over k, as before. Remember that E is generated by k and finitely many elements $\alpha_1, \ldots, \alpha_n$, as mentioned in Section 19.9. We also have that for each $j = 1, \ldots, n$ there is a nonnegative integer m_j such that

$$\alpha_j^{p^{m_j}} \in E_0, \tag{21.1.9}$$

as in (21.1.4). This implies that there is a nonnegative integer m such that

$$\alpha_j^{p^m} \in E_0 \tag{21.1.10}$$

for each $j = 1, \ldots, m$. It follows that

$$E^{p^m} \subseteq E_0 \tag{21.1.11}$$

under these conditions, as in [11].

If (21.1.7) holds, then (21.1.8) holds with $r = m$, and we get that

$$E = E_0. \tag{21.1.12}$$

This means that E is separable over k, as in Corollary 4 on p188 of [11].

If E is any algebraic extension of k again, then let F be the compositum of E^p and k, so that F is a subfield of E that contains k. Clearly

$$E \text{ is purely inseparable over } F, \tag{21.1.13}$$

as mentioned in [11].

Suppose that E is separable over k. It is easy to see that

$$E \text{ is separable over } F. \tag{21.1.14}$$

This is part of the fact that the class of separable extensions is distinguished, as in Section 20.6.

Combining (21.1.13) and (21.1.14), we get that

$$E = F, \tag{21.1.15}$$

as in (21.1.5). This means that (21.1.7) holds, so that (21.1.8) holds, as in Corollary 4 on p188 of [11].

Suppose that k is perfect, so that $k^p = k$, as in Section 20.13. This implies that E is separable over k, as before. This means that (21.1.8) holds, as in the preceding paragraph. This is the same as saying that the compositum of E^p and k^p is equal to E in this case, so that

$$E^p = E. \tag{21.1.16}$$

Thus E is perfect too, as in the corollary on p190 of [11].

If k is perfect, then it is easy to see that $k^{p^r} = k$ for every positive integer r. In this case, if E is purely inseparable over k, then it follows that $E = k$. One can use also this and Proposition 12 on p189 of [11] to get that any algebraic extension over k is separable, as in the corollary on p190 of [11].

21.2 Some additional corollaries

Let k be a field of positive characteristic p again, and let K be a normal algebraic extension of k. If K_0 is the maximal separable extension of k contained in K, then Corollary 2 on p188 of [11] says that

$$K_0 \text{ is normal over } k. \tag{21.2.1}$$

To see this, let $\overline{K}$ be an algebraic closure of K, which may be considered as an algebraic closure of k, as in Section 19.13. Also let σ_0 be an embedding of K_0 into $\overline{K}$ over k. There is an extension of σ_0 to an embedding σ of K into $\overline{K}$ over k, as in Section 19.13.

Because K is normal over k, we have that $\sigma(K) = K$, as in Section 19.15. It is easy to see that

$$\sigma(K_0) \text{ is separable over } k, \tag{21.2.2}$$

because K_0 is separable over k. This implies that

$$\sigma(K_0) \subseteq K_0, \tag{21.2.3}$$

because

$$\sigma(K_0) \subseteq \sigma(K) = K, \tag{21.2.4}$$

and because K_0 is the maximal separable extension of k contained in K. Similarly,

$$\sigma^{-1}(K_0) \text{ is separable over } k, \tag{21.2.5}$$

and

$$\sigma^{-1}(K_0) \subseteq \sigma^{-1}(K) = K, \tag{21.2.6}$$

so that

$$\sigma^{-1}(K_0) \subseteq K_0. \tag{21.2.7}$$

It follows that

$$\sigma_0(K_0) = \sigma(K_0) = K_0, \tag{21.2.8}$$

so that (21.2.1) holds, as in Section 19.15.

If E is a finite extension of k, then

$$[E : k]_i = \frac{[E : k]}{[E : k]_s} \tag{21.2.9}$$

is called the *inseparable degree* or *degree of inseparability* of E over k, as on p180 of [11]. This is a nonnegative integer power of p, as in Corollary 1 on p180 of [11]. This follows from (20.14.2) when E is generated by a single element, as an extension of k. Otherwise, one can repeat the process, using basic properties of the ordinary and separable degrees of finite extensions, as in Sections 19.9 and 20.5. This can also be defined for extensions of fields of characteristic 0, in which case it is equal to 1.

Note that E is separable over k if and only if

$$[E : k]_i = 1, \tag{21.2.10}$$

as in Corollary 2 on p180 of [11]. This uses the characterization of separability mentioned in Section 20.5. Similarly, E is purely inseparable over k if and only if

$$[E : k]_i = [E : k], \tag{21.2.11}$$

as in (20.15.2).

Let F be a subfield of E that contains k. Corollary 3 on p180 of [11] states that

$$[E : k]_i = [E : F]_i\, [F : k]_i, \tag{21.2.12}$$

which follows from the analogous properties of the ordinary and separable degrees.

Now let E, F be finite extensions of k such that E is separable over k, and F is purely inseparable over k. Suppose that E and F are subfields of a larger field, so that their compositum $E\,F$ is a subfield of that larger field, as in Section 19.10. Under these conditions, Corollary 3 on p188 of [11] states that

$$[E\,F : F] = [E : k] = [E\,F : k]_s \tag{21.2.13}$$

and

$$[E\,F : E] = [F : k] = [E\,F : k]_i. \tag{21.2.14}$$

We also have that

$$E\,F \text{ is separable over } F \tag{21.2.15}$$

and

$$E\,F \text{ is purely inseparable over } E, \tag{21.2.16}$$

as mentioned in [11].

Of course,

$$[E\,F : k]_s = [E\,F : F]_s\,[F : k]_s \tag{21.2.17}$$

and

$$[E\,F : k]_s = [E\,F : E]_s\,[E : k]_s, \tag{21.2.18}$$

as in Section 20.5. It follows from (21.2.17) that

$$[E\,F : k]_s = [E\,F : F]_s = [E\,F : F], \tag{21.2.19}$$

using the hypothesis that F be purely inseparable over k in the first step, and (21.2.15) in the second step. One can get the second step in (21.2.13) using (21.2.18), (21.2.16), and the hypothesis that E be separable over k.

Similarly,

$$[E\,F : k]_i = [E\,F : F]_i\,[F : k]_i \tag{21.2.20}$$

and

$$[E\,F : k]_i = [E\,F : E]_i\,[E : k]_i, \tag{21.2.21}$$

as in (21.2.12). Using (21.2.21), we get that

$$[E\,F : k]_i = [E\,F : E]_i = [E\,F : E], \tag{21.2.22}$$

because E is separable over k by hypothesis, and because of (21.2.16). The second step in (21.2.14) can be obtained from (21.2.20), (21.2.15), and the hypothesis that F be purely inseparable over k.

21.3 Some remarks about Gal(K/k)

Let k and K be fields, with $k \subseteq K$, and remember that Gal(K/k) is the Galois group of automorphisms of K over k, as in Section 19.15. If L is a subfield of K that contains k, then

$$\mathrm{Gal}(K/L) \text{ is a subgroup of } \mathrm{Gal}(K/k). \tag{21.3.1}$$

Consider

$$K_1 = \{u \in K : \theta(u) = u \text{ for every } \theta \in \mathrm{Gal}(K/k)\}, \tag{21.3.2}$$

which is a subfield of K that contains k. Observe that

$$\mathrm{Gal}(K/K_1) = \mathrm{Gal}(K/k). \tag{21.3.3}$$

If $K_1 = k$, then K is normal over k in the sense defined on p459 of [12], as mentioned in Section 19.15. Note that

$$K_1 = \{u \in K : \theta(u) = u \text{ for every } \theta \in \mathrm{Gal}(K/K_1)\}, \tag{21.3.4}$$

because of (21.3.3). This means that K is automatically normal over K_1 in the sense of [12].

Suppose from now on in this section that K is algebraic over k. Let $\overline{K}$ be an algebraic closure of K, which may be considered as an algebraic closure of k, as in Section 19.13. Let L be a subfield of K that contains k again. If $\theta \in \mathrm{Gal}(K/k)$, then the restriction of θ to L may be considered as an embedding of L into $\overline{K}$. If L is normal over k, then it follows that

(21.3.5) $$\theta(L) = L,$$

as in Section 19.15.

Suppose that k has positive characteristic p. If L is a subfield of K that contains k, and if L is separable over k, then

(21.3.6) $$\theta(L) \text{ is separable over } k$$

for every $\theta \in \mathrm{Gal}(K/k)$. Let K_0 be the maximal separable extension of k contained in K, as in Section 21.1. It is easy to see that

(21.3.7) $$\theta(K_0) = K_0$$

for every $\theta \in \mathrm{Gal}(K/k)$.

21.4 Separability of K over K_1

Let k be a field of positive characteristic p, and let K be an algebraic extension of k. If K_1 is as in (21.3.2), then we would like to show that

(21.4.1) $$K \text{ is separable over } K_1.$$

This corresponds to part of the conclusion of Proposition 12 on p189 of [11]. Although the proposition is stated with the additional hypothesis that K be normal over k, this does not appear to be needed for this part of the conclusion.

If K is a finite extension of k, then K is a finite extension of K_1. In this case, one could use Theorem 12 on p459 of [12] to get (21.4.1). More precisely, condition (ii) in that theorem is that K be normal over K_1 in the sense of [12], which corresponds to (21.3.4). Condition (iii) in that theorem is that (21.4.1) hold, and that K be normal in the sense used here, as in [2, 11]. Part of the theorem is that these conditions are equivalent for finite extensions.

In fact, the proof in [12] that condition (ii) implies condition (iii) seems to work as well when K is an algebraic extension of k, as mentioned in Section 20.8. This is related to Exercise 3 on p461 of [12], as before.

The argument in [11] also begins with the case where K is finite over k, with a suitable reduction afterwards. Let $\alpha \in K$ be given. One would like to find a maximal collection $\sigma_1, \ldots, \sigma_r$ of the Galois group $\mathrm{Gal}(K/k)$ of K over k such that

(21.4.2) $$\sigma_1(\alpha), \ldots, \sigma_r(\alpha)$$

are distinct elements of K. If K is a finite extension of k, then

(21.4.3) $$[K : k]_s < \infty,$$

as in Section 20.5. This means that Gal(K/k) has only finitely many elements, so that it is easy to find $\sigma_1, \ldots, \sigma_r \in \mathrm{Gal}(K/k)$ as before.

However, if $\sigma \in \mathrm{Gal}(K/k)$, then $\sigma(\alpha)$ is a root of the minimal polynomial of α over k, and of course there are only finitely many roots of this polynomial. One can use this to get $\sigma_1, \ldots, \sigma_r \in \mathrm{Gal}(K/k)$, without asking that Gal(K/k) have only finitely many elements. Note that α should be one of the elements of K in the list (21.4.2), because otherwise one could also include the identity mapping on K. Equivalently, one may as well take the identity mapping on K to be one of the elements of Gal(K/k) being used.

Let X be an indeterminate, and consider the polynomial

$$\prod_{j=1}^{r}(X - \sigma_j(\alpha)). \tag{21.4.4}$$

Note that α is a root of this polynomial, as in the preceding paragraph, and that the roots of this polynomial are distinct, by construction. Although the coefficients of this polynomial may be taken initially to be in K, one can check that they are in K_1, because the set of roots of the polynomial is invariant under Gal(K/k), by construction. This implies that α is separable over K_1.

The argument in [12] is somewhat analogous, although it considers any irreducible polynomial with coefficients in K_1 and a root in K, and shows that it should be a multiple of a polynomial of the same type as in the previous paragraph.

Let K_0 be the maximal separable extension of k contained in K, as in Section 21.1, and consider the compositum $K_0\, K_1$ of K_0 and K_1, which is another subfield of K that contains k. Remember that K is purely inseparable over K_0, as in Section 21.1, which implies that K is purely inseparable over $K_0\, K_1$. We also have that K is separable over $K_0\, K_1$, because of (21.4.1). It follows that

$$K = K_0\, K_1, \tag{21.4.5}$$

as in Section 21.1. This is another part of Proposition 12 on p189 of [11].

21.5 Normality and separability

Let k be a field of positive characteristic p, and let K be a normal algebraic extension of k, as in Section 19.15. If K_1 is as in (21.3.2) again, then

$$K_1 \text{ is purely inseparable over } k, \tag{21.5.1}$$

as in Proposition 12 on p189 of [11].

To see this, let $\alpha \in K_1$ be given. Let $\overline{K}$ be an algebraic closure of K, which may be considered as an algebraic closure of k, as in Section 19.13. Remember that $k(\alpha)$ is the subfield of K generated by k and α, and let τ be an embedding of $k(\alpha)$ into $\overline{K}$ over k. This can be extended to an embedding of K into $\overline{K}$ over k, as in Section 19.13, and we let τ denote this extension as well. Note that

$$\tau(K) = K, \tag{21.5.2}$$

because K is normal over k, as in Section 19.15.

Thus $\tau \in \mathrm{Gal}(K/k)$, so that

$$\tau(\alpha) = \alpha, \tag{21.5.3}$$

because $\alpha \in K_1$, by hypothesis. This implies that

$$\tau \text{ is the identity mapping on } k(\alpha). \tag{21.5.4}$$

This means that

$$[k(\alpha) : k]_s = 1, \tag{21.5.5}$$

as in Section 20.5. It follows that α is purely inseparable over k, as in Section 20.15. This shows that (21.5.1) holds.

If K is also a separable extension of k, then K_1 is separable over k, and (21.5.1) implies that

$$K_1 = k, \tag{21.5.6}$$

as in Section 21.1. Alternatively, one can use the argument in the previous paragraphs, and observe that (21.5.5) is the same as saying that $[k(\alpha) : k] = 1$ in this case. This implies that $\alpha \in k$, so that (21.5.6) holds. This works as well for fields of characteristic 0, which corresponds to the first part of Theorem 1 on p192 of [11], as in Section 20.8.

Let K_0 be the maximal separable extension of k that is contained in K, as in Section 21.1. Observe that $K_0 \cap K_1$ is separable over k, and purely inseparable over k, because of (21.5.1). This implies that

$$K_0 \cap K_1 = k, \tag{21.5.7}$$

as in Section 21.1. This is another part of Proposition 12 on p189 of [11].

Chapter 22

Some more rings and modules

22.1 Some formal series

Let A be a ring with a multiplicative identity element e_A, and let T be an indeterminate. Consider the space $A((T))$ of formal series of the form

$$f(T) = \sum_{j=j_0}^{\infty} f_j \, T^j, \tag{22.1.1}$$

where $j_0 \in \mathbf{Z}$, and $f_j \in A$ for each $j \geq j_0$. More precisely, we consider f_j to be defined as an element of A for every integer j, with $f_j = 0$ when $j < j_0$. This space is discussed on p27 of [4], and on p285 of [12], with A taken to be a field. A formal series of this type is called an *extended formal power series* in [12].

The space $c(\mathbf{Z}, A)$ of all A-valued functions on $\mathbf{Z}$ may be considered as a both a left and right module over A, with respect to pointwise addition of functions, and pointwise multiplication of functions by elements of A on the left or the right, as appropriate. We may define $A((T))$ as the subset of $c(\mathbf{Z}, A)$ whose elements are equal to 0 at all but finitely many negative integers, and which may be expressed as in (22.1.1). This is a submodule of $c(\mathbf{Z}, A)$, as both a left and right module over A. An element of $A((T))$ be also be expressed as

$$f(T) = \sum_{j >> -\infty} f_j \, T^j, \tag{22.1.2}$$

to indicate that $f_j = 0$ for all but finitely many $j < 0$, as on p27 of [4].

Let

$$g(T) = \sum_{l >> -\infty} g_l \, T^l \tag{22.1.3}$$

be another element of $A((T))$. Put

$$h_n = \sum_{j+l=n} f_j\, g_l, \tag{22.1.4}$$

where more precisely the sum is taken over all $j, l \in \mathbf{Z}$ such that $j + l = n$. It is easy to see that all but finitely many terms in the sum are equal to 0, because $f_j = 0$ for all but finitely many $j < 0$, and $g_l = 0$ for all but finitely many $l < 0$. Thus the sum is defined as an element of A for every $n \in \mathbf{Z}$, and one can check that it is equal to 0 for all but finitely many $n < 0$. It follows that

$$h(T) = \sum_{n>>-\infty} h_n\, T^n \tag{22.1.5}$$

defines an element of $A((T))$, and we put

$$f(T)\, g(T) = h(T). \tag{22.1.6}$$

One can check that this definition of multiplication on $A((T))$ is associative. If A is a commutative ring, then multiplication on $A((T))$ is commutative as well.

Remember that $A[[T]]$ is the space of formal power series in T with coefficients in A, as in Section 4.3. This may be identified with the subset of $A((T))$ consisting of formal series for which the coefficient of T^j is equal to 0 for every $j < 0$. We can also identify A with the subset of $A((T))$ consisting of formal series for which the coefficient of T^j is equal to 0 when $j \neq 0$, which is compatible with the previous identification of A with a subring of $A[[T]]$. Using these identifications, e_A corresponds to the multiplicative identity element of $A((T))$. In fact, $A((T))$ is a ring, which contains $A[[T]]$ as a subring.

In particular, $A((T))$ contains A as a subring too. Left and right multiplication on $A((T))$ by elements of A correspond exactly to termwise multiplication by elements of A, as before.

Of course, invertible elements of $A[[T]]$ are invertible in $A((T))$. Remember that an element of $A[[T]]$ is invertible in $A[[T]]$ when the coefficient of T^j is invertible in A for $j = 0$, as in Section 4.5. If $f(T) \in A((T))$ is as in (22.1.1), and if f_{j_0} is invertible as an element of A, then it follows that

$$f(T) \text{ is invertible in } A((T)). \tag{22.1.7}$$

More precisely, $f(T)\, T^{-j_0}$ is invertible as an element of $A[[T]]$ in this case, and

$$f(T)^{-1} = (f(t)\, T^{-j_0})^{-1}\, T^{-j_0}. \tag{22.1.8}$$

If A is a division ring, then it follows that

$$A((T)) \text{ is a division ring} \tag{22.1.9}$$

as well.

22.2 Some local rings

Let A be a division ring, with multiplicative identity element $e_A \neq 0$, and let T be an indeterminate. Thus the corresponding ring $A[[T]]$ of formal power series in T with coefficients in A may be defined as in Section 4.3. Consider

$$B_1 = \left\{ f(T) = \sum_{j=0}^{\infty} f_j\, T^j \in A[[T]] : f_1 = 0 \right\}, \tag{22.2.1}$$

as in Exercise 10 on p160 of [3]. It is easy to see that

$$B_1 \text{ is a subring of } A[[T]]. \tag{22.2.2}$$

Note that $e_A \in B_1$.

Put

$$\mathcal{I}_1 = \left\{ f(T) = \sum_{j=0}^{\infty} f_j\, T^j \in B_1 : f_0 = 0 \right\} = (A[[T]])\, T^2. \tag{22.2.3}$$

This is a two-sided ideal in $A[[T]]$ that is contained in B_1, and thus a two-sided ideal in B_1 in particular. Of course,

$$B_1 \setminus \mathcal{I}_1 = \left\{ f(T) = \sum_{j=0}^{\infty} f_j\, T^j \in A[[T]] : f_0 \neq 0,\ f_1 = 0 \right\}. \tag{22.2.4}$$

Remember that an element of $A[[T]]$ is invertible in $A[[T]]$ when the constant term is invertible in A, as in Section 4.5. In this case, it suffices to ask that the constant term be nonzero, because A is a division ring, by hypothesis. If

$$f(T) \in B_1 \setminus \mathcal{I}_1, \tag{22.2.5}$$

then it follows that $f(T)$ is invertible in $A[[T]]$. In fact, one can check that

$$f(T)^{-1} \in B_1, \tag{22.2.6}$$

using the description of the multiplicative inverse in Section 4.5.

Note that the elements of $\mathcal{I}_1$ have neither a left or right inverse in $A[[T]]$, as in Section 4.14. In particular, this means that they do not have a left or right inverse in B_1. It follows that

$$B_1 \text{ is a local ring}, \tag{22.2.7}$$

as in Section 4.13. This corresponds to the first part of Exercise 10 on p160 of [3].

Remember that

$$f(T) \mapsto f_0 \tag{22.2.8}$$

defines a ring homomorphism from $A[[T]]$ onto A, as in Section 4.3. Of course, this homomorphism also maps B_1 on A. The kernel of the restriction of this homomorphism to B_1 is equal to $\mathcal{I}_1$. This leads to a ring isomorphism from $B_1/\mathcal{I}_1$ onto A.

22.3 Some remarks about $a(T)\,B_1$

Let us continue with the same notation and hypotheses as in the previous section. Let $a(T) = \sum_{m=0}^{\infty} a_m\,T^m$ be a nonzero element of $A[[T]]$, and let m_0 be the smallest nonnegative integer such that

$$a_{m_0} \neq 0. \tag{22.3.1}$$

Thus

$$a(T) = a_{m_0}\,T^{m_0}\,(e_A - c(T)), \tag{22.3.2}$$

where $c(T) = \sum_{r=0}^{\infty} c_r\,T^r \in A[[T]]$, with

$$c_0 = 0 \text{ and } c_1 = -a_{m_0}^{-1}\,a_{m_0+1}. \tag{22.3.3}$$

It is easy to see that

$$(e_A - c(T))^{-1} - e_A - c_1\,T \in (A[[T]])\,T^2, \tag{22.3.4}$$

using the description of $(e_A - c(T))^{-1}$ in Section 4.5.

Consider

$$(e_A - c(T))\,B_1 = \{(e_A - c(T))\,f(T) : f(T) \in B_1\} \tag{22.3.5}$$

which is a subset of $A[[T]]$. This is contained in

$$\Big\{g(T) = \sum_{l=0}^{\infty} g_l\,T^l \in A[[T]] : g_1 = -c_1\,g_0\Big\}. \tag{22.3.6}$$

If $g(T)$ is an element of this set, then one can check that

$$(e_A - c(T))^{-1}\,g(T) \in B_1, \tag{22.3.7}$$

using (22.3.4). This implies that (22.3.6) is contained in (22.3.5). Thus (22.3.5) is the same as (22.3.6).

If follows that

$$\begin{aligned} a(T)\,B_1 \quad = \quad & \Big\{h(T) = \sum_{n=0}^{\infty} h_n\,T^n \in A[[T]] : h_n = 0 \text{ when } n < m_0, \\ & \qquad\qquad\qquad\qquad h_{m_0+1} = -c_1\,h_{m_0}\Big\}. \end{aligned} \tag{22.3.8}$$

Another part of Exercise 10 on p160 of [3] says that $\mathcal{I}_1$ is not free as a right module over B_1. Using (22.3.8), we get that $\mathcal{I}_1$ is not generated by a single element, as a right module over B_1. Any set of generators of $\mathcal{I}_1$, as a right module over B_1, should include at least one element for which the coefficient of T^2 is not 0. Another generator is needed to get arbitrary coefficients of T^3 of elements of $\mathcal{I}_1$. However, one can verify that $\mathcal{I}_1$ cannot be freely generated in this way, as a right module over B_1.

22.4 Some right modules over B_1

We continue with the same notation and hypotheses as in the previous two sections. Of course, $A[[T]]$ may be considered as a right module over B_1, and B_1 and $\mathcal{I}_1$ may be considered as submodules of $A[[T]]$, as a right module over B_1. If m_0 is a nonnegative integer, then

$$(A[[T]])\, T^{m_0} \tag{22.4.1}$$

may be considered as a submodule of $A[[T]]$, as a right module over B_1, and (22.3.8) is a submodule of $A[[T]]$, as a right module over B_1. Note that $c_1 \in A$ may be arbitrary in (22.3.8).

Let V be a submodule of $A[[T]]$, as a right module over B_1, and suppose that $V \neq \{0\}$. Let $m_0(V)$ be the largest nonnegative integer such that

$$V \subseteq (A[[T]])\, T^{m_0(V)}. \tag{22.4.2}$$

Equivalently, $m_0(V)$ is the smallest nonnegative integer such that V contains an element of $(A[[T]])\, T^{m_0(V)}$ for which the coefficient of $T^{m_0(V)}$ is nonzero. This implies that V contains a submodule of the form (22.3.8) with $m_0 = m_0(V)$ for some $c_1 \in A$. If V is not equal to this module, then one can check that

$$V = (A[[T]])\, T^{m_0(V)}. \tag{22.4.3}$$

It follows that

$$A[[T]] \text{ is Noetherian as a right module over } B_1, \tag{22.4.4}$$

as in Section 9.7. More precisely, every submodule of $A[[T]]$, as a right module over B_1, is generated by one or two elements. One could also look at the Noetherian property in terms of the ascending chain condition. Thus $\mathcal{I}_1$ is Noetherian as a right module over B_1, and B_1 is right Noetherian as a ring.

Another part of Exercise 10 on p160 of [3] states that

$$\mathcal{I}_1 \text{ is not projective, as a right module over } B_1. \tag{22.4.5}$$

More precisely, if $\mathcal{I}_1$ were projective as a right module over B_1, then one could get that $\mathcal{I}_1$ is free as a right module over B_1, as in Section 13.9. To see this, let $\mathcal{I}_1 \cdot \mathcal{I}_1$ be the subset of $\mathcal{I}_1$ consisting of sums of products of pairs of elements of $\mathcal{I}_1 = (A[[T]])\, T^2$, as in Section 13.4. In this case,

$$\mathcal{I}_1 \cdot \mathcal{I}_1 = (A[[T]])\, T^4. \tag{22.4.6}$$

This is a submodule of $\mathcal{I}_1$, as a right module over B_1, as in Section 13.7.

The quotient

$$\mathcal{I}_1/(\mathcal{I}_1 \cdot \mathcal{I}_1) \tag{22.4.7}$$

may be defined in the usual way as a right module over B_1. In fact, this may be considered as a right module over $B_1/\mathcal{I}_1$, as in Section 13.7. This corresponds to considering (22.4.7) as a right module over A here. As a right module over A, (22.4.7) is freely generated by the images of T^2 and T^3 in the quotient.

Note that $\mathcal{I}_1$ is generated by T^2 and T^3, as a right module over B_1. If $\mathcal{I}_1$ were projective as a right module over B_1, then $\mathcal{I}_1$ should be freely generated by T^2 and T^3, as a right module over B_1, as in Section 13.9. However, $\mathcal{I}_1$ is not freely generated by T^2 and T^3, as a right module over B_1, and so $\mathcal{I}_1$ is not projective as a right module over B_1.

Let us consider the Cartesian product $B_1 \times B_1$ as a right module over B_1, which is the same as the direct sum of two copies of B_1, as a right module over itself. As in Exercise 10 on p160 of [3],

$$\phi((b_1(T), b_2(T))) = b_1(T)\, T^2 - b_2(T)\, T^3 \tag{22.4.8}$$

defines a homomorphism from $B_1 \times B_1$ into $\mathcal{I}_1$, as right modules over B_1. In fact,

$$\phi(B_1 \times B_1) = \mathcal{I}_1. \tag{22.4.9}$$

We also have that

$$\psi(b(T)) = (b(T)\, T, b(T)) \tag{22.4.10}$$

defines an injective homomorphism from $\mathcal{I}_1$ into $B_1 \times B_1$, as right modules over B_1. Note that

$$\phi \circ \psi = 0 \tag{22.4.11}$$

on $\mathcal{I}_1$.

One can check that

$$\ker \phi = \psi(\mathcal{I}_1), \tag{22.4.12}$$

as mentioned in [3]. One can use this to get a projective resolution of $\mathcal{I}_1$ as a right module over B_1, as in Section 10.2.

22.5 Some more local rings

Let A be a division ring again, with multiplicative identity element $e_A \neq 0$, and let T_1, T_2 be commuting indeterminates. Consider the corresponding ring $A[[T_1, T_2]]$ of formal power series in T_1, T_2 with coefficients in A, as in Section 4.3. Put

$$B_2 = \Big\{ f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^2} f_\alpha\, T^\alpha \in A[[T_1, T_2]] : f_\alpha = 0 \text{ when } \alpha_1 + \alpha_2 \text{ is odd} \Big\}, \tag{22.5.1}$$

as in Exercise 11 on p160 of [3]. Observe that

$$B_2 \text{ is a subring of } A[[T_1, T_2]], \tag{22.5.2}$$

with $e_A \in B_2$.

It is easy to see that

$$\mathcal{I}_2 = \Big\{ f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^2} f_\alpha\, T^\alpha \in B_2 : f_0 = 0 \Big\} \tag{22.5.3}$$

is a two-sided ideal in B_2. Remember that $f(T) \mapsto f_0$ is a ring homomorphism from $A[[T_1, T_2]]$ onto A, as in Section 4.3. Equivalently, $\mathcal{I}_2$ is the intersection of the kernel of this homomorphism with B_2, which is the same as the kernel of the restriction of this homomorphism to B_2. This homomorphism also maps B_2 onto A, which leads to a ring isomorphism from $B_2/\mathcal{I}_2$ onto A.

An element of $A[[T_1, T_2]]$ is invertible in $A[[T_1, T_2]]$ when its constant term is invertible in A, as in Section 4.5, which means that the constant term is nonzero in this case, because A is a division ring. If

$$f(T) \in B_2 \setminus \mathcal{I}_2, \tag{22.5.4}$$

then $f(T)$ is invertible in $A[[T_1, T_2]]$, and one can check that

$$f(T)^{-1} \in B_2, \tag{22.5.5}$$

using the description of $f(T)^{-1}$ in Section 4.5.

The elements of $\mathcal{I}_2$ do not have one-sided inverses in $A[[T_1, T_2]]$, as in Section 4.14, and in particular they do not have local inverses in B_2. This implies that

$$B_2 \text{ is a local ring,} \tag{22.5.6}$$

as in Section 4.13. This corresponds to the first part of Exercise 11 on p160 of [3].

Of course, $A[[T_1, T_2]]$ may be considered as a right module over B_2, and B_2, $\mathcal{I}_2$ may be considered as submodules of $A[[T_1, T_2]]$, as a right module over B_2. Consider

$$\begin{aligned} W_2 \quad = \quad & \Big\{ f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^2} f_\alpha\, T^\alpha \in A[[T_1, T_2]] : \\ & \qquad f_\alpha = 0 \text{ when } \alpha_1 + \alpha_2 \text{ is even} \Big\}. \end{aligned} \tag{22.5.7}$$

It is easy to see that this is also a submodule of $A[[T_1, T_2]]$, as a right module over B_2, as in Exercise 11 on p160 of [3].

22.6 More on W_2

Let us continue with the same notation and hypotheses as in the previous section. Observe that

$$W_2 \text{ is generated by } T_1, T_2, \text{ as a right module over } B_2. \tag{22.6.1}$$

Let $W_2 \cdot \mathcal{I}_2$ be the subset of W_2 consisting of finite sums of products of elements of W_2 and $\mathcal{I}_2$, as in Section 13.4. This is a submodule of W_2, as a right module over B_2, as in Section 13.7. One can check that

$$W_2 \cdot \mathcal{I}_2 = \Big\{ f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^2} f_\alpha \, T^\alpha \in W_2 : f_\alpha = 0 \text{ when } \alpha_1 + \alpha_2 = 1 \Big\}. \tag{22.6.2}$$

The quotient

$$W_2/(W_2 \cdot \mathcal{I}_2) \tag{22.6.3}$$

may be defined as a right module over B_2 in the usual way. This may also be considered as a right module over $B_2/\mathcal{I}_2$, as in Section 13.7. This means that (22.6.3) may be considered as a right module over A here.

One can check that (22.6.3) is freely generated by the images of T_1 and T_2 in the quotient, as a right module over A, using (22.6.2). If W_2 were projective as a right module over B_2, then W_2 should be freely generated by T_1 and T_2, as a right module over B_2, as in Section 13.9. However, it is easy to see that W_2 is not freely generated by T_1 and T_2, as a right module over B_2. This implies that

$$W_2 \text{ is not projective, as a right module over } B_2. \tag{22.6.4}$$

This is another part of Exercise 11 on p160 of [3].

Let us consider $B_2 \times B_2$ as a right module over B_2, which is the same as the direct sum of two copies of B_2, as a right module over itself. Observe that

$$\phi((b_1(T), b_2(T))) = b_1(T)\, T_2 - b_2(T)\, T_1 \tag{22.6.5}$$

defines a homomorphism from $B_1 \times B_2$ into W_2, as right modules over B_2, as in Exercise 11 on p160 of [3]. We also have that

$$\phi(B_2 \times B_2) = W_2, \tag{22.6.6}$$

as in (22.6.1). Clearly

$$\psi(a(T)) = (a(T)\, T_1, a(T)\, T_2) \tag{22.6.7}$$

defines an injective homomorphism from W_2 into $B_1 \times B_2$, as right modules over B_2. It is easy to see that $\phi \circ \psi = 0$ on W_2. In fact, one can verify that

$$\ker \phi = \psi(W_2), \tag{22.6.8}$$

as mentioned in Exercise 11 on p160 of [3]. This can be used to get a projective resolution of W_2 as a right module over B_2, as in Section 10.2.

22.7 Another family of local rings

Let A be a division ring, with multiplicative identity element $e_A \neq 0$, and let T_1, T_2 be commuting indeterminates again. Consider the corresponding rings $A[[T_1]]$ and $A[[T_2]]$ of formal power series in T_1 and T_2, respectively, with coefficients in A, as in Section 4.3. The Cartesian product

$$(A[[T_1]]) \times (A[[T_2]]) \tag{22.7.1}$$

of $A[[T_1]]$ and $A[[T_2]]$ is also a ring, with respect to coordinatewise addition and scalar multiplication, which is the same as the direct sum of these two rings. Note that (e_A, e_A) is the multiplicative identity element in (22.7.1).

Consider

$$B_3 = \{(f(T_1), g(T_2)) \in (A[[T_1]]) \times (A[[T_2]]) : f(0) = g(0)\}, \tag{22.7.2}$$

as in Exercise 12 on p160 of [3], where $f(0)$, $g(0)$ are the same as the constant terms in $f(T_1)$, $g(T_2)$. It is easy to see that

$$B_3 \text{ is a subring of } (A[[T_1]]) \times (A[[T_2]]). \tag{22.7.3}$$

Put

$$\mathcal{I}_3 = \{(f(T), g(T)) \in B_3 : f(0) = g(0) = 0\}. \tag{22.7.4}$$

One can check that this is a two-sided ideal in B_3. In fact, this is a two-sided ideal in (22.7.1).

Of course, the Cartesian product $A \times A$ of A with itself is a ring with respect to coordinatewise addition and multiplication too, and (e_A, e_A) is the multiplicative identity element in $A \times A$. Clearly

$$(f(T_1), g(T_2)) \mapsto (f(0), g(0)) \tag{22.7.5}$$

defines a ring homomorphism from (22.7.1) onto $A \times A$, whose kernel is equal to $\mathcal{I}_3$.

Note that

$$\{(a, a) : A \in A\} \tag{22.7.6}$$

is a subring of $A \times A$, and that

$$a \mapsto (a, a) \tag{22.7.7}$$

is a ring isomorphism from A onto (22.7.6). The restriction of (22.7.5) to B_3 corresponds to a ring homomorphism

$$(f(T_1), g(T_2)) \mapsto f(0) = g(0) \tag{22.7.8}$$

from B_3 onto A in this way. The kernel of this homomorphism is equal to $\mathcal{I}_3$ as well, so that $B_3/\mathcal{I}_3$ is isomorphic to A as a ring.

It is easy to see that an element $(f(T_1), g(T_2))$ of (22.7.1) has a multiplicative inverse in (22.7.1) if and only if $f(T_1)$ and $g(T_2)$ have multiplicative inverses in $A[[T_1]]$ and $A[[T_2]]$, respectively, with

$$(f(T_1), g(T_2))^{-1} = (f(T_1)^{-1}, g(T_2)^{-1}). \tag{22.7.9}$$

This happens exactly when $f(0)$ and $g(0)$ are invertble in A, as in Section 4.5, which is the same as saying that $f(0), g(0) \neq 0$ in this case, because A is a division ring. Under these conditions, the constant terms in $f(T_1)^{-1}$ and $g(T_2)^{-1}$ are the same as the multiplicative inverses of the constant terms in $f(T_1)$, $g(T_2)$, respectively. If $(f(T_1), g(T_2)) \in B_3$ and

$$f(0) = g(0) \neq 0, \tag{22.7.10}$$

then it follows that

$$(f(T_1), g(T_2))^{-1} \in B_3. \tag{22.7.11}$$

Similarly, an element $(f(T_1), g(T_2))$ of (22.7.1) has a one-sided inverse in (22.7.1) if and only if $f(T_1)$ and $g(T_2)$ have the same type of one-sided inverses in $A[[T_1]]$ and $A[[T_2]]$, respectively. In this case, $f(0)$ and $g(0)$ have the same type of one-sided inverses in A, as in Section 4.14, so that $f(0), g(0) \neq 0$. Thus the elements of $\mathcal{I}_3$ do not have one-sided inverses in (22.7.1), and in particular they do not have one-sided inverses in B_3. This means that

$$B_3 \text{ is a local ring,} \tag{22.7.12}$$

as in Section 4.13. This corresponds to the first part of Exercise 12 on p160 of [3].

Observe that

$$(f(T_1), g(T_2)) \mapsto f(T_1) \tag{22.7.13}$$

and

$$(f(T_1), g(T_2)) \mapsto g(T_2) \tag{22.7.14}$$

define ring homomorphisms from (22.7.1) onto $A[[T_1]]$ and $A[[T_2]]$, respectively. The restrictions of these homomorphisms to B_2 map B_2 on $A[[T_1]]$ and $A[[T_2]]$, respectively, as well.

22.8 Some right modules over B_3

Let us continue with the same notation and hypotheses as in the previous section. We may consider (22.7.1) as a right module over B_3, and B_3, $\mathcal{I}_3$ are submodules of (22.7.1), as a right module over B_3.

Put

$$V_3 = \{(f(T_1)\, T_1, 0) : f(T_1) \in A[[T_1]]\} \tag{22.8.1}$$

and

$$W_3 = \{(0, g(T_2)\, T_2) : g(T_2) \in A[[T_2]]\}. \tag{22.8.2}$$

It is easy to see that these are submodules of (22.7.1), as a right module over B_3, as on p161 of [3]. In fact, V_3 is the same as the kernel of the restriction of (22.7.14) to B_2, and W_3 is the same as the kernel of the restriction of (22.7.13) to B_2.

Observe that V_3 is generated, as a right module over B_3, by any element $(f(T_1)\,T_1, 0)$ with $f(0) \neq 0$. Any collection of generators of V_3, as a right module over B_3, has to contain an element of this type. However, one can check that V_3 is not freely generated as a right module over B_3. Similarly, W_3 is not freely generated as a right module over B_3. This is another part of Exercise 12 on p160 of [3].

Let $V_3 \cdot \mathcal{I}_3$ be the subset of V_3 consisting of finite sums of propducts of elements of V_3 and $\mathcal{I}_3$, as in Section 13.4, and similarly for $W_3 \cdot \mathcal{I}_3$. These are submodules of V_3 and W_3, respectively, as right modules over B_3, as in Section 13.7. One can verify that

$$V_3 \cdot \mathcal{I}_3 = \{(f(T_1)\,T_1^2, 0) : f(T_1) \in A[[T_1]]\} \tag{22.8.3}$$

and

$$W_3 \cdot \mathcal{I}_3 = \{(0, g(T_2)\,T_2^2) : g(T_2) \in A[[T_2]]\}. \tag{22.8.4}$$

The quotients

$$V_3/(V_3 \cdot \mathcal{I}_3) \tag{22.8.5}$$

and

$$W_3/(W_3 \cdot \mathcal{I}_3) \tag{22.8.6}$$

may be defined as right modules over B_3, as usual. These may also be considered as right modules over $B_3/\mathcal{I}_3$, as in Section 13.7. This means that (22.8.5) and (22.8.6) may be considered as right modules over A.

It is easy to see that (22.8.5) is freely generated by the image of $(T_2, 0)$ in the quotient, as a right module over A, because of (22.8.3). Similarly, (22.8.6) is freely generated by the image of $(0, T_2)$ in the quotient, as a right module over A, because of (22.8.4). If V_3 were projective as a right module over B_3, then V_3 should be freely generated by $(T_1, 0)$, as a right module over B_3, as in Section 9.10. Similarly, if W_3 were projective as a right module over B_3, then it should be freely generated by $(0, T_2)$, as a right module over B_3. It follows that

(22.8.7) V_3 and W_3 are not projective, as right modules over B_3,

because V_3 and W_3 are not freely generated in this way, as on p161 of [3].

Observe that

$$\phi((f(T_1), g(T_2))) = (0, g(T_2)\,T_2) \tag{22.8.8}$$

defines a homomorphism from B_3 onto W_3, as right modules over B_3, as on p161 of [3]. Similarly,

$$\phi'((f(T_1), g(T_2))) = (f(T_1)\,T_1, 0) \tag{22.8.9}$$

defines a homomorphism from B_2 onto V_3, as right modules over B_3.

One can check that

$$\ker \phi = V_3 \tag{22.8.10}$$

and

$$\ker \phi' = W_3, \tag{22.8.11}$$

as mentioned on p161 of [3]. One can use ϕ and ϕ' to get projective resolutions of V_3 and W_3 are right modules over B_3, as in Section 10.2.

22.9 Direct systems of sets

Let $(I, \preceq)$ be a nonempty pre-directed set, and let E_j be a set for each $j \in I$. Suppose that for every $j, l \in I$ with $j \preceq l$ we have a mapping $\theta_{j,l}$ from E_j into E_l. We ask that $\theta_{j,j}$ be the identity mapping on E_j for each $j \in I$, and that for every $j, l, r \in I$ with $j \preceq l \preceq r$, we have that

$$\theta_{l,r} \circ \theta_{j,l} = \theta_{j,r}. \tag{22.9.1}$$

In this case, the family of sets E_j and mappings $\theta_{j,l}$ is said to form a *direct* or *inductive system* over $(I, \preceq)$, as usual. Note that if $j, l \in I$, $j \preceq l$, and $l \preceq j$, then $\theta_{j,l}$ and $\theta_{l,j}$ are inverses of each other.

Let $\widetilde{E}_j$ be a nonempty set that contains E_j for each $j \in I$. If $x, y \in \prod_{j \in I} \widetilde{E}_j$, then put $x \sim y$ when there is an $l \in I$ such that

$$x_j = y_j \tag{22.9.2}$$

for each $j \in I$ with $l \preceq j$. One can check that this defines an equivalence relation on $\prod_{j \in I} \widetilde{E}_j$. This permits us to define to corresponding quotient set

$$\Big(\prod_{j \in I} \widetilde{E}_j\Big) / \sim \tag{22.9.3}$$

as the set of equivalence classes in $\prod_{j \in I} A_j$ with respect to $\sim$.

If $l \in I$ and $w_l \in E_l$, then there are elements x of $\prod_{j \in I} \widetilde{E}_j$ such that

$$x_j = \theta_{l,j}(w_l) \tag{22.9.4}$$

for every $j \in I$ with $l \preceq j$. Any two such elements of $\prod_{j \in I} \widetilde{E}_j$ are clearly equivalent with respect to $\sim$. This defines a mapping θ_l from E_l into the quotient set (22.9.3). If $r \in I$ and $l \preceq r$, then it is easy to see that

$$\theta_l = \theta_r \circ \theta_{l,r}. \tag{22.9.5}$$

Consider

$$\varinjlim E_j = \bigcup_{l \in I} \theta_l(E_l), \tag{22.9.6}$$

which is a subset of the quotient set (22.9.3). This is the *direct* or *inductive limit* of the direct system of E_j's, $j \in I$. More precisely, the direct limit consists of (22.9.6) together with the mappings θ_l from E_l into this set for each $l \in I$. If $l, r \in I$ and $l \preceq r$, then

$$\theta_l(E_l) \subseteq \theta_r(E_r), \tag{22.9.7}$$

by (22.9.5).

Let $l_1, l_2 \in I$, $w_{1,l_1} \in E_{l_1}$, and $w_{2,l_2} \in E_{l_2}$ be given. If

$$\theta_{l_1}(w_{1,l_1}) = \theta_{l_2}(w_{2,l_2}), \tag{22.9.8}$$

then one can check that there is an $r \in I$ such that

$$l_1, l_2 \preceq r \tag{22.9.9}$$

and

$$\theta_{l_1,r}(w_{1,l_1}) = \theta_{l_2,r}(w_{2,l_2}). \tag{22.9.10}$$

Conversely, if there is an $r \in I$ such that (22.9.9) and (22.9.10) hold, then

$$\theta_{l_1,t}(w_{1,l_1}) = \theta_{l_2,t}(w_{2,l_2}) \tag{22.9.11}$$

for every $t \in I$ such that $r \preceq t$, by (22.9.1). Under these conditions, (22.9.8) holds, by construction.

Let C be another set, and let γ_l be a mapping from E_l into C for each $l \in I$. Suppose that for every $l, r \in I$ with $l \preceq r$, we have that

$$\gamma_l = \gamma_r \circ \theta_{l,r}. \tag{22.9.12}$$

Under these conditions, one can check that there is a unique mapping γ from (22.9.6) into C such that

$$\gamma_l = \gamma \circ \theta_l \tag{22.9.13}$$

for every $l \in I$. More precisely, the main point is to verify that γ is well defined in this way. This determines the direct limit uniquely, up to suitable equivalence.

Let $\widetilde{E}$ be a set, and suppose now that E_j is a subset of $\widetilde{E}$ for every $j \in I$. If $j, l \in I$ and $j \preceq l$, then we ask that

$$E_j \subseteq E_l, \tag{22.9.14}$$

and we let $\theta_{j,l}$ be the obvious inclusion mapping from E_j into E_l. These mappings satisfy the conditions in the definition of a direct system of sets, and the direct limit is equivalent to

$$\bigcup_{j \in I} E_j. \tag{22.9.15}$$

Of course, this uses the obvious inclusion mapping from E_l into (22.9.15) for each $l \in I$.

22.10 Some limits of free modules

Let us continue with the same notation and hypotheses as at the beginning of the previous section, and put

$$E = \varinjlim E_j. \tag{22.10.1}$$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a nonzero multiplicative identity element e_A. If $j \in I$, then let F_{E_j} be the free left or right module over A generated by E_j. This may be defined precisely as the space $c_{00}(E_j, A)$ of A-valued functions on E_j with finite support, where each element of E_j is identified with the A-valued function on E_j equal to e_A at that element, and to 0 otherwise. Similarly, let F_E be the free left or right module over A generated by E, which may be defined precisely as $c_{00}(E, A)$.

If $j, l \in I$ and $j \preceq l$, then there is a unique homomorphism $\Theta_{j,l}$ from F_{E_j} into F_{E_l}, as modules over A, corresponding to $\theta_{j,l}$. This mapping is the identity mapping on F_{E_j} when $j = l$, and it is easy to see that

$$\Theta_{l,r} \circ \Theta_{j,l} = \Theta_{j,r} \tag{22.10.2}$$

for all $j, l, r \in I$ with $j \preceq l \preceq r$, by (22.9.1). In particular, if $j, l \in I$, $j \preceq l$, and $l \preceq j$, then $\Theta_{j,l}$ and $\Theta_{l,j}$ are inverses of each other. Thus the F_{E_j}'s, $j \in I$, together with the homomorphisms $\Theta_{j,l}$, form a direct system of modules over $(I, \preceq)$.

Similarly, if $l \in I$, then there is a unique homomorphism Θ_l from F_{E_l} into F_E, as modules over A, corresponding to θ_l. If $r \in I$ and $l \preceq r$, then

$$\Theta_l = \Theta_r \circ \Theta_{l,r}, \tag{22.10.3}$$

because of (22.9.5).

Let Z be another left or right module over A, as appropriate, and let ζ_l be a homomorphism from F_{E_l} into Z, as modules over A, for each $l \in I$. Note that ζ_l is determined by its values on on the elements of E_l, as generators of F_{E_l}, as a module over A. Suppose that for each $l, r \in I$ with $l \preceq r$, we have that

$$\zeta_l = \zeta_r \circ \Theta_{l,r}. \tag{22.10.4}$$

If $l \in I$, then let γ_l be the restriction of ζ_l to E_l, considered as a subset of F_{E_l}, as before. It follows from (22.10.4) that (22.9.12) holds for every $l, r \in I$ with $l \preceq r$. This implies that there is a unique mapping γ from E into Z such that (22.9.13) holds for every $l \in I$, as in the previous section.

This leads to a unique homomorphism ζ from F_E into Z, as modules over A, that corresponds to γ on E. If $l \in I$, then

$$\zeta_l = \zeta \circ \Theta_l, \tag{22.10.5}$$

by (22.9.13).

This shows that the direct limit of the F_{E_j}'s, $j \in I$, may be identified with F_E, using the homomorphisms Θ_l, $l \in I$, as in Section 3.2.

22.11 More on systems of modules

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let $(I, \preceq)$ be a nonempty pre-directed set, and let V_j be a left module over A for every $j \in I$. Suppose that for every $j, l \in I$ with $j \preceq l$ we have a homomorphism $\nu_{j,l}$ from V_j into V_l, as left modules over A, that satisfies the conditions mentioned in Section 3.2. This means that $\nu_{j,j}$ is the identity mapping on V_j for every $j \in I$, and that if $j, l, r \in I$ satisfy $j \preceq l \preceq r$, then $\nu_{l,r} \circ \nu_{j,l} = \nu_{j,r}$. The direct limit $V = \varinjlim V_j$ of the V_j's can be defined as a left module over A, as before.

Of course, $\prod_{j \in I} V_j$ may be considered as a left module over A, with respect to coordinatewise addition and left multiplication by elements of A. Let W be the set of $w \in \prod_{j \in I} V_j$ for which there is an $l \in I$ such that

$$w_j = 0 \tag{22.11.1}$$

for each $j \in I$ with $l \preceq j$. It is easy to see that W is a submodule of $\prod_{j \in I} V_j$, as a left module over A. Thus the quotient

$$\Big(\prod_{j \in I} V_j\Big)/W \tag{22.11.2}$$

is defined as a left module over A.

