Some topics in analysis related to bi-invariant semimetrics and semi-ultrametrics

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Preface

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Chapter 1

Semimetrics and seminorms

1.1 Semimetrics and semi-ultrametrics

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$. As usual, $d(\cdot, \cdot)$ is said to be a *semimetric* on X if it satisfies the following three properties. First,

$$d(x, x) = 0 \quad \text{for every } x \in X. \tag{1.1.1}$$

Second,

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in X. \tag{1.1.2}$$

Third,

$$d(x, z) \le d(x, y) + d(y, z) \quad \text{for every } x, y, z \in X, \tag{1.1.3}$$

which is the *triangle inequality*. If we also have that

$$d(x, y) > 0 \quad \text{for every } x, y \in X \text{ with } x \neq y, \tag{1.1.4}$$

then $d(\cdot, \cdot)$ is said to be a *metric* on X. The *discrete metric* is defined on X by putting $d(x, y)$ equal to 1 when $x \neq y$, and equal to 0 when $x = y$.

Similarly, $d(\cdot, \cdot)$ is said to be a *semi-ultrametric* on X if it satisfies (1.1.1), (1.1.2), and

$$d(x, z) \le \max(d(x, y), d(y, z)) \quad \text{for every } x, y \in X. \tag{1.1.5}$$

Note that (1.1.5) implies (1.1.3), so that a semi-ultrametric on X is a semimetric in particular. If a semi-ultrametric $d(\cdot, \cdot)$ on X satisfies (1.1.4) too, then $d(\cdot, \cdot)$ is said to be an *ultrametric* on X. It is easy to see that the discrete metric on X is an ultrametric.

If a is a positive real number with $a \le 1$, then it is well known that

$$(r + t)^a \le r^a + t^a \tag{1.1.6}$$

for all nonnegative real numbers r, t. To see this, observe first that

$$\max(r,t) \le (r^a + t^a)^{1/a} \tag{1.1.7}$$

for every $a > 0$ and $r, t \ge 0$. If $a \le 1$, then it follows that

$$r + t \le \max(r,t)^{1-a}\,(r^a + t^a) \le (r^a + t^a)^{(1-a)/a+1} = (r^a + t^a)^{1/a} \tag{1.1.8}$$

for every $r, t \ge 0$. This implies (1.1.6), as desired.

If $d(x,y)$ is a semimetric on X, then it is easy to see that

$$d(x,y)^a \tag{1.1.9}$$

is a semimetric on X when $0 < a \le 1$, using (1.1.6). If $d(x,y)$ is a semi-ultrametric on X, then (1.1.9) is a semi-ultrametric on X for every $a > 0$.

Let $d(x,y)$ be a semimetric on X again, and let t be a positive real number. One can verify that

$$d_t(x,y) = \min(d(x,y), t) \tag{1.1.10}$$

also defines a semimetric on X. If $d(x,y)$ is a semi-ultrametric on X, then (1.1.10) defines a semi-ultrametric on X too.

If $d_1(x,y), \ldots, d_n(x,y)$ are finitely many semimetrics on X, then one can check that

$$d'(x,y) = \sum_{j=1}^{n} d_j(x,y) \tag{1.1.11}$$

and

$$d(x,y) = \max_{1 \le j \le n} d_j(x,y) \tag{1.1.12}$$

are semimetrics on X too. If $d_j(x,y)$ is a semi-ultrametric on X for each $j = 1, \ldots, n$, then (1.1.12) is a semi-ultrametric on X as well. Observe that

$$d(x,y) \le d'(x,y) \le n\, d(x,y) \tag{1.1.13}$$

for every $x, y \in X$.

Let I be a nonempty set, let X_j be a set for each $j \in I$, and consider their Cartesian product

$$X = \prod_{j \in I} X_j. \tag{1.1.14}$$

If $x \in X$ and $l \in I$, then let x_l be the lth coordinate of x in X_l. Suppose that $d_l(x_l, y_l)$ is a semimetric on X_l for some $l \in I$, and put

$$\widetilde{d}_l(x,y) = d_l(x_l, y_l) \tag{1.1.15}$$

for every $x, y \in X$. It is easy to see that this defines a semimetric on X, which is a semi-ultrametric when $d_l(x_l, y_l)$ is a semi-ultrametric on X_l.

1.2 Open and closed balls

Let X be a set, and let $d(\cdot,\cdot)$ be a semimetric on X. The *open ball* in X centered at $x \in X$ with radius $r > 0$ with respect to $d(\cdot,\cdot)$ is defined as usual by

$$B(x,r) = B_d(x,r) = \{y \in X : d(x,y) < r\}. \tag{1.2.1}$$

Similarly, the *closed ball* in X centered at x with radius $r \geq 0$ with respect to $d(\cdot,\cdot)$ is defined by

$$\overline{B}(x,r) = \overline{B}_d(x,r) = \{y \in X : d(x,y) \leq r\}. \tag{1.2.2}$$

We may also use the notation $B_X(x,r) = B_{X,d}(x,r)$ for (1.2.1), and $\overline{B}_X(x,r) = \overline{B}_{X,d}(x,r)$ for (1.2.2).

Suppose for the moment that $d(\cdot,\cdot)$ is a semi-ultrametric on X. If $x, w \in X$ satisfy $d(w,x) < r$ for some $r > 0$, then one can check that

$$B(w,r) = B(x,r). \tag{1.2.3}$$

Similarly, if $d(w,x) \leq r$ for some $r \geq 0$, then

$$\overline{B}(w,r) = \overline{B}(x,r). \tag{1.2.4}$$

Suppose now that $d(x,y)^a$ is a semimetric on X for some $a > 0$. Observe that

$$B_{d^a}(x,r^a) = B_d(x,r) \tag{1.2.5}$$

for every $x \in X$ and $r > 0$, and that

$$\overline{B}_{d^a}(x,r^a) = \overline{B}_d(x,r) \tag{1.2.6}$$

for every $x \in X$ and $r \geq 0$.

Let $t > 0$ be given, and let $d_t(\cdot,\cdot)$ be defined on X as in (1.1.10). If $x \in X$, then it is easy to see that

$$\begin{aligned} B_{d_t}(x,r) &= B_d(x,r) \quad \text{when } r \leq t \\ &= X \quad \text{when } r > t. \end{aligned} \tag{1.2.7}$$

Similarly,

$$\begin{aligned} \overline{B}_{d_t}(x,r) &= \overline{B}_d(x,r) \quad \text{when } r < t \\ &= X \quad \text{when } r \geq t. \end{aligned} \tag{1.2.8}$$

Let $d_1, \ldots, d_n$ be finitely many semimetrics on X, and remember that their maximum defines a semimetric d on X as well, as in (1.1.12). It is easy to see that

$$B_d(x,r) = \bigcap_{j=1}^{n} B_{d_j}(x,r) \tag{1.2.9}$$

for every $x \in X$ and $r > 0$, and that

$$\overline{B}_d(x, r) = \bigcap_{j=1}^{n} \overline{B}_{d_j}(x, r) \tag{1.2.10}$$

for every $x \in X$ and $r \geq 0$.

Let $X_1, \ldots, X_n$ be finitely many sets, and let $X = \prod_{j=1}^{n} X_j$ be their Cartesian product. Suppose that d_j is a semimetric on X_j for each $j = 1, \ldots, n$, so that $\widetilde{d}_j(x, y) = d_j(x_j, y_j)$ defines a semimetric on X for every $j = 1, \ldots, n$, as in (1.1.15). Thus

$$d(x, y) = \max_{1 \leq j \leq n} \widetilde{d}_j(x, y) = \max_{1 \leq j \leq n} d_j(x_j, y_j) \tag{1.2.11}$$

defines a semimetric on X, which is a metric on X when d_j is a metric on X_j for each $j = 1, \ldots, n$. It is easy to see that

$$B_{X,d}(x, r) = \prod_{j=1}^{n} B_{X_j, d_j}(x_j, r) \tag{1.2.12}$$

for every $x \in X$ and $r > 0$, and that

$$\overline{B}_{X,d}(x, r) = \prod_{j=1}^{n} \overline{B}_{X_j, d_j}(x, r) \tag{1.2.13}$$

for every $x \in X$ and $r \geq 0$.

Let $d(\cdot, \cdot)$ be a semimetric on a set X again, and let Y be a subset of X. Note that the restriction of $d(x, y)$ to $x, y \in Y$ defines a semimetric on Y, which is a metric or semi-ultrametric when $d(\cdot, \cdot)$ has the same property on X. If $x \in Y$, then

$$B_Y(x, r) = B_X(x, r) \cap Y \tag{1.2.14}$$

for every $r > 0$, and

$$\overline{B}_Y(x, r) = \overline{B}_X(x, r) \cap Y \tag{1.2.15}$$

for every $r \geq 0$.

1.3 Absolute value functions

Let k be a field. A nonnegative real-valued function $|\cdot|$ on k is said to be an *absolute value function* on k if it satisfies the following three conditions. First,

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{1.3.1}$$

Second,

$$|x\, y| = |x|\, |y| \quad \text{for every } x, y \in k. \tag{1.3.2}$$

Third,

$$|x + y| \leq |x| + |y| \quad \text{for every } x, y \in k. \tag{1.3.3}$$

It is well known that the standard absolute value functions on the real numbers **R** and the complex numbers **C** satisfy these conditions. The *trivial absolute value function* may be defined on any field k by putting $|x|$ equal to 1 when $x \neq 0$, and to 0 when $x = 0$.

If $|\cdot|$ is an absolute value function on k, then it is easy to see that $|1| = 1$, where more precisely the first 1 is the multiplicative identity element in k, and the second 1 is the multiplicative identity element in **R**. Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n, then $|x| = 1$. One can check that

$$d(x, y) = |x - y| \tag{1.3.4}$$

defines a metric on k, using the fact that $|-1| = 1$ to get that this is symmetric in x and y.

A nonnegative real-valued function $|\cdot|$ on k is said to be an *ultrametric absolute value function* on k if it satisfies (1.3.1), (1.3.2), and

$$|x + y| \le \max(|x|, |y|) \quad \text{for every } x, y \in k. \tag{1.3.5}$$

Of course, (1.3.5) implies (1.3.3), so that ultrametric absolute value functions are absolute value functions in particular. If $|\cdot|$ is an ultrametric absolute value function on k, the (1.3.4) is an ultrametric on k. It is easy to see that the trivial absolute value function on k is an ultrametric absolute value function, for which the associated ultrametric is the discrete metric.

If p is a prime number, then the *p-adic absolute value* $|x|_p$ of a rational number x is defined as follows. This is equal to 0 when $x = 0$, and otherwise $x = p^j\,(a/b)$ for some integers a, b, and j, where $a, b \neq 0$ and neither a nor b is an integer multiple of p. In this case, we put

$$|x|_p = p^{-j}, \tag{1.3.6}$$

and one can check that this defines an ultrametric absolute value function on the rational numbers **Q**.

If k is a field with an absolute value function $|\cdot|$, and k is not complete with respect to the associated metric (1.3.4), then one can pass to a completion, by standard arguments. More precisely,

(1.3.7) the completion of k is a field, $|\cdot|$ extends to an absolute value function on the completion of k, and k is dense in its completion,

with respect to the associated metric. The completion of k is also unique, up to a suitable isomorphic equivalence. If p is a prime number, then the field $\mathbf{Q}_p$ of *p-adic numbers* is obtained by completing **Q** using the p-adic absolute value function.

If $|\cdot|$ is an absolute value function on a field k, and if $0 < a \le 1$, then it is easy to see that $|x|^a$ defines an absolute value function on k too, using (1.1.6). If $|\cdot|$ is an ultrametric absolute value function on k, then $|\cdot|^a$ defines an ultrametric absolute value function on k for every $a > 0$. A pair $|\cdot|_1$, $|\cdot|_2$ of absolute value functions on k are said to be *equivalent* if there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{1.3.8}$$

for every $x \in k$. One can check that the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ determine the same topology on k in this case. Conversely, it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k when their associated metrics determine the same topology on k.

If k is a field, $x \in k$, and n is in the set $\mathbf{Z}_+$ of positive integers, then let $n \cdot x$ be the sum of n x's in k. An absolute value function $|\cdot|$ on k is said to be *archimedean* if there are $n \in \mathbf{Z}_+$ such that $|n \cdot 1|$ is arbitrarily large. Thus $|\cdot|$ is *non-archimedean* on k if $|n \cdot 1|$, $n \in \mathbf{Z}_+$, is bounded. If $|\cdot|$ is an ultrametric absolute value function on k, then $|n \cdot 1| \le 1$ for every $n \in \mathbf{Z}_+$, so that $|\cdot|$ is non-archimedean on k. Conversely, it is well known that

(1.3.9) non-archimedean absolute value functions are ultrametric absolute value functions.

If k is a field, $|\cdot|$ is an absolute value function on k, and k_0 is a subfield of k, then the restriction of $|x|$ to $x \in k_0$ defines an absolute value function on k_0. If $|\cdot|$ is an ultrametric absolute value function on k_0, then its restriction to k_0 is an ultrametric absolute value function on k_0. More precisely, $|\cdot|$ is non-archimedean on k if and only if its restriction to k_0 is non-archimedean.

A famous theorem of Ostrowski states that an absolute value function on $\mathbf{Q}$ is either the trivial absolute value function, or equivalent to the standard Euclidean absolute value function on $\mathbf{Q}$, or equivalent to the p-adic absolute value function on $\mathbf{Q}$ for some prime number p. Let k be a field with an archimedean absolute value function $|\cdot|$, and suppose that k is complete with respect to the associated metric. Another famous theorem of Ostrowski states that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to an absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard absolute value function.

Let k be field with an absolute value function $|\cdot|$ again, and observe that

$$\{|x| : x \in k \setminus \{0\}\} \tag{1.3.10}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. If 1 is not a limit point of (1.3.10) with respect to the standard topology on the real line, then $|\cdot|$ is said to be *discrete* on k. In this case, it is not too difficult to show that (1.3.10) consists of the integer powers of a positive real number, which is equal to 1 exactly when $|\cdot|$ is trivial on k. One can also show that discrete absolute value functions are ultrametric absolute value functions.

1.4 Seminorms

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *seminorm* on V with respect to $|\cdot|$ on k if

$$N(t\,v) = |t|\,N(v) \quad \text{for every } t \in k \text{ and } v \in V, \tag{1.4.1}$$

and

$$N(v + w) \le N(v) + N(w) \quad \text{for every } v, w \in V. \tag{1.4.2}$$

If we also have that

$$N(v) > 0 \quad \text{for every } v \in V \text{ with } v \ne 0, \tag{1.4.3}$$

then N is said to be a *norm* on V with respect to $|\cdot|$ on V. If N satisfies (1.4.1) and

$$N(v + w) \le \max(N(v), N(w)) \quad \text{for every } v, w \in V, \tag{1.4.4}$$

then N is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k. Clearly (1.4.4) implies (1.4.2), so that a semi-ultranorm on V is a seminorm in particular. A semi-ultranorm on V that satisfies (1.4.3) is said to be an *ultranorm* on V. Note that $|\cdot|$ may be considered as a norm on k, as a one-dimensional vector space over itself, which is an ultranorm when $|\cdot|$ is an ultrametric absolute value function on k.

If N is a seminorm on V with respect to $|\cdot|$ on k, then

$$d(v, w) = d_N(v, w) = N(v - w) \tag{1.4.5}$$

is a semimetric on V, which is a metric when N is a norm. Similarly, (1.4.5) is a semi-ultrametric on V when N is a semi-ultranorm on V, and (1.4.5) is an ultrametric on V when N is an ultranorm on V.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k. Consider the nonnegative real-valued function N defined on V by putting $N(v)$ equal to 1 when $v \ne 0$, and to 0 when $v = 0$. It is easy to see that this defines an ultranorm on V, which may be called the *trivial ultranorm* on V. The corresponding ultrametric (1.4.5) is the same as the discrete metric on V.

If N is a semi-ultranorm on V with respect to $|\cdot|$ on k and $N(v) > 0$ for some $v \in V$, then it is easy to see that $|\cdot|$ is an ultrametric absolute value function on V. If N is a seminorm on V with respect to $|\cdot|$ on k and $0 < a \le 1$, then one can verify that

$$N(v)^a \tag{1.4.6}$$

is a seminorm on V with respect to $|\cdot|^a$ as an absolute value function on k, using (1.1.6). Similarly, if $|\cdot|$ is an ultrametric absolute value function on k, and N is a semi-ultranorm on V with respect to $|\cdot|$ on k, then (1.4.6) is a semi-ultranorm on V with respect to $|\cdot|^a$ as an ultrametric absolute value function on k for every $a > 0$.

Let n be a positive integer, and consider the space k^n of n-tuples of elements of k, which is a vector space over k with respect to coordinatewise addition and scalar multiplication. If $v \in k^n$, then put

$$\|v\|_1 = \sum_{j=1}^{n} |v_j| \tag{1.4.7}$$

and

(1.4.8)
$$\|v\|_\infty = \max_{1 \le j \le n} |v_j|.$$

One can check that these define norms on k^n with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then (1.4.8) is an ultranorm on k^n. We also have that

(1.4.9)
$$\|v\|_\infty \le \|v\|_1 \le n\, \|v\|_\infty$$

for every $v \in k^n$.

1.5 Formal series

Let k be a field, and let T be an indeterminate. As in [4, 11], we shall normally use upper-case letters for indeterminates, and lower case letters for elements of k or related spaces. Let $k((T))$ be the space of formal series of the form

(1.5.1)
$$f(T) = \sum_{j=j_0}^{\infty} f_j\, T^j,$$

where j_0 is in the set $\mathbf{Z}$ of integers, and $f_j \in k$ for every integer $j \ge j_0$. More precisely, we can take $f_j = 0$ when $j < j_0$, and define $k((T))$ to be the space of k-valued functions on $\mathbf{Z}$ that are equal to 0 for all but finitely many $j < 0$. As in [4], an element of $k((T))$ may be expressed as

(1.5.2)
$$f(T) = \sum_{j >> -\infty} f_j\, T^j$$

to indicate that $f_j = 0$ for all but finitely many $j < 0$.

We may consider $k((T))$ as a vector space over k with respect to termwise addition and scalar multiplication, which corresponds to pointwise addition and scalar multiplication of k-valued functions on $\mathbf{Z}$. The space $k[[T]]$ of *formal power series* in T with coefficients in k may be identified with the linear subspace of $k((T))$ consisting of $f(T)$ as in (1.5.1) with $j_0 \ge 0$. Similarly, the space $k[T]$ of *formal polynomials* in T with coefficients in k may be identified with the linear subspace of $k[[T]]$ consisting of $f(T)$ as in (1.5.1) with $j_0 \ge 0$ and $f_j = 0$ for all but finitely many j.

Let $f(T)$ and

(1.5.3)
$$g(T) = \sum_{l >> -\infty} g_l\, T^l$$

in $k((T))$ be given, and put

(1.5.4)
$$h_n = \sum_{j=-\infty}^{\infty} f_j\, g_{n-j}$$

for every $n \in \mathbf{Z}$. More precisely, one can check that all but finitely many of the terms in the sum on the right are equal to 0, so that the sum defines an element of k. One can also verify that for all but finitely many $n < 0$, all of the terms on the right side of (1.5.4) are equal to 0, so that $h_n = 0$. Thus

$$h(T) = \sum_{n>>-\infty} h_n\, T^n \tag{1.5.5}$$

defines an element of $k((T))$, and we put

$$f(T)\, g(T) = h(T). \tag{1.5.6}$$

It is well known and not difficult to check that $k((T))$ is a commutative associative algebra over k with respect to this definition of multiplication. Of course, $k[T]$ and $k[[T]]$ are subalgebras of $k((T))$. If we identify elements of k with formal polynomials with the given coefficient of T^j when $j = 0$ and all other coefficients equal to 0, then the multiplicative identity element 1 of k is the multiplicative identity element in $k((T))$ as well.

If $a(T) \in k[[T]]$, then $\sum_{l=0}^\infty a(T)^l\, T^l$ can be defined as an element of $k[[T]]$ in a standard way, where $a(T)^l$ is interpreted as being equal to 1 when $l = 0$. One can check that

$$(1 - a(T)\, T) \sum_{l=0}^{\infty} a(T)^l\, T^l = 1, \tag{1.5.7}$$

so that $1 - a(T)\, T$ is invertible in $k[[T]]$, with

$$(1 - a(T)\, T)^{-1} = \sum_{l=0}^{\infty} a(T)^l\, T^l. \tag{1.5.8}$$

One can use this to show that $k((T))$ is a field.

More precisely, if $f(T) \in k[[T]]$, then $f(T)$ is invertible in $k[[T]]$ if and only if $f_0 \neq 0$. This also uses the fact that $f(T) \mapsto f_0$ defines an algebra homomorphism from $k[[T]]$ onto k.

1.6 Absolute values on $k((T))$

Let us continue with the same notation and hypotheses as in the previous section. If $f(T) = \sum_{j>>-\infty} f_j\, T^j$ is a nonzero element of $k((T))$, then there is an integer $j_0(f(T))$ such that

$$f_{j_0(f(T))} \neq 0,\ f_j = 0 \text{ when } j < j_0(f(T)). \tag{1.6.1}$$

We can interpret $j_0(f(T))$ as being $+\infty$ when $f(T) = 0$. It is easy to see that

$$j_0(f(T)\, g(T)) = j_0(f(T)) + j_0(g(T)), \tag{1.6.2}$$
$$j_0(f(T) + g(T)) \geq \min(j_0(f(T)), j_0(f(T))) \tag{1.6.3}$$

for every $f(t), g(T) \in k((T))$, with suitable interpretations when either $f(T)$ or $g(T)$ is 0.

Let r be a positive real number with $r \le 1$, and put

$$|f(T)|_r = r^{j_0(f(T))} \tag{1.6.4}$$

when $f(T) \in k((T))$ and $f(T) \ne 0$, and $|0|_r = 0$. Observe that

$$|f(T)\, g(T)|_r = |f(T)|_r\, |g(T)|_r, \tag{1.6.5}$$

$$|f(T) + g(T)|_r \le \max(|f(T)|_r, |g(T)|_r) \tag{1.6.6}$$

for every $f(T), g(T) \in k((T))$, by (1.6.2) and (1.6.3). Thus $|\cdot|_r$ is an ultrametric absolute value function on $k((T))$, which is the trivial absolute value function when $r = 1$. If a is a positive real number, then $0 < r^a \le 1$ and

$$|f(T)|_r^a = |f(T)|_{r^a} \tag{1.6.7}$$

for every $f(T) \in k((T))$.

Suppose from now on in this section that $r < 1$. If $n \in \mathbf{Z}$, then

$$T^n\, k[[T]] = \{f(T)\, T^n : f(T) \in k[[T]]\} \tag{1.6.8}$$

is the same as the closed ball in $k((T))$ centered at 0 with radius r^n with respect to the ultrametric associated to $|\cdot|_r$. We can also identify (1.6.8) with the Cartesian product of a family of copies of k, indexed by integers $j \ge n$. One can check that the topology determined on (1.6.8) by the restriction of the ultrametric on $k((T))$ associated to $|\cdot|_r$ to (1.6.8) is the same as the product topology, using the discrete topology on k.

It is not too difficult to show that

(1.6.9) $k((T))$ is complete with respect to the ultrametric associated to $|\cdot|_r$.

More precisely, any Cauchy sequence in $k((T))$ with respect to the ultrametric associated to $|\cdot|_r$ is bounded, which means that there is an $n \in \mathbf{Z}$ such that the terms of the Cauchy sequence are contained in (1.6.8). One can verify that for each integer $j \ge n$, the coefficients of T^j of the terms of the Cauchy sequence are eventually constant. The eventual constant values of the coefficients of T^j, $j \ge n$, can be used to define an element of (1.6.8). The given Cauchy sequence converges to this element of (1.6.8) with respect to the metric associated to $|\cdot|_r$, by the remark in the preceding paragraph.

Note that $|\cdot|_r$ may be considered as a norm on $k((T))$, as a vector space over k, with respect to the trivial absolute value function on k. Of course, (1.6.8) is a linear subspace of $k((T))$ for every $n \in \mathbf{Z}$, as a vector space over k.

1.7 Translation-invariant semimetrics

Let G be a group, and let $d(\cdot,\cdot)$ be a semimetric on G. We say that $d(\cdot,\cdot)$ is *invariant under left translations* on G if

$$d(a\,x, a\,y) = d(x, y) \tag{1.7.1}$$

for every $a, x, y \in G$. Similarly, $d(\cdot, \cdot)$ is said to be *invariant under right translations* on G if

$$d(x\,a, y\,a) = d(x, y) \tag{1.7.2}$$

for every $a, x, y \in G$. Of course,

$$d(x^{-1}, y^{-1}) \tag{1.7.3}$$

is a semimetric on G too. It is easy to see that $d(\cdot, \cdot)$ is invariant under left or right translations on G if and only if (1.7.3) is invariant under right or left translations on G, respectively.

If $d(\cdot, \cdot)$ is invariant under either left or right translations on G, then one can check that

$$d(x, e) = d(x^{-1}, e) \tag{1.7.4}$$

for every $x \in G$, where e is the identity element in G. If $d(\cdot, \cdot)$ is invariant under both left and right translations on G, then one can verify that

$$d(x, y) = d(x^{-1}, y^{-1}) \tag{1.7.5}$$

for every $x, y \in G$. If $d(\cdot, \cdot)$ is invariant under either left or right translations on G and (1.7.5) holds, then it follows that $d(\cdot, \cdot)$ is invariant under both left and right translations on G.

If $a, b \in G$ and $A, B \subseteq G$, then put

$$a\,B = \{a\,y : y \in B\}, \tag{1.7.6}$$

$$A\,b = \{x\,b : x \in A\}, \tag{1.7.7}$$

$$A\,B = \{x\,y : x \in A,\ y \in B\} \tag{1.7.8}$$

$$A^{-1} = \{x^{-1} : x \in A\}. \tag{1.7.9}$$

Equivalently,

$$A\,B = \bigcup_{a \in A} a\,B = \bigcup_{b \in B} A\,b. \tag{1.7.10}$$

If $A = A^{-1}$, then A is said to be *symmetric* about e in G. If $d(\cdot, \cdot)$ is invariant under left or right translations on G, then open and closed balls in G centered at e with respect to $d(\cdot, \cdot)$ are symmetric about e, by (1.7.4).

If $d(\cdot, \cdot)$ is invariant under left translations on G, then

$$a\,B(x, r) = B(a\,x, r) \tag{1.7.11}$$

for every $a, x \in G$ and $r > 0$, and

$$a\,\overline{B}(x, r) = \overline{B}(a\,x, r) \tag{1.7.12}$$

for every $a, x \in G$ and $r \geq 0$. We also have that

$$d(e, x\,y) \leq d(e, x) + d(x, x\,y) = d(e, x) + d(e, y) \tag{1.7.13}$$

for every $x, y \in G$ in this case. This implies that

$$B(e, r)\, B(e, t) \subseteq B(e, r + t) \tag{1.7.14}$$

for every $r, t > 0$, and

$$\overline{B}(e, r)\, \overline{B}(e, t) \subseteq \overline{B}(e, r + t) \tag{1.7.15}$$

for every $r, t \geq 0$.

Similarly, if $d(\cdot, \cdot)$ is invariant under right translations on G, then

$$B(x, r)\, a = B(x\, a, r) \tag{1.7.16}$$

for every $a, x \in G$ and $r > 0$,

$$\overline{B}(x, r)\, a = \overline{B}(x\, a, r) \tag{1.7.17}$$

for every $a, x \in G$ and $r \geq 0$, and

$$d(e, x\, y) \leq d(e, y) + d(y, x\, y) = d(e, x) + d(e, y) \tag{1.7.18}$$

for every $x, y \in G$. This means that (1.7.14) and (1.7.15) hold too, as before.

Suppose now that $d(\cdot, \cdot)$ is a semi-ultrametric on G. If $d(\cdot, \cdot)$ is invariant under left translations on G, then

$$d(e, x\, y) \leq \max(d(e, x), d(x, x\, y)) = \max(d(e, x), d(e, y)) \tag{1.7.19}$$

for every $x, y \in G$. Similarly, if $d(\cdot, \cdot)$ is invariant under right translations on G, then

$$d(e, x\, y) \leq \max(d(e, y), d(y, x\, y)) = \max(d(e, x), d(e, y)). \tag{1.7.20}$$

In both cases, we get that

(1.7.21) open and closed balls in G centered at e with respect to $d(\cdot, \cdot)$ are subgroups of G.

This also uses the fact that these balls are symmetric about e, as before.

If $d(\cdot, \cdot)$ is a semimetric on G that is invariant under left or right translations on G, then $\overline{B}(e, 0)$ is a subgroup of G. This follows from (1.7.15) and the fact that $\overline{B}(e, 0)$ is symmetric about e in G.

1.8 Conjugations and subgroups

Let G be a group again, and put

$$C_a(x) = a\, x\, a^{-1} \tag{1.8.1}$$

for every $a, x \in G$, which defines *conjugation* by a on G. Let us say that a semimetric $d(\cdot, \cdot)$ on G is *invariant under conjugations* on G if

$$d(a\, x\, a^{-1}, a\, y\, a^{-1}) = d(x, y) \tag{1.8.2}$$

for every $a, x, y \in G$. This implies that open and closed balls in G centered at e with respect to $d(\cdot,\cdot)$ are invariant under conjugations on G. If $d(\cdot,\cdot)$ is invariant under left and right translations on G, then $d(\cdot,\cdot)$ is invariant under conjugations on G. If $d(\cdot,\cdot)$ is invariant under left or right translations on G, and if $d(\cdot,\cdot)$ is invariant under conjugations on G, then $d(\cdot,\cdot)$ is invariant under both left and right translations on G.

If $d(\cdot,\cdot)$ is a semi-ultrametric on G that is invariant under left or right translations on G, then open and closed balls in G centered at e with respect to $d(\cdot,\cdot)$ are subgroups of G, as in the previous section. If $d(\cdot,\cdot)$ is invariant under left and right translations on G, and thus conjugations on G, then it follows that

(1.8.3) open and closed balls in G centered at e with respect to $d(\cdot,\cdot)$ are normal subgroups of G.

If $d(\cdot,\cdot)$ is a semimetric on G and $a \in G$, then

$$d_a(x, y) = d(a\,x\,a^{-1}, a\,y\,a^{-1}) \tag{1.8.4}$$

is a semimetric on G as well, and a semi-ultrametric on G when $d(\cdot,\cdot)$ is a semi-ultrametric on G. If $d(\cdot,\cdot)$ is invariant under left or right translations on G, then one can check that (1.8.4) has the same property. Note that

$$B_{d_a}(x, r) = a^{-1} B_d(a\,x\,a^{-1}, r)\,a \tag{1.8.5}$$

for every $x \in G$ and $r > 0$, and

$$\overline{B}_{d_a}(x, r) = a^{-1}\,\overline{B}_d(a\,x\,a^{-1}, r)\,a \tag{1.8.6}$$

for every $x \in G$ and $r \geq 0$.

Let A be a subgroup of G, and for each $x, y \in G$, put

$$\begin{aligned} d_L(x, y) = d_{A,L}(x, y) &= 0 \quad \text{when } x\,A = y\,A \\ &= 1 \quad \text{when } x\,A \neq y\,A \end{aligned} \tag{1.8.7}$$

and

$$\begin{aligned} d_R(x, y) = d_{A,R}(x, y) &= 0 \quad \text{when } A\,x = A\,y \\ &= 1 \quad \text{when } A\,x \neq A\,y. \end{aligned} \tag{1.8.8}$$

One can check that these define semi-ultrametrics on G. It is easy to see that (1.8.7) is invariant under left translations on G, and that (1.8.8) is invariant under right translations on G. We also have that (1.8.7) is invariant under right translations on G by elements of A, and that (1.8.8) is invariant under left translations on G by elements of A.

One can verify that

$$d_{A,R}(x, y) = d_{A,L}(x^{-1}, y^{-1}) \tag{1.8.9}$$

for every $x, y \in G$. By construction, the open balls in G centered at e with radius $r \le 1$ with respect to (1.8.7) and (1.8.8) are equal to A, as are the closed balls in G centered at e with radius $r < 1$ with respect to (1.8.7) and (1.8.8).

If A is a normal subgroup of G, then (1.8.7) and (1.8.8) are the same. Their common value may be denoted $d_A(x, y)$ in this case, which is a semi-ultrametric on G that is invariant under both left and right translations.

Let $d_1, \ldots, d_n$ be finitely many semimetrics on G, and remember that their sum d' and maximum d define semimetrics on G as well, as in Section 1.1. If d_j is invariant under left translations on G for each $j = 1, \ldots, n$, then d and d' are invariant under left translations too. Similarly, if d_j is invariant under right translations on G for every $j = 1, \ldots, n$, then d and d' are invariant right translations on G.

Let $A_1, \ldots, A_n$ be finitely many subgroups of G, so that

$$A = \bigcap_{j=1}^{n} A_j \tag{1.8.10}$$

is also a subgroup of G. It is easy to see that

$$\max_{1 \le j \le n} d_{A_j,L}(x, y) = d_{A,L}(x, y) \tag{1.8.11}$$

$$\max_{1 \le j \le n} d_{A_j,R}(x, y) = d_{A,R}(x, y) \tag{1.8.12}$$

for every $x, y \in G$.

1.9 Sequences of semimetrics

Let X be a set, and let $d_1, d_2, d_3, \ldots$ be a sequence of semimetrics on X. Also let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, so that

$$d_j'(x, y) = \min(d_j(x, y), a_j) \tag{1.9.1}$$

defines a semimetric on X for every $j \ge 1$, as in Section 1.1. If $x \in X$ and $r > 0$, then

$$\begin{aligned} B_{d_j'}(x, r) &= B_{d_j}(x, r) \quad \text{when } r \le a_j \\ &= X \quad \text{when } r > a_j \end{aligned} \tag{1.9.2}$$

for every $j \ge 1$, as in Section 1.2. Similarly, if $r \ge 0$, then

$$\begin{aligned} \overline{B}_{d_j'}(x, r) &= \overline{B}_{d_j}(x, r) \quad \text{when } r < a_j \\ &= X \quad \text{when } r \ge a_j \end{aligned} \tag{1.9.3}$$

for every $j \ge 1$. Remember that (1.9.1) is an ultrametric on X when d_j is an ultrametric on X.

Suppose that the a_j's are bounded in $\mathbf{R}$, and put

$$d(x, y) = \sup_{j \ge 1} d_j'(x, y) \tag{1.9.4}$$

for every $x, y \in X$. One can check that this defines a semimetric on X too, and a semi-ultrametric on X when d_j is a semi-ultrametric on X for every $j \ge 1$. Observe that

$$\overline{B}_d(x, r) = \bigcap_{j=1}^{\infty} \overline{B}_{d_j'}(x, r) \tag{1.9.5}$$

for every $x \in X$ and $r \ge 0$. If X is a group and d_j is invariant under left translations for every j, then d_j' is invariant under left translations for every j, and d is invariant under left translations as well. Of course, the analogous statement for invariance under right translations holds as well.

Let us say that the collection of d_j's, $j \ge 1$, is *nondegenerate* on X if for every $x, y \in X$ with $x \ne y$, we have that $d_j(x, y) > 0$ for some j. This implies that the collection of d_j''s, $j \ge 1$, is also nondegenerate on X. In this case, it is easy to see that d is a metric on X.

Suppose now that

$$\lim_{j \to \infty} a_j = 0, \tag{1.9.6}$$

which implies in particular that the a_j's are bounded. One can check that the supremum on the right side of (1.9.4) is attained for every $x, y \in X$ under these conditions. It follows that

$$B_d(x, r) = \bigcap_{j=1}^{\infty} B_{d_j'}(x, r) \tag{1.9.7}$$

for every $x \in X$ and $r > 0$. Note that the intersections on the right sides of (1.9.5) and (1.9.7) can be reduced to finite intersections, because of (1.9.2) and (1.9.3).

Let G be a group, and let $A_1, A_2, A_3, \ldots$ be a sequence of subgroups of G. If $\bigcap_{j=1}^{\infty} A_j = \{e\}$, then the sequences of semimetrics $d_{A_j,L}$ and $d_{A_j,R}$ defined in the previous section for $j \ge 1$ are nondegerate.

1.10 Subadditive functions

Let G be a group, and let N be a nonnegative real-valued function on G. Let us say that N is *subadditive* on G if

$$N(x\,y) \le N(x) + N(y) \tag{1.10.1}$$

for every $x, y \in G$. Similarly, let us say that N is *ultra-subadditive* on G if

$$N(x\,y) \le \max(N(x), N(y)) \tag{1.10.2}$$

for every $x, y \in G$. Thus ultra-subadditivity implies subadditivity. If

$$N(x^{-1}) = N(x) \tag{1.10.3}$$

for every $x \in G$, then we say that N is *symmetric* on G. We shall normally be concerned with subadditive functions that satisfy

$$N(e) = 0. \tag{1.10.4}$$

If $N(x) > 0$ when $x \neq e$, then we say that N is *nondegenerate*.

If N is subadditive and symmetric on G and satisfies (1.10.4), then one can check that

$$d_{N,L}(x, y) = N(x^{-1}\, y) = N(y^{-1}\, x) \tag{1.10.5}$$

is a semimetric on G that is invariant under left translations, and that

$$d_{N,R}(x, y) = N(x\, y^{-1}) = N(y\, x^{-1}) \tag{1.10.6}$$

is a semimetric on G that is invariant under right translations. These are semi-ultrametrics when N is ultra-subadditive, and metrics when N is nondegerate. Conversely, if $d(\cdot, \cdot)$ is a semimetric on G that is invariant under left or right translations, then

$$N_d(x) = d(x, e) \tag{1.10.7}$$

is subadditive, symmetric, and equal to 0 at e. Moreover, N_d is ultra-subadditive when $d(\cdot, \cdot)$ is a semi-ultrametric that is invariant under left or right translations, and N_d is nondegenerate when $d(\cdot, \cdot)$ is a metric.

If N is subadditive on G, then

$$N(x)^a \tag{1.10.8}$$

is subadditive on G for $0 < a \leq 1$, because of (1.1.6). If N is ultra-subadditive on G, then (1.10.8) is ultra-subadditive for every $a > 0$. Of course, if N is symmetric on G, satisfies (1.10.4), or is nondegenerate, then (1.10.8) has the same property.

If N is subadditive on G, then it is easy to see that

$$N_t(x) = \min(N(x), t) \tag{1.10.9}$$

is subadditive on G for every $t > 0$. Similarly, if N is ultra-subadditive, symmetric, satisfies (1.10.4), or is nondegenerate, then (1.10.9) has the same property. Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k. If N is a seminorm on V with respect to $|\cdot|$ on k, then N is subadditive and symmetric on V, as a commutative group with respect to addition. If $|\cdot|$ is the trivial absolute value function on k, then (1.10.9) is a seminorm on V too. Otherwise, we have that

$$N_t(\alpha\, v) \leq N_t(v) \tag{1.10.10}$$

for every $v \in V$ and $\alpha \in k$ with $|\alpha| \leq 1$. In particular,

$$N_t(\alpha\, v) = N_t(v) \tag{1.10.11}$$

for every $v \in V$ and $\alpha \in k$ with $|\alpha| = 1$.

If A is a subgroup of G and $x \in G$, then put

$$N_A(x) = 0 \quad \text{when } x \in A \qquad (1.10.12)$$
$$= 1 \quad \text{when } x \notin A.$$

It is easy to see that N_A is ultra-subadditive and symmetric on G, and of course $N_A(e) = 0$. In this case, (1.10.5) and (1.10.6) are the same as (1.8.7) and (1.8.8), respectively.

If $N_1, \ldots, N_l$ are finitely many subadditive functions on G, then their sum and maximum are subadditive as well. If N_j is ultra-subadditive for each $j = 1, \ldots, l$, then

$$N(x) = \max_{1 \le j \le l} N_j(x) \qquad (1.10.13)$$

is ultra-subadditive on G. If N_j is symmetric on G for each $j = 1, \ldots, n$, or $N_j(e) = 0$ for each $j = 1, \ldots, n$, then the sum and maximum have the same property. Let $A_1, \ldots, A_l$ be finitely many subgroups of G, so that $A = \bigcap_{j=1}^{l} A_j$ is a subgroup of G too. Observe that

$$N_A(x) = \max_{1 \le j \le l} N_{A_j}(x) \qquad (1.10.14)$$

for every $x \in G$.

To say that N is invariant under conjugations on G means that

$$N(a\,x\,a^{-1}) = N(x) \qquad (1.10.15)$$

for every $a, x \in G$. One can check that this is equivalent to the condition that

$$N(x\,y) = N(y\,x) \qquad (1.10.16)$$

for every $x, y \in G$. This is the same as saying that the right sides of (1.10.5) and (1.10.6) are the same. If A is a subgroup of G, then N_A is invariant under conjugations on G exactly when A is a normal subgroup of G.

1.11 Bounded linear mappings

Let k be a field with an absolute value function $|\cdot|$, let V, W be vector spaces over k, and let N_V, N_W be seminorms on V, W, respectively, with respect to $|\cdot|$ on k. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if

$$N_W(T(v)) \le C\,N_V(v) \qquad (1.11.1)$$

for some $C \ge 0$ and every $v \in V$. In this case, the *operator seminorm* of T is defined by

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \ge 0 : (1.11.1) \text{ holds}\}. \qquad (1.11.2)$$

One can check that the space $\mathcal{BL}(V, W)$ of bounded linear mappings from V into W is a vector space over k with respect to pointwise addition and scalar multiplication, and that $\|T\|_{op}$ defines a seminorm on $\mathcal{BL}(V,W)$ with respect to $|\cdot|$ on k. Similarly, $\|T\|_{op}$ is a norm on $\mathcal{BL}(V,W)$ when N_W is a norm on W, and $\|T\|_{op}$ is a semi-ultranorm on $\mathcal{BL}(V,W)$ when N_W is a semi-ultranorm on W. If N_W is a norm on W, and W is complete with respect to the metric associated to N_W, then $\mathcal{BL}(V,W)$ is complete with respect to the metric associated to $\|\cdot\|_{op}$, by standard arguments.

Let Z be another vector space over k, with a seminorm N_Z with respect to $|\cdot|$ on k. If T_1 is a bounded linear mapping from V into W, and T_2 is a bounded linear mapping from W into Z, then their composition $T_2 \circ T_1$ is bounded as a linear mapping from V into Z, with

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW}\,\|T_2\|_{op,WZ}. \tag{1.11.3}$$

Let n be a positive integer, and let us take $V = k^n$ for the moment. Let $e_1, \ldots, e_n$ be the standard basis vectors in k^n, which means that the jth coordinate of e_l is 1 when $j = l$, and 0 when $j \ne l$. If T is any linear mapping from k^n into W, then

$$N_W(T(v)) \le \sum_{l=1}^{n} |v_l|\, N_W(T(e_l)) \tag{1.11.4}$$

for every $v \in k^n$. If we take k^n to be equipped with the norm $\|v\|_1$ in (1.4.7), then T is bounded, with

$$\|T\|_{op} = \max_{1\le l\le n} N_W(T(e_l)). \tag{1.11.5}$$

More precisely, (1.11.4) implies that T is bounded, with $\|T\|_{op}$ less than or equal to the right side, and the opposite inequality can be verified directly. If N_W is a semi-ultranorm on W, then

$$N_W(T(v)) \le \max_{1\le l\le n} (|v_l|\, N_W(T(e_l))) \tag{1.11.6}$$

for every $v \in k^n$. If we take k^n to be equipped with the norm $\|v\|_\infty$ in (1.4.8), then we get that T is bounded again, and that (1.11.5) holds.

Let V be any vector space over k with a seminorm N_V with respect to $|\cdot|$ on k again, and let $\mathcal{BL}(V) = \mathcal{BL}(V,V)$ be the space of bounded linear mappings from V into itself. This is an associative algebra over k, with respect to composition of linear mappings. Note that the identity mapping $I = I_V$ on V is bounded, with

$$\|I\|_{op} = 1 \tag{1.11.7}$$

when $N_V(v) > 0$ for some $v \in V$.

A linear mapping T from V into W is said to be an *isometry* if

$$N_W(T(v)) = N_V(v) \tag{1.11.8}$$

for every $v \in V$. This implies that T is injective when N_V is a norm on V. A one-to-one linear mapping T from V onto W is an isometry if and only if T and T^{-1} are bounded, with

$$\|T\|_{op,VW},\ \|T^{-1}\|_{op,WV} \le 1. \tag{1.11.9}$$

If T_1 is an isometric linear mapping from V into W, and T_2 is an isometric linear mapping from W into Z, then $T_2 \circ T_1$ is an isometric linear mapping from V into Z. The collection of one-to-one isometric linear mappings from V onto itself is a subgroup of the group of invertible elements of $\mathcal{BL}(V)$.

1.12 Submultiplicativity

Let $\mathcal{A}$ be a ring, and let N be a nonnegative real-valued function on $\mathcal{A}$ that is subadditive and symmetric on $\mathcal{A}$, as a commutative group with respect to addition. This means that

$$N(x + y) \le N(x) + N(y) \tag{1.12.1}$$

and

$$N(-x) = N(x) \tag{1.12.2}$$

for every $x, y \in \mathcal{A}$, and we ask that $N(0) = 0$ too. Suppose that

$$N(x\,y) \le N(x)\,N(y) \tag{1.12.3}$$

for every $x, y \in \mathcal{A}$, so that N is *submultiplicative* on $\mathcal{A}$. Let us also suppose from now on in this section that $\mathcal{A}$ has a multiplicative identity element e, and observe that $N(e) \le N(e)^2$, by (1.12.3). It follows that either $N(e) = 0$, which implies that $N(x) = 0$ for every $x \in \mathcal{A}$, or $N(e) \ge 1$.

If $x, y \in \mathcal{A}$ are invertible, then

$$x^{-1} - y^{-1} = x^{-1}\,(y - x)\,y^{-1}, \tag{1.12.4}$$

and hence

$$N(x^{-1} - y^{-1}) \le N(x^{-1})\,N(y^{-1})\,N(x - y). \tag{1.12.5}$$

In particular,

$$\begin{aligned} N(y^{-1}) &\le N(x^{-1}) + N(x^{-1} - y^{-1}) \\ &\le N(x^{-1}) + N(x^{-1})\,N(x - y)\,N(y^{-1}), \end{aligned} \tag{1.12.6}$$

so that

$$(1 - N(x^{-1})\,N(x - y))\,N(y^{-1}) \le N(x^{-1}). \tag{1.12.7}$$

If

$$N(x^{-1})\,N(x - y) < 1, \tag{1.12.8}$$

then we get that

$$N(y^{-1}) \le (1 - N(x^{-1})\,N(x - y))^{-1}\,N(x^{-1}). \tag{1.12.9}$$

This implies that

$$N(x^{-1} - y^{-1}) \le (1 - N(x^{-1})\,N(x - y))^{-1}\,N(x^{-1})^2\,N(x - y) \tag{1.12.10}$$

when (1.12.8) holds, by (1.12.5).

Suppose for the moment that N is ultra-subadditive on $\mathcal{A}$, as a commutative group with respect to addition, so that

$$N(u + v) \le \max(N(u), N(v)) \tag{1.12.11}$$

for every $u, v \in \mathcal{A}$. This implies that

$$\begin{aligned} N(y^{-1}) &\le \max(N(x^{-1}), N(x^{-1} - y^{-1})) \\ &\le \max(N(x^{-1}), N(x^{-1})\, N(x - y)\, N(y^{-1})), \end{aligned} \tag{1.12.12}$$

because of (1.12.5). It follows that

$$N(y^{-1}) \le N(x^{-1}) \tag{1.12.13}$$

when (1.12.8) holds. Thus

$$N(x^{-1} - y^{-1}) \le N(x^{-1})^2\, N(x - y) \tag{1.12.14}$$

when (1.12.8) holds, by (1.12.5).

Let us suppose from now on in this section that

$$N(e) = 1. \tag{1.12.15}$$

Let $G(\mathcal{A})$ be the group of invertible elements of $\mathcal{A}$, and put

$$U(\mathcal{A}) = U_N(\mathcal{A}) = \{x \in G(\mathcal{A}) : N(x),\ N(x^{-1}) \le 1\}. \tag{1.12.16}$$

It is easy to see that this is a subgroup of $G(\mathcal{A})$. More precisely, if $x \in U(\mathcal{A})$, then

$$N(x) = N(x^{-1}) = 1, \tag{1.12.17}$$

because $N(e) \le N(x)\, N(x^{-1})$.

If $a \in U(\mathcal{A})$, then one can check that

$$N(a\, x) = N(x\, a) = N(x) \tag{1.12.18}$$

for every $x \in \mathcal{A}$. This means that the semimetric

$$d_N(x, y) = N(x - y) \tag{1.12.19}$$

on $\mathcal{A}$ is invariant under left and right multiplication by a on $\mathcal{A}$. It follows that the restriction of (1.12.19) to $x, y \in U(\mathcal{A})$ is invariant under left and right translations on $U(\mathcal{A})$, as a group with respect to multiplication.

1.13 Completeness and invertibility

Let $\mathcal{A}$ be a ring with a multiplicative identity element e again, and let N be a nonnegative real-valued function on $\mathcal{A}$ that is subadditive and symmetric on $\mathcal{A}$, as a commutative group with respect to addition, as in the previous section. Suppose that $N(0) = 0$ and that N is nondegenerate on $\mathcal{A}$, so that (1.12.19) is a metric on $\mathcal{A}$, and that N is submultiplicative on $\mathcal{A}$. In this section, we also ask that $\mathcal{A}$ be complete with respect to this metric.

If $a \in \mathcal{A}$, then

$$(1.13.1) \qquad (e - a) \sum_{j=0}^{n} a^j = \Big(\sum_{j=0}^{n} a^j\Big)(e - a) = e - a^{n+1}$$

for every nonnegative integer n, where a^j is interpreted as being e when $j = 0$. Note that

$$(1.13.2) \qquad N(a^j) \le N(a)^j$$

for every $j \ge 1$. Suppose that

$$(1.13.3) \qquad N(a) < 1,$$

so that $N(a^j) \to 0$ as $j \to \infty$, and $\sum_{j=0}^{\infty} N(a^j)$ converges as an infinite series of nonnegative real numbers. One can check that the sequence of partial sums $\sum_{j=0}^{n} a^j$ is a Cauchy sequence in $\mathcal{A}$ with respect to the metric associated to N, using the analogous property of $\sum_{j=0}^{n} N(a^j)$ in the real line. This means that this sequence converges in $\mathcal{A}$ with respect to the metric associated to N, because $\mathcal{A}$ is complete, by hypothesis. Let $\sum_{j=0}^{\infty} a^j$ be the limit of the sequence of partial sums in $\mathcal{A}$, as usual, and observe that

$$(1.13.4) \qquad (e - a) \sum_{j=0}^{\infty} a^j = \Big(\sum_{j=0}^{\infty} a^j\Big)(e - a) = e,$$

by (1.13.1). This implies that $e - a$ is invertible in $\mathcal{A}$, with

$$(1.13.5) \qquad (e - a)^{-1} = \sum_{j=0}^{\infty} a^j.$$

Suppose now that $x \in \mathcal{A}$ is invertible, and that $y \in \mathcal{A}$ satisfies (1.12.8). Thus

$$(1.13.6) \qquad N(x^{-1}(x - y)) \le N(x^{-1})\, N(x - y) < 1,$$

and

$$(1.13.7) \qquad y = x - (x - y) = x\,(e - x^{-1}(x - y)).$$

It follows that y is invertible in $\mathcal{A}$, because $e - x^{-1}(x - y)$ is invertible in $\mathcal{A}$, as in the preceding paragraph. In particular, the group $G(\mathcal{A})$ of invertible elements of $\mathcal{A}$ is an open set with respect to the metric associated to N.

If C is a nonnegative real number, then one can check that

$$(1.13.8) \qquad \{x \in G(\mathcal{A}) : N(x^{-1}) \le C\}$$

is a closed set in $\mathcal{A}$ with respect to the metric associated to N. Indeed, if $\{x_j\}_{j=1}^\infty$ is a sequence of elements of this set that converges to $x \in \mathcal{A}$ with respect to the metric associated to N, then it is easy to see that $\{x_j^{-1}\}_{j=1}^\infty$ is a Cauchy sequence, using (1.12.5). It follows that $\{x_j^{-1}\}_{j=1}^\infty$ converges to an element of $\mathcal{A}$, because $\mathcal{A}$ is complete, by hypothesis. One can verify that the limit of $\{x_j^{-1}\}_{j=1}^\infty$ is the multiplicative inverse of x in $\mathcal{A}$.

Suppose that $N(e) = 1$, and let $U(\mathcal{A})$ be as in (1.12.16). This is a closed set in $\mathcal{A}$ with respect to the metric associated to N, by the remarks in the previous paragraph. Suppose that N is ultra-subadditive on $\mathcal{A}$ as a commutative group with respect to addition. If $x \in U(\mathcal{A})$, $y \in \mathcal{A}$, and $N(x - y) < 1$, then y is invertible in $\mathcal{A}$, as before. We also have that $y \in U(\mathcal{A})$ under these conditions, by (1.12.11) and (1.12.13).

1.14 Invertible linear mappings

Let k be a field with an absolute value function $|\cdot|$, let V, W be vector spaces over k with seminorms N_V, N_W, respectively, with respect to $|\cdot|$ on k, and let T be a linear mapping from V into W. We may be interested in situations in which there is a positive real number c such that

$$c\,N_V(v) \le N_W(T(v)) \tag{1.14.1}$$

for every $v \in V$. If T is a one-to-one mapping from V onto W, then (1.14.1) is the same as saying that T^{-1} is bounded as a linear mapping from W into V, with

$$\|T^{-1}\|_{op,WV} \le 1/c. \tag{1.14.2}$$

If N_V is a norm on V, then (1.14.1) implies that T is injective. Of course, if V and W are finite-dimensional vector spaces over k of the same dimension, then injective linear mappings from V into W are automatically surjective.

Suppose that (1.14.1) holds, and let R be a linear mapping from V into W such that $R - T$ is bounded, with

$$\|R - T\|_{op,VW} < c. \tag{1.14.3}$$

Observe that

$$\begin{aligned} N_W(T(v)) &\le N_W(R(v)) + N_W(R(v) - T(v)) \\ &\le N_W(R(v)) + \|R - T\|_{op,VW}\, N_V(v) \end{aligned} \tag{1.14.4}$$

for every $v \in V$. This implies that

$$(c - \|R - T\|_{op,VW})\, N_V(v) \le N_W(R(v)) \tag{1.14.5}$$

for every $v \in V$. If N_W is a semi-ultranorm on W, then

$$\begin{aligned} N_W(T(v)) &\le \max(N_W(R(v)), N_W(R(v) - T(v))) \\ &\le \max(N_W(R(v)), \|R - T\|_{op,VW}\, N_V(v)) \end{aligned} \tag{1.14.6}$$

for every $v \in V$. In this case, one can check that

$$c\, N_V(v) \le N_W(R(v)) \tag{1.14.7}$$

for every $v \in V$.

Suppose that T is an isometric linear mapping from V into W, so that T is a bounded linear mapping with $\|T\|_{op,VW} \le 1$, and (1.14.1) holds with $c = 1$. If N_W is a semi-ultranorm on W, and if R is a bounded linear mapping from V into W that satisfies (1.14.3) with $c = 1$, then (1.14.7) holds with $c = 1$. We also have that

$$\|R\|_{op} \le \max(\|R - T\|_{op,VW}, \|T\|_{op,VW}) \le 1, \tag{1.14.8}$$

because $\|\cdot\|_{op,VW}$ is a semi-ultranorm on $\mathcal{BL}(V, W)$, as in Section 1.11. This means that R is an isometric linear mapping from V into W too under these conditions.

Let us now take $V = W$, $N_V = N_W$, and let $G(\mathcal{BL}(V))$ be the group of invertible elements in the algebra $\mathcal{BL}(V)$ of bounded linear mappings from V into itself, as in Section 1.12. Put

$$U(\mathcal{BL}(V)) = \{T \in G(\mathcal{BL}(V)) : \|T\|_{op},\, \|T^{-1}\|_{op} \le 1\}, \tag{1.14.9}$$

as before. This is the same as the group of one-to-one isometric linear mappings from V onto itself, as in Section 1.11.