If $l \in I$ and $v_l \in V_l$, then there are elements y of $\prod_{j \in I} V_j$ such that

$$y_j = \nu_{l,j}(v_l) \tag{22.11.3}$$

for each $j \in I$ with $l \preceq j$. The difference between any two such elements of $\prod_{j \in I} V_j$ is an element of W. This defines a mapping η_l from V_l into the quotient (22.11.2). It is easy to see that η_l is a homomorphism from V_l into (22.11.2), as modules over A. If $r \in I$ and $l \preceq r$, then

$$\eta_l = \eta_r \circ \nu_{l,r}. \tag{22.11.4}$$

Remember that the direct limit V is equipped with a homomorphism ν_l from V_l into V, as modules over A, for each $l \in I$, as in Section 3.2. It follows from the remarks in the preceding paragraph that there is a unique homomorphism η from V into the quotient (22.11.2), as modules over A, such that

$$\eta_l = \eta \circ \nu_l \tag{22.11.5}$$

for every $l \in I$, as in Section 3.2. Remember that

$$V = \bigcup_{l \in I} \nu_l(V_l), \tag{22.11.6}$$

as in Section 3.2. This implies that

$$\eta(V) = \bigcup_{l \in I} \eta(\nu_l(V_l)) = \bigcup_{l \in I} \eta_l(V_l). \tag{22.11.7}$$

Suppose that $\eta(v) = 0$ for some $v \in V$. As in (22.11.6), $v = \nu_l(v_l)$ for some $l \in I$ and $v_l \in V_l$. Thus we get that

$$\eta_l(v_l) = \eta(\nu_l(v_l)) = \eta(v) = 0. \tag{22.11.8}$$

This means that if y is an element of $\prod_{j \in I} V_j$ that satisfies (22.11.3), then $y \in W$. It follows that there is an $r \in I$ such that $y_j = 0$ for each $j \in I$ with $r \preceq j$, by definition of W. If $j \in I$ and $l, r \preceq j$, then we get that

$$\nu_{l,j}(v_l) = y_j = 0. \tag{22.11.9}$$

In this case, we obtain that

$$v = \nu_l(v_l) = \nu_j(\nu_{l,j}(v_l)) = 0, \tag{22.11.10}$$

where the second step is as in Section 3.2. This shows that η is injective, so that the definition of the direct limit of the V_j's as modules in Section 3.2 is compatible with the definition of the direct limit of the V_j's as sets in Section 22.9.

22.12 Projective resolutions of direct limits

Let us continue with the same notation and hypotheses as at the beginning of the previous section. Under these conditions, Lemma 9.5* on p100 of [3] says that one can find a projective resolution X_j of V_j for each $j \in I$, such that the family of X_j's, $j \in I$, form a direct system, whose limit $X = \varinjlim X_j$ is a projective resolution of V. Remember that direct systems of graded modules and complexes were discussed in Section 8.15. Of course, there are analogous statements for right modules.

If $j \in I$, then let $X_{0,j}$ be the free left module F_{V_j} over A generated by V_j, as in Section 22.10. Similarly, let X_0 be the free left module F_V over A generated by V. We may consider the V_j's, $j \in I$, as a direct system of sets, so that the $X_{0,j}$'s, $j \in I$, form a direct system of left modules over A, as before. In fact, we may identify V with the direct limit of the V_j's, $j \in I$, as sets, as in the previous section. This permits us to identify X_0 with the direct limit of the $X_{0,j}$'s, as modules over A, as in Section 22.10.

If $j \in I$, then there is a natural homomorphism ϕ_j from $X_{0,j}$ onto V_j, as modules over A. This homomorphism sends an element of V_j, considered as an element of $X_{0,j}$, to itself in V_j. The family of ϕ_j's, $j \in I$, defines a homomorphism between the direct systems of $X_{0,j}$'s and V_j's, as in Section 3.4. Similarly, there is a homomorphism ϕ from X_0 onto V, as modules over A, which sends an element of V, considered as an element of X_0, to itself in V. This corresponds to the direct limit $\varinjlim \phi_j$ of the ϕ_j's, as before.

Let R_j be the kernel of ϕ_j for each $j \in I$, and let R be the kernel of ϕ. It is easy to see that the family of R_j's, $j \in I$, is a direct system of left modules over A. More precisely, if $j, l \in I$ and $j \preceq l$, then the homomorphism from $X_{0,j}$

into $X_{0,l}$ obtained from $\nu_{j,l}$ as in Section 22.10 maps R_j into R_l, and satisfies the conditions in the definition of a direct system of modules. The family of inclusion mappings from the R_j's into the $X_{0,j}$'s, $j \in I$, defines a homomorphism between the direct systems of R_j's and $X_{0,j}$'s. The direct limit of these inclusion mappings defines an isomorphism from $\varinjlim R_j$ onto R, as modules over A, as in Section 3.4.

Thus R can be identified with the direct limit of the R_j's, $j \in I$. We can repeat the process to get the desired projective resolutions, as in Section 10.2.

Chapter 23

Some satellites

23.1 Some basic comparisons

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, V_1, P, and P_1 be all left or all right modules over A, let β, β_1 be homomorsphisms from P, P_1 onto V, V_1, respectively, and let g be a homomorphism from V into V_1.

Thus $g \circ \beta$ is a homomorphism from P into V_1, as modules over A. We would like to consider some related mappings and their properties, as in the discussion starting on p33 of [3].

Suppose that P is projective, so that there is a homomorphism f from P into P_1, as modules over A, such that

(23.1.1) $$\beta_1 \circ f = g \circ \beta.$$

Note that

(23.1.2) $$f(\ker \beta) \subseteq \ker \beta_1$$

in this case. This means that the restriction of f to $\ker \beta$ defines a homomorphism from $\ker \beta$ into $\ker \beta_1$, as modules over A.

If $\widetilde{f}$ is another homomorphism from P into P_1, then

(23.1.3) $$\beta_1 \circ \widetilde{f} = g \circ \beta$$

if and only if

(23.1.4) $$(f - \widetilde{f})(P) \subseteq \ker \beta_1.$$

This is the same as saying that $f - \widetilde{f}$ may be considered as the composition of a homomorphism from P into $\ker \beta_1$ with the obvious inclusion mapping from $\ker \beta_1$ into P_1.

If $t \in k$, then $t\,g$ is a homomorphism from V into V_1 as well, as modules over A. Of course, $t\,f$ is a homomorphism from P into P_1, whose composition with β_1 is equal to the composition of β with $t\,g$.

Let $\widehat{g}$ be another homomorphism from V into V_1, as modules over A, so that $\widehat{g} \circ \beta$ is a homomorphism from P into V_1. As before, there is a homomorphism $\widehat{f}$ from P into P_1, as modules over A, such that

$$\beta_1 \circ \widehat{f} = \widehat{g} \circ \beta, \tag{23.1.5}$$

because P is projective. It follows that

$$\beta_1 \circ (f + \widehat{f}) = (g + \widehat{g}) \circ \beta. \tag{23.1.6}$$

Let V_2 and P_2 be left or right modules over A, as appropriate, let β_2 be a homomorphism from P_2 onto V_2, and let g_1 be a homomorphism from V_1 into V_2. Note that $g_1 \circ \beta_1$ is a homomorphism from P_1 into V_2, as modules over A. Suppose now that P_1 is projective as well, so that there is a homomorphism f_1 from P_1 into P_2, as modules over A, such that

$$\beta_2 \circ f_1 = g_1 \circ \beta_1. \tag{23.1.7}$$

Of course, $g_1 \circ g$ is a homomorphism from V into V_2, and $g_1 \circ g \circ \beta$ is a homomorphism from P into V_2. Under these conditions, $f_1 \circ f$ is a homomorphism from P into P_2, with

$$\beta_2 \circ f_1 \circ f = g_1 \circ \beta_1 \circ f = g_1 \circ g \circ \beta. \tag{23.1.8}$$

Suppose that $V_2 = V$, $P_2 = P$, $\beta_2 = \beta$, and $g_1 \circ g$ is the identity mapping on V. In this case, $f_1 \circ f$ is a homomorphism from P into itself, with

$$\beta \circ f_1 \circ f = \beta, \tag{23.1.9}$$

by (23.1.8). Equivalently, this means that $f_1 \circ f$ minus the identity mapping on P maps P into the kernel of β.

23.2 Comparisons and tensor products

Let us continue with the same notations and hypotheses as in the previous section. Put

$$M = \ker \beta, \ M_1 = \ker \beta_1, \tag{23.2.1}$$

which are submodules of P, P_1, respectively, as modules over A. Let α, α_1 be the obvious inclusion mappings from M, M_1 into P, P_1, respectively. Remember that f maps M into M_1, as in (23.1.2), and let f' be the restriction of f to M, considered as a homomorphism from M into M_1, as modules over A. Thus

$$\alpha_1 \circ f' = f \circ \alpha, \tag{23.2.2}$$

as homomorphisms from M into P_1, by construction.

Suppose now that V, V_1, P, P_1, and so on are right modules over A, and let W be a left module over A. Let $M \bigotimes_A W$, $P \bigotimes_A W$, $M_1 \bigotimes_A W$, and $P_1 \bigotimes_A W$

be tensor products of M, P, M_1, and P_1 with W over A, respectively. Consider the homomorphisms

(23.2.3) α^W, α_1^W from $M \bigotimes_A W$, $M_1 \bigotimes_A W$ into $P \bigotimes_A W$, $P_1 \bigotimes_A W$,

respectively, as modules over k, corresponding to α, α_1 and the identity mapping on W, as in Section 1.9. Similarly, we get homomorphisms

(23.2.4) f'_W, f_W from $M \bigotimes_A W$, $P \bigotimes_A W$ into $M_1 \bigotimes_A W$, $P_1 \bigotimes_A W$,

respectively, as modules over k, corresponding to f', f and the identity mapping on W. Note that

(23.2.5) $$\alpha_1^W \circ f'_W = f_W \circ \alpha^W,$$

as homomorphisms from $M \bigotimes_A W$ into $P_1 \bigotimes_A W$, because of (23.2.2), as in (2) on p34 of [3].

Using (23.2.5), we get that

(23.2.6) $$f'_W(\ker \alpha^W) \subseteq \ker \alpha_1^W.$$

Consider the homomorphism

(23.2.7) $$\theta_1(g) \text{ from } \ker \alpha^W \text{ into } \ker \alpha_1^W,$$

as modules over k, corresponding to the restriction of f'_W to $\ker \alpha^W$, as on p34 of [3].

Let us check that

(23.2.8) $$\theta_1(g) \text{ does not depend on the choice of } f$$

from the previous section, as in Proposition 1.1 on p34 of [3]. Let $\widetilde{f}$ be another homomorphism from P into P_1, as modules over A, such that (23.1.3) holds. Thus $f - \widetilde{f}$ maps P into the kernel of β_1, as in (23.1.4). Let h be the homomorphism from P into $\ker \beta_1$, as modules over A, that is equal to $f - \widetilde{f}$. This means that

(23.2.9) $$f - \widetilde{f} = \alpha_1 \circ h,$$

as homomorphisms from P into P_1.

Let $\widetilde{f}'$ be the restriction of $\widetilde{f}$ to M, considered as a homomorphism from M into M_1. Note that

(23.2.10) $$f' - \widetilde{f}' = h \circ \alpha.$$

Let $\widetilde{f}'_W$ be the homomorphism from $M \bigotimes_A W$ into $M_1 \bigotimes_A W$ corresponding to $\widetilde{f}'$ and the identity mapping on W, and let h_W be the homomorphism from $P \bigotimes_A W$ into $M_1 \bigotimes_A W$ that corresponds to h and the identity mapping on W. Thus

(23.2.11) $$f'_W - \widetilde{f}'_W = h_W \circ \alpha^W,$$

as homomorphisms from $M \bigotimes_A W$ into $M_1 \bigotimes_A W$, because of (23.2.10). This implies that

(23.2.12) $$f'_W = \widetilde{f}'_W \text{ on } \ker \alpha^W,$$

as desired.

If $t \in k$, then $t\,g$ is a homomorphism from V into V_1, and $t\,f$ is a homomorphism from P into P_1 that is related to $t\,g$ in the usual way. It is easy to see that

(23.2.13) $$\theta_1(t\,g) = t\,\theta_1(g),$$

as homomorphisms from $\ker \alpha^W$ into $\ker \alpha_1^W$.

Let $\widehat{g}$ be another homomorphism from V into V_1, as in the previous section, and let $\widehat{f}$ be a homomorphism from P into P_1 such that (23.1.5) holds. Under these conditions, $g + \widehat{g}$ is a homomorphism from V into V_1, and $f + \widehat{f}$ is a homomorphism from P into P_1 that satisfies (23.1.6). This means that homomorphisms

(23.2.14) $$\theta_1(\widehat{g}) \text{ and } \theta_1(g + \widehat{g}) \text{ from } \ker \alpha^W \text{ into } \ker \alpha_1^W$$

may be defined using $\widehat{f}$ and $f + \widehat{f}$, respectively, as before. It follows that

(23.2.15) $$\theta_1(g + \widehat{g}) = \theta_1(g) + \theta_1(\widehat{g}),$$

as in Proposition 1.1 on p34 of [3].

Suppose for the moment that $V_1 = V$, $P_1 = P$, $\beta_1 = \beta$, and g is the identity mapping I_V on V. In this case, we can take f to be the identity mapping I_P on P, so that f' is the identity mapping I_M on $M = M_1$. This means that f'_W is the identity mapping on $M \bigotimes_A W = M_1 \bigotimes_A W$. Note that $\alpha = \alpha_1$, so that $\alpha_1^W = \alpha^W$, and thus $\ker \alpha_1^W = \ker \alpha^W$. Under these conditions, we get that

(23.2.16) $$\theta_1(g) \text{ is the identity mapping on } \ker \alpha^W.$$

Let V_1, P_1, β_1, and g be as before, and suppose now that P_1 is projective as a module over A. Also let V_2, P_2, β_2, g_1, and f_1 be as in the previous section. Put

(23.2.17) $$M_2 = \ker \beta_2,$$

which is a submodule of P_2, and let α_2 be the obvious inclusion mapping from M_2 into P_2. Let $M_2 \bigotimes_A W$ and $P_2 \bigotimes_A W$ be tensor products of M_2 and P_2 with W over A, and consider the homomorphism

(23.2.18) $$\alpha_2^W \text{ from } M_2 \bigotimes_A W \text{ into } P_2 \bigotimes_A W$$

corresponding to α_2 and the identity mapping on W. We can define a homomorphism

(23.2.19) $$\theta_1(g_1) \text{ from } \ker \alpha_1^W \text{ into } \ker \alpha_2^W,$$

as modules over k, using f_1, as before. Similarly, we can define a homomorphism

(23.2.20) $$\theta_1(g_1 \circ g) \text{ from } \ker \alpha^W \text{ into } \ker \alpha_2^W,$$

as modules over k, using $f_1 \circ f$, because of (23.1.8). One can check that

$$\theta_1(g_1 \circ g) = \theta_1(g_1) \circ \theta_1(g), \tag{23.2.21}$$

using the analogous property for induced mappings between tensor products. This is another part of Proposition 1.1 on p34 of [3].

Suppose that $V_2 = V$, $P_2 = P$, and $\beta_2 = \beta$, so that $M_2 = M$, $\alpha_2 = \alpha$, $\alpha_2^W = \alpha^W$, and thus $\ker \alpha_2^W = \ker \alpha^W$. If $g_1 \circ g$ is the identity mapping on V, then

$$\theta_1(g_1 \circ g) \text{ is the identity mapping on } \ker \alpha^W, \tag{23.2.22}$$

as in (23.2.16). Similarly, if $g \circ g_1$ is the identity mapping on V_1, then

$$\theta_1(g \circ g_1) \text{ is the identity mapping on } \ker \alpha_1^W. \tag{23.2.23}$$

If g and g_1 are inverses of each other, then it follows that

$$\theta_1(g) \text{ and } \theta_1(g_1) \text{ are inverses of each other,} \tag{23.2.24}$$

because of (23.2.21).

Of course, there are analogous statements when we start with left modules over A as in the previous section, and consider their tensor products with a right module W over A.

23.3 Left satellites of tensor products

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V be a right module over A, let W be a left module over A, and let

$$T(V) = V \bigotimes\nolimits_A W \tag{23.3.1}$$

be a tensor product of V and W over A.

Suppose that P is a projective right module over A, and that β is a homomorphism from P onto V, as modules over A. Put $M = \ker \beta$, and let α be the obvious inclusion mapping from M into P, as in the previous section. Let $M \bigotimes_A W$, $P \bigotimes_A W$ be tensor products of M, P with W again, respectively, and consider the homomorphism

$$\alpha^W \text{ from } M \bigotimes\nolimits_A W \text{ into } P \bigotimes\nolimits_A W, \tag{23.3.2}$$

as modules over k, corresponding to α and the identity mapping on W, as before. Put

$$S_1 T(V) = \ker \alpha^W, \tag{23.3.3}$$

which is a submodule of $M \bigotimes_A W$, and a module over k in particular. This corresponds to (5) on p35 of [3].

Let P_1 be another projective right module over A, and let β_1 be a homomorphism from P_1 onto V, as modules over A. Put $M_1 = \ker \beta_1$, and let α_1 be the obvious inclusion mapping from M_1 into P_1, as before. Let $M_1 \bigotimes_A W$, $P_1 \bigotimes_A W$ be tensor products of M_1, P_1 with W, respectively, and consider the homomorphism

(23.3.4) $$\alpha_1^W \text{ from } M_1 \bigotimes_A W \text{ into } P_1 \bigotimes_A W,$$

as modules over k, corresponding to α_1 and the identity mapping on W. This is the same as in the previous section, with $V = V_1$.

If we take g to be the identity mapping on V, then we get a homomorphism $\theta_1(g)$ from $\ker \alpha^W$ into $\ker \alpha_1^W$, as modules over k, as in (23.2.8). Of course, this depends on P, P_1 and β, β_1.

Let us take $V_2 = V$, $P_2 = P$, and $\beta_2 = \beta$, so that we have $M_2 = M$, $\alpha_2 = \alpha$, and $\alpha_2^W = \alpha^W$, as in the previous section. If we take g_1 to be the identity mapping on $V_1 = V$, then we get a homomorphism $\theta_1(g_1)$ from $\ker \alpha_1^W$ into $\ker \alpha_2^W = \ker \alpha^W$, as modules over k, as in (23.2.19). This depends on P_1, $P_2 = P$ and β_1, $\beta_2 = \beta$, as before.

Because $g_1 \circ g$ is the identity mapping on V, and $g \circ g_1$ is the identity mapping on $V_1 = V$, we get that $\theta_1(g)$ and $\theta_1(g_1)$ are inverses of each other, as in (23.2.24). This shows that (23.3.3) is unique up to natural isomorphisms, as on p35f of [3]. This is called the *left satellite* of (23.3.1), as on p36 of [3].

If V is projective as a right module over A, the we can take $P = V$ and β to be the identity mapping on V, so that $M = \{0\}$, $\alpha = 0$, $\alpha^W = 0$, and

(23.3.5) $$S_1 T(V) = \{0\}.$$

This corresponds to part of Proposition 1.3 on p37 of [3].

If W is projective as a left module over A, then α^W is injective, because α is injective, as mentioned near the end of Section 2.7. This implies that (23.3.5) holds in this case too. This corresponds to part of Proposition 1.2 on p36 of [3].

Of course, there are analogous statements when V is a left module over A, W is a right module over A, and $T(V) = W \bigotimes_A V$ is a tensor product of W and V, as modules over A.

23.4 Comparisons and $\mathrm{Hom}_A(\cdot, W)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. As in Section 23.1, we let V, V_1, P, and P_1 be all left or all right modules over A, we let β, β_1 be homomorphisms from P, P_1 into V, V_1, respectively, and we let g be a homomorphism from V into V_1. We also let M, M_1 be the kernels of β, β_1, respectively, and let α, α_1 be the obvious inclusion mappings from M, M_1 into P, P_1, respectively.

Suppose that P is projective again, so that there is a homomorphism f from P into P_1 such that $\beta_1 \circ f = g \circ \beta$. Remember that

(23.4.1) $$f(M) \subseteq M_1,$$

and let f' be the restriction of f to M, considered as a homomorphism from M into M_1.

Let W be another left or right module over A, depending on whether the other modules are left or right modules. Consider the homomorphisms

$$\mathrm{Hom}(\alpha, I_W) \text{ from } \mathrm{Hom}_A(P, W) \text{ into } \mathrm{Hom}_A(M, W) \tag{23.4.2}$$

and

$$\mathrm{Hom}(\alpha_1, I_W) \text{ from } \mathrm{Hom}_A(P_1, W) \text{ into } \mathrm{Hom}_A(M_1, W) \tag{23.4.3}$$

associated to α, α_1 and the identity mapping I_W on W as in Section 6.3. Similarly, we get homomorphisms

$$\mathrm{Hom}(f', I_W) \text{ from } \mathrm{Hom}_A(M_1, W) \text{ into } \mathrm{Hom}_A(M, W) \tag{23.4.4}$$

and

$$\mathrm{Hom}(f, I_W) \text{ from } \mathrm{Hom}_A(P_1, W) \text{ into } \mathrm{Hom}_A(P, W). \tag{23.4.5}$$

It is easy to see that

$$\mathrm{Hom}(f', I_W) \circ \mathrm{Hom}(\alpha_1, I_W) = \mathrm{Hom}(\alpha, I_W) \circ \mathrm{Hom}(f, I_W), \tag{23.4.6}$$

as homomorphisms from $\mathrm{Hom}_A(P_1, W)$ into $\mathrm{Hom}_A(M, W)$, because $\alpha_1 \circ f' = f \circ \alpha$. This corresponds to a remark on p34 of [3].

Remember that the cokernel of $\mathrm{Hom}(\alpha, I_W)$ is the quotient of $\mathrm{Hom}_A(M, W)$ by the image of $\mathrm{Hom}_A(P, W)$ under $\mathrm{Hom}(\alpha, I_W)$, as a module over k. Similarly, the cokernel of $\mathrm{Hom}(\alpha_1, I_W)$ is the quotient of $\mathrm{Hom}_A(M_1, W)$ by the image of $\mathrm{Hom}_A(P_1, W)$ under $\mathrm{Hom}(\alpha_1, I_W)$. It follows from (23.4.6) that $\mathrm{Hom}(f', I_W)$ induces a homomorphism

$$\begin{aligned} &\theta^1(g) \text{ from the cokernel of } \mathrm{Hom}(\alpha_1, I_W) \\ &\text{into the cokernel of } \mathrm{Hom}(\alpha, I_W), \end{aligned} \tag{23.4.7}$$

as modules over k, as on p34 of [3].

We would like to verify that

$$\theta^1(g) \text{ does not depend on the choice of } f, \tag{23.4.8}$$

as in Proposition 1.1 on p34 of [3]. If $\widetilde{f}$ is another homomorphism from P into P_1 with $\beta_1 \circ \widetilde{f} = g \circ \beta$, then $f - \widetilde{f}$ maps P into the kernel of β_1, as before. If h is the homomorphism from P into $M_1 = \ker \beta_1$ defined by $f - \widetilde{f}$, then $f - \widetilde{f} = \alpha_1 \circ h$, as homomorphisms from P into P_1. Let $\widetilde{f}'$ be the restriction of $\widetilde{f}$ to M, as a homomorphism into M_1, so that $f' - \widetilde{f}' = h \circ \alpha$, as before.

Using $\widetilde{f}'$, we get a homomorphism

$$\mathrm{Hom}(\widetilde{f}', I_W) \text{ from } \mathrm{Hom}_A(M_1, W) \text{ into } \mathrm{Hom}_A(M, W). \tag{23.4.9}$$

Observe that

$$\begin{aligned} \mathrm{Hom}(f', I_W) - \mathrm{Hom}(\widetilde{f}', I_W) &= \mathrm{Hom}(f' - \widetilde{f}', I_W) = \mathrm{Hom}(h \circ \alpha, I_W) \\ &= \mathrm{Hom}(\alpha, I_W) \circ \mathrm{Hom}(h, I_W), \end{aligned} \tag{23.4.10}$$

where $\mathrm{Hom}(h, I_W)$ is a homomorphism from $\mathrm{Hom}_A(M_1, W)$ into $\mathrm{Hom}_A(P, W)$, as modules over k. In particular,

(23.4.11) $\mathrm{Hom}(f' - \widetilde{f}', I_W)$ maps $\mathrm{Hom}_A(M_1, W)$ into the image of $\mathrm{Hom}_A(P, W)$ under $\mathrm{Hom}(\alpha, W)$.

This implies that $\mathrm{Hom}(f', I_W)$ and $\mathrm{Hom}(\widetilde{f}', I_W)$ induce the same homomorphisms into the cokernel of $\mathrm{Hom}(\alpha, I_W)$, as desired.

If $t \in k$, then $t\,g$ is a homomorphism from V into V_1, and $t\,f$ is a homomorphism from P into P_1 that is related to $t\,g$ in the same way as before. Using this, we get that

(23.4.12) $$\theta^1(t\,g) = t\,\theta^1(g).$$

Let $\widehat{g}$ be another homomorphism from V into V_1, and let $\widehat{f}$ be a homomorphism from P into P_1 such that $\beta_1 \circ \widehat{f} = \widehat{g} \circ \beta$. Thus $g + \widehat{g}$ is a homomorphism from V into V_1, $f + \widehat{f}$ is a homomorphism from P into P_1, and $\beta_1 \circ (f + \widehat{f}) = (g + \widehat{g}) \circ \beta$. It follows that homomorphisms $\theta^1(\widehat{g})$ and $\theta^1(g + \widehat{g})$ may be defined using $\widehat{f}$ and $f + \widehat{f}$, respectively, as before. This implies that

(23.4.13) $$\theta^1(g + \widehat{g}) = \theta^1(g) + \theta^1(\widehat{g}),$$

as in Proposition 1.1 on p34 of [3].

If $V_1 = V$, $P_1 = P$, $\beta_1 = \beta$, and g is the identity mapping I_V on V, then we can take f to be the identity mapping I_P on P, so that f' is the identity mapping I_M on $M = M_1$. Under these conditions,

(23.4.14) $$\mathrm{Hom}(f', I_W) = \mathrm{Hom}(I_M, I_W)$$

is the identity mapping on $\mathrm{Hom}_A(M, W) = \mathrm{Hom}_A(M_1, W)$. We also have that $\alpha = \alpha_1$, so that $\mathrm{Hom}(\alpha, I_W) = \mathrm{Hom}(\alpha_1, I_W)$, and thus their cokernels are the same. This means that

(23.4.15) $\theta^1(g)$ is the identity mapping on the cokernel of $\mathrm{Hom}(\alpha, I_W)$

in this case.

Let V_1, P_1, β_1, and g be as before, and let V_2, P_2 be left or right modules over A, depending on whether the other modules are left or right modules over A. Also let β_2 be a homomorphism from P_2 onto V_2, and let g_1 be a homomorphism from V_1 into V_2, as modules over A. Put $M_2 = \ker \beta_2$ again, and let α_2 be the obvious inclusion mapping from M_2 into P_2. Suppose that P_1 is projective as a module over A, so that there is a homomorphism f_1 from P_1 into P_2, as modules over A, such that $\beta_2 \circ f_1 = g_1 \circ \beta_1$, as before.

As usual, we get a homomorphism

(23.4.16) $\mathrm{Hom}(\alpha_2, I_W)$ from $\mathrm{Hom}_A(P_2, W)$ into $\mathrm{Hom}_A(M_2, W)$,

as modules over k. We can define a homomorphism

(23.4.17) $\theta^1(g_1)$ from the cokernel of $\mathrm{Hom}(\alpha_2, I_W)$ into the cokernel of $\mathrm{Hom}(\alpha_1, I_W)$, as modules over k, using f_1, as before. Similarly, we can define a homomorphism

(23.4.18) $\theta^1(g_1 \circ g)$ from the cokernel of $\mathrm{Hom}(\alpha_2, I_W)$ into the cokernel of $\mathrm{Hom}(\alpha, I_W)$,

as modules over k, using $f_1 \circ f$. One can verify that

$$\theta^1(g_1 \circ g) = \theta^1(g) \circ \theta^1(g_1), \tag{23.4.19}$$

as in Proposition 1.1 on p34 of [3] again.

Suppose that $V_2 = V$, $P_2 = P$, and $\beta_2 = \beta$, so that $M_2 = M$, $\alpha_2 = \alpha$, $\mathrm{Hom}(\alpha_2, I_W) = \mathrm{Hom}(\alpha, I_W)$, and the cokernels of the latter homomorphisms are the same. If $g_1 \circ g$ is the identity mapping on V, then

(23.4.20) $\theta^1(g_1 \circ g)$ is the identity mapping on the cokernel of $\mathrm{Hom}(\alpha, I_W)$,

as in (23.4.15). Similarly, if $g \circ g_1$ is the identity mapping on V_1, then

(23.4.21) $\theta^1(g \circ g_1)$ is the identity mapping on the cokernel of $\mathrm{Hom}(\alpha_1, I_W)$.

If g and g_1 are inverse of each other, then we get that

(23.4.22) $\theta^1(g)$ and $\theta^1(g_1)$ are inverses of each other,

using (23.4.19).

23.5 The right satellite of $\mathrm{Hom}_A(\cdot, W)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative agebra over k with a multiplicative identity element e_A. Also let V, W be both left or both right modules over A, and put

$$T(V) = \mathrm{Hom}_A(V, W). \tag{23.5.1}$$

Suppose that P is a projective left or right module over A, as appropriate, and that β is a homomorphism from P onto V, as modules over A. Put $M = \ker \beta$, and let α be the obvious inclusion mapping from M into P, as before. Using α, we get a homomorphism $\mathrm{Hom}(\alpha, I_W)$ as in (23.4.2). Under these conditions, we put

$$\begin{aligned} S^1 T(V) &= \text{the cokernel of } \mathrm{Hom}(\alpha, I_W) \\ &= \mathrm{Hom}_A(M, W)/\mathrm{Hom}(\alpha, I_W)\big(\mathrm{Hom}_A(P, W)\big), \end{aligned} \tag{23.5.2}$$

which is a module over k. This corresponds to (6a) on p36 of [3].

Let P_1 be another projective left or right module over A, as appropriate, and let β_1 be a homomorphism from P_1 onto V, as modules over A. This corresponds to taking $V_1 = V$ in the previous section. Put $M_1 = \ker \beta_1$ again, and let α_1

be the obvious inclusion mapping from M_1 into P_1. We can use α_1 to get a homomorphism $\mathrm{Hom}(\alpha_1, I_W)$ as in (23.4.3).

If we take g to be the identity mapping on V, then we get a homomorphism $\theta^1(g)$ from the cokernel of $\mathrm{Hom}(\alpha_1, I_W)$ into the cokernel of $\mathrm{Hom}(\alpha, I_W)$, as modules over k, as in (23.4.7). Note that this depends on P, P_1, β, and β_1.

Let us take $V_2 = V$, $P_2 = P$, and $\beta_2 = \beta$, so that $M_2 = M$, $\alpha_2 = \alpha$, $\mathrm{Hom}(\alpha_2, I_W) = \mathrm{Hom}(\alpha, I_W)$, and the cokernels of the latter homomorphisms are the same, as in the previous section. We can take g_1 to be the identity mapping on $V_1 = V$, to get a homomorphism $\theta^1(g_1)$ from the cokernel of $\mathrm{Hom}(\alpha, I_W)$ into the cokernel of $\mathrm{Hom}(\alpha_1, I_W)$, as in (23.4.17). Of course, this depends on P_1, $P_2 = P$, β_1, and $\beta_2 = \beta$, as before.

In particular, $g_1 \circ g$ is the identity mapping on V, and $g \circ g_1$ is the identity mapping on $V_1 = V$, so that $\theta^1(g)$ and $\theta^1(g_1)$ are inverses of each other, as in (23.4.22). This means that (23.5.2) is unique up to natural isomorphisms, as in [3]. This is the *right satellite* of (23.5.1), as on p36 of [3].

If V is projective as a module over A, then we can take $P = V$ and β to be the identity mapping on V, so that $M = \{0\}$, $\alpha = 0$, $\mathrm{Hom}(\alpha, I_W) = 0$, and

(23.5.3) $$S^1T(V) = \{0\}.$$

This corresponds to another part of Proposition 1.3 on p37 of [3].

If W is injective as a module over A, then $\mathrm{Hom}(\alpha, I_W)$ maps $\mathrm{Hom}_A(P, W)$ onto $\mathrm{Hom}_A(M, W)$. This means that (23.5.3) holds in this case as well, as in Proposition 1.2 on p36 of [3]. This also corresponds to the sufficiency of injectivity in Corollary 2.2a on p111 of [3].

23.6 Necessity of injectivity

Let us continue with the same notation and hypotheses as in the previous section. Note that (23.5.3) holds if and only if

(23.6.1) $$\mathrm{Hom}(\alpha, I_W)\big(\mathrm{Hom}_A(P, W)\big) = \mathrm{Hom}_A(M, W),$$

by (23.5.2). More precisely, we have seen that this does not depend on the choice of projective module P over A, or homomorphism β from P onto V.

Let P be any projective left or right module over A, as appropriate, and let M be a submodule of P. Suppose that V is the quotient module

(23.6.2) $$V = P/M,$$

and β is the natural quotient mapping from P onto V. If (23.5.3) holds, then (23.6.1) holds, as in the preceding paragraph.

Supposee that

(23.6.3) (23.5.3) holds for every left or right module V over A,

as appropriate. This implies that (23.6.1) holds for every projective left or right module P over A, as appropriate, and every submodule M of P, as before.

Under these conditions, we get that

$$W \text{ is injective, as a module over } A, \tag{23.6.4}$$

as in Section 9.2. This corresponds to the necessity of injectivity in Corollary 2.2a on p111 of [3].

23.7 Some more comparisons

Let k be a commutative ring with a mutliplicative identity element again, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let W, W_1, Q, and Q_1 be all left or all right modules over A, let β, β_1 be injective homomorphisms from W, W_1 into Q, Q_1, respectively, and let g be a homomorphism from W_1 into W.

Note that $g \circ \beta$ is a homomorphism from W_1 into Q, as modules over A. Let us consider some related mappings and their properties, as mentioned beginning on p34 of [3].

Suppose that Q is injective, as a module over A. This implies that there is a homomorphism f from Q_1 into Q, as modules over A, such that

$$f \circ \beta_1 = \beta \circ g. \tag{23.7.1}$$

In particular,

$$f(\beta_1(W_1)) = \beta(g(W_1)) \subseteq \beta(W). \tag{23.7.2}$$

Put

$$N = Q/\beta(W), \ N_1 = Q_1/\beta_1(W_1), \tag{23.7.3}$$

which are modules over A, and let α, α_1 be the natural quotient mappings from Q, Q_1 onto N, N_1, respectively. It follows from (23.7.2) that f induces a homomorphism

$$f' \text{ from } N_1 \text{ into } N, \tag{23.7.4}$$

as modules over A, such that

$$f' \circ \alpha_1 = \alpha \circ f. \tag{23.7.5}$$

If $\widetilde{f}$ is another homomorphism from Q_1 into Q, then

$$\widetilde{f} \circ \beta_1 = \beta \circ g \tag{23.7.6}$$

if and only if

$$\beta_1(W_1) \subseteq \ker(f - \widetilde{f}). \tag{23.7.7}$$

This holds exactly when there is a homomorphism h from N_1 into Q such that

$$f - \widetilde{f} = h \circ \alpha_1. \tag{23.7.8}$$

Let $\widetilde{f}'$ be the homomorphism from N_1 into N, as modules over A, induced by $\widetilde{f}$, so that

$$\widetilde{f}' \circ \alpha_1 = \alpha \circ \widetilde{f}, \tag{23.7.9}$$

as in (23.7.5). Using (23.7.5), (23.7.8), and (23.7.9), we get that

$$(f' - \widetilde{f'}) \circ \alpha_1 = \alpha \circ (f - \widetilde{f}) = \alpha \circ h \circ \alpha_1. \tag{23.7.10}$$

This implies that

$$f' - \widetilde{f'} = \alpha \circ h, \tag{23.7.11}$$

as homomorphisms from N_1 into N, because α_1 maps Q_1 onto N_1.

If $t \in k$, then $t\,g$ is a homomorphism from W_1 into W too, as modules over k. In this case, $t\,f$ is a homomorphism from Q_1 into Q, and the composition of β_1 with $t\,f$ is equal to the composition of $t\,g$ with β.

Let $\widehat{g}$ be another homomorphism from W_1 into W, as modules over A, so that $\beta \circ \widehat{g}$ is a homomorphism from W_1 in Q. Because Q is injective, there is a homomorphism $\widehat{f}$ from Q_1 into Q, as modules over A, such that

$$\widehat{f} \circ \beta_1 = \beta \circ \widehat{g}, \tag{23.7.12}$$

as before. Thus

$$(f + \widehat{f}) \circ \beta_1 = \beta \circ (g + \widehat{g}). \tag{23.7.13}$$

Let W_2 and Q_2 be left or right modules over A, as appropriate, let β_2 be an injective homomorphism from W_2 into Q_2, and let g_1 be a homomorphism from W_2 into W_1. Thus $\beta_1 \circ g_1$ is a homomorphism from W_2 into Q_1, as modules over A. If Q_1 is injective as a module over A, then there is a homomorphism f_1 from Q_2 into Q_1, as modules over A, such that

$$f_1 \circ \beta_2 = \beta_1 \circ g_1. \tag{23.7.14}$$

Note that $g \circ g_1$ is a homomorphism from W_2 into W, and that $\beta \circ g \circ g_1$ is a homomorphism from W_2 into Q. Similarly, $f \circ f_1$ is a homomorphism from Q_2 into Q, with

$$f \circ f_1 \circ \beta_2 = f \circ \beta_1 \circ g_1 = \beta \circ g \circ g_1. \tag{23.7.15}$$

Suppose now that $W_2 = W$, $Q_2 = Q$, $\beta_2 = \beta$, and $g \circ g_1$ is the identity mapping on W. Under these conditions, $f \circ f_1$ is a homomorphism from Q into itself, with

$$f \circ f_1 \circ \beta = \beta, \tag{23.7.16}$$

as in (23.7.15). This is the same as saying that $\beta(W)$ is contained in the kernel of $f \circ f_1$ minus the identity mapping on Q.

23.8 Comparisons and $\mathrm{Hom}_A(V, \cdot)$

Let us continue with the same notation and hypotheses as in the previous section, and let V be a left or right module over A, depending on whether the other modules are left or right modules. Consider the homomorphisms

$$\mathrm{Hom}(I_V, \alpha) \text{ from } \mathrm{Hom}_A(V, Q) \text{ into } \mathrm{Hom}_A(V, N) \tag{23.8.1}$$

and

$$\text{(23.8.2)} \qquad \mathrm{Hom}(I_V, \alpha_1) \text{ from } \mathrm{Hom}_A(V, Q_1) \text{ into } \mathrm{Hom}_A(V, N_1)$$

associated to the identity mapping I_V on V and α, α_1 as in Section 6.3. Similarly, we have homomorphisms

$$\text{(23.8.3)} \qquad \mathrm{Hom}(I_V, f) \text{ from } \mathrm{Hom}_A(V, Q_1) \text{ into } \mathrm{Hom}_A(V, Q)$$

and

$$\text{(23.8.4)} \qquad \mathrm{Hom}(I_V, f') \text{ from } \mathrm{Hom}_A(V, N_1) \text{ into } \mathrm{Hom}_A(V, N).$$

Observe that

$$\text{(23.8.5)} \qquad \mathrm{Hom}(I_V, f') \circ \mathrm{Hom}(I_V, \alpha_1) = \mathrm{Hom}(I_V, \alpha) \circ \mathrm{Hom}(I_V, f),$$

as homomorphisms from $\mathrm{Hom}_A(V, Q_1)$ into $\mathrm{Hom}_A(V, N)$, because of (23.7.5), as in (2a) on p35 of [3].

Remember that the cokernel of $\mathrm{Hom}(I_V, \alpha_1)$ is the quotient of $\mathrm{Hom}_A(V, N_1)$ by the image of $\mathrm{Hom}_A(V, Q_1)$ under $\mathrm{Hom}(I_V, \alpha_1)$, and similarly that the cokernel of $\mathrm{Hom}(I_V, \alpha)$ is the quotient of $\mathrm{Hom}_A(V, N)$ by the image of $\mathrm{Hom}_A(V, Q)$ under $\mathrm{Hom}(I_V, \alpha)$. Using (23.8.5), we get that $\mathrm{Hom}(I_V, f')$ induces a homomorphism

$$\text{(23.8.6)} \qquad \begin{array}{l} \theta^1(g) \text{ from the cokernel of } \mathrm{Hom}(I_V, \alpha_1) \\ \text{into the cokernel of } \mathrm{Hom}(I_V, \alpha), \end{array}$$

as modules over k, as on p35 of [3]. We would like to check that

$$\text{(23.8.7)} \qquad \theta^1(g) \text{ does not depend on the choice of } f,$$

as in Proposition 1.1a on p35 of [3].

Let $\widetilde{f}$ be another homomorphism from Q_1 into Q that satisfies (23.7.6), so that (23.7.7) holds. Remember that $\widetilde{f}$ induces a homomorphism $\widetilde{f}'$ from N_1 into N as in (23.7.9). We have seen that there is a homomorphism h from N_1 into Q, as modules over A, such that (23.7.11) holds.

We can use $\widetilde{f}'$ to get a homomorphism

$$\text{(23.8.8)} \qquad \mathrm{Hom}(I_V, \widetilde{f}') \text{ from } \mathrm{Hom}_A(V, N_1) \text{ into } \mathrm{Hom}_A(V, N).$$

We also have that

$$\begin{aligned} \mathrm{Hom}(I_V, f') - \mathrm{Hom}(I_V, \widetilde{f}') &= \mathrm{Hom}(I_V, f' - \widetilde{f}') = \mathrm{Hom}(I_V, \alpha \circ h) \\ \text{(23.8.9)} \qquad &= \mathrm{Hom}(I_V, \alpha) \circ \mathrm{Hom}(I_V, h), \end{aligned}$$

using (23.7.11) in the second step. This implies that

$$\text{(23.8.10)} \qquad \begin{array}{l} \mathrm{Hom}(I_V, f' - \widetilde{f}') \text{ maps } \mathrm{Hom}_A(V, N_1) \text{ into} \\ \text{the image of } \mathrm{Hom}_A(V, Q) \text{ under } \mathrm{Hom}(I_V, \alpha), \end{array}$$

because $\mathrm{Hom}(I_V, h)$ maps $\mathrm{Hom}_A(V, N_1)$ into $\mathrm{Hom}_A(V, Q)$. Using this, we get that $\mathrm{Hom}(I_V, f')$ and $\mathrm{Hom}(I_V, \widetilde{f}')$ induce the same homomorphisms into the cokernel of $\mathrm{Hom}(I_V, \alpha)$, as desired.

If $t \in k$, then $t\,g$ is a homomorphism from W_1 into W, $t\,f$ is a homomorphism from Q_1 into Q, and they are related as in (23.7.1). One can verify that

$$\theta^1(t\,g) = t\,\theta^1(g). \tag{23.8.11}$$

Let $\widehat{g}$ be another homomorphism from W_1 into W, and let $\widehat{f}$ be a homomorphism from Q_1 into Q such that (23.7.12) holds. This means that $g + \widehat{g}$ is a homomorphism from W_1 into W, $f + \widehat{f}$ is a homomorphism from Q_1 into Q, and that (23.7.13) holds. Thus $\theta^1(\widehat{g})$ and $\theta^1(g + \widehat{g})$ may be defined using $\widehat{f}$ and $f + \widehat{f}$, respectively, as before. It follows that

$$\theta^1(g + \widehat{g}) = \theta^1(g) + \theta^1(\widehat{g}), \tag{23.8.12}$$

as in Proposition 1.1a pn p35 of [3].

If $W_1 = W$, $Q_1 = Q$, $\beta_1 = \beta$, and g is the identity mapping I_W on W, then we can take f to be the identity mapping I_Q on Q, so that f' is the identity mapping I_N on N. In this case,

$$\mathrm{Hom}(I_V, f') = \mathrm{Hom}(I_V, I_N) \tag{23.8.13}$$

is the identity mapping on $\mathrm{Hom}_A(V, N_1) = \mathrm{Hom}_A(V, N)$. Note that $\alpha_1 = \alpha$, so that $\mathrm{Hom}(I_V, \alpha_1) = \mathrm{Hom}(I_V, \alpha)$, and their cokernels are the same. Thus

(23.8.14) $\theta^1(g)$ is the identity mapping on the cokernel of $\mathrm{Hom}(I_V, \alpha_1)$

under these conditions.

Let W_1, Q_1, β_1, and g be as before, and let W_2, Q_2, β_2, and g_1 be as in the previous section. Put

$$N_2 = Q_2/\beta_2(W_2), \tag{23.8.15}$$

and let α_2 be the natural quotient mapping from Q_2 onto N_2. Suppose that Q_1 is injective as a module over A, so that there is a homomorphism f_1 from Q_2 into Q_1, as modules over A, such that (23.7.14) holds.

Using α_2, we get a homomorphism

(23.8.16) $\mathrm{Hom}(I_V, \alpha_2)$ from $\mathrm{Hom}_A(V, Q_2)$ into $\mathrm{Hom}_A(V, N_2)$,

as modules over k. We can use f_1 to define a homomorphism

(23.8.17) $\theta^1(g_1)$ from the cokernel of $\mathrm{Hom}(I_V, \alpha_2)$
into the cokernel of $\mathrm{Hom}(I_V, \alpha_1)$,

as modules over k, as before. Similarly, we can use $f \circ f_1$ to define a homomorphism

(23.8.18) $\theta^1(g \circ g_1)$ from the cokernel of $\mathrm{Hom}(I_V, \alpha_2)$
into the cokernel of $\mathrm{Hom}(I_V, \alpha)$,

as modules over k. One can check that

$$\theta^1(g \circ g_1) = \theta^1(g_1) \circ \theta^1(g), \tag{23.8.19}$$

as in Proposition 1.1a on p35 of [3].

Suppose now that $W_2 = W$, $Q_2 = Q$, and $\beta_2 = \beta$, so that $N_2 = N$, $\alpha_2 = \alpha$, $\mathrm{Hom}(I_V, \alpha_2) = \mathrm{Hom}(I_V, \alpha)$, and the cokernels of the latter homomorphisms are the same. If $g \circ g_1$ is the identity mapping on W, then

(23.8.20) $\theta^1(g \circ g_1)$ is the identity mapping on the cokernel of $\mathrm{Hom}(I_V, \alpha)$,

as in (23.8.14). Simiarly, if $g_1 \circ g$ is the identity mapping on W_1, then

(23.8.21) $\theta^1(g_1 \circ g)$ is the identity mapping on the cokernel of $\mathrm{Hom}(I_V, \alpha_1)$.

If g and g_1 are inverses of each other, then it follows that

(23.8.22) $\theta^1(g)$ and $\theta^1(g_1)$ are inverses of each other,

because of (23.8.19).

23.9 The right satellite of $\mathrm{Hom}_A(V, \cdot)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, W be both left or both right modules over A, and put

$$T(W) = \mathrm{Hom}_A(V, W). \tag{23.9.1}$$

Suppose that Q is an injective left or right module over A, depending on whether V, W are left or right modules over A, and let β be an injective homomorphism from W into Q, as modules over A. Put $N = Q/\beta(W)$, and let α be the natural quotient mapping from Q onto N, as before. We can use α to get a homomorphism $\mathrm{Hom}(I_V, \alpha)$ as in (23.8.1). Put

$$\begin{aligned} S^1T(W) &= \text{the cokernel of } \mathrm{Hom}(I_V, \alpha) \\ &= \mathrm{Hom}_A(V, N)/\mathrm{Hom}(I_V, \alpha)\big(\mathrm{Hom}_A(V, Q)\big), \end{aligned} \tag{23.9.2}$$

which is a module over k. This corresponds to (6) on p35 of [3].

Let Q_1 be another injective left or right module over A, as appropriate, and let β_1 be an injective homomorphism from W into Q_1. This corresponds to taking $W_1 = W$ in the previous two sections. Put $N_1 = Q_1/\beta(W)$, and let α_1 be the natural quotient mapping from Q_1 onto N_1, as before. Using α_1, we get a homomorphism $\mathrm{Hom}(I_V, \alpha_1)$ as in (23.8.2).

If we take g to be the identity mapping on $W_1 = W$, then we get a homomorphism $\theta^1(g)$ from the cokernel of $\mathrm{Hom}(I_V, \alpha_1)$ into the cokernel of $\mathrm{Hom}(I_V, \alpha)$, as modules over k, as in (23.8.6). This depends on Q, Q_1, β, and β_1.

Let us take $W_2 = W$, $Q_2 = Q$, and $\beta_2 = \beta$, so that $N_2 = N$, $\alpha_2 = \alpha$, $\mathrm{Hom}(I_V, \alpha_2) = \mathrm{Hom}(I_V, \alpha)$, and the cokernels of the latter homomorphisms are the same, as in the previous section. Let us also take g_1 to be the identity mapping on $W_2 = W$, as a mapping into $W_1 = W$, to get a homomorphism $\theta^1(g_1)$ from the cokernel of $\mathrm{Hom}(I_v, \alpha)$ into the cokernel of $\mathrm{Hom}(I_V, \alpha_1)$, as in (23.8.17). This depends on Q_1, $Q_2 = Q$, β_1, and $\beta_2 = \beta$, as before.

Of course, $g \circ g_1$ is the identity mapping on $W_2 = W$, and $g_1 \circ g$ is the identity mapping on $W_1 = W$. Thus $\theta^1(g)$ and $\theta^1(g_1)$ are inverses of each other, as in (23.8.22). This shows that (23.9.2) is unique up to natural isomorphisms, as on p36 of [3]. This is the *right satellite* of (23.9.1), as in [3].

If W is injective as a module over A, then we can take $Q = W$ and β to be the identity mapping on W, so that $N = \{0\}$, $\alpha = 0$, $\mathrm{Hom}(I_V, \alpha) = 0$, and

(23.9.3) $$S^1T(W) = \{0\}.$$

This corresponds to part of Proposition 1.3 on p37 of [3].

If V is projective as a module over A, then $\mathrm{Hom}(I_V, \alpha)$ maps $\mathrm{Hom}_A(V, Q)$ onto $\mathrm{Hom}_A(V, N)$. It follows that (23.9.3) holds in this case too, as in Proposition 1.2 on p36 of [3]. This corresponds to the sufficiency of projectivity in Corollary 2.2 on p110 of [3] as well.

23.10 Necessity of projectivity

We continue with the same notation and hypotheses as in the previous section. Of course, (23.9.3) holds if and only if

(23.10.1) $$\mathrm{Hom}(I_V, \alpha)\big(\mathrm{Hom}_A(V, Q)\big) = \mathrm{Hom}_A(V, N),$$

by (23.9.2). This does not depend on the choice of injective module Q over A or injective homomorphism β from W into Q, as before.

Suppose now that Q is any injective left or right module over A, as appropriate, and that α is a homomorphism from Q onto another module N over A. Put

(23.10.2) $$W = \ker \alpha,$$

and let β be the obvious inclusion mapping from W into Q. If (23.9.3) holds, then it follows that (23.10.1) holds.

Suppose that

(23.10.3) (23.9.3) holds for every left or right module W over A,

as appropriate. In this case, (23.10.1) holds for every injective left or right module Q over A, as appropriate, and every homomorphism α from Q onto another module N over A, as in the preceding paragraph.

This implies that

(23.10.4) V is projective, as a module over A,

as in Section 9.1. This corresponds to the necessity of projectivity in Corollary 2.2 on p110 of [3].

23.11 A type of double satellite

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, P_V be right modules over A, and let W, P_W be left modules over A.

Suppose that P_V, P_W are projective as modules over A, and that

$$\beta_V, \beta_W \text{ are homomorphisms from } P_V, P_W \text{ onto } V, W, \tag{23.11.1}$$

respectively, as modules over A. Put

$$M_V = \ker \beta_V,\ M_W = \ker \beta_W, \tag{23.11.2}$$

which are submodules of P_V, P_W, respectively, as modules over A. Consider the obvious inclusion mappings

$$\alpha_V, \alpha_W \text{ from } M_V, M_W \text{ into } P_V, P_W, \tag{23.11.3}$$

respectively.