Suppose for the rest of the section that V is finite-dimensional as a vector space over k, and that N_V is a norm on V. If $T \in G(\mathcal{BL}(V))$ and $R \in \mathcal{BL}(V)$ satisfy

$$\|R - T\|_{op} < 1/\|T^{-1}\|_{op}, \tag{1.14.10}$$

then R is injective on V, by the earlier argument. More precisely, R is a one-to-one linear mapping from V onto itself with bounded inverse, and

$$\begin{aligned} \|R^{-1}\|_{op} &\le ((1/\|T^{-1}\|_{op}) - \|R - T\|_{op})^{-1} \\ &= (1 - \|T^{-1}\|_{op}\, \|R - T\|_{op})^{-1}\, \|T^{-1}\|_{op}. \end{aligned} \tag{1.14.11}$$

If N_V is a ultranorm on V, then we get that

$$\|R^{-1}\|_{op} \le \|T^{-1}\|_{op}. \tag{1.14.12}$$

In particular, if $T \in U(\mathcal{BL}(V))$, $R \in \mathcal{BL}(V)$, and

$$\|R - T\|_{op} < 1, \tag{1.14.13}$$

then $R \in U(\mathcal{BL}(V))$ when N_V is an ultranorm on V.

1.15 Rings of matrices

Let $\mathcal{A}$ be a ring, and let n be a positive integer. The space $M_n(\mathcal{A})$ of $n \times n$ matrices with entries in $\mathcal{A}$ is also a ring, with respect to entrywise addition of matrices, and matrix multiplication.

Let N be a nonnegative real-valued function on $\mathcal{A}$, and suppose that N is symmetric and ultra-subadditive on $\mathcal{A}$ as a commutative group with respect to addition, that N is submultiplicative on $\mathcal{A}$, and that $N(0) = 0$. Let $a = (a_{j,l})$ be an element of $M_n(\mathcal{A})$, so that $a_{j,l} \in \mathcal{A}$ for every $j, l = 1, \ldots, n$, and put

$$N_n(a) = \max_{1 \le j,l \le n} N(a_{j,l}). \tag{1.15.1}$$

This defines a nonnegative real-valued function on $M_n(\mathcal{A})$, and one can check that N_n is symmetric and ultra-subadditive on $M_n(\mathcal{A})$ as a commutative group with respect to addition, that N_n is submultiplicative on $M_n(\mathcal{A})$ with respect to matrix multiplication, and that $M_n(0) = 0$. If N is nondegenerate on $\mathcal{A}$, then N_n is nondegenerate on $M_n(\mathcal{A})$.

Suppose that $\mathcal{A}$ has a multiplicative identity element e, and let $I = I_n$ be the identity matrix in $M_n(\mathcal{A})$, with diagonal entries equal to e and all other entries equal to 0. This is the multiplicative identity element in $M_n(\mathcal{A})$. The group $G(M_n(\mathcal{A}))$ of invertible elements in $M_n(\mathcal{A})$ is also denoted $GL_n(\mathcal{A})$.

Observe that

$$\mathcal{A}_1 = \{x \in \mathcal{A} : N(x) \le 1\} \tag{1.15.2}$$

is a subring of $\mathcal{A}$. Suppose that $N(e) = 1$, so that $e \in \mathcal{A}_1$. It is easy to see that $x \in \mathcal{A}_1$ is invertible in $\mathcal{A}_1$ if and only if x is invertible in $\mathcal{A}$, with $N(x^{-1}) \le 1$. Thus the subgroup $U(\mathcal{A})$ of the group $G(\mathcal{A})$ of invertible elements of $\mathcal{A}$ defined in (1.12.16) is the same as the group $G(\mathcal{A}_1)$ of invertible elements of $\mathcal{A}_1$.

Of course, the ring $M_n(\mathcal{A}_1)$ of $n \times n$ matrices with entries in $\mathcal{A}_1$ may be considered as a subring of $M_n(\mathcal{A})$. In fact,

$$M_n(\mathcal{A}_1) = \{a \in M_n(\mathcal{A}) : N_n(a) \le 1\}, \tag{1.15.3}$$

by the definition of N_n. The subgroup $U_{N_n}(M_n(\mathcal{A}))$ of $G(M_n(\mathcal{A})) = GL_n(\mathcal{A})$ defined as in (1.12.16) is the same as the group $G(M_n(\mathcal{A}_1)) = GL_n(\mathcal{A}_1)$ of invertible elements in $M_n(\mathcal{A}_1)$, as in the preceding paragraph.

Suppose that $\mathcal{A}$ is a commutative ring, so that the determinant $\det a$ of $a \in M_n(\mathcal{A})$ can be defined as an element of $\mathcal{A}$ in the usual way. It is well known that a is invertible in $M_n(\mathcal{A})$ if and only if $\det a$ is invertible in $\mathcal{A}$. If $a \in M_n(\mathcal{A}_1)$ satisfies

$$N_n(a - I) < 1, \tag{1.15.4}$$

then one can check that

$$N(\det a - e) < 1. \tag{1.15.5}$$

Suppose now that $\mathcal{A}$ is a field k, and that N is an ultrametric absolute value function $|\cdot|$ on k. In this case, $\mathcal{A}_1$ consists of $x \in k$ such that $|x| \le 1$, and the group of invertible elements of $\mathcal{A}_1$ consists of $x \in k$ with $|x| = 1$. In particular, one can verify that $x \in k$ satisfies $|x| = 1$ when $|x - 1| < 1$.

Chapter 2

Semimetrics and topologies

2.1 Collections of semimetrics

Let X be a set, and let $\mathcal{M}$ be a nonempty collection of semimetrics on X. A subset U of X is said to be an *open set* with respect to $\mathcal{M}$ if for every $x \in U$ there are finitely many elements $d_1, \ldots, d_n$ of $\mathcal{M}$ and positive real numbers $r_1, \ldots, r_n$ such that

$$\bigcap_{j=1}^{n} B_{d_j}(x, r) \subseteq U. \tag{2.1.1}$$

One can check that this defines a topology on X. Of course, if $\mathcal{M}$ consists of a single semimetric, then this is the usual topology determined by that semimetric.

In particular, if $U \subseteq X$ is an open set with respect to any element of $\mathcal{M}$, then U is an open set with respect to $\mathcal{M}$. It is well known that open balls are open sets with respect to a semimetric. Using this, it is easy to see that the open balls in X with respect to elements of $\mathcal{M}$ form a sub-base for the topology determined by $\mathcal{M}$.

If $\mathcal{M}$ consists of finitely many semimetrics, then the topology determined on X by $\mathcal{M}$ is the same as the topology determined by the sum or maximum of the elements of $\mathcal{M}$. If $\mathcal{M}$ consists of an infinite sequence of semimetrics, then one can get a single semimetric that determines the same topology on X as in Section 1.9.

Let us say that $\mathcal{M}$ is *nondegenerate* on X if for every $x, y \in X$ with $x \neq y$ there is a $d \in \mathcal{M}$ such that $d(x, y) > 0$. In this case, one can check that X is Hausdorff with respect to the topology determined by $\mathcal{M}$.

One can check that closed balls with respect to a semimetric are closed sets with respect to the topology determined by that semimetric. It is easy to see that closed balls of positive radius with respect to a semi-ultrametric are open sets with respect to that semi-ultrametric. One can also verify that open balls with respect to a semi-ultrametric are closed sets with respect to the semi-ultrametric.

Let us say that a topological space Y is *regular in the strict sense* if for every $y \in Y$ and closed set $E \subseteq Y$ with $y \notin E$ there are disjoint open sets $U, V \subseteq Y$ such that $y \in U$ and $E \subseteq V$. Equivalently, this means that for every $y \in Y$ and open set $W \subseteq Y$ with $y \in W$ there is an open set $U \subseteq Y$ such that $y \in U$ and the closure $\overline{U}$ of U in Y is contained in W. If Y is regular in the strict sense and satisfies the first or zeroth separation condition, then Y is *regular in the strong sense*. It is easy to see that this implies that Y is Hausdorff. If the topology on Y is determined by a nonempty collection of semimetrics, then one can check that Y is regular in the strict sense.

If Y is a subset of X, then the restriction to Y of a semimetric on X is a semimetric on Y. Let $\mathcal{M}$ be a nonempty collection of semimetrics on X, and let $\mathcal{M}_Y$ be the collection of the restrictions to Y of the elements of $\mathcal{M}$. One can check that the topology determined on Y by $\mathcal{M}_Y$ is the same as the topology induced on Y by the topology determined on X by $\mathcal{M}$.

2.2 Topological groups

Let G be a group that is also equipped with a topology. We say that G is a *topological group* if the group operations are continuous. More precisely, this means that $x \mapsto x^{-1}$ is continuous as a mapping from G into itself, and that multiplication on G is continuous as a mapping from $G \times G$ into G, using the associated product topology on $G \times G$. If e is the identity element in G, then the condition that $\{e\}$ be a closed set is sometimes included in the definition of a topological group.

If G is a topological group, then it is easy to see that left and right translations on G are continuous. In fact, they are homeomorphisms from G onto itself, because the inverse of a left or right translation on G is a left or right translation too, respectively. If $\{e\}$ is a closed set, then it follows that G satisfies the first separation condition. It is well known that G is Hausdorff in this case. More precisely, topological groups are regular in the strict sense, as in Section 2.4.

If there is a local base for the topology of G at e with only finitely or countably many elements, then a famous theorem states that there is a semimetric on G that is invariant under left translations and determines the same topology. Equivalently, there is a semimetric on G that is invariant under right translations and determines the same topology. If $\{e\}$ is a closed set, then these semimetrics are metrics. Otherwise, if G is any topological group, then there is a nonempty collection of semimetrics on G that determines the same topology and whose elements are invariant under left translations. Of course, there is also a nonempty collection of semimetrics that determines the same topology whose elements are invariant under right translations.

If G is equipped with the topology determined by any nonempty collection of semimetrics that are invariant under left translations, then left translations on G are automatically continuous, and one can check that the group operations on G are continuous at e, as in Section 1.7. In order for G to be a topological group with respect to this topology, it would be enough to know that right translations are continuous as well. In particular, this holds when $x \mapsto x^{-1}$ is continuous with respect to this topology. If conjugations on G are continuous, then continuity of right translations is equivalent to continuity of left translations. Of course, there are analogous statements for collections of semimetrics that are invariant under right translations.

If G is equipped with the topology determined by a nonempty collection of semimetrics that are invariant under both left and right translations, then G is a topological group. In this case, there is a local base for the topology of G at e consisting of open sets that are invariant under conjugations. Conversely, if G is a topological group, and if there is a local base for the topology of G at e that consists of open sets that are invariant under conjugations, then it is well known that there is a nonempty collection of semimetrics on G that determines the same topology, and whose elements are invariant under both left and right translations. If there is a local base for the topology of G at e with only finitely or countably many elements too, then there is a semimetric on G that determines the same topology and is invariant under both left and right translations.

Of course, any group is a topological group with respect to the discrete topology. It is easy to see that a subgroup of a topological group is a topological group as well, with respect to the induced topology. One can verify that the closure of a subgroup of a topological group is a subgroup too.

If I is a nonempty set and G_j is a group for each $j \in I$, then the Cartesian product $G = \prod_{j \in I} G_j$ is a group too, where the group operations are defined coordinatewise. If G_j is a topological group for each $j \in I$, then one can check that G is a topological group with respect to the product topology. Suppose that d_l is a semimetric on G_l for some $l \in I$, and consider $\widetilde{d_l}(x, y) = d_l(x_l, y_l)$ on G, as in Section 1.1. If d_l is invariant under left or right translations on G_l, then it is easy to see that $\widetilde{d_l}$ has the same property on G.

2.3 Nice families of subgroups

Let G be a group. If G is a topological group and U is an open subgroup of G, then U is a closed set, because the complement of U can be expressed as a union of translates of U. If the topology on G is determined by a collection of semi-ultrametrics that are invariant under left or right translations, then there is a local base for the topology of G at the identity element e consisting of open subgroups of G. If the topology on G is determined by a collection of semi-ultrametrics that are invariant under both left and right translations, then there is a local base for the topology of G at e consisting of open normal subgroups of G.

Conversely, if there is a local base for the topology of G at e consisting of open subgroups, then one can get collections of semi-ultrametrics that are invariant under either left or right translations and determine the same topology on G, using the semimetrics defined in Section 1.8. Similarly, if there is a local base for the topology of G at e consisting of open normal subgroups of G, then one can get a collection of semi-ultrametrics that are invariant under both left and right translations and determine the same topology. In both cases, if the local base has only finitely or countably many elements, then one can get a single semi-ultrametric that determines the same topology, as in Section 1.9. One could also start with a local sub-base for the topology of G at e consisting of open subgroups.

If $\mathcal{B}$ is any nonempty collection of subgroups of G, then we can define topologies $\tau_L(\mathcal{B})$, $\tau_R(\mathcal{B})$ on G such that $\mathcal{B}$ is a local sub-base at e, and either left or right translations are continuous, respectively. More precisely, a subset U of G is an open set with respect to $\tau_L(\mathcal{B})$ if for every $x \in U$ there are finitely many elements $A_1, \ldots, A_n$ of $\mathcal{B}$ such that

$$\bigcap_{j=1}^{n} (x\, A_j) \subseteq U. \tag{2.3.1}$$

Similarly, U is an open set with respect to $\tau_R(\mathcal{B})$ if for every $x \in U$ there are finitely many elements $A_1, \ldots, A_n$ of $\mathcal{B}$ such that

$$\bigcap_{j=1}^{n} (A_j\, x) \subseteq U. \tag{2.3.2}$$

It is easy to see that these define topologies on G with the properties mentioned before. In particular, note that

$$\mathcal{B} \subseteq \tau_L(\mathcal{B}),\ \tau_R(\mathcal{B}). \tag{2.3.3}$$

If

$$\tau_L(\mathcal{B}) = \tau_R(\mathcal{B}), \tag{2.3.4}$$

then one can check that G is a topological group with respect to this topology. This condition is also necessary for G to be a topological group with respect to $\tau_L(\mathcal{B})$ or $\tau_R(\mathcal{B})$. Let us say that $\mathcal{B}$ is *nice* when (2.3.4) holds. One can verify that this happens if and only if for every $x \in G$ and $A \in \mathcal{B}$ there are finitely many elements $A_1, \ldots, A_n$ of $\mathcal{B}$ such that

$$\bigcap_{j=1}^{n} A_j \subseteq x\, A\, x^{-1}. \tag{2.3.5}$$

If the elements of $\mathcal{B}$ are normal subgroups of G, then $\mathcal{B}$ is automatically nice.

Let us say that $\mathcal{B}$ is *nondegenerate* if $\bigcap_{A \in \mathcal{B}} A = \{e\}$. One can check that G is Hausdorff with respect to $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$ in this case.

2.4 Regularity and topological groups

Let G be a topological group, and let A, B be subsets of G. If $a, b \in G$ and A, B are open sets, then $A\, b$ and $a\, B$ are open sets, by continuity of translations.

If A or B is an open set, then $A\,B$ is an open set, because it is a union of open sets. If A is an open set, then A^{-1} is an open set too.

If W is an open subset of G that contains the identity element e, then there are open subsets U, V of G such that $e \in U, V$ and

$$U\,V \subseteq W, \tag{2.4.1}$$

because of continuity of multiplication. More precisely, we can reduce to the case where $U = V$, by replacing U, V with their intersection.

Let $x \in G$ and $E \subseteq G$ be given, and observe that x is an element of the closure $\overline{E}$ of E in G if and only if for every open set $U_0 \subseteq G$ with $x \in U_0$, we have that

$$(U_0\,x) \cap E \neq \emptyset. \tag{2.4.2}$$

It is easy to see that (2.4.2) holds if and only if

$$x \in U_0^{-1}\,E. \tag{2.4.3}$$

Thus

$$\overline{E} = \bigcap \{U\,E : U \subseteq G \text{ is an open set, with } e \in U\}, \tag{2.4.4}$$

and similarly

$$\overline{E} = \bigcap \{E\,V : V \subseteq G \text{ is an open set, with } e \in V\}. \tag{2.4.5}$$

In particular, if U, V are as in (2.4.1), then

$$\overline{U}, \overline{V} \subseteq W. \tag{2.4.6}$$

One can use this and continuity of translations to get that G is regular in the strict sense.

Let K be a compact subset of G, let W be an open subset of G, and suppose that $K \subseteq W$. If $x \in K$, then $x^{-1}\,W$ is an open subset of G that contains e, and there is an open subset $U(x)$ of G such that $e \in U(x)$ and

$$U(x)\,U(x) \subseteq x^{-1}\,W, \tag{2.4.7}$$

as in (2.4.1). The open sets $U(x)\,x$, $x \in K$, form an open covering of K, and so there are finitely many elements $x_1, \ldots, x_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} U(x_j)\,x_j, \tag{2.4.8}$$

by compactness. Note that

$$U = \bigcap_{j=1}^{n} U(x_j) \tag{2.4.9}$$

is an open subset of G that contains e, and that

$$U\,K \subseteq \bigcup_{j=1}^{n} U\,U(x_j)\,x_j \subseteq \bigcup_{j=1}^{n} U(x_j)\,U(x_j)\,x_j \subseteq W. \tag{2.4.10}$$

Similarly, there is an open subset V of G such that $e \in V$ and

$$K\,V \subseteq W, \tag{2.4.11}$$

which could also be obtained from the previous statement and continuity of $x \mapsto x^{-1}$.

Let A be a subset of G that contains e and is symmetric about e, which can always be arranged by replacing A with its intersection with A^{-1}. Put $A^1 = A$ and $A^{j+1} = A^j\,A$ for each $j \in \mathbf{Z}_+$, so that A^j consists of products of j elements of A. It is easy to see that

$$A^j\,A^l \subseteq A^{j+l} \tag{2.4.12}$$

for every $j, l \geq 1$, and that

$$(A^j)^{-1} = A^j \tag{2.4.13}$$

for every $j \geq 1$. This implies that

$$\bigcup_{j=1}^{\infty} A^j \tag{2.4.14}$$

is a subgroup of G. If A is an open subset of G, then (2.4.14) is an open set as well.

2.5 U-Separated sets

Let G be a topological group, and suppose that U is an open subset of G that contains the identity element e. A pair A, B of subsets of G is said to be *left-invariant U-separated* if

$$(A\,U) \cap B = \emptyset. \tag{2.5.1}$$

Equivalently, this means that

$$A \cap (B\,U^{-1}) = \emptyset. \tag{2.5.2}$$

Similarly, A, B are *right-invariant U-separated* if

$$(U\,A) \cap B = \emptyset, \tag{2.5.3}$$

which is the same as saying that

$$A \cap (U^{-1}B) = \emptyset. \tag{2.5.4}$$

We also have that (2.5.3) holds if and only if

$$(A^{-1}\,U^{-1}) \cap B^{-1} = \emptyset, \tag{2.5.5}$$

which means that A^{-1}, B^{-1} are left-invariant U^{-1}-separated.

Let U_1 be an open subset of G such that $e \in U_1$ and

$$U_1\, U_1^{-1} \subseteq U. \tag{2.5.6}$$

If A, B are left-invariant U-separated, then

$$(A\, U_1\, U_1^{-1}) \cap B = \emptyset. \tag{2.5.7}$$

Equivalently, this means that

$$(A\, U_1) \cap (B\, U_1) = \emptyset. \tag{2.5.8}$$

It follows in particular that

$$\overline{A} \cap \overline{B} = \emptyset, \tag{2.5.9}$$

by (2.4.5). If A is compact, B is a closed set, and

$$A \cap B = \emptyset, \tag{2.5.10}$$

then there is an open subset U of G such that $e \in U$ and A, B are left-invariant U separated sets, because of (2.4.11).

If A is compact and open, then there is an open subset U of G such that $e \in U$ and

$$A\, U \subseteq A. \tag{2.5.11}$$

This can be obtained from the remarks in the preceding paragraph, or directly from (2.4.11).

Suppose that A, U are subsets of G such that $e \in U$, U is an open set, and (2.5.11) holds. We may also suppose that U is symmetric about e, by replacing it with ints intersection with U^{-1}. It is easy to see that

$$A\, U^j \subseteq A \tag{2.5.12}$$

for every $j \in \mathbf{Z}_+$, where U^j is as in the previous section, using (2.5.11). Put

$$U_0 = \bigcup_{j=1}^{\infty} U^j, \tag{2.5.13}$$

so that

$$A\, U_0 \subseteq A, \tag{2.5.14}$$

by (2.5.12). More precisely,

$$A\, U_0 = A, \tag{2.5.15}$$

because $e \in U$, so that $A \subseteq A\, U \subseteq A\, U_0$. Remember that U_0 is an open subgroup of G under these conditions, as in the previous section. If $e \in A$, then

$$U_0 \subseteq A, \tag{2.5.16}$$

by (2.5.14).

In particular, if there is a local base for the topology of G at e consisting of compact open sets, then there is a local base consisting of open subgroups. More precisely, these open subgroups are also compact in this case, because they are closed sets contained in compact sets.

2.6 Totally bounded sets

Let X be a set with a semimetric $d(x,y)$. As usual, a subset E of X is said to be *totally bounded* with respect to $d(\cdot,\cdot)$ if for every $r > 0$, E is contained in the union of finitely many open balls in X of radius r. In particular, this happens when E is compact with respect to the topology determined on X by $d(\cdot,\cdot)$.

Now let G be a topological group, and let E be a subset of G. Let us say that E is *left-invariant totally bounded* in G if for every open subset U of G that contains the identity element e, there are finitely many elements $a_1, \ldots, a_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} (a_j\, U). \tag{2.6.1}$$

Similarly, we say that E is *right-invariant totally bounded* in G if for every open subset U of G with $e \in U$ there are finitely many elements $b_1, \ldots, b_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} (U\, b_j). \tag{2.6.2}$$

Of course, left and right-invariant total boundedness are the same when G is commutative.

One can check that

(2.6.3) E is left-invariant totally bounded in G if and only if E^{-1} is right-invariant totally bounded in G.

If E is compact, then one can verify that E is both left and right-invariant totally bounded in G.

Suppose for the moment that the topology on G is determined by a semimetric $d(\cdot,\cdot)$. If $d(\cdot,\cdot)$ is invariant under left translations, then one can verify that

(2.6.4) E is left-invariant totally bounded if and only if E is totally bounded with respect to $d(\cdot,\cdot)$.

Similarly, if $d(\cdot,\cdot)$ is invariant under right translations, then

(2.6.5) E is right-invariant totally bounded if and only if E is totally bounded with respect to $d(\cdot,\cdot)$.

If $d(\cdot,\cdot)$ is invariant under both left and right translations, then it follows that left and right-invariant total boundedness are equivalent.

Let ϕ be a homomorphism from G into another topological group. If ϕ is continuous at e, then one can check that ϕ is continuous at every point, using continuity of translations. In this case, if E is left or right-invariant totally bounded, then one can verify that $\phi(E)$ has the same property.

If E is symmetric about e, then

(2.6.6) left and right-invariant total boundedness of E are equivalent,

and we may simply refer to the total boundedness of E. If G is totally bounded, then

(2.6.7) every open subgroup of G has finite index in G.

If there is a local base $\mathcal{B}$ for the topology of G at e consisting of open subgroups of G, then

(2.6.8) G is totally bounded if and only if every element of $\mathcal{B}$ has finite index in G.

If A_1, A_2 are subgroups of G and A_1 has finite index in G, then one can check that $A_1 \cap A_2$ has finite index in A_2. If A_2 has finite index in G as well, then it follows that $A_1 \cap A_2$ has finite index in G.

If A is any subgroup of G, then the intersection of all of the conjugates of A in G is a normal subgroup of G. If A has finite index in G, then one can verify that there are only finitely many distinct conjugates of A in G. This implies that the intersection of the conjugates of A in G has finite index in G, as in the preceding paragraph.

If A is an open subgroup of G, then the conjugates of A in G are open subgroups. If A also has finite index in G, then it follows that the intersection of all of the conjugates of A in G is an open subgroup.

If G is totally bounded, and there is a local base for the topology of G at e consisting of open subgroups, then there is a local base for the topology of G at e consisting of open normal subgroups.

If A is a closed subgroup of G of finite index, then A is an open set, because the complement of A is the union of finitely many translates of A.

2.7 Equicontinuous families of conjugations

Let G be a topological group, and note that $C_a(x) = a\,x\,a^{-1}$ is a homeomorphism from G into itself for each $a \in G$. If E is a subset of G, then we say that

(2.7.1) $$\mathcal{C}(E) = \{C_a : a \in E\}$$

is *equicontinuous* at the identity element e if for every open subset W of G that contains e there is an open subset V of G such that $e \in V$ and

(2.7.2) $$C_a(V) \subseteq W$$

for every $a \in E$. If E has only finitely many elements, then this can be obtained from the continuity of conjugations on G. If A is a subgroup of G, and there is a local base for the topology of G at e consisting of open sets that are invariant under conjugations by elements of A, then $\mathcal{C}(A)$ is equicontinuous at e.

Conversely, if A is a subgroup of G, and $\mathcal{C}(A)$ is equicontinuous at e, then there is a local base for the topology of G at e consisting of open sets that are invariant under conjugations by elements of A. To see this, let an open subset W of G that contains e be given, and let V be an open subset of G such that $e \in V$ and (2.7.2) holds for every $a \in A$. This means that

$$V \subseteq a^{-1}\, W\, a \tag{2.7.3}$$

for every $a \in A$. Note that

$$\bigcap_{a \in A} (a^{-1}\, W\, a) \tag{2.7.4}$$

is automatically invariant under conjugations by elements of A. This set contains e and is contained in W, by construction. More precisely, V is contained in (2.7.4), by (2.7.3), which implies that V is contained in the interior of W. Of course, the interior of W is invariant under conjugations by elements of A too, as desired. If W is an open subgroup of G, then (2.7.4) is an open subgroup of G as well, because it contains V.

Suppose now that E is a right-invariant totally bounded subset of G, and let us verify that $\mathcal{C}(E)$ is equicontinuous at e. If W is an open subset of G that contains e, then there are open subsets U_1, U_2, U_3 of G that contain e and satisfy

$$U_1\, U_2\, U_3 \subseteq W. \tag{2.7.5}$$

This implies that

$$y\, U_2\, y^{-1} \subseteq W \tag{2.7.6}$$

for every $y \in U_1 \cap U_3^{-1}$. Using the hypothesis that E be right-invariant totally bounded, we get that there are finitely many elements $b_1, \ldots, b_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} ((U_1 \cap U_3^{-1})\, b_j). \tag{2.7.7}$$

Observe that

$$V = \bigcap_{j=1}^{n} (b_j^{-1}\, U_2\, b_j) \tag{2.7.8}$$

is an open subset of G that contains e. If $x \in E$, then $x = y\, b_j$ for some $y \in U_1 \cap U_3^{-1}$ and $1 \le j \le n$. It follows that

$$x\, V\, x^{-1} = y\, b_j\, V\, b_j^{-1}\, y^{-1} \subseteq y\, U_2\, y^{-1} \subseteq W, \tag{2.7.9}$$

as desired.

If A is a subgroup of G that is totally bounded, then $\mathcal{C}(A)$ is equicontinuous at e, as in the preceding paragraph. This implies that there is a local base for the topology of G at e that is invariant under conjugations by elements of A, as before.

2.8 Small sets

Let X be a set with a semimetric $d(x, y)$. Let us say that a subset A of X is r*-small* with respect to $d(\cdot, \cdot)$ for some $r > 0$ if

$$d(x, y) < r \tag{2.8.1}$$

for every $x, y \in A$. Equivalently, this means that

$$A \subseteq B_d(w, r) \tag{2.8.2}$$

for every $w \in A$. It is easy to see that open balls in X of radius r with respect to $d(\cdot, \cdot)$ are $2\,r$-small, and r-small if $d(\cdot, \cdot)$ is a semi-ultrametric on X. It follows that a subset E of X is totally bounded with respect to $d(\cdot, \cdot)$ if and only if for every $r > 0$, E is contained in the union of finitely many r-small sets.