Let $M_V \bigotimes_A M_W$, $P_V \bigotimes_A M_W$, and $M_V \bigotimes_A P_W$ be tensor products of the indicated modules over A. We can use α_V and the identity mapping on M_W to get a homomorphism

$$\overline{\alpha}_V \text{ from } M_V \bigotimes_A M_W \text{ into } P_V \bigotimes_A M_W, \tag{23.11.4}$$

as modules over k. Similarly, we can use the identity mapping on M_V and α_W to get a homomorphism

$$\overline{\alpha}_W \text{ from } M_V \bigotimes_A M_W \text{ into } M_V \bigotimes_A P_W, \tag{23.11.5}$$

as modules over k. Using $\overline{\alpha}_V$ and $\overline{\alpha}_W$, we get a homomorphism

$$\text{from } M_V \bigotimes_A M_W \text{ into } \big(P_V \bigotimes_A M_W\big) \bigoplus \big(M_V \bigotimes_A P_W\big), \tag{23.11.6}$$

as modules over k. Of course, the kernel of this homomorphism is equal to

$$(\ker \overline{\alpha}_V) \cap (\ker \overline{\alpha}_W), \tag{23.11.7}$$

which is a submodule of $M_V \bigotimes_A M_W$.

This corresponds to taking the left satellite of

$$T(V, W) = V \bigotimes_A W \tag{23.11.8}$$

in the first variable, and then in the second variable, as in (6) on p50 of [3]. This also corresponds to taking the left satellite of (23.11.8) in the second variable, and then the first variable, as in (7) on p50 of [3].

23.12 Another type of double satellite

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let V, P_V, W, and Q_W be all left or all right modules over A.

Suppose that P_V is projective as a module over A, and that

(23.12.1) β_V is a homomorphism from P_V onto V,

as modules over A. Similarly, suppose that Q_W is injective as a module over A, and that

(23.12.2) β_W is an injective homomorphism from W into Q_W,

as modules over A. Put

(23.12.3) $$M_V = \ker \beta_V,\ N_W = Q_W/\beta_W(W),$$

which are modules over A. Thus we get the obvious inclusion mapping

(23.12.4) α_V from M_V into P_V,

and the natural quotient mapping

(23.12.5) α_W from Q_W onto N_W.

We can use α_V and the identity mapping I_{N_W} on N_W to get a homomorphism

(23.12.6) $\mathrm{Hom}(\alpha_V, I_{N_W})$ from $\mathrm{Hom}_A(P_V, N_W)$ into $\mathrm{Hom}_A(M_V, N_W)$,

as modules over k, as in Section 6.3. Similarly, we can use the identity mapping I_{M_V} on M_V and α_W to get a homomorphism

(23.12.7) $\mathrm{Hom}(I_{M_V}, \alpha_W)$ from $\mathrm{Hom}_A(M_V, Q_W)$ into $\mathrm{Hom}_A(M_V, N_W)$,

as modules over k. Using these two homomorphisms, we get a homomorphism

(23.12.8) from $\mathrm{Hom}_A(P_V, N_W) \bigoplus \mathrm{Hom}_A(M_V, Q_W)$ into $\mathrm{Hom}_A(M_V, N_V)$,

as modules over k. The image of this homomorphism is the same as the submodule

(23.12.9) $$\begin{aligned} &\mathrm{Hom}(\alpha_V, I_{N_W})\big(\mathrm{Hom}_A(P_V, N_W)\big) \\ &+\mathrm{Hom}(I_{M_V}, \alpha_W)\big(\mathrm{Hom}_A(M_V, Q_W)\big) \end{aligned}$$

of $\mathrm{Hom}_A(M_V, N_W)$ generated by the images of (23.12.6) and (23.12.7). Thus the cokernel of the homomorphism as in (23.12.8) is the same as the quotient of $\mathrm{Hom}_A(M_V, N_W)$ by the submodule (23.12.9).

This corresponds to taking the right satellite of

(23.12.10) $$T(V, W) = \mathrm{Hom}_A(V, W)$$

in the first variable, and then in the second variable, as in (6a) on p50 of [3], adjusted for the fact that (23.12.10) is contravariant in the first variable, as in [3]. This also corresponds to taking the right satellite of (23.12.10) in the second variable, and then in the first variable, as in (7a) on p50 of [3], suitably adjusted for this case, as before.

23.13 Left and right ideals

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let $\mathcal{I}_L$, $\mathcal{I}_R$ be left and right ideals in A, respectively. In particular, $\mathcal{I}_L$, $\mathcal{I}_R$ are submodules of A, as a module over k, so that their interesection is a submodule of A as a module over k as well.

Let $\mathcal{I}_R\,\mathcal{I}_L$ be the subset of A consisting of finite sums of products of elements of $\mathcal{I}_R$ and $\mathcal{I}_L$. This is another submodule of A, as a module over k, with

$$\mathcal{I}_R\,\mathcal{I}_L \subseteq \mathcal{I}_L \cap \mathcal{I}_R. \tag{23.13.1}$$

Of course, $\mathcal{I}_L$ and $\mathcal{I}_R$ may be considered as left and right modules over A, respectively. Let $\mathcal{I}_R \bigotimes_A \mathcal{I}_L$ be a tensor product of $\mathcal{I}_R$ and $\mathcal{I}_L$ over A. There is an obvious homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A \mathcal{I}_L \text{ into } A, \tag{23.13.2}$$

as modules over k. This homomorphism is obtained from the restriction of multiplication on A to $\mathcal{I}_R \times \mathcal{I}_L$, as a mapping into A. The image of the homomorphism as in (23.13.2) is the same as $\mathcal{I}_R\,\mathcal{I}_L$.

The quotients $A/\mathcal{I}_L$ and $A/\mathcal{I}_R$ may be defined as left and right modules over A in the usual way. Let $\mathcal{I}_R \bigotimes_A (A/\mathcal{I}_L)$ be a tensor product of $\mathcal{I}_R$ and $A/\mathcal{I}_L$ over A. There is an obvious homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A (A/\mathcal{I}_L) \text{ into } A/\mathcal{I}_L, \tag{23.13.3}$$

as modules over k. This homomorphism is obtained from the restriction of the action of A on $A/\mathcal{I}_L$ on the left to $\mathcal{I}_R \times (A/\mathcal{I}_L)$, as a mapping into $A/\mathcal{I}_L$.

There is a natural homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A A \text{ onto } \mathcal{I}_R \bigotimes_A (A/\mathcal{I}_L), \tag{23.13.4}$$

as modules over k. This homomorphism is obtained from the identity mapping on $\mathcal{I}_R$, and the natural quotient mapping from A onto $A/\mathcal{I}_L$. This may be identified with a homomorphism

$$\text{from } \mathcal{I}_R \text{ onto } \mathcal{I}_R \bigotimes_A (A/\mathcal{I}_L), \tag{23.13.5}$$

as modules over k, using the usual identification of $\mathcal{I}_R \bigotimes_A A$ with $\mathcal{I}_R$.

The composition of the homomorphism as in (23.13.5) with the homomorphism as in (23.13.3) is a homomorphism

$$\text{from } \mathcal{I}_R \text{ into } A/\mathcal{I}_L, \tag{23.13.6}$$

as modules over k. This homomorphism is the same as the restriction to $\mathcal{I}_R$ of the natural quotient mapping from A onto $A/\mathcal{I}_L$. Thus the kernel of this homomorphism is $\mathcal{I}_L \cap \mathcal{I}_R$.

It follows that the kernel of the homomorphism as in (23.13.3) is equal to the image of $\mathcal{I}_L \cap \mathcal{I}_R$ under the homomorphism as in (23.13.5). This uses the fact that the homomorphism as in (23.13.5) is surjective.

There is a natural homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A \mathcal{I}_L \text{ into } \mathcal{I}_R \bigotimes_A A, \tag{23.13.7}$$

as modules over k. This homomorphism is obtained from the identity mapping on $\mathcal{I}_R$, and the obvious inclusion mapping from $\mathcal{I}_L$ into A. This homomorphism maps $\mathcal{I}_R \bigotimes_A \mathcal{I}_L$ onto the kernel of the homomorphism as in (23.13.4), as discussed in Section 2.5.

The homomorphism as in (23.13.7) corresponds to a homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A \mathcal{I}_L \text{ into } \mathcal{I}_R, \tag{23.13.8}$$

as modules over k, because of the usual identification of $\mathcal{I}_R \bigotimes_A A$ with $\mathcal{I}_R$. This homomorphism is the same as the one obtained from the restriction of multiplication on A to $\mathcal{I}_R \times \mathcal{I}_L$, as in (23.13.2), considered as a homomorphism into $\mathcal{I}_R$. The image of this homomorphism is the same as $\mathcal{I}_R \mathcal{I}_L$, as before.

This means that the kernel of the homomorphism as in (23.13.5) is equal to $\mathcal{I}_R \mathcal{I}_L$. It follows that the image of $\mathcal{I}_L \cap \mathcal{I}_R$ under the homomorphism as in (23.13.5) is isomorphic to the quotient

$$(\mathcal{I}_L \cap \mathcal{I}_R)/(\mathcal{I}_R \mathcal{I}_L), \tag{23.13.9}$$

as modules over k. This implies that the kernel of the homomorphism as in (23.13.3) is isomorphic to (23.13.9), as modules over k.

This corresponds to a satellite, as in Section 23.3. This is basically the same as the first part of Exercise 19 on p126 of [3], considered in terms of satellites.

23.14 More on left, right ideals

Let us continue with the same notation and hypotheses as in the previous section. There is a natural homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A \mathcal{I}_L \text{ into } A \bigotimes_A \mathcal{I}_L, \tag{23.14.1}$$

as modules over k. This homomorphism is obtained from the obvious inclusion mapping from $\mathcal{I}_R$ into A, and the identity mapping on $\mathcal{I}_L$. This corresponds to a homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A \mathcal{I}_L \text{ into } \mathcal{I}_L, \tag{23.14.2}$$

as modules over k, because of the usual identification of $A \bigotimes_A \mathcal{I}_L$ with $\mathcal{I}_L$.

Of course, these homormophisms are analogous to those as in (23.13.7) and (23.13.8). As before, the homomorphism as in (23.14.2) is the same as the one obtained from the restriction of multiplication on A to $\mathcal{I}_R \times \mathcal{I}_L$, as in (23.13.2), considered as a homomorphism into $\mathcal{I}_L$. This may also be considered as a homomorphism

$$\text{from } \mathcal{I}_R \bigotimes_A \mathcal{I}_L \text{ onto } \mathcal{I}_R\,\mathcal{I}_L, \tag{23.14.3}$$

as modules over k.

Consider

$$\text{the kernel of the homomorphism as in (23.14.3),} \tag{23.14.4}$$

which is a submodule of $\mathcal{I}_R \bigotimes_A \mathcal{I}_L$, as a module over k. This corresponds to a double satellite, as in Section 23.11. This is basically the same as the second part of Exercise 19 on p126 of [3], considered in terms of satellites.

Part VII

Augmented rings and associative algebras

Chapter 24

Augmented rings

24.1 Left augmentations

Let A be a ring with a multiplicative identity element e_A. A *left augmentation* of A consists of a left module Q over A and a homomorphism ε from A onto Q, as left modules over A. Equivalently, A is said to be a *left augmented ring* when it is equipped with a left augmentation, as on p143 of [3]. In this case, Q is called the *augmentation module*, and ε is called the *augmentation homomorphism (or epimorphism)*, as in [3]. The kernel $\mathcal{I} = \mathcal{I}_\varepsilon$ of ε is called the *augmentation ideal*, as in [3], and is a left ideal in A.

Under these conditions, if V is a right module over A, then we put

(24.1.1) $$T(V) = T^L(V) = V \bigotimes_A Q,$$

which is a commutative group. Suppose for the moment that k is a commutative ring with a multiplicative identity element, and that A is an associative algebra over k. This means that modules over A may be considered as modules over k, and (24.1.1) is defined as a module over k too.

Of course,

(24.1.2) $$(v, b) \mapsto v \cdot b$$

defines a mapping from $V \times A$ into V that is bilinear over $\mathbf{Z}$. If $a, b \in A$ and $v \in V$, then the values of this mapping at $(v \cdot a, b)$ and $(v, a\, b)$ are the same. This mapping can be used to show that V satisfies the requirements of $V \bigotimes_A A$, as a commutative group, and in fact as a right module over A, as in Section 1.10.

We may also restrict (24.1.2) to $V \times \mathcal{I}$, to get a mapping into V that is bilinear over $\mathbf{Z}$. This leads to a homomorphism

(24.1.3) $$\text{from } V \bigotimes_A \mathcal{I} \text{ into } V,$$

as commutative groups. This can be identified with the homomorphism

(24.1.4) $$\text{from } V \bigotimes_A \mathcal{I} \text{ into } V \bigotimes_A A,$$

as commutative groups, that corresponds to the identity mapping on V and the obvious inclusion mapping from $\mathcal{I}$ into V, as in Section 1.9.

By hypothesis,

$$0 \longrightarrow \mathcal{I} \longrightarrow A \stackrel{\varepsilon}{\longrightarrow} Q \longrightarrow 0 \tag{24.1.5}$$

is an exact sequence of left modules over A, using the obvious inclusion mapping from $\mathcal{I}$ into A. This leads to an exact sequence

$$V \bigotimes_A \mathcal{I} \longrightarrow V \bigotimes_A A \longrightarrow V \bigotimes_A Q \longrightarrow 0, \tag{24.1.6}$$

as commutative groups, as in Section 2.5. More precisely, this uses the mappings obtained from those in (24.1.5), and the identity mapping on V.

This leads to an isomorphism between (24.1.1) and

$$\text{the cokernel of the homomorphism as in (24.1.4),} \tag{24.1.7}$$

which is to say the quotient of $V \bigotimes_A A$ by the image of $V \bigotimes_A \mathcal{I}$ under the homomorphism as in (24.1.4). This corresponds to (2) on p144 of [3]. This is also mentioned in Exercise 2 on p31 of [1], in the case of commutative rings. This is essentially the same as in Section 13.15 as well.

If W is a left module over A, then put

$$U(W) = U_L(W) = \mathrm{Hom}_A(Q, W), \tag{24.1.8}$$

which is a commutative group. As before, if A is an algebra over a commutative ring k with a multiplicative identity element, then modules over A may be considered as modules over k, and (24.1.8) is defined as a module over k.

Of course, A may be considered as a left module over itself, so that the space $\mathrm{Hom}_A(A, W)$ of homomorphisms from A into W, as left modules over A, is defined as a commutative group. There is a natural isomorphism between $\mathrm{Hom}_A(A, W)$ and W, as commutative groups, as in Section 1.7, and in fact as left modules over A, as in Section 1.8. There is an obvious homomorphism

$$\text{from } \mathrm{Hom}_A(A, W) \text{ into } \mathrm{Hom}_A(\mathcal{I}, W), \tag{24.1.9}$$

which sends a homomorphism from A into W to its restriction to $\mathcal{I}$. This may be identified with a homomorphism

$$\text{from } W \text{ into } \mathrm{Hom}_A(\mathcal{I}, W), \tag{24.1.10}$$

using the natural isomorphism between $\mathrm{Hom}_A(A, W)$ and W mentioned before.

Homomorphisms from Q into W, as left modules over A, correspond to homomorphisms from A into W, as left modules over A, whose kernels contain $\mathcal{I}$. Thus (24.1.8) is isomorphic in a natural way to

$$\text{the kernel of the homomorphism as in (24.1.9).} \tag{24.1.11}$$

This corresponds to (2a) on p144 of [3].

24.2 Right augmentations, and related matters

Let A be a ring with a multiplicative identity element e_A again. A *right augmentation* of A consists of a right module Q over A and a homomorphism ε from A onto Q, as right modules over A. If A is equipped with a right augmentation, then A is said to be a *right augmented ring*. As before, Q is called the *augmentation module*, and ε is called the *augmentation homomorphism (or epimorphism)*. The kernel $\mathcal{I} = \mathcal{I}_\varepsilon$ is called the *augmentation ideal*, and is a right ideal in A.

In this case, if V is a left module over A, then we put

$$T(V) = T^R(V) = Q \bigotimes_A V, \tag{24.2.1}$$

which is a commutative group. If W is a right module over A, then we put

$$U(W) = U_R(W) = \mathrm{Hom}_A(Q, W), \tag{24.2.2}$$

which is a commutative group. There are analogues of the statements in the previous section for right augmentations, as on p145 of [3].

Suppose now that $\mathcal{I}$ is a two-sided ideal in A. Thus

$$Q = A/\mathcal{I} \tag{24.2.3}$$

is a ring, which may be considered as both a left and right module over A. This means that A may be considered as both a left and right augmented ring, as on p145 of [3].

If V is a right module over A, then (24.1.1) may be considered as a right module over A too, as in Section 1.10. Similarly, if V is a left module over A, then (24.2.1) may considered as a left module over A.

If W is a left module over A, then (24.1.8) may be considered as a left module over A, as in Section 1.8. Similarly, if W is a right module over A, then (24.2.2) may be considered as a right module over A.

Let V be a right module over A again. Consider the subset

$$V \cdot \mathcal{I} \tag{24.2.4}$$

of V consisting of finite sum of elements of V of the form $v \cdot b$, where $v \in V$ and $b \in \mathcal{I}$. This is a subgroup of V, as a commutative group with respect to addition, which is the same as the image of $V \bigotimes_A \mathcal{I}$ under the homomorphism as in (24.1.3). Thus the quotient

$$V/(V \cdot \mathcal{I}) \tag{24.2.5}$$

can be identified with (24.1.7), which is isomorphic to (24.1.1), as a commutative group, as before. Note that this works when $\mathcal{I}$ is any left ideal in A.

More precisely, (24.2.4) is a submodule of V here, as a right module over A, because $\mathcal{I}$ is also a right ideal in A. This means that the quotient (24.2.5) is a right module over A as well. Under these conditions, the homomorphisms as in

(24.1.3) and (24.1.4) are homomorphisms between right modules over A. Similarly, the isomorphism between (24.1.1) and (24.1.7) mentioned in the previous section is an isomorphism between right modules over A.

Remember that the homomorphism from $V \bigotimes_A A$ onto $V \bigotimes_A Q$ in (24.1.6) is obtained from ε and the identity mapping on V. This is also a homomorphism between right modules over A in this case. It follows that (24.2.5) is isomorphic to (24.1.1), as right modules over A. This corresponds to (1) on p154 of [3].

If we take $V = Q$, considered as a right module over A, then (24.2.4) is equal to $\{0\}$. Of course, this means that (24.2.5) is equal to Q, so that (24.1.1) is isomorphic to Q, as a right module over A. This corresponds to (8) on p146 of [3].

24.3 Augmentations and homology

Let A be a ring with a multiplicative identity element e_A. Suppose that A is equipped with a left augmentation, with augmentation module Q, homomorphism ε, and ideal $\mathcal{I} = \mathcal{I}_\varepsilon$, as in Section 24.1. If V is a right module over A, then we put $T(V) = T^L(V) = V \bigotimes_A Q$, as before.

If n is a nonnegative integer, then the *nth homology group* $T_n(V)$ of A as a left augmented ring with coefficients in V is defined as a commutative group on p143 of [3]. Another characterization of $T_n(V)$, up to isomorphism, is given in Theorem 1.1 on p145 of [3]. In particular,

$$T_0(V) = T(V). \tag{24.3.1}$$

If V is projective as a right module over A, then

$$T_n(V) = \{0\} \text{ for every } n \geq 1. \tag{24.3.2}$$

As mentioned on p143 of [3], V may have additional actions that commute with the action of A on the right, which lead to suitable actions on $T_n(V)$. Suppose for instance that k is a commutative ring with a multiplicative identity element, and that A is an associative algebra over k. This means that modules over A may be considered as modules over k, and $T(V)$ is defined as a module over k. Similarly, $T_n(V)$ is a module over k for each $n \geq 0$, as in [3].

Remember that V satisfies the requirements of $V \bigotimes_A A$, as a commutative group, and in fact as a right module over A, as in Section 1.10. This uses the mapping $(v, b) \mapsto v \cdot b$ from $V \times A$ into V. We can use the restriction of this mapping to $V \times \mathcal{I}$, to get a mapping into V that is bilinear over $\mathbf{Z}$, and satisfies the usual compatibility condition with respect to the actions of A on V on the right and on $\mathcal{I}$ on the left.

This leads to a homomorphism

$$\text{from } V \bigotimes_A \mathcal{I} \text{ into } V, \tag{24.3.3}$$

as commutative groups, as in Section 24.1. This can be identified with the homomorphism

$$\text{from } V \bigotimes_A \mathcal{I} \text{ into } V \bigotimes_A A, \tag{24.3.4}$$

as commutative groups, that corresponds to the identity mapping on V and the obvious inclusion mappng from $\mathcal{I}$ into A, as in Section 1.9. It can be shown that

$$T_1(V) \text{ is isomorphic to the kernel of the homomorphism as in (24.3.3),} \tag{24.3.5}$$

as a commutative group, as in (3) on p144 of [3].

Suppose that $X = X(V)$ is a projective resolution of V, as a right module over A, as in Section 10.2. We may consider $X \bigotimes_A Q$ as a graded module over $\mathbf{Z}$, with

$$\Big(X \bigotimes\nolimits_A Q\Big)^j = X^j \bigotimes\nolimits_A Q \tag{24.3.6}$$

for each $j \in \mathbf{Z}$, as in Section 7.5. More precisely, $X \bigotimes_A Q$ may be considered as a complex over $\mathbf{Z}$, as before. The homology of this complex can be used to obtain $T_n(V)$, as on p143 of [3]. Namely,

$$T_n(V) = H\Big(X \bigotimes\nolimits_A Q\Big)_n = H\Big(X \bigotimes\nolimits_A Q\Big)^{-n} \tag{24.3.7}$$

for each $n \geq 0$.

Alternatively, suppose that $X = X(Q)$ is a projective resolution of Q, as a left module over A. As before, $V \bigotimes_A X$ may be considered as a graded module over $\mathbf{Z}$, with

$$\Big(V \bigotimes\nolimits_A X\Big)^j = V \bigotimes\nolimits_A X^j \tag{24.3.8}$$

for each $j \in \mathbf{Z}$. In fact, $V \bigotimes_A X$ may be considered as a complex over $\mathbf{Z}$, as in Section 7.5 again. The homology of this complex can be used to obtain $T_n(V)$, with

$$T_n(V) = H\Big(V \bigotimes\nolimits_A X\Big)_n = H\Big(V \bigotimes\nolimits_A X\Big)^{-n} \tag{24.3.9}$$

for each $n \geq 0$, as on p144 of [3].

Remember that A is automatically free as a left module over itself, and thus projective. This means that the augmentation homomorphism ε may be considered as a homomorphism from a projective left module over A onto Q. This can be used as the initial step in the construction of a projective resolution of Q, as in Section 10.2, and mentioned on p144 of [3].

Now let $X = X(V)$ be a projective resolution of V, as a right module over A, and let $Y = Y(Q)$ be a projective resolution of Q, as a left module over A, as in Section 10.2 again. Under these conditions, $X \bigotimes_A Y$ can be defined as a double complex over $\mathbf{Z}$, as in Section 6.2. In particular, $X \bigotimes_A Y$ is doubly-graded as a module over $\mathbf{Z}$, as in Section 6.1. One can obtain a single grading on $X \bigotimes_A Y$ from the double grading in the usual way, as in Section 5.13. One can also combine the two differentiation operators on $X \bigotimes_A Y$, as a double complex, to get a single differentiation operator, so that $X \bigotimes_A Y$ becomes a single complex, as in Section 5.14.

The homology of $X \bigotimes_A Y$ as a single complex can be used to get $T_n(V)$, with

$$T_n(V) = H\Big(X \bigotimes_A Y\Big)_n = H\Big(X \bigotimes_A Y\Big)^{-n} \tag{24.3.10}$$

for each $n \geq 0$. This uses one of the equivalent definitions of $T_n(V)$ indicated on p143 of [3], which is based on a definition on p107 of [3]. The equivalence of the two definitions of $T_n(V)$ on p143 of [3], as well as the descriptions in (24.3.7) and (24.3.9), are mentioned on p107 of [3] too.

The definition on p107 of [3] mentioned in the preceding paragraph is based on another definition on p84 of [3], which is related to the discussion in Section 10.11. The equivalence with (24.3.7) and (24.3.9) is related to the discussion in Section 10.12. The other equivalent definition of $T_n(V)$ on p143 of [3] is based on "satellites", which are also mentioned in Section 10.12.

Of course, there are analogous notions and statements for right augmented rings, as mentioned on p145 of [3].

24.4 Two-sided ideals and homology

Let us continue with the same notation and hypotheses as in the previous section. Suppose now that $\mathcal{I}$ is a two-sided ideal in A, so that $Q = A/\mathcal{I}$ is a ring. In this case, $T_n(V)$ may be considered as a right module over A for every $n \geq 0$, because Q is a right module over A, as on p145 of [3].

We may also take $V = Q$ in the previous section, because Q is a right module over A. Remember that $T_0(Q) = T(Q)$ was disccused in Section 24.2.

Of course, if $q \in Q$ and $a \in \mathcal{I}$, then

$$q \cdot a = 0 \tag{24.4.1}$$

in Q, by construction. This implies that the homomorphism

$$\text{from } Q \bigotimes_A \mathcal{I} \text{ into } Q \tag{24.4.2}$$

as in (24.3.3), with $V = Q$, is equal to 0. This means that

$$T_1(Q) \text{ is isomorphic to } Q \bigotimes_A \mathcal{I}, \tag{24.4.3}$$

as in (24.3.5). This corresponds to part of (9) on p146 of [3].

In this case,

$$0 \longrightarrow \mathcal{I} \longrightarrow A \xrightarrow{\varepsilon} Q \longrightarrow 0 \tag{24.4.4}$$

may be considered as an exact sequence of left and right modules over A, using the obvious inclusion mapping from $\mathcal{I}$ into A. This leads to an exact sequence

$$\mathcal{I} \bigotimes_A \mathcal{I} \longrightarrow A \bigotimes_A \mathcal{I} \longrightarrow Q \bigotimes_A \mathcal{I} \longrightarrow 0, \tag{24.4.5}$$

as in Section 2.5. This uses the mappings obtained from those in (24.4.5), and the identity mapping on $\mathcal{I}$.

As usual, $\mathcal{I}$ satisfies the requirements of $A \bigotimes_A \mathcal{I}$, using the map $(a, b) \mapsto a\, b$ from $A \times \mathcal{I}$ into $\mathcal{I}$. Thus the mapping in the first step in (24.4.5) corresponds to the homomorphism

$$\text{from } \mathcal{I} \bigotimes_A \mathcal{I} \text{ into } \mathcal{I} \tag{24.4.6}$$

with

$$a \otimes b \mapsto a\, b \tag{24.4.7}$$

for every $a, b \in \mathcal{I}$. The image

$$\mathcal{I}^2 = \mathcal{I}\,\mathcal{I} \tag{24.4.8}$$

of the homomorphism as in (24.4.6) consists of finite sums of products of the form $a\, b$, with $a, b \in \mathcal{I}$. This is a submodule of $\mathcal{I}$, as both a left and right module over A.

Using the exactness of (24.4.5), we get an isomorphism

$$\text{from } \mathcal{I}/\mathcal{I}^2 \text{ onto } Q \bigotimes_A \mathcal{I}, \tag{24.4.9}$$

as both left and right modules over A. This corresponds to another part of (9) on p146 of [3].

It can be shown that

$$\begin{array}{l} T_2(Q) \text{ is isomorphic to the kernel} \\ \text{of the homomorphism as in (24.4.6),} \end{array} \tag{24.4.10}$$

as in (10) on p146 of [3].

24.5 Augmentations and cohomology

Let A be a ring with a multiplicative identity element e_A. Suppose that A is a left augmented ring, with augmentation module Q, homomorphism ε, and ideal $\mathcal{I} = \mathcal{I}_\varepsilon$, as in Section 24.1. If W is a left module over A, then we put $U(W) = U_L(W) = \mathrm{Hom}_A(Q, W)$, as before.

If n is a nonnegative integer, then the *nth cohomology group* $U^n(W)$ of A as a left augmented ring with coefficients in W is defined as a commutative group on p143 of [3]. Theorem 1.1a on p145 of [3] gives another characterization of $U^n(W)$, up to isomorphism. In particular,

$$U^0(W) = U(W). \tag{24.5.1}$$

If W is injective as a left module over A, then

$$U^n(W) = \{0\} \text{ for every } n \geq 1. \tag{24.5.2}$$

As on p143 of [3], W may have additional actions that commute with the action of A on the left, and these lead to suitable actions on $U^n(W)$. In particular, if k is a commutative ring with a multiplicative identity element, and A is an associative algebra over k, then modules over A may be considered as modules over k, and $U(W)$ is defined as a module over k. In this case, $U^n(W)$ is a module over k for every $n \geq 0$, as in [3].

Remember that the space $\mathrm{Hom}_A(A, W)$ of homomorphisms from A into W, as left modules over A, is isomorphic to W in a natural way, as commutative groups, and as left modules over A, as in Section 24.1. Using this isomorphism, the obvious homomorphism

$$\text{from } \mathrm{Hom}_A(A, W) \text{ into } \mathrm{Hom}_A(\mathcal{I}, W), \tag{24.5.3}$$

which sends a homomorphism from A into W to its restriction to $\mathcal{I}$, may be identified with a homomorphism

$$\text{from } W \text{ into } \mathrm{Hom}_A(\mathcal{I}, W), \tag{24.5.4}$$

as before. It can be shown that

$$\begin{gathered} U^1(W) \text{ is isomorphic to the cokernel} \\ \text{of the homomorphism as in (24.5.3),} \end{gathered} \tag{24.5.5}$$

as a commutative group, as in (3a) on p144 of [3].

Suppose that $Y = Y(W)$ is an injective resolution of W, as a left module over A, as in Section 10.7. In particular, Y is a complex over A, so that $\mathrm{Hom}_A^{gr}(Q, Y)$ may be defined as a complex over $\mathbf{Z}$, as in Section 7.12. Under these conditions,

$$U^n(W) = H\big(\mathrm{Hom}_A^{gr}(Q, Y)\big)^n \tag{24.5.6}$$

for each $n \geq 0$, as on p143 of [3].

Alternatively, suppose that $X = X(Q)$ is a projective resolution of Q, as a left module over A, as in Section 10.2. Thus X is a complex over A in particular, and $\mathrm{Hom}_A^{gr}(X, W)$ may be defined as a complex over $\mathbf{Z}$, as in Section 8.4. In this case,

$$U^n(W) = H\big(\mathrm{Hom}_A^{gr}(X, W)\big)^n \tag{24.5.7}$$

for each $n \geq 0$, as on p144 of [3].

Now let $X = X(Q)$ be a projective resolution of Q, as a left module over A, and let $Y = Y(W)$ be an injective resolution of W as a left module over A, as in Sections 10.2 and 10.7. Under these conditions, $\mathrm{Hom}_A^{gr}(X, Y)$ can be defined as a double complex over $\mathbf{Z}$, as in Section 6.5. This uses the definition of $\mathrm{Hom}_A^{gr}(X, Y)$ as a doubly-graded module over $\mathbf{Z}$ in Section 6.3. As usual, one can get a single grading on $\mathrm{Hom}_A^{gr}(V, W)$ from the double grading as in Section 5.13. The two differentiation operators on $\mathrm{Hom}_A^{gr}(X, Y)$, as a double complex, can be combined to get a single differentiation operator, so that $\mathrm{Hom}_A^{gr}(X, Y)$ becomes a single complex, as in Section 5.14.

The homology of $\mathrm{Hom}_A^{gr}(X, Y)$ as a single complex can be used to obtain $U^n(W)$, with

$$U^n(W) = H\big(\mathrm{Hom}_A^{gr}(X, Y)\big)^n \tag{24.5.8}$$

for each $n \geq 0$. This uses one of the equivalent definitions of $U^n(W)$ mentioned on p143 of [3], which uses another definition on p107 of [3]. The equivalence of the two definitions of $U^n(W)$ on p143 of [3] is mentioned on p107 of [3], as well as the equivalence with the descriptions in (24.5.6) and (24.5.7).

The definition on p107 of [3] mentioned in the preceding paragraph uses another definition on p83 of [3], which is related to the discussion in Section 10.13. The equivalence with (24.5.6) and (24.5.7) is related to the discussions in Sections 10.14 and 10.15. The equivalence with the descriptions in terms of "satellites" is mentioned in those sections as well.

There are analogous notions and statements for right augmented rings, as mentioned on p145 of [3].

If $\mathcal{I}$ is a two-sided ideal in A, then Q may be considered as a right module over A too, as in Section 24.2. In this case, $U^n(W)$ may be considered as a left module over A for every $n \geq 0$, as on p145 of [3].

24.6 Dual numbers and augmentations

Let A be a ring with a multiplicative identity element e_A, and let $A[d]$ be the corresponding ring of dual number, as in Section 5.4. Also let ε be the mapping from $A[d]$ onto A defined by

$$\varepsilon(a_1 + a_2\, d) = a_1 \tag{24.6.1}$$

for every $a_1, a_2 \in A$. This is a ring homomorphism, which is the same as in Section 9.14, with $A[d]$ considered as a graded ring.

Thus A may be considered as a left and right module over $A[d]$, using ε. Equivalently, the actions of d on A on the left and right are equal to 0.

We may consider $A[d]$ as a left and right augmented ring, with augmentation homomorphism ε, and augmentation module A. The augmentation ideal is $A\, d$, which may be considered as a left and right module over $A[d]$, because it is a two-sided ideal in $A[d]$.

Observe that

$$a \mapsto a\, d \tag{24.6.2}$$

defines an isomorphism

$$\text{from } A \text{ onto } A\, d, \tag{24.6.3}$$

as left and right modules over $A[d]$. More precisely, the actions of d on $A\, d$ on the left and on the right are equal to 0.

It is easy to see that

$$a_1 + a_2\, d \mapsto a_1\, d \tag{24.6.4}$$

is a homomorphism

$$\text{from } A[d] \text{ onto } A\, d, \tag{24.6.5}$$

as left and right modules over $A[d]$. This is the same as the composition of ε with the isomorphism as in (24.6.2).

Consider the graded left and right module X over $A[d]$ defined by

$$\begin{aligned} X^j &= \{0\} \quad \text{when } j > 0 \\ &= A[d] \quad \text{when } j \leq 0. \end{aligned} \tag{24.6.6}$$

Let d_X^j be the mapping from X^j into X^{j+1} which is given by (24.6.4) when $j < 0$, and which is of course equal to 0 when $j \ge 0$. Note that

$$d_X^{j+1} \circ d_X^j = 0 \tag{24.6.7}$$

for every j, by construction. Let d_X be the homomorphism from X into itself of degree 1 whose jth component is equal to d_X^j for each j. It follows that (X, d_X) is a graded left and right module over $A[d]$ with differentiation that is a complex, as in Section 5.10.

We may consider A as a graded left and right module over $A[d]$ with differentiation that is a complex by taking $A^0 = A$, $A^j = \{0\}$ when $j \ne 0$, and differentiation operator $d_A = 0$, as before. Let ε_X be the homomorphism from X into A of degree 0 defined by taking ε_X^0 to be the map from $X^0 = A[d]$ into $A^0 = A$ given by ε, and with $\varepsilon_X^j = 0$ when $j \ne 0$. Observe that

$$\varepsilon_X^0 \circ d_X^{-1} = 0, \tag{24.6.8}$$

by construction. This implies that ε_X is a homomorphism from X into A, as modules with differentiation, as in Section 10.1. More precisely, ε_X is a map from X into A, as complexes over $A[d]$.

This makes X a left complex over A, as a left or right module over $A[d]$, with augmentation ε_X, as in Section 10.1. Clearly X is projective over $A[d]$, because $A[d]$ is projective as a left or right module over itself. It is easy to see that X is acyclic as a left complex over A, using the remarks in Section 10.1. Thus X is a projective resolution of A, as a left or right module over $A[d]$, as in Section 10.2. This corresponds to some remarks on p147 of [3].

24.7 Dual numbers and homology

Let us continue with the same notation and hypotheses as in the previous section. Also let V be a right module over $A[d]$. Equivalently, this means that V is a right module over A with differentiation, where the action of d on V on the right is the same as the differentiation operator d_V, as in Section 5.4.

Let us take $Y = V \bigotimes_{A[d]} X$, which may be considered as a graded right module over $A[d]$. More precisely,

$$\begin{aligned} Y^j = V \bigotimes_{A[d]} X^j &= \{0\} && \text{when } j > 0 \\ &= V \bigotimes_{A[d]} A[d] && \text{when } j \le 0, \end{aligned} \tag{24.7.1}$$

by (24.6.6). We can identify Y^j with V when $j \le 0$, as a right module over $A[d]$, in the usual way.

In fact, Y is a right module over $A[d]$ with differentiation, where the differentiation operator d_Y is obtained from d_X and the identity mapping on V in the usual way, as in Section 7.5. More precisely, Y is a complex, and d_Y^j is obtained from d_X^j and the identity mapping on V in the usual way for each j. Of course, $d_Y^j = 0$ when $j \geq 0$.

If $j < 0$, then Y^j and Y^{j+1} can be identified with V, as right modules over $A[d]$, as before. Using this identification, one can check that d_Y^j corresponds exactly to d_V. This uses the fact that d_X^j corresponds to the mapping from $A[d]$ into itself given by (24.6.4), as in the previous section.

If $n \geq 1$, then we get that

$$T_n(V) = H(Y)_n = H(Y)^{-n} = H(V), \tag{24.7.2}$$

as in (24.3.9). Here $H(V)$ is the usual homology of V as a module with differentation, as in Section 5.1.

We also have that

$$T(V) = T_0(V) = V \bigotimes_{A[d]} A \text{ is isomorphic to } Z'(V), \tag{24.7.3}$$

where $Z'(V)$ is as in Section 5.1, for V as a module with differentiation. This can be obtained from the description of $T(V)$ in Section 24.2, when the augmentation ideal $\mathcal{I}$ is a two-sided ideal. Here $\mathcal{I} = A\,d$, and it is easy to see that

$$V \cdot \mathcal{I} = V \cdot (A\,d) = V \cdot d = d_V(V). \tag{24.7.4}$$

One could obtain (24.7.3) from the fact that $T_0(V) = H(Y)^0$, as in Section 24.3, as well.

These remarks correspond to some of those on p147 of [3].

24.8 Dual numbers and cohomology

Let us continue with the same notation and hypotheses as in Section 24.6 again. Now let W be a left module over $A[d]$. Thus W may be considered as a left module over A with differentiation, where the differentiation operator d_W corresponds to the action of d on W on the left, as in Section 5.4.

Here we take $Y = \mathrm{Hom}^{gr}_{A[d]}(X, W)$, which may be considered initially as a graded module over $\mathbf{Z}$, as in Section 8.4. In fact, Y is a graded left module over $A[d]$, because X is a graded left and right module over $A[d]$. Using (24.6.6), we get that

$$\begin{aligned} Y^j = \mathrm{Hom}_{A[d]}(X^{-j}, W) &= \mathrm{Hom}_{A[d]}(A[d], W) && \text{when } j \geq 0 \\ &= \{0\} && \text{when } j < 0. \end{aligned} \tag{24.8.1}$$

We can identify Y^j with W when $j \geq 0$, as a left module over $A[d]$, as in Section 1.8.

Let $\phi \in Y$ be given, so that $\phi^j \in \mathrm{Hom}_{A[d]}(X^{-j}, W)$ for each $j \in \mathbf{Z}$. Under these conditions, $d_Y(\phi) \in Y$ is defined by putting

$$d_Y(\phi)^{j+1} = \phi^j \circ d_X^{-j-1} \tag{24.8.2}$$

for each j, as in Section 8.4. Remember that Y is a complex with respect to d_Y, as before. Equivalently, d_Y^j is defined on Y^j by composition with d_X^{-j-1} for each j. If $j < 0$, then $-j-1 \geq 0$, so that $d_X^{-j-1} = 0$, and thus $d_Y^j = 0$.

If $j \geq 0$, then Y^j and Y^{j+1} can be identified with W, as left modules over $A[d]$, as before. In this case, one can check that d_Y^j corresponds to d_W with respect to this identification. More precisely, this uses the fact that d_X^{-j-1} corresponds to (24.6.4) as a mapping from $A[d]$ into itself, by construction.

If $n \geq 1$, then we obtain that

$$U^n(W) = H(Y)^n = H(W), \tag{24.8.3}$$

as in (24.5.7). This is the usual homology of W as a module with differentiation, as in Section 5.1.

We also get that

$$U(W) = U^0(W) = \mathrm{Hom}_{A[d]}(A, W) \text{ is isomorphic to } Z(W), \tag{24.8.4}$$

where $Z(W)$ is as in Section 5.1. Indeed, $\mathrm{Hom}_{A[d]}(A, W)$ is isomorphic in a natural way to the submodule of $\mathrm{Hom}_{A[d]}(A[d], W)$ consisting of homomorphisms from $A[d]$ into W, as left modules over $A[d]$, that are equal to 0 on the augmentation ideal $A\,d$ in $A[d]$. If we identify $\mathrm{Hom}_{A[d]}(A[d], W)$ with W in the usual way, then this submodule corresponds to $Z(W)$. One could obtain (24.8.4) from the fact that $U^0(W) = H(Y)^0$, as in Section 24.5, too.

These remarks correspond to some more of those on p147 of [3].

24.9 Augmentations and homomorphisms

Let A, B be rings with multiplicative identity elements e_A, e_B, respectively. Suppose that A, B are both left augmented rings, with augmentation modules Q_A, Q_B, augmentation homomorphisms ε_A, ε_B, and augmentation ideals $\mathcal{I}_A$, $\mathcal{I}_B$, respectively. Of course, one could consider right augmentations as well.

Suppose that ϕ is a homomorphism from A into B as rings, with $\phi(e_A) = e_B$. If

$$\phi(\mathcal{I}_A) \subseteq \mathcal{I}_B, \tag{24.9.1}$$

then ϕ is said to be a *map or homomorphism of augmented rings*, as on p149 of [3].

In this case, there is a unique mapping ψ from Q_A into Q_B such that

$$\psi \circ \varepsilon_A = \varepsilon_B \circ \phi. \tag{24.9.2}$$

If $a \in A$ and $x \in Q_A$, then it follows that

$$\psi(a \cdot x) = \phi(a) \cdot \psi(x). \tag{24.9.3}$$

Note that Q_B may be considered as a left module over A, where the action of $a \in A$ on Q_B on the left is given by the action of $\phi(a) \in B$ on Q_B on the left, as in Section 2.9. Using this, ψ may be considered as a homomorphism from Q_A into Q_B, as left modules over A, as on p149 of [3].

Similarly, B may be considered as a right module over itself, and thus a right module over A, using ϕ, as in Section 2.9. Let

$$ {}_{(\phi)}Q_A = B \bigotimes_A Q_A \tag{24.9.4}$$

be a tensor product of B and Q_A as modules over A. This may be considered as a left module over B, which is the covariant ϕ-extension of Q_A, as in Section 2.9.

Consider the mapping from $B \times Q_A$ into Q_B defined by

$$ (b, x) \mapsto b \cdot \psi(x) \tag{24.9.5}$$

for every $b \in B$ and $x \in Q_A$. This mapping is bilinear over $\mathbf{Z}$, and compatible with the actions of A on B on the right and on Q_A on the left, by (24.9.3). It follows that there is a unique homomorphism

$$ g \text{ from (24.9.4) into } Q_B, \tag{24.9.6}$$

as commutative groups with respect to addition, such that

$$ g(b \otimes x) = b \cdot \psi(x) \tag{24.9.7}$$

for every $b \in B$ and $x \in Q_A$, as on p149f of [3]. More precisely, g is a homomorphism as in (24.9.6), as left modules over B.

Let X_A be a projective resolution of Q_A, as a left module over A, and let X_B be a projective resolution of Q_B, as a left module over B. Also let

$$ {}_{(\phi)}X_A = B \bigotimes_A X_A \tag{24.9.8}$$

be a tensor product of B and X_A, as modules over A, which is the covariant ϕ-extension of X_A. This may be considered as a complex over $\mathbf{Z}$, as in Section 7.5. More precisely, this may be considered as a left module over B, as before, and in fact as a complex over B. Note that

$$ ({}_{(\phi)}X_A)^j = B \bigotimes_A X_A^j = {}_{(\phi)}\, X_A^j \tag{24.9.9}$$

is the covariant ϕ-extension of X_A^j, as a left module over A, for each integer j.

One can check that

$$ \text{(24.9.8) is a left complex over (24.9.4),} \tag{24.9.10}$$

as left modules over B, in a natural way. The augmentation of (24.9.8) as a left complex can be obtained from the augmentation of X_A, as a left complex over Q_A, and the identity mapping on B. Observe that

$$ \text{(24.9.9) is projective as a left module over } B \tag{24.9.11}$$

for each j, because X_A^j is projective as a left module over A for each j, by hypothesis, as in Section 2.10. This means that

$$\text{(24.9.12)} \qquad \text{(24.9.8) is projective as a left complex over (24.9.4),}$$

as left modules over B, as in Section 10.1. This is mentioned at the top of p150 of [3].

It follows that there is a map

$$\text{(24.9.13)} \qquad G \text{ from (24.9.8) into } X_B \text{ over } g,$$

as in Section 10.3. This map is unique up to homotopy, as in Section 10.4.

Let V be a right module over B, so that V may be considered as a right module over A using ϕ, as in Section 2.9. If we consider V as a right module over A, then a tensor product $V \bigotimes_A X_A$ of V and X_A over A may be considered as a complex over $\mathbf{Z}$, as in Section 7.5. The homology of this complex may be used to obtain the homology of A as a left augmented ring with coefficients in V, as in Section 24.3.

Remember that V satisfies the requirements of a tensor product $V \bigotimes_B B$. This permits us to identify $V \bigotimes_A X_A$ with a tensor product of the form

$$\text{(24.9.14)} \qquad \Big(V \bigotimes_B B\Big) \bigotimes_A X_A.$$

There is a natural isomorphism between a tensor product of this form and a tensor product of the form

$$\text{(24.9.15)} \qquad V \bigotimes_B \Big(B \bigotimes_A X_A\Big),$$

as in Section 1.12. This was also mentioned in connection with covariant ϕ-extensions in Section 2.10. More precisely, one can check that this is an isomorphism between these tensor products as complexes.

A tensor product $V \bigotimes_B X_B$ of V and X_B over B may be considered as a complex over $\mathbf{Z}$ as well, as in Section 7.5 again. The homology of this complex may be used to obtain the homology of B as a left augmented ring with coefficients in V, as in Section 24.3. Using G and the identity mapping on V, we get a map

$$\text{(24.9.16)} \qquad \text{from (24.9.15) into } V \bigotimes_B X_B,$$

as complexes, as in Section 7.6. Note that this map is uniquely determined up to homotopy by g. We may consider this as a map

$$\text{(24.9.17)} \qquad \text{from } V \bigotimes_A X_A \text{ into } V \bigotimes_B X_B,$$

as complexes, using the identification and isomorphism mentioned in the preceding paragraph.

The map as in (24.9.17) leads to a homomorphism

$$\text{(24.9.18)} \qquad \text{from } H\Big(V \bigotimes_A X_A\Big) \text{ into } H\Big(V \bigotimes_B X_B\Big)$$

of degree 0, as graded modules over $\mathbf{Z}$, as in Section 5.11. This homomorphism is uniquely determined by g, because induced maps on homology are preserved by homotopies. This corresponds to (2) on p150 of [3]. This determines a homomorphism

(24.9.19) from the homology of A into the homology of B, as left augmented rings, with coefficients in V,

as in (1) on p149 of [3].

Similarly, let Z be a left module over B, which may be considered as a left module over A too, as in Section 2.9. Thus $\mathrm{Hom}^{gr}_A(X_A, Z)$ may be defined as a complex over Z, as in Section 8.4. The homology of this complex can be used to get the cohomology of A as a left augmented ring with coefficients in Z, as in Section 24.5.

We also have that $\mathrm{Hom}^{gr}_B(X_B, Z)$ and

$$\mathrm{Hom}^{gr}_B({}_{(\phi)}X_A, Z) = \mathrm{Hom}^{gr}_B\big(B \bigotimes\nolimits_A X_A\big) \tag{24.9.20}$$

are defined as complexes over $\mathbf{Z}$, as in Section 8.4. Remember that G is in particular a map from (24.9.8) into X_B as complexes over B, as in Section 10.3. This implies that G induces a map

(24.9.21) from $\mathrm{Hom}^{gr}_B(X_B, Z)$ into (24.9.20),

as complexes over $\mathbf{Z}$, as in Section 8.5. This map is uniquely determined up to homotopy by g.

There is a natural isomorphism

(24.9.22) between (24.9.20) and $\mathrm{Hom}^{gr}_A(X_A, Z)$,

as in Section 2.10. More precisely, one can reduce to the previous statements for ordinary spaces of homomorphisms between modules without gradings, to deal with graded modules here. One can also check that this is an isomorphism between complexes over $\mathbf{Z}$. Using this, the map in (24.9.21) may be considered as a map

(24.9.23) from $\mathrm{Hom}^{gr}_B(X_B, Z)$ into $\mathrm{Hom}^{gr}_A(X_A, Z)$,

as complexes over $\mathbf{Z}$.

This leads to a homomorphism

(24.9.24) from $H\big(\mathrm{Hom}^{gr}_B(X_B, Z)\big)$ into $H\big(\mathrm{Hom}^{gr}_A(X_A, Z)\big)$

of degree 0, as graded modules over $\mathbf{Z}$, as in Section 5.11 again. This homomorphism is uniquely determined by g, and corresponds to (2a) on p150 of [3]. This determines a homomorphism

(24.9.25) from the cohomology of B into the cohomology of A, as left augmented rings, with coefficients in Z,

as in (1a) on p149 of [3].

24.10 The Mapping Theorem

Let us continue with the same notation and hypotheses as in the previous section. One may be interested in having the homomorphisms as in (24.9.19) and (24.9.25) be isomorphisms. This is addressed in Theorem 3.1 on p150 of [3], which is called the Mapping Theorem. Of course, this is equivalent to the homomorphisms as in (24.9.18) and (24.9.24) being isomorphisms.

Suppose for the moment that $V = B$, considered as a right module over B. Thus

$$V \bigotimes_A X_A = B \bigotimes_A X_A =_{(\phi)} X_A, \tag{24.10.1}$$

and X_B satisfies the requirements of $V \bigotimes X_B = B \bigotimes_B X_B$. Suppose also that the homomorphism as in (24.9.18) is an isomorphism in this case, so that we have an isomorphism

$$\text{from } H\big(B \bigotimes_A X_A\big) \text{ onto } H(X_B) \tag{24.10.2}$$

of degree 0, as graded modules over $\mathbf{Z}$. This determines an isomorphism

$$\text{from } H\big(B \bigotimes_A X_A\big)_n \text{ onto } H(X_B)_n \tag{24.10.3}$$

for each $n \geq 0$.

Remember that $H\big(B \bigotimes_A X_A\big)_n$ is isomorphic to

$$\begin{aligned}&\text{the } n\text{th homology group of } A, \text{ as a left augmented ring}\\ &\text{with coefficients in } B, \text{ as a right module over } A \text{ using } \phi,\end{aligned} \tag{24.10.4}$$

for each $n \geq 0$, as in Section 24.3.

The condition that we have an isomorphism as in (24.10.3) when $n = 0$ means that

$$\text{the homomorphism } g \text{ in (24.9.6) is an isomorphism.} \tag{24.10.5}$$

This is the same as (i) on p150 of [3].

Of course,

$$H(X_B)_n = \{0\} \tag{24.10.6}$$

when $n > 0$, because X_B is acyclic as a left complex over Q_B, by hypothesis, as in Section 10.1. Thus the condition that we have an isomorphism as in (24.10.3) when $n > 0$ means that

$$H\big(B \bigotimes_A X_A\big)_n = \{0\} \tag{24.10.7}$$

when $n > 0$. This is the same as saying that

$$\text{(24.10.4) is equal to } \{0\} \tag{24.10.8}$$

when $n > 0$, as in (ii) on p150 of [3].

Suppose now that (24.10.5) holds, and that (24.10.8) holds for every $n > 0$. If X_A is any projective resolution of Q_A, as a left module over A, then (24.10.8) implies that (24.10.7) holds for every $n > 0$. We can use g to consider $B \bigotimes_A X_A$ as a left complex over Q_B, instead of $B \bigotimes_A Q_A$, as before. Our hypotheses imply $B \bigotimes_A X_A$ is acyclic as a left complex over Q_B, as in Section 10.1. It follows that

(24.10.9) $$B \bigotimes_A X_A \text{ is a projective resolution of } Q_B,$$

as a left module over B, because of (24.9.11), as in Theorem 3.1 on p150 of [3].

This permits us to take

$$X_B = B \bigotimes_A X_A \tag{24.10.10}$$

in this case. Under these conditions, the map G as in (24.9.13) is the identity mapping on (24.10.10). This implies that the homomorphisms as in (24.9.18) and (24.9.24) are isomorphisms, so that the homomorphisms as in (24.9.19) and (24.9.25) are isomorphisms, as in Theorem 3.1 on p150 of [3].

24.11 Some interesting conditions

Let A be a left augmented ring, with multiplicative identity element e_A, augmentation module Q, augmentation module Q, augmentation homomorphism ε, and augmentation ideal $\mathcal{I}$, as in Section 24.1. Also let n be a positive integer, and let $x_1, \ldots, x_n$ be n commuting elements of A. Suppose that

(24.11.1) $$\mathcal{I} \text{ is generated by } x_1, \ldots, x_n, \text{ as a left ideal in } A.$$

Put $\mathcal{I}_0 = \{0\}$, and for each $l = 1, \ldots, n$, let

(24.11.2) $$\mathcal{I}_l \text{ be the left ideal in } A \text{ generated by } x_1, \ldots, x_l.$$

Thus $\mathcal{I}_n = \mathcal{I}$, by (24.11.1). Consider the following condition:

(24.11.3) $$\text{if } a \in A,\ 1 \le l \le n, \text{ and } a\, x_l \in \mathcal{I}_{l-1}, \text{ then } a \in \mathcal{I}_{l-1}.$$

This is the condition (i) in Theorem 4.2 on p150 of [3].

Let us mention a couple of classes of examples where this holds, as on p151 of [3]. Let A_0 be a ring with a multiplicative identity element, and let $x_1, \ldots, x_n$ be commuting indeterminates. The corresponding ring $A_0[x_1, \ldots, x_n]$ of formal polynomials in $x_1, \ldots, x_n$ with coefficients in A_0, as in Section 4.3, may be considered as a left augmented ring, where the augmentation homomorphism sends an element of $A_0[x_1, \ldots, x_n]$ to its constant term in A_0. The augmentation ideal consists of elements of $A_0[x_1, \ldots, x_n]$ with constant term equal to 0, and it is easy to see that this is generated by $x_1, \ldots, x_n$, as a left ideal in $A_0[x_1, \ldots, x_n]$. One can check that (24.11.3) holds in this case.

Similarly, the ring $A_0[[x_1, \ldots, x_n]]$ of formal power series in $x_1, \ldots, x_n$, as in Section 4.3, may be considered as a left augmented ring, where the augmentation homomorphism sends an element of $A_0[[x_1, \ldots, x_n]]$ to its constant term. As before, the augmentation ideal consists of elements of $A_0[[x_1, \ldots, x_n]]$ with constant term equal to 0, and one can check that this is generated by $x_1, \ldots, x_n$, as a left ideal in $A_0[[x_1, \ldots, x_n]]$. One can verify that (24.11.3) holds in this case too.

Let us return now to the same notation and hypotheses as at the beginning of the section. Let $T_1, \ldots, T_n$ be commuting indeterminates, so that the ring $\mathbf{Z}[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in $\mathbf{Z}$ may be defined as in Section 4.3 again. As on p151 of [3],

(24.11.4) A may be considered as a right module over $\mathbf{Z}[T_1, \ldots, T_n]$,

where for each $a \in A$ and $l = 1, \ldots, n$,

$$a \cdot T_l = a\, x_l. \tag{24.11.5}$$

Put $\mathcal{J}_0 = \{0\}$, and for each $l = 1, \ldots, n$, let

(24.11.6) $\mathcal{J}_l$ be the ideal in $\mathbf{Z}[T_1, \ldots, T_n]$ generated by $T_1, \ldots, T_l$.

Observe that

$$\mathcal{I}_l = A \cdot \mathcal{J}_l \tag{24.11.7}$$

for each $l = 0, 1, \ldots, n$, as on p151 of [3].

Similarly, let V be any right module over $\mathbf{Z}[T_1, \ldots, T_n]$. Note that

$$V \cdot \mathcal{J}_l \tag{24.11.8}$$

is a submodule of V, as a module over $\mathbf{Z}[T_1, \ldots, T_n]$, for each $l = 0, 1, \ldots, n$. Of course, if $0 \le l_1 \le l_2 \le n$, then

$$\mathcal{J}_{l_1} \subseteq \mathcal{J}_{l_2}. \tag{24.11.9}$$

This implies that

$$V \cdot \mathcal{J}_{l_1} \subseteq V \cdot \mathcal{J}_{l_2}. \tag{24.11.10}$$

Consider the following condition:

(24.11.11) if $v \in V$, $1 \le l \le n$, and $v \cdot T_l \in V \cdot \mathcal{J}_{l-1}$, then $v \in V \cdot \mathcal{J}_{l-1}$.

This is the same as the condition (i') in Proposition 4.3 on p151 of [3]. If $V = A$, considered as a right module over $\mathbf{Z}[T_1, \ldots, T_n]$, then this is the same as (24.11.3).

If $1 \le l \le n$, then the quotient modules $V/(V \cdot \mathcal{J}_{l-1})$ and $(V \cdot \mathcal{J}_l)/(V \cdot \mathcal{J}_{l-1})$ are defined as modules over $\mathbf{Z}[T_1, \ldots, T_n]$. The mapping

$$v \mapsto v \cdot T_l \tag{24.11.12}$$

from V into $V \cdot \mathcal{J}_l$ induces a homomorphism

(24.11.13) from $V/(V \cdot \mathcal{J}_{l-1})$ into $(V \cdot \mathcal{J}_l)/(V \cdot \mathcal{J}_{l-1})$,

as modules over $\mathbf{Z}[T_1, \ldots, T_n]$. It is easy to see that this homomorphism is surjective. The condition (24.11.11) says that this induced homomorphism is injective, and thus an isomorphism, as mentioned at the top of p152 of [3].

24.12 A related left complex

Let n be a positive integer, and let $T_1, \ldots, T_n$ be n commuting indeterminates, so that the ring

$$B = \mathbf{Z}[T_1, \ldots, T_n] \tag{24.12.1}$$

of formal polynomials in $T_1, \ldots, T_n$ with coefficients in $\mathbf{Z}$ may be defined as in Section 4.3, as before. Also let $\mathcal{J}_l$ be the ideal in $\mathbf{Z}[T_1, \ldots, T_n]$ defined in the previous section for each $l = 0, 1, \ldots, n$, and let V be a right module over $\mathbf{Z}[T_1, \ldots, T_n]$.

Let $y_1, \ldots, y_n$ be another collection of n indeterminates, and let

$$E_B(y_1, \ldots, y_n) \tag{24.12.2}$$

be the corresponding exterior ring in $y_1, \ldots, y_n$ with coefficients in B, as in Section 4.15. Similarly, we get a right module

$$\mathcal{E}_V(y_1, \ldots, y_n) \tag{24.12.3}$$

over $E_B(y_1, \ldots, y_n)$, as before. More precisely, $E_B(y_1, \ldots, y_n)$ may be considered as a graded ring, as in Section 9.14, and $\mathcal{E}_V(y_1, \ldots, y_n)$ may be considered as a graded module over $E_B(y_1, \ldots, y_n)$, as a graded ring, as in Section 9.15. Note that this does not use a grading on B.

Let us take

$$X = \mathcal{E}_V(y_1, \ldots, y_n), \tag{24.12.4}$$

as a right module over $E_B(y_1, \ldots, y_n)$, and thus a right module over B in particular. We may consider X as a graded module over B in the usual sense, without a grading on B, with

$$X_j = (\mathcal{E}_V(y_1, \ldots, y_n))^j \tag{24.12.5}$$

for each $j \in \mathbf{Z}$, as on p151 of [3]. Remember that

$$X_j = X^{-j} \tag{24.12.6}$$

for each $j \in \mathbf{Z}$, as in Section 5.9.

Let $1 \leq r \leq n$ be given, and let us define a homomorphism d_r from X_r into X_{r-1}, as modules over B, as follows. Let $I = \{j_1, \ldots, j_r\}$ be a subset of $\{1, \ldots, n\}$ with r elements, where $j_1 < j_2 < \cdots < j_r$. Thus $y_I = y_{j_1} \wedge \cdots \wedge y_{j_r}$, as in (4.15.2). If $v \in V$, then we put

$$d_r(v\, y_I) = \sum_{l=1}^{r} (-1)^{l+1}\, (v \cdot T_{j_l})\, y_{(I \setminus \{j_l\})}, \tag{24.12.7}$$

as on p151 of [3]. Of course,

$$y_{(I \setminus \{j_l\})} = y_{j_1} \wedge \cdots \wedge \widehat{y}_{j_l} \wedge \cdots \wedge y_{j_r}, \tag{24.12.8}$$

where $\widehat{y}_{j_l}$ indicates that y_{j_l} should be omitted from the right side.

It is easy to see that this defines a homomorphism from X_r into X_{r-1}, as modules over B, because B is commutative. If $r \le 0$ or $r > n$, then we take $d_r = 0$. One can check that

$$d_{r-1} \circ d_r = 0 \tag{24.12.9}$$

for every $r \in \mathbf{Z}$, as on p151 of [3].

This makes X a complex, as a module over B, where the restriction of the differentiation operator d on X to X_r is equal to d_r for each $r \in \mathbf{Z}$. Note that X is negative as a graded module over B, because $X^j = X_{-j} = \{0\}$ when $j > 0$.

Remember that the quotient module $V/(V \cdot \mathcal{J}_n)$ is defined as a module over B, as in the previous section. We may consider this module as a complex in the usual way, where the jth submodule is the same module when $j = 0$ and is $\{0\}$ otherwise, and where the differentiation operator is equal to 0.

In fact,

(24.12.10) we may consider X as a left complex over $V/(V \cdot \mathcal{J}_n)$,

as on p151 of [3]. To do this, we should choose a map ε_X from X into $V/(V \cdot \mathcal{J}_n)$, as complexes, as in Section 10.1. Of course, ε_X is determined by its restriction $\varepsilon_{X,0}$ to X_0 in this case, as before. We can identify X_0 with V in an obvious way, as a module over B. Using this identification, we take $\varepsilon_{X,0}$ to be the natural quotient mapping from V onto $V/(V \cdot \mathcal{J}_n)$, as in [3].

It is easy to see that

$$\varepsilon_{X,0} \circ d_1 = 0, \tag{24.12.11}$$

by construction, as in [3]. This implies that ε_X is a homomorphism from X into $V/(V \cdot \mathcal{J}_n)$, as modules with differentiation, as in Section 10.1.

If V satisfies the condition (24.11.11), then Proposition 3.4 on p151 of [3] states that

(24.12.12) X is acyclic as a left complex over $V/(V \cdot \mathcal{J}_n)$,

as in Section 10.1. To show this, we shall consider some more left complexes, as in [3].

24.13 Some more left complexes

Let us continue with the same notation and hypotheses as in the previous section. If $1 \le l \le n$, then

$$E_B(y_1, \ldots, y_l) \tag{24.13.1}$$

can be identified with a subring of $E_B(y_1, \ldots, y_n)$ in an obvious way. If $l = 0$, then we interpret (24.13.1) as being B. Note that (24.13.1) is homogeneous as a subgroup of $E_B(y_1, \ldots, y_n)$, as a graded commutative group with respect to addition, and that the grading induced on (24.13.1) by the one on $E_B(y_1, \ldots, y_n)$ is the same as the grading defined on (24.13.1) as in Section 9.14. In particular, $\mathcal{E}_V(y_1, \ldots, y_n)$ may be considered as a graded right module over (24.13.1), as a graded ring, as in Section 9.15.

Similarly,

(24.13.2) $$\mathcal{E}_V(y_1, \ldots, y_l)$$

may be considered as a submodule of $\mathcal{E}_V(y_1, \ldots, y_n)$, as a module over (24.13.1), when $1 \le l \le n$. If $l = 0$, then we interpret (24.13.2) as being V. As before, (24.13.2) is homogeneous as a subgroup of $\mathcal{E}_V(y_1, \ldots, y_n)$, as a graded commutative group with respect to addition, and the grading induced on (24.13.2) by the one on $\mathcal{E}_V(y_1, \ldots, y_n)$ is the same as the grading defined on (24.13.2) as in Section 9.15.

If $1 \le l \le n$, then we take

(24.13.3) $$X^{(l)} = \mathcal{E}_V(y_1, \ldots, y_l),$$

as a right module over (24.13.1), and thus a right module over B in particular, as on p152 of [3]. This is the same as (24.12.4) when $l = n$. As before, we may consider $X^{(l)}$ as a graded module over B in the usual sense, without a grading on B, with

(24.13.4) $$X^{(l)}_j = (\mathcal{E}_V(y_1, \ldots, y_l))^j$$

for each $j \in \mathbf{Z}$. If $l = 0$, then we interpret (24.13.3) as being V, as in the preceding paragraph, and we interpret (24.13.4) as being V when $j = 0$, and $\{0\}$ otherwise.

Note that $X^{(l)}$ may be considered as a homogeneous submodule of X for each l, as in Section 5.9. Similarly, $X^{(l-1)}$ may be considered as a homogeneous submodule of $X^{(l)}$ when $l \ge 1$.

The differentiation operator d defined on X in the previous section maps $X^{(l)}$ into itself for each l, so that $X^{(l)}$ may be considered as a submodule of X, as a module with differentiation. It follows that $X^{(l)}$ is a complex with respect to the restriction of d to $X^{(l)}$ for each l. The restriction of d_r to $X^{(l)}_r$ is equal to 0 when $r > l$. In particular, the restriction of d to $X^{(0)}$ is equal to 0.

The quotient module $V/(V \cdot \mathcal{J}_l)$ is defined as a module over B for each $l = 0, 1, \ldots, n$, as in Section 24.11. As usual, we may consider this module as a complex, where the jth submodule is the same module when $j = 0$ and is $\{0\}$ otherwise, and with differentiation operator equal to 0. If $l = 0$, then this is the same as $X^{(0)}$.

We would like to choose a map $\varepsilon_{X^{(l)}}$ from $X^{(l)}$ into $V/(V \cdot \mathcal{J}_l)$, as complexes, for each $l = 0, 1, \ldots, n$, as in Section 10.1. As before, $\varepsilon_{X^{(l)}}$ is determined by its restriction $\varepsilon_{X^{(l)},0}$ to $X^{(l)}_0$ in this case, and we can identify $X^{(l)}_0$ with V, as a module over B. We take $\varepsilon_{X^{(l)},0}$ to be the natural quotient mapping from V onto $V/(V \cdot \mathcal{J}_l)$, as on p152 of [3].

One can check that

(24.13.5) the composition of the restriction of d_1 to $X^{(l)}_1$ with $\varepsilon_{X^{(l)},0}$ is equal to 0.

This means that $\varepsilon_{X^{(l)}}$ is a homomorphism from $X^{(l)}$ into $V/(V \cdot \mathcal{J}_l)$, as modules with differentiation, as in Section 10.1. Thus

(24.13.6) we may consider $X^{(l)}$ as a left complex over $V/(V \cdot \mathcal{J}_l)$,

as on p152 of [3].

Suppose that V satisfies (24.11.11). We would like to show that

$$X^{(l)} \text{ is acyclic as a left complex over } V/(V \cdot \mathcal{J}_l) \tag{24.13.7}$$

for each $l = 0, 1, \ldots, n$, as on p152 of [3]. This is the same as (24.12.12) when $l = n$, because $X = X^{(n)}$. It is easy to see that (24.13.7) holds when $l = 0$, because $\mathcal{J}_0 = \{0\}$, as in Section 24.11, and $X^{(0)} = V$.

If $0 \le l \le n$, then put

$$\begin{aligned} Y_j^{(l)} &= X_j^{(l)} \quad \text{when } j \ne 0 \\ &= V \cdot \mathcal{J}_l \quad \text{when } j = 0, \end{aligned} \tag{24.13.8}$$

as on p152 of [3]. This defines $Y^{(l)}$ as a graded module over B, which is in fact a homogeneous submodule of $X^{(l)}$, as in Section 5.9. More precisely, $Y^{(l)}$ is a submodule of $X^{(l)}$, as a module with differentiation, because

$$d_1(X_1^{(l)}) \subseteq V \cdot \mathcal{J}_l. \tag{24.13.9}$$

In particular, $Y^{(l)}$ is a complex, with respect to the restriction of d to $Y^{(l)}$.

One can verify that (24.13.7) holds if and only if

$$H(Y^{(l)}) = \{0\}, \tag{24.13.10}$$

as on p152 of [3]. This uses some remarks about acyclicity in Section 10.1, and the fact that $\varepsilon_{X^{(l)},0}$ maps $X_0^{(l)}$ onto $V/(V \cdot \mathcal{J}_l)$ with kernel equal to $V \cdot \mathcal{J}_l$.

In order to show (24.13.7), we use induction on l, as in [3]. The base case $l = 0$ has already been mentioned. Thus we suppose that $l \ge 1$, and that the analogue of (24.13.7) for $l - 1$ holds. This means that

$$H(Y^{(l-1)}) = \{0\}, \tag{24.13.11}$$

as in the preceding paragraph.

Note that $Y^{(l-1)}$ is a homogeneous submodule of $Y^{(l)}$, as a graded module over B. We also have that $Y^{(l-1)}$ is a submodule of $Y^{(l)}$, as a module with differentiation. Thus the quotient $Y^{(l)}/Y^{(l-1)}$ can be defined as a complex in a natural way.

Using the obvious inclusion mapping from $Y^{(l-1)}$ into $Y^{(l)}$, and the natural quotient mapping from $Y^{(l)}$ onto $Y^{(l)}/Y^{(l-1)}$, we get a sequence of induced mappings

$$H(Y^{(l-1)}) \longrightarrow H(Y^{(l)}) \longrightarrow H(Y^{(l)}/Y^{(l-1)}). \tag{24.13.12}$$

This sequence is exact, as in some remarks near the beginning of Section 5.3. If we can show that

$$H(Y^{(l)}/Y^{(l-1)}) = \{0\}, \tag{24.13.13}$$

then it will follow that (24.13.10) holds, because of (24.13.11), as on p152 of [3].

Of course,

$$(Y^{(l)}/Y^{(l-1)})_j = X_j^{(l)}/X_j^{(l-1)} \quad \text{when } j \neq 0 \tag{24.13.14}$$
$$= (V \cdot \mathcal{J}_l)/(V \cdot \mathcal{J}_{l-1}) \quad \text{when } j = 0,$$

by construction. Multiplication by y_l on the right defines a homomorphism

$$\text{from } X_{j-1}^{(l-1)} \text{ into } X_j^{(l)}, \tag{24.13.15}$$

as modules over B, for each j. We can compose this with the natural quotient mapping from $X_j^{(l)}$ onto $X_j^{(l)}/X_j^{(l-1)}$ to get a homomorphism

$$\text{from } X_{j-1}^{(l-1)} \text{ into } X_j^{(l)}/X_j^{(l-1)}, \tag{24.13.16}$$

as modules over B, for each j. One can check that

(24.13.17) the homomorphism as in (24.13.16) is an isomorphism when $j \neq 0$.

Remember that we also have an isomorphism

$$\text{from } V/(V \cdot \mathcal{J}_{l-1}) \text{ onto } (Y^{(l)}/Y^{(l-1)})_0 = (V \cdot \mathcal{J}_l)/(V \cdot \mathcal{J}_{l-1}), \tag{24.13.18}$$

as modules over B. This is the homomorphism as in (24.11.13), which is an isomorphism because of the hypothesis (24.11.11). This homomorphism was induced by the homomorphism from V into $V \cdot \mathcal{J}_l$ defined by $v \mapsto v \cdot T_l$.

The acyclicity of $X^{(l-1)}$ as a left complex over $V/(V \cdot \mathcal{J}_{l-1})$ is equivalent to the exactness of the sequence

$$0 \longrightarrow X_{l-1}^{(l-1)} \xrightarrow{d_{l-1}} X_{l-2}^{(l-1)} \xrightarrow{d_{l-2}} \cdots \xrightarrow{d_1} X_0^{(l-1)} \xrightarrow{\varepsilon_{X^{(l-1)},0}} V/(V \cdot \mathcal{J}_{l-1}) \longrightarrow 0, \tag{24.13.19}$$

as in Section 10.1. More precisely, this uses the fact that $X_j^{(l-1)} = \{0\}$ when $j \geq l$, by construction. The condition (24.13.13) is equivalent to the exactness of the sequence

$$0 \longrightarrow X_l^{(l)}/X_l^{(l-1)} \longrightarrow X_{l-1}^{(l)}/X_{l-1}^{(l-1)} \longrightarrow \cdots \tag{24.13.20}$$
$$\longrightarrow X_1^{(l)}/X_1^{(l-1)} \longrightarrow (V \cdot \mathcal{J}_l)/(V \cdot \mathcal{J}_{l-1}) \longrightarrow 0,$$

where the homomorphisms are given by differentiation on $Y^{(l)}/Y^{(l-1)}$. This uses the fact that $(Y^{(l)}/Y^{(l-1)})_j = \{0\}$ when $j > l$ or $j < 0$.

We have just seen that we have isomomorphisms between the modules in these two sequences. One can verify that these isomorphisms are compatible with the homomorphisms in these two sequences, as on p152 of [3].

This means that the exactness of (24.13.20) follows from the exactness of (24.13.19), as on p152 of [3]. Thus (24.13.13) follows from our induction hypothesis. This implies that (24.13.10) holds, as before.

This is the same as saying that (24.13.7) holds, as before. Thus (24.13.7) holds for every $l = 0, 1, \ldots, n$, as desired.

Remember that (24.12.12) is the same as (24.13.7) with $l = n$. This shows that (24.12.12) holds when V satisfies (24.11.11), as mentioned in the previous section.

24.14 Some nice projective resolutions

Let us now see how the results mentioned in the previous two sections can be used to deal with augmented rings as in Section 24.11, as on p152 of [3]. Let A be a left augmented ring again, with multiplicative identity element e_A, augmentation module Q, augmentation homomorphism ε, and augmentation ideal $\mathcal{I}$. Also let n be a positive integer, let $x_1, \ldots, x_n$ be n commuting elements of A, and suppose that $\mathcal{I}$ is generated by $x_1, \ldots, x_n$, as a left ideal in A, as before.

Let $T_1, \ldots, T_n$ be n commuting indeterminates, and let us consider A as a right module over the ring $\mathbf{Z}[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in $\mathbf{Z}$, as before. Thus $a \cdot T_l = a\, x_l$ for each $a \in A$ and $l = 1, \ldots, n$. Remember that

(24.14.1) $$\mathcal{I} = A \cdot \mathcal{J}_n,$$

where $\mathcal{J}_n$ is the ideal in $\mathbf{Z}[T_1, \ldots, T_n]$ generated by $T_1, \ldots, T_n$.

Let us take

(24.14.2) $$V = A,$$

as a right module over $\mathbf{Z}[T_1, \ldots, T_n]$. Note that $\mathcal{I}$ is a submodule of V, as a module over $\mathbf{Z}[T_1, \ldots, T_n]$. This means that

(24.14.3) $$V/\mathcal{I} = V/(V \cdot \mathcal{J}_n)$$

may be considered as a module over $\mathbf{Z}[T_1, \ldots, T_n]$. However, the action of T_l on (24.14.3) on the right is equal to 0 for each $l = 1, \ldots, n$, by construction.

Let $y_1, \ldots, y_n$ be another collection of n indeterminates, as in Section 24.12. Let us take

(24.14.4) $$X = \mathcal{E}_V(y_1, \ldots, y_n) = \mathcal{E}_A(y_1, \ldots, y_n),$$

as before. More precisely, this may be considered as a right module over $B = \mathbf{Z}[T_1, \ldots, T_n]$, because V is a right module over B, as in Section 4.15. Similarly, X may be considered as a left module over A, because A is a left module over itself.

Put

(24.14.5) $$X_j = (\mathcal{E}_V(y_1, \ldots, y_n))^j = (\mathcal{E}_A(y_1, \ldots, y_n))^j$$

for each $j \in \mathbf{Z}$, as in Section 24.12. This is a submodule of X, as both a right module over B, and a left module over A. This defines a grading on X, as a right module over B, and a left module over A.

We can define a homomorphism d_r from X_r into X_{r-1}, as right modules over B, for each $r \in \mathbf{Z}$, as in Section 24.12. It is easy to see that d_r is a homomorphism from X_r into X_{r-1} as left modules over A as well. This means that X is a complex, as a right module over B, and as a left module over A.

Remember that X is a left complex over $V/(V \cdot \mathcal{J}_n)$, as right modules over B, with the augmentation map ε_X defined in Section 24.12. More precisely, the restriction $\varepsilon_{X,0}$ of ε_X to X_0 corresponds exactly to the natural quotient mapping from V onto $V/(V \cdot \mathcal{J}_n)$. Of course,

$$Q = A/\mathcal{I} = V/(V \cdot \mathcal{J}_n), \tag{24.14.6}$$

as commutative groups with respect to addition, and this may be considered as both a left module over A, and a right module over B. Using this, it is easy to see that

(24.14.7) we may consider X as a left complex over Q,

as left modules over A.

If $I \subseteq \{1, \ldots, n\}$, then let y_I be as in (4.15.2). If $0 \le j \le n$, then $(\mathcal{E}_A(y_1, \ldots, y_n))^j$ is freely generated as a left module over A by y_I, where I has exactly j elements. Of course, $X_j = \{0\}$ when $j < 0$ or $j > n$. It follows that X_j is projective as a left module over A for each $j \in \mathbf{Z}$, so that X is projective as a left complex of modules over A, as in Section 10.1.

Suppose now that (24.11.3) holds, which means that (24.11.11) holds in this case. This implies that X is acyclic as a left complex over $V/(V \cdot \mathcal{J}_n)$, as right modules over B, as in (24.12.12). Using this, we get that

(24.14.8) X is acyclic as a left complex over Q, as left modules over A.

It follows that

(24.14.9) X is a projective resolution of Q, as a left module over A,

as in Section 10.2. This corresponds to some remarks on p152 of [3], and to part of Theorem 4.2 on p150 of [3].

24.15 Using these projective resolutions

Let us continue with the same notation and hypotheses as in the previous section. Let W be a right module over A, and let

$$W \bigotimes_A X = W \bigotimes_A \mathcal{E}_A(y_1, \ldots, y_n) \tag{24.15.1}$$

be a tensor product of W and $X = \mathcal{E}_A(y_1, \ldots, y_n)$ over A. This is a graded module over $\mathbf{Z}$, with

$$\Big(W \bigotimes_A X\Big)_j = W \bigotimes_A X_j = W \bigotimes_A (\mathcal{E}_A(y_1, \ldots, y_n))^j \tag{24.15.2}$$

for each $j \in \mathbf{Z}$. It is easy to see that

$$W \bigotimes_A X = \mathcal{E}_W(y_1, \ldots, y_n), \tag{24.15.3}$$

where $\mathcal{E}_W(y_1, \ldots, y_n)$ is as in Section 4.15. Similarly,

$$\left(W \bigotimes_A X\right)_j = (\mathcal{E}_W(y_1, \ldots, y_n))^j \tag{24.15.4}$$

for each j, where $(\mathcal{E}_W(y_1, \ldots, y_n))^j$ is as in Section 9.15.

More precisely, $W \bigotimes_A X$ may be considered as a complex over $\mathbf{Z}$, as in Section 7.5. Remember that the differentiation operator $d_{W \bigotimes_A X}$ on $W \bigotimes_A X$ is obtained from the differentiation operator $d = d_X$ on X and the identity mapping on W in the usual way. If $r \in \mathbf{Z}$, then the corresponding homomorphism

$$d_{W \bigotimes_A X, r} \text{ from } \left(W \bigotimes_A X\right)_r \text{ into } \left(W \bigotimes_A X\right)_{r-1} \tag{24.15.5}$$

is obtained from the homomorphism d_r from X_r into X_{r-1} and the identity mapping on W in the same way. In particular, this is equal to 0 when $d_r = 0$, which happens when $r \leq 0$ and when $r > n$.

If $1 \leq r \leq n$, then (24.15.5) can be described as follows. Let $I = \{j_1, \ldots, j_r\}$ be a subset of $\{1, \ldots, n\}$ with r elements, where $j_1 < j_2 < \cdots < j_r$, so that $y_I = y_{j_1} \wedge \cdots \wedge y_{j_r}$, as before. If $w \in W$, then

$$d_{W \bigotimes_A X, r}(w\, y_I) = \sum_{l=1}^{r} (-1)^{l+1} (w \cdot x_{j_l})\, y_{(I \setminus \{j_l\})}, \tag{24.15.6}$$

as in (2) on p152 of [3]. Remember that the homology of this complex may be used to obtain the homology of A, as a left augmented ring, with coefficients in W, as in Section 24.3.

This type of complex was first found by J. L. Koszul, in connection with cohomology theory of Lie groups, as mentioned on p153 of [3].

Similarly, let Z be a left module over A, so that

$$\mathrm{Hom}_A^{gr}(X, Z) = \mathrm{Hom}_A^{gr}(\mathcal{E}_A(y_1, \ldots, y_n), Z) \tag{24.15.7}$$

may be defined as a graded module over $\mathbf{Z}$ as in Section 8.4. More precisely,

$$\left(\mathrm{Hom}_A^{gr}(X, Z)\right)^j = \mathrm{Hom}_A(X_j, Z) = \mathrm{Hom}_A((\mathcal{E}_A(y_1, \ldots, y_n))^j, Z) \tag{24.15.8}$$

for each $j \in \mathbf{Z}$. Of course, this reduces to $\{0\}$ when $j < 0$, and when $j > n$. If $0 \leq j \leq n$, then the elements of (24.15.8) may be indentified with Z-valued functions $f(I)$ defined for subsets I of $\{1, \ldots, n\}$ such that the number $|I|$ of elements of I is equal to j.

In fact, (24.15.7) may be considered as a complex over $\mathbf{Z}$, as in Section 8.4. Let

$$\delta = d_{\mathrm{Hom}_A^{gr}(X, Z)} \tag{24.15.9}$$

be the corresponding differentiation operator on (24.15.7), and let δ_j be its restriction to (24.15.8) for each j. This is the mapping from (24.15.8) into

$$\begin{aligned} \left(\mathrm{Hom}_A^{gr}(X, Z)\right)^{j+1} &= \mathrm{Hom}_A(X_{j+1}, Z) \\ &= \mathrm{Hom}_A((\mathcal{E}_A(y_1, \ldots, y_n))^{j+1}, Z) \end{aligned} \tag{24.15.10}$$

defined by composing d_{j+1} with a homomorphism from X_j into Z, as modules over A, to get a homomorphism from X_{j+1} into Z, as modules over A. Note that $\delta_j = 0$ when $j < 0$ or $j \geq n$.

Suppose that $0 \leq j < n$, and that f is a Z-valued function on the set of all subsets of $\{1, \ldots, n\}$ with exactly j elements, which is identified with an element of (24.15.8), as before. Let $\delta_j(f)$ be the Z-valued function on the set of all subsets of $\{1, \ldots, n\}$ with exactly $j+1$ elements that corresponds to δ_j of the element of (24.15.8) corresponding to f. If $I = \{m_1, \ldots, m_{j+1}\}$ is a subset of $\{1, \ldots, n\}$ with $j+1$ elements, where $m_1 < m_2 < \cdots < m_{j+1}$, then

$$(\delta_j(f))(I) = \sum_{l=1}^{j+1} (-1)^{l+1} \, (x_l \cdot f(I \setminus \{m_l\})). \tag{24.15.11}$$

This corresponds to (2a) on p153 of [3]. The homology of this complex may be used to obtain the cohomology of A, as a left augmented ring, with coefficients in Z, as in Section 24.5.

Chapter 25

Some helpful facts

25.1 Projectivity and tensor products

Let k be a commutative ring with a multiplicative identity element, and let A, B be associative algebras over k with multiplicative identity elements e_A, e_B, respectively. Also let V, W be modules over k, and suppose that

(25.1.1) V is a right module over A, and W is a left module over A.

Thus we can let $V \bigotimes_A W$ be a tensor product of V and W over A, which is a module over k.

Suppose for the moment that

(25.1.2) $$W \text{ is a right module over } B,$$

where the actions of A and B on W commute with each other. Thus $V \bigotimes_A W$ may be considered as a right module over B too, as in Section 1.10. If

(25.1.3) $$V \text{ is projective as a right module over } A$$

and

(25.1.4) $$W \text{ is projective as a right module over } B,$$

then

(25.1.5) $$V \bigotimes_A W \text{ is projective as a right module over } B,$$

as in Proposition 5.3 on p28 of [3].

If V is free as a right module over A, then $V \bigotimes_A W$ corresponds to a direct sum of copies of W, as a right module over B. This means that $V \bigotimes_A W$ is projective as a right module over B, because W is projective.

If V is projective as a right module over A, then there is a free right module over A that is isomorphic to the direct sum of V and another right module U over A, as in Section 2.7. This implies that the direct sum of $V \bigotimes_A W$ and a tensor product $U \bigotimes_A W$ of U and W over A is projective as a right module over B, as in the preceding paragraph. It follows that $V \bigotimes_A W$ is projective as a right module over B, as in Section 2.7.

Alternatively, let Y and Z be right modules over B, and let ψ be a homomorphism from Y onto Z, as modules over B. Consider the homomorphism

$$\Psi \text{ from } \mathrm{Hom}_B(W,Y) \text{ into } \mathrm{Hom}_B(W,Z) \tag{25.1.6}$$

defined by composing a homomorphism from W into Y, as modules over B, with ψ to get a homomorphism from W into Z, as modules over B. Note that

$$\Psi \text{ maps } \mathrm{Hom}_B(W,Y) \text{ onto } \mathrm{Hom}_B(W,Z), \tag{25.1.7}$$

because W is projective as a module over B, by hypothesis. More precisely, $\mathrm{Hom}_B(W,Y)$ and $\mathrm{Hom}_B(W,Z)$ may be considered as right modules over A, because W is a left module over A, and the actions of A and B on W commute, as in Section 1.8. It is easy to see that Ψ is a homomorphism as in (25.1.6), as right modules over A.

Using Ψ, we get a homomorphism

$$\widehat{\Psi} \text{ from } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W,Y)\big) \text{ into } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W,Z)\big), \tag{25.1.8}$$

as modules over k. As before, $\widehat{\Psi}$ is defined by composing a homomorphism from V into $\mathrm{Hom}_B(W,Y)$, as modules over A, with Ψ to get a homomorphism from V into $\mathrm{Hom}_B(W,Z)$, as modules over A. We also have that

$$\widehat{\Psi} \text{ maps } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W,Y)\big) \text{ onto } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W,Z)\big), \tag{25.1.9}$$

because V is projective as a right module over A, by hypothesis.

Consider the homomorphism

$$\widetilde{\Psi} \text{ from } \mathrm{Hom}_B\big((V \bigotimes_A W), Y\big) \text{ into } \mathrm{Hom}_B\big((V \bigotimes_A W), Z\big), \tag{25.1.10}$$

as modules over k, defined by composing a homomorphism from $V \bigotimes_A W$ into Y, as modules over B, with ψ to get a homomorphism from $V \bigotimes_A W$ into Z, as modules over B.

There is a natural isomorphism

$$\text{from } \mathrm{Hom}_B\big((V \bigotimes_A W), Z\big) \text{ onto } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W,Z)\big), \tag{25.1.11}$$

as modules over k, as in Section 1.13. Of course, there is an analogous isomorphism with Z replaced by Y. One can check that $\widetilde{\Psi}$ corresponds to $\widehat{\Psi}$ with respect to these isomorphisms. Thus the surjectivity of $\widetilde{\Psi}$ follows from the surjectivity of $\widehat{\Psi}$. This implies (25.1.5).

Suppose now that

$$V \text{ is a left module over } B, \tag{25.1.12}$$

where the actions of A and B on V commute with each other. This means that $V \bigotimes_A W$ may be considered as a left module over B as well, as in Section 1.10. If

$$V \text{ is projective as a left module over } B \tag{25.1.13}$$

and

$$W \text{ is projective as a left module over } A, \tag{25.1.14}$$

then

$$V \bigotimes_A W \text{ is projective as a left module over } B, \tag{25.1.15}$$

as on p28 of [3].

One can start with the case where W is free as a left module over A, and then reduce to that case using direct sums, as before. Alternatively, if Z is a left module over B, then there is a natural isomorphism

$$\text{from } \mathrm{Hom}_B\big((V \bigotimes_A W), Z\big) \text{ onto } \mathrm{Hom}_A\big(W, \mathrm{Hom}_B(V, Z)\big), \tag{25.1.16}$$

as modules over k, as in Section 1.13 again. This can be used to obtain (25.1.15), as before.

25.2 Injectivity of $\mathrm{Hom}_A(V, Z)$

Let k be a commutative ring with a multiplicative identity element, and let A, B be associative algebras over k with multiplicative identity elements e_A, e_B, respectively. Also let V, Z be modules over k, and suppose that

$$V \text{ and } Z \text{ are both left or both right modules over } A. \tag{25.2.1}$$

This means that $\mathrm{Hom}_A(V, Z)$ is defined as a module over k.

Suppose too that

$$Z \text{ is a left or right module over } B, \tag{25.2.2}$$

where the actions of A and B on Z commute. This implies that $\mathrm{Hom}_A(V, Z)$ may be considered as a left or right module over B, as appropriate, as well, as in Section 1.8. If

$$V \text{ is projective as a module over } A, \tag{25.2.3}$$

and

$$Z \text{ is injective as a module over } B, \tag{25.2.4}$$

then

$$\mathrm{Hom}_A(V, Z) \text{ is injective as a module over } B, \tag{25.2.5}$$

as in Exercise 5 on p32 of [3].

Suppose for the moment that V is free as a module over A, and thus isomorphic to a direct sum of copies of A. This implies that $\mathrm{Hom}_A(V, Z)$ is isomorphic to a direct product of copies of $\mathrm{Hom}_A(A, Z)$, as in Section 1.7. This means that $\mathrm{Hom}_A(V, Z)$ is isomorphic to a direct product of copies of Z, as modules over B. It follows that (25.2.5) holds, because Z is injective as a module over B, as in Section 2.8.

If (25.2.3) holds, then there is a left or right module U over A, as appropriate, such that the direct sum of U and V is free as a module over A, as in Section

2.7. In this case, $\mathrm{Hom}_A\big(U \bigoplus V, Z\big)$ is injective as a module over B, as in the preceding paragraph. Of course, $\mathrm{Hom}_A\big(U \bigoplus V, Z\big)$ is isomorphic to the direct sum of $\mathrm{Hom}_A(U, Z)$ and $\mathrm{Hom}_A(V, Z)$, as modules over B. It follows that (25.2.5) holds, as in Section 2.8 again.

Alternatively, let W be a left or right module over B, as appropriate, and let W_0 be a submodule of W. There is an obvious homomorphism

$$\text{from } \mathrm{Hom}_B\big(W, \mathrm{Hom}_A(V, Z)\big) \text{ into } \mathrm{Hom}_B\big(W_0, \mathrm{Hom}_A(V, Z)\big), \tag{25.2.6}$$

as modules over k, which sends a homomorphism from W into $\mathrm{Hom}_A(V, Z)$ to its restriction to W_0. We would like to verify that this homomorphism is surjective, under the conditions considered here.

Note that $\mathrm{Hom}_B(W, Z)$ is a module over k which may be considered as a left or right module over A, as appropriate, as in Section 1.8 again. There is a natural isomorphism

$$\text{from } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W, Z)\big) \text{ onto } \mathrm{Hom}_B\big(W, \mathrm{Hom}_A(V, Z)\big), \tag{25.2.7}$$

as modules over k, as in Section 1.14. Similarly, there is a natural isomorphism

$$\text{from } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W_0, Z)\big) \text{ onto } \mathrm{Hom}_B\big(W_0, \mathrm{Hom}_A(V, Z)\big), \tag{25.2.8}$$

as modules over k. Using these isomorphisms, the homomorphism as in (25.2.6) corresponds to a homomorphism

$$\text{from } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W, Z)\big) \text{ into } \mathrm{Hom}_A\big(V, \mathrm{Hom}_B(W_0, Z)\big), \tag{25.2.9}$$

as modules over k.

More precisely, there is an obvious homomorphism

$$\text{from } \mathrm{Hom}_B(W, Z) \text{ into } \mathrm{Hom}_B(W_0, Z), \tag{25.2.10}$$

as modules over k or A, which sends a homomorphism from W into Z to its restriction to W_0. One can check that the homomorphism as in (25.2.9) sends a homomorphism from V into $\mathrm{Hom}_B(W, Z)$ to its composition with the homomorphism as in (25.2.10).

The homomorphism as in (25.2.10) is surjective, because of (25.2.4). This implies that the homomorphism as in (25.2.9) is surjective, because of (25.2.3). This means that the homomorphism as in (25.2.6) is surjective, as desired.

25.3 Another injectivity property

Let k be a commutative ring with a multiplicative identity element, and let A, B be associative algebras over k with multiplicative identity elements e_A, e_B, respectively. Also let V, Z be modules over k, and suppose that

$$V \text{ and } Z \text{ are both left or both right modules over } B, \tag{25.3.1}$$

so that $\mathrm{Hom}_B(V,Z)$ is defined as a module over k.

In this section, we ask that

$$V \text{ be a left or right module over } A \tag{25.3.2}$$

too, where the actions of A and B on V commute. This means that $\mathrm{Hom}_B(V,Z)$ may be considered as a right or left module over A, respectively, as in Section 1.8. If

$$V \text{ is projective as a module over } A \tag{25.3.3}$$

and

$$Z \text{ is injective as a module over } B, \tag{25.3.4}$$

then

$$\mathrm{Hom}_B(V,Z) \text{ is injective as a module over } A, \tag{25.3.5}$$

as in Proposition 1.4 on p107 of [3].

Suppose for the sake of simplicity that V is a right module over A, so that $\mathrm{Hom}_B(V,Z)$ is a left module over A. Let W be a left module over A, and let W_0 be a submodule of W. There is an obvious homomorphism

$$\text{from } \mathrm{Hom}_A\big(W, \mathrm{Hom}_B(V,Z)\big) \text{ into } \mathrm{Hom}_A\big(W_0, \mathrm{Hom}_B(V,Z)\big), \tag{25.3.6}$$

as modules over k, which sends a homomorphism from W into $\mathrm{Hom}_B(V,Z)$, as modules over A, to its restriction to W_0. We would like to check that this homomorphism is a surjection.

Let $V \bigotimes_A W$, $V \bigotimes_A W_0$ be tensor products of V with W, W_0, respectively, over A. These modules over k may also be considered as left or right modules over B, as appropriate, as in Section 1.10. There is a natural isomorphism

$$\text{from } \mathrm{Hom}_A\big(W, \mathrm{Hom}_B(V,Z)\big) \text{ onto } \mathrm{Hom}_B\big(\big(V \bigotimes_A W\big), Z\big), \tag{25.3.7}$$

as modules over k, as in Section 1.13. Similarly, there is a natural isomorphism

$$\text{from } \mathrm{Hom}_A\big(W_0, \mathrm{Hom}_B(V,Z)\big) \text{ onto } \mathrm{Hom}_B\big(\big(V \bigotimes_A W_0\big), Z\big), \tag{25.3.8}$$

as modules over k. Using these isomomorphisms, the homomorphism as in (25.3.6) corresponds to a homomorphism

$$\text{from } \mathrm{Hom}_B\big(\big(V \bigotimes_A W\big), Z\big) \text{ into } \mathrm{Hom}_B\big(\big(V \bigotimes_A W_0\big), Z\big), \tag{25.3.9}$$

as modules over k.

There is a natural homomorphism

$$\text{from } V \bigotimes_A W_0 \text{ into } V \bigotimes_A W, \tag{25.3.10}$$

as modules over B, corresponding to the identity mapping on V, and the obvious inclusion mapping from W_0 into W. One can verify that the homomorphism as in (25.3.9) sends a homomorphism from $V \bigotimes_A W$ to Z, as modules over B, to its composition with the homomorphism as in (25.3.10). The homomorphism as in (25.3.10) is injective, because of (25.3.3), as mentioned near the end of Section 2.7. This implies that the homomorphism as in (25.3.9) is surjective, because of (25.3.4). It follows that the homomorphism as in (25.3.6) is surjective, as desired.

25.4 An associativity isomorphism

Let k be a commutative ring with a multiplicative identity element, and let A, B, and C be associative algebras over k, with multiplicative identity elements e_A, e_B, and e_C, respectively. Also let $A \bigotimes_k B$ and $B \bigotimes_k C$ be tensor products of A, B and B, C, respectively, over k. These may be considered as associative algebras over k with multiplicative identity elements too, as in Section 4.1.

Let V, W, and Z be modules over k. Suppose that

$$\text{V is a right module over A and B, where the actions of A and B on V commute with each other,} \tag{25.4.1}$$

and that

$$\text{W is a left module over A and a right module over C, where the actions of A and C on W commute with each other.} \tag{25.4.2}$$

Let $V \bigotimes_A W$ be a tensor product of V and W over A which is a module over k. Remember that $V \bigotimes_A W$ may be considered as a right module over each of B and C, as in Section 1.10. It is easy to see that the actions of B and C on $V \bigotimes_A W$ commute with each other.

Similarly, suppose that

$$\text{Z is a left module over B and C, where the actions of B and C on Z commute with each other.} \tag{25.4.3}$$

Let $W \bigotimes_C Z$ be a tensor product of W and Z over C, which is a module over k. As before, $W \bigotimes_C Z$ may be considered as a left module over A and B, where the actions of A and B on $W \bigotimes_C Z$ commute with each other.

Note that V may be considered as a right module over $A \bigotimes_k B$, and Z may be considered as a left module over $B \bigotimes_k C$, as in Section 4.2. Similarly, $V \bigotimes_A W$ may be considered as a right module over $B \bigotimes_k C$, and $W \bigotimes_C Z$ may be considered as a left module over $A \bigotimes_k B$. Let

$$(V \bigotimes_A W) \bigotimes_{B \bigotimes_k C} Z \tag{25.4.4}$$

be a tensor product of $V \bigotimes_A W$ and Z over $B \bigotimes_k C$, and let

$$V \bigotimes_{A \bigotimes_k B} (W \bigotimes_C Z) \tag{25.4.5}$$

be a tensor product of V and $W \bigotimes_C Z$ over $A \bigotimes_k B$.

Under these conditions, there is a unique homomorphism from (25.4.4) into (25.4.5), as modules over k, which sends

$$(v \otimes_A w) \otimes_{B \bigotimes_k C} z \tag{25.4.6}$$

to

$$v \otimes_{A \bigotimes_k B} (w \otimes_C z) \tag{25.4.7}$$

for every $v \in V$, $w \in W$, and $z \in Z$, as in Proposition 2.1 on p165 of [3]. More precisely, this homomorphism is an isomorphism, as in [3]. A simpler version of this was discussed in Section 1.12.

In particular, this homomorphism can be obtained directly using the definition of a tensor product, as before. One can start by fixing $z \in Z$, and considering the mapping from $V \times W$ into (25.4.5) that sends $(v, w) \in V \times W$ to (25.4.7). This leads to a unique homomorphism from $V \bigotimes_A W$ into (25.4.5), as modules over k, that sends $v \otimes_A w$ to (25.4.7) for every $v \in V$ and $w \in W$. This may be considered as a mapping from $\big(V \bigotimes_A W\big) \times Z$ into (25.4.5), which is bilinear over k and has other nice properties. Using this, we get a homomorphism from (25.4.4) into (25.4.5), as modules over k.

Similarly, there is a unique homomorphism from (25.4.5) into (25.4.4) that sends (25.4.7) to (25.4.6) for every $v \in V$, $w \in W$, and $z \in Z$. This homomorphism is the inverse of the previous one, so that these homomorphisms are isomorphisms.

25.5 Some more isomorphisms

Let k, A, B, and C be as in the previous section, as well as $A \bigotimes_k B$ and $B \bigotimes_k C$. Also let V, W, and Z be modules over k again, and suppose that (25.4.1) and (25.4.2) hold. As before, we let $V \bigotimes_A W$ be a tensor product of V and W over A, which is a module over k that may be considered as a right module over B and C, and where the actions of B and C on $V \bigotimes_A W$ commute with each other.

Suppose now that

$$\begin{aligned} &Z \text{ is a right module over } B \text{ and } C, \text{ where the actions} \\ &\text{of } B \text{ and } C \text{ on } Z \text{ commute with each other.} \end{aligned} \tag{25.5.1}$$

Thus $\mathrm{Hom}_C(W, Z)$ is defined as a module over k. This may be considered as a right module over A and B in this case, as in Section 1.8. It is easy to see that the actions of A and B on $\mathrm{Hom}_C(W, Z)$ commute with each other.

Remember that V may be considered as a right module over $A \bigotimes_k B$, and that $V \bigotimes_A W$ may be considered as a right module over $B \bigotimes_k C$, as in the previous section. Similarly, $\mathrm{Hom}_C(W, Z)$ may be considered as a right module over $A \bigotimes_k B$, and Z may be considered as a right module over $B \bigotimes_k C$, as in Section 4.2. Thus

$$\mathrm{Hom}_{A \bigotimes_k B}\big(V, \mathrm{Hom}_C(W, Z)\big) \tag{25.5.2}$$

and

$$\mathrm{Hom}_{B \bigotimes_k C}\big(\big(V \bigotimes_A W\big), Z\big) \tag{25.5.3}$$

are defined as modules over k.