Let $d_1, \ldots, d_n$ be finitely many semimetrics on X, and let d be their maximum, which is also a semimetric on X. A subset A of X is r-small with respect to d if and only if A is r-small with respect to d_j for each $j = 1, \ldots, n$. If A_j is an r-small subset of X with respect to d_j for each $j = 1, \ldots, n$, then $\bigcap_{j=1}^n A_j$ is r-small with respect to d. Of course, if a subset E of X is totally bounded with respect to d, then E is totally bounded with respect to d_j for every $j = 1, \ldots, n$. One can check that the converse holds too, using the previous remark about intersections of r-small sets.

Let G be a topological group, and suppose for the moment that the topology on G is determined by a nonempty collection $\mathcal{M}$ of semimetrics. If the elements of $\mathcal{M}$ are invariant under left translations, then a subset E of G is left-invariant totally bounded if and only if E is totally bounded with respect to each element of $\mathcal{M}$. This uses the fact that if E is totally bounded with respect to each element of $\mathcal{M}$, then E is totally bounded with respect to the maximum of any finite subset of $\mathcal{M}$, as in the preceding paragraph. Similarly, if the elements of $\mathcal{M}$ are invariant under right translations, then E is right-invariant totally bounded if and only if E is totally bounded with respect to every element of $\mathcal{M}$.

Let U be an open subset of G that contains the identity element e. Let us say that a subset A of G is *left-invariant* U*-small* if

$$A \subseteq a\,U \tag{2.8.3}$$

for every $a \in A$, which is the same as saying that $a^{-1}\,A \subseteq U$ for every $a \in A$. Similarly, let us say that A is *right-invariant* U*-small* if

$$A \subseteq U\,a \tag{2.8.4}$$

for every $a \in A$, which means that $A\,a^{-1} \subseteq U$ for every $a \in A$. Equivalently, A is left-invariant U-small when

$$A^{-1}\,A \subseteq U, \tag{2.8.5}$$

and A is right-invariant U-small when

$$A\,A^{-1} \subseteq U. \tag{2.8.6}$$

Thus A is left-invariant U-small if and only if A^{-1} is right-invariant U-small. It is easy to see that $A^{-1}\,A$ and $A\,A^{-1}$ are automatically symmetric about e. This implies that A is left or right-invariant U-small if and only if A is left or right-invariant U^{-1}-small, respectively.

If V is an open subset of G that contains e, is symmetric about e, and satisfies

$$V\,V \subseteq U, \tag{2.8.7}$$

then V is both left and right-invariant U-small. This implies that left translates of V are left-invariant U-small, and right translates of V are right-invariant U-small. If a subset E of G is left or right-invariant totally bounded, then E can be covered by finitely many left or right translates of V, and thus by finitely many left or right-invariant U-small sets, respectively. Conversely, if E can be covered by finitely many left or right-invariant U-small sets, then E can be covered by finitely many left or right translates of U, respectively. This means that E is left or right-invariant totally bounded if and only if for every open subset U of G that contains e, E can be covered by finitely many left or right-invariant U-small sets, respectively.

2.9 Total boundedness and submultiplicativity

Let $\mathcal{A}$ be a ring, and let N be a nonnegative real-valued function on $\mathcal{A}$ that is subadditive and symmetric on $\mathcal{A}$, as a commutative group with respect to addition, and also submultiplicative. Thus $N(x-y)$ defines a semimetric on $\mathcal{A}$, and one can check that multiplication on $\mathcal{A}$ is continuous as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, with respect to the topology determined on $\mathcal{A}$ by $N(x-y)$ and the associated product topology on $\mathcal{A} \times \mathcal{A}$. Suppose from now on in this section that $\mathcal{A}$ has a multiplicative identity element e, and let $G(\mathcal{A})$ be the group of invertible elements in $\mathcal{A}$, as before. One can verify that $x \mapsto x^{-1}$ is continuous with respect to the topology induced on $G(\mathcal{A})$ by the topology determined on $\mathcal{A}$ by the semimetric associated to N, using some of the remarks in Section 1.12. It follows that $G(\mathcal{A})$ is a topological group with respect to this topology.

If $a \in \mathcal{A}$ and B is a subset of $\mathcal{A}$, then put

$$a\,B = \{a\,b : b \in B\}, \quad B\,a = \{b\,a : b \in B\}. \tag{2.9.1}$$

Note that $x \mapsto a\,x$ and $x \mapsto x\,a$ define continuous mappings from $\mathcal{A}$ into itself, with respect to the topology determined by the semimetric associated to N. If $a \in G(\mathcal{A})$, then these mappings are homeomorphisms from $\mathcal{A}$ onto itself.

In the following, we let $B(x,r)$ be the open ball in $\mathcal{A}$ centered at $x \in \mathcal{A}$ with radius $r > 0$ with respect to the semimetric associated to N. Note that the sets $B(e,r) \cap G(\mathcal{A})$, $r > 0$, form a local base at e for the topology induced on $G(\mathcal{A})$ by the topology determined on $\mathcal{A}$ by the semimetric associated to N.

A subset E of $G(\mathcal{A})$ is left-invariant totally bounded in $G(\mathcal{A})$ if and only if for every $r > 0$ there are finitely many elements $x_1, \ldots, x_n$ of $G(\mathcal{A})$ such that

$$E \subseteq \bigcup_{j=1}^{n} x_j \, B(e, r). \tag{2.9.2}$$

Similarly, E is right-invariant totally bounded in $G(\mathcal{A})$ if and only if for every $r > 0$ there are finitely many elements $x_1, \ldots, x_n$ of $G(\mathcal{A})$ such that

$$E \subseteq \bigcup_{j=1}^{n} B(e, r) \, x_j. \tag{2.9.3}$$

Note that it suffices to consider small r here.

Let us suppose for the rest of the section that $N(e) > 0$, so that $N(x) > 0$ for every $x \in \mathcal{G}(\mathcal{A})$. If $a \in \mathcal{A}$ satisfies $N(a) > 0$, then

$$a \, B(e, r), \; B(e, r) \, a \subseteq B(a, r \, N(a)) \tag{2.9.4}$$

for every $r > 0$. Note that

$$B(a, r \, N(a)) \subseteq B(0, (1 + r) \, N(a)) \tag{2.9.5}$$

for every $r > 0$. If E is left or right-invariant totally bounded in $G(\mathcal{A})$, then it follows that N is bounded on E.

If E is left or right-invariant totally bounded in $G(\mathcal{A})$, then we can take $x_1, \ldots, x_n \in G(\mathcal{A})$ in (2.9.2) or (2.9.3), as appropriate, to be elements of E. This can be obtained from the characterization of total boundedness in terms of U-small sets in the previous section. In particular, this means that $N(x_1), \ldots, N(x_n)$ are bounded, as in the preceding paragraph. Alternatively, we may ask that E intersects $x_j \, B(e, r)$ or $B(e, r) \, x_j$, as appropriate, for each $j = 1, \ldots, n$. This can be used to get an uppoer bound for $N(x_j)$ when $r < 1$.

If E is left or right-invariant totally bounded in $G(\mathcal{A})$, then E is totally bounded in $\mathcal{A}$, with respect to the semimetric associated to N. This follows from (2.9.4) and the fact that we can take the x_j's in (2.9.2) or (2.9.3), as appropriate, with $N(x_j)$ bounded. If we use the second argument mentioned in the previous paragraph, then we can take $r \leq 1/2$ here.

If $y \in G(\mathcal{A}) \cap B(e, r)$, $0 < r < 1/N(e)$, then

$$N(y^{-1}) \leq (1 - N(e) \, r)^{-1} \, N(e), \tag{2.9.6}$$

as in Section 1.12. If E is left or right-invariant totally bounded in $G(\mathcal{A})$, then it follows that N is bounded on E^{-1}, by taking $r = 1/2 \, N(e)$ in (2.9.2) or (2.9.3), as appropriate.

If $x \in G(\mathcal{A})$, then

$$x^{-1} \, B(x, t), \; B(x, t) \, x^{-1} \subseteq B(e, t \, N(x^{-1})) \tag{2.9.7}$$

for every $t > 0$. This implies that

$$B(x,t) \subseteq x\,B(e, t\,N(x^{-1})),\ B(e, t\,N(x^{-1}))\,x \tag{2.9.8}$$

for every $t > 0$.

If E is totally bounded in $\mathcal{A}$ with respect to the semimetric associated to N, and if N is bounded on E^{-1}, then E is both left and right-invariant totally bounded in $G(\mathcal{A})$. To see this, let $t > 0$ be given, so that there are finitely many elements $x_1, \ldots, x_n$ of $\mathcal{A}$ such that

$$E \subseteq \bigcup_{j=1}^{n} B(x_j, t). \tag{2.9.9}$$

More precisely, we can take $x_1, \ldots, x_n \in E$, using the characterization of total boundedness in terms of coverings by small sets in the previous section. In particular, $x_1, \ldots, x_n \in G(\mathcal{A})$, so that

$$E \subseteq \bigcup_{j=1}^{n} x_j\,B(e, t\,N(x_j^{-1})),\ \bigcup_{j=1}^{n} B(e, t\,N(x_j^{-1}))\,x_j, \tag{2.9.10}$$

by (2.9.8). One can use this to get that E is left and right-invariant totally bounded in $G(\mathcal{A})$, because $N(x_1^{-1}), \ldots, N(x_n^{-1})$ are bounded.

2.10 Local total boundedness conditions

A topological space is said to be *locally compact* if every point is contained in an open set that is contained in a compact set. Let us say that a set X with a semimetric $d(\cdot,\cdot)$ is *locally totally bounded* if for every $x \in X$ there is an $r > 0$ such that $B(x,r)$ is totally bounded. If X is locally compact with respect to the topology determined by $d(\cdot,\cdot)$, then X is locally totally bounded, because compact subsets of X are totally bounded.

Now let G be a topological group. If the identity element e is contained in an open set that is contained in a compact set, then it is easy to see that G is locally compact, because of continuity of translations. Let us say that G is *locally totally bounded* if there is an open subset U of G that contains e and is either left or right-invariant totally bounded. In this case, $U \cap U^{-1}$ is an open set that contains e and is both left and right-invariant totally bounded. If G is locally compact, then G is locally totally bounded, because compact subsets of G are both left and right-invariant totally bounded.

Let k be a field with an absolute value function $|\cdot|$. We shall refer to a subset E of k as being totally bounded if E is totally bounded with respect to the metric associated to $|\cdot|$, which is the same as saying that E is totally bounded in k as a commutative topological group with respect to addition and the topology determined by this metric. Similarly, k is locally totally bounded as a metric space or a commutative topological group with respect to addition if and only if $B(0,r)$ is totally bounded for some $r > 0$.

Put

$$t\,E = \{t\,x : x \in E\} \tag{2.10.1}$$

for every $t \in k$. If E is compact, then $t\,E$ is compact for every $t \in k$, because multiplication by t is continuous as a mapping from k into itself. If E is totally bounded, then it is easy to see that $t\,E$ is totally bounded for every $t \in k$ too. Observe that

$$t\,B(0,r) = B(0,|t|\,r) \tag{2.10.2}$$

for every $r > 0$ when $t \neq 0$, and

$$t\,\overline{B}(0,r) = \overline{B}(0,|t|\,r) \tag{2.10.3}$$

for every $r \geq 0$.

Suppose for the moment that $|\cdot|$ is not the trivial absolute value function on k, so that there is an $x \in k$ with $x \neq 0$ and $|x| \neq 1$. This implies that there are $y, z \in k$ with $0 < |y| < 1$ and $|z| > 1$, using x and $1/x$. Thus

$$|y^j| = |y|^j \to 0 \quad \text{and} \quad |z^j| = |z|^j \to \infty \tag{2.10.4}$$

as $j \to \infty$. If $\overline{B}(0,r_0)$ is compact for some $r_0 > 0$, then it follows that $\overline{B}(0,r)$ is compact for some arbitrarily large values of r, because of (2.10.3). This implies that all subsets of k that are both closed and bounded are compact. Similarly, if $\overline{B}(0,r_0)$ is totally bounded for some $r_0 > 0$, then $\overline{B}(0,r)$ is totally bounded for some arbitrarily large values of r, which implies that all bounded subsets of k are totally bounded. Of course, if k is complete with respect to the metric associated to $|\cdot|$, then subsets of k that are both closed and totally bounded are compact.

2.11 Residue fields

Let k be a field with an ultrametric absolute value function $|\cdot|$. In this case, $B(0,r)$ is a subgroup of k as a commutative group with respect to addition for every $r > 0$, and $\overline{B}(0,r)$ is a subgroup for every $r \geq 0$. In fact, $\overline{B}(0,1)$ is a subring of k, and $B(0,r)$, $\overline{B}(0,r)$ are ideals in $\overline{B}(0,1)$ when $r \leq 1$. Thus the quotients

$$\overline{B}(0,1)/\overline{B}(0,r) \tag{2.11.1}$$

and

$$\overline{B}(0,1)/B(0,r) \tag{2.11.2}$$

are defined as commutative rings when $0 < r \leq 1$. It is easy to see that $\overline{B}(0,1)$ is totally bounded if and only if (2.11.1) has only finitely many elements for every $0 < r \leq 1$, which happens if and only if (2.11.2) has only finitely many elements for every $0 < r \leq 1$.

It is well known that the quotient

$$\overline{B}(0,1)/B(0,1) \tag{2.11.3}$$

is a field, which is known as the *residue field* associated to $|\cdot|$ on k. More precisely, nonzero elements of (2.11.3) come from $x \in \overline{B}(0,1)$ with $|x| = 1$, which means that $1/x \in \overline{B}(0,1)$ too. If $|\cdot|$ is the trivial absolute value function on k, then the residue field reduces to k.

Put

$$\rho_1 = \sup\{|x| : x \in k,\ |x| < 1\}, \tag{2.11.4}$$

and note that $0 \le \rho_1 \le 1$. One can verify that $\rho_1 = 0$ if and only if $|\cdot|$ is the trivial absolute value function on k, and that $\rho_1 < 1$ if and only if $|\cdot|$ is discrete on k, in which case the supremum is attained. If $0 < \rho_1 < 1$, then one can check that the positive values of $|\cdot|$ on k are the same as the integer powers of ρ_1. If $B(0,1)$ is totally bounded in k, then one can verify that $\rho_1 < 1$, so that $|\cdot|$ is discrete on k.

Suppose for the moment that $|\cdot|$ is nontrivial and discrete on k, so that $0 < \rho_1 < 1$. This means that $B(0,1) = \overline{B}(0,\rho_1)$, so that the residue field is the same as (2.11.1), with $r = \rho_1$. If the residue field has only finitely many elements, then $\overline{B}(0,1)$ is the union of finitely many closed balls of radius ρ_1. If $j \in \mathbf{Z}$, then it follows that any closed ball in k of radius ρ_1^j can be expressed as the union of the same number of closed balls of radius ρ_1^{j+1}. If $l \in \mathbf{Z}_+$, then it follows that any closed ball of radius ρ_1^j can be expressed as the union of finitely many closed balls of radius ρ_1^{j+l}, so that $\overline{B}(0,1)$ is totally bounded.

Let a be a positive real number, so that $|\cdot|^a$ also defines an ultrametric absolute value function on k. Of course, open and closed balls in k of radius r with respect to the metric associated to $|\cdot|$ are the same as open and closed balls of radius r^a with respect to the metric associated to $|\cdot|^a$. In particular, the open and closed unit balls in k with respect to these metrics are the same. This implies that the residue field associated to $|\cdot|^a$ is the same as the one associated to $|\cdot|$.

2.12 p-Adic integers

Let k be a field with an absolute value function $|\cdot|$. If $x \in k$ and n is a nonnegative integer, then

$$(1-x)\sum_{j=0}^{n} x^j = 1 - x^{n+1}, \tag{2.12.1}$$

so that

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x} \tag{2.12.2}$$

when $x \neq 1$. This implies that

$$\sum_{j=0}^{n} x^j \to \frac{1}{1-x} \tag{2.12.3}$$

as $n \to \infty$ when $|x| < 1$, with respect to the metric associated to $|\cdot|$.

Let p be a prime number, and let $|\cdot|_p$ be the p-adic absolute value on $\mathbf{Q}_p$. If $y \in \mathbf{Z}$ and $x = p\,y$, then $|x|_p = |y|_p/p < 1$, and the remarks in the preceding paragraph imply that $1/(1-x)$ can be approximated by integers with respect to the p-adic metric. Suppose now that $w \in \mathbf{Q}$ satisfies $|w|_p \le 1$, so that $w = a/b$ for some $a, b \in \mathbf{Z}$ with $b \ne 0$ and b not an integer multiple of p. This implies that there are $c, y \in \mathbf{Z}$ such that $b\,c = 1 - p\,y$, so that

$$w = a\,c/b\,c = a\,c\,(1 - p\,y)^{-1}. \tag{2.12.4}$$

It follows that w can be approximated by integers with respect to the p-adic metric, as before.

Note that

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \le 1\} \tag{2.12.5}$$

is a subring of $\mathbf{Q}_p$, as in the previous section. The elements of $\mathbf{Z}_p$ are called *p-adic integers*. Of course, $\mathbf{Z} \subseteq \mathbf{Z}_p$, and $\mathbf{Z}_p$ is a closed set in $\mathbf{Q}_p$. Let $z \in \mathbf{Z}_p$ be given, and let us check that z can be approximated by integers with respect to the p-adic metric. Remember that $\mathbf{Q}$ is dense in $\mathbf{Q}_p$, by construction. If $w \in \mathbf{Q}$ and $|z - w|_p \le 1$, then $|w|_p \le 1$, by the ultrametric version of the triangle inequality. This implies that w can be approximated by integers with respect to the p-adic metric, as before. It follows that z can be approximated by integers with respect to the p-adic metric, by first approximating z by $w \in \mathbf{Q} \cap \mathbf{Z}_p$. This means that $\mathbf{Z}_p$ is the same as the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$.

If $j \in \mathbf{Z}$, then $p^j\,\mathbf{Z}_p \subseteq \mathbf{Q}_p$ can be defined as in (2.10.1). This is the same as the closed ball in $\mathbf{Q}_p$ centered at 0 with radius p^{-j}. Note that if $x \in \mathbf{Q}_p$ and $x \ne 0$, then $|x|_p$ is an integer power of p. This can be verified using the fact that $\mathbf{Q}$ is dense in $\mathbf{Q}_p$.

If $j \in \mathbf{Z}_+$, then $p^j\,\mathbf{Z}_p$ is an ideal in $\mathbf{Z}_p$, as in the previous section, and thus the quotient

$$\mathbf{Z}_p/p^j\,\mathbf{Z}_p \tag{2.12.6}$$

is defined as a commutative ring. The composition of the natural inclusion of $\mathbf{Z}$ in $\mathbf{Z}_p$ with the quotient mapping from $\mathbf{Z}_p$ onto (2.12.6) defines a ring homomorphism from $\mathbf{Z}$ into (2.12.6). One can check that this homomorphism is surjective, because $\mathbf{Z}$ is dense in $\mathbf{Z}_p$. The kernel of this homomorphism is

$$\mathbf{Z} \cap (p^j\,\mathbf{Z}_p) = p^j\,\mathbf{Z}. \tag{2.12.7}$$

This leads to a ring isomorphism from $\mathbf{Z}/p^j\,\mathbf{Z}$ onto (2.12.6).

2.13 Total boundedness and products

Let G be a topological group, and let $U_1, \ldots, U_n$ be finitely many open subsets of G that contain the identity element e. Put $U = \bigcap_{j=1}^n U_j$, which is also an open set that contains e. If A_j is a subset of G that is left-invariant U_j-small for each $j = 1, \ldots, n$, then it is easy to see that

$$\bigcap_{j=1}^{n} A_j \tag{2.13.1}$$

is left-invariant U-small. Similarly, if A_j is right-invariant U_j-small for each $j = 1, \ldots, n$, then (2.13.1) is right-invariant U-small.

Let I be a nonempty set, and let G_j be a topological group for each $j \in I$. Thus $G = \prod_{j \in I} G_j$ is a topological group with respect to the product topology, where the group operations are defined coordinatewise. Let E_j be a subset of G_j for each $j \in I$, and put $E = \prod_{j \in I} E_j$. Suppose that E_j is left-invariant totally bounded for each j, and let us check that E is left-invariant totally bounded as well.

If U is an open subset of G that contains the identity element, then we would like to show that E can be covered by finitely many left-invariant U-small sets. It suffices to consider open sets U in a local base for the product topology on G at the identity element.

If $l \in I$, then let π_l be the standard coordinate projection from G onto G_l. Let $l_1, \ldots, l_n$ be finitely many elements of I, and let U_{l_r} be an open subset of G_{l_r} that contains the identity element for each $r = 1, \ldots, n$. Thus $\pi_{l_r}^{-1}(U_{l_r})$ is an open subset of G that contains the identity element for each $r = 1, \ldots, n$. Consider

$$U = \bigcap_{r=1}^{n} \pi_{l_r}^{-1}(U_{l_r}), \tag{2.13.2}$$

which is also an open subset of the identity element. Note that open subsets of G of this type form a local base for the product topology at the identity element.

Of course, E_{l_r} can be covered by finitely many left-invariant U_{l_r}-small subsets of G_{l_r} for each $r = 1, \ldots, n$, because E_{l_r} is left-invariant totally bounded, by hypothesis. If A_{l_r} is a left-invariant U_{l_r}-small subset of G_{l_r}, then it is easy to see that $\pi_{l_r}^{-1}(A_{l_r})$ is $\pi_{l_r}^{-1}(U_{l_r})$-small in G. If this happens for each $r = 1, \ldots, n$, then it follows that

$$\bigcap_{r=1}^{n} \pi_{l_r}^{-1}(A_{l_r}) \tag{2.13.3}$$

is left-invariant U-small in G, as before. One can verify that E can be covered by finitely many sets of this type, using the analogous coverings of E_{l_r} for each $r = 1, \ldots, n$. This implies that E is left-invariant totally bounded, because open sets as in (2.13.2) form a local base for the product topology at the identity element.

Similarly, if E_j is right-invariant totally bounded in G_j for every $j \in I$, then E is right-invariant totally bounded in G.

If E is any left or right-invariant totally bounded subset of G, then $\pi_l(E)$ has the same property in G_l for every $l \in I$, because π_l is a continuous group homomorphism from G onto G_l. Of course,

$$E \subseteq \prod_{l \in I} \pi_l(E). \tag{2.13.4}$$

2.14 Profinite groups

A topological space X is said to be *zero dimensional* at a point $x \in X$ if there is a local base for the topology of X consisting of open sets that are also closed. If this holds at every point in X, then X is said to be zero dimensional. Equivalently, this means that there is a base for the topology of X consisting of open sets that are closed as well.

As usual, X is said to be *totally disconnected* if the only connected subsets of X have at most one element. If X is zero dimensional and Hausdorff, then it is easy to see that X is totally disconnected. If X is locally compact, Hausdorff, and totally disconnected, then it is well known that X is zero dimensional. In this case, it follows that there is a local base for the topology of X at each point consisting of compact open sets.

Let G be a topological group, and suppose that $\{e\}$ is a closed set, so that G is Hausdorff as a topological space. If G is locally compact and totally disconnected, then there is a local base for the topology of G at e consisting of compact open sets, as in the preceding paragraph. This implies that there is a local base for the topology of G at e consisting of compact open subgroups, as in Section 2.5.

Note that $\mathbf{Q}$ is a commutative topological group with respect to addition and the topology induced by the standard topology on $\mathbf{R}$. It is easy to see that $\mathbf{Q}$ is zero dimensional as a topological space. However, one can check that $\mathbf{Q}$ is the only open subgroup of itself.

A compact topological group G is said to be *profinite* if $\{e\}$ is a closed set, and the open subgroups of G form a local base for the topology at e. Note that open subgroups of G have finite index, because G is compact. This implies that open subgroups have only finitely many conjugates, whose intersection is an open normal subgroup. It follows that the open normal subgroups form a local base for the topology at e.

Of course, finite groups may be considered as profinite groups, with respect to the discrete topology. One can check that the Cartesian product of any nonempty family of profinite groups is profinite, with respect to the product topology. In particular, the product of a nonempty family of finite groups is profinite. One can also verify that closed subgroups of profinite groups are profinite, with respect to the induced topology.

Suppose now that G is a totally bounded topological group, and that the open subgroups of G form a local base for the topology at e. As before, open subgroups of G have finite index, and thus only finitely many conjugates. The intersections of their conjugates are open normal subgroups, so that the open normal subgroups form a local base for the topology at e.

Let $\mathcal{B}$ be a local sub-base for the topology of G at e consisting of open normal subgroups. If $A \in \mathcal{B}$, then A has finite index in G, as before, so that G/A is a finite group. Let us consider G/A as a topological group with respect to the discrete topology. It is easy to see that the quotient mapping q_A from G onto G/A is continuous, because A is an open set.

Put

$$K = \prod_{A \in \mathcal{B}} (G/A), \tag{2.14.1}$$

which is a compact Hausdorff topological group with respect to the product topology, and where the group operations are defined coordinatewise, as usual. If $A \in \mathcal{B}$, then let π_A be the coordinate mapping from K onto G/A, which is a continuous group homomorphism. There is a natural group homomorphism q from G into K such that

$$\pi_A \circ q = q_A \tag{2.14.2}$$

for every $A \in \mathcal{B}$. Note that q is continuous, because (2.14.2) is continuous for every $A \in \mathcal{B}$.

Suppose that $\{e\}$ is a closed set, so that G is Hausdorff, and

$$\bigcap_{A \in \mathcal{B}} A = \{e\}, \tag{2.14.3}$$

because $\mathcal{B}$ is a local sub-base for the topology of G at e. This means that the kernel of q is trivial, so that q is injective. One can check that q is a homeomorphism from G onto $q(G)$, with respect to the topology induced on $q(G)$ by the product topology on K.

The closure $\overline{q(G)}$ of $q(G)$ in K is a profinite group, as before. Of course, if G is compact, then $q(G)$ is compact, and thus closed, because K is Hausdorff.

2.15 Invertible matrices

Let $\mathcal{A}$, $\mathcal{B}$ be rings with multiplicative identity elements $e_{\mathcal{A}}$, $e_{\mathcal{B}}$, respectively, and let ϕ be a ring homomorphism from $\mathcal{A}$ into $\mathcal{B}$ such that

$$\phi(e_{\mathcal{A}}) = e_{\mathcal{B}}. \tag{2.15.1}$$

Under these conditions, the restriction of ϕ to the group $G(\mathcal{A})$ of invertible elements of $\mathcal{A}$ is a group homomorphism into $G(\mathcal{B})$. Of course, if $G(\mathcal{B})$ has only finitely many elements, then the kernel of ϕ in $G(\mathcal{A})$ has finite index.

Let n be a positive integer, and let $M_n(\mathcal{A})$, $M_n(\mathcal{B})$ be the rings of $n \times n$ matrices with entries in $\mathcal{A}$, $\mathcal{B}$, respectively, with respect to matrix multiplication. Using ϕ, we get a mapping ϕ_n from $M_n(\mathcal{A})$ into $M_n(\mathcal{B})$, defined by evaluating ϕ at the entries of an element of $M_n(\mathcal{A})$. It is easy to see that ϕ_n is a ring homomorphism from $M_n(\mathcal{A})$ into $M_n(\mathcal{B})$, which sends the identity matrix in $M_n(\mathcal{A})$ to the identity matrix in $M_n(\mathcal{B})$. Thus the restriction of ϕ_n to $GL_n(\mathcal{A}) = G(M_n(\mathcal{A}))$ is a group homomorphism into $GL_n(\mathcal{B})$. If $\mathcal{B}$ has only finitely many elements, then $M_n(\mathcal{B})$ and thus $GL_n(\mathcal{B})$ have only finitely many elements, so that the kernel of ϕ_n in $GL_n(\mathcal{A})$ has finite index.

If $\mathcal{A}$ and $\mathcal{B}$ are commutative rings, then the determinant can be defined on $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ in the usual way. In this case,

$$\det \phi_n(a) = \phi(\det a) \tag{2.15.2}$$

for every $a \in M_n(\mathcal{A})$.

Now let k be a field with an ultrametric absolute value function $|\cdot|$. Observe that $M_n(k)$ is an associative algebra over k with respect to matrix multiplication, and that

$$N(a) = N_n(a) = \max_{1 \le j,l \le n} |a_{j,l}| \tag{2.15.3}$$

is an ultranorm on $M_n(k)$, as a vector space over k, with respect to $|\cdot|$ on k. More precisely, N is submultiplicative on $M_n(k)$, as in Section 1.15, and the norm of the identity matrix is equal to 1. It is easy to see that the determinant is continuous as a mapping from $M_n(k)$ into k, with respect to the metrics associated to $|\cdot|$ and N on k and $M_n(k)$, respectively. In particular,

$$GL_n(k) = \{a \in M_n(k) : \det a \neq 0\} \tag{2.15.4}$$

is an open subset of $M_n(k)$.

Remember that the closed unit ball $\overline{B}(0,1)$ in k with respect to the metric associated to $|\cdot|$ is a subring of k, so that $M_n(\overline{B}(0,1))$ is a subring of $M_n(k)$. Equivalently,

$$M_n(\overline{B}(0,1)) = \{a \in M_n(k) : N(a) \le 1\}, \tag{2.15.5}$$

and

$$GL_n(\overline{B}(0,1)) = \{a \in M_n(\overline{B}(0,1)) : |\det a| = 1\}, \tag{2.15.6}$$

because an element of $\overline{B}(0,1)$ is invertible in $\overline{B}(0,1)$ exactly when its absolute value is equal to 1. If $0 < r \le 1$, then $B(0,r)$ and $\overline{B}(0,r)$ are ideals in $\overline{B}(0,1)$, so that the corresponding quotients are commutative rings too. Thus

$$M_n(\overline{B}(0,1)/B(0,r)) \tag{2.15.7}$$

and

$$M_n(\overline{B}(0,1)/\overline{B}(0,r)) \tag{2.15.8}$$

are rings with respect to matrix multiplication. We also get ring homomorphisms from $M_n(\overline{B}(0,1))$ onto (2.15.7) and (2.15.8), from the corresponding quotient homomorphisms on $\overline{B}(0,1)$.

The restrictions of these ring homomorphisms to $GL_n(\overline{B}(0,1))$ are group homomorphisms from $GL_n(\overline{B}(0,1))$ into

$$GL_n(\overline{B}(0,1)/B(0,r)) \tag{2.15.9}$$

and

$$GL_n(\overline{B}(0,1)/\overline{B}(0,r)), \tag{2.15.10}$$

respectively. Of course, (2.15.8) and (2.15.10) have only one element when $r = 1$. If $\overline{B}(0,1)$ is totally bounded in k with respect to the metric associated to $|\cdot|$, then the quotients of $\overline{B}(0,1)$ by $B(0,r)$ and $\overline{B}(0,r)$ have only finitely many elements, as in Section 2.11. This implies that (2.15.7) and (2.15.8) have only finitely many elements, so that (2.15.9) and (2.15.10) have only finitely many elements. If $\overline{B}(0,1)$ is compact in k, then $M_n(\overline{B}(0,1))$ and $GL_n(\overline{B}(0,1))$ are compact with respect to the metric associated to N.

As usual, (2.15.9) and (2.15.10) consist of elements of (2.15.7) and (2.15.8), respectively, whose determinant is invertible in the appropriate quotient ring. If $x \in \overline{B}(0,1)$ maps to an invertible element of the quotient by $B(0,r)$, then one can check that $|x| = 1$. Similarly, if x maps to an invertible element of the quotient by $\overline{B}(0,r)$ and $r < 1$, then $|x| = 1$. One can use this to check that the group homomorphisms from $GL_n(\overline{B}(0,1))$ into (2.15.9) and (2.15.10) mentioned in the preceding paragraph are surjective. This also uses (2.15.2) and (2.15.6).

If $a \in M_n(\overline{B}(0,1))$ satisfies $N(a - I) < 1$, then

$$|\det a - 1| < 1, \tag{2.15.11}$$

as in Section 1.15. This implies that $|\det a| = 1$, so that $a \in GL_n(\overline{B}(0,1))$. It follows that the kernel of the group homomorphism from $GL_n(\overline{B}(0,1))$ onto (2.15.9) mentioned earlier is equal to

$$\{a \in M_n(\overline{B}(0,1)) : N(a - I) < r\}. \tag{2.15.12}$$

If $r < 1$, then the kernel of the group homomorphism from $GL_n(\overline{B}(0,1))$ onto (2.15.10) mentioned earlier is

$$\{a \in M_n(\overline{B}(0,1)) : N(a - I) \le r\}. \tag{2.15.13}$$

Note that these are open sets with respect to the metric associated to N.

Chapter 3

Some filtrations of groups

3.1 Subgroups and subadditivity

Let G be a group, and let A be a subgroup of G. Also let N_A be a nonnegative real-valued function on A, and let ρ be a positive real number. Consider the nonnegative real-valued function N defined on G by

$$\begin{aligned} N(x) &= N_A(x) \quad \text{when } x \in A \\ &= \rho \qquad\quad \text{when } x \notin A. \end{aligned} \tag{3.1.1}$$

If N_A is subadditive on A and

$$N_A(x) \le 2\,\rho \tag{3.1.2}$$

for every $x \in A$, then one can check that N is subadditive on G. Similarly, if N_A is ultra-subadditive on A and

$$N_A(x) \le \rho \tag{3.1.3}$$

for every $x \in A$, then N is ultra-subadditive on G. Clearly N is symmetric on G when N_A is symmetric on A, and $N(e) = 0$ when $N_A(e) = 0$. If N_A is nondegenerate on A, then N is nondegenerate on G.

Now let $\mathcal{A}$ be a ring with a multiplicative identity element e, and let $N_{\mathcal{A}}$ be a nonnegative real-valued function on $\mathcal{A}$ that is subadditive and symmetric on $\mathcal{A}$ as a commutative group with respect to addition. Suppose that $N_{\mathcal{A}}(0) = 0$, $N_{\mathcal{A}}(e) = 1$, and that $N_{\mathcal{A}}$ is submultiplicative on $\mathcal{A}$. Let $G(\mathcal{A})$ be the group of invertible elements of $\mathcal{A}$, and let $U(\mathcal{A})$ be the subgroup of $G(\mathcal{A})$ consisting of $x \in G(\mathcal{A})$ such that $N_{\mathcal{A}}(x), N_{\mathcal{A}}(x^{-1}) \le 1$, as before. Remember that $N_{\mathcal{A}}(x-y)$ defines a semimetric on $\mathcal{A}$ that is invariant under left and right multiplication by elements of $U(\mathcal{A})$, as in Section 1.12. This means that the restriction of $N_{\mathcal{A}}(x-y)$ to $x, y \in U(\mathcal{A})$ is invariant under left and right translations on $U(\mathcal{A})$, as a group with respect to multiplication, as before.

Put

$$N_{U(\mathcal{A})}(x) = N_{\mathcal{A}}(x - e) \tag{3.1.4}$$

for every $x \in U(\mathcal{A})$. It follows that $N_{U(\mathcal{A})}$ is subadditive on $U(\mathcal{A})$, as a group with respect to multiplication, as in Section 1.7. Of course, it is easy to verify this directly as well. Similarly, if $N_{\mathcal{A}}$ is ultra-subadditive on $\mathcal{A}$, as a commutative group with respect to addition, then $N_{U(\mathcal{A})}$ is ultra-subadditive on $U(\mathcal{A})$, as a group with respect to multiplication. One can check that $N_{U(\mathcal{A})}$ is symmetric on $U(\mathcal{A})$, as a group with respect to multiplication, because $N_{\mathcal{A}}$ is symmetric on $\mathcal{A}$, as a group with respect to addition. By construction,

$$N_{U(\mathcal{A})}(e) = 0. \tag{3.1.5}$$

If $N_{\mathcal{A}}$ is nondegenerate on $\mathcal{A}$, then $N_{U(\mathcal{A})}$ is nondegenerate on $U(\mathcal{A})$. Note that $N_{U(\mathcal{A})}$ is invariant under conjugations on $U(\mathcal{A})$, because $N_{\mathcal{A}}$ is invariant under left and right multiplication by elements of $U(\mathcal{A})$. We also have that

$$N_{U(\mathcal{A})}(x) \le 2 \tag{3.1.6}$$

for every $x \in U(\mathcal{A})$, and

$$N_{U(\mathcal{A})}(x) \le 1 \tag{3.1.7}$$

for every $x \in U(\mathcal{A})$ when $N_{\mathcal{A}}$ is ultra-subadditive on $\mathcal{A}$.

Let ρ be a positive real number again, and consider the nonnegative real-valued function N defined on $G(\mathcal{A})$ by

$$\begin{aligned} N(x) &= N_{U(\mathcal{A})}(x) \quad \text{when } x \in U(\mathcal{A}) \\ &= \rho \qquad\qquad\;\; \text{when } x \notin U(\mathcal{A}). \end{aligned} \tag{3.1.8}$$

If $\rho \ge 1$, then N is subadditive on $G(\mathcal{A})$, as before. Similarly, if $N_{\mathcal{A}}$ is ultra-subadditive on $\mathcal{A}$, and $\rho \ge 1$, then N is ultra-subadditive on $G(\mathcal{A})$. Observe that N is symmetric on $G(\mathcal{A})$, because $N_{U(\mathcal{A})}$ is symmetric on $U(\mathcal{A})$, as in the preceding paragraph. If $N_{\mathcal{A}}$ is nondegenerate on $\mathcal{A}$, then N is nondegenerate on $G(\mathcal{A})$.

3.2 Basic filtration functions

Let G be a group. Let us say that a function μ on G with values in $\mathbf{R} \cup \{+\infty\}$ is a *basic filtration function* on G if it satisfies the following three conditions. First,

$$\mu(e) = +\infty, \tag{3.2.1}$$

where e is the identity element in G. Second, μ should be symmetric on G, so that

$$\mu(x^{-1}) = \mu(x) \tag{3.2.2}$$

for every $x \in G$. Third,

$$\mu(x\,y) \ge \min(\mu(x), \mu(y)) \tag{3.2.3}$$

for every $x, y \in G$. This corresponds to conditions (1) and (3) in Definition 2.1 on p7 of [26]. If we also have that

$$\mu(x) < +\infty \tag{3.2.4}$$

for every $x \in G$ with $x \neq e$, then we say that μ is *nondegenerate* on G.

More precisely, condition (3) on p7 of [26] asks that

$$\mu(x\,y^{-1}) \ge \min(\mu(x), \mu(y)) \tag{3.2.5}$$

for every $x, y \in G$. One can get (3.2.2) from (3.2.5), using (3.2.1). Once one has (3.2.2), (3.2.5) is equivalent to (3.2.3).

Let μ be a basic filtration function on G. If $t \in \mathbf{R}$, then

$$G_t = \{x \in G : \mu(x) \ge t\} \tag{3.2.6}$$

and

$$G_t^+ = \{x \in G : \mu(x) > t\} \tag{3.2.7}$$

are subgroups of G, as on p7 of [26]. More precisely, one can also take $t = +\infty$ in (3.2.6).

As usual, μ is said to be invariant under conjugations on G if

$$\mu(u\,x\,u^{-1}) = \mu(x) \tag{3.2.8}$$

for every $u, x \in G$. Equivalently, this means that

$$\mu(x\,y) = \mu(y\,x) \tag{3.2.9}$$

for every $x, y \in G$. In this case, (3.2.6) and (3.2.7) are normal subgroups of G.

Let A be a subgroup of G, and let μ_A be a basic filtration function on A. Also let τ be a real number, and let μ be the function defined on G with values in $\mathbf{R} \cup \{+\infty\}$ by

$$\begin{aligned} \mu(x) &= \mu_A(x) && \text{when } x \in A \\ &= \tau && \text{when } x \notin A. \end{aligned} \tag{3.2.10}$$

If

$$\mu_A(x) \ge \tau \tag{3.2.11}$$

for every $x \in A$, then one can check that μ is a basic filtration function on G. If μ_A is nondegenerate on A, then μ is nondegenerate on G.

3.3 Connections with ultra-subadditivity

Let G be a group, and suppose that μ is a basic filtration function on G. Also let r be a positive real number strictly less than 1, and put

$$N_r(x) = N_{\mu,r}(x) = r^{\mu(x)} \tag{3.3.1}$$

for every $x \in G$, which is interpreted as being equal to 0 when $\mu(x) = +\infty$. It is easy to see that this defines a symmetric ultra-subadditive function on G, with

$$N_r(e) = 0. \tag{3.3.2}$$

If μ is nondegenerate on G, then N_r is nondegenerate too, in the sense that $N_r(x) > 0$ for every $x \in G$ with $x \neq e$.

Conversely, let N be a nonnegative real-valued function on G that is symmetric, ultra-subadditive, and satisfies $N(e) = 0$. If $0 < r < 1$, then there is a unique function $\mu_r = \mu_{N,r}$ on G with values in $\mathbf{R} \cup \{+\infty\}$ such that

$$r^{\mu_r(x)} = N(x) \tag{3.3.3}$$

for every $x \in G$, which means that $\mu_r(x) = +\infty$ when $N(x) = 0$. One can check that μ_r is a basic filtration function on G, which is nondegenerate when N is nondegenerate.

Let a be a positive real number, and let $0 < r < 1$ be given again, so that $0 < r^a < 1$. If μ is a basic filtration function on G, then $a\,\mu$ is a basic filtration function too, and

$$N_{a\,\mu,r} = N_{\mu,r^a} = N_{\mu,r}^a. \tag{3.3.4}$$

Similarly, if N is as in the preceding paragraph, then N^a has the same properties, and

$$\mu_{N^a,r} = \mu_{N,r^a} = a\,\mu_{N,r}. \tag{3.3.5}$$

Let N_r be as in (3.3.1) for some $0 < r < 1$. If $t \in \mathbf{R}$, then (3.2.6) and (3.2.7) are the same as

$$\{x \in G : N_r(x) \le r^t\} \tag{3.3.6}$$

and

$$\{x \in G : N_r(x) < r^t\}, \tag{3.3.7}$$

respectively. If $t = +\infty$, then (3.2.6) corresponds to (3.3.7), with r^t interpreted as being 0. Note that (3.3.7) and (3.3.6) are the same as the open and closed balls of radius r^t, respectively, determined by the left and right-invariant semimetrics on G associated to N_r as in Section 1.10.

Of course, N_r is invariant under conjugations exactly when μ is invariant under conjugations. Remember that this happens exactly when the left and right-invariant semimetrics associated to N_r are the same.

3.4 Regular filtration functions

Let G be a group. If $x, y \in G$, then put

$$x^y = y^{-1}\,x\,y \tag{3.4.1}$$

and

$$(x, y) = x^{-1}\,y^{-1}\,x\,y, \tag{3.4.2}$$

which is the *commutator* of x and y in G. Thus $x \mapsto x^y$ is an automorphism of G, and

$$(x^y)^z = x^{y\,z} \tag{3.4.3}$$

for every $x, y, z \in G$, as on p6 of [26]. Note that

$$(x, y)^{-1} = (y, x) \tag{3.4.4}$$

for every $x, y \in G$.

Let μ be a basic filtration function on G. Let us say that μ is *regular* on G if it satisfies the following two additional conditions. First,

$$\mu(x) > 0 \tag{3.4.5}$$

for every $x \in G$. Second,

$$\mu((x, y)) \ge \mu(x) + \mu(y) \tag{3.4.6}$$

for every $x, y \in G$. These conditions correspond to (2) and (4) in Definition 2.1 on p7 of [26].

If μ is any basic filtration function on G, then

$$\mu((x, y)) \ge \min(\mu(x), \mu(y)) \tag{3.4.7}$$

for every $x, y \in G$. Consider the condition that

$$\mu((x, y)) \ge \max(\mu(x), \mu(y)) \tag{3.4.8}$$

for every $x, y \in G$. If μ satisfies (3.4.6), and if $\mu(x) \ge 0$ for every $x \in G$, then (3.4.8) holds. Note that (3.4.6) and (3.4.8) hold automatically when G is commutative, by (3.2.1).

Suppose that μ satisfies (3.4.8), and let $x, y \in G$ be given. Thus

$$\mu(x^{-1}\, x^y) = \mu((x, y)) \ge \mu(x). \tag{3.4.9}$$

This implies that

$$\mu(x^y) \ge \min(\mu(x), \mu(x^{-1}\, x^y)) = \mu(x). \tag{3.4.10}$$

Similarly,

$$\mu(x) = \mu((x^y)^{y^{-1}}) \ge \mu(x^y). \tag{3.4.11}$$

It follows that $\mu(x^y) = \mu(x)$, which is to say that μ is invariant under conjugations.

Conversely, if μ is invariant under conjugations, then

$$\mu((x, y)) = \mu(x^{-1}\, x^y) \ge \min(\mu(x^{-1}), \mu(x^y)) = \mu(x). \tag{3.4.12}$$

Similarly,

$$\mu((x, y)) = \mu((y, x)) \ge \mu(y). \tag{3.4.13}$$

Thus (3.4.8) is equivalent to μ being invariant under conjugations.

3.5 Regularity and ultra-subadditivity

Let G be a group, and let μ be a function defined on G with values in $\mathbf{R} \cup \{+\infty\}$. If $0 < r < 1$, then $N(x) = r^{\mu(x)}$ defines a nonnegative real-valued function on G, with $N(x) = 0$ when $\mu(x) = +\infty$. Conversely, if N is a nonnegative real-valued function on G, then we get a function μ on G with values in $\mathbf{R} \cup \{+\infty\}$ in this way. Remember that basic filtration functions μ correspond exactly to functions N that are symmetric, ultra-subadditive, and satisfy $N(e) = 0$, as in Section 3.3. Under these conditions, μ is regular if and only if

(3.5.1) $$N(x) < 1$$

for every $x \in G$, and

(3.5.2) $$N((x, y)) \le N(x)\, N(y)$$

for every $x, y \in G$.

If N is symmetric and ultra-subadditive on G, then

(3.5.3) $$N((x, y)) \le \max(N(x), N(y))$$

for every $x, y \in G$, which corresponds to (3.4.7). The analogue of (3.4.8) is that

(3.5.4) $$N((x, y)) \le \min(N(x), N(y))$$

for every $x, y \in G$. In particular, this holds when N satisfies (3.5.2), and $N(x) \le 1$ for every $x \in G$, as before. If G is commutative and $N(e) = 0$, then (3.5.2) and (3.5.4) hold automatically. We also have that (3.5.4) holds if and only if N is invariant under conjugations, as in the previous section.

Let $\mathcal{A}$ be a ring with a multiplicative identity element e, and let $N_{\mathcal{A}}$ be a nonnegative real-valued function on $\mathcal{A}$ that is ultra-subadditive and symmetric on $\mathcal{A}$, as a commutative group with respect to addition. Suppose that $N_{\mathcal{A}}(0) = 0$, $N_{\mathcal{A}}(e) = 1$, and that $N_{\mathcal{A}}$ is submultiplicative on $\mathcal{A}$. Let $G(\mathcal{A})$ be the group of invertible elements in $\mathcal{A}$, and let $U(\mathcal{A})$ be the subgroup of $G(\mathcal{A})$ consisting of $x \in G(\mathcal{A})$ with $N_{\mathcal{A}}(x), N_{\mathcal{A}}(x^{-1}) \le 1$, as in Section 1.12. If $x \in U(\mathcal{A})$, then put

(3.5.5) $$N_{U(\mathcal{A})}(x) = N_{\mathcal{A}}(x - e),$$

as in Section 3.1. Remember that $N_{U(\mathcal{A})}$ is ultra-subadditive, symmetric, and invariant under conjugations on $U(\mathcal{A})$, as a group with respect to multiplication.

Let ρ be a real number with $\rho \ge 1$, and let N be the nonnegative real-valued function defined on $G(\mathcal{A})$ by

(3.5.6) $$\begin{aligned} N(x) &= N_{U(\mathcal{A})}(x) \quad \text{when } x \in U(\mathcal{A}) \\ &= \rho \quad\quad\quad\quad \text{when } x \notin U(\mathcal{A}), \end{aligned}$$

as before. Remember that N is ultra-subadditive and symmetric on $G(\mathcal{A})$, and that $N(e) = N_{U(\mathcal{A})}(e) = 0$. If $0 < r < 1$, then there is a unique function μ on $G(\mathcal{A})$ with values in $\mathbf{R} \cup \{+\infty\}$ such that

(3.5.7) $$r^{\mu(x)} = N(x)$$

for every $x \in G(\mathcal{A})$, as usual. This defines a basic filtration function on $G(\mathcal{A})$, as in Section 3.3. If $N_{\mathcal{A}}$ is nondegenerate on $\mathcal{A}$ as a commutative group with respect to addition, then $N_{U(\mathcal{A})}$ is nondegenerate on $U(\mathcal{A})$ as a group with respect to multiplication, which implies that N and μ are nondegenerate on $G(\mathcal{A})$.

Of course, the restriction of μ to $U(\mathcal{A})$ is a basic filtration function on $U(\mathcal{A})$. The restriction of μ to $U(\mathcal{A})$ is also invariant under conjugations on $U(\mathcal{A})$, because $N_{U(\mathcal{A})}$ is invariant under conjugations on $U(\mathcal{A})$. If $x, y \in U(\mathcal{A})$, then $(x, y) \in U(\mathcal{A})$, and

$$N_{U(\mathcal{A})}((x,y)) = N_{\mathcal{A}}(x^{-1}\,y^{-1}\,x\,y - e) = N_{\mathcal{A}}(x\,y - y\,x), \tag{3.5.8}$$

because $N_{\mathcal{A}}$ is invariant under multiplication by elements of $U(\mathcal{A})$, as in Section 1.12. Observe that

$$(x - e)\,(y - e) - (y - e)\,(x - e) = x\,y - y\,x \tag{3.5.9}$$

for every $x, y \in \mathcal{A}$. It follows that

$$N_{\mathcal{A}}(x\,y - y\,x) \le N_{\mathcal{A}}(x - e)\,N_{\mathcal{A}}(y - e) \tag{3.5.10}$$

for every $x, y \in \mathcal{A}$, by ultra-subadditivity with respect to addition and submultiplicativity of $N_{\mathcal{A}}$ on $\mathcal{A}$. This means that

$$N_{U(\mathcal{A})}((x,y)) \le N_{U(\mathcal{A})}(x)\,N_{U(\mathcal{A})}(y) \tag{3.5.11}$$

for every $x, y \in U(\mathcal{A})$. This implies the analogous condition (3.4.6) for μ on $U(\mathcal{A})$, as before.

Consider

$$U_0(\mathcal{A}) = \{x \in G(\mathcal{A}) : N_{\mathcal{A}}(x - e) < 1\}. \tag{3.5.12}$$

If $x \in U_0(\mathcal{A})$, then $N_{\mathcal{A}}(x) \le 1$ by ultra-subadditivity of $N_{\mathcal{A}}$ with respect to addition on $\mathcal{A}$. We also have that $N_{\mathcal{A}}(x^{-1}) \le 1$, as in Section 1.12. This implies that $U_0(\mathcal{A})$ is contained in $U(\mathcal{A})$, so that

$$U_0(\mathcal{A}) = \{x \in U(\mathcal{A}) : N_{U(\mathcal{A})}(x) < 1\}. \tag{3.5.13}$$

This is a subgroup of $U(\mathcal{A})$, because $N_{U(\mathcal{A})}$ is ultra-subadditive and symmetric on $U(\mathcal{A})$. The restriction of μ to $U_0(\mathcal{A})$ is a regular filtration function, because of (3.5.11). This is related to Theorem 4.1 on p9 of [26].

Remember that

$$\mathcal{A}_1 = \{x \in \mathcal{A} : N_{\mathcal{A}}(x) \le 1\} \tag{3.5.14}$$

is a subring of $\mathcal{A}$, as in Section 1.15. It is easy to see that

$$\mathcal{A}_0 = \{x \in \mathcal{A} : N_{\mathcal{A}}(x) < 1\} \tag{3.5.15}$$

is a two-sided ideal in $\mathcal{A}_1$, so that the quotient $\mathcal{A}_1/\mathcal{A}_0$ is a ring too. Note that $e \in \mathcal{A}_1$, and that the image of e in $\mathcal{A}_1/\mathcal{A}_0$ under the natural quotient mapping is the multiplicative identity element in $\mathcal{A}_1/\mathcal{A}_0$. Remember that $U(\mathcal{A})$ is the same as the group $G(\mathcal{A}_1)$ of invertible elements in $\mathcal{A}_1$, as in Section 1.15. Let $G(\mathcal{A}_1/\mathcal{A}_0)$ be the group of invertible elements of $\mathcal{A}_1/\mathcal{A}_0$, as usual. The restriction of the natural quotient mapping from $\mathcal{A}_1$ onto $\mathcal{A}_1/\mathcal{A}_0$ to $G(\mathcal{A}_1)$ is a group homomorphism from $G(\mathcal{A}_1)$ into $G(\mathcal{A}_1/\mathcal{A}_0)$. It is easy to see that $U_0(\mathcal{A})$ is the same as the kernel of that group homomorphism.

3.6 Regularity and matrices

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let $0 < r < 1$ be given. Thus there is a function ν on k with values in $\mathbf{R} \cup \{+\infty\}$ such that

$$r^{\nu(x)} = |x| \tag{3.6.1}$$

for every $x \in k$, with $\nu(0) = +\infty$. This may be considered as a basic filtration function on k as a commutative group with respect to addition, and we also have that

$$\nu(x\,y) = \nu(x) + \nu(y) \tag{3.6.2}$$

for every $x, y \in k$. Remember that the closed unit ball $\overline{B}(0,1)$ in k is a subring of k, and that $B(0,t)$ and $\overline{B}(0,t)$ are ideals in $\overline{B}(0,1)$ when $0 < t \le 1$.

Let n be a positive integer, and remember that the space $M_n(k)$ of $n \times n$ matrices with entries in k is an associative algebra over k with respect to matrix multiplication. If $a = (a_{j,l}) \in M_n(k)$, then put

$$\|a\| = \max_{1 \le j,l \le n} |a_{j,l}|, \tag{3.6.3}$$

which defines a submultiplicative ultranorm on $M_n(k)$ with respect to $|\cdot|$ on k, as in Section 2.15. Thus

$$M_n(\overline{B}(0,1)) = \{a \in M_n(k) : \|a\| \le 1\}, \tag{3.6.4}$$

which is a subring of $M_n(k)$, and the group $GL_n(\overline{B}(0,1))$ of invertible elements in $M_n(\overline{B}(0,1))$ consists of $a \in M_n(\overline{B}(0,1))$ such that $|\det a| = 1$, as before. If we take $\mathcal{A} = M_n(k)$ and $N_{\mathcal{A}}$ to be (3.6.3), then $G(\mathcal{A}) = GL_n(k)$, and $U(\mathcal{A}) \subseteq G(\mathcal{A})$ is the same as $GL_n(\overline{B}(0,1))$. In particular,

$$\|a - I\| \tag{3.6.5}$$

is the same as (3.5.5) here, and is ultra-subadditive, symmetric, nondegenerate, and invariant under conjugations on $GL_n(\overline{B}(0,1))$, as a group with respect to matrix multiplication.

If $a \in M_n(k)$, then put

$$\nu_n(a) = \min_{1 \le j,l \le n} \nu(a_{j,l}), \tag{3.6.6}$$

which defines ν_n as a function on $M_n(k)$ with values in $\mathbf{R} \cup \{+\infty\}$. This may be considered as a basic filtration function on $M_n(k)$, as a commutative group with respect to addition, and one can check that

$$\nu_n(a\,b) \ge \nu_n(a) + \nu_n(b) \tag{3.6.7}$$

for every $a, b \in M_n(k)$. Equivalently,

$$r^{\nu_n(a)} = \|a\| \tag{3.6.8}$$

for every $a \in M_n(k)$. If $a \in GL_n(\overline{B}(0,1))$, then put

$$\mu(a) = \nu_n(a - I), \tag{3.6.9}$$

so that

$$r^{\mu(a)} = \|a - I\|. \tag{3.6.10}$$

This defines a nondegerate basic filtration function on $GL_n(\overline{B}(0,1))$ that is invariant under conjugations. If $a, b \in GL_n(\overline{B}(0,1))$, then

$$\|(a,b) - I\| \le \|a - I\|\,\|b - I\|, \tag{3.6.11}$$

as in (3.5.11). Equivalently, this means that

$$\mu((a,b)) \ge \mu(a) + \mu(b) \tag{3.6.12}$$

for every $a, b \in GL_n(\overline{B}(0,1))$.