Proposition 2.2 on p165 of [3] states that there is a unique homomorphism from (25.5.2) into (25.5.3), as modules over k, with the following property. If ϕ is an element of (25.5.2), then this homomorphism sends ϕ to an element of (25.5.3) with

$$v \otimes_A w \mapsto (\phi(v))(w) \tag{25.5.4}$$

for every $v \in V$ and $w \in W$. More precisely, if $v \in V$, then $\phi(v) \in \mathrm{Hom}_C(W, Z)$, so that $(\phi(v))(w) \in Z$ for each $w \in W$.

If ϕ is an element of (25.5.2), then

$$(v, w) \mapsto (\phi(v))(w) \tag{25.5.5}$$

defines a mapping from $V \times W$ into Z that is bilinear over k. This leads to a unique homomorphism from $V \bigotimes_A W$ into Z, as modules over k, that satisfies (25.5.4) for every $v \in V$ and $w \in W$. One can check that this homomorphism is an element of (25.5.3). This defines a mapping from (25.5.2) into (25.5.3), which is easily seen to be linear over k.

Conversely, any element of (25.5.3) can be used to get a mapping from $V \times W$ into Z that is bilinear over k and has other relevant properties. This can be used to get an element of (25.5.2). This defines a homomorphism from (25.5.3) into (25.5.2), as modules over k, which is the inverse of the previous homomorphism from (25.5.2) into (25.5.3). Thus we get an isomorphism between these modules over k, as in [3]. A simpler version of this was discussed in Section 1.13.

There is an analogous statement with actions on the left and right exchanged, and using a tensor product $W \bigotimes_A V$, as mentioned on p165 of [3]. The corresponding simplified version was discussed in Section 1.13.

25.6 Another projectivity property

Let us continue with the same notation and hypotheses as in the previous section. Suppose that

$$V \text{ is projective as a right module over } A \bigotimes_k B, \tag{25.6.1}$$

and that

$$W \text{ is projective as a right module over } C. \tag{25.6.2}$$

Under these conditions,

$$V \bigotimes_A W \text{ is projective as a right module over } B \bigotimes_k C, \tag{25.6.3}$$

as in Proposition 2.3 on p165 of [3].

To see this, let Y and Z be modules over k that are right modules over $B \bigotimes_k C$. Equivalently, this means that (25.5.1) holds, and similarly

$$\begin{aligned} &Y \text{ is a right module over } B \text{ and } C, \text{ where the actions} \\ &\text{of } B \text{ and } C \text{ on } Y \text{ commute with each other.} \end{aligned} \tag{25.6.4}$$

Let ψ be a homomorphism from Y onto Z, as right modules over $B \bigotimes_k C$. Consider the homomorphism

$$\text{(25.6.5)}\quad \widetilde{\Psi} \text{ from } \mathrm{Hom}_{B \bigotimes_k C}\big((V \bigotimes_A W), Y\big) \text{ into } \mathrm{Hom}_{B \bigotimes_k C}\big((V \bigotimes_A W), Z\big),$$

as modules over k, defined by composing a homomorphism from $V \bigotimes_A W$ into Y, as right modules over $B \bigotimes_k C$, with ψ to get a homomorphism from $V \bigotimes_A W$ into Z, as right modules over $B \bigotimes_k C$. We would like to show that $\widetilde{\Psi}$ is surjective.

As in the previous section, V and $\mathrm{Hom}_C(W, Z)$ may be considered as right modules over $A \bigotimes_k B$, and (25.5.2) is isomorphic to (25.5.3), as modules over k, in a natural way. Similarly, $\mathrm{Hom}_C(W, Y)$ may be considered as a right module over $A \bigotimes_k B$, and

$$\text{(25.6.6)}\qquad \mathrm{Hom}_{A \bigotimes_k B}\big(V, \mathrm{Hom}_C(W, Y)\big)$$

is isomorphic to

$$\text{(25.6.7)}\qquad \mathrm{Hom}_{B \bigotimes_k C}\big((V \bigotimes_A W), Y\big),$$

as modules over k, in a natural way.

Consider the homomorphism

$$\text{(25.6.8)}\qquad \Psi \text{ from } \mathrm{Hom}_C(W, Y) \text{ into } \mathrm{Hom}_C(W, Z)$$

that sends a homomorphism from W into Y, as modules over C, to its composition with ψ. The composition is a homomorphism from W into Z, as modules over C, because ψ is a homomorphism from Y onto Z, as modules over C in particular. Observe that

$$\text{(25.6.9)}\qquad \Psi \text{ maps } \mathrm{Hom}_C(W, Y) \text{ onto } \mathrm{Hom}_C(W, Z),$$

because of (25.6.2). One can check that Ψ is a homomorphism as in (25.6.8), as right modules over $A \bigotimes_k B$. Equivalently, Ψ is a homomorphism as in (25.6.8), as modules over each of A and B.

This leads to a homomorphism

$$\text{(25.6.10)}\qquad \begin{aligned} &\widehat{\Psi} \text{ from } \mathrm{Hom}_{A \bigotimes_k B}\big(V, \mathrm{Hom}_C(W, Y)\big) \\ &\text{into } \mathrm{Hom}_{A \bigotimes_k B}\big(V, \mathrm{Hom}_C(W, Z)\big), \end{aligned}$$

as modules over k, which sends a homomorphism from V into $\mathrm{Hom}_C(W, Y)$, as right modules over $A \bigotimes_k B$, to its composition with Ψ. In fact,

$$\text{(25.6.11)}\qquad \begin{aligned} &\widehat{\Psi} \text{ maps } \mathrm{Hom}_{A \bigotimes_k B}\big(V, \mathrm{Hom}_C(W, Y)\big) \\ &\text{onto } \mathrm{Hom}_{A \bigotimes_k B}\big(V, \mathrm{Hom}_C(W, Z)\big), \end{aligned}$$

because of (25.6.1) and (25.6.9).

One can verify that $\widetilde{\Psi}$ corresponds to $\widehat{\Psi}$ with respect to the isomorphisms between (25.5.2) and (25.5.3), and between (25.6.6) and (25.6.7). Thus the surjectivity of $\widetilde{\Psi}$ follows from (25.6.11), as desired.

Of course, a simpler version of this was discussed in Section 25.1.

25.7 A couple of corollaries

Let k be a commutative ring with a multiplicative identity element, and let A, B, and C be associative algebras over k, with multiplicative identity elements e_A, e_B, and e_C, respectively. Also let $A \bigotimes_k B$ and $B \bigotimes_k C$ be tensor products of A, B and B, C over k again, considered as associative algebras with multiplicative identity elements, as in Section 4.1.

Suppose that V is a module over k that is a right module over A and B, where the actions of A and B on V commute with each other. Equivalently, this means that V is a right module over $A \bigotimes_k B$, as in Section 4.2. If

$$V \text{ is projective as a right module over } A\bigotimes_k B \tag{25.7.1}$$

and

$$A \text{ is projective as a module over } k, \tag{25.7.2}$$

then

$$V \text{ is projective as a right module over } B. \tag{25.7.3}$$

This follows from the remarks in the previous section, with $C = k$, and $W = A$, considered as a left module over A and a right module over C. In this case, B satisfies the requirements of a tensor product of B and C over k, and V satisfies the requirements of a tensor product of V and W over A. More precisely, it is easy to see that multiplication on B is the same as multiplication on $B \bigotimes_k C$. Similarly, the given actions of B and k on V are the same as the actions of B and C on $V \bigotimes_A W$ considered previously.

This corresponds to Corollary 2.4 on p166 of [3], which is stated in terms of the opposite algebra A^{op} of A. Of course, A^{op} is the same as A as a module over k, so that (25.7.2) is the same as for A^{op}.

Let C be any associative algebra over k with a multiplicative identity element e_C again. Suppose now that V, W are modules over k, with V a right module over B, and W a right module over C. Let $V \bigotimes_k W$ be a tensor product of V and W over k, which may be considered as a right module over B and C, as in Section 1.10. One can check that the actions of B and C on $V \bigotimes_k W$ commute with each other, as before.

If (25.7.3) holds, and

$$W \text{ is projective as a right module over } C, \tag{25.7.4}$$

then

$$V\bigotimes_k W \text{ is projective as a right module over } B\bigotimes_k C. \tag{25.7.5}$$

This follows from the remarks in the previous section, with $A = k$. More precisely, B satisfies the requirements of $A \bigotimes_k B$, and multiplication on B is the same as multiplication on $A \bigotimes_k B$. Thus (25.6.1) reduces to (25.7.3). This corresponds to Corollary 2.5 on p166 of [3].

25.8 Some alternate approaches

We would like to consider some alternative arguments for the projectivity conditions discussed in the previous two sections, which are analogous to the first argument in Section 25.1. Let k be a commutative ring with a multiplicative identity element again, and let A, B, and C be associative algebras over k, with multiplicative identity elements e_A, e_B, and e_C, respectively. Also let $A \bigotimes_k B$ and $B \bigotimes_k C$ be tensor products of of A, B and B, C, respectively, considered as associative algebras over k with multiplicative identity elements, as before.

Suppose that

(25.8.1) $\qquad A$ is projective as a module over k.

This implies that

(25.8.2) $\qquad A \bigotimes_k B$ is projective as a right module over B,

as in Section 25.1. Let V be a module over k that is a right module over A and B, where the actions of A and B on V commute, so that V may be considered as a right module over $A \bigotimes_k B$. If V is free as a right module over $A \bigotimes_k B$, then

(25.8.3) $\qquad V$ is projective as a right module over B.

Indeed, V corresponds to the direct sum of copies of $A \bigotimes_k B$, and thus a direct sum of projective right modules over B.

Suppose now that

(25.8.4) $\qquad V$ is projective as a right module over $A \bigotimes_k B$.

This implies that there is a free right module over $A \bigotimes_k B$ that is isomorphic to the direct sum of V and another right module U over $A \bigotimes_k B$, as in Section 2.7. This means that the direct sum of V and U is projective as a right module over B, as in the preceding paragraph. It follows that (25.8.3) holds, as in Section 2.7 again.

Let V, W be modules over k, with V a right module over B, and W a right module over C, and let $V \bigotimes_k W$ be a tensor product of V and W over k. As in the previous section, $V \bigotimes_k W$ may be considered as a right module over B and C, where the actions of B and C on $V \bigotimes_k W$ commute, so that $V \bigotimes_k W$ may be considered as a right module over $B \bigotimes_k C$. If V and W are free as right modules over B and C, respectively, then $V \bigotimes_k W$ is free as a right module over $B \bigotimes_k C$.

Suppose that (25.8.4) holds, and that

(25.8.5) $\qquad W$ is projective as a right module over C.

Thus there are right modules U, Z over B, C, respectively, such that $V \bigoplus U$ and $W \bigoplus Z$ are free as right modules over B and C, respectively, as in Section 2.7. This implies that a tensor product

(25.8.6) $$(V \bigoplus U) \bigotimes_k (W \bigoplus Z)$$

is free as a right module over $B \bigotimes_k C$, as in the preceding paragraph. This tensor product is isomorphic to the direct sum of $V \bigotimes_k W$ and tensor products $U \bigotimes_k W$, $V \bigotimes_k Z$, and $U \bigotimes_k Z$, as a module over k, right modules over B and C, and hence over $B \bigotimes_k C$. This means that

$$V \bigotimes_k W \text{ is projective as a right module over } B \bigotimes_k C, \tag{25.8.7}$$

as in Section 2.7.

Let W be a module over k that is a left module over A and a right module over C, where the actions of A and C on W commute with each other. Let us consider $A \bigotimes_k B$ as a right module over A for the moment, and let

$$(A \bigotimes_k B) \bigotimes_A W \tag{25.8.8}$$

be a tensor product of $A \bigotimes_k B$ and W over A. This may be considered as a right module over B and C, where the actions of B and C commute, and thus as a right module over $B \bigotimes_k C$. We would like to check that

$$(25.8.8) \text{ is projective as a right module over } B \bigotimes_k C. \tag{25.8.9}$$

Although this may be considered as a particular case of (25.6.3), we would like to use this to give another argument for (25.6.3).

Remember that $A \bigotimes_k B$ is isomorphic to $B \bigotimes_k A$ in a natural way, as in Section 1.4. Thus (25.8.8) is isomorphic to

$$(B \bigotimes_k A) \bigotimes_A W \tag{25.8.10}$$

in a natural way. This is isomorphic to a tensor product of the form

$$B \bigotimes_k (A \bigotimes_A W) \tag{25.8.11}$$

in a natural way, as in Section 1.12. This reduces to a tensor product of the form $B \bigotimes_k W$, because W satisfies the requirements of $A \bigotimes_A W$. This means that (25.8.9) follows from (25.8.7), which is a bit simpler in this case.

Suppose now that V is a module over k that is a right module over A and B, where the actions of A and B on V commute, so that V may be considered as a right module over $A \bigotimes_k B$. If

$$V \text{ is projective as a right module over } A \bigotimes_k B, \tag{25.8.12}$$

and W is as in the previous two paragraphs, then

$$V \bigotimes_A W \text{ is projective as a right module over } B \bigotimes_k C, \tag{25.8.13}$$

as in Section 25.6. Alternatively, if V is free as a right module over $A \bigotimes_k B$, then $V \bigotimes_A W$ corresponds to the direct sum of modules of the form (25.8.8), and is thus projective as a right module over $B \bigotimes_k C$. Otherwise, there is a free right module over $A \bigotimes_k B$ that is isomorphic to the direct sum of V and another right module U over $A \bigotimes_k B$, as in Section 2.7. This implies that the direct sum of $V \bigotimes_A W$ and a tensor product $U \bigotimes_A W$ is projective as a right module over $B \bigotimes_k C$, so that (25.8.13) holds, as in Section 2.7 again.

25.9 Another result about injectivity

Let k be a commutative ring with a multiplicative identity element, and let A, B, and C be associative algebras over k, with multiplicative identity elements e_A, e_B, and e_C, respectively. As before, we let $A \bigotimes_k B$ and $B \bigotimes_k C$ be tensor products of A, B and B, C over k, respectively, considered as associative algebras over k with multiplicative identity elements, as in Section 4.1.

Let W and Z be modules over k. Suppose that

(25.9.1) W is a left module over A and a right module over C, where the actions of A and C on W commute with each other,

and that

$$\text{(25.9.2)} \qquad Z \text{ is a right module over } B \text{ and } C, \text{ where the actions of } B \text{ and } C \text{ on } Z \text{ commute with each other.}$$

In particular, $\mathrm{Hom}_C(W, Z)$ is defined as a module over k, and may be considered as a right module over A and B, as in Section 1.8. One can check that the actions of A and B on $\mathrm{Hom}_C(W, Z)$ commute with each other, as before. Remember that Z may be considered as a right module over $B \bigotimes_k C$, and that $\mathrm{Hom}_C(W, Z)$ may be considered as a right module over $A \bigotimes_k B$, as in Section 4.2.

Suppose that

$$\text{(25.9.3)} \qquad W \text{ is projective as a left module over } A,$$

and that

$$\text{(25.9.4)} \qquad Z \text{ is injective as a right module over } B \bigotimes_k C.$$

Under these conditions,

$$\text{(25.9.5)} \qquad \mathrm{Hom}_C(W, Z) \text{ is injective as a right module over } A \bigotimes_k B,$$

as in Proposition 2.3a on p166 of [3]. Simpler versions of this were discussed in Sections 25.2 and 25.3.

Let V be a module over k that is a right module over $A \bigotimes_k B$. Thus V may be considered as a right module over A and B, where the actions of A and B on V commute with each other. Let $V \bigotimes_A W$ be a tensor product of V and W over A, which is a module over k that may be considered as a right module over B and C. As before, the actions of B and C on $V \bigotimes_A W$ commute with each other, so that $V \bigotimes_A W$ may be considered as a right module over $B \bigotimes_k C$. Remember that there is a natural isomorphism

$$\text{(25.9.6)} \quad \text{from } \mathrm{Hom}_{A \bigotimes_k B}\big(V, \mathrm{Hom}_C(W, Z)\big) \text{ onto } \mathrm{Hom}_{B \bigotimes_k C}\big(\big(V \bigotimes_A W\big), Z\big),$$

as modules over k, as in Section 25.5.

Let V_0 be a submodule of V as a right module over $A \bigotimes_k B$, and thus as a module over A and B. Also let $V_0 \bigotimes_A W$ be a tensor product of V_0 and W over

A, which is another module over k that may be considered as a right module over B and C. The actions of B and C on $V_0 \bigotimes_A W$ commute with each other, so that $V \bigotimes_A W$ may be considered as a right module over $B \bigotimes_k C$. There is a natural isomorphism

$$\text{(25.9.7) from } \operatorname{Hom}_{A \bigotimes_k B}\big(V_0, \operatorname{Hom}_C(W, Z)\big) \text{ onto } \operatorname{Hom}_{B \bigotimes_k C}\big((V_0 \bigotimes_A W), Z\big),$$

as modules over k, as in Section 25.5 again.

There is a natural homomorphism

$$\text{(25.9.8)} \qquad \begin{aligned} &\text{from } \operatorname{Hom}_{A \bigotimes_k B}\big(V, \operatorname{Hom}_C(W, Z)\big) \\ &\text{into } \operatorname{Hom}_{A \bigotimes_k B}\big(V_0, \operatorname{Hom}_C(W, Z)\big), \end{aligned}$$

as modules over k, that sends a homomorphism from V into $\operatorname{Hom}_C(W, Z)$, as right modules over $A \bigotimes_k B$, to its restriction to V_0. We would like to show that this homomorphism is surjective.

Using the isomorphisms as in (25.9.6) and (25.9.7), the homomorphism as in (25.9.8) corresponds to a homomorphism

$$\text{(25.9.9)} \qquad \begin{aligned} &\text{from } \operatorname{Hom}_{B \bigotimes_k C}\big((V \bigotimes_A W), Z\big) \\ &\text{into } \operatorname{Hom}_{B \bigotimes_k C}\big((V_0 \bigotimes_A W), Z\big), \end{aligned}$$

as modules over k. It suffices to show that this homomorphism is surjective.

There is a natural homomorphism

$$\text{(25.9.10)} \qquad \text{from } V_0 \bigotimes_A W \text{ into } V \bigotimes_A W,$$

as modules over k, corresponding to the obvious inclusion mapping of V_0 into V and the identity mapping on W. It is easy to see that this is a homomorphism as in (25.9.10), as modules over B and C. This homomorphism is injective, because of (25.9.3), as discussed near the end of Section 2.7.

One can check that the homomorphism as in (25.9.9) sends a homomorphism from $V \bigotimes_A W$ into Z, as right modules over $B \bigotimes_k C$, to its composition with the homomorphism as in (25.9.10). It follows that the homomorphism as in (25.9.9) is surjective, because of (25.9.4), and the injectivity of the homomorphism as in (25.9.10).

25.10 Two corollaries about injectivity

Let k, A, B, C, $A \bigotimes_k B$, and $B \bigotimes_k C$ be as in the previous section. Also let Z be a module over k that is a right module over B and C, where the actions of B and C commute. Thus Z may be considered as a right module over $B \bigotimes_k C$, as in Section 4.2. If (25.9.4) holds, and

$$\text{(25.10.1)} \qquad C \text{ is projective as a module over } k,$$

then

(25.10.2) Z is injective as a module over B.

This follows from the remarks in the previous section, with $A = k$, and $W = C$, considered as a left module over A and a right module over C.

More precisely, this uses the fact that B satisfies the requirements of a tensor product of A and B over k in this case, where multiplication on B is the same as muldiplication on $A \bigotimes_k B$. This also uses the usual identification of $\mathrm{Hom}_C(C, Z)$ with Z. Note that Z is the same as $\mathrm{Hom}_C(C, Z)$ as modules over k, and right modules over B, with respect to this identification. This corresponds to Corollary 2.4a on p166 of [3].

Let A be any associative algebra over k with a multiplicative identity element e_A again, and let W, Z be modules over k, with W a left module over A, and Z a right module over V. If (25.9.3) and (25.10.2) hold, then

$$\mathrm{Hom}_k(W, Z) \text{ is injective as a right module over } A \bigotimes_k B, \tag{25.10.3}$$

as in Corollary 2.5a on p166 of [3]. This follows from the remarks in the previous section, with $C = k$. In this case, B satisfies the requirements of a tensor product of B and C over k, and multiplication on B is the same as multiplication on $B \bigotimes_k C$. This means that (25.9.4) redcues to (25.10.2).

25.11 Augmentations and tensor products

Let k be a commutative ring with a multiplicative identity element, and let A, C be associative algebras over k, with multiplicative identity elements e_A, e_C, respectively. Suppose that A has a left or right augmentation, with augmentation module Q_A and augmentation homomorphism ε_A. More precisely, we take Q_A to be a module over k that is a left or right module over A, as appropriate, and ε_A to be a homomorphism from A onto Q_A, as modules over k, and left or right modules over A, as appropriate.

Let $C \bigotimes_k A$ be a tensor product of C and A over k, considered as an associative algebra over k, as in Section 4.1. Also let

$$Q_{C \bigotimes_k A} = C \bigotimes_k Q_A \tag{25.11.1}$$

be a tensor product of C and Q_A over k, which may be considered initially as a module over k. Using the action of A on Q_A, we get an action of A on $C \bigotimes_k Q_A$, so that $C \bigotimes_k Q_A$ may be considered as a module over A on the left or right, as appropriate, as in Section 1.10. Similarly, $C \bigotimes_k Q_A$ may be considered as a module over C on the left and right, using the actions of C on itself on the left and right. Note that these actions of C and A on $C \bigotimes_k Q_A$ commute with each other.

Using ε_A and the identity mapping on C, we get a homomorphism

$$\varepsilon_{C \bigotimes_k A} \text{ from } C \bigotimes_k A \text{ onto } C \bigotimes_k Q_A, \tag{25.11.2}$$

as modules over k. This may be considered as a homomorphism as in (25.11.2), as left and right modules over C, and left or right modules over A, as appropriate. We may consider (25.11.1) as a left or right module over $C \bigotimes_k A$, as appropriate, as in Section 4.2. It is easy to see that (25.11.2) is a homomorphism between left or right modules over $C \bigotimes_k A$, as appropriate, where $C \bigotimes_k A$ is considered as a left and right module over itself in the usual way. Thus

(25.11.3) (25.11.1) and (25.11.2) define a left or right augmentation on $C \bigotimes_k A$, as appropriate,

as on p163 of [3].

Observe that

$$\phi(a) = e_C \otimes a \tag{25.11.4}$$

defines a homomorphism ϕ from A into $C \bigotimes_k A$, as algebras over k. Similarly,

$$\psi(x) = e_C \otimes x \tag{25.11.5}$$

defines a homomorphism from Q_A into (25.11.1), as modules over k, and left or right modules over A, as appropriate. It is easy to see that

$$\varepsilon_{C \bigotimes_k A} \circ \phi = \psi \circ \varepsilon_A, \tag{25.11.6}$$

as on p163f of [3]. In particular, this implies that

$$\phi(\ker \varepsilon_A) \subseteq \ker \varepsilon_{C \bigotimes_k A}. \tag{25.11.7}$$

This means that ϕ is a homomorphism from A into $C \bigotimes_k A$ as augmented rings, as in Section 24.9.

Remember that $C \bigotimes_k A$ may be considered as a left and right module over A, using ϕ, as in Section 2.9. In this case, the actions of A on $C \bigotimes_k A$ on the left and right using ϕ are the same as the usual actions obtained from the actions of A on itself on the left and right. Similarly, (25.11.1) may be considered as a left or right module over A, as appropriate, using ϕ and the left or right action of $C \bigotimes_k A$, as appropriate. This action of A is the same as the one defined directly, as before.

Suppose that Q_A is a left module over A, and let

$$_{(\phi)}Q_A = \big(C \bigotimes\nolimits_k A\big) \bigotimes\nolimits_A Q_A \tag{25.11.8}$$

be a tensor product of $C \bigotimes_k A$ and Q_A, as modules over A. This may be considered as a left module over $C \bigotimes_k A$, which is the covariant ϕ-extension of Q_A, as in Section 2.9. There is a natural isomorphism between (25.11.8) and

$$C \bigotimes\nolimits_k \big(A \bigotimes\nolimits_A Q_A\big), \tag{25.11.9}$$

as in Section 1.12. This reduces to (25.11.1), because Q_A satisfies the requirements of $A \bigotimes_A Q_A$. This corresponds to the condition (i) on p164 of [3].

Similarly, let X_A be any left module over A, and let

$$(25.11.10) \qquad {}_{(\phi)}X_A = \Big(C \bigotimes_k A\Big) \bigotimes_A X_A$$

be a tensor product of $C \bigotimes_k A$ and X_A, as modules over A. This may be considered as a left module over $C \bigotimes_k A$, which is the covariant ϕ-extension of X_A. There is a natural isomorphism between this and

$$(25.11.11) \qquad C \bigotimes_k \Big(A \bigotimes_A X_A\Big),$$

as before. This reduces to $C \bigotimes_k X_A$, because X_A satisfies the requirements of $A \bigotimes_A X_A$. Of course, there are analogous statements for right augmentations.

25.12 Homomorphisms induced by ϕ

Let us continue with the same notations and hypotheses as in the previous section, including that Q_A be a left module over A. Let V be a module over k that is a right module over $C \bigotimes_k A$. Equivalently, this means that V is a right module over A and C, where the actions of A and C on V commute with each other. This action of A on V is the same as the one obtained from the action of $C \bigotimes_k A$ on V using ϕ, as in Section 2.9. Using ϕ, we get a homomorphism

(25.12.1) from the homology of A into the homology of $C \bigotimes_k A$,
as left augmented rings, with coefficients in V,

as in Section 24.9, and on p164 of [3].

Similarly, let Z be a module over k that is a left module over $C \bigotimes_k A$. This means that Z is a left module over A and C, where the actions of A and C on Z commute with each other, as usual. The action of A on Z is the same as the one obtained from the action of $C \bigotimes_k A$ on Z using ϕ, as before. Using ϕ, we get a homomorphism

(25.12.2) from the cohomology of $C \bigotimes_k A$ into the cohomology of A,
as left augmented rings, with coefficients in Z,

as in Section 24.9 again. This is mentioned on p164 of [3] too.

One may be interested in having the homomorphisms as in (25.12.1) and (25.12.2) be isomorphisms, as in Proposition 1.1 on p164 of [3]. This holds under two additional conditions, as in Section 24.10. These conditions are (24.10.5) and (24.10.8), in the earlier notation, which correspond to (i) and (ii) on p164 of [3]. Note that the augmented ring B in Section 24.10 corresponds to $C \bigotimes_k A$ here. The first condition holds automatically here, as in [3], because of the isomorphism between (25.11.8) and (25.11.1) discussed in the previous section.

If n is a nonnegative integer, then consider

(25.12.3) the nth homology group of A, as a left augmented ring
with coefficients in $C \bigotimes_k A$, as a right module over A,

as in Section 24.3. The second condition (24.10.8) mentioned in the preceding paragraph is that

$$(25.12.3) \text{ is equal to } \{0\} \tag{25.12.4}$$

when $n > 0$.

Let X_A be a projective resolution of Q_A, as a left module over A, as in Section 10.2. A tensor product

$$(C \bigotimes_k A) \bigotimes_A X_A \tag{25.12.5}$$

of $C \bigotimes_k A$, as a right module over A, and X_A over A, may be considered as a complex over k, as in Section 7.5. Remember that (25.12.3) is given by

$$H\Big((C \bigotimes_k A) \bigotimes_A X_A\Big)_n \tag{25.12.6}$$

for each $n \geq 0$, as in Section 24.3. Thus (25.12.4) is the same as saying that

$$(25.12.6) \text{ is equal to } \{0\} \tag{25.12.7}$$

when $n > 0$.

As in the previous section, (25.12.5) is isomorphic to a tensor product $C \bigotimes_k X_A$, as a module over k. We may consider $C \bigotimes_k X_A$ as a complex over k, as before, and the isomorphism just mentioned is in fact an isomorphism between complexes. Thus (25.12.6) is isomorphic to

$$H\big(C \bigotimes_k X_A\big)_n \tag{25.12.8}$$

for each $n \geq 0$. This means that (25.12.4) and (25.12.7) are the same as saying that

$$(25.12.8) \text{ is equal to } \{0\} \tag{25.12.9}$$

when $n > 0$.

If (25.12.9) holds for every $n > 0$, then the homomorphisms as in (25.12.1) and (25.12.2) are isomorphisms, as before. In this case, (25.12.5) is a projective resolution of (25.11.1), as a left module over $C \bigotimes_k A$, as in Section 24.10. This means that

$$C \bigotimes_k X_A \text{ is a projective resolution of (25.11.1),} \tag{25.12.10}$$

as a left module over $C \bigotimes_k A$. More precisely, this uses the fact that (25.12.5) is isomorphic to $C \bigotimes_k X_A$ as left modules over A and C, and thus over $C \bigotimes_k A$. This corresponds to another part of Proposition 1.1 on p164 of [3].

Suppose now that

$$A \text{ is projective as a module over } k. \tag{25.12.11}$$

This implies that projective modules over A are projective as modules over k, as in Section 2.12. Alternatively, a module over A may be identified with its tensor product with A, over A. This permits one to get the projectivity over k from projectivity of tensor products, as in Section 25.1.

If (25.12.11) holds, then it follows that

(25.12.12) X_A is a projective resolution of Q_A, as a module over k,

as mentioned on p164 of [3]. This implies that (25.12.8) does not depend on the chioce of X_A, up to isomorphism, as in Section 10.12. In particular, this means that the condition (25.12.9) does not depend on the choice of X_A.

More precisely, let Y_A be any projective resolution of Q_A, as a module over k. A tensor product $C \bigotimes_k Y_A$ of C and Y_A over k may be considered as a complex over k, as before, so that

$$H\big(C \bigotimes_k Y_A\big)_n \tag{25.12.13}$$

is defined for each n. If (25.12.11) holds, and thus (25.12.12) holds, then

(25.12.14) (25.12.8) is isomorphic to (25.12.13)

for each n, as in Section 10.12. This means that (25.12.9) is the same as saying that

(25.12.15) (25.12.13) is equal to $\{0\}$

when $n > 0$.

Suppose now that

(25.12.16) Q_A is projective as a module over k.

This permits us to take $(Y_A)_0 = Q_A$, $(Y_A)_j = \{0\}$ when $j \neq 0$, as in Section 10.2. This means that

$$\big(C \bigotimes_k Y_A\big)_0 = C \bigotimes_k Q_A \text{ and } \big(C \bigotimes_k Y_A\big)_j = \{0\} \text{ when } j \neq 0. \tag{25.12.17}$$

In particular, (25.12.15) holds for every $n > 0$ under these conditions.

25.13 Some remarks about matrices

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let n be a positive integer, and let A^n be the space of n-tuples of elements of A. This may be considered as a module over k with respect to coordinatewise addition and scalar multiplication, and as a left and right module over A, where A act coordinatewise on the left and on the right.

Similarly, let $M_n(A)$ be the space of $n \times n$ matrices with entries in A. This may be considered as a module over k, with respect to entrywise addition and scalar multiplication, and as an associative algebra over k, with respect to matrix multiplication. The multiplicative identity element in $M_n(A)$ is the usual identity matrix I_n, whose entries are equal to e_A along the diagonal, and are 0 otherwise.

Remember that the algebra of homomorphisms from A^n into itself, as a right module over A, can be identified with $M_n(A)$ in a standard way, as in Section 4.7. We may consider A^n as a left module over $M_n(A)$, using this action of $M_n(A)$ on A^n.

It is easy to see that

(25.13.1) A^n is projective, as a left module over $M_n(A)$.

Indeed, $M_n(A)$ is projective as a left module over itself. One can check that $M_n(A)$ is isomorphic to the direct sum of n copies of A^n, as a left module over $M_n(A)$. This implies (25.13.1), as in Section 2.7.

25.14 Semigroups and tensor products

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Also let Σ be a semigroup, with the semigroup operation expressed multiplicatively, and with an identity element e_Σ. Thus we get semigroup algebras $k(\Sigma)$ and $A(\Sigma)$ of Σ with coefficients in k and A, respectively, as in Section 4.9.

There is an obvious mapping

(25.14.1) from $A \times k(\Sigma)$ into $A(\Sigma)$,

defined by multiplying an element of A by an element of $k(\Sigma)$ to get an element of $A(\Sigma)$, using scalar multiplication on A. This mapping is bilinear over k, and it is easy to see that $A(\Sigma)$ satisfies the requirements of a tensor product

(25.14.2) $$A \bigotimes_k k(\Sigma)$$

of A and $k(\Sigma)$ over k, as a module over k, with respect to this mapping. One can check that multiplication on (25.14.2), as in Section 4.1, corresponds to multiplication in $A(\Sigma)$. This corresponds to a remark on p188 of [3].

Now let Σ_1, Σ_2 be semigroups, with the semigroup operations expressed multiplicative, and with identity elements $e_1 = e_{\Sigma_1}$, $e_2 = e_{\Sigma_2}$, respectively. The product

(25.14.3) $$\Sigma_1 \times \Sigma_2$$

of Σ_1 and Σ_2 is a semigroup as well, where the semigroup operation is defined coordinatewise, and with identity element (e_1, e_2). Consider the semigroup algebras $k(\Sigma_1)$, $k(\Sigma_2)$, and $k(\Sigma_1 \times \Sigma_2)$ of these semigroups, with coefficients in k.

There is an obvious mapping

(25.14.4) from $k(\Sigma_1) \times k(\Sigma_2)$ into $k(\Sigma_1 \times \Sigma_2)$, which is bilinear over k, and is basically the identity mapping on (25.14.3). One can check that $k(\Sigma_1 \times \Sigma_2)$ satisfies the requirements of a tensor product

$$k(\Sigma_1) \bigotimes_k k(\Sigma_2) \tag{25.14.5}$$

of $k(\Sigma_1)$ and $k(\Sigma_2)$ over k, as a module over k, with respect to this mapping. More precisely, one can verify that multiplication on (25.14.5), as in Section 4.1, corresponds to multiplication in $k(\Sigma_1 \times \Sigma_2)$.

Let V be a module over k. Suppose that V is a left module over Σ_1 and Σ_2, as in Section 4.8, where the actions of Σ_1 and Σ_2 commute with each other. Under these conditions, the actions of Σ_1 and Σ_2 on V can be combined to get an action of (25.14.3) on V on the left. Conversely, if V is a left module over (25.14.3), then V may be considered as a left module over Σ_1 and Σ_2, where the actions of Σ_1 and Σ_2 on V commute with each other. Of course, this uses the obvious embeddings of Σ_1 and Σ_2 into (25.14.3). Similarly, if V is a right module over Σ_1 and Σ_2, where the actions of Σ_1 and Σ_2 commute with each other, then V may be considered as a right module over (25.14.3). Conversely, if V is a right module over (25.14.3), then V may be considered as a right module over Σ_1 and Σ_2, where the actions of Σ_1 and Σ_2 commute with each other.

Chapter 26

A family of augmentations

26.1 The enveloping algebra

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Remember that the corresponding opposite algebra A^{op} is the same as A as a module over k, but with multiplication in the opposite order, as in Section 1.11. If $x \in A$, then we may use x^{op} to indicate that x is being considered as an element of A^{op}, as before.

Let

$$A^{en} = A \bigotimes_k A^{op} \tag{26.1.1}$$

be a tensor product of A and A^{op}, as modules over k. We can define multiplication on A^{en}, in such a way that

$$(a_1 \otimes a_2^{op})\,(b_1 \otimes b_2^{op}) = (a_1\,b_1) \otimes (a_2^{op}\,b_2^{op}) = (a_1\,b_1) \otimes ((b_2\,a_2)^{op}) \tag{26.1.2}$$

for every $a_1, b_1 \in A$ and $a_2^{op}, b_2^{op} \in A^{op}$, as in Section 4.1. This makes A^{en} an associative algebra, with multilicative identity element $e_A \otimes e_A^{op}$, as before. This is called the *enveloping algebra* of A, as on p167 of [3].

Let us say that a module V over k is a *two-sided module* over A if V is both a left and right module over A, and the actions of A on V on the left and right commute with each other. This means that

$$a_1 \cdot (v \cdot a_2) = (a_1 \cdot v) \cdot a_2 \tag{26.1.3}$$

for every $a_1, a_2 \in A$ and $v \in V$. Using the action of A on V on the right, we may consider V as a left module over A^{op}, as in Section 1.11. We may also consider V as a left module over A^{en}, with

$$(a_1 \otimes a_2^{op}) \cdot v = a_1 \cdot (v \cdot a_2) = (a_1 \cdot v) \cdot a_2 \tag{26.1.4}$$

for every $a_1, a_2 \in A$ and $v \in V$, as in Section 4.2. This corresponds to a remark on p167 of [3].

Similarly, we may consider V as a right module over A^{op}, using the action of A on V on the left. We may consider V as a right module over A^{en} too, with

$$v \cdot (a_1 \otimes a_2^{op}) = a_2 \cdot (v \cdot a_1) = (a_2 \cdot v) \cdot a_1 \tag{26.1.5}$$

for every $a_1, a_2 \in A$ and $v \in V$, as in Section 4.2 again. This corresponds to another remark on p167f of [3].

In particular, A may be considered as a two-sided module over itself. Thus we may consider A as a left module over A^{en}, with

$$(a_1 \otimes a_2^{op}) \cdot a = a_1\, a\, a_2 \tag{26.1.6}$$

for every $a_1, a_2, a \in A$, as on p168 of [3].

Note that $(a_1, a_2^{op}) \mapsto a_1\, a_2$ is bilinear over k, as a map from $A \times A^{op}$ into A. This implies that there is a unique homomorphism

$$\rho \text{ from } A^{en} \text{ into } A, \tag{26.1.7}$$

as modules over k, such that

$$\rho(a_1 \otimes a_2^{op}) = a_1\, a_2 \tag{26.1.8}$$

for every $a_1, a_2 \in A$. Equivalently,

$$\rho(x) = x \cdot e_A \tag{26.1.9}$$

for every $x \in A^{en}$, as on p168 of [3].

More precisely, ρ is a homomorphism from A^{en} into A, as left modules over A^{en}. It is easy to see that

$$\rho(A^{en}) = A. \tag{26.1.10}$$

This means that

$$A^{en} \text{ is a left augmented ring,} \tag{26.1.11}$$

as in Section 24.1, with augmentation module A, as a left module over A^{en}, and augmentation homomorphism ρ, as on p168 of [3].

26.2 The augmentation ideal J

Let us continue with the same notation and hypotheses as in the previous section, and let

$$J = \ker \rho \tag{26.2.1}$$

be the augmentation ideal of A^{en}, as an augmented ring, as on p168 of [3]. This is a left ideal in A^{en}, as an associative algebra over k.

If $a \in A$, then

$$\rho(a \otimes e_A^{op} - e_A \otimes a^{op}) = a\, e_A - e_A\, a = 0, \tag{26.2.2}$$

so that

$$a \otimes e_A^{op} - e_A \otimes a^{op} \in J. \tag{26.2.3}$$

Because J is a left module over A^{en}, J may be considered as a two-sided module over A. Proposition 3.1 on p168 of [3] states that

$$\begin{aligned}&J \text{ is generated by elements as in (26.2.3), with } a \in A,\\ &\text{as a left module over } A.\end{aligned} \tag{26.2.4}$$

To see this, let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be elements of A, with

$$\sum_{l=1}^n a_l \otimes b_l^{op} \in J. \tag{26.2.5}$$

This means that

$$0 = \rho\Big(\sum_{l=1}^n a_l \otimes b_l^{op}\Big) = \sum_{l=1}^n a_l\, b_l. \tag{26.2.6}$$

Using this, we get that

$$\begin{aligned}\sum_{l=1}^n a_l \otimes b_l^{op} &= \sum_{l=1}^n (a_l \otimes e_A^{op})\,(e_A \otimes b_l^{op})\\ &= \sum_{l=1}^n (a_l \otimes e_A^{op})\,(e_A \otimes b_l^{op} - b_l \otimes e_A^{op}).\end{aligned} \tag{26.2.7}$$

This shows that $\sum_{l=1}^n a_l \otimes b_l^{op}$ is in the submodule of J, as a left module over A, generated by elements as in (26.2.3), with $a \in A$.

If $a \in A$, then put

$$j(a) = a \otimes e_A^{op} - e_A \otimes a^{op}. \tag{26.2.8}$$

This defines j as a homomorphism from A into J, as modules over k. If $a, b \in A$, then

$$\begin{aligned}j(a\,b) &= (a\,b) \otimes e_A^{op} - e_A \otimes (a\,b)^{op} = (a\,b) \otimes e_A^{op} - e_A \otimes b^{op}\,a^{op}\\ &= (a \otimes e_A^{op})\,(b \otimes e_A^{op}) - (e_a \otimes b^{op})\,(e_A \otimes a^{op})\\ &= (a \otimes e_A^{op})\,j(b) + (e_A \otimes b^{op})\,j(a).\end{aligned} \tag{26.2.9}$$

Equivalently, this means that

$$j(a\,b) = a \cdot j(b) + j(a) \cdot b, \tag{26.2.10}$$

when J is considered as a two-sided module over A, as on p168 of [3].

26.3 Crossed homomorphisms

Let us continue with the same notation and hypotheses as in the previous two sections, and let V be a module over k that is a two-sided module over A. A homomorphism f from A into V, as modules over k, is said to be a *crossed homomorphism* or *derivation* if

(26.3.1) $$f(a\,b) = a \cdot f(b) + f(a) \cdot b$$

for every $a, b \in A$, as on p168 of [3]. This implies that

(26.3.2) $$f(e_A) = 0,$$

as in [3]. Note that the set of all crossed homomorphisms from A into V is a submodule of the space $\mathrm{Hom}_k(A, V)$ of all homomorphisms from A into V, as modules over k.

Let U be another module over k that is a two-sided module over A. A homomorphism h from U into V, as modules over k, is considered to be a *homomorphism* from U into V as two-sided modules over A, if h is a homomorphism from U into V, as both left and right modules over A. This is the same as saying that h is a homomorphism from U into V, as left modules over A^{en}. The space of these homomorphisms may be denoted $\mathrm{Hom}_{A^{en}}(U, V)$, as usual.

If g is a crossed homomorphism from A into U, and h is a homomorphism from U into V, as modules over k and two-sided modules over A, then it is easy to see that

(26.3.3) $h \circ g$ is a crossed homomorphism from A into V.

Remember that J may be considered as a two-sided module over A, as in the previous section. The mapping j from A into J defined in (26.2.8) is a crossed homomorphism, as in (26.2.10). If h is a homomorphism from J into V, as modules over k and two-sided modules over A, then $h \circ j$ is a crossed homomorphism from A into V. This defines a homomorphism

(26.3.4) from $\mathrm{Hom}_{A^{en}}(J, V)$ into the space
of crossed homomorphisms from A into V,

as modules over k.

Proposition 3.2 on p168 of [3] states that this homomorphism is an isomorphism. The injectivity of this homomorphism can be obtained from (26.2.4).

Let f be a crossed homomorphism from A into V. We would like to find a homomorphism h from J into V, as modules over k and two-sided modules over A, such that

(26.3.5) $$f = h \circ j.$$

Observe that

(26.3.6) $$(a, b^{op}) \mapsto -a \cdot f(b)$$

defines a mapping from $A \times A^{op}$ into V that is bilinear over k. This leads to a unique homomorphism

$$\text{from } A^{en} = A \bigotimes_k A^{op} \text{ into } V, \tag{26.3.7}$$

as modules over k, such that

$$a \otimes b^{op} \mapsto -a \cdot f(b) \tag{26.3.8}$$

for every $a, b \in A$. Let h be the restriction of the homomorphism as in (26.3.7) to J, which is a homomorphism from J into V, as modules over k. If $a \in A$, then

$$h(j(a)) = h(a \otimes e_A^{op} - e_A \otimes a^{op}) = -a \cdot f(e_A) + e_A \cdot f(a) = f(a), \tag{26.3.9}$$

because of (26.3.2). This definition of h was motivated by (26.2.7), as mentioned on p168 of [3].

It is easy to see that the homomorphism as in (26.3.7) is a homomorphism

$$\text{from } A^{en} \text{ into } V, \text{ as left modules over } A. \tag{26.3.10}$$

In particular, h is a homomorphism from J into V, as left modules over A.

We would like to show that h is also a homomorphism from J into V, as right modules over A. Let $x \in A^{en}$ be given, so that x can be expressed as

$$x = \sum_{l=1}^{n} a_l \otimes b_l^{op}, \tag{26.3.11}$$

where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are elements of A. If $c \in A$, then

$$\begin{aligned} x \cdot c = (e_A \otimes c^{op})\, x &= (e_A \otimes c^{op}) \sum_{l=1}^{n} a_l \otimes b_l^{op} \\ &= \sum_{l=1}^{n} a_l \otimes (c^{op}\, b_l^{op}) = \sum_{l=1}^{n} a_l \otimes (b_l\, c)^{op}. \end{aligned} \tag{26.3.12}$$

Thus

$$\begin{aligned} h(x \cdot c) &= h((e_A \otimes c^{op})\, x) = -\sum_{l=1}^{n} a_l \cdot f(b_l\, c) \\ &= -\sum_{l=1}^{n} a_l \cdot (b_l \cdot f(c)) - \sum_{l=1}^{n} a_l \cdot (f(b_l) \cdot c) \\ &= -\sum_{l=1}^{n} (a_l\, b_l) \cdot f(c) - \sum_{l=1}^{n} (a_l \cdot f(b_l)) \cdot c \\ &= -\Big(\sum_{l=1}^{n} a_l\, b_l\Big) \cdot f(c) + h(x) \cdot c. \end{aligned} \tag{26.3.13}$$

If $x \in J$, then

$$0 = \rho(x) = \sum_{l=1}^{n} a_l \, b_l, \tag{26.3.14}$$

and we get that

$$h(x \cdot c) = h(x) \cdot c, \tag{26.3.15}$$

as desired.

26.4 Principal crossed homomorphisms

Let us continue with the same notation and hypotheses as in the previous three sections. If $v \in V$, then

$$x \mapsto x \cdot v \tag{26.4.1}$$

defines a homomorphism from A^{en} into V, as modules over k, and left modules over A^{en}. This is the unique homomorphism from A^{en} into V, as modules over k, such that

$$a \otimes b^{op} \mapsto (a \otimes b^{op}) \cdot v = a \cdot (v \cdot b) \tag{26.4.2}$$

for every $a, b \in A$. Let h_v be the restriction of this homomorphism to J, which is a homomorphism from J into V, as modules over k, and left modules over A^{en}. Note that

$$v \mapsto h_v \tag{26.4.3}$$

is linear over k, as a map from V into the space $\mathrm{Hom}_{A^{en}}(J, V)$ of homomorphisms from A^{en} into V, as left modules over A^{en}.

As in the previous section,

$$f_v = h_v \circ j \tag{26.4.4}$$

is a crossed homomorphism from A into V. If $a \in A$, then

$$\begin{aligned} f_v(a) = h_v(j(a)) &= h_v(a \otimes e_A^{op} - e_A \otimes a^{op}) \\ &= a \cdot (v \cdot e_A) - e_A \cdot (v \cdot a) = a \cdot v - v \cdot a. \end{aligned} \tag{26.4.5}$$

A crossed homomorphism of this form is said to be *principal*, or an *inner derivation*, as on p169 of [3]. Of course,

$$v \mapsto f_v \tag{26.4.6}$$

defines a homomorphism from V into the space of crossed homomorphisms from A into V, as modules over k. This homomorphism maps V onto the set of principal crossed homomorphisms from A into V, which is a submodule of the space of crossed homomorphisms from A into V, as a module over k.

Remember that $e_A \otimes e_A^{op}$ is the multiplicative identity element of A^{en}. Every homomorphism from A^{en} into V, as left modules over A^{en}, is of the form (26.4.1) for a unique $v \in V$, which is the image of $e_A \otimes e_A^{op}$ under the homomorphism. It follows that a crossed homomorphism f from A into V is principal exactly when it can be expressed as $h \circ j$, where h is the restriction to J of a homomorphism from A^{en} into V, as left modules over A^{en}.

The space $\mathrm{Hom}_{A^{en}}(A^{en}, V)$ of all homomorphisms from A^{en} into V, as left modules over A^{en}, is isomorphic to V, as modules over k, in a natural way, as in the preceding paragraph. There is a natural homomorphism

$$\text{from } \mathrm{Hom}_{A^{en}}(A^{en}, V) \text{ into } \mathrm{Hom}_{A^{en}}(J, V), \tag{26.4.7}$$

as modules over k, which sends a homomorphism from A^{en} into V, as left modules over A^{en}, to its restriction to J. Remember that

$$h \mapsto h \circ j \tag{26.4.8}$$

defines an isomorphism from $\mathrm{Hom}_{A^{en}}(J, V)$ onto the space of crossed homomorphisms from A into V, as modules over k, as in the previous section. Using this isomorphism, the space of principal crossed homomorphisms from A into V corresponds to the image of $\mathrm{Hom}_{A^{en}}(A^{en}, V)$ under the homomorphism as in (26.4.7), as on p169 of [3].

26.5 Homology and cohomology of A

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k, with a multiplicative identity element e_A. Using A, we get the enveloping algebra $A^{en} = A \bigotimes_k A^{op}$ of A, as in Section 26.1. Let ρ be the homomorphism from A^{en} onto A, as left modules over A^{en}, defined in Section 26.1. This makes A^{en} a left augmented ring, with augmentation ideal $J = \ker \rho$, as before.

Let V be a two-sided module over A. Remember that we may consider V as a left or right module over A^{en}, as in Section 26.1.

Let us consider V as a right module over A^{en} for the moment. If n is a nonnegative integer, then the *nth homology group*

$$H_n(A, V) \tag{26.5.1}$$

of A with coefficients in V may be defined as in Section 24.3, with respect to A^{en} as a left augmented ring. More precisely, this is a module over k for each n.

Let us now consider V as a left module over A^{en}. If n is a nonnegative integer, then the *nth cohomology group*

$$H^n(A, V) \tag{26.5.2}$$

of A with coefficients in V may be defined as in Section 24.5, with respect to A^{en} as a left augmented ring. This is also a module over k for each n.

This follows the definitions of homology and cohomology of associative algebras on p169 of [3]. If k is a field, then these cohomology groups agree with those of Hochschild, as in [3].

If we consider V as a right module over A^{en}, then

$$H_0(A, V) = V \bigotimes_{A^{en}} A, \tag{26.5.3}$$

where A is considered as a left module over A^{en}, as in Section 24.3. If V is considered as a left module over A^{en}, then

$$H^0(A, V) = \mathrm{Hom}_{A^{en}}(A, V), \tag{26.5.4}$$

where A is considered as a left module over A^{en}, as in Section 24.5. These modules will be discussed further in the next section.