Consider

$$G = \{a \in M_n(\overline{B}(0,1)) : \|a - I\| < 1\}. \tag{3.6.13}$$

If $a \in G$, then $|\det a - 1| < 1$, so that $|\det a| = 1$, as in Sections 1.15 and 2.15. This means that G is contained in $GL_n(\overline{B}(0,1))$, and in fact it is a normal subgroup. Remember that the quotient mapping from $\overline{B}(0,1)$ onto its quotient by $B(0,1)$ leads to a group homomorphism from $GL_n(\overline{B}(0,1))$ onto

$$GL_n(\overline{B}(0,1)/B(0,1)), \tag{3.6.14}$$

as in Section 2.15. The kernel of this homomorphism is G, as before. Equivalently, G consists of $a \in GL_n(\overline{B}(0,1))$ such that $\mu(a) > 0$. It follows that the restriction of μ to G is a regular filtration function, as in Theorem 4.1 on p9 of [26].

3.7 Some remarks about invertibility

Let $\mathcal{A}$ be a ring with a multiplicative identity element e. Suppose that $x \in \mathcal{A}$ has left and right multiplicative inverses in $\mathcal{A}$, so that there are $a, b \in \mathcal{A}$ such that

$$a\,x = x\,b = e. \tag{3.7.1}$$

Under these conditions,

$$a = a\,(x\,b) = (a\,x)\,b = b, \tag{3.7.2}$$

so that x is invertible in $\mathcal{A}$.

Suppose now that there are $y, z \in \mathcal{A}$ such that $y\,x$ and $x\,z$ are invertible in $\mathcal{A}$. This means that

$$(y\,x)^{-1}\,y\,x = x\,z\,(x\,z)^{-1} = e, \tag{3.7.3}$$

so that x has left and right inverses in $\mathcal{A}$. It follows that x is invertible in $\mathcal{A}$, with

$$x^{-1} = (y\,x)^{-1}\,y = z\,(x\,z)^{-1}, \tag{3.7.4}$$

as in the preceding paragraph.

Let $N_{\mathcal{A}}$ be a nonnegative real-valued function on A that is symmetric and ultra-subadditive on $\mathcal{A}$ as a comutative group with respect to addition, submultiplicative on $\mathcal{A}$, and satisfies $N_{\mathcal{A}}(0) = 0$ and $N_{\mathcal{A}}(e) = 1$. Suppose that if $a \in \mathcal{A}$ satisfies

$$N_{\mathcal{A}}(a - e) < 1, \tag{3.7.5}$$

then a is invertible in $\mathcal{A}$. Remember that this happens when $N_{\mathcal{A}}$ is nondegenerate on $\mathcal{A}$, and $\mathcal{A}$ is complete with respect to the metric associated to $N_{\mathcal{A}}$, as in Section 1.13. This also holds with $\mathcal{A} = M_n(k)$ and $N_{\mathcal{A}} = \|\cdot\|$ as in the previous section.

If $a \in \mathcal{A}$ satisfies (3.7.5), then

$$N_{\mathcal{A}}(a^{-1}) \le 1, \tag{3.7.6}$$

as in Section 1.12. Note that $N_{\mathcal{A}}(a) \le 1$ in this case.

If there are $y, z \in \mathcal{A}$ such that

$$N_{\mathcal{A}}(y\,x - e),\ N_{\mathcal{A}}(x\,z - e) < 1, \tag{3.7.7}$$

then $y\,x$ and $x\,z$ are invertible in $\mathcal{A}$, by hypothesis. This implies that x is invertible in $\mathcal{A}$, as before. We also have that

$$N_{\mathcal{A}}((y\,x)^{-1}),\ N_{\mathcal{A}}((x\,z)^{-1}) \le 1, \tag{3.7.8}$$

as in (3.7.6). It follows that

$$N_{\mathcal{A}}(x^{-1}) \le N_{\mathcal{A}}(y),\ N_{\mathcal{A}}(z), \tag{3.7.9}$$

by (3.7.4).

Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the open and closed unit balls in $\mathcal{A}$ with respect to $N_{\mathcal{A}}$, as in (3.5.14) and (3.5.15). Thus $\mathcal{A}_1$ is a subring of $\mathcal{A}$, and $\mathcal{A}_0$ is a two-sided ideal in $\mathcal{A}_1$, as before, so that $\mathcal{A}_1/\mathcal{A}_0$ is a ring as well. If $x \in \mathcal{A}_1$ is mapped to an invertible element of $\mathcal{A}_1/\mathcal{A}_0$, then there are $y, z \in \mathcal{A}_1$ that satisfy (3.7.7). This implies that x is invertible in $\mathcal{A}$, as before. More precisely,

$$x^{-1} \in \mathcal{A}_1, \tag{3.7.10}$$

by (3.7.9).

If $\mathcal{A} = M_n(k)$ and $N_{\mathcal{A}} = \|\cdot\|$ are as in the previous section, then $\mathcal{A}_1 = M_n(\overline{B}(0,1))$ and $\mathcal{A}_0 = M_n(B(0,1))$. In this case, $\mathcal{A}_1/\mathcal{A}_0$ is isomorphic to $M_n(\overline{B}(0,1)/B(0,1))$ in a natural way, as associative algebras over k.

3.8 Some identities

Let G be a group, with identity element e. If $x, y \in G$, then we put $x^y = y^{-1}\,x\,y$ and $(x, y) = x^{-1}\,y^{-1}\,x\,y$, as in Section 3.4. Remember that $(x^y)^z = x^{y\,z}$ for every $z \in G$, and that $(x, y)^{-1} = (y, x)$. We also have that

$$x\,y = y\,x^y = y\,x\,(x, y) \tag{3.8.1}$$

and

$$x^y = x\,(x, y), \tag{3.8.2}$$

as in (1) on p6 of [26].

Let us check that

$$(x, y\,z) = (x, z)\,(x, y)^z \tag{3.8.3}$$

and

$$(x\,y, z) = (x, z)^y\,(y, z), \tag{3.8.4}$$

as in (2) and (2') on p6 of [26], respectively. To get (3.8.3), observe that

$$\begin{aligned} x\,(x, y\,z) = x^{y\,z} = (x^y)^z &= (x\,(x, y))^z \\ &= x^z\,(x, y)^z = x\,(x, z)\,(x, y)^z. \end{aligned} \tag{3.8.5}$$

Similarly,

$$\begin{aligned} x\,y\,(x\,y, z) = (x\,y)^z &= x^z\,y^z \\ &= (x\,(x, z))\,(y\,(y, z)) = x\,y\,(x, z)^y\,(y, z), \end{aligned} \tag{3.8.6}$$

which implies (3.8.4). One could also obtain (3.8.4) from (3.8.3).

Now let us verify that

$$(x^y, (y, z))\,(y^z, (z, x))\,(z^x, (y, x)) = e, \tag{3.8.7}$$

as in (3) on p6 of [26], and restated at the top of p7. Observe that

$$\begin{aligned} (x^y, (y, z)) &= y^{-1}\,x^{-1}\,y\,(y, z)^{-1}\,y^{-1}\,x\,y\,(y, z) \\ &= y^{-1}\,x^{-1}\,y\,z^{-1}\,y^{-1}\,z\,y\,y^{-1}\,x\,y\,y^{-1}\,z^{-1}\,y\,z \\ &= y^{-1}\,x^{-1}\,y\,z^{-1}\,y^{-1}\,z\,x\,z^{-1}\,y\,z. \end{aligned} \tag{3.8.8}$$

Put

$$u = z\,x\,z^{-1}\,y\,z, \quad v = x\,y\,x^{-1}\,z\,x, \quad w = y\,z\,y^{-1}\,x\,y, \tag{3.8.9}$$

so that

$$(x^y, (y, z)) = w^{-1}\,u, \tag{3.8.10}$$

by (3.8.8). One can check that

$$(y^z, (z, x)) = u^{-1}\,v, \quad (z^x, (y, x)) = v^{-1}\,w, \tag{3.8.11}$$

by permuting x, y, z cyclically. This implies (3.8.7), as desired.

If A, B are subsets of G, then we let $A\,B$ be the set of products $a\,b$, with $a \in A$ and $b \in B$, as usual. If A or B is invariant under conjugations, then it is easy to see that

$$A\,B = B\,A. \tag{3.8.12}$$

If A and B are subgroups of G, at least one of which is normal, then $A\,B$ is a subgroup of G too. If A and B are normal subgroups of G, then $A\,B$ is a normal subgroup as well.

If A and B are subgroups of G, then we let (A, B) be the subgroup of G generated by the commutators (a, b) with $a \in A$ and $b \in B$. If A and B are normal subgroups of G, then it is easy to see that (A, B) is a normal subgroup. If A, B, and C are normal subgroups of G, then one can verify that

$$(A, (B, C)) \subseteq (B, (C, A))\,(C, (A, B)), \tag{3.8.13}$$

using (3.8.7). This corresponds to some remarks on p7 of [26].

3.9 Lie algebras

Let k be a commutative ring with a multiplicative identity element, and let A, B be modules over k. A module homomorphism from A into B is also said to be *linear over* k.

Let A be a commutative group, with the group operations expressed additively. If $a \in A$ and $n \in \mathbf{Z}_+$, then we let $n \cdot a$ be the sum of n a's in A. If we put $0 \cdot a = 0$ and $(-n) \cdot a = -(n \cdot a)$, then A becomes a module over $\mathbf{Z}$. If B is another commutative group, then any group homomorphism from A into B is linear over $\mathbf{Z}$.

Let k be a commutative ring with a multiplicative identity element again, and let A, B, and C be modules over k. A mapping β from $A \times B$ into C is said to be *bilinear over* k if $\beta(a, b)$ is linear over k in each variable.

Let β be a bilinear mapping from $A \times A$ into C. If

$$\beta(a, b) = \beta(b, a) \tag{3.9.1}$$

for every $a, b \in A$, then β is said to be *symmetric* on $A \times A$. If

$$\beta(a, b) = -\beta(b, a) \tag{3.9.2}$$

for every $a, b \in A$, then β is said to be *antisymmetric* on $A \times A$. If

$$\beta(a, a) = 0 \tag{3.9.3}$$

for every $a \in A$, then one can check that β is antisymmetric on $A \times A$, by considering $\beta(a + b, a + b)$ for each $a, b \in A$. If β is antisymmetric on $A \times A$, and $1 + 1$ has a multiplicative inverse in k, then (3.9.3) holds for every $a \in A$.

A module A over k is said to be an *algebra in the strict sense* over k if it is equipped with a mapping from $A \times A$ into A that is bilinear over k. If this bilinear mapping is symmetric, then A is said to be *commutative* as an algebra over k. Similarly, if this bilinear mapping satisfies the associative law, then A is said to be *associative* as an algebra over k.

Let A be a module over k, and let $[a, b]$ be a mapping from $A \times A$ into A that is bilinear over k. Suppose that

(3.9.4) $$[a, a] = 0$$

for every $a \in A$, and that

(3.9.5) $$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for every $a, b, c \in A$, which is known as the *Jacobi identity*. Under these conditions, $(A, [\cdot, \cdot])$ is said to be a *Lie algebra over k*, as in Definition 1 on p2 of [26].

Let A be an associative algebra over k, where multiplication of $a, b \in A$ is expressed as $a\, b$, as usual. One can check that A is a Lie algebra with respect to the commutator bracket $[a, b] = a\, b - b\, a$, as in Example (iii) on p2 of [26].

3.10 The quotients G_t/G_t^+

Let G be a group, and let μ be a regular filtration function on G, as in Section 3.4. If t is a nonnegative real number, then put

(3.10.1) $$G_t = \{x \in G : \mu(x) \geq t\},$$

(3.10.2) $$G_t^+ = \{x \in G : \mu(x) > t\},$$

as on p7 of [26]. These are subgroups of G, as in Section 3.2. More precisely, these are normal subgroups, because μ is invariant under conjugations, as in Section 3.4.

Of course, $G_t^+ \subseteq G_t$ for every $t \geq 0$. Put

(3.10.3) $$\mathrm{gr}_t G = G_t/G_t^+$$

for every $t \geq 0$, as in Definition 2.2 on p7 of [26]. This is a group for each $t \geq 0$, because G_t^+ is normal as a subgroup of G_t in particular. Note that $G_0^+ = G$, by the definition of a regular filtration function, so that (3.10.3) has only one element when $t = 0$.

If $x, y \in G$, then it is easy to see that

(3.10.4) $$\mu((x, y)) > \min(\mu(x), \mu(y)),$$

because μ is a regular filtration function on G. In particular, if $x, y \in G_t$ for some $t \geq 0$, then

(3.10.5) $$(x, y) \in G_t^+.$$

This implies that

(3.10.6) $$\mathrm{gr}_t G \text{ is commutative for every } t \geq 0,$$

as in the first part of Proposition 2.3 on p7 of [26].

More precisely, if $x, y \in G$, then

$$\mu((x,y)) > \max(\mu(x), \mu(y)), \tag{3.10.7}$$

because μ is a regular filtration function on G. This implies that

(3.10.8) (3.10.5) holds when $x \in G_t$ for some $t \geq 0$ and $y \in G$.

If $x \in G_t$ and $y \in G$, then $x^y \in G_t$, because G_t is a normal subgroup of G, as before. In fact,

(3.10.9) x and x^y are mapped to the same element of $\mathrm{gr}_t G$

by the natural quotient mapping, because $x^{-1}\, x^y$ is an element of G_t^+, as in (3.10.8). This is the second part of Proposition 2.3 on p7 of [26].

If r is another nonnegative real number, $x \in G_r$, and $y \in G_t$, then

$$(x,y) \in G_{r+t}, \tag{3.10.10}$$

because μ is a regular filtration function on G. Similarly,

$$(u,y) \in G_{r+t}^+ \tag{3.10.11}$$

when $u \in G_r^+$, and

$$(x,v) \in G_{r+t}^+ \tag{3.10.12}$$

when $v \in G_t^+$. Let $c_{r,t}$ be the mapping from $G_r \times G_t$ into G_{r+t} defined by

$$c_{r,t}(x,y) = (x,y). \tag{3.10.13}$$

This leads to a mapping from $G_r \times G_t$ into $\mathrm{gr}_{r+t}\, G$, by composing $c_{r,t}$ with the natural quotient mapping from G_{r+t} onto $\mathrm{gr}_{r+t}\, G$. Part of the third part of Proposition 2.3 on p8 of [26] is that this induces a mapping from $(\mathrm{gr}_r\, G) \times (\mathrm{gr}_t\, G)$ into $\mathrm{gr}_{r+t}\, G$.

To see this, we use the fact that

$$(x\,u, y) = (x,y)^u\,(u,y), \tag{3.10.14}$$

$$(x, y\,v) = (x,v)\,(x,y)^v, \tag{3.10.15}$$

as in Section 3.8. If $u \in G_r^+$, then it follows that $(x\,u, y)$ and $(x,y)^u$ are mapped to the same element of $\mathrm{gr}_{r+t}\, G$, because of (3.10.11). Note that (x,y) and $(x,y)^u$ are mapped to the same element of $\mathrm{gr}_{r+t}\, G$ too, as before. This means that $(x\,u, y)$ and (x,y) are mapped to the same element of $\mathrm{gr}_{r+t}\, G$. Similarly, if $v \in G_t^+$, then $(x, y\,v)$ and (x,y) are mapped to the same element of $\mathrm{gr}_{r+t}\, G$. This shows that we get a well-defined mapping $\overline{c}_{r,t}$ from $(\mathrm{gr}_r\, G) \times (\mathrm{gr}_t\, G)$ into $\mathrm{gr}_{r+t}\, G$, as in the preceding paragraph. More precisely, the composition of $\overline{c}_{r,t}$ with the natural quotient mappings from G_r, G_t onto $\mathrm{gr}_r\, G$, $\mathrm{gr}_t\, G$, respectively, is the same as the composition of $c_{r,t}$ with the natural quotient mapping from G_{r+t} onto $\mathrm{gr}_{r+t}\, G$.

Another part of the third part of Proposition 2.3 on p8 of [26] is that $\overline{c}_{r,t}$ is bilinear over $\mathbf{Z}$, as a mapping from $(\mathrm{gr}_r\, G) \times (\mathrm{gr}_t\, G)$ into $\mathrm{gr}_{r+t}\, G$. Let $x' \in G_r$ and $y' \in G_t$ be given, and observe that

$$(x\,x', y) = (x, y)^{x'}\,(x', y), \tag{3.10.16}$$

$$(x, y'\,y) = (x, y)\,(x, y')^y, \tag{3.10.17}$$

as in Section 3.8 again. Remember that (x, y) and $(x, y)^{x'}$ are mapped to the same element of $\mathrm{gr}_{r+t}\, G$, and that (x, y') and $(x, y')^y$ are mapped to the same element of $\mathrm{gr}_{r+t}\, G$, as before. This implies that $(x\,x', y)$ and

$$(x, y)\,(x', y) \tag{3.10.18}$$

are mapped to the same element of $\mathrm{gr}_{r+t}\, G$, and similarly that $(x, y'\,y)$ and

$$(x, y)\,(x, y') \tag{3.10.19}$$

are mapped to the same element of $\mathrm{gr}_{r+t}\, G$. It follows that $\overline{c}_{r,t}$ is bilinear over $\mathbf{Z}$, as desired.

3.11 Regularity and Lie algebras

Let us continue with the same notation and hypotheses as in the previous section. Let $\mathrm{gr}\, G$ be the direct sum of $\mathrm{gr}_t\, G$ over $t \geq 0$, as a direct sum of commutative groups. One may wish to use additive notation for the group structure on $\mathrm{gr}\, G$, even if we use multiplicative notation for the group structure on G.

If $r, t \geq 0$, then we may consider $\mathrm{gr}_r\, G$, $\mathrm{gr}_t\, G$, and $\mathrm{gr}_{r+t}\, G$ as subgroups of $\mathrm{gr}\, G$. It is easy to see that there is a unique mapping c from $(\mathrm{gr}\, G) \times (\mathrm{gr}\, G)$ into $\mathrm{gr}\, G$ that is bilinear over $\mathbf{Z}$, and which agrees with $\overline{c}_{r,t}$ on $(\mathrm{gr}_r\, G) \times (\mathrm{gr}_{r+t}\, G)$. The fourth part of Proposition 2.3 on p8 of [26] states that $\mathrm{gr}\, G$ is a Lie algebra over $\mathbf{Z}$ with respect to c.

Let $\xi \in \mathrm{gr}\, G$ be given, so that ξ can be expressed as the sum of finitely many terms of the form $\xi_t \in \mathrm{gr}_t\, G$, $t \geq 0$. We would like to check that $c(\xi, \xi)$ is the identity element in $\mathrm{gr}\, G$. To do this, it suffices to verify that $c(\xi_t, \xi_t)$ is the identity element for each $t \geq 0$ in the sum, and that $c(\xi_r, \xi_t)$ is the inverse of $c(\xi_t, \xi_r)$ for all $r, t \geq 0$ in the sum. This is the same as saying that $\overline{c}_{t,t}(\xi_t, \xi_t)$ is the identity element in $\mathrm{gr}_{2\,t}\, G$ for every $t \geq 0$ in the sum, and that $\overline{c}_{r,t}(\xi_r, \xi_t)$ is the inverse of $\overline{c}_{t,r}(\xi_t, \xi_r)$ in $\mathrm{gr}_{r+t}\, G$ for every $r, t \geq 0$ in the sum.

Let us choose $x_t \in G_t$ so that x_t is mapped to ξ_t by the natural quotient mapping from G_t onto $\mathrm{gr}_t\, G$ for each $t \geq 0$ in the sum. Of course, the commutator (x_t, x_t) is the identity element in G for each $t \geq 0$ in the sum. By construction, (x_t, x_t) is mapped to $\overline{c}_{t,t}(\xi_t, \xi_t)$ by the natural quotient mapping from $G_{2\,t}$ onto $\mathrm{gr}_{2\,t}\, G$ for every $t \geq 0$ in the sum. This means that $\overline{c}_{t,t}(\xi_t, \xi_t)$ is the same as the identity element in $\mathrm{gr}_{2\,t}\, G$ for every $t \geq 0$ in the sum.

Similarly, the commutators (x_r, x_t) and (x_t, x_r) are mapped to $\overline{c}_{r,t}(\xi_r, \xi_t)$ and $\overline{c}_{t,r}(\xi_t, \xi_r)$, respectively, by the natural quotient mapping from G_{r+t} onto

$\mathrm{gr}_{r+t} G$ for every $r, t \geq 0$ in the sum. Of course, (x_r, x_t) is the inverse of (x_t, x_r) in G for every $r, t \geq 0$ in the sum. This implies that $\overline{c}_{r,t}(\xi_r, \xi_t)$ is the inverse of $\overline{c}_{t,r}(\xi_t, \xi_r)$ in $\mathrm{gr}_{r+t} G$ for every $r, t \geq 0$ in the sum.

Now let $\xi, \eta, \zeta \in \mathrm{gr}\, G$ be given. To show that the Jacobi identity holds, we want to check that the combination of

$$c(\xi, c(\eta, \zeta)), \tag{3.11.1}$$

$$c(\eta, c(\zeta, \xi)), \tag{3.11.2}$$

$$c(\zeta, c(\xi, \eta)) \tag{3.11.3}$$

is the identity element in $\mathrm{gr}\, G$. Because of the linearity over $\mathbf{Z}$ in ξ, η, and ζ of these expressions, we can reduce to the case where $\xi \in \mathrm{gr}_{t_1} G$, $\eta \in \mathrm{gr}_{t_2} G$, and $\zeta \in \mathrm{gr}_{t_3} G$ for some nonnegative real numbers t_1, t_2, and t_3. Let us now choose $x \in G_{t_1}$, $y \in G_{t_2}$, and $z \in G_{t_2}$ so that they are mapped to ξ, η, and ζ, respectively, by the natural quotient mappings from G_{t_j} onto $\mathrm{gr}_{t_j} G$ for $j = 1, 2, 3$. Put

$$t = t_1 + t_2 + t_3, \tag{3.11.4}$$

and note that (3.11.1), (3.11.2), and (3.11.3) are elements of $\mathrm{gr}_t G$.

Of course,

$$(x, (y, z)), \tag{3.11.5}$$

$$(y, (z, x)), \tag{3.11.6}$$

$$(z, (x, y)) \tag{3.11.7}$$

are elements of G_t. Observe that (3.11.5), (3.11.6), and (3.11.7) are mapped to (3.11.1), (3.11.2), and (3.11.3), respectively, by the natural quotient mapping from G_t onto $\mathrm{gr}_t G$. Thus it suffices to show that the product of (3.11.5), (3.11.6), and (3.11.7) is mapped to the identity element by the natural quotient mapping from G_t onto $\mathrm{gr}_t G$.

The product of

$$(x^y, (y, z)), \tag{3.11.8}$$

$$(y^z, (z, x)), \tag{3.11.9}$$

$$(z^x, (x, y)) \tag{3.11.10}$$

is the identity element in G, as in Section 3.8. Remember that x, y, and z are mapped to the same elements of $\mathrm{gr}_{t_j} G$ as x^y, y^z, and z^x, respectively, by the natural quotient mapping from G_{t_j} onto $\mathrm{gr}_{t_j} G$ for $j = 1, 2, 3$, as in the previous section. This implies that (3.11.5), (3.11.6), and (3.11.7) are mapped to the same elements of $\mathrm{gr}_t G$ as (3.11.8), (3.11.9), and (3.11.10), respectively, by the natural quotient mapping from G_t onto $\mathrm{gr}_t G$. It follows that the product of (3.11.5), (3.11.6), and (3.11.7) is mapped to the identity element by the natural quotient mapping from G_t onto $\mathrm{gr}_t G$, as desired.

3.12 Integral filtration functions

Let G be a group, and let μ be a basic filtration function on G. Let us say that μ is *integral* on G if μ takes values in $\mathbf{Z} \cup \{+\infty\}$. If $n \in \mathbf{Z}$, then

$$G_n = \{x \in G : \mu(x) \geq n\} \tag{3.12.1}$$

is a subgroup of G, as before. Note that

$$G_{n+1} \subseteq G_n \tag{3.12.2}$$

for every $n \in \mathbf{Z}$, and that

$$\bigcup_{n=-\infty}^{\infty} G_n = G. \tag{3.12.3}$$

If μ is invariant under conjugations on G, then G_n is a normal subgroup of G for every $n \in \mathbf{Z}$, as before.

Conversely, suppose that G_n is a subgroup of G for each $n \in \mathbf{Z}$, and that these subgroups satisfy (3.12.2) and (3.12.3). Put $\mu(x)$ equal to the largest integer n such that $x \in G_n$ when there is such an n, and $\mu(x) = +\infty$ when $x \in G_n$ for every $n \in \mathbf{Z}_+$. One can check that this defines a basic filtration function on G, which corresponds to part of Proposition 3.1 on p8 of [26]. If G_n is also a normal subgroup of G for every $n \in \mathbf{Z}$, then μ is invariant under conjugations on G.

Similarly, a regular filtration function μ on G is said to be *integral* if it takes values in $\mathbf{Z}_+ \cup \{+\infty\}$. In particular,

$$G_1 = G \tag{3.12.4}$$

in this case, which implies (3.12.3). We also get that

$$(G_n, G_m) \subseteq G_{n+m} \tag{3.12.5}$$

for every $n, m \in \mathbf{Z}_+$, where the left side is as defined in Section 3.8.

Conversely, suppose that G_n is a subgroup of G for every $n \in \mathbf{Z}_+$, and that these subgroups satisfy (3.12.2), (3.12.4), and (3.12.5). Let μ be defined on G as before, so that μ now takes values in $\mathbf{Z}_+ \cup \{+\infty\}$. One can verify that μ is a regular filtration function on G, as in Proposition 3.1 on p8 of [26].

As a basic class of examples, let us consider the *descending* or *lower central series*, as on p9 of [26]. Thus we put $G_1 = G$, and define G_n for $n \geq 2$ by

$$G_{n+1} = (G_n, G). \tag{3.12.6}$$

Note that G_n is a normal subgroup of G for every $n \geq 1$, and that (3.12.2) holds by construction. We would like to check that (3.12.5) holds for every $n, m \geq 1$. Of course, (3.12.5) holds when $n = 1$ or $m = 1$.

Let $m > 1$ be given, and suppose by induction that the analogue of (3.12.5) with m replaced by $m - 1$ holds for every $n \geq 1$. Observe that

$$(G_n, G_m) = (G_n, (G_{m-1}, G)) \subseteq (G_{m-1}, (G, G_n))\,(G, (G_n, G_{m-1})), \tag{3.12.7}$$

where the second step is as in Section 3.8. This implies that

$$(G_n, G_m) \subseteq (G_{m-1}, G_{n+1})\,(G, G_{n+m-1}), \tag{3.12.8}$$

using the definition of G_n and the induction hypothesis. It follows that

$$(G_n, G_m) \subseteq G_{n+m}\, G_{n+m}, \tag{3.12.9}$$

for the same reasons. This implies (3.12.5), because G_{n+m} is a subgroup of G.

Let $K_1, K_2, K_3, \ldots$ be a sequence of subgroups of G such that $K_1 = G$, $K_{n+1} \subseteq K_n$ for every $n \geq 1$, and

$$(K_n, G) \subseteq K_{n+1} \tag{3.12.10}$$

for every $n \geq 1$. Under these conditions,

$$G_n \subseteq K_n \tag{3.12.11}$$

for every $n \geq 1$, as mentioned on p9 of [26]. Indeed, if this holds for some $n \geq 1$, then

$$G_{n+1} = (G_n, G) \subseteq (K_n, G) \subseteq K_{n+1}. \tag{3.12.12}$$

3.13 Rings and quotients

Let $\mathcal{A}$ be a ring with a multiplicative identity element e, and let $N_{\mathcal{A}}$ be a nonnegative real-valued function on $\mathcal{A}$ that is symmetric and ultra-subadditive on $\mathcal{A}$, as a commutative group with respect to addition. Suppose also that $N_{\mathcal{A}}(0) = 0$, $N_{\mathcal{A}}(e) = 1$, and that $N_{\mathcal{A}}$ is submultiplicative on $\mathcal{A}$. Let $G(\mathcal{A})$ be the group of invertible elements of $\mathcal{A}$, as before, and let $U(\mathcal{A})$ be the subgroup of $G(\mathcal{A})$ consisting of $x \in G(\mathcal{A})$ with $N_{\mathcal{A}}(x), N_{\mathcal{A}}(x^{-1}) \leq 1$, as in Section 1.12. Remember that

$$N_{U(\mathcal{A})}(x) = N_{\mathcal{A}}(x - e) \tag{3.13.1}$$

is ultra-subadditive, symmetric, and invariant under conjugations on $U(\mathcal{A})$, as a group with respect to multiplication, as in Section 3.1.

Let $r \in (0, 1)$ be given, and let μ be the function on $U(\mathcal{A})$ with values in $[0, +\infty]$ such that

$$r^{\mu(x)} = N_{U(\mathcal{A})}(x) \tag{3.13.2}$$

for every $x \in U(\mathcal{A})$. This is the same as the restriction to $U(\mathcal{A})$ of the function defined on $G(\mathcal{A})$ in Section 3.5. Thus μ is a basic filtration function on $U(\mathcal{A})$ that is invariant under conjugations, as before. If $x, y \in U(\mathcal{A})$, then $(x, y) \in U(\mathcal{A})$, and we have seen that

$$N_{U(\mathcal{A})}((x, y)) = N_{\mathcal{A}}(x\,y - y\,x) \leq N_{U(\mathcal{A})}(x)\, N_{U(\mathcal{A})}(y). \tag{3.13.3}$$

Let $B_{\mathcal{A}}(x, t)$ and $\overline{B}_{\mathcal{A}}(x, t)$ be the open an closed balls in $\mathcal{A}$ centered at $x \in \mathcal{A}$ with radius t with respect to the semimetric $N_{\mathcal{A}}(y - z)$. Suppose from now on in this section that

$$B_{\mathcal{A}}(e, 1) \subseteq G(\mathcal{A}), \tag{3.13.4}$$

as in Section 3.7. Under these conditions, $B_{\mathcal{A}}(e,1)$ is the same as the subgroup $U_0(\mathcal{A})$ of $U(\mathcal{A})$ defined in Section 3.5. In particular, the restriction of μ to $B_{\mathcal{A}}(e,1)$ is a regular filtration function, as before.

Let us take $G = B_{\mathcal{A}}(e,1)$, so that the subgroups G_t, G_t^+ can be defined as in Section 3.10 for $t \ge 0$. If $t > 0$, then

$$G_t = \overline{B}_{\mathcal{A}}(e, r^t), \tag{3.13.5}$$

$$G_t^+ = B_{\mathcal{A}}(e, r^t), \tag{3.13.6}$$

by the definition of μ. If $t = 0$, then G_t, G_t^+ are the same as G. Remember that G_t, G_t^+ are normal subgroups of G for every $t \ge 0$, and we put $\mathrm{gr}_t\, G = G_t/G_t^+$. Thus

$$\mathrm{gr}_t G = \overline{B}_{\mathcal{A}}(e, r^t)/B_{\mathcal{A}}(e, r^t) \tag{3.13.7}$$

when $t > 0$.

Remember that open and closed balls in $\mathcal{A}$ centered at 0 are subgroups of $\mathcal{A}$ as a commutative group with respect to addition. Let $x, y \in \overline{B}_{\mathcal{A}}(0, r^t)$ be given for some $t > 0$, so that $e + x, e + y \in \overline{B}_{\mathcal{A}}(e, r^t)$. Similarly,

$$(e + x)\,(e + y) = e + x + y + x\,y \tag{3.13.8}$$

and $e + x + y$ are elements of $\overline{B}_{\mathcal{A}}(e, r^t)$, and

$$N_{\mathcal{A}}((e + x)\,(e + y) - (e + x + y)) = N_{\mathcal{A}}(x\,y) \le N_{\mathcal{A}}(x)\,N_{\mathcal{A}}(y) \le r^{2\,t}. \tag{3.13.9}$$

This means that

$$N_{\mathcal{A}}((e + y)^{-1}\,(e + x)^{-1}\,(e + x + y) - e) \le r^{2\,t}, \tag{3.13.10}$$

because $N_{\mathcal{A}}$ is invariant under multiplication by elements of $U(\mathcal{A})$ on $\mathcal{A}$. It follows that

$$(e + y)^{-1}\,(e + x)^{-1}\,(e + x + y) \in \overline{B}_{\mathcal{A}}(e, r^{2\,t}) \subseteq B_{\mathcal{A}}(e, r^t), \tag{3.13.11}$$

because $t > 0$.

Thus (3.13.8) and $e + x + y$ are mapped to the same element of $\mathrm{gr}_t\, G$. This shows that we get a group homomorphism from $\overline{B}_{\mathcal{A}}(0, r^t)$, as a commutative group with respect to addition, into $\mathrm{gr}_t\, G$, by sending $x \in \overline{B}_{\mathcal{A}}(0, r^t)$ to $e + x$ in $\overline{B}_{\mathcal{A}}(e, r^t)$, and mapping that into the quotient $\mathrm{gr}_t\, G$. This homomorphism is surjective, because $x \mapsto e + x$ maps $\overline{B}_{\mathcal{A}}(0, r^t)$ onto $\overline{B}_{\mathcal{A}}(e, r^t)$. The kernel of this homomorphism is $B_{\mathcal{A}}(0, r^t)$, so that we get a group isomorphism from

$$\overline{B}_{\mathcal{A}}(0, r^t)/B_{\mathcal{A}}(0, r^t), \tag{3.13.12}$$

as a commutative group with respect to addition, onto $\mathrm{gr}_t\, G$.

Now let $x \in \overline{B}_{\mathcal{A}}(0, r^{t_1})$ and $y \in \overline{B}_{\mathcal{A}}(0, r^{t_2})$ be given for some $t_1, t_2 > 0$, and observe that

$$(e + x, e + y) = (e + x)^{-1}\,(e + y)^{-1}\,(e + x)\,(e + y) \in \overline{B}_{\mathcal{A}}(e, r^{t_1 + t_2}), \tag{3.13.13}$$

by (3.13.3). Clearly

$$e + x\,y - y\,x \in \overline{B}_{\mathcal{A}}(e, r^{t_1+t_2}) \tag{3.13.14}$$

as well, and we would like to show that $(e+x, e+y)$ and $e+x-y\,x$ are mapped to the same element of the quotient $\mathrm{gr}_{t_1+t_2} G$. Using the invariance of $N_{\mathcal{A}}$ under multiplication by elements of $U(\mathcal{A})$, we get that

$$\begin{aligned} (3.13.15)\quad & N_{\mathcal{A}}((e + x, e + y)^{-1} (e + x\,y - y\,x) - e) \\ &= N_{\mathcal{A}}((e + y, e + x) (e + x\,y - y\,x) - e) \\ &= N_{\mathcal{A}}((e + y) (e + x) (e + x\,y - y\,x) - (e + x) (e + y)). \end{aligned}$$

It is easy to see that

$$\begin{aligned} (3.13.16)\quad & (e + y) (e + x) (e + x\,y - y\,x) - (e + x) (e + y) \\ &= (e + y + x + y\,x) (e + x\,y - y\,x) - (e + x + y + x\,y) \\ &= (y + x + y\,x) (x\,y - y\,x). \end{aligned}$$

This implies that (3.13.15) is less than or equal to

$$r^{t_1+t_2} \max(r^{t_1}, r^{t_2}) < r^{t_1+t_2}, \tag{3.13.17}$$

as desired.

Of course, we may consider $[x, y] = x\,y - y\,x$ as a mapping from

$$\overline{B}_{\mathcal{A}}(0, r^{t_1}) \times \overline{B}_{\mathcal{A}}(0, r^{t_2}) \tag{3.13.18}$$

into $\overline{B}_{\mathcal{A}}(0, r^{t_1+t_2})$ that is bilinear over $\mathbf{Z}$. We can compose this mapping with the natural quotient mapping from $\overline{B}_{\mathcal{A}}(0, r^{t_1+t_2})$ onto

$$\overline{B}_{\mathcal{A}}(0, r^{t_1+t_2})/B_{\mathcal{A}}(0, r^{t_1+t_2}), \tag{3.13.19}$$

to get a mapping from (3.13.18) into (3.13.19) that is bilinear over $\mathbf{Z}$. It is easy to see that this leads to a mapping from

$$(\overline{B}_{\mathcal{A}}(0, r^{t_1})/B_{\mathcal{A}}(0, r^{t_1})) \times (\overline{B}_{\mathcal{A}}(0, r^{t_2})/B_{\mathcal{A}}(0, r^{t_2})) \tag{3.13.20}$$

into (3.13.19) that is bilinear over $\mathbf{Z}$. More precisely, the previous mapping from (3.13.18) into (3.13.19) is the same as the new mapping from (3.13.20) into (3.13.19) composed with the appropriate quotient mapping in each variable.

We can also use $(e + x, e + y)$ to get a mapping from (3.13.18) into

$$\overline{B}_{\mathcal{A}}(e, r^{t_1+t_2}), \tag{3.13.21}$$

as before. We can compose this mapping with the natural quotient mapping from (3.13.21) onto $\mathrm{gr}_{t_1+t_2} G$, to get a mapping from (3.13.18) into $\mathrm{gr}_{t_1+t_2} G$. Remember that we have a group homomorphism from $\overline{B}_{\mathcal{A}}(0, r^{t_1+t_2})$, as a group with respect to addition, onto $\mathrm{gr}_{t_1+t_2} G$, defined by adding e to map onto (3.13.21), and composing with the natural quotient mapping from (3.13.21) onto $\mathrm{gr}_{t_1+t_2} G$. The earlier remarks show that the mapping from (3.13.18) into $\mathrm{gr}_{t_1+t_2} G$ obtained from $(e+x, e+y)$ is the same as the composition of the mapping obtained using $[x, y]$ to map (3.13.18) into $\overline{B}_{\mathcal{A}}(0, r^{t_1+t_2})$ with the group homomorphism from $\overline{B}_{\mathcal{A}}(0, r^{t_1+t_2})$ onto $\mathrm{gr}_{t_1+t_2} G$ just mentioned.

3.14 Algebras and quotients

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let $\mathcal{A}$ be an associative algebra over k with a multiplicative identity element e. Thus $\mathcal{A}$ is a vector space over k in particular, and we let $N_{\mathcal{A}}$ be a semi-ultranorm on $\mathcal{A}$ with respect to $|\cdot|$ on k that is submultiplicative and satisfies $N_{\mathcal{A}}(e) = 1$. Of course, $\mathcal{A}$ may be considered as a ring, and $N_{\mathcal{A}}$ satisfies the conditions mentioned at the beginning of the previous section. Let us continue with the same notation as in the previous section, for open and closed balls in $\mathcal{A}$ in particular. If (3.13.4) holds, then we have seen that some multiplicative groups are related to open and closed balls in $\mathcal{A}$ centered at 0, as commutative groups with respect to addition.

In this section, we can use scalar multiplication on $\mathcal{A}$ to relate open and closed balls in $\mathcal{A}$ centered at 0 of different radii. If $\alpha \in k$ and $E \subseteq \mathcal{A}$, then put

$$\alpha\, E = \{\alpha\, x : x \in E\}. \tag{3.14.1}$$

If $\alpha \neq 0$ and $\rho > 0$, then

$$\alpha\, \overline{B}_{\mathcal{A}}(0, \rho) = \overline{B}_{\mathcal{A}}(0, |\alpha|\, \rho) \tag{3.14.2}$$

and

$$\alpha\, B_{\mathcal{A}}(0, \rho) = B_{\mathcal{A}}(0, |\alpha|\, \rho). \tag{3.14.3}$$

More precisely, multiplication by α defines a group isomorphism from $\overline{B}_{\mathcal{A}}(0, \rho)$ onto $\overline{B}_{\mathcal{A}}(0, |\alpha|\, \rho)$, and from $B_{\mathcal{A}}(0, \rho)$ onto $B_{\mathcal{A}}(0, |\alpha|\, \rho)$, as commutative groups with respect to addition. This leads to a group isomorphism from

$$\overline{B}_{\mathcal{A}}(0, \rho)/B_{\mathcal{A}}(0, \rho) \tag{3.14.4}$$

onto

$$\overline{B}_{\mathcal{A}}(0, |\alpha|\, \rho)/B_{\mathcal{A}}(0, |\alpha|\, \rho). \tag{3.14.5}$$

If $\rho = |\beta|$ for some $\beta \in k$, then multiplication by β defines a group isomorphism from $\overline{B}_{\mathcal{A}}(0, 1)$ onto $\overline{B}_{\mathcal{A}}(0, \rho)$, as commutative groups with respect to addition, which sends $B_{\mathcal{A}}(0, 1)$ onto $B_{\mathcal{A}}(0, \rho)$. This leads to a group isomorphism from

$$\overline{B}_{\mathcal{A}}(0, 1)/B_{\mathcal{A}}(0, 1) \tag{3.14.6}$$

onto (3.14.4), as commutative groups with respect to addition. Of course, if there is no $x \in \mathcal{A}$ such that $N_{\mathcal{A}}(x) = \rho$, then

$$\overline{B}_{\mathcal{A}}(0, \rho) = B_{\mathcal{A}}(0, \rho), \tag{3.14.7}$$

so that

$$\overline{B}_{\mathcal{A}}(0, \rho)/B_{\mathcal{A}}(0, \rho) = \{0\}. \tag{3.14.8}$$

In some cases, the values of $N_{\mathcal{A}}$ on $\mathcal{A}$ are the same as the values of $|\cdot|$ on k, so that for every $x \in \mathcal{A}$ there is an $\alpha \in k$ such that $N_{\mathcal{A}}(x) = |\alpha|$. This means that (3.14.8) holds when there is no $\beta \in k$ such that $\rho = |\beta|$.

If $\rho_1, \rho_2 > 0$, then $[x, y] = x\,y - y\,x$ defines a mapping from

$$\overline{B}_{\mathcal{A}}(0, \rho_1) \times \overline{B}_{\mathcal{A}}(0, \rho_2) \tag{3.14.9}$$

into

$$\overline{B}_{\mathcal{A}}(0, \rho_1\,\rho_2) \tag{3.14.10}$$

that is bilinear over k, as before. Similarly, if $\alpha_1, \alpha_2 \in k$, $\alpha_1, \alpha_2 \neq 0$, then $[x, y]$ defines a mapping from

$$\overline{B}_{\mathcal{A}}(0, |\alpha_1|\,\rho_1) \times \overline{B}_{\mathcal{A}}(0, |\alpha_2|\,\rho_2) \tag{3.14.11}$$

into

$$\overline{B}_{\mathcal{A}}(0, |\alpha_1|\,|\alpha_2|\,\rho_1\,\rho_2) \tag{3.14.12}$$

that is bilinear over **Z**. Multiplication by α_j defines a group isomorphism from $\overline{B}_{\mathcal{A}}(0, \rho_j)$ onto $\overline{B}_{\mathcal{A}}(0, |\alpha_j|\,\rho_j)$, as commutative groups with respect to addition, for $j = 1, 2$, and multiplication by $\alpha_1\,\alpha_2$ defines a group isomorphism from (3.14.10) onto (3.14.12). The composition of the mapping from (3.14.11) into (3.14.12) with the mappings from $\overline{B}_{\mathcal{A}}(0, \rho_j)$ onto $\overline{B}_{\mathcal{A}}(0, |\alpha_j|\,\rho_j)$ defined by multiplication by α_j for $j = 1, 2$ in each variable is the same as the composition of the mapping from (3.14.9) into (3.14.10) with multiplication by $\alpha_1\,\alpha_2$, as a mapping from (3.14.10) onto (3.14.12). Of course, this leads to analogous statements for mappings involving quotients as in (3.14.4).

Let n be a positive integer, and let us now take $\mathcal{A}$ to be the algebra $M_n(k)$ of $n \times n$ matrices with entries in k. If $N_{\mathcal{A}}$ is the ultranorm on $M_n(k)$ defined by taking the maximum of the absolute values of the entries of an element of $M_n(k)$, then $N_{\mathcal{A}}$ is submultiplicative, and it takes values in the set of values of $|\cdot|$ on k. Let $B(0, \rho)$, $\overline{B}(0, \rho)$ be the open and closed balls in k centered at 0 with radius $\rho > 0$ with respect to the ultrametric associated to $|\cdot|$. It is easy to see that

$$\overline{B}_{\mathcal{A}}(0, \rho) = M_n(\overline{B}(0, \rho)), \tag{3.14.13}$$

$$B_{\mathcal{A}}(0, \rho) = M_n(B(0, \rho)), \tag{3.14.14}$$

where the right sides are the subsets of $M_n(k)$ consisting of matrices with entries in $\overline{B}(0, \rho)$, $B(0, \rho)$, respectively. Note that these are subgroups of $M_n(k)$, as a commutative group with respect to addition. Their quotient (3.14.4) corresponds to the space

$$M_n(\overline{B}(0, \rho)/B(0, \rho)) \tag{3.14.15}$$

of $n \times n$ matrices with entries in the quotient group $\overline{B}(0, \rho)/B(0, \rho)$, which is a commutative group with respect to entrywise addition of matrices. This is related to Exercise 1 on p10 of [26].

3.15 Multiplication and quotients

Let $\mathcal{A}$ be a ring with a multiplicative identity element e and a nonnegative real-valued function $N_{\mathcal{A}}$ that is symmetric and ultra-subadditive on $\mathcal{A}$, as a commutative group with respect to addition. As before, we also ask that $N_{\mathcal{A}}$ be submultiplicative on $\mathcal{A}$, and that $N_{\mathcal{A}}(0) = 0$, $N_{\mathcal{A}}(e) = 1$. If $\rho_1, \rho_2 > 0$ and $x_j \in \overline{B}_{\mathcal{A}}(0, \rho_j)$ for $j = 1, 2$, then

$$x_1\, x_2,\; x_2\, x_1 \in \overline{B}_{\mathcal{A}}(0, \rho_1\, \rho_2). \tag{3.15.1}$$

Thus multiplication on $\mathcal{A}$ in either order defines a mapping from

$$\overline{B}_{\mathcal{A}}(0, \rho_1) \times \overline{B}_{\mathcal{A}}(0, \rho_2) \tag{3.15.2}$$

into

$$\overline{B}_{\mathcal{A}}(0, \rho_1\, \rho_2) \tag{3.15.3}$$

that is bilinear over $\mathbf{Z}$. Similarly, $[x_1, x_2] = x_1\, x_2 - x_2\, x_1$ defines a mapping from (3.15.2) into (3.15.3) that is bilinear over $\mathbf{Z}$, as before.

We can compose these mappings with the natural quotient mapping from (3.15.3) onto the quotient group

$$\overline{B}_{\mathcal{A}}(0, \rho_1\, \rho_2)/B_{\mathcal{A}}(0, \rho_1\, \rho_2) \tag{3.15.4}$$

to get mappings from (3.15.2) into (3.15.4) that are bilinear over $\mathbf{Z}$ as well. This leads to mappings from

$$(\overline{B}_{\mathcal{A}}(0, \rho_1)/B_{\mathcal{A}}(0, \rho_1)) \times (\overline{B}_{\mathcal{A}}(0, \rho_2)/B_{\mathcal{A}}(0, \rho_2)) \tag{3.15.5}$$

into (3.15.4) that are bilinear over $\mathbf{Z}$. More precisely, the previous mappings from (3.15.2) into (3.15.4) are the same as the new mappings from (3.15.5) into (3.15.4) composed with the appropriate quotient mappings in each variable. Remember that $\overline{B}_{\mathcal{A}}(0, 1)$ is a subring of $\mathcal{A}$, and that $B_{\mathcal{A}}(0, 1)$ is a two-sided ideal in $\overline{B}_{\mathcal{A}}(0, 1)$, so that the quotient

$$\overline{B}_{\mathcal{A}}(0, 1)/B_{\mathcal{A}}(0, 1) \tag{3.15.6}$$

is a ring. If $\rho_1 = \rho_2 = 1$, then the previous bilinear mappings from (3.15.5) into (3.15.4) can be defined in terms of the ring operations on (3.15.6).

Let k be a field with an ultrametric absolute value function $|\cdot|$ again, and suppose now that $\mathcal{A}$ is an associative algebra over k with a multiplicative identity element e. Also let $N_{\mathcal{A}}$ be a semi-ultranorm on $\mathcal{A}$ with respect to $|\cdot|$ on k that is submultiplicative and satisfies $N_{\mathcal{A}}(e) = 1$, so that $N_{\mathcal{A}}$ satisfies the same conditions as before on $\mathcal{A}$ as a ring. Suppose that $\rho_j = |\beta_j|$ where $\beta_j \in k$ and $\beta_j \neq 0$ for $j = 1, 2$, so that multiplication by β_j defines a group isomorphism from $\overline{B}_{\mathcal{A}}(0, 1)$ onto $\overline{B}_{\mathcal{A}}(0, \rho_j)$, as commutative groups with respect to addition, for $j = 1, 2$. Multiplication by β_j also maps $B_{\mathcal{A}}(0, 1)$ onto $B_{\mathcal{A}}(0, \rho_j)$ for $j = 1, 2$, which induces an isomorphism from (3.15.6) onto

$$\overline{B}_{\mathcal{A}}(0, \rho_j)/B_{\mathcal{A}}(0, \rho_j), \tag{3.15.7}$$

as commutative groups with respect to addition, for $j = 1, 2$. Similarly, multiplication by $\beta_1\, \beta_2$ defines a group isomorphism from $\overline{B}_{\mathcal{A}}(0, 1)$ onto (3.15.3) that maps $B_{\mathcal{A}}(0,1)$ onto $B_{\mathcal{A}}(0,\rho_1\,\rho_2)$, which induces an isomorphism from (3.15.6) onto (3.15.4), as commutative groups with respect to addition.

Consider the mapping from

$$\overline{B}_{\mathcal{A}}(0,1) \times \overline{B}_{\mathcal{A}}(0,1) \tag{3.15.8}$$

onto (3.15.2) defined by multiplication by β_1 in the first coordinate, and multiplication by β_2 in the second coordinate. The composition of this mapping with any of the mappings from (3.15.2) into (3.15.3) defined by $x_1\,x_2$, $x_2\,x_1$, or $[x_1, x_2]$, is the same as the analogous mapping from (3.15.8) into $\overline{B}_{\mathcal{A}}(0,1)$ composed with the mapping from $\overline{B}_{\mathcal{A}}(0,1)$ onto (3.15.3) defined by multiplication by $\beta_1\,\beta_2$. We also get a mapping from

$$(\overline{B}_{\mathcal{A}}(0,1)/B_{\mathcal{A}}(0,1)) \times (\overline{B}_{\mathcal{A}}(0,1)/B_{\mathcal{A}}(0,1)) \tag{3.15.9}$$

onto (3.15.5) using the group isomorphism from (3.15.6) onto (3.15.7) obtained from multiplication by β_j in the jth coordinate for $j = 1, 2$. The composition of this mapping with any of the mappings from (3.15.5) into (3.15.4) associated to $x_1\,x_2$, $x_2\,x_1$, or $[x_1, x_2]$ is the same as the analogous mapping from (3.15.9) into (3.15.6) composed with the group isomorphism from (3.15.6) onto (3.15.4) obtained from multiplication by $\beta_1\,\beta_2$ as in the preceding paragraph. Of course, the mappings from (3.15.9) into (3.15.6) associated to $x_1\,x_2$, $x_2\,x_1$, and $[x_1, x_2]$ can be defined in terms of the ring operations on (3.15.6), as before.

Let n be a positive integer, let $\mathcal{A}$ be the algebra $M_n(k)$ of $n \times n$ matrices with entries in k, and let $N_{\mathcal{A}}$ be the ultranorm on $M_n(k)$ defined by taking the maximum of the absolute values of the entries of an element of $M_n(k)$, as in the previous section. Thus

$$\overline{B}_{\mathcal{A}}(0,1) = M_n(\overline{B}(0,1)), \tag{3.15.10}$$

$$B_{\mathcal{A}}(0,1) = M_n(B(0,1)), \tag{3.15.11}$$

as before. Their quotient is isomorphic as a ring to the ring

$$M_n(\overline{B}(0,1)/B(0,1)) \tag{3.15.12}$$

of $n \times n$ matrices with entries in the residue field $\overline{B}(0,1)/B(0,1)$. More precisely, the natural quotient homomorphism from $\overline{B}(0,1)$ onto $\overline{B}(0,1)/B(0,1)$ leads to a ring homomorphism from (3.15.10) onto (3.15.12), as in Section 2.15. The kernel of this homomorphism is clearly (3.15.11).

Chapter 4

Continuity conditions and Haar measure

4.1 Uniform continuity and semimetrics

Let X, Y be sets with semimetrics d_X, d_Y, respectively, and let f be a mapping from X into Y. We say that f is *uniformly continuous along a subset A of X* if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d_Y(f(x), f(w)) < \epsilon \tag{4.1.1}$$

for every $x \in A$ and $w \in X$ with $d_X(x, w) < \delta$. Equivalently, this means that

$$f(B_X(x, \delta)) \subseteq B_Y(f(x), \epsilon) \tag{4.1.2}$$

for every $x \in A$. Of course, this implies that f is continuous at every element of A, with respect to the topologies determined on X, Y by d_X, d_Y, respectively. If $A = X$, then we simply say that f is uniformly continuous on X. If f is uniformly continuous along a subset A of X, then the restriction of f to A is uniformly continuous, with respect to the restriction of d_X to elements of A. If A has only finitely many elements, and f is continuous at every point in A, then it is easy to see that f is uniformly continuous along A.

Suppose that f is continuous at every point in a compact set $A \subseteq X$, and let us check that f is uniformly continuous along A. Let $\epsilon > 0$ be given, and for each $a \in A$, let $\delta(a)$ be a positive real number such that

$$d_Y(f(a), f(w)) < \epsilon/2 \tag{4.1.3}$$

for every $w \in X$ with $d_X(a, w) < \delta(a)$. The collection of open balls

$$B_X(a, \delta(a)/2) \tag{4.1.4}$$

in X with $a \in A$ forms an open covering of A, and so there are finitely many elements $a_1, \ldots, a_n$ of A such that

$$A \subseteq \bigcup_{j=1}^{n} B_X(a_j, \delta(a_j)/2), \tag{4.1.5}$$

by compactness. Put

$$\delta = \min_{1 \le j \le n} (\delta(a_j)/2), \tag{4.1.6}$$

and let $x \in A$ and $w \in X$ be given, with $d_X(x, w) < \delta$. Thus $d_X(a_j, x) < \delta(a_j)/2$ for some $j \in \{1, \ldots, n\}$, by (4.1.5). This implies that

$$d_X(a_j, w) \le d_X(a_j, x) + d_X(x, w) < \delta(a_j)/2 + \delta \le \delta(a_j), \tag{4.1.7}$$

by the definition of δ. It follows that

$$d_Y(f(a_j), f(x)),\ d_Y(f(a_j), f(w)) < \epsilon/2, \tag{4.1.8}$$

as in (4.1.3). This means that

$$d_Y(f(x), f(w)) \le d_Y(f(x), f(a_j)) + d_Y(f(a_j), f(w)) < \epsilon/2 + \epsilon/2 = \epsilon, \tag{4.1.9}$$

as desired.

Let f be any mapping from X into Y that is uniformly continuous along a set $A \subseteq X$. Also let Z be another set with a semimetric d_Z, and let g be a mapping from Y into Z. If g is uniformly continuous along a set $B \subseteq Y$, and $f(A) \subseteq B$, then it is easy to see that

(4.1.10) the composition $g \circ f$ is uniformly continuous along A

as a mapping from X into Z.

If f is a uniformly continuous mapping from X into Y, and $E \subseteq X$ is totally bounded with respect to d_X, then one can check that

(4.1.11) $f(E)$ is totally bounded with respect to d_Y in Y.

More precisely, this also holds when f is uniformly continuous on E, with respect to the restriction of d_X to E. This can be verified using the characterization of totally bounded sets in terms of small sets, as in Section 2.8.