Let X be a projective resolution of A as a left module over A^{en}, as in Section 10.2. If we consider V as a right module over A^{en} again, then

$$V \bigotimes_{A^{en}} X \tag{26.5.5}$$

may be considered as a complex over k, as in Sections 7.5 and 24.3. The homology of this complex can be used to obtain (26.5.1), as in Section 24.3. Namely,

$$H_n(A, V) = H\Big(V \bigotimes_{A^{en}} X\Big)_n = H\Big(V \bigotimes_{A^{en}} X\Big)^{-n} \tag{26.5.6}$$

for each $n \geq 0$, as on p169 of [3].

Similarly, if we consider V as a left module over A^{en}, then

$$\mathrm{Hom}^{gr}_{A^{en}}(X, V) \tag{26.5.7}$$

may be defined as a complex over k, as in Sections 8.4 and 24.5. The homology of this complex can be used to obtain (26.5.2), as in Section 24.5. Namely,

$$H^n(A, V) = H\Big(\mathrm{Hom}^{gr}_{A^{en}}(X, V)\Big)^n \tag{26.5.8}$$

for each $n \geq 0$, as on p169 of [3].

26.6 Some basic cases

Let us continue with the same notation and hypotheses as in the previous section. We would like to consider some basic cases of the homology and cohomology groups.

Remember that J is a left ideal in A^{en}, and thus a left module over A^{en} in particular. Let us consider V as a right module over A^{en} again, and let $V \bigotimes_{A^{en}} J$ be a tensor product of V and J over A^{en}. Using the mapping

$$(v, x) \mapsto v \cdot x \tag{26.6.1}$$

from $V \times J$ into V, we get a unique homomorphism

$$\text{from } V \bigotimes_{A^{en}} J \text{ into } V, \tag{26.6.2}$$

as modules over k, with

$$v \otimes x \mapsto v \cdot x \tag{26.6.3}$$

for every $v \in V$ and $x \in J$, as usual. Let $V \cdot J$ be the image of the homomorphism as in (26.6.2), which is a submodule of V, as a module over k. Equivalently, $V \cdot J$ consists of finite sums of elements of the form $v \cdot x$, with $v \in V$ and $x \in J$.

If $a \in A$, then $j(a) = a \otimes e_A^{op} - e_A \otimes a^{op} \in J$, and

$$v \cdot j(a) = v \cdot a - a \cdot v \tag{26.6.4}$$

is an element of $V \cdot J$ for every $v \in V$. This uses the actions of A on V on the left and right on the right side. In fact,

(26.6.5) every element of $V \cdot J$ can be expressed as a finite sum of elements of the form $v \cdot a - a \cdot v$, with $a \in A, v \in V$,

because of (26.2.4). This is mentioned just after (3a) on p170 of [3].

Because $V \cdot J$ is a submodule of V, as a module over k, the quotient

$$V/(V \cdot J) \tag{26.6.6}$$

can be defined as a module over k. We also have that (26.5.3) is isomorphic to (26.6.6), as a module over k, in a natural way, as in Sections 24.1 and 24.2. This corresponds to (2) and the first part of Proposition 4.1 on p170 of [3].

Remember that homomorphisms from A into V, as left modules over A^{en}, correspond to homomorphisms from A^{en} into V, as left modules over A^{en}, whose kernels contain J. If $v \in V$, then $x \mapsto x \cdot v$ is a homomorphism from A^{en} into V, as left modules over A^{en}, and every such homomorphism is of this form. Thus (26.5.4) is isomorphic to

$$\{v \in V : x \cdot v = 0 \text{ for every } x \in J\}, \tag{26.6.7}$$

as modules over k, in a natural way. This corresponds to (2a) on p170 of [3].

It is easy to see that (26.6.7) is equal to

$$\{v \in V : j(a) \cdot v = 0 \text{ for every } a \in A\}, \tag{26.6.8}$$

using (26.2.4). If $a \in A$ and $v \in V$, then

$$j(a) \cdot v = (a \otimes e_A^{op} - e_A \otimes a^{op}) \cdot v = a \cdot v - v \cdot a, \tag{26.6.9}$$

using the actions of A on V on the left and right on the right side. This means that (26.6.7) is equal to

$$\{v \in V : a \cdot v = v \cdot a \text{ for every } a \in A\}, \tag{26.6.10}$$

as in the second part of Proposition 4.1 on p170 of [3]. The elements of (26.6.10) are said to be *invariant elements* of V, as in [3].

Consider the natural homomorphism

$$\text{from } \mathrm{Hom}_{A^{en}}(A^{en}, V) \text{ into } \mathrm{Hom}_{A^{en}}(J, V), \tag{26.6.11}$$

as modules over k, that sends a homomorphism from A^{en} into V, as left modules over A^{en}, to its restriction to J. We have that

$$\begin{aligned} &H^1(A, V) \text{ is isomorphic to the cokernel} \\ &\text{of the homomorphism as in (26.6.11),} \end{aligned} \tag{26.6.12}$$

as modules over k. This corresponds to (3a) on p170 of [3], and the analogous statement for arbitrary augmented rings was mentioned in Section 24.5.

We have seen that $h \mapsto h \circ j$ defines an isomorphism from $\mathrm{Hom}_{A^{en}}(J, V)$ onto the space of crossed homomorphisms from A into V, as modules over k, as in Section 26.3. This isomorphism maps the image of the homomorphism as in (26.6.11) onto the space of principal crossed homomorphisms from A into V, as in Section 26.4. It follows that this isomorphism induces an isomorphism from the cokernel of the homomorphism as in (26.6.11) onto

$$\begin{aligned} &\text{the quotient of the space of all crossed homomorphisms} \\ &\text{from } A \text{ into } V \text{ by the submodule consisting of all} \\ &\text{principal crossed homomorphisms from } A \text{ into } V, \end{aligned} \tag{26.6.13}$$

as modules over k. This means that

$$H^1(A, V) \text{ is isomorphic to (26.6.13),} \tag{26.6.14}$$

as modules over k, as in Proposition 4.1 on p170 of [3].

26.7 A right augmentation on A^{en}

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. This leads to the enveloping algebra A^{en} of A, as in Section 26.1.

Of course, A may be considered as a two-sided module over itself, and thus a left or right module over A^{en}, as before. If we consider A as a right module over A^{en}, then we get that

$$a \cdot (a_1 \otimes a_2^{op}) = a_2\, a\, a_1 \tag{26.7.1}$$

for every $a, a_1, a_2 \in A$, as in Section 26.1.

Observe that $(a, a_2^{op}) \mapsto a_2\, a_1$ is bilinear over k, as a map from $A \times A^{op}$ into A. This leads to a unique homomorphism

$$\rho' \text{ from } A^{en} \text{ into } A, \tag{26.7.2}$$

as modules over k, such that

$$\rho'(a_1 \otimes a_2^{op}) = a_2\, a_1 \tag{26.7.3}$$

for every $a_1, a_2 \in A$. Equivalently,

$$\rho'(x) = e_A \cdot x \tag{26.7.4}$$

for every $x \in A^{en}$. This defines a homomorphism from A^{en} into A, as right modules over A^{en}. In fact,

$$\rho'(A^{en}) = A, \tag{26.7.5}$$

so that

$$A^{en} \text{ is a right augmented ring,} \tag{26.7.6}$$

as in Section 24.2, with augmentation module A, as a right module over A^{en}, and augmentation homomorphism ρ'.

This augmentation is discussed in Remark 1 on p171 of [3]. It is mentioned there that one gets the same homology and cohomology groups using this right augmentation as for the left augmentation discussed earlier. More precisely, the descriptions of $H_0(A, V)$ and $H^0(A, V)$ are the same for both augmentations, and the other homology and cohomology groups are satellites of these.

Note that

$$\rho' = \rho \tag{26.7.7}$$

when A is commutative, as on p171 of [3].

26.8 Some modules $S_n(A)$

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Thus we get the corresponding enveloping algebra A^{en}, as in Section 26.1. We would like to define a left complex $S(A)$ over A as a left module over A^{en}, as on p174 of [3]. We begin with some modules $S_n(A)$ that will be used to get $S(A)$.

If $n \ge -1$ is an integer, then we take $S_n(A)$ to be the tensor product of $n+2$ copies of A over k. This means that

$$S_{-1}(A) = A, \tag{26.8.1}$$

and we can take $S_{n+1}(A)$ to be a tensor product $A \bigotimes_k S_n(A)$ of A and $S_n(A)$, as modules over k, for each $n \ge -1$. If $n \ge 1$, then the $(n+2)$-fold tensor products can be arranged into products of pairs in other ways, up to isomorphism, as in Section 1.12. In particular, if $a_0, a_1, \ldots, a_n, a_{n+1} \in A$, then

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \tag{26.8.2}$$

defines an element of $S_n(A)$, which is linear over k in each variable. Note that every element of $S_n(A)$ can be expressed as a finite sum of elements of this form.

We may consider $S_n(A)$ as a left module over A for each $n \ge -1$, using the action of A on itself on the left on the first factor of A in $S_n(A)$, as in Section 1.10. If $a, a_0, a_1, \ldots, a_n, a_{n+1} \in A$, then

$$a \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = (a\, a_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}. \tag{26.8.3}$$

Similarly, we may consider $S_n(A)$ as a right module over A, using the action of A on itself on the right on the last factor of A in $S_n(A)$. If $a_0, a_1, \ldots, a_n, a_{n+1}, b$ are elements of A, then

$$(26.8.4)\qquad (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) \cdot b = a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1}\, b).$$

It is easy to see that the actions of A on $S_n(A)$ on the left and on the right commute with each other, so that $S_n(A)$ may be considered as a two-sided module over A. Equivalently, $S_n(A)$ may be considered as a left module over A^{en}. If $a, b, a_0, a_1, \ldots, a_n, a_{n+1} \in A$, then

$$(26.8.5)\qquad \begin{aligned} &(a \otimes b^{op}) \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) \\ &\quad = \quad (a\, a_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1}\, b), \end{aligned}$$

as on p174 of [3]. Of course, this should be interpreted a bit carefully when $n = -1$.

Consider the homomorphism

$$(26.8.6)\qquad s_n \text{ from } S_n(A) \text{ into } S_{n+1}(A),$$

as modules over k, defined by

$$(26.8.7)\qquad s_n(x) = e_A \otimes x$$

for each $x \in S_n(A)$. More precisely, this is a homomorphism as in (26.8.6), as right modules over A, as on p174 of [3].

Of course, $(a, x) \mapsto a \cdot x$ defines a mapping from $A \times S_n(A)$ into $S_n(A)$ that is bilinear over k. This leads to a unique homomorphism

$$(26.8.8)\qquad t_n \text{ from } S_{n+1}(A) \text{ into } S_n(A),$$

as modules over k, such that

$$(26.8.9)\qquad t_n(a \otimes x) = a \cdot x$$

for every $a \in A$ and $x \in S_n(A)$.

Thus

$$(26.8.10)\qquad t_n(s_n(x)) = x$$

for every $x \in S_n(A)$. In particular, this means that

$$(26.8.11)\qquad s_n \text{ is injective on } S_n(A),$$

as on p174 of [3]. Note that

$$(26.8.12)\quad S_{n+1}(A) \text{ is generated by } s_n(S_n(A)), \text{ as a left module over } A.$$

26.9 Some homomorphisms d_n

We continue with the same notation and hypotheses as in the previous section. If n is a nonnegative integer, then we would like to define a homomorphism

$$d_n \text{ from } S_n(A) \text{ into } S_{n-1}(A), \tag{26.9.1}$$

as modules over k, and left modules over A, with the following two properties, as on p174 of [3]. First,

$$d_0(a_0 \otimes a_1) = a_0\, a_1 \tag{26.9.2}$$

for every $a_0, a_1 \in A$. Second,

$$d_{n+1}(s_n(x)) + s_{n-1}(d_n(x)) = x \tag{26.9.3}$$

for every $x \in S_n(A)$ and $n \geq 0$.

Of course, $(a_0, a_1) \mapsto a_0\, a_1$ is bilinear over k as a mapping from $A \times A$ into A, so that there is a unique homomorphism d_0 from $S_0(A)$ into $S_{-1}(A)$, as modules over k, that satisfies (26.9.2). It is easy to see that d_0 is a homomorphism from $S_0(A)$ into $S_{-1}(A)$, as left and right modules over A, as usual.

One can check that such a family of homomorphisms is unique, as on p174 of [3]. More precisely, the restriction of d_{n+1} to $s_n(S_n(A))$ is uniquely determined by d_n, because of (26.9.3). We also have that d_{n+1} is uniquely determined on $S_{n+1}(A)$ by its restriction to $s_n(S_n(A))$, because of (26.8.12), and because d_{n+1} is supposed to be a homomorphism from $S_{n+1}(A)$ into $S_n(A)$, as left modules over A.

In fact, we can take

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{l=0}^{n} (-1)^l\, a_0 \otimes \cdots \otimes (a_l\, a_{l+1}) \otimes \cdots \otimes a_{n+1} \tag{26.9.4}$$

for every $n \geq 0$ and $a_0, \ldots, a_{n+1} \in A$, as in (3) on p174 of [3]. More precisely, the right side is clearly linear over k in each of the a_i's, $i = 0, \ldots, n+1$, and one can use this to get a unique homomorphism d_n from $S_n(A)$ into $S_{n-1}(A)$, as modules over k, that satisfies (26.9.4). Note that (26.9.4) reduces to (26.9.2) when $n = 0$. One can verify directly that (26.9.4) satisfies (26.9.3) for every $n \geq 0$.

It is easy to see from (26.9.4) that d_n is a homomorphism from $S_n(A)$ into $S_{n-1}(A)$, as left modules over A. Similarly, d_n is a homomorphism from $S_n(A)$ into $S_{n-1}(A)$, as right modules over A, as on p174 of [3]. Thus d_n may be considered as a homomorphism from $S_n(A)$ into $S_{n-1}(A)$, as left modules over A^{en}.

We would like to check that

$$d_{n-1} \circ d_n = 0 \tag{26.9.5}$$

for every $n \geq 1$, as on p174 of [3]. If $n = 1$, then this can be verified using associativity of multiplication on A. Suppose that (26.9.5) holds for some $n \geq 1$, and let us show that the analogous statement holds for $n+1$. Using (26.9.3) and its analogue for $n-1$, we get that

$$d_n \circ d_{n+1} \circ s_n = d_n - d_n \circ s_{n-1} \circ d_n = s_{n-2} \circ d_{n-1} \circ d_n = 0. \tag{26.9.6}$$

This implies that $d_n \circ d_{n+1} = 0$, because of (26.8.12), as desired.

Of course, $A^{en} = A \bigotimes_k A^{op}$ is a left module over itself. Let us identify

$$S_0(A) = A \bigotimes_k A \tag{26.9.7}$$

with A^{en}, as a module over k, in the obvious way. More precisely, we can identify $S_0(A)$ with A^{en} as a left module over A^{en}, as on p175 of [3].

Using this identification,

(26.9.8) $\quad d_0$ corresponds to the augmentation homomorphism ρ,

as a homomorphism from A^{en} into A, as left modules over A^{en}, as in Section 26.1. This is also mentioned on p175 of [3].

26.10 The standard complex $S(A)$

We continue with the same notation and hypotheses as in the previous two sections. We would like to define a left complex $S(A)$ over A, as a left module over A^{en}, as mentioned at the beginning of Section 26.8. We take $S(A)$ to be the direct sum of $S_n(A)$, $n \geq 0$, as a module over k and a left module over A^{en}, with the grading defined by

$$\begin{aligned} S(A)_n = S(A)^{-n} &= S_n(A) \quad \text{when } n \geq 0 \\ &= \{0\} \qquad \text{when } n < 0. \end{aligned} \tag{26.10.1}$$

Note that $S(A)$ is negative as a graded module, as in Section 5.9.

Put

$$\begin{aligned} d_{S(A),n} &= d_n \quad \text{when } n \geq 1 \\ &= 0 \quad \text{when } n \leq 0, \end{aligned} \tag{26.10.2}$$

which is a homomorphism from $S(A)_n$ into $S(A)_{n-1}$, as left modules over A^{en}. Observe that

$$d_{S(A),n-1} \circ d_{S(A),n} = 0 \tag{26.10.3}$$

for every n, using (26.9.5) when $n \geq 2$. This defines a differentiation operator $d_{S(A)}$ on $S(A)$, which makes $S(A)$ into a complex, as in Section 5.10.

In order for $S(A)$ to be a left complex over A, as a left module over A^{en}, we need to choose an augmentation $\varepsilon = \varepsilon_{S(A)}$, as in Section 10.1. This is determined by $\varepsilon_0 = \varepsilon_{S(A),0}$, which is a homomorphism from $S(A)_0$ into A, as left modules over A^{en}, as before. Here we take

$$\varepsilon_0 = d_0, \tag{26.10.4}$$

which is a homomorphism from $S(A)_0 = S_0(A)$ into $A = S_{-1}(A)$, as left modules over A^{en}, as on p175 of [3]. Thus

$$\varepsilon_0 \circ d_{S(A),1} = d_0 \circ d_1 = 0, \tag{26.10.5}$$

by (26.9.5). This shows that $S(A)$ is a left complex over A, with respect to the augmentation ε.

In fact,

(26.10.6) $S(A)$ is acyclic as a left complex over A,

as on p175 of [3]. This is equivalent to the exactness of the sequence

$$\begin{aligned} \cdots \longrightarrow S_n(A) \xrightarrow{d_n} S_{n-1}(A) \longrightarrow \cdots \\ \longrightarrow S_1(A) \xrightarrow{d_1} S_0(A) \xrightarrow{d_0} S_{-1}(A) \longrightarrow 0, \end{aligned} \tag{26.10.7}$$

as in Section 10.1. Of course, this also uses the way that $S(A)$ and ε are defined here. Note that d_0 maps $S_0(A)$ onto $S_{-1}(A)$. The exactness of the rest of this sequence can be obtained from (26.9.3).

Let $\widetilde{S}_n(A)$ be a tensor product of n copies of A over k when $n \geq 1$, and put $\widetilde{S}_0(A) = k$. We may consider $S_n(A)$ as a tensor product of the form

$$S_n(A) = A \bigotimes_k \widetilde{S}_n(A) \bigotimes_k A \tag{26.10.8}$$

for each $n \geq 0$, as on p175 of [3]. The actions of A on the left and right on $S_n(A)$ are obtained from the actions of A on the left and right on the first and last factors of A on the right side of (26.10.8), respectively, as in Section 26.8.

We can identify (26.10.8) with a tensor product of the form

$$A \bigotimes_k A \bigotimes_k \widetilde{S}_n(A), \tag{26.10.9}$$

as a module over k, by interchanging the second and third factors on the right side of (26.10.8). Using this identification, the actions of A on the left and on the right on $S_n(A)$ are obtained from the actions of A on the left and on the right on the first and second factors of A in (26.10.9), respectively.

We can identify (26.10.9), and thus $S_n(A)$, with

$$A^{en} \bigotimes_k \widetilde{S}_n(A), \tag{26.10.10}$$

as a module over k, as on p175 of [3]. We may consider (26.10.10) as a left module over A^{en}, where the action of A^{en} on the left on (26.10.10) is obtained from the action of A^{en} on the left on itself, as the first factor in (26.10.10). This corresponds exactly to the actions of A^{en} on the left on $S_n(A)$ or (26.10.9) obtained from the actions of A on the left and right mentioned before.

If

(26.10.11) A is projective as a module over k, then it follows that

$$\widetilde{S}_n(A) \text{ is projective as a module over } k \tag{26.10.12}$$

for every $n \geq 0$, as in Section 25.1. In this case, we get that

$$\text{(26.10.10) is projective as a left module over } A^{en}, \tag{26.10.13}$$

as in Section 25.1 again. Using the identifications mentioned in the previous paragraphs, we obtain that

$$S_n(A) \text{ is projective as a left module over } A^{en} \tag{26.10.14}$$

for every $n \geq 0$, as on p175 of [3].

Combining this with (26.10.6), we get that

$$S(A) \text{ is a projective resolution of } A, \text{ as a left module over } A^{en}, \tag{26.10.15}$$

as in Section 10.2. This is called the *standard complex of* A, as on p175 of [3].

26.11 Using the standard complex

We continue with the same notation and hypotheses as in the previous three sections. Let V be a two-sided module over A. If we consider V as a right module over A^{en}, then a tensor product

$$V \bigotimes\nolimits_{A^{en}} S(A) \tag{26.11.1}$$

of V and $S(A)$ over A^{en} may be considered as a complex over k, as in Section 7.5. In particular, this is a graded module over k, with

$$\begin{aligned} \big(V \bigotimes\nolimits_{A^{en}} S(A)\big)_n = V \bigotimes\nolimits_{A^{en}} S(A)_n &= V \bigotimes\nolimits_{A^{en}} S_n(A) \quad \text{when } n \geq 0 \\ &= \{0\} \quad \text{when } n < 0. \end{aligned} \tag{26.11.2}$$

If A is projective as a module over k, then

$$H_n(A, V) = H\Big(V \bigotimes\nolimits_{A^{en}} S(A)\Big)_n \tag{26.11.3}$$

for every $n \geq 0$, because of (26.10.15), as in Section 26.5.

Using the identification of $S_n(A)$ with (26.10.10), we get that

$$V \bigotimes\nolimits_{A^{en}} S_n(A) \tag{26.11.4}$$

may be identified with

$$V \bigotimes\nolimits_{A^{en}} \big(A^{en} \bigotimes\nolimits_k \widetilde{S}_n(A)\big), \tag{26.11.5}$$

as a module over k, for each $n \geq 0$. This is isomorphic to

$$\Big(V \bigotimes_{A^{en}} A^{en}\Big) \bigotimes_k \widetilde{S}_n(A) \tag{26.11.6}$$

in a natural way, as modules over n, as in Section 1.12. Of course, V satisfies the requirements of a tensor product of V with A^{en} over A^{en}, so that (26.11.4) may be identified with

$$V \bigotimes_k \widetilde{S}_n(A), \tag{26.11.7}$$

as a module over k, for each $n \geq 0$. This corresponds to a remark on p175 of [3].

Let $v \in V$ and $a_1, \ldots, a_n \in A$ be given, for some $n \geq 1$. If we consider (26.11.7) as a tensor product of V and n copies of A over k, then

$$v \otimes a_1 \otimes \cdots \otimes a_n \tag{26.11.8}$$

may be considered as an element of (26.11.7). Using the identification of (26.11.4) and (26.11.7), the differentiation operator on (26.11.1) sends (26.11.8) to

$$\begin{aligned} & (v \cdot a_1) \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{j=1}^{n-1} (-1)^j \, v \otimes a_1 \otimes \cdots \otimes (a_j \, a_{j+1}) \otimes \cdots \otimes a_n \\ & + (-1)^n \, (a_n \cdot v) \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{aligned} \tag{26.11.9}$$

as on 175 of [3].

If we consider V as a left module over A^{en}, then

$$\mathrm{Hom}^{gr}_{A^{en}}(S(A), V) \tag{26.11.10}$$

may be defined as a complex over k, as in Section 8.4. Note that

$$\begin{aligned} \big(\mathrm{Hom}^{gr}_{A^{en}}(S(A), V)\big)^n &= \mathrm{Hom}_{A^{en}}(S(A)_n, V) \\ &= \mathrm{Hom}_{A^{en}}(S_n(A), V) \quad \text{when } n \geq 0 \\ &= \{0\} \qquad\qquad\qquad\quad \text{when } n < 0. \end{aligned} \tag{26.11.11}$$

If A is projective as a module over k, then

$$H^n(A, V) = H\Big(\mathrm{Hom}^{gr}_{A^{en}}(S(A), V)\Big)^n \tag{26.11.12}$$

for every $n \geq 0$, because of (26.10.15), as in Section 26.5.

Using the identification of $S_n(A)$ with (26.10.10) again, we get that

$$\mathrm{Hom}_{A^{en}}(S_n(A), V) \tag{26.11.13}$$

may be identified with

$$\mathrm{Hom}_{A^{en}}\big(A^{en} \bigotimes_k \widetilde{S}_n(A), V\big) \tag{26.11.14}$$

as a module over k, for each $n \ge 0$. This may be identified with

$$\mathrm{Hom}_k\big(\widetilde{S}_n(A), \mathrm{Hom}_{A^{en}}(A^{en}, V)\big), \tag{26.11.15}$$

as in Section 1.13. Of course, $\mathrm{Hom}_{A^{en}}(A^{en}, V)$ may be identified with V, as usual. This means that (26.11.15) may be identified with

$$\mathrm{Hom}_k(\widetilde{S}_n(A), V). \tag{26.11.16}$$

Thus (26.11.13) may be identified with (26.11.16), as on p175 of [3].

The elements of (26.11.16) are called *n-dimensional cochains*, as on p175 of [3]. They can be identified with functions on the Cartesian product A^n of n copies of A with values in V that are linear over k in each variable when $n \ge 1$, as in [3]. If $n = 0$, then (26.11.16) can be identified with V.

Remember that (26.11.10) has a differentiation operator, obtained from the one on $S(A)$. Let f be a function on A^n with values in V that is linear over k in each variable for some $n \ge 1$, or an element of V when $n = 0$. Thus f corresponds to an element of (26.11.13), and we let $\delta(f)$ be the "coboundary" of f, which is the function on A^{n+1} with values in V that is linear over k in each variable, and which corresponds to the differentiation operator on (26.11.10) acting on f. If $a_1, \ldots, a_{n+1} \in A$, then

$$\begin{aligned}(\delta(f))(a_1, \ldots, a_{n+1}) &= a_1 \cdot f(a_2, \ldots, a_{n+1}) \\ &\quad + \sum_{j=1}^{n} (-1)^j f(a_1, \ldots, a_j\, a_{j+1}, \ldots, a_{n+1}) \\ &\quad + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1},\end{aligned} \tag{26.11.17}$$

as on p175 of [3]. This corresponds to the classical definition of Hochschild, as in [3].

26.12 Some modules $N_n(A)$

Let us continue with the same notation and hypotheses as in the previous four sections. Of course, $t \mapsto t\, e_A$ defines a natural homomorphism from k into A. Let A' be the cokernel of this homomorphism, as a module over k. Thus A' is given by the quotient

$$A' = A/\{t\, e_A : t \in k\}, \tag{26.12.1}$$

as a module over k.

Let $\widetilde{N}_n(A)$ be a tensor product of n copies of A' over k when $n \ge 1$, and put $\widetilde{N}_0(A) = k$. If n is a nonnegative integer, then we take

$$N_n(A) = A^{en} \bigotimes\nolimits_k \widetilde{N}_n(A), \tag{26.12.2}$$

as a module over k, as on p176 of [3]. This may be considered as a left module over A^{en}, where the action of A^{en} on the left is obtained from the action of A^{en} on the left on itself, in the first factor on the right side of (26.12.2).

Let q be the natural quotient mapping from A onto A'. This leads to a natural homomorphism

$$q_n \text{ from } \widetilde{S}_n(A) \text{ onto } \widetilde{N}_n(A), \tag{26.12.3}$$

as modules over k, for each $n \geq 0$, which is the identity mapping on k when $n = 0$.

Remember that $S_n(A)$ can be identified with (26.10.10) for each $n \geq 0$, as a module over k, and as a left module over A^{en}. Using the identity mapping on A^{en} and q_n, we get a homomorphism

$$\text{from } S_n(A) \text{ onto } N_n(A), \tag{26.12.4}$$

as modules over k, and as left modules over A^{en}, for every $n \geq 0$, as on p176 of [3]. Note that $N_0(A)$ is the same as $S_0(A)$ with this identification, and that the homomorphism as in (26.12.4) is the identity mapping when $n = 0$.

As in Section 26.10, we can identify $N_n(A)$ with a tensor product of the form

$$A \bigotimes_k A \bigotimes_k \widetilde{N}_n(A), \tag{26.12.5}$$

as a module over k, for each $n \geq 0$. This may be considered as a two-sided module over A, where the actions of A on the left and right are obtained from the actions of A on the left and right on the first and second factors of A in (26.12.5), respectively. Using this, (26.12.5) may be considered as a left module over A^{en}, which corresponds exactly to $N_n(A)$ as a left module over A^{en}.

Similarly, we can identify (26.12.5) with a tensor product of the form

$$A \bigotimes_k \widetilde{N}_n(A) \bigotimes_k A, \tag{26.12.6}$$

as a module over k, by interchanging the second and third factors in (26.12.5). Of course, the actions of A on the left and right on (26.12.5) mentioned in the preceding paragraph correspond to those obtained from the actions of A on the left and right on the first and last factors of A in (26.12.6).

We may consider a tensor product

$$A' \bigotimes_k \widetilde{N}_n(A) \bigotimes_k A \tag{26.12.7}$$

as a right module over A, using the action of A obtained from the action of A on the right on the last factor of A. There is a natural homomorphism

$$\text{from (26.12.6) onto (26.12.7)}, \tag{26.12.8}$$

as modules over k, corresponding to q in the first factor, and the identity mappings on $\widetilde{N}_n(A)$ and A in the other two factors. More precisely, this is a homomorphism between right modules over A.

We may identify (26.12.7) with a tensor product

$$\widetilde{N}_{n+1}(A) \bigotimes_k A, \tag{26.12.9}$$

as modules over k, in an obvious way. As usual, this tensor product may be considered as a right module over A, using the action of A obtained from the action of A on the right on the second factor of A. This corresponds to the action of A on the right on (26.12.6) mentioned before.

We may identify $N_{n+1}(A)$ with a tensor product

$$A \bigotimes_k \widetilde{N}_{n+1}(A) \bigotimes_k A, \tag{26.12.10}$$

as modules over k, in the same way that $N_n(A)$ is identified with (26.12.6). As usual, this may be considered as a two-sided module over A, where the actions of A on the left and right are obtained from the actions of A on the left and right on the first and last factors of A, respectively. Thus (26.12.10) may be considered as a left module over A^{en}, which corresponds exactly to $N_{n+1}(A)$ as a left module over A^{en}.

26.13 Analogues of s_n, d_n

We continue with the same notation and hypotheses as in the previous sections. Let us identify (26.12.10) with a tensor product of A and (26.12.9) over k. Using this identification, we get a homomorphism

$$\text{from (26.12.9) into (26.12.10),} \tag{26.13.1}$$

which sends an element x of (26.12.9) to $e_A \otimes x$, as an element of (26.12.10). This is a homomorphism as in (26.13.1), as modules over k, and right modules over A.

We can compose the homomorphisms as in (26.12.8) and (26.13.1) to get a homomorphism from (26.12.6) into (26.12.10), as modules over k, and right modules over A. This uses the identification of (26.12.7) with (26.12.9) mentioned earlier. This homomorphism may be considered as a homomorphism

$$\text{from } N_n(A) \text{ into } N_{n+1}(A), \tag{26.13.2}$$

as modules over k, and right modules over A, using identifications mentioned before.

The homomorphism as in (26.13.2) is analogous to the homomorphism s_n from $S_n(A)$ into $S_{n+1}(A)$ defined in Section 26.8, as on p176 of [3]. More precisely, this homomorphism corresponds exactly to s_n with respect to the homomorphism as in (26.12.4), and the analogous homomorphism from $S_{n+1}(A)$ onto $N_{n+1}(A)$.

Suppose that $a_0, a_1, \ldots, a_{n+1} \in A$ for some $n \geq 1$, so that

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) \in S_{n-1}(A) \tag{26.13.3}$$

can be expressed as in (26.9.4). Consider

$$\text{the image of } d_n(a_0 \otimes \cdots \otimes a_{n+1}) \text{ in } N_{n-1}(A), \tag{26.13.4}$$

under the homomorphism from $S_{n-1}(A)$ onto $N_{n-1}(A)$ as in (26.12.4). If

$$a_r = t_r\, e_A \text{ for some } r = 1, \ldots, n \text{ and } t_r \in k, \tag{26.13.5}$$

then one can check that

$$\text{(26.13.4) is equal to 0.} \tag{26.13.6}$$

More precisely, the terms on the right side of (26.9.4) corresponding to $l \neq r-1, r$ are mapped to 0 in $N_{n-1}(A)$, because $q(a_r) = 0$, by hypothesis. One can verify that the terms corresponding to $l = r-1, r$ on the right side of (26.9.4) cancel in this case.

This shows that (26.13.4) depends only on $a_0, q(a_1), \ldots, q(a_n), a_{n+1}$. Thus we get a mapping

$$\text{from } A \times A' \times \cdots \times A' \times A \text{ into } N_{n-1}(A), \tag{26.13.7}$$

with n factors of A' in the domain, that is linear over k in each variable. This leads to a homomorphism

$$\text{from (26.12.6) into } N_{n-1}(A), \tag{26.13.8}$$

as modules over k, in the usual way. This may be considered as a homomorphism

$$\text{from } N_n(A) \text{ into } N_{n-1}(A), \tag{26.13.9}$$

using the identification of $N_n(A)$ with (26.12.6) mentioned earlier. It is easy to see that this is a homomorphism between left and right modules over A, or equivalently left modules over A^{en}.

This homomorphism is the analogue of d_n in Section 26.9 for $N_n(A)$, as on p176 of [3]. One can check that this homomorphism corresponds exactly to d_n with respect to the homomorphism as in (26.12.4), and the analgous homomorphism from $S_{n-1}(A)$ onto $N_{n-1}(A)$. We can include $n = 0$ here by taking

$$N_{-1}(A) = S_{-1}(A) = A, \tag{26.13.10}$$

and using d_0 for the homomorphism as in (26.13.9).

Observe that the analogue of (26.9.5) holds on $N_n(A)$ for every $n \geq 1$, for the homomorphisms as in (26.13.9) in place of the d_n's. Similarly, the analogue of (26.9.3) holds on $N_n(A)$ for every $n \geq 0$, for the homomorphisms as in (26.13.2) and (26.13.9) in place of the s_n's and d_n's, as on p176 of [3].

26.14 The normalized standard complex $N(A)$

Let us continue with the same notation and hypotheses as in the last few sections. Let $N(A)$ be the direct sum of $N_n(A)$ over $n \geq 0$, as a module over k and a left module over A^{en}, as on p176 of [3]. This is a graded module over A^{en}, with grading defined by

(26.14.1)
$$\begin{aligned} N(A)_n = N(A)^{-n} &= N_n(A) \quad \text{when } n \geq 0 \\ &= \{0\} \quad\quad \text{when } n < 0. \end{aligned}$$

Put

(26.14.2)
$$\begin{aligned} d_{N(A),n} &= \text{the homomorphism as in (26.13.9)} \quad \text{when } n \geq 1 \\ &= 0 \quad\quad \text{when } n \leq 0. \end{aligned}$$

This is a homomorphism from $N(A)_n$ into $N(A)_{n-1}$, as left modules over A^{en}. Note that

(26.14.3)
$$d_{N(A),n-1} \circ d_{N(A),n} = 0$$

for every n, as mentioned near the end of the previous section. This defines a differentiation operator $d_{N(A)}$ on $N(A)$, which makes $N(A)$ into a negative complex, as in Sections 5.9 and 5.10.

We can make $N(A)$ a left complex over A, as a left module over A^{en}, by choosing a suitable augmentation $\varepsilon_{N(A)}$, as in Section 10.1. This is determined by $\varepsilon_{N(A),0}$, which is a homomorphism

(26.14.4)
$$\text{from } N(A)_0 = N_0(A) = S_0(A) \text{ into } A,$$

as left modules over A^{en}. As in Section 26.10, we take

(26.14.5)
$$\varepsilon_{N(A),0} = d_0.$$

One can check that

(26.14.6)
$$\varepsilon_{N(A),0} \circ d_{N(A),1} = 0,$$

as in (26.10.5). This implies that $N(A)$ is a left complex over A, with respect to the augmentation $\varepsilon_{N(A)}$, as on p176 of [3].

More precisely,

(26.14.7)
$$N(A) \text{ is acyclic as a left complex over } A,$$

as on p176 of [3]. This is analogous to the corresponding statement (26.10.6) for $S(A)$.

Suppose now that

(26.14.8)
$$A' \text{ is projective as a module over } k.$$

This implies that

(26.14.9)
$$\widetilde{N}(A) \text{ is projective as a module over } k$$

for every $n \geq 0$, as in Section 25.1. Using this, we get that

(26.14.10)
$$N_n(A) \text{ is projective as a left module over } A^{en}$$

for every $n \geq 0$, as in Section 25.1 again. It follows that

(26.14.11) $N(A)$ is a projective resolution of A, as a left module over A^{en},

because of (26.14.7). This is called the *normalized standard complex of* A, as on p176 of [3].

26.15 Some notation

We continue with the same notation and hypotheses as in the previous sections. Let n be a nonnegative integer. In this section, we shall use the identification of $N_n(A)$ with a tensor product of the form (26.12.6), which is more compatible with the initial definition of $S_n(A)$ in Section 26.8. More precisely, this is analogous to the identification of $S_n(A)$ with a tensor product of the form (26.10.8). Using these identifications, the homomorphism from $S_n(A)$ onto $N_n(A)$, as modules over k, as in (26.12.4), is obtained using the identity mapping on A in the first and last factors in the tensor products, and the homomorphism q_n from $\widetilde{S}_n(A)$ onto $\widetilde{N}_n(A)$ as in (26.12.3) in the middle factors.

It is a bit simpler to consider $S_n(A)$ as a tensor product of $n+2$ copies of A over k, as in Section 26.8. Similarly, $N_n(A)$ may be considered as a tensor product of $n+2$ modules over k, where the first module is A, the next n modules are copies of A' as in (26.12.1), and the last module is another copy of A. Using these identifications, the homomorphism from $S_n(A)$ onto $N_n(A)$, as modules over k, as in (26.12.4), is obtained using the identity mapping on the first and last copies of A, and the natural quotient mapping q' from A onto A' on the n copies of A in the middle.

If $a_0, a_1, \ldots, a_n, a_{n+1} \in A$, then

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \tag{26.15.1}$$

defines an element of $S_n(A)$, as in Section 26.8. Similarly,

$$a_0 \otimes q(a_1) \otimes \cdots \otimes q(a_n) \otimes a_{n+1} \tag{26.15.2}$$

defines an element of $N_n(A)$, using the identification mentioned in the preceding paragraph. Of course, the homomorphism from $S_n(A)$ onto $N_n(A)$ as in (26.12.4) sends (26.15.1) to (26.15.2), with these identifications.

As on p176 of [3], one may use the notation

$$a_0\,[a_1, \ldots, a_n]\,a_{n+1} \tag{26.15.3}$$

for (26.15.2). If $a_0 = e_A$, then this may be expressed as

$$[a_1, \ldots, a_n]\,a_{n+1}. \tag{26.15.4}$$

Similarly, if $a_{n+1} = e_A$, then (26.15.3) may be expressed as

$$a_0\,[a_1, \ldots, a_n]. \tag{26.15.5}$$

If $a_0 = a_{n+1} = e_A$, then (26.15.3) may be expressed as

$$[a_1, \ldots, a_n]. \tag{26.15.6}$$

If $n = 0$, then (26.15.3) becomes

$$a_0\,[\,]\,a_1, \tag{26.15.7}$$

which is the same as $a_0 \otimes a_1$ as an element of $S_0(A) = N_0(A)$.

Suppose now that $n \geq 1$, and let $d_{N(A),n}$ be as in the previous section. Using the notation in the preceding paragraph, we have that

$$\begin{aligned} d_{N(A),n}(a_0\,[a_1, \ldots, a_n]\,a_{n+1}) &= (a_0\,a_1)\,[a_2, \ldots, a_n]\,a_{n+1} \\ &\quad + \sum_{l=1}^{n-1} (-1)^l\, a_0\,[a_1, \ldots, a_l\,a_{l+1}, \ldots, a_n]\,a_{n+1} \\ &\quad + (-1)^n\, a_0\,[a_1, \ldots, a_{n-1}]\,(a_n\,a_{n+1}). \end{aligned} \tag{26.15.8}$$

This is mentioned on p176 of [3], with $a_0 = a_{n+1} = e_A$. Note that $d_{N(A),n}$ is determined by its values in that case, because $d_{N(A),n}$ is a homomorphism as in (26.13.9), as left and right modules over A.

Of course, if $n = 1$, then the sum in the middle of the right side of (26.15.8) should be interpreted as being equal to 0. In this case, (26.15.8) reduces to

$$d_{N(A),1}(a_0\,[a_1]\,a_2) = (a_0\,a_1)\,[\,]\,a_2 - a_0\,[\,]\,(a_1\,a_2). \tag{26.15.9}$$

This is mentioned on p176 of [3] too, with $a_0 = a_2 = e_A$, as before.

It may be convenient to sometimes use the notation (26.15.3) for (26.15.1), as an element of $S_n(A)$, as mentioned on p176 of [3]. It should always be clear from the particular discussion which interpretation is intended.

Chapter 27

A family of augmentations, 2

27.1 Homomorphisms between algebras

Let k, k_1 be commutative rings with multiplicative identity elements 1_k, 1_{k_1}, respectively. Also let A, B be associative algebras over k, k_1, with multiplicative identity elements e_A, e_B, respectively. Suppose that ψ is a ring homomorphism from k into k_1, with $\psi(1_k) = 1_{k_1}$.

Let ϕ be a ring homomorphism from A into B, with $\phi(e_A) = e_B$. Suppose that

(27.1.1) $$\phi(t\,a) = \psi(t)\,\phi(a)$$

for every $a \in A$ and $t \in k$. If $k = k_1$ and ψ is the identity mapping on k, then this means that ϕ is a homomorphism from A into B, as algebras over k. Alternatively, one may consider B as an algebra over k, using ψ.

Remember that A^{op}, B^{op} are the opposite algebras corresponding to A, B, respectively, and let

(27.1.2) $$A^{en} = A \bigotimes\nolimits_k A^{op}, \quad B^{en} = B \bigotimes\nolimits_{k_1} B^{op}$$

be enveloping algebras of A, B, as algebras over k, k_1, respectively, as in Section 26.1. Note that B^{en} may be considered as an algebra over k, using ψ.

Of course, ϕ may be considered as a ring homomorphism from A^{op} into B^{op}. Consider the mapping

(27.1.3) $$(a_1, a_2^{op}) \mapsto \phi(a_1) \otimes_{k_1} \phi(a_2)^{op}$$

from $A \times A^{op}$ into B^{en}. It is easy to see that this mapping is bilinear over k, when B^{en} is considered as a module over k using ψ. Thus we get a unique homomorphism

(27.1.4) $$\phi^{en} \text{ from } A^{en} \text{ into } B^{en},$$

as modules over k, such that

$$\phi^{en}(a_1 \otimes_k a_2^{op}) = \phi(a_1) \otimes_{k_1} \phi(a_2)^{op} \tag{27.1.5}$$

for every $a_1, a_2 \in A$. One can check that ϕ^{en} is a ring homomorphism.

Let ρ_A, ρ_B be the homomorphisms from A^{en}, B^{en} onto A, B, as left modules over A^{en}, B^{en}, respectively, as in Section 26.1. One can verify that

$$\rho_B \circ \phi^{en} = \phi \circ \rho_A, \tag{27.1.6}$$

as on p171 of [3]. In particular,

$$\phi^{en}(\ker \rho_A) \subseteq \rho_B. \tag{27.1.7}$$

This shows that ϕ^{en} is a homomorphism from A^{en} into B^{en}, as left augmented rings with respect to the augmentations defined by ρ_A and ρ_B, respectively, as in Section 24.9.

Let V be a module over k_1 that is a two-sided module over B, as an algebra over k_1. Thus V may be considered as a module over k using ψ, and as a two-sided module over A using ϕ, as in Section 2.9. Alternatively, V may be considered as a left or right module over B^{en}, as an algebra over k_1, and thus as a left or right module over A^{en}, as an algebra over k, using ϕ^{en}.

Using ϕ^{en}, we get induced homomorphisms

$$\text{from } H_n(A, V) \text{ into } H_n(B, V) \tag{27.1.8}$$

and

$$\text{from } H^n(B, V) \text{ into } H^n(A, V) \tag{27.1.9}$$

for each $n \geq 0$, as in Section 24.9. These homomorphisms correspond to those in (1), (2) on p171 of [3].

27.2 The case where $B = k_1 \otimes_k A$

Let us continue with the same notation and hypotheses as in the previous section, with the following additions. In this section, we consider k_1 to be an algebra over k, with $\psi(t) = t \cdot 1_{k_1}$ for every $t \in k$. We also take B to be a tensor product

$$B = k_1 \bigotimes_k A \tag{27.2.1}$$

of k_1 and A over k, which may be considered as an algebra over k and k_1. Thus

$$\phi(a) = 1_{k_1} \otimes a \tag{27.2.2}$$

defines a homomorphism from A into B, as algebras over k, where $\phi(e_A) = 1_{k_1} \otimes e_A$ is the multiplicative identity element in B.

Consider

$$B^{en} = \big(k_1 \bigotimes_k A\big) \bigotimes_{k_1} \big(k_1 \bigotimes_k A\big)^{op}. \tag{27.2.3}$$

As a module over k, $(k_1 \bigotimes_k A)^{op}$ is the same as $k_1 \bigotimes_k A^{op}$, and in fact they are the same as algebras over k and k_1. Thus B^{en} can be expressed as

$$B^{en} = \big(k_1 \bigotimes_k A\big) \bigotimes_{k_1} \big(k_1 \bigotimes_k A^{op}\big). \tag{27.2.4}$$

This is isomorphic in a natural way to a tensor product of the form

$$\big((k_1 \bigotimes_k A) \bigotimes_{k_1} k_1\big) \bigotimes_k A^{op}, \tag{27.2.5}$$

as in Section 1.12. This reduces to

$$\big(k_1 \bigotimes_k A\big) \bigotimes_k A^{op}, \tag{27.2.6}$$

because any module over k_1 satisfies the requirements of its tensor product with k_1, over k_1. It follows that B^{en} can be identified with a tensor product of the form

$$k_1 \bigotimes_k A^{en}, \tag{27.2.7}$$

up to a natural isomorphism, as in Section 1.12 again. This corresponds to some remarks near the top of p172 of [3].

Of course, $k_1 \bigotimes_k A$ may be considered as a two-sided module over k_1 and A, because k_1 and A are two-sided modules over themselves. This corresponds to considering B as a two-sided module over itself, and thus as a left or right module over B^{en}. Alternatively, we may consider $k_! \bigotimes_k A$ as a left or right module over k_1 and A^{en}, where the actions of k_1 and A^{en} commute with each other. This means that $k_1 \bigotimes_k A$ may be considered as a left or right module over (27.2.7). This is equivalent to considering $k_1 \bigotimes_k A$ as a left or right module over B^{en}, respectively, with respect to the identification between B^{en} and (27.2.7) mentioned in the preceding paragraph.

Remember that A^{en}, B^{en} may be considered as left augmented rings, with augmentation modules A, B and augmentation homomorphisms ρ_A, ρ_B, as in Section 26.1. Alternatively, using the identification of B^{en} with (27.2.7), we can get a left augmentation with augmentation module $k_1 \bigotimes_k A$, and augmentation homomorphism obtained from the identity mapping on k_1 and ρ_A, as in Section 25.11. These are equivalent descriptions of the same augmentation, because of the remarks in the preceding paragraph.

Similarly,

$$x \mapsto 1_{k_1} \otimes x \tag{27.2.8}$$

defines a homomorphism from A^{en} into (27.2.7), as algebras over k. This corresponds to the homomorphism ϕ^{en} from A^{en} into B^{en} mentioned in the previous section, with respect to the identification between B^{en} and (27.2.7), because ϕ is defined here as in (27.2.2).

This means that we are in the same type of situation as in Section 25.11, with A, Q_A, and ε_A taken to be A^{en}, A, and ρ_A, respectively, and with $C = k_1$, as mentioned on p172 of [3]. The homomorphism considered in Section 25.11 is given by (27.2.8), which corresponds to ϕ^{en}, as in the preceding paragraph.

Let V be a module over k_1 that is a two-sided module over B, as an algebra over k_1, as in the previous section. Thus V may be considered as a module over k, because k_1 is an algebra over k, and as a two-sided module over A, as an algebra over k. Let us emphasize that the actions of k_1 on B on the right and on the left, coming from the actions of B on V on the right and on the left, are supposed to be the same, and the same as the action of k_1 on B as a module over k_1, as on p172 of [3].

We may consider V as a left or right module over B^{en}, as an algebra over k_1, and as a left or right module over A^{en}, as an algebra over k, as in the previous section. We may also consider V as a left or right module over (27.2.7), as an algebra over k_1. This is equivalent to considering V as a left or right module over B^{en}, with respect to the identification of B^{en} with (27.2.7) mentioned earlier.

Under these conditions, we get induced homomorphisms as in (27.1.8) and (27.1.9) for each $n \geq 0$, using ϕ^{en}. These are the same as the homomorphisms from the homology of A^{en} into the homology of B^{en}, as left augmented rings, with coefficients in V, and from the cohomology of B^{en} into the cohomology of A^{en} with coefficients in V, as in Section 24.9.

Using the identification of B^{en} with (27.2.7), we may identify these homomorphisms with the homomorphisms from the homology of A^{en} into the homology of (27.2.7), as left augmented rings, with coefficients in V, and from the cohomology of (27.2.7) into the cohomology of A^{en} with coefficients in V, corresponding to the homomorphism from A^{en} into (27.2.7) defined by (27.2.8). These homomorphisms were discussed in Section 25.12.

Suppose now that

(27.2.9) $\qquad A$ is projective as a module over k.

This corresponds to the condition that Q_A be projective as a module over k near the end of Section 25.12. This also implies that A^{en} is projective as a module over k, as in Section 25.1. In this case, it follows that the homomorphisms mentioned in the preceding paragraph are isomorphisms, as in Section 25.12.

This means that

(27.2.10) the homomorphisms as in (27.1.8) and (27.1.9) are isomorphisms

when (27.2.9) holds. If X is a projective resolution of A, as a left module over A^{en}, then we get that

(27.2.11) $\qquad k_1 \bigotimes_k X$ is a projective resolution of B, as a left module over B^{en},

when (27.2.9) holds, as in Section 25.12 again. This corresponds to Proposition 5.1 on p172 of [3].

27.3 Projectivity of A over A^{en}

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k, with a multiplicative identity element e_A.

Remember that A^{op} is the opposite algebra of A, and let $A^{en} = A \bigotimes_k A^{op}$ be a tensor product of A and A^{op}, as in Section 26.1.

Of course, A may be considered as a two-sided module over itself, and thus as a left module over A^{en}, as before. One may wish to know more about conditions under which

$$A \text{ is projective as a left module over } A^{en}, \tag{27.3.1}$$

as on p179 of [3].

Let ρ be the homomorphism from A^{en} onto A, as modules over k, and left modules over A^{en}, defined in Section 26.1. If (27.3.1) holds, then

$$\begin{aligned} &\text{there is a homomorphism } f \text{ from } A \text{ into } A^{en}, \\ &\text{as left modules over } A^{en}, \text{ such that } \rho \circ f = I_A, \end{aligned} \tag{27.3.2}$$

where I_A is the identity mapping on A.

Conversely, suppose that (27.3.2) holds, so that

$$\begin{aligned} &A^{en} \text{ corresponds to the direct sum of } f(A) \text{ and } \ker \rho, \\ &\text{as a left module over } A^{en}. \end{aligned} \tag{27.3.3}$$

This implies that (27.3.1) holds, because A^{en} is projective as a module over itself, and f is injective, as in Section 2.7.

Let $A \bigotimes_k A$ be a tensor product of A with itself, as a module over k. Of course, this is the same as A^{en}, as a module over k, but it is convenient to refer to it a bit differently. We may consider $A \bigotimes_k A$ as a two-sided module over A, where the action of A on the left is obtained from the action of A on the left on the first factor of A, and the action of A on the right is obtained from the action of A on the right on the second factor of A. This corresponds to A^{en} as a left module over itself.

Using multiplication on A, we get a unique homomorphism $\widetilde{\rho}$ from $A \bigotimes_k A$ into A, as modules over k, such that

$$\widetilde{\rho}(a_1 \otimes a_2) = a_1\, a_2 \tag{27.3.4}$$

for every $a_1, a_2 \in A$. More precisely, $\widetilde{\rho}$ is a homomorphism from $A \bigotimes_k A$ onto A, as left and right modules over A. This corresponds to the homomorphism ρ from A^{en} onto A, as left modules over A^{en}, from Section 26.1.

Thus (27.3.2) is the same as saying that

$$\begin{aligned} &\text{there is a homomorphism } \widetilde{f} \text{ from } A \text{ into } A \bigotimes_k A, \\ &\text{as left and right modules over } A, \text{ such that } \widetilde{\rho} \circ \widetilde{f} = I_A. \end{aligned} \tag{27.3.5}$$

This is equivalent to (27.3.1), as before.

Remember that an element x of $A \bigotimes_K A$, as a two-sided module over A, is said to be invariant if

$$a \cdot x = x \cdot a \tag{27.3.6}$$

for every $a \in A$, as in Section 26.6. Proposition 7.7 on p179 of [3] states that (27.3.1) holds if and only if

(27.3.7) there is an invariant element x of $A \bigotimes_k A$ such that $\widetilde{\rho}(x) = e_A$.

If (27.3.1) holds, then (27.3.5) holds, and one can take

$$x = \widetilde{f}(e_A). \tag{27.3.8}$$

Conversely, suppose that (27.3.7) holds, and put

$$\widetilde{f}(a) = a \cdot x = x \cdot a \tag{27.3.9}$$

for every $a \in A$. This defines a homomorphism from A into $A \bigotimes_k A$, as modules over k, and in fact as left and right modules over A. If $a \in A$, then

$$\widetilde{\rho}(\widetilde{f}(a)) = \widetilde{\rho}(a \cdot x) = a\,\widetilde{\rho}(x) = a\,e_A = a. \tag{27.3.10}$$

This means that (27.3.5) holds, so that (27.3.1) holds.

27.4 Some examples with matrices

Let k be a commutative ring with a multiplicative identity element 1_k, and let n be a positive integer. The space $M_n(k)$ of $n \times n$ matrices with entries in k is an associative algebra over k with respect to matrix multiplication, and with the usual identity matrix I_n as the multiplicative identity element.

If $1 \le j, l \le n$, then let $a_{j,l}$ be the element of $M_n(k)$ whose (j,l) entry is equal to 1_k, and whose other entries are equal to 0. Thus

$$\sum_{j=1}^{n} a_{j,j} = I_n. \tag{27.4.1}$$

Of course, $M_n(k)$ is freely generated by $a_{j,l}$, $1 \le j, l \le n$, as a module over k. If $1 \le j, l, m, r \le n$, then

$$\begin{aligned} a_{j,l}\,a_{m,r} &= a_{j,r} \quad \text{when } l = m \\ &= 0 \quad\;\;\, \text{when } l \ne m. \end{aligned} \tag{27.4.2}$$

Let us take

$$A = M_n(k) \tag{27.4.3}$$

in the previous section. We would like to verify that (27.3.7) holds, as in Proposition 7.8 on p179 of [3]. To do this, we take

$$x = \sum_{j=1}^{n} a_{j,1} \otimes a_{1,j}. \tag{27.4.4}$$

Let us first check that x is invariant as an element of $A \bigotimes_k A$.

If $1 \le m, r \le n$, then

$$a_{m,r} \cdot x = \sum_{j=1}^{n} (a_{m,r}\, a_{j,1}) \otimes a_{1,j} = a_{m,1} \otimes a_{1,r}. \tag{27.4.5}$$

Similarly,

$$x \cdot a_{m,r} = \sum_{j=1}^{n} a_{j,1} \otimes (a_{1,j}\, a_{m,r}) = a_{m,1} \otimes a_{1,r}. \tag{27.4.6}$$

It follows that x is invariant in $A \bigotimes_k A$, because $M_n(k)$ is generated by the $a_{m,r}$'s, $1 \le m, r \le n$, as a module over k.

Observe that

$$\widetilde{\rho}(x) = \sum_{j=1}^{n} a_{j,1}\, a_{1,j} = \sum_{j=1}^{n} a_{j,j} = I_n. \tag{27.4.7}$$

Thus (27.3.7) holds, as desired.

27.5 Some remarks about polynomial algebras

Let k be a commutative ring with a multiplicative identity element, let n be a positive integer, and let $T_1, \ldots, T_n$ be n commuting indeterminates. Thus the space $k[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in k may be defined as in Section 4.3, and is a commutative associative algebra over k. Remember that k may be identified with the subalgebra of $k[T_1, \ldots, T_n]$ consisting of formal polynomials for which the coefficient of T^α is 0 when $\alpha \neq 0$. The multiplicative identity element 1_k in k corresponds to the multiplicative identity element of $k[T_1, \ldots, T_n]$ in this way.

Let ε_0 be the mapping from $k[T_1, \ldots, T_n]$ that sends a formal polynomial $f(T)$ to its constant term f_0. This is a homomorphism from $k[T_1, \ldots, T_n]$ onto k, as algebras over k, with $\varepsilon_0(1_k) = 1_k$.

Let ε be any homomorphism from $k[T_1, \ldots, T_n]$ into k, as algebras over k, such that

$$\varepsilon(1_k) = 1_k. \tag{27.5.1}$$

Note that ε is uniquely determined by its vales at $T_1, \ldots, T_n$, which may be arbitrary elements of k.

It is easy to see that there is a unique homomorphism ϕ from $k[T_1, \ldots, T_n]$ into itself, as an algebra over k, such that $\phi(1_k) = 1_k$ and

$$\phi(T_j) = T_j - \varepsilon(T_j) \tag{27.5.2}$$

for each $j = 1, \ldots, n$. More precisely, ϕ is an automorphism of $k[T_1, \ldots, T_n]$, whose inverse is the analogous homomorphism that sends T_j to $T_j + \varepsilon(T_j)$ for each $j = 1, \ldots, n$.

By construction,

$$\varepsilon(\phi(T_j)) = \varepsilon(T_j) - \varepsilon(T_j) = 0 \tag{27.5.3}$$

for $j = 1, \ldots, n$. This implies that

$$\varepsilon \circ \phi = \varepsilon_0 \tag{27.5.4}$$

on $k[T_1, \ldots, T_n]$. This corresponds to some remarks on p180 of [3].

27.6 More on polynomial algebras

Let k be a commutative ring with a multiplicative identity element, let n be a positive integer, and let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be commuting indeterminates. Consider

$$A = k[X_1, \ldots, X_n], \tag{27.6.1}$$

which is a commutative associative algebra over k, as before. Of course, A is isomorphic to $k[Y_1, \ldots, Y_n]$ in an obvious way.

Because A is commutative, $A^{op} = A$, and we may take

$$A^{en} = A \bigotimes_k A^{op} = A \bigotimes_k A \tag{27.6.2}$$

to be a tensor product of A with itself over k, as in Section 26.1. This is isomorphic to a tensor product

$$(k[X_1, \ldots, X_n]) \bigotimes_k (k[Y_1, \ldots, Y_n]) \tag{27.6.3}$$

of $k[X_1, \ldots, X_n]$ and $k[Y_1, \ldots, Y_n]$ over k, using the isomorphism mentioned in the preceding paragraph.

Consider the algebra

$$k[X_1, \ldots, X_n, Y_1, \ldots, Y_n] \tag{27.6.4}$$

of formal polynomials in $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ with coefficients in k. There is an obvious mapping

$$\text{from } (k[X_1, \ldots, X_n]) \times (k[Y_1, \ldots, Y_n]) \text{ into (27.6.4)} \tag{27.6.5}$$

defined by multiplication, and which is bilinear over k. Using this mapping, (27.6.4) satisfies the requirements of a tensor product as in (27.6.3). More precisely, multiplication in (27.6.4) corresponds to multiplication in (27.6.3), as a tensor product of associative algebras, as in Section 4.1.

Let ρ be the mapping from A^{en} onto A defined in Section 26.1. This is an algebra homomorphism in this case, because A is commutative. If A^{en} is identified with (27.6.4) as before, then ρ corresponds to the mapping

$$\eta \text{ from (27.6.4) into } k[X_1, \ldots, X_n] \tag{27.6.6}$$

in which Y_j is replaced with X_j for each $j = 1, \ldots, n$, as on p180 of [3].

The space

$$A[Y_1, \ldots, Y_n] = (k[X_1, \ldots, X_n])[Y_1, \ldots, Y_n] \tag{27.6.7}$$

of formal polynomials in $Y_1, \ldots, Y_n$ with coefficients in (27.6.1) may be defined as in Section 4.3. This is a commutative associative algebra over A, as well as over k. This can be identified with (27.6.4), as an algebra over k, in a reasonable way. With this identification, η corresponds to the homomorphism

(27.6.8) from (27.6.7) into (27.6.1)

in which an element of (27.6.7) is evaluated at $(X_1, \ldots, X_n)$, as an element of A^n.

We also have a homomorphism

(27.6.9) η_0 from (27.6.7) into (27.6.1)

which sends an element of (27.6.7) to its constant term. Observe that there is an automorphism

(27.6.10) Φ on (27.6.7),

as an algebra over A, such that

(27.6.11) $$\eta \circ \Phi = \eta_0,$$

as in the previous section. Here η is considered as a homomorphism as in (27.6.8).

This tells us a lot about A^{en} as a left augmented ring with respect to ρ, as on p180 of [3].

27.7 Some remarks about free algebras

Let k be a commutative ring with a multiplicative identity element, and let

(27.7.1) $$E = \{x_1, \ldots, x_m\}$$

be a finite set with m distinct elements $x_1, \ldots, x_m$ for some positive integer m. Using E, we can get the free semigroup $\Sigma(E)$ generated by E, as in Section 4.10. This leads to the corresponding semigroup algebra $k(\Sigma(E))$, as in Sections 4.9 and 4.10, which is an associative algebra over k. This is the same as the free algebra

(27.7.2) $$F_k(x_1, \ldots, x_m)$$

over k generated by $x_1, \ldots, x_m$, as on p146, 148 of [3].

Let V be a module over k that is a two-sided module over (27.7.2). Suppose that ϕ is a crossed homomorphism from (27.7.2) into V, as in Section 26.3. It is easy to see that

(27.7.3) ϕ is uniquely determined by its values at $x_1, \ldots, x_m$.

One can check that

(27.7.4) the values of ϕ at $x_1, \ldots, x_m$ may be arbitrary elements of V.

This corresponds to the first part of Exercise 2 on p181 of [3].

Let A be (27.7.2), and let A^{en} be a corresponding enveloping algebra, as in Section 26.1. Also let ρ be the homomorphism from A^{en} onto A, as left modules over A^{en}, defined previously, and put $J = \ker \rho$, as in Section 26.2. We may consider J as a two-sided module over A, because it is a left ideal in A^{en}, and thus a left module over A^{en}. Remember that $j(a) = a \otimes e_A^{op} - e_A \otimes a^{op}$ defines a crossed homomorphism from A into J, as a two-sided module over A. Here e_A is the multiplicative identity element in A, which corresponds to the identity element in $\Sigma(E)$.

If h is a homomorphism from J into V, as two-sided modules over A, then $h \circ j$ is a crossed homomorphism from A into V, as in Section 26.3. We have also seen that h is uniquely determined by $h \circ j$, because J is generated by $j(A)$ as a left module over A, as in Section 26.2. It follows that h is uniquely determined by its values at the elements

$$j(x_l) = x_l \otimes e_A^{op} - e_A \otimes x_l \tag{27.7.5}$$

of J, with $l = 1, \ldots, m$, because of (27.7.3).

Remember that every crossed homomorphism from A into V can be expressed as $h \circ j$ for some homomorphism h from J into V, as two-sided modules over A, as in Section 26.3. This and (27.7.4) imply that the values of such a homomorphism h at the elements (27.7.5) of J may be arbitrary elements of V.

The second part of Exercise 2 on p181 of [3] states that J is freely generated, as a left module over A^{en}, by the elements of the form (27.7.5). To see that J is generated by these elements, as a left module over A^{en}, one can consider the quotient of J by the submodule, as a left module over A^{en}, generated by these elements. The corresponding quotient mapping is equal to 0 at these elements, by construction, and is thus equal to 0, because it is uniquely determined by its values at these elements, as before. Alternatively, one can check that $j(A)$ is contained in this submodule, because j is a crossed homomorphism from A into J.

Of course, A^{en} is a left module over itself, and one can use this to consider A^{en} as a two-sided module over A. The values of a homomorphism h from J into A^{en}, as two-sided modules over A, or equivalently as left modules over A^{en}, at the elements (27.7.5) of J may be arbitrary elements of A^{en}, as before. In particular, for each $r = 1, \ldots, m$, there is such a homomorphism h_r that is equal to $e_A \otimes e_A^{op}$ at (27.7.5) when $l = r$, and equal to 0 otherwise. This implies that J is freely generated by these elements, as a left module over A^{en}.

27.8 More on semigroup algebras

Let k be a ring with a multiplicative identity element, and let Σ be a semigroup, with the semigroup operation expressed multiplicatively, and with an identity element e_Σ. Also let $k(\Sigma)$ be the corresponding semigroup algebra of Σ with coefficients in k, as in Section 4.9, and let V be a module over k.

Remember that an action of Σ on V on the left or right with suitable properties makes V into a left or right module over Σ, as in Section 4.8. This corresponds to V being a left or right module over $k(\Sigma)$, as appropriate, as in Section 4.9.

Let us say that V is a *two-sided module over* Σ if V is both a left and right module over Σ, and that the actions of Σ on V on the left and on the right commute with each other. This corresponds exactly to V being a two-sided module over $k(\Sigma)$.

Let Σ^{op} be the opposite semigroup of Σ, as in Section 4.8. Also let A be an asociative algebra over k with a multiplicative identity element e_A, and let A^{op} be the corresponding opposite algebra. It is easy to seee that the opposite algebra

$$(A(\Sigma))^{op} \tag{27.8.1}$$

of $A(\Sigma)$ can be identified with the semigroup algebra

$$A^{op}(\Sigma^{op}) \tag{27.8.2}$$

of Σ^{op} with coefficients in A^{op}. In particular, the opposite algebra $(k(\Sigma))^{op}$ of $k(\Sigma)$ can be identified with the semigroup algebra

$$k(\Sigma^{op}) \tag{27.8.3}$$

of Σ^{op} with coefficients in k, because k is commutative.

If V is a two-sided module over Σ, then V may be considered as a left module over Σ and Σ^{op}, where the actions of Σ and Σ^{op} on V commute with each other. These actions can be combined to get an action of the product semigroup

$$\Sigma \times \Sigma^{op} \tag{27.8.4}$$

on V on the left, as in Section 25.14. Similarly, V may be considered as a right module over Σ and Σ^{op}, where these actions on V commute with each other. These actions can be combined, to get an action of (27.8.4) on V on the right. Conversely, if V is a left or right module over (27.8.4), then we get commuting actions of Σ and Σ^{op} on the left or right, as appropriate, and V may be considered as a two-sided module over Σ.

Remember that an enveloping algebra $k(\Sigma)^{en}$ of $k(\Sigma)$ is obtained from a tensor product of $k(\Sigma)$ and $k(\Sigma)^{op}$ over k, as in Section 26.1. This may be identified with a tensor product of $k(\Sigma)$ and $k(\Sigma^{op})$, as before. The semigroup algebra

$$k(\Sigma \times \Sigma^{op}) \tag{27.8.5}$$

of (27.8.4) with coefficients in k satisfies the requirements of such a tensor product, as in Section 25.14.

Let V be a two-sided module over Σ, and thus over $k(\Sigma)$. Let us say that $v \in V$ is *invariant* as an element of V, as a two-sided module over Σ, if

$$x \cdot v = v \cdot x \tag{27.8.6}$$

for every $x \in \Sigma$. This is equivalent to the condition that v be invariant as an element of V, as a two-sided module over $k(\Sigma)$, as in Section 26.6.

Note that a homomorphism f from $k(\Sigma)$ into any module V over k, as modules over k, is uniquely determined by its restriction to Σ, and that any V-valued function on Σ occurs in this way. If V is a two-sided module over Σ, and thus $k(\Sigma)$, then f is a crossed homomorphism, as in Section 26.3, if and only if

$$f(x\,y) = x \cdot f(y) + f(x) \cdot y \tag{27.8.7}$$

for every $x, y \in \Sigma$.

Part VIII

Supplemented algebras and semigroup rings

Chapter 28

Supplemented algebras

28.1 Basic notions

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Of course, k may be considered as an algebra over itself.

Suppose that ε is a homomorphism from A into k, as algebras over k, with $\varepsilon(e_A) = 1$. Under these conditions, A together with ε is called a *supplemented algebra*, as on p182 of [3].

Note that

$$\varepsilon(t\, e_A) = t\, \varepsilon(e_A) = t \tag{28.1.1}$$

for every $t \in k$. In particular, $\varepsilon(A) = k$, so that A may be considered as a left and right augmented ring, with augmentation homomorphism ε, as in Sections 24.1 and 24.2.

Using (28.1.1), we get that $t \mapsto t\, e_A$ is a one-to-one mapping from k into A. Let $\mathcal{I}$ be the kernel of ε, which is a two-sided ideal in A. Observe that A corresponds to the direct sum of

$$\{t\, e_A : t \in k\} \tag{28.1.2}$$

and $\mathcal{I}$, as a module over k, as on p182 of [3].

The augmentation module of A as a left or right augmented ring is k, considered as a left or right module over A, as appropriate, using ε. If V is any module over k, then V may be considered as a left module ${}_{\varepsilon}V$ over A, with

$$a \cdot v = \varepsilon(a)\, v \tag{28.1.3}$$

for every $a \in A$ and $v \in V$, as on p183 of [3]. Similarly, V may be considered as a right module V_{ε} over A, with

$$v \cdot a = \varepsilon(a)\, v \tag{28.1.4}$$

for every $a \in A$ and $v \in V$. In both cases, we say that the corresponding actions or operators of A on V are *trivial*, as on p183 of [3]. Thus the actions of A on k mentioned earlier are trivial in this sense.

If V is already a right module over A, then ${}_\varepsilon V$ is a two-sided module over A, as in Section 26.1. Similarly, if V is already a left module over A, then V_ε is a two-sided module over A. More precisely, the actions of A on V should be linear over k in both cases, and thus commute with the trivial actions of A on V. It follows that ${}_\varepsilon V$ or V_ε may be considered as a module over an enveloping algebra A^{en} of A, as appropriate, as in Section 26.1. This corresponds to some remarks on p183 of [3].

One can use this to reinterpret previous definitions for two-sided modules in the case of one-sided modules, as on p183 of [3]. Some instances of this will be discussed in the next sections.

28.2 Some related modules

Let us continue with the same notation and hypotheses as in the previous section. Let W be a module over k that is a left module over A, so that W_ε is a two-sided module over A. Let us say that $w \in W$ is *invariant* if w is invariant as an element of W_ε, as a two-sided module over A, as in Section 26.6. This means that for every $a \in A$,

(28.2.1) $$a \cdot w = w \cdot a = \varepsilon(a)\, w,$$

as before.

Let W^A be the collection of invariant elements of W. It is easy to see that this is a submodule of W, as a module over k, and in fact as a module over A, as on p183 of [3]. More precisely, A acts trivially on W^A, in the sense described in the previous section, by construction. This is the largest submodule of W on which A acts trivially.

Observe that

(28.2.2) $$\mathcal{I} = \{a - \varepsilon(a)\, e_A : a \in A\}.$$

One can use this to check that

(28.2.3) $$W^A = \{w \in W : a \cdot w = 0 \text{ for every } a \in \mathcal{I}\},$$

as on p183 of [3].

Let

(28.2.4) $$\mathrm{Hom}_A(k, W)$$

be the space of homomorphisms from k into W, as modules over k, and left modules over A. These homomorphisms correspond to homomorphisms from A into W, as modules over k, and left modules over A, that are equal to 0 on $\mathcal{I}$. Of course, homomorphisms from A into W, as left modules over A, correspond to elements of W in a natural way. It follows that (28.2.4) corresponds to W^A in a natural way, because of (28.2.3), as on p183 of [3]. This could also be considered as a version of a remark in Section 24.1, as in [3].

Let $H^0(A, W_\varepsilon)$ be as in Section 26.5. This is isomorphic to W^A, as a module over k, as in Section 26.6. This corresponds to (1) on p183 of [3].

Now let V be a module over k that is a right module over A. Consider the subset $V \cdot \mathcal{I}$ of V consisting of finite sums of elements of the form $v \cdot a$, where $v \in V$ and $a \in \mathcal{I}$. This is a submodule of V, as a module over k, and in fact as a module over A. The quotient

$$V_A = V/(V \cdot \mathcal{I}) \tag{28.2.5}$$

is a module over k, and a right module over A.

More precisely, A acts trivially on V_A, in the sense described in the previous section, by construction. In fact, $V \cdot \mathcal{I}$ is the smallest submodule of V, as a module over A, such that A acts trivially on the quotient, as on p183 of [3]. Let

$$V \bigotimes\nolimits_A k \tag{28.2.6}$$

be a tensor product of V and k over A, where k is considered as a left module over A, as in the previous section. Note that (28.2.6) is isomorphic to V_A as a module over k, as in Sections 24.1 and 24.2.

Remember that ${}_\varepsilon V$ is a two-sided module over A, as in the previous section. Thus V may be considered as a right module over an enveloping algebra A^{en} of A, as in Section 26.1. Let J be the augmentation ideal of A^{en}, as an augmented ring, as in Section 26.2. One can check that

$$V \cdot \mathcal{I} = ({}_\varepsilon V) \cdot J, \tag{28.2.7}$$

as on p183 of [3]. This uses (28.2.2), and the description of J in Section 26.2.

Let $H_0(A, {}_\varepsilon V)$ be as in Section 26.5. This is isomorphic to

$$({}_\varepsilon V)/(({}_\varepsilon V) \cdot J), \tag{28.2.8}$$

as a module over k, as in Section 26.6. Note that (28.2.8) is isomorphic to (28.2.5), as a module over k, because of (28.2.7). This corresponds to (2) on p183 of [3].

28.3 Crossed homomorphisms into W_ε

We continue with the same notation and hypotheses as in the previous two sections. Let W be a module over k that is a left module over A again, so that W_ε is a two-sided module over A. A homomorphism f from A into W_ε, as modules over k, is a crossed homomorphism if

$$f(a\,b) = a \cdot f(b) + f(a) \cdot b = a \cdot f(b) + \varepsilon(b)\, f(a) \tag{28.3.1}$$

for every $a, b \in A$, as in Section 26.3. We may now consider this as a *crossed homomorphism* from A into W, as on p183 of [3].

Let U be another module over k that is a left module over U, so that U_ε is a two-sided module over A. If g is a homomorphism from U into W, as modules over k and left modules over A, then g is a homomorphism from U_ε into W_ε, as two-sided modules over A. If f_0 is a crossed homomorphism from A into U, then

(28.3.2) $g \circ f_0$ is a crossed homomorphism from A into W,

as in Section 26.3.

If $a \in A$, then put

$$p(a) = a - \varepsilon(a)\, e_A, \tag{28.3.3}$$

which is an element of $\mathcal{I}$. Note that $p(a) = a$ when $a \in \mathcal{I}$. If $a, b \in \mathcal{I}$, then

$$p(a\,b) = a\,b - \varepsilon(a\,b)\, e_A = a\,b - \varepsilon(a)\,\varepsilon(b)\, e_A = a\,p(b) + \varepsilon(b)\,p(a). \tag{28.3.4}$$

Of course, $\mathcal{I}$ may be considered as a two-sided module over A, because $\mathcal{I}$ is a two-sided ideal in A. Let $\mathcal{I}^L$ be $\mathcal{I}$, considered as a module over k, and a left module over A. Thus $(\mathcal{I}^L)_\varepsilon$ is a two-sided module over A, although the action of A on the right is not normally the usual one.

It follows from (28.3.4) that

(28.3.5) p is a crossed homomorphism from A into $\mathcal{I}^L$.

Let g be a homomorphism from $\mathcal{I}^L$ into W, as left modules over A. Observe that

(28.3.6) $g \circ p$ is a crossed homomorphism from A into W,

by (28.3.2).

If f is a crossed homomorphism from A into W, then $f(e_A) = 0$, as in Section 26.3. This implies that

$$f(p(a)) = f(a) - \varepsilon(a)\, f(e_A) = f(a) \tag{28.3.7}$$

for every $a \in A$. In particular,

(28.3.8) f is uniquely determined by its restriction to $\mathcal{I}$.

If f is a crossed homomorphism from A into W, then

(28.3.9) the restriction of f to $\mathcal{I}$ defines a homomorphism from $\mathcal{I}^L$ into W,

as modules over k, and left modules over A. Combining this with (28.3.7), we get that f can be expressed as

$$f = g \circ p, \tag{28.3.10}$$

where g is a homomorphism from $\mathcal{I}^L$ into W, as modules over k, and left modules over A. This representation is unique, because it implies that

$$g(a) = g(p(a)) = f(a) \tag{28.3.11}$$

for every $a \in \mathcal{I}$. This corresponds to a remark on p183 of [3].

If $w \in W$, then $a \mapsto a \cdot w$ defines a homomorphism from A into W, as modules over k, and left modules over A. This implies that

$$a \mapsto p(a) \cdot w = a \cdot w - \varepsilon(a)\, w \tag{28.3.12}$$

defines a crossed homomorphism from A into W. A crossed homomorphism of this form is said to be *principal*, as in Section 26.4. This is the same as a principal crossed homomorphism from A into W_ε, as a two-sided module over A.

Remember that a homomorphism from A into W, as left modules over A, is of the form $a \mapsto a \cdot w$ for a unique $w \in W$. Thus a crossed homomorphism from A into W is principal exactly when it can be expressed as in (28.3.10), where g is a homomorphism from A into W, as left modules over A. This corresponds to another remark on p183 of [3].

The set of all crossed homomorphisms from A into W is a submodule of the space $\mathrm{Hom}_k(A, W)$ of all homomorphisms from A into W, as modules over k, as in Section 26.3. We have an isomorphism from

$$\mathrm{Hom}_A(\mathcal{I}^L, W) \tag{28.3.13}$$

onto the space of crossed homomorphisms from A into W, as modules over k, defined by composition with p, as before. Similarly, there is an isomorphism from

$$\mathrm{Hom}_{A^{en}}(J, W_\varepsilon) \tag{28.3.14}$$

onto the space of crossed homomorphisms from A into W_ε, as modules over k, as in Section 26.3. It follows that (28.3.13) and (28.3.14) are isomorphic as modules over k, as on p184 of [3].

The map from $w \in W$ to (28.3.12) is clearly linear over k. In particular, the set of all principal crossed homomorphisms from A into W is a submodule of the space of all crossed homomorphisms from A into W, as a module over k.

Consider the obvious homomorphism

$$\text{from } \mathrm{Hom}_A(A, W) \text{ into } \mathrm{Hom}_A(\mathcal{I}^L, W), \tag{28.3.15}$$

which sends a homomorphism from A into W, as left modules over A, to its restriction to $\mathcal{I}$. The image of this homomorphism corresponds to the space of all principal crossed homomorphisms from A into W, under the isomorphism from (28.3.13) onto the space of all crossed homomorphisms from A into W mentioned earlier.

This leads to an isomorphism from the cokernel of the homomorphism as in (28.3.15) onto

(28.3.16) the quotient of the space of all crossed homomorphisms from A into W by the submodule consisting of all principal crossed homomorphisms from A into W, as a module over k. Note that the cokernel of the homomorphism as in (28.3.15) is also isomorphic to

(28.3.17) the first cohomology group of A as a left augmented ring with coefficients in W,

as in Section 24.5. Thus (28.3.16) is isomorphic to (28.3.17), as modules over k, as mentioned on p184 of [3].

Let $H^1(A, W_\varepsilon)$ be as in Section 26.5. This is isomorphic to (28.3.16), as modules over k, as in Section 26.6. It follows that (28.3.17) is isomorphic to $H^1(A, W_\epsilon)$, as modules over k, as in (3) on p184 of [3].

28.4 Modules with trivial operators

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k, with multiplicative identity element e_A. Suppose that A is a supplemented algebra over k, with corresponding homomorphism ε from A onto k, as in Section 28.1. Let $\mathcal{I}$ be the kernel of ε, as before.

Let V be a module over k that is a right module over A with trivial operators, so that $v \cdot a = \varepsilon(a)\, v$ for every $a \in A$ and $v \in V$, as in Section 28.1. Let $V \bigotimes_A k$ be a tensor product of V and k over A, where k is considered as a left module over A using ε, as before. This is isomorphic to $V_A = V/(V \cdot \mathcal{I})$, as a module over k, as in Section 28.2. In this case,

$$V \cdot \mathcal{I} = \{0\}, \tag{28.4.1}$$

by hypothesis. This means that $V \bigotimes_A k$ is isomorphic to V, as a module over k. This is the same as saying that the 0th homology group of A as a left augmented ring with coefficients in V is isomorphic to V. This corresponds to a remark on p184 of [3].

Similarly, let W be a module over k that is a left module over A with trivial operators, so that the submodule W^A of invariant elements of W is equal to W. This implies that the space $\mathrm{Hom}_A(k, W)$ of homomorphisms from k into W, as modules over k, and left modules over A, is isomorphic to W in a natural way, as a module over k, as in Section 28.2. This means that the 0th cohomology group of A as a left augmented ring with coefficients in W is isomorphic to W, as in Section 24.5. This corresponds to another remark on p184 of [3].

In this case, a homomorphism f from A into W, as modules over k, is a crossed homomorphism if

$$f(a\, b) = \varepsilon(a)\, f(b) + \varepsilon(b)\, f(a) \tag{28.4.2}$$

for every $a, b \in A$. The only principal crossed homomorphism is equal to 0 on A. Thus the quotient (28.3.16) is the same as the space of all crossed homomorphisms from A into W, as on p184 of [3].

Let V be a module over k that is a right module over A with trivial operators again, and let

$$V \bigotimes_A \mathcal{I} \tag{28.4.3}$$

be a tensor product of V and $\mathcal{I}$ over A, where $\mathcal{I}$ is considered as a left module over A. There is a natural homomorphism from (28.4.3) into V, as modules over k, that comes from the mapping $(v, b) \mapsto v \cdot b$ from $V \times \mathcal{I}$ into V, as in Section 24.3. This homomorphism is equal to 0 here, because V is a right module over A with trivial operators. This means that the one-dimensional homology of A as a left augmented ring with coefficients in V is isomorphic to (28.4.3), as in Section 24.3.

Remember that V satisfies the requirements of a tensor product $V \bigotimes_k k$ of V and k over k, as a module over k. One can check that this is compatible with the action of A on the right on $V \bigotimes_k k$ that is obtained from the action of A on k using ε. Thus (28.4.3) may be identified with a tensor product of the form

$$\Big(V \bigotimes_k k\Big) \bigotimes_A \mathcal{I}. \tag{28.4.4}$$

There is a natural isomorphism between (28.4.4) and a tensor product of the form

$$V \bigotimes_k \Big(k \bigotimes_A \mathcal{I}\Big), \tag{28.4.5}$$

where k is considered as a right module over A, as in Section 1.12. Remember that $\mathcal{I}^2 = \mathcal{I}\,\mathcal{I}$ consists of finite sums of products of the form $a\,b$, with $a, b \in \mathcal{I}$. This is a submodule of $\mathcal{I}$, as a left and right module over A, and we have seen that there is an isomorphism

$$\text{from } \mathcal{I}/\mathcal{I}^2 \text{ onto } k \bigotimes_A \mathcal{I}, \tag{28.4.6}$$

as left and right modules over A, as in Section 24.4. Thus we get that (28.4.3) is isomorphic to

$$V \bigotimes_k (\mathcal{I}/\mathcal{I}^2), \tag{28.4.7}$$

as on p184 of [3].

28.5 Homomorphisms and supplemented algebras

Let k, k_1 be commutative rings with multiplicative identity elements 1_k, 1_{k_1}, respectively, and let A, B be associative algebras over k, k_1, with multiplicative identity elements e_A, e_B, respectively. Suppose that ψ is a ring homomorphism from k into k_1, with $\psi(1_k) = 1_{k_1}$, and that ϕ is a ring homomorphism from A into B, with $\phi(e_A) = e_B$, and

$$\phi(t\,a) = \psi(t)\,\phi(a) \tag{28.5.1}$$

for every $a \in A$ and $t \in k$.

Suppose that A, B are in fact supplemented algebras, with corresponding homomorphisms ε_A, ε_B from A, B onto k, k_1, respectively, as in Section 28.1. If

$$\varepsilon_B \circ \phi = \psi \circ \varepsilon_A, \tag{28.5.2}$$

then the pair ϕ, ψ is considered as a *map* or *homomorphism* between supplemented algebras, as on p184 of [3]. Of course, this implies that

$$\phi(\ker \varepsilon_A) \subseteq \ker \varepsilon_B. \tag{28.5.3}$$

This means that we get a homomorphism from A into B as left or right augmented rings, with respect to the augmentations defined by ε_A, ε_B, as in Section 24.9. This corresponds to a remark on p184 of [3].

If V is a right module over B, then V may be considered as a right module over A, using ϕ. In this case, we can use ϕ to get a homomorphism from the homology of A into the homology of B, as left augmented rings, with coefficients in V, as in Section 24.3. Similarly, if Z is a left module over B, then Z may be considered as a left module over A, using ϕ. This leads to a homomorphism from the cohomology of B into the cohomology of A, as left augmented rings, with coefficients in Z, as before. This corresponds to another remark on p184 of [3].

In particular, we may be interested in the case where $k = k_1$ and ψ is the identity mapping on k, as on p184 of [3].

28.6 Homomorphisms from tensor products

Let k be a commutative ring with a multiplicative identity element, and let A, C be associative algebras over k with multiplicative identity elements e_A, e_C, respectively. Suppose that A is a supplemented algebra, with corresponding homomorphism ε_A from A onto k, as in Section 28.1.

Let $C \bigotimes_k A$ be a tensor product of C and A over k, which may be considered as an associative algebra over k, as in Section 4.1. Remember that C satisifes the requirements of a tensor product of C and k over k, as a module over k. One can check that this is also compatible with multiplication on C and on $C \bigotimes_k k$, as a tensor product of associative algebras over k. Using this, the identity mapping on C, and ε_A, we get a homomorphism

$$\varepsilon_{C \bigotimes_k A} \text{ from } C \bigotimes_k A \text{ onto } C, \tag{28.6.1}$$

as algebras over k. Alternatively,

$$\varepsilon_A \, e_C \tag{28.6.2}$$

may be considered as a homomorphism from A into the center of C, as algebras over k. One can use the identity mapping on C and (28.6.2) to get a unique homomorphism as in (28.6.1), as algebras over k, such that

$$\varepsilon_{C \bigotimes_k A}(c \otimes a) = \varepsilon_A(a) \, c \tag{28.6.3}$$

for every $a \in A$ and $c \in C$, as in Section 4.1. Remember that $e_C \otimes e_A$ is the multiplicative identity element in $C \bigotimes_k A$, and observe that

$$\varepsilon_{C \bigotimes_k A}(e_C \otimes e_A) = e_C, \tag{28.6.4}$$

because $\varepsilon_A(e_A) = 1$.

We may consider $C \bigotimes_k A$ as a left or right augmented ring, with augmentation homomorphism $\varepsilon_{C \bigotimes_k A}$ and augmentation module C, as in Section 25.11. This is mentioned at the bottom of p184 of [3].

As before,

$$\phi(a) = e_C \otimes a \tag{28.6.5}$$

defines a homomorphism from A into $C \bigotimes_k A$, as algebras over k. Of course,

$$\psi(t) = t\, e_C \tag{28.6.6}$$

defines a homomorphism from k into C, as algebras over k. If $a \in A$, then

$$\varepsilon_{C \bigotimes_k A}(\phi(a)) = \varepsilon_{C \bigotimes_k A}(e_C \otimes a) = \varepsilon_A(a)\, e_C = \psi(\varepsilon_A(a)). \tag{28.6.7}$$

In particular, ϕ maps the kernel of ε_A into the kernel of $\varepsilon_{C \bigotimes_k A}$. This implies that ϕ is a homomorphism from A into $C \bigotimes_k A$ as augmented rings, as in Section 24.9.

Let V be a module over k that is a right module over $C \bigotimes_k A$. This means that V is a right module over A and C, where the actions of A and C commute with each other. We can use ϕ to get a homomorphism

(28.6.8) from the homology of A into the homology of $C \bigotimes_k A$,
as left augmented rings, with coefficients in V,

as in Section 25.12. This corresponds to (5) on p185 of [3].

Similarly, let Z be a module over k that is a left module over $C \bigotimes_k A$, which means that Z is a left module over A and C, where the actions of A and C on Z commute with each other. We can use ϕ to get a homomorphism

(28.6.9) from the cohomology of $C \bigotimes_k A$ into the cohomology of A,
as left augmented rings, with coefficients in Z,

as in Section 25.12 again. This corresponds to (5a) on p185 of [3].

Suppose now that

(28.6.10) A is projective as a module over k.

Note that the augmentation module $\varepsilon_A(A) = k$ of A is automatically projective as a module over k in this case. This implies that

(28.6.11) the homomorphisms as in (28.6.8) and (28.6.9) are isomorphisms, as in Section 25.12. If X is a projective resolution of k, as a left module over A using ε_A, then we also get that

$$C \bigotimes_k X \text{ is a projective resolution of } C, \text{ as a left module over } C \bigotimes_k A, \tag{28.6.12}$$

as before. This corresponds to Proposition 1.1 on p185 of [3].

If C is a commutative algebra over k, then $C \bigotimes_k A$ may be considered as a commutative algebra over C. In this case,

$$C \bigotimes_k A \text{ may be considered as a supplemented algebra,} \tag{28.6.13}$$

as an algebra over C, using the homomorphism $\varepsilon_{C \bigotimes_k A}$, as on p185 of [3].

28.7 Connection with A^{en}

Let k be a commutative ring with a multilpicative identity element, and let A be an associative algebra over k, with a multiplicative identity element e_A. Suppose that A is a supplemented algebra over k, with corresponding homomorphism ε from A onto k, as in Section 28.1.

Remember that A^{op} is the opposite algebra of A, and let $A^{en} = A \bigotimes_k A^{op}$ be a tensor product of A and A^{op} over k, considered as an algebra over k, as in Section 26.1. Also let ρ be the homomorphism from A^{en} onto A, as left modules over A^{en}, defined in Section 26.1.

Observe that

$$(a_1, a_2^{op}) \mapsto \varepsilon(a_2)\, a_1 \tag{28.7.1}$$

defines a mapping from $A \times A^{op}$ into A that is bilinear over k. This leads to a unique homomorphism

$$\phi \text{ from } A^{en} \text{ into } A, \tag{28.7.2}$$

as modules over k, such that

$$\phi(a_1 \otimes a_2^{op}) = \varepsilon(a_2)\, a_1 \tag{28.7.3}$$

for every $a_1, a_2 \in A$. It is easy to see that this defines a homomorphism as in (28.7.2), as algebras over k.

If $a_1, a_2 \in A$, then

$$\begin{aligned} \varepsilon(\phi(a_1 \otimes a_2^{op})) &= \varepsilon(\varepsilon(a_2)\, a_1) = \varepsilon(a_1)\, \varepsilon(a_2) \\ &= \varepsilon(a_1\, a_2) = \varepsilon(\rho(a_1 \otimes a_2^{op})). \end{aligned} \tag{28.7.4}$$

This implies that

$$\varepsilon \circ \phi = \varepsilon \circ \rho, \tag{28.7.5}$$

as mappings from A^{en} into k. It follows that

$$\phi(\ker \rho) \subseteq \ker \varepsilon. \tag{28.7.6}$$

This means that ϕ is a homomorphism as in (28.7.2), as augmented rings with respect to the augmentations defined by ρ and ε, as in Section 24.9. This corresponds to some remarks on p185 of [3].

Let V be a module over k that is a right module over A. We may consider V as a right module over A^{en}, using ϕ, as in Section 2.9. The action of A^{en} on V on the right corresponds to commuting actions of A and A^{op} on V on the right, as in Section 4.2. It is easy to see that the action of A on V on the right obtained from this action of A^{en} on V on the right is the same as the action of A on V as a right module over A. The action of A^{op} on V on the right obtained from this action of A^{en} on V on the right is the same as the trivial action on V defined using ε, as in Section 28.1.

This action of A^{op} on V on the right corresponds exactly to an action of A on V on the left, as usual. This action of A on V on the left is the same as the trivial action defined using ε, as before. This makes V into a two-sided module over A, corresponding to V as a right module over A^{en}, as in Section 26.1. This is the same as ${}_{\varepsilon}V$ as a two-sided module over A, as in Section 28.1, with the trivial action of A on V on the left, and the action of A on V as a right module over A. Thus we may use ${}_{\varepsilon}V$ to refer to V as a right module over A^{en} in this way.

If n is a nonnegative integer, then we can use ϕ to get a homomorphism

$$\text{from } H_n(A, {}_{\varepsilon} V) \text{ into the } n\text{th homology group of } A, \quad \text{as a left augmented ring, with coefficients in } V, \tag{28.7.7}$$

where $H_n(A, {}_{\varepsilon} V)$ is as in Section 26.5. This follows from the discussion in Section 24.9, because $H_n(A, {}_{\varepsilon} V)$ is the same as the nth homology group of A^{en}, as a left augmented ring, with coefficients in ${}_{\varepsilon}V$, as a right module over A^{en}. This corresponds to some more remarks on p185 of [3].

Similarly, let Z be a module over k that is a left module over A. We may consider Z as a left module over A^{en}, using ϕ, as in Section 2.9 again. The action of A^{en} on Z on the left corresponds to commuting actions of A and A^{op} on Z on the left, as before. The action of A on Z on the left obtained from the action of A^{en} on Z on the left is the same as the action of A on Z as a left module over A. The action of A^{op} on Z on the left obtained from the action of A^{en} on Z on the left is the same as the trivial action on Z defined using ε, because of the way that ϕ is defined.

This action of A^{op} on Z on the left corresponds to an action of A on Z on the right. This action of A on Z on the right is the same as the trivial action of A on V defined using ε. This makes Z into a two-sided module over A, which corresponds to Z as a left module over A^{en}. This is the same as Z_{ε} as a two-sided module over A, as in Section 28.1. We may use Z_{ε} to refer to Z as a left module over A^{en} in this way, as before.

We can use ϕ to get a homomorphism

$$\text{from the } n\text{th cohomology group of } A, \text{ as a left augmented ring, with coefficients in } Z, \text{ into } H^n(A, Z_{\varepsilon}), \tag{28.7.8}$$

for each $n \geq 0$, where $H^n(A, Z_\varepsilon)$ is as in Section 26.5. This follows from the discussion in Section 24.9 again, as on p185 of [3].

If

$$A \text{ is projective as a module over } k, \tag{28.7.9}$$

then Theorem 2.1 on p185 of [3] states that

(28.7.10) the homomorphisms as in (28.7.7) and (28.7.8) are isomorphisms.

We shall discuss this further in the next two sections.

28.8 A helpful isomorphism

Let us continue with the same notation and hypotheses as in the previous section. Also let ${}_\varepsilon A$ be as in Section 28.1, where A is considered as a module over k, and a right module over itself. Thus ${}_\varepsilon A$ is a two-sided module over A, where the left action of A on ${}_\varepsilon A$ is the trivial action. Of course, we may consider ${}_\varepsilon A$ as a left or right module over A^{en}, as in Section 26.1.

Suppose that W is another module over k that is a two-sided module over A, so that W may be considered as a left or right module over A^{en} as well. Let

$${}_\varepsilon A \bigotimes_{A^{en}} W \tag{28.8.1}$$

be a tensor product of ${}_\varepsilon A$, as a right module over A^{en}, and W, as a left module over A^{en}, over A^{en}. Remember that k may be considered as a left or right module over A, using ε, as in Section 28.1. Let

$$W \bigotimes_A k \tag{28.8.2}$$

be a tensor product of W, as a right module over A, and k, as a left module over A, over A.

Consider the mapping from ${}_\varepsilon A \times W$ into $W \bigotimes_A k$ defined by

$$(a, w) \mapsto (a \cdot_A w) \otimes_A 1_k, \tag{28.8.3}$$

where 1_k is the multiplicative identity element in k. More precisely, if $a \in {}_\varepsilon A$ and $w \in W$, then $a \cdot_A w \in W$ is defined using the action of a as an element of A on W on the left. The mapping defined by (28.8.3) is clearly bilinear over k. If $x \in A^{en}$, then we would like to check that

$$((a \cdot_{A^{en}} x) \cdot_A w) \otimes_A 1_k = (a \cdot_A (x \cdot_{A^{en}} w)) \otimes_A 1_k, \tag{28.8.4}$$

where $a \cdot_{A^{en}} x$ is defined using the action of A^{en} on A on the right, and $x \cdot_{A^{en}} w$ is defined using the action of A^{en} on W on the left.

It suffices to verify (28.8.4) when $x \in A^{en}$ is of the form $x_1 \otimes_k x_2^{op}$, where $x_1, x_2 \in A$. In this case, (28.8.4) is the same as saying that

$$((x_2 \cdot_A (a \cdot_A x_1)) \cdot_A w) \otimes_A 1_k = (a \cdot_A ((x_1 \cdot_A w) \cdot x_2)) \otimes_A 1_k, \tag{28.8.5}$$

using the various actions of A on ${}_\varepsilon A$ and W on the left and right. The left side is equal to

$$(\varepsilon(x_2)\,(a\,x_1)\cdot_A w)\otimes_A 1_k \;=\; ((a\,x_1)\cdot_A w)\otimes_A \varepsilon(x_1) \;=\; ((a\,x_1)\cdot_A w)\otimes_A (x_1\cdot_A 1_k), \tag{28.8.6}$$

because the actions of A on ${}_\varepsilon A$ and k on the left are trivial. This is equal to

$$(((a\,x_1)\cdot_A w)\cdot_A x_1)\otimes_A 1_k, \tag{28.8.7}$$

because of the way that $W \bigotimes_A k$ is defined. It is easy to see that this is equal to the right side of (28.8.5), as desired.

It follows that there is a unique homomorphism

$$\tau \text{ from } {}_\varepsilon A \bigotimes_{A^{en}} W \text{ into } W \bigotimes_A k, \tag{28.8.8}$$

as modules over k, such that

$$\tau(a\otimes_{A^{en}} w) = (a\cdot_A w)\otimes_A 1_k \tag{28.8.9}$$

for every $a\in {}_\varepsilon A$ and $w \in W$. Lemma 2.2 on p186 of [3] states that

$$\tau \text{ is an isomorphism.} \tag{28.8.10}$$

To show this, we shall find an inverse mapping σ.

Consider the mapping from $W\times k$ into ${}_\varepsilon A \bigotimes_{A^{en}} W$ defined by

$$(w,t)\mapsto (t\,e_A)\otimes_{A^{en}} w. \tag{28.8.11}$$

Note that this mapping is bilinear over k. We would like to check that

$$(t\,e_A)\otimes_{A^{en}} (w\cdot_A a) = ((a\cdot_A t)\,e_A)\otimes_{A^{en}} w \tag{28.8.12}$$

for every $a\in A$, $w\in W$, and $t\in k$.

Of course, $e_A\otimes_k a^{op}\in A^{en}$, and

$$w\cdot_A a = (e_A\otimes_k a^{op})\cdot_{A^{en}} w. \tag{28.8.13}$$

Using this, we get that the left side of (28.8.12) is equal to

$$(t\,e_A)\otimes_{A^{en}} ((e_A\otimes_k a^{op})\cdot w) = ((t\,e_A)\cdot(e_A\otimes_k a^{op}))\otimes_{A^{en}} w, \tag{28.8.14}$$

because of the way that ${}_\varepsilon A \bigotimes_{A^{en}} W$ is defined. This is equal to

$$(t\,\varepsilon(a)\,e_A)\otimes_{A^{en}} w, \tag{28.8.15}$$

because the action of A on ${}_\varepsilon A$ on the left is trivial. This is the same as the right side of (28.8.12), as desired.

Thus there is a unique homomorphism

$$\sigma \text{ from } W \bigotimes_A k \text{ into } {}_\varepsilon A \bigotimes_{A^{en}} W, \tag{28.8.16}$$

as modules over k, such that

$$\sigma(w \otimes_A t) = (t\, e_A) \otimes_{A^{en}} w \tag{28.8.17}$$

for every $w \in W$ and $t \in k$. If $a \in {}_\varepsilon A$ and $w \in W$, then

$$\sigma(\tau(a \otimes_{A^{en}} w)) = \sigma((a \cdot_A w) \otimes_A 1_k) = e_A \otimes_{A^{en}} (a \cdot_A w). \tag{28.8.18}$$

Observe that $a \otimes e_A^{op} \in A^{en}$, and that

$$a \cdot_A w = (a \otimes_k e_A^{op}) \cdot_{A^{en}} w. \tag{28.8.19}$$

It follows that (28.8.18) is equal to

$$\begin{aligned} e_A \otimes_{A^{en}} ((a \otimes_k e_A^{op}) \cdot_{A^{en}} w) &= (e_A \cdot_{A^{en}} (a \otimes e_A^{op})) \otimes_{A^{en}} w \\ &= a \otimes_{A^{en}} w. \end{aligned} \tag{28.8.20}$$

This implies that $\sigma \circ \tau$ is the identity mapping on ${}_\varepsilon A \bigotimes_{A^{en}} W$.

Similarly, if $w \in W$ and $t \in k$, then

$$\begin{aligned} \tau(\sigma(w \otimes_A t)) &= \tau((t\, e_A) \otimes_{A^{en}} w) \\ &= ((t\, e_A) \cdot_A w) \otimes_A 1_k = w \otimes_A t. \end{aligned} \tag{28.8.21}$$

This means that $\tau \circ \sigma$ is the identity mapping on $W \bigotimes_A k$. Thus σ and τ are inverses of each other, as on p186 of [3].

28.9 Some verifications

We continue with the same notation and hypotheses as in the previous two sections. In other to show (28.7.10), we want to use the Mapping Theorem, as in Section 24.10. In this discussion, A^{en} plays the role of A in Sections 24.9 and 24.10, and A plays the role of B before. Similarly, A plays the role of Q_A before, as a left module over A^{en}, and k plays the role of Q_B, as a left module over A.

The homomorphism ϕ from A^{en} into A in Section 28.7 plays the same role as the homomorphism ϕ from A into B in Section 24.3. The mapping ψ from Q_A into Q_B before corresponds to ε here.

In Sections 24.9 and 24.10, B was sometimes considered as a right module over A, using the homomorphism ϕ, as in Section 2.9. This corresponds to considering A as a right module over A^{en} using ϕ here. It is easy to see that this is the same as ${}_\varepsilon A$, considered as a right module over A^{en}, as in the previous section. Of course, this is not normally the same as considering A as a right module over A^{en} in the usual way.

Thus the analogue of $B \bigotimes_A Q_A$ in Section 24.9 is a tensor product

$$ {}_\varepsilon A \bigotimes_{A^{en}} A, \tag{28.9.1}$$

where A is considered as a left module over A^{en} in the usual way. If we take $W = A$ in the previous section, then we get an isomorphism τ from (28.9.1) onto $A \bigotimes_A k$, which can be identified with k as a left module over A in the usual way. This corresponds to the first condition that we need from Section 24.10, as on p186 of [3].

To deal with the second condition that we need from Section 24.10, we let X be a projective resolution of A, as a left module over A^{en}. This corresponds to X_A in Section 24.9. Also let

$$ {}_\varepsilon A \bigotimes_{A^{en}} X \tag{28.9.2}$$

be a tensor product of ${}_\varepsilon A$ and X over A^{en}, which corresponds to $B \bigotimes_A X_A$ in Section 24.9. This may be considered as a complex over k, as in Section 7.5. The second condition that we need from Section 24.10 is that

$$ H\Big({}_\varepsilon A \bigotimes_{A^{en}} X\Big)_n = \{0\} \tag{28.9.3}$$

for every $n > 0$.

Of course, we may consider X as a two-sided module over A, because it is a left module over A^{en}. Let

$$ X \bigotimes_A k \tag{28.9.4}$$

be a tensor product of X, as a right module over A, and k, as a left module over A, over A. This may be considered as a complex over k too, as in Section 7.5.

If we take $W = X$ in the previous section, then we get an isomorphism τ from (28.9.2) onto (28.9.4), as modules over k. It is easy to see that τ is of degree 0, and in fact an isomorphism from (28.9.2) onto (28.9.4), as complexes over k. This induces an isomorphism between the homology groups of these complexes, so that (28.9.3) is equivalent to

$$ H\big(X \bigotimes_A k\big)_n = \{0\} \tag{28.9.5}$$

for every $n > 0$, as on p186 of [3].

Note that X^j is projective as a left module over A^{en} for each j, by hypothesis. If

$$ A \text{ is projective as a module over } k, \tag{28.9.6}$$

then it follows that X^j is projective as a left module over A^{op} for each j, as in Section 25.7, because $A^{en} = A \bigotimes_k A^{op}$. Equivalently, this means that

$$ X^j \text{ is projective as a right module over } A \tag{28.9.7}$$

for each j, as on p186 of [3].

By hypothesis, X is an acyclic left complex over A, as a left module over A^{en}. It is easy to see that X may be considered as an acyclic left complex over A, as a left module over A^{op}, and thus as a right module over A. It follows from this and (28.9.7) that X may be considered as a projective resolution of A, as a right module over itself.

Of course, A is projective as a right module over itself, and so we can get another projective resolution Y of A, as a right module over itself, by taking $Y^0 = A$ and $Y^j = \{0\}$ when $j \neq 0$. Under these conditions, the homology groups of $X \bigotimes_A k$ are isomorphic to those of $Y \bigotimes_A k$, as in Section 10.12. Clearly

$$H\big(Y \bigotimes\nolimits_A k\big)_n = \{0\} \tag{28.9.8}$$

for every $n > 0$. This implies that the second condition that we need from Section 24.10 holds when (28.9.6) holds, as on p186 of [3].

Thus (28.7.10) holds when (28.9.6) holds, as in Section 24.10.

28.10 Some related projective resolutions

Let us continue with the same notation and hypotheses as in the previous three sections. Let $\widetilde{A}$ be A, considered as a module over k in the usual way, as well as a left module over A and a right module over A^{en}, as follows. The action of A on itself on the left is taken to be the usual one, using multiplication in the algebra. The action of A^{en} on the right is obtained using the homomorphism ϕ from A^{en} onto A defined in Section 28.7, as in Section 2.9. This corresponds to ${}_\varepsilon A$ as a right module over A^{en}, as in the previous section.

One can check that these actions of A on $\widetilde{A}$ on the left, and of A^{en} on $\widetilde{A}$ on the right, commute with each other. Of course, the usual actions of A on itself on the left and right commute with each other, by associativity. The commutativity of the two actions on $\widetilde{A}$ just mentioned basically involves the commutativity of the usual left action of A on itself with the trivial action of A on itself defined using ε as well. The latter can be verified using the linearity over k of multiplication on A.

Let X be a projective resolution of A, as a left module over A^{en} in the usual way, as in the previous section. A tensor product

$$\widetilde{A} \bigotimes\nolimits_{A^{en}} X \tag{28.10.1}$$

of $\widetilde{A}$, as a right module over A^{en}, and X, as a left module over A^{en}, over A^{en}, is the same as (28.9.2), as a module over k. We may consider (28.10.1) as a left module over A, using the action of A on $\widetilde{A}$ on the left, because that commutes with the action of A^{en} on $\widetilde{A}$ on the right, as in the preceding paragraph. This is not normally the same as the analogous action of A on (28.9.2) on the left.

Remember that X may be considered as a two-sided module over A, and that $X \bigotimes_A k$ is a tensor product of X, as a right module over A, and k, as a left module over A, over A, as in (28.9.4). We may consider $X \bigotimes_A k$ as a left module over A, using the left action of A on X.

As in the previous section, we can take $W = X$ in Section 28.8, to get an isomorphism τ from (28.10.1) onto $X \bigotimes_A k$, as modules over k. One can check that τ is an isomorphism from (28.10.1) onto $X \bigotimes_A k$, as left modules over A.

More precisely, (28.10.1) and $X \bigotimes_A k$ may be considered as complexes over k, as in the previous section, and in fact as complexes over A. We also have that τ is an isomorphism from (28.10.1) onto $X \bigotimes_A k$ as complexes over k, as before, and thus as complexes over A.

Let us suppose from now on in this section that A is projective as a module over k, as in (28.9.6). This means that the second as well as the first of the usual conditions from Section 24.10 are satisfied, as in the previous section. It follows that

(28.10.2) (28.10.1) is a projective resolution of k, as a left module over A,

as in Section 24.10. One can use this and the fact that τ is an isomorphism from (28.10.1) onto $X \bigotimes_A k$, as complexes over A, to get that

(28.10.3) $X \bigotimes_A k$ is a projective resolution of k, as a left module over A.

This corresponds to the second part of Theorem 2.1 on p185 of [3].

One can also get (28.10.3) a bit more directly, using the same type of arguments. Remember that k satisfies the requirements of $A \bigotimes_A k$, as a module over k, and as a left module over A, as in Section 1.10. This implies that $X \bigotimes_A k$ is a left complex over k, as a left module over A, because X is a left complex over A, as a left module over A^{en}, and thus as a left module over A.

If (28.9.6) holds, then one can check that $X \bigotimes_A k$ is acyclic as a left complex over k, as a left module over A. This uses (28.9.5) and the acyclicity of X, as a left complex over A, as a left module over A^{en}, and thus over A.

Remember that X^j is projective as a left module over A^{en} for each j, by hypothesis. We may consider X^j as a two-sided module over A for each j, or equivalently a right module over A^{en}. One can check that X^j is projective as a right module over A^{en} for each j as well.

It follows that

$$\left(X \bigotimes_A k\right)^j = X^j \bigotimes_A k \tag{28.10.4}$$

is projective as a right module over $A^{op} \bigotimes_k k = A^{op}$ for each j, as in Sections 25.6 and 25.8, because k is projective as a module over itself. Equivalently, this means that (28.10.4) is projective as a left module over A.

28.11 Standard complexes and resolutions

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that A is a supplemented algebra over k, with corresponding homomorphism ε from A onto k, as in Sections 28.1 and 28.7. Let $A^{en} = A \bigotimes_k A^{op}$ be a tensor product of A and the opposite algebra A^{op} over k, considered as an algebra over k, as in Section 26.1.

Remember that A corresponds to the direct sum of

$$\{t\, e_A : t \in k\} \tag{28.11.1}$$

and the kernel $\mathcal{I}$ of ε, as a module over k, as in Section 28.1. Put

$$A' = A/\{t\, e_A : t \in k\}, \tag{28.11.2}$$

considered as a module over k, as in Section 26.12. It follows that

$$A' \text{ is isomorphic to } \mathcal{I}, \tag{28.11.3}$$

as a module over k.

Suppose for the rest of the section that

$$A \text{ is projective as a module over } k. \tag{28.11.4}$$

This implies that $\mathcal{I}$ is projective as a module over k, as in Section 2.7. Equivalently, this means that

$$A' \text{ is projective as a module over } k, \tag{28.11.5}$$

by (28.11.3). This corresponds to a remark on p186 of [3].

If X is a projective resolution of A, as a left module over A^{en}, then a tensor product $X \bigotimes_A k$ of X, as a right module over A, and k, as a left module over A, over A, is a projective resolution of k, as a left module over A, as in the previous section. Remember that the standard complex $S(A)$ of A is a projective resolution of A, as a left module over A^{en}, as in Section 26.10, because of (28.11.4). Similarly, the normalized standard complex $N(A)$ of A is a projective resolution of A, as a left module over A^{en}, as in Section 26.14, because of (28.11.5). Let

$$S(A, \varepsilon) = S(A) \bigotimes_A k \tag{28.11.6}$$

and

$$N(A, \varepsilon) = N(A) \bigotimes_A k \tag{28.11.7}$$

be tensor products of $S(A)$, $N(A)$, as right modules over A, and k, as a left module over A, over A, respectively, as on p186 of [3]. Of course, we would like to say more about how these tensor products behave, as in [3].

In particular, $S(A)$ is a graded left module over A^{en}, with $S(A)_n = S_n(A)$ when $n \geq 0$, and $S(A)_n = \{0\}$ when $n \leq 0$, as in Section 26.10, where $S_n(A)$ is as in Section 26.8. Let $\widetilde{S}_n(A)$ be a tensor product of n copies of A over k when $n \geq 1$, and put $\widetilde{S}_0(A) = k$, as in Section 26.10. We may consider $S_n(A)$ as a tensor product of the form

$$S_n(A) = A \bigotimes_k \widetilde{S}_n(A) \bigotimes_k A \tag{28.11.8}$$

for each $n \geq 0$, where the actions of A on the left and right on $S_n(A)$ are obtained from the actions of A on the left and on the right on the first and last factors of A on the right side of (28.11.8), as before.

More precisely, $S(A, \varepsilon)$ is a graded left module over A, with

$$S(A,\varepsilon)_n = S(A)_n \bigotimes_A k = S_n(A) \bigotimes_A k \quad \text{when } n \geq 0 \tag{28.11.9}$$
$$= \{0\} \quad \text{when } n < 0,$$

as in Section 7.5. If $n \geq 0$, then $S(A, \varepsilon)_n$ is isomorphic to

$$A \bigotimes_k \widetilde{S}_n(A) \bigotimes_k \Big(A \bigotimes_A k\Big) \tag{28.11.10}$$

in a natural way, as in Section 1.12. Thus we may take $S(A, \varepsilon)_n$ to be a tensor product of the form

$$S(A,\varepsilon)_n = A \bigotimes_k \widetilde{S}_n(A) \tag{28.11.11}$$

when $n \geq 0$, because k satisfies the requirements of $A \bigotimes_A k$, and similarly for tensor products with k, over k. Note that the action of A on the left on $S(A, \varepsilon)_n$ corresponds to the action of A on the left on the first factor of A on the right side of (28.11.11).

Similarly, $N(A)$ is a graded left module over A^{en}, with $N(A)_n = N_n(A)$ when $n \geq 0$, and $N(A)_n = \{0\}$ when $n \leq 0$, as in Section 26.14. As before, we let $\widetilde{N}_n(A)$ be a tensor product of n copies of A' over k when $n \geq 1$, and we take $\widetilde{N}_0(A) = k$. We can identify $N_n(A)$ with a tensor product of the form

$$N_n(A) = A \bigotimes_k \widetilde{N}_n(A) \bigotimes_k A, \tag{28.11.12}$$

where the actions of A on the left and on the right on $N_n(A)$ are obtained from the actions of A on the left and on the right on the first and last factors of A on the right side of (28.11.12), as in Section 26.12.

We also have that $N(A, \varepsilon)$ is a graded left module over A, with

$$N(A,\varepsilon)_n = N(A)_n \bigotimes_A k = N_n(A) \bigotimes_A k \quad \text{when } n \geq 0$$
$$= \{0\} \quad \text{when } n < 0, \tag{28.11.13}$$

as in Section 7.5. If $n \geq 0$, then we can take $N(A, \varepsilon)$ to be a tensor product of the form

$$N(A,\varepsilon)_n = A \bigotimes_k \widetilde{N}_n(A), \tag{28.11.14}$$

as before. With this identification, the action of A on the left on $N(A, \varepsilon)_n$ corresponds to the action of A on the left on the first factor of A on the right side of (28.11.14). This corresponds to the description of $N(A, \varepsilon)$ on p186 of [3].

28.12 Differentiation on these resolutions

Let us continue with the same notation and hypotheses as in the previous section. Remember that the differentiation operators $d_{S(A)}$ and $d_{N(A)}$ on $S(A)$ and $N(A)$ were defined in Sections 26.10 and 26.14, respectively. This leads to differentiation operators $d_{S(A,\varepsilon)}$ and $d_{N(A,\varepsilon)}$ on $S(A,\varepsilon)$ and $N(A,\varepsilon)$, respectively, as in Section 7.5.

More precisely, for each $n \in \mathbf{Z}$, $d_{S(A,\varepsilon),n}$ is the homomorphism

$$\text{from } S(A,\varepsilon)_n \text{ into } S(A,\varepsilon)_{n-1}, \tag{28.12.1}$$

as left modules over A, that corresponds to $d_{S(A),n}$ and the identity mapping on k. Similarly, $d_{N(A,\varepsilon),n}$ is the homomorphism

$$\text{from } N(A,\varepsilon)_n \text{ into } N(A,\varepsilon)_{n-1}, \tag{28.12.2}$$

as left modules over A, that corresponds to $d_{N(A),n}$ and the identity mapping on k. Note that these homomorphisms are equal to 0 when $n \le 0$. To describe these differentiation operators more precisely when $n \ge 1$, we shall use some notation analogous to that of Section 26.15.

If $n \ge 0$, then it will be helpful to consider $S(A,\varepsilon)_n$ as a tensor product of $n+1$ copies of A over k. Similarly, it will be helpful to consider $N(A,\varepsilon)_n$ as a tensor product of $n+1$ modules over k, where the first module is A, and the next n modules are copies of A'. Thus, if $a_0, a_1, \ldots, a_n \in A$, then

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \tag{28.12.3}$$

defines an element of $S(A,\varepsilon)_n$. We also get that

$$a_0 \otimes q(a_1) \otimes \cdots \otimes q(a_n) \tag{28.12.4}$$

defines an element of $N(A,\varepsilon)_n$, where q is the natural quotient mapping from A onto A', as in Section 26.12.

We may use the notation

$$a_0\,[a_1, \ldots, a_n] \tag{28.12.5}$$

for (28.12.4) as on p186f of [3]. If $a_0 = e_A$, then this may be expressed as

$$[a_1, \ldots, a_n]. \tag{28.12.6}$$

If $n = 0$, then (28.12.5) becomes

$$a_0\,[\,], \tag{28.12.7}$$

which is the same as a_0 as an element of $S(A,\varepsilon)_0 = N(A,\varepsilon)_0 = A$. We may sometimes multiply expressions like these on the right by elements of k as well.

If $n \ge 1$, then

$$\begin{aligned} d_{N(A,\varepsilon),n}(a_0\,[a_1,\ldots,a_n]) &= (a_0\,a_1)\,[a_2,\ldots,a_n] \\ &\quad + \sum_{l=1}^{n-1} (-1)^l\, a_0\,[a_1,\ldots,a_l\,a_{l+1},\ldots,a_n] \\ &\quad + (-1)^n\, a_0\,[a_1,\ldots,a_{n-1}]\,\varepsilon(a_n). \end{aligned} \tag{28.12.8}$$

This is mentioned on p186 of [3], with $a_0 = e_A$. Of course, one can reduce to that case, because $d_{N(A,\varepsilon),n}$ is a homomorphism as in (28.12.2), as left module over A. If $n = 1$, then the sum in the middle of the right side of (28.12.8) should be interpreted as being 0, and we get that

$$d_{N(A,\varepsilon),1}(a_0\,[a_1]) = (a_0\,a_1)\,[\,] - a_0\,[\,]\varepsilon(a_1) = a_0\,a_1 - a_0\,\varepsilon(a_1). \tag{28.12.9}$$

This is also mentioned on p186 of [3], with $a_0 = e_A$.

If we are working with $S(A,\epsilon)_n$, then we may use the notation (28.12.5) for (28.12.3), as in Section 26.15. In this case, we get the same expressions for $d_{S(A,\varepsilon),n}$ as in the preceding paragraph, as mentioned on p187 of [3].

One could also consider k as a right module over A, and A as a right augmented ring, as mentioned on p187 of [3]. If A is projective as a module over k, then tensor products

$$S(\varepsilon, A) = k \bigotimes\nolimits_A S(A) \tag{28.12.10}$$

and

$$N(\varepsilon, A) = k \bigotimes\nolimits_A N(A) \tag{28.12.11}$$

are projective resolutions of k as a right module over A, as in [3].

28.13 Some related complexes

Let k be a commutative ring with a multiplicative identity element, and let A be an associative algebra over k with a multiplicative identity element e_A. Suppose that A is a supplemented algebra over k, with corresponding homomorphism ε from A onto k, as in Section 28.1.

Let X be a projective resolution of k, as a left module over A. Consider a tensor product

$$\overline{X} = k \bigotimes\nolimits_A X \tag{28.13.1}$$

of k, as a right module over A, and X, over A, as in the statement of Proposition 2.3 on p187 of [3]. Note that $\overline{X}$ may be considered as a complex over k, as in Section 7.5.

Let V be a module over k, which leads to a right module V_ε over A with trivial action of A on the right, as in Section 28.1. Alternatively, a tensor product

$$V \bigotimes\nolimits_k k \tag{28.13.2}$$

of V and k over k may be considered as a right module over A, using the action of A on k on the right defined by ε. Of course, V satisifes the requirements of a tensor product as in (28.13.2), as a module over k, in the usual way. If V is considered as a tensor product (28.13.2), as a module over k, in this way, then it is easy to see that V_ε is the same as (28.13.2), considered as a right module over A, using the action of A on k on the right.

Consider a tensor product

$$V_\varepsilon \bigotimes\nolimits_A X \tag{28.13.3}$$

of V_ε and X over A. This corresponds to a tensor product of the form

$$\Big(V \bigotimes_k k\Big) \bigotimes_A X, \tag{28.13.4}$$

where k is considered as a right module over A, as in the preceding paragraph. This is isomorphic to a tensor product of the form

$$V \bigotimes_k \Big(k \bigotimes_A X\Big) = V \bigotimes_k \overline{X}, \tag{28.13.5}$$

in a natural way, as in Section 1.12. More precisely, these tensor products may be considered as complexes over k, as in Section 7.5, and they are isomorphic as complexes.

It follows that

$$H\Big(V_\varepsilon \bigotimes_A X\Big)_n \tag{28.13.6}$$

is isomorphic to

$$H\Big(V \bigotimes_k \overline{X}\Big)_n \tag{28.13.7}$$

for each n. Note that (28.13.6) corresponds to the nth homology of A, as a left augmented ring, with coefficients in V_ε for every $n \geq 0$, as in Section 24.3. This corresponds to the first part of Proposition 2.3 on p187 of [3].

Similarly, let W be a module over k, so that ${}_\varepsilon W$ is a left module over A with trivial action of A on the left, as in Section 28.1. Alternatively, the space

$$\mathrm{Hom}_k(k, W) \tag{28.13.8}$$

of homomorphisms from k into W, as modules over k, may be considered as a left module over A, using the action of A on k on the right, as in Section 1.8. If we identify (28.13.8) with W, as modules over k, in the usual way, then one can check that ${}_\varepsilon W$ corresponds to (28.13.8), considered as a left module over A in this way.

Remember that

$$\mathrm{Hom}_A^{gr}(X, {}_\varepsilon W) \tag{28.13.9}$$

may be defined as a complex over k as in Section 8.4. We may consider

$$\mathrm{Hom}_k^{gr}(\overline{X}, W) = \mathrm{Hom}_k^{gr}\Big(k \bigotimes_A X, W\Big) \tag{28.13.10}$$

as a complex over k as well. This is isomorphic to

$$\mathrm{Hom}_A^{gr}\big(X, \mathrm{Hom}_k(k, W)\big) \tag{28.13.11}$$

in a natural way, where (28.13.8) is considered as a left module over A, as in the preceding paragraph. This uses isomorphisms as in Section 1.13 for modules without gradings. Of course, we can identify (28.13.11) with (28.13.9), by identifying (28.13.8) with ${}_\varepsilon W$, as left modules over A, as in the preceding paragraph.

More precisely, (28.13.11) may be considered as a complex over k, which can be identified with (28.13.9), as a complex over k. We also have that (28.13.10) and (28.13.11) are isomorphic as complexes, so that (28.13.9) and (28.13.10) are isomorphic as complexes. This means that

$$H\big(\mathrm{Hom}_A^{gr}(X, {}_\varepsilon W)\big)^n \tag{28.13.12}$$

is isomorphic to

$$H\big(\mathrm{Hom}_k^{gr}(\overline{X}, W)\big)^n \tag{28.13.13}$$

for each n. Remember that (28.13.12) corresponds to the nth cohomology of A, as a left augmented ring, with coefficients in ${}_\varepsilon W$ for every $n \geq 0$, as in Section 24.5. This corresponds to the second part of Proposition 2.3 on p187 of [3].

28.14 Standard complexes and $\overline{X}$

Let us continue with the same notation and hypotheses as in the previous section, and suppose now that

$$A \text{ is projective as a module over } k. \tag{28.14.1}$$

Remember that the complexes $S(A, \varepsilon)$ and $N(A, \varepsilon)$ defined in Section 28.11 are proejective resolutions of k, as a left module over A, under these conditions. Thus we can take X to be either of these two complexes in the previous section. The corresponding complexes $\overline{X}$ as in (28.13.1) are

$$S(\varepsilon, A, \varepsilon) = k \bigotimes\nolimits_A S(A, \varepsilon) \tag{28.14.2}$$

and

$$N(\varepsilon, A, \varepsilon) = k \bigotimes\nolimits_A N(A, \varepsilon), \tag{28.14.3}$$

as on p187 of [3].

The grading on $S(\varepsilon, A, \varepsilon)$ is given by

$$S(\varepsilon, A, \varepsilon)_n = k \bigotimes\nolimits_A S(A, \varepsilon)_n \tag{28.14.4}$$

for each n, as in Section 7.5. This is $\{0\}$ when $n < 0$, and it can be identified with a tensor product of the form

$$k \bigotimes\nolimits_A \big(A \bigotimes\nolimits_k \widetilde{S}_n(A)\big) \tag{28.14.5}$$

when $n \geq 0$, using the description of $S(A, \varepsilon)_n$ in Section 28.11. It follows that we can take

$$S(\varepsilon, A, \varepsilon)_n = \widetilde{S}_n(A) \tag{28.14.6}$$

when $n \geq 0$, with suitable identifications. This uses an isomorphism as in Section 1.12, and the fact that k satisfies the requirements of $k \bigotimes_A A$, as a module over k.

Similarly,

$$N(\varepsilon, A, \varepsilon)_n = k \bigotimes\nolimits_A N(A, \varepsilon)_n \tag{28.14.7}$$

for each n, which is equal to $\{0\}$ when $n < 0$. If $n \geq 0$, then we can take

$$N(\varepsilon, A, \varepsilon)_n = \widetilde{N}_n(A), \tag{28.14.8}$$

with suitable identifications, as on p187 of [3]. This uses the description of $N(A, \varepsilon)_n$ in Section 28.11, and the same type of isomorphisms as in the preceding paragraph.

Using the differentiation operators

$$d_{S(A,\varepsilon)} \text{ and } d_{N(A,\varepsilon)} \tag{28.14.9}$$

on $S(A, \varepsilon)$ and $N(A, \varepsilon)$, respectively, as in Section 28.12, we get differentiation operators

$$d_{S(\varepsilon,A,\varepsilon)} \text{ and } d_{N(\varepsilon,A,\varepsilon)} \tag{28.14.10}$$

on $S(\varepsilon, A, \varepsilon)$ and $N(\varepsilon, A, \varepsilon)$, respectively, as in Section 7.5. More precisely, for each $n \in \mathbf{Z}$, we get homomorphisms

$$d_{S(\varepsilon,A,\varepsilon),n} \text{ from } S(\varepsilon, A, \varepsilon)_n \text{ into } S(\varepsilon, A, \varepsilon)_{n-1} \tag{28.14.11}$$

and

$$d_{N(\varepsilon,A,\varepsilon),n} \text{ from } N(\varepsilon, A, \varepsilon)_n \text{ into } N(\varepsilon, A, \varepsilon)_{n-1}, \tag{28.14.12}$$

as modules over k. These homomorphisms are obtained from the identity mapping on k and the corresponding homomorphisms

$$d_{S(A,\varepsilon),n} \text{ and } d_{N(A,\varepsilon),n}, \tag{28.14.13}$$

respectively, in the usual way. Of course, (28.14.11) and (28.14.12) are equal to 0 when $n \leq 0$. We can describe these homomorphisms more precisely when $n \geq 1$ using the same type of notation as in Sections 26.15 and 28.12.

Remember that $\widetilde{S}_n(A)$, $\widetilde{N}_n(A)$ are tensor products of n copies of A, A', respectively, when $n \geq 1$, and that they are equal to k when $n = 0$, as in Section 28.11. If $a_1, \ldots, a_n \in A$ for some $n \geq 1$, then

$$a_1 \otimes \cdots \otimes a_n \tag{28.14.14}$$

defines an element of $\widetilde{S}_n(A)$, and

$$q(a_1) \otimes \cdots \otimes q(a_n) \tag{28.14.15}$$

defines an element of $\widetilde{N}_n(A)$, where q is the natural quotient mapping from A onto A', as in Section 26.12. We may use the notation

$$[a_1, \ldots, a_n] \tag{28.14.16}$$

for (28.14.15), as in Sections 26.15 and 28.12. If we are working with $\widetilde{S}_n(A)$, then we may use (28.14.16) for (28.14.14), as before. In both cases, $[\,]$ may be interpreted as being the multiplicative identity element in k, as on p176 of [3].

If $n \geq 1$, then

$$\begin{aligned} d_{N(\varepsilon,A,\varepsilon),n}([a_1,\ldots,a_n]) &= \varepsilon(a_1)\,[a_2,\ldots,a_n] \\ &\quad + \sum_{l=1}^{n-1} (-1)^l\,[a_1,\ldots,a_l\,a_{l+1},\ldots,a_n] \\ &\quad + (-1)^n [a_1,\ldots,a_{n-1}]\,\varepsilon(a_n), \end{aligned} \tag{28.14.17}$$

as on p187 of [3]. If $n = 1$, then the sum in the middle of the right side should be interpreted as being 0, and we get that

$$d_{N(\varepsilon,A,\varepsilon),1}([a_1]) = \varepsilon(a_1)\,[\,] - [\,]\,\varepsilon(a_1) = 0, \tag{28.14.18}$$

as in [3].

If we are working with $S(\varepsilon, A, \varepsilon)_n$, then we may use the notation (28.14.16) for (28.14.14), and we get the same expressions for $d_{S(\varepsilon,A,\varepsilon),n}$ as in the preceding paragraph.

Chapter 29

Semigroups and augmentations

29.1 Semigroup rings and augmentations

Let A be a ring with a multiplicative identity element e_A, and let Σ be a semigroup, with the semigroup operation expressed multiplicatively, and with an identity element e_Σ. Remember that the corresponding semigroup ring $A(\Sigma)$ may be defined as in Section 4.9. The element of $A(\Sigma)$ corresponding to e_Σ is the multiplicative identity element of $A(\Sigma)$, which may be expressed as $e_A\, e_\Sigma$, to be precise.

Let ε be a ring homomorphism from $A(\Sigma)$ onto A, so that

$$\varepsilon(e_\Sigma) = e_A. \tag{29.1.1}$$

Using ε, A may be considered as a left and right module over $A(\Sigma)$, as in Section 2.9. Thus $A(\Sigma)$ may be considered as a left and right augmented ring with respect to ε, as in Sections 24.1 and 24.2.

Put

$$\mu(x) = \varepsilon(x) \tag{29.1.2}$$

for every $x \in \Sigma$, where x is considered as an element of $A(\Sigma)$ on the right side, which could be expressed as $e_A\, x$, to be more precise. Thus μ defines a mapping from Σ into A, with

$$\mu(x\,y) = \mu(x)\,\mu(y) \tag{29.1.3}$$

for every $x, y \in \Sigma$. Of course,

$$\mu(e_\Sigma) = e_A, \tag{29.1.4}$$

by (29.1.1).

Let us say that ε is a *multiplicative augmentation* of $A(\Sigma)$ if

$$\varepsilon(a\, e_\Sigma) = a \tag{29.1.5}$$

for every $a \in A$, as on p148 of [3]. We shall normally be concerned only with multiplicative augmentations of $A(\Sigma)$, as in [3]. In this case,

$$\varepsilon(a\,x) = \varepsilon(a\,e_\Sigma\,x) = \varepsilon(a\,e_\Sigma)\,\varepsilon(x) = a\,\varepsilon(x) \tag{29.1.6}$$

for every $a \in A$ and $x \in \Sigma$. This implies that

$$\varepsilon \text{ is uniquely determined on } A(\Sigma) \text{ by } \mu, \tag{29.1.7}$$

as on p148 of [3].

Let

$$Z(A) = \{a \in A : a\,b = b\,a \text{ for every } b \in A\} \tag{29.1.8}$$

be the *center* of A as a ring, which is a subring of A that contains e_A. If ε is a multiplicative augmentation of $A(\Sigma)$, then

$$\mu(x) \in Z(A) \text{ for every } x \in \Sigma, \tag{29.1.9}$$

as on p148 of [3]. This follows from the fact that elements of A and Σ commute with each other in $A(\Sigma)$, by construction.

Conversely, suppose that μ is a mapping from Σ into A that satisfies (29.1.3), (29.1.4), and (29.1.9). Consider the mapping ε from $A(\Sigma)$ into A defined by

$$\varepsilon\Big(\sum_{j=1}^n a_j\,x_j\Big) = \sum_{j=1}^n a_j\,\mu(x_j) \tag{29.1.10}$$

for every $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in \Sigma$. It is easy to see that ε defines a ring homomorphism from $A(\Sigma)$ onto A that is a multiplicative augmentation of $A(\Sigma)$ under these conditions, as on p148 of [3].

The semigroup Σ together with a mapping μ from Σ into A that satisfies (29.1.3), (29.1.4), and (29.1.9) may be called an *augmented semigroup* (with multiplicative identity element), as on p188 of [3].

29.2 Semigroup algebras and augmentations

Let k be a commutative ring with a multiplicative identity element 1_k, and let Σ be a semigroup, with semigroup operation expressed multiplicatively, and with an identity element e_Σ. The corresponding semigroup ring $k(\Sigma)$, as in Section 4.9, may be considered as an algebra over k.

Suppose that ε_k is a ring homomorphism from $k(\Sigma)$ onto k, so that $\varepsilon_k(e_\Sigma) = 1_k$. We can define a mapping μ_k from Σ into k by (29.1.2), which satisfies (29.1.3) and

$$\mu_k(e_\Sigma) = 1_k, \tag{29.2.1}$$

as before.

The condition that ε_k be a multiplicative augmentation of $k(\Sigma)$ is equivalent to asking that ε_k be a homomorphism from $k(\Sigma)$ onto k, as algebras over k. If μ_k

is any mapping from Σ into k that satisfies (29.1.3) and (29.2.1), then (29.1.10) defines a homomorphism from $k(\Sigma)$ onto k, as algebras over k, as before.

Let A be an associative algebra over k, with multiplicative identity element e_A, and let $A(\Sigma)$ be the corresponding semigroup algebra of Σ over A, as in Section 4.9. Suppose that μ_k is a mapping from Σ into k that satisfies (29.1.3) and (29.2.1), and put

$$\mu_A(x) = \mu_k(x)\, e_A \tag{29.2.2}$$

for every $x \in \Sigma$. This defines a mapping from Σ into A that satisfies (29.1.3), (29.1.4), and (29.1.9). It follows that (29.1.10) defines a ring homomorphism ε_A from $A(\Sigma)$ onto A that is a multiplicative augmentation of $A(\Sigma)$, as before. Note that ε_A is also a homomorphism from $A(\Sigma)$ into A, as algebras over k.

Any ring with a multiplicative identity element may be considered as an associative algebra over its center. Using this, a multiplicative augmentation of the corresponding group ring may be considered to be as in the preceding paragraph, as on p188 of [3].

Of course, $k(\Sigma)$ is freely generated by elements of Σ, as a module over k. It is easy to see that $A(\Sigma)$ corresponds to a tensor product

$$A \bigotimes_k k(\Sigma) \tag{29.2.3}$$

of A and $k(\Sigma)$ over k, as an algebra over k, as in Section 4.1. This was mentioned in Section 25.14, and corresponds to a remark on p188 of [3].

Let ε_k be a homomorphism from $k(\Sigma)$ onto k, as algebras over k, and let μ_k be the corresponding mapping from Σ into k, as in (29.1.2). This leads to a mapping μ_A from Σ into A as in (29.2.2), and a ring homomorphism ε_A from $A(\Sigma)$ onto A that is a multiplicative augmentation of $A(\Sigma)$, as in (29.1.10). Remember that A satisfies the requirements of $A \bigotimes_k k$, as a module over k. One can check that multiplication on A is compatible with multiplication on $A \bigotimes_k k$, as a tesor product of algebras over k, as in Section 4.1. Using this identification, one can verify that ε_A corresponds to the homomorphism from (29.2.3) into $A \bigotimes_k k$ obtained from the identity mapping on A and ε_k, as on p188 of [3].

Alternatively,

$$\varepsilon_k\, e_A \tag{29.2.4}$$

defines a homomorphism from $k(\Sigma)$ into the center of A, as algebras over k. It is easy to see that ε_A corresponds to the homomorphism from (29.2.3) into A obtained from the identity mapping on A and (29.2.4) as in Section 4.1. Of course, this is basically the same as the description in the preceding paragraph.

This means that we are in the same type of situation as in Section 28.6, with $k(\Sigma)$, A in the roles of A, C before. Similarly, $C \bigotimes_k A$ in the earlier discussion corresponds to (29.2.3) here, which we identify with $A(\Sigma)$. The homomorphism ϕ from A into $C \bigotimes_k A$ before now corresponds to the homomorphism

$$\phi \text{ from } k(\Sigma) \text{ into } A(\Sigma), \tag{29.2.5}$$

as algebras over k, that sends an element of $k(\Sigma)$ to its product with e_A.

Of course,

(29.2.6) $$\psi(t) = t\, e_A$$

defines a homomorphism from k into A, as algebras over k, which corresponds to the analogous homomorphism from k into C in Section 28.6. Note that the homomorphisms ε_A, $\varepsilon_{C \bigotimes_k A}$ before correspond to ε_k, ε_A, respectively, here. It is easy to see that

(29.2.7) $$\varepsilon_A \circ \phi = \psi \circ \varepsilon_k,$$

as before. In particular,

(29.2.8) $$\phi(\ker \varepsilon_k) \subseteq \ker \varepsilon_A,$$

so that ϕ is a homomorphism from $k(\Sigma)$ into $A(\Sigma)$, as augmented rings, as in Section 24.9.

Let V be a module over k that is a right module over $A(\Sigma)$. We may consider V as a right module over $k(\Sigma)$, using the identification of $A(\Sigma)$ with (29.2.3), or equivalently using ϕ. We can use ϕ to get a homomorphism

(29.2.9) from the homology of $k(\Sigma)$ into the homology of $A(\Sigma)$, as left augmented rings, with coefficients in V,

as in Section 28.6.

Similarly, let Z be a module over k that is a left module over $A(\Sigma)$, and which may be considered as a left module over $k(\Sigma)$, as before. We can use ϕ to get a homomorphism

(29.2.10) from the cohomology of $A(\Sigma)$ into the cohomology of $k(\Sigma)$, as left augmented rings, with coefficients in Z,

as in Section 28.6 again.

Note that $k(\Sigma)$ is projective as a module over k, because it is free as a module over k. It follows that

(29.2.11) the homomorphisms as in (29.2.9) and (29.2.10) are isomorphisms,

as in Section 28.6. This corresponds to (1) and (1a) on p188 of [3].

Observe that $k(\Sigma)$ is a supplemented algebra over k with respect to ε_k, as on p188 of [3].

29.3 Semigroups and homomorphisms

Let Σ, Σ' be semigroups, with the semigroup operations expressed multiplicatively, and with identity elements e_Σ, $e_{\Sigma'}$, respectively. Also let ϕ be a homomorphism from Σ' into Σ, as semigroups, with

(29.3.1) $$\phi(e_{\Sigma'}) = e_\Sigma.$$

Let k be a commutative ring with a multiplicative identity element, and let $k(\Sigma)$, $k(\Sigma')$ be the semigroup algebras of Σ, Σ' with coefficients in k, as in

Section 4.9. If we identify Σ, Σ' with subsets of $k(\Sigma)$, $k(\Sigma')$ in the obvious way, then ϕ can be extended to a homomorphism

$$\text{from } k(\Sigma') \text{ into } k(\Sigma), \tag{29.3.2}$$

as modules over k, and in fact as algebras over k.

Suppose that $\mu = \mu_\Sigma$, $\mu' = \mu_{\Sigma'}$ are augmentations of Σ, Σ', respectively, as semigroups, with values in k, as in Section 29.1. If

$$\mu' = \mu \circ \phi \tag{29.3.3}$$

on Σ', then ϕ is said to be a *map or homomorphism of augmented semigroups*, as on p189 of [3].

Let $\varepsilon = \varepsilon_\Sigma$, $\varepsilon' = \varepsilon_{\Sigma'}$ be the homomorphisms from $k(\Sigma)$, $k(\Sigma')$ onto k, as algebras over k, corresponding to μ, μ', respectively, as in (29.1.10). Suppose that (29.3.3) holds, which implies that

$$\varepsilon' = \varepsilon \circ \phi \tag{29.3.4}$$

on $k(\Sigma')$. This means that ϕ defines a homomorphism from $k(\Sigma')$ into $k(\Sigma)$, as supplemented algebras with respect to ε', ε, respectively, as in Section 28.5. This corresponds to a remark on p189 of [3].

Let V be a module over k that is a right module over $k(\Sigma)$. This means that V may be considered as a right module over $k(\Sigma')$, using ϕ, as in Section 2.9. We can use ϕ to get a homomorphism

$$\begin{aligned} &\text{from the homology of } k(\Sigma') \text{ into the homology of } k(\Sigma), \\ &\text{as left augmented rings, with coefficients in } V, \end{aligned} \tag{29.3.5}$$

as in Section 28.5.

Similarly, let Z be a module over k that is a left module over $k(\Sigma)$, so that Z may be considered as a left module over $k(\Sigma')$, using ϕ. We can use ϕ to get a homomorphism

$$\begin{aligned} &\text{from the cohomology of } k(\Sigma), \text{ into the cohomology of } k(\Sigma'), \\ &\text{as left augmented rings, with coefficients in } Z, \end{aligned} \tag{29.3.6}$$

as in Section 28.5 again. These homomorphisms are mentioned on p189 of [3].

One may be interested in conditions under which these homomorphisms are isomorphisms, as in Section 24.10. We have seen that the combination of two conditions is sufficient for this, and we would like to restate these conditions for this case. These conditions are necessary to get an isomorphism as in (29.3.5) when $V = k(\Sigma)$, considered as a right module over itself, as before.

Note that the augmentation modules of $k(\Sigma)$, $k(\Sigma')$ are equal to k, considered as left modules over $k(\Sigma)$, $k(\Sigma')$ using ε, ε', respectively. The augmentation module of $k(\Sigma)$ may be considered as a left module over $k(\Sigma')$, using ϕ, as in Section 24.9. This is the same as considering k as a left module over $k(\Sigma')$ using ε', because of (29.3.4).

Similarly, $k(\Sigma)$ may be considered as a right module over itself, and thus a right module over $k(\Sigma')$, using ϕ, as in Section 24.9. Let

$$k(\Sigma)\bigotimes\nolimits_{k(\Sigma')} k \tag{29.3.7}$$

be a tensor product of $k(\Sigma)$, as a right module over $k(\Sigma')$, and k, as a left module over $k(\Sigma')$, over $k(\Sigma')$.

The mapping ψ from the augmentation module of $k(\Sigma')$ into the augmentation module of $k(\Sigma)$, as in Section 24.9, corresponds to the identity mapping on k here, because of (29.3.4). There is an obvious mapping

$$\text{from } k(\Sigma) \times k \text{ into } k, \tag{29.3.8}$$

corresponding to the action of $k(\Sigma)$ on k on the left using ε. This mapping is bilinear over k, and satisfies the appropriate compatibility condition with the actions of $k(\Sigma')$ on $k(\Sigma)$ on the right, and on k on the left, to get a homomorphism

$$\text{from (29.3.7) into } k, \tag{29.3.9}$$

as modules over k. More precisely, this compatibility condition uses (29.3.4). This homomorphism corresponds to the one called g in Section 24.9.

The first of the two conditions that we need from Section 24.10 is that

$$\text{the homomorphism as in (29.3.9) is an isomorphism.} \tag{29.3.10}$$

This corresponds to condition (i) in Proposition 3.1 on p189 of [3].

Remember that

$$\begin{aligned}&\text{the } n\text{th homology group of } k(\Sigma'), \text{ as a left augmented ring}\\ &\quad\text{with coefficients in } k(\Sigma), \text{ as a right module over } k(\Sigma') \text{ using } \phi\end{aligned} \tag{29.3.11}$$

is as in Section 24.3 for each $n \geq 0$. The second condition that we need from Section 24.10 is that

$$\text{(29.3.11) is equal to } \{0\} \tag{29.3.12}$$

when $n > 0$. This corresponds to condition (ii) in Proposition 3.1 on p189 of [3].

Suppose that (29.3.10) holds, and that (29.3.12) holds for every $n > 0$. This implies that the homomorphisms as in (29.3.5) and (29.3.6) are isomorphisms, as in Section 24.10. This is part of Proposition 3.1 on p189 of [3].

Let X be a projective resolution of k, as a left module over $k(\Sigma')$. Also let

$$k(\Sigma)\bigotimes\nolimits_{k(\Sigma')} X \tag{29.3.13}$$

be a tensor product of $k(\Sigma)$, as a right module over $k(\Sigma')$, and X, as a left module over $k(\Sigma')$, over $k(\Sigma')$. This may be considered as a complex over k, as in Section 7.5. Remember that (29.3.11) is isomorphic to the nth homology group of (29.3.13) for each $n \geq 0$, as in Section 24.3.

More precisely, (29.3.13) may be considered as a left module over $k(\Sigma)$, and a complex over $k(\Sigma)$. In fact,

(29.3.14) (29.3.13) is a projective resolution of k, as a left module over $k(\Sigma)$

under these conditions, as in Section 24.10. This is another part of Proposition 3.1 on p189 of [3].

29.4 Some related surjectivity conditions

Let us continue with the same notation and hypotheses as in the previous section. Observe that

(29.4.1) the homomorphism as in (29.3.9) is a surjection.

This can be seen by considering e_Σ as an element of $k(\Sigma)$, and using the fact that $\varepsilon(e_\Sigma) = \mu(e_\Sigma)$ is the multiplicative identity element in k.

Consider a tensor product

$$k(\Sigma')\bigotimes_{k(\Sigma')} k \tag{29.4.2}$$

of $k(\Sigma')$, as a right module over itself, and k, as a left module over $k(\Sigma')$ using ε', as in the previous section, over $k(\Sigma')$. Of course, this is isomorphic to k, as a modul over itself, in the usual way, and we shall use this in a moment.

Remember that $k(\Sigma)$ may be considered as a right module over $k(\Sigma')$, as in the previous section. Using this, it is easy to see that ϕ is a homomorphism from $k(\Sigma')$ into $k(\Sigma)$, as right modules over $k(\Sigma')$. This was also mentioned in Section 2.9, for homomorphisms between arbitrary associative algebras.

Using ϕ and the identity mapping on k, we get a homomorphism

(29.4.3) from (29.4.2) into (29.3.7),

as modules over k. The composition of this homomorphism with the one as in (29.3.9) is a homomorphism

(29.4.4) from (29.4.2) into k,

as modules over k. This homomorphism is the same as the one that can be obtained from the mapping from $k(\Sigma') \times k$ into k corresponding to the action of $k(\Sigma')$ on k on the left using ε'. More precisely, this uses the analogous description of the homomorphism as in (29.3.9), and (29.3.4).

It follows that

(29.4.5) the homomorphism as in (29.4.4) is an isomorphism,

because k satisfies the requirements of a tensor product as in (29.4.2). In particular, this implies (29.4.1).

Suppose now that

(29.4.6) $$\phi(\Sigma') = \Sigma.$$

This implies that

(29.4.7) $$\phi(k(\Sigma')) = k(\Sigma).$$

This means that

(29.4.8) the homomorphism as in (29.4.3) is a surjection.

One can check that (29.3.10) holds in this case, because of (29.4.5). This corresponds to a remark after the statement of Proposition 3.1 on p189 of [3].

29.5 Some augmentations and homomorphisms

Let k be a commutative ring with a multiplicative identity element 1_k, and let Σ be a semigroup, with semigroup operation expressed multiplicatively, and with an identity element e_Σ. Also let $k(\Sigma)$ be the corresponding semigroup algebra, with coefficients in k, as in Section 4.9.

If we take

(29.5.1) $$\mu_k(x) = 1_k \text{ for every } x \in \Sigma,$$

then μ_k satisfies (29.1.3) and (29.2.1), and thus defines a k-valued augmentation of Σ. This is called the *unit augmentation* of Σ, as on p188 of [3].

Suppose for the moment that if $x, y \in \Sigma$ and

(29.5.2) $$x\,y = e_\Sigma,$$

then

(29.5.3) $$x = y = e_\Sigma.$$

Let μ_k be the k-valued function on Σ defined by

(29.5.4) $$\begin{aligned} \mu_k(x) &= 1_k \quad \text{when } x = e_\Sigma \\ &= 0 \quad \text{when } x \neq e_\Sigma. \end{aligned}$$

It is easy to see that this satisfies (29.1.3) and (29.2.1), and thus defines an augmentation of Σ. This is called the *zero augmentation* of Σ, as on p188 of [3].

Let Σ be any semigroup with an identity element again, and let θ be any k-valued augmentation of Σ. There is a unique homomorphism Θ from $k(\Sigma)$ into itself, as a module over k, such that

(29.5.5) $$\Theta(x) = \theta(x)\,x$$

for every $x \in \Sigma$. It is easy to see that Θ is a homomorphism from $k(\Sigma)$ into itself, as an algebra over k, under these conditions.

Let ε be the augmentation of $k(\Sigma)$ that corresponds to the unit augmentation of Σ, and let ε_θ be the augmentation of $k(\Sigma)$ that corresponds to θ as in Section 29.1. Observe that

(29.5.6) $$\varepsilon \circ \Theta = \varepsilon_\theta$$

on $k(\Sigma)$. This corresponds to a remark on p189 of [3].

If $\theta(x)$ has a multiplicative inverse in k for each $x \in \Sigma$, then $1/\theta$ defines an augmentation on Σ too. In this case, Θ is an automorphism of $k(\Sigma)$.

Note that this condition holds when Σ is a group. Because of this, one often restricts one's attention to the unit augmentation on Σ when Σ is a group, as in [3].

Let Σ be any semigroup with identity element again. It is easy to see that the kernel of ε is freely generated by elements of the form

$$x - e_\Sigma, \tag{29.5.7}$$

with $x \in \Sigma$ and $x \neq e_\Sigma$, as a module over k, as on p189 of [3].

29.6 More on the unit augmentation

Let k be a commutative ring with a multiplicative identity element, and let Σ be a semigroup, with the semigroup operation expressed multiplicatively, and with an identity element e_Σ. Also let $k(\Sigma)$ be the corresponding semigroup algebra again, with coefficients in k, as in Section 4.9. In this section, we take $k(\Sigma)$ to be equipped with the augmentation ε that corresponds to the unit augmentation of Σ.

Let V be a module over k. Remember that an action of Σ on V on the left or right makes V into a left or right module over Σ, as appropriate, and under suitable conditions, as in Section 4.8. This corresponds exactly to V being a left or right module over $k(\Sigma)$, as appropriate, as in Section 4.9.

We may always consider V as a left or right module over Σ, where the action of each element of Σ is the identity mapping. In this case, we may say that V is *trivial* as a left or right module over Σ, as appropriate, or that the corresponding action of Σ on V is trivial. This corresponds exactly to trivial actions of $k(\Sigma)$ on V with respect to ε, as in Section 28.1.

Let W be a module over k that is a left module over Σ, and thus a left module over $k(\Sigma)$. We may consider W as a right module over Σ, using the trivial action of Σ on W. This makes W into a two-sided module over Σ, because the trivial action of Σ on W commutes with any other action on W. This corresponds to W_ε as a two-sided module over $k(\Sigma)$, as in Section 28.1.

Let us say that $w \in W$ is *invariant* if w is invariant as an element of W, as a two-sided module over Σ, as in Section 27.8. This means that

$$x \cdot w = w \tag{29.6.1}$$

for every $x \in \Sigma$, as before. This is the same as saying that w is invariant in W in the sense of Section 28.2.

A homomorphism f from $k(\Sigma)$ into W, as modules over k, is uniquely determined by its restriction to Σ, and any W-valued function on Σ can occur in this way. It is easy to see that f is a crossed homomorphism, as in Section 28.3, if and only if

$$f(x\,y) = x \cdot f(y) + f(x) \tag{29.6.2}$$

for every $x, y \in \Sigma$.

If W is trivial as both a left and right module over Σ, then (29.6.2) reduces to

$$f(x\,y) = f(x) + f(y) \tag{29.6.3}$$

for every $x, y \in \Sigma$.

29.7 Free semigroups and augmentations

Let E be a nonempty set, and let $\Sigma(E)$ be the free semigroup generated by E, as in Section 4.10. Also let k be a commutative ring with a multiplicative identity element, and let $k(\Sigma(E))$ be the corresponding semigroup algebra with coefficients in k, as in Section 4.9.

Let μ be a k-valued augmentation of $\Sigma(E)$, and let ε be the corresponding augmentation of $k(\Sigma(E))$, as in Section 29.1. Note that μ is uniquely determined by its values on E, which may be arbitrary elements of k.

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