Let $x_0 \in X$ be given, and put

$$f_0(x) = d_X(x, x_0) \tag{4.1.12}$$

for every $x \in X$. One can check that

$$|f_0(x) - f_0(w)| \le d_X(x, w) \tag{4.1.13}$$

for every $x, w \in X$, using the triangle inequality. This uses the standard absolute value function on the real line on the left side, and it implies that f_0 is uniformly continuous as a real-valued function on X, with respect to the standard Euclidean metric on $\mathbf{R}$.

4.2 Uniform continuity and topological groups

Let G be a topological group, let Y be a set with a semimetric d_Y, and let f be a mapping from G into Y. We say that f is *left-invariant uniformly continuous along a subset A of G* if for every $\epsilon > 0$ there is an open subset U of G such that U contains the identity element e and

$$d_Y(f(a), f(a\,x)) < \epsilon \tag{4.2.1}$$

for every $a \in A$ and $x \in U$. This is the same as saying that

$$f(a\,U) \subseteq B_Y(f(a), \epsilon) \tag{4.2.2}$$

for every $a \in A$. Similarly, f is *right-invariant uniformly continuous along A* if for every $\epsilon > 0$ there is an open set U in G such that $e \in U$ and

$$d_Y(f(a), f(x\,a)) < \epsilon \tag{4.2.3}$$

for every $a \in A$ and $x \in U$. This means that

$$f(U\,a) \subseteq B_Y(f(a), \epsilon) \tag{4.2.4}$$

for every $a \in A$, as before.

If f is left or right-invariant uniformly continuous along A, then f is continuous at every element of A, with respect to the topology determined on Y by d_Y. If f is continuous at every element of A, and if A has only finitely many elements, then one can check that f is left and right-invariant uniformly continuous along A. If A is the whole group G, then we simply say that f is left or right-invariant uniformly continuous on G, as appropriate. It is easy to see that f is left-invariant uniformly continuous along a subset A of G if and only if

$$\widetilde{f}(x) = f(x^{-1}) \tag{4.2.5}$$

is right-invariant uniformly continuous along A^{-1}. Of course, if G is commutative, then left and right-invariant uniform continuity are the same.

If f is continuous at every point in A, and A is compact, then f is left and right-invariant uniformly continuous along A. Let us check that f is left-invariant uniformly continuous along A, the argument for right-invariant uniform continuity being analogous. One could also reduce to the left-invariant case, using (4.2.5). Let $\epsilon > 0$ be given, and for each $a \in A$, let $U(a)$ be an open subset of G such that $e \in U(a)$ and

$$d_Y(f(a), f(a\,x)) < \epsilon/2 \tag{4.2.6}$$

for every $x \in U(a)$. If $a \in A$, then we can use continuity of multiplication on G at e to get an open subset $U_1(a)$ of G such that $e \in U_1(a)$ and

$$U_1(a)\,U_1(a) \subseteq U(a). \tag{4.2.7}$$

Thus A is covered by the open sets $a\,U_1(a)$, $a \in A$. If A is compact, then there are finitely many elements $a_1, \ldots, a_n$ of A such that

$$A \subseteq \bigcup_{j=1}^{n} a_j\, U_1(a_j). \tag{4.2.8}$$

Let us take

$$U = \bigcap_{j=1}^{n} U_1(a_j), \tag{4.2.9}$$

which is an open set that contains e. If $a \in A$ and $x \in U$, then we would like to verify that (4.2.1) holds.

Using (4.2.8), we get that

$$a = a_j\, w \tag{4.2.10}$$

for some $j \in \{1, \ldots, n\}$ and $w \in U_1(a_j)$. It follows that

$$w\,x \in U_1(a_j)\,U \subseteq U_1(a_j)\,U_1(a_j) \subseteq U(a_j). \tag{4.2.11}$$

This implies that

$$d_Y(f(a_j), f(a\,x)) = d_Y(f(a_j), f(a_j\,w\,x)) < \epsilon/2. \tag{4.2.12}$$

Similarly,

$$d_Y(f(a_j), f(a)) = d_Y(f(a_j), f(a_j\,w)) < \epsilon/2, \tag{4.2.13}$$

because $U_1(a_j) \subseteq U(a_j)$. This means that

$$\begin{aligned} d_Y(f(a), f(a\,x)) &\le d_Y(f(a), f(a_j)) + d_Y(f(a_j), f(a\,x)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned} \tag{4.2.14}$$

as desired.

4.3 Some properties of uniform continuity

Let G be a topological group again, let Y be a set with a semimetric d_Y, and let f be a mapping from G into Y. If E is a left-invariant totally bounded subset of G, and f is left-invariant uniformly continuous on G, then it is easy to see that $f(E)$ is totally bounded in Y with respect to d_Y. Similarly, if E is right-invariant totally bounded, and f is right-invariant uniformly continuous, then $f(E)$ is totally bounded in Y.

Let Z be another set with a semimetric d_Z, and let g be a mapping from Y into Z that is uniformly continuous along $B \subseteq Y$. Also let A be a subset of G such that $f(A) \subseteq B$. If f is left-invariant uniformly continuous along A, then one can check that $g \circ f$ is left-invariant uniformly continuous along A as a mapping into Z. Similarly, if f is right-invariant uniformly continuous along A, then $g \circ f$ is right-invariant uniformly continuous along A.

Suppose for the moment that the topology on G is determined by a semimetric d, and let A be a subset of G again. If d is invariant under left translations, then one can verify that f is left-invariant uniformly continuous along A if and only if f is uniformly continuous along A with respect to d. Similarly, if d is invariant under right translations, then f is right-invariant uniformly continuous along A if and only if f is uniformly continuous along A with respect to d.

Suppose now that f is left-invariant uniformly continuous on G, and that d_Y is a semi-ultrametric on Y. Let $\epsilon > 0$ be given, and let U be an open subset of G such that $e \in U$ and (4.2.1) holds for every $a \in G$ and $x \in U$. We may also ask that U be symmetric about e, by replacing U with $U \cap U^{-1}$, if necessary. If $a \in G$ and $x_1, \ldots, x_n \in U$ for some positive integer n, then

$$d_Y(f(a\,x_1 \cdots x_{j-1}), f(a\,x_1 \cdots x_{j-1}\,x_j)) < \epsilon \tag{4.3.1}$$

for every $j = 1, \ldots, n$. It follows that

$$d_Y(f(a), f(a\,x_1 \cdots x_n)) < \epsilon, \tag{4.3.2}$$

because d_Y is a semi-ultrametric on Y. Equivalently, this means (4.2.1) holds for every $a \in G$ and $x \in U^n$, where U^n consists of products of n elements of U, as in Section 2.4. Under these conditions,

$$U_0 = \bigcup_{n=1}^{\infty} U^n \tag{4.3.3}$$

is an open subgroup of G, as before. Using (4.3.2), we get that (4.2.1) holds for every $a \in G$ and $x \in U_0$.

Similarly, if f is right-invariant uniformly continuous on G, then for each $\epsilon > 0$ there is an open subgroup U_0 of G such that (4.2.3) holds for every $a \in G$ and $x \in U_0$. This can be shown using the same type of argument as in the preceding paragraph, or by reducing to the previous case using (4.2.5).

4.4 Haar measures

Let X be a locally compact Hausdorff topological space, and let μ be a nonnegative Borel measure on X. Note that compact subsets of X are closed sets, and thus Borel sets. If for every Borel set $E \subseteq X$ we have that

$$\mu(E) = \inf\{\mu(U) : U \subseteq X \text{ is an open set, and } E \subseteq U\}, \tag{4.4.1}$$

then μ is said to be *outer regular* on X. We may also be concerned with the *inner regularity* condition

$$\mu(E) = \sup\{\mu(K) : K \subseteq X \text{ is compact, and } K \subseteq E\}. \tag{4.4.2}$$

In particular, we may be interested in situations where this inner regularity condition holds for open sets, and for Borel sets E such that $\mu(E) < +\infty$.

Let G be a locally compact topological group such that $\{e\}$ is a closed set, which implies that G is Hausdorff, as before. A nonnegative Borel measure H_L on G is said to be a *left-invariant Haar measure* if it satisfies the following four conditions. First,

$$H_L(U) > 0 \tag{4.4.3}$$

for every nonempty open subset U of G. Second,

$$H_L(K) < +\infty \tag{4.4.4}$$

for every compact subset K of G. Third, H_L is *invariant under left translations*, in the sense that

$$H_L(a\,E) = H_L(E) \tag{4.4.5}$$

for every Borel subset E of G and $a \in G$. Note that translates of Borel subsets of G are Borel sets too, by continuity of translations. Fourth, H_L is outer regular, and the inner regularity condition (4.4.2) holds when E is an open set, or a Borel set with $H_L(E) < +\infty$.

Similarly, a nonnegative Borel measure H_R on G is said to be a *right-invariant Haar measure* if it satisfies the first, second, and fourth conditions in the preceding paragraph, and is *invariant under right translations*. This means that

$$H_R(E\,a) = H_R(E) \tag{4.4.6}$$

for every Borel subset E of G and $a \in G$. One can check that H_L is a left-invariant Haar measure on G if and only if

$$H_R(E) = H_L(E^{-1}) \tag{4.4.7}$$

is a right-invariant Haar measure on G. Of course, left and right-invariant Haar measures are the same when G is commutative. The product of a left or right-invariant Haar measure by a positive real number is a left or right-invariant Haar measure as well, respectively.

It is well known that left and right-invariant Haar measures on G exist, and are unique, up to multiplication by a positive real number. If G is any group equipped with the discrete topology, then counting measure on G is both a left and right-invariant Haar measure. If $G = \mathbf{R}^n$ for some positive integer n, as a commutative topological group with respect to addition and the standard topology, then n-dimensional Lebesgue measure is a Haar measure.

4.5 Haar integrals

If X and Y are topological spaces, then we let $C(X, Y)$ be the space of all continuous mappings from X into Y. In particular, $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ are the spaces of continuous real and complex-valued functions on X, using the standard topologies on $\mathbf{R}$ and $\mathbf{C}$. These are vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, with respect to pointwise addition and scalar multiplication of functions. The *support* of a real or complex-valued function f on X is defined as usual to be the closure in X of the set of $x \in X$ such that $f(x) \neq 0$. The spaces of continuous real and complex-valued functions on X with compact support are denoted $C_{com}(X, \mathbf{R})$ and $C_{com}(X, \mathbf{C})$, respectively, and are linear subspaces of $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$, respectively.

Let G be a locally compact topological group such that $\{e\}$ is a closed set. A *left-invariant Haar integral* on G is a linear functional I_L on $C_{com}(G, \mathbf{R})$ that satisfies the following properties. First, I_L should be nonnegative, in the sense that

$$I_L(f) \geq 0 \tag{4.5.1}$$

for every $f \in C_{com}(G, \mathbf{R})$ such that $f(x) \geq 0$ for every $x \in G$. More precisely, if we also have that $f(x) > 0$ for some $x \in G$, then we ask that

$$I_L(f) > 0. \tag{4.5.2}$$

In addition, I_L should be *invariant under left translations*, in the following sense. If $a \in G$ and $f \in C_{com}(G, \mathbf{R})$, then put

$$(L_a(f))(x) = f(a\,x) \tag{4.5.3}$$

for every $x \in G$. It is easy to see that

$$L_a(f) \in C_{com}(G, \mathbf{R}) \tag{4.5.4}$$

too, because of continuity of translations, and we ask that

$$I_L(L_a(f)) = I_L(f). \tag{4.5.5}$$

Similarly, a *right-invariant Haar integral* on G is a linear functional I_R on $C_{com}(G, \mathbf{R})$ that satisfies the same positivity condition as in the preceding paragraph, and which is *invariant under right translations*, in the following sense. If $a \in G$ and $f \in C_{com}(G, \mathbf{R})$, then put

$$(R_a(f))(x) = f(x\,a) \tag{4.5.6}$$

for every $x \in G$, and observe that

$$R_a(f) \in C_{com}(G, \mathbf{R}). \tag{4.5.7}$$

In this case, we ask that

$$I_R(R_a(f)) = I_R(f). \tag{4.5.8}$$

Invariance under left and right translations are equivalent when G is commutative. The product of a left or right-invariant Haar integral is a left or right-invariant Haar integral too, respectively.

It is well known that left and right-invariant Haar integrals on G exist, and are unique, up to multiplication by a positive real number. If H_L is a left-invariant Haar measure on G, then

$$I_L(f) = \int_G f\,dH_L \tag{4.5.9}$$

defines a left-invariant Haar integral on G. Conversely, if I_L is a left-invariant Haar integral on G, then there is a unique left-invariant Haar measure H_L on G such that (4.5.9) holds for every $f \in C_{com}(G, \mathbf{R})$, by the Riesz representation theorem. Similarly, if H_R is a right-invariant Haar measure on G, then

$$I_R(f) = \int_G f \, dH_R \tag{4.5.10}$$

defines a right-invariant Haar integral on G. Conversely, if I_R is a right-invariant Haar integral on G, then there is a unique right-invariant Haar measure H_R on G such that (4.5.10) holds for every $f \in C_{com}(G, \mathbf{R})$, by the Riesz representation theorem.

If $f \in C_{com}(G, \mathbf{R})$, then it is easy to see that $\widetilde{f}(x) = f(x^{-1})$ defines an element of $C_{com}(G, \mathbf{R})$ as well. One can check that I_L is a left-invariant Haar integral on G exactly when

$$I_R(f) = I_L(\widetilde{f}) \tag{4.5.11}$$

is a right-invariant Haar integral on G.

4.6 Left and right translations

Let G be a locally compact topological group such that $\{e\}$ is a closed set, and let H_L be a left-invariant Haar measure on G. If $b \in G$, then put

$$H_{L,b}(E) = H_L(E\,b) \tag{4.6.1}$$

for every Borel subset E of G. One can check that this also defines a left-invariant Haar measure on G. Thus the uniqueness of left-invariant Haar measure implies that $H_{L,b}$ can be expressed as a positive real number times H_L.

Suppose that A is a Borel subset of G such that

$$0 < H_L(A) < +\infty \tag{4.6.2}$$

and

$$H_L(A\,b) = H_L(A). \tag{4.6.3}$$

Under these conditions, the positive real number mentioned in the preceding paragraph is equal to 1, so that

$$H_{L,b} = H_L. \tag{4.6.4}$$

Of course, $H_L(b^{-1}\,A\,b) = H_L(A\,b)$, so that (4.6.3) is equivalent to

$$H_L(b^{-1}\,A\,b) = H_L(A). \tag{4.6.5}$$

In particular, this holds when

$$b^{-1}\,A\,b = A. \tag{4.6.6}$$

If G is compact, then we get that (4.6.4) holds for every $b \in G$. This means that H_L is invariant under right translations, and thus may be considered as right-invariant Haar measure on G too. We may refer to this simply as Haar measure on G, and denote it H. In this case, it is customary to normalize Haar measure by

$$H(G) = 1. \tag{4.6.7}$$

Suppose that G has a Haar measure H that is invariant under left and right translations, such as when G is commutative or compact. Let us check that

$$H(E^{-1}) = H(E) \tag{4.6.8}$$

for every Borel subset E of G. The uniqueness of Haar measure implies that there is a positive real number c such that

$$H(E^{-1}) = c\, H(E) \tag{4.6.9}$$

for every Borel subset of G, because $H(E^{-1})$ is also a Haar measure that is invariant under left and right translations. To show that $c = 1$, it suffices to verify that (4.6.8) holds for some Borel set E with $H(E)$ positive and finite. Of course, (4.6.8) holds automatically when E is symmetric about e. If E has nonempty interior and E is contained in a compact set, then $H(E)$ is positive and finite. We can replace E with $E \cup E^{-1}$ if necessary to get E to be symmetric about e.

Of course, we could also have considered the effect of left translations on a right-invariant Haar measure. There are analogous arguments for left and right-invariant Haar integrals as well.

4.7 Some remarks about regularity conditions

Let X be a locally compact Hausdorff topological space, let μ be a nonnegative Borel measure on X, and suppose that

$$\mu(K) < +\infty \tag{4.7.1}$$

for every compact subset K of X. If every Borel subset E of X satisfies the inner regularity condition (4.4.2), then μ is said to be *inner regular* on X. We say that μ is *regular* on X if μ is both inner and outer regular on X.

A subset E of X is said to be *σ-compact* if E can be expressed as the union of a sequence of compact sets. If E is σ-compact, then E is a Borel set, and E satisfies the inner regularity condition (4.4.2). Indeed, if E is σ-compact, then it is easy to see that E can be expressed as the union of an increasing sequence of compact sets, because the union of finitely many compact sets is compact as well. The measures of these compact sets with respect to μ tends to $\mu(E)$ in this case, by a standard argument.

If E is a Borel subset of X, then a milder *inner regularity* condition is that

$$\mu(E) = \sup\{\mu(A) : A \subseteq X \text{ is a closed set, and } A \subseteq E\}. \tag{4.7.2}$$

If X is compact, then (4.7.2) implies (4.4.2), because closed subsets of X are compact. If X is σ-compact, then one can check that closed subsets of X are σ-compact as well. Using this, one can verify that (4.7.2) implies (4.4.2) in this case too. If $\mu(X) < +\infty$, then one can check that (4.7.2) holds for every Borel subset E of X if and only if μ is outer regular on X, by considering outer regularity for $X \setminus E$.

A subset of X that can be expressed as the union of a sequence of closed sets is said to be an F_σ *set*. Note that σ-compact sets are F_σ sets, and that F_σ sets are Borel sets. More precisely, an F_σ E set can be expressed as the union of an increasing sequence of closed sets, because finite unions of closed sets are closed sets. The measures of these closed sets with respect to μ tends to $\mu(E)$, as before, which implies that E satisfies (4.7.2). If X is σ-compact, then F_σ subsets of X are σ-compact too.

Similarly, a subset of X that can be expressed as the intersection of a sequence of open sets is said to be a G_δ *set*. Any G_δ set is a Borel set, and G_δ sets in X are the same as the complements of F_σ sets in X. If the topology on X is determined by a metric d, then it is well known that every closed set in X is a G_δ set, so that open subsets of X are F_σ sets. More precisely, if A is any subset of X, then

$$U_j = \bigcup_{x \in A} B(x, 1/j) \tag{4.7.3}$$

is an open subset of X that contains A for every $j \geq 1$, and one can check that

$$\overline{A} = \bigcap_{j=1}^{\infty} U_j. \tag{4.7.4}$$

This argument also works for semimetrics, although we are only considering Hausdorff spaces in this section.

If every open subset of X is σ-compact, then it is well known that μ is automatically regular on X, as in Theorem 2.18 on p50 of [21]. If X is σ-compact, and every open subset of X is an F_σ set, then it follows that every open set in X is σ-compact, as before. In particular, if the topology on X is determined by a metric, and X is σ-compact, then every open subset of X is σ-compact.

Suppose for the moment that there is a base for the topology of X with only finitely or countably many elements. In this case, one can use Lindelöf's theorem to get that X is σ-compact, because X is locally compact. It is well known that X is also regular as a topological space, because X is locally compact and Hausdorff. One can use this and Lindelöf's theorem to get that every open subset of X is σ-compact. More precisely, if x is an element of an open subset W of X, then there is an open subset U of X such that $x \in U$, $\overline{U} \subseteq W$, and $\overline{U}$ is compact, because X is locally compact and regular.

It is well known that a separable metric space has a base for its topology with only finitely or countably many elements, and that compact metric spaces are separable. Similarly, one can check that σ-compact metric spaces are separable.

If a topological space has a base for its topology with only finitely or countably many elements, and the space is regular in the strong sense, then there is a metric on the space that determines the same topology, by famous theorems of Urysohn and Tychonoff.

4.8 Products and product measures

Let $G_1, \ldots, G_n$ be finitely many locally compact topological groups, and suppose that the set containing only the identity element is a closed set in each G_j. Observe that

$$G = \prod_{j=1}^{n} G_j \tag{4.8.1}$$

is a locally compact topological group in which the set containing only the identity element is a closed set, with respect to the product topology, and where the group operations are defined coordinatewise. If there is a base for the topology of G_j with only finitely or countably many elements for each $j = 1, \ldots, n$, then there is a base for the product topology on G with only finitely or countably many elements, consisting of products of elements of the bases for the G_j's. In this case, one can get left or right-invariant Haar measures on G from left or right-invariant Haar measures on the G_j's, respectively, using the standard product measure construction. More precisely, open subsets of G are measurable with respect to the standard product measure construction under these conditions, so that Borel subsets of G are measurable with respect to the standard product measure construction as well.

Alternatively, one can get left or right-invariant Haar integrals on G from left or right-invariant Haar integrals on the G_j's, respectively. More precisely, if f is a continuous real-valued function on G with compact support, then one can define left or right-invariant Haar integrals of f using left or right-invariant Haar integrals in each coordinate, respectively.

Suppose now that G_j is a compact topological group for each element j of a nonempty set I, and that the set containing only the identity element is a closed set in G_j. This implies that

$$G = \prod_{j \in I} G_j \tag{4.8.2}$$

is a compact topological group in which the set containing only the identity element is a closed set, with respect to the product topology, and where the group operations are defined coordinatewise. If I is countably infinite, and there is a base for the topology of each G_j with only finitely or countably many elements, then one can get a base for the topology of G with only finitely or countably many elements as well. This base consists of products of open subsets of the G_j's, where these open sets are equal to G_j for all but finitely many j, and the others are elements of the given bases for the G_j's. If $j \in I$, then we can take H_j to be the Haar measure on G_j that is invariant under left and right translations and normalized so that

$$H_j(G_j) = 1. \tag{4.8.3}$$

One can use these measures to get Haar measure H on G, by a standard product measure construction. As before, open subsets of G are measurable with respect to this product measure construction, because they can be expressed as unions of elements of the base for the topology of G mentioned earlier.

Alternatively, let I_j be the Haar integral on G_j that is invariant under left and right translations and normalized so that it is equal to 1 for the constant function on G_j equal to 1. If f is a continuous real-valued function on G, then one can use these Haar integrals on f in the jth variable for any finite set of $j \in I$. To define the Haar integral of f, one can pass to a suitable limit. Note that f is left and right-invariant uniformly continuous on G, because G is compact. In particular, this implies that f approximately depends on only finitely many coordinates.

These constructions can be simplified when G_j has only finitely many elements for each $j \in I$. This will be considered more broadly in the next section.

4.9 Haar measure on profinite groups

Let G be a compact topological group for which $\{e\}$ is a closed set, and let H be Haar measure on G normalized so that $H(G) = 1$, which is invariant under both left and right translations in this case. If U is an open subgroup of G, then $H(U)$ is the reciprocal of the index of U in G, which is the number of left or equivalently right cosets of U in G.

Suppose from now on in this section that G is profinite, so that the open subgroups of G form a local base for the topology of G at e. More precisely, the open normal subgroups of G form a local base for the topology of G at e, as in Section 2.14.

Let $\mathcal{A}$ be the collection of all subsets of G that can be expressed as the union of finitely many translates of open normal subgroups of G. Remember that open subgroups of G are closed sets, which means that they are compact, because G is compact. Of course, every open subset W of G can be expressed as a union of translates of open normal subgroups, because the open normal subgroups form a local base for the topology of G at e. If W is compact too, then it follows that W can be expressed as the union of finitely many translates of open normal subgroups. Thus $\mathcal{A}$ is the same as the collection of subsets of G that are open and closed, because closed subsets of G are compact.

If $U_1, \ldots, U_n$ are finitely many open normal subgroups of G, then

$$U = \bigcap_{j=1}^{n} U_j \tag{4.9.1}$$

is an open normal subgroup of G too. Of course, U has finite index in G, and thus in U_j for each $j = 1, \ldots, n$. This implies that U_j can be expressed as the union of finitely many translates of U for each $j = 1, \ldots, n$. It follows that every element of $\mathcal{A}$ can be expressed as the union of finitely many translates of a single open normal subgroup of G.

Of course, left and right translates of a normal subgroup are the same. Thus the elements of $\mathcal{A}$ can be expressed as the union of finitely many left translates of a single open normal subgroup of G. Remember that the left translates of any subgroup are pairwise disjoint. This implies that the Haar measure of the union of finitely many distinct left cosets of an open subgroup U is equal to the number of cosets times $H(U)$. This determines the Haar measure of elements of $\mathcal{A}$.

Let W be an open subset of G, let K be a compact subset of G, and suppose that $K \subseteq W$. If $x \in K$, then there is an open normal subgroup U of G such that $x\,U \subseteq W$. Because K is compact, K is contained in the union of finitely many such cosets. Equivalently, this means that there is an element of $\mathcal{A}$ that contains K and is contained in W.

This shows that Haar measure is uniquely determined on open subsets of G, because Haar measure is uniquely determined on $\mathcal{A}$. More precisely, this uses the inner regularity of Haar measure on open sets. It follows that Haar measure is uniquely determined on G, by outer regularity.

Note that $\mathcal{A}$ is an algebra of subsets of G, which is to say that it contains G and the empty set as elements, and it is closed under finite unions, intersections, and complements. This follows from the earlier characterization of $\mathcal{A}$ as the collection of subsets of G that are open and closed. Alternatively, if $A, B \in \mathcal{A}$, then one can check that A and B can be expressed as the union of finitely many left cosets of a single open normal subgroup of G. This implies that the union, intersection, and complements of A and B can be expressed as the union of finitely many left cosets of the same open normal subgroup of G.

One can define Haar measure directly on $\mathcal{A}$ as before, with the normalization $H(G) = 1$. One can verify directly that Haar measure is finitely additive on $\mathcal{A}$. This uses the fact that pairs of elements of $\mathcal{A}$ can be expressed as finite unions of left cosets of a single open normal subgroup of G, as in the preceding paragraph.

Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of pairwise-disjoint elements of $\mathcal{A}$ such that $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$. Thus $\bigcup_{j=1}^\infty A_j$ is compact, which implies that

$$\bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{\infty} A_j \tag{4.9.2}$$

for some positive integer n, because A_j is an open set for each j. This means that $A_j = \emptyset$ when $j > n$, so that

$$H\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} H(A_j), \tag{4.9.3}$$

because H is finitely additive on $\mathcal{A}$.

Under these conditions, a famous theorem of E. Hopf implies that H can be extended to a countably-additive measure on a σ-algebra of subsets of G that contains $\mathcal{A}$ in a natural way. More precisely, one can define an outer measure on G associated to H on $\mathcal{A}$, and use Carathéodory's notion of measurable sets with respect to that outer measure. This outer measure is invariant under left and right translations on G, because of the analogous property of H on $\mathcal{A}$. This implies that the corresponding σ-algebra of measurable sets is invariant under left and right translations as well.

Suppose that there is a local base for the topology of G at e with only finitely or countably many elements. This implies that there is a local base for the topology of G at e consisting of only finitely or countably many open normal subgroups. It follows that there are only finitely or countably many cosets of these open normal subgroups of G, which form a base for the topology of G. In particular, every open subset of G can be expressed as the union of finitely or countable many elements of $\mathcal{A}$ in this case.

4.10 Haar measure and absolute values

Let k be a field with an absolute value function $|\cdot|$, and suppose that k is locally compact with respect to the topology determined by the associated metric. Thus k may be considered as a commutative locally compact topological group with respect to addition.

If $|\cdot|$ is the trivial absolute value function on k, then the associated metric is the discrete metric, and counting measure on k satisfies the requirements of Haar measure. Let us suppose from now on in this section that $|\cdot|$ is not the trivial absolute value function on k.

Remember that closed and bounded subsets of k are compact, because k is locally compact and $|\cdot|$ is nontrivial on k, as in Section 2.10. In particular, this implies that k is complete with respect to the metric associated to $|\cdot|$. This uses the well-known facts that a Cauchy sequence of elements of any metric space is bounded, and that a Cauchy sequence of elements of a compact subset of a metric space converges to an element of that subset.

If $|\cdot|$ is archimedean on k, then a famous theorem of Ostrowski implies that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to an absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard absolute value function, as mentioned in Section 1.3. One-dimensional Lebesgue measure on $\mathbf{R}$ satisfies the requirements of Haar measure. As a topological group with respect to addition, $\mathbf{C}$ is the same as $\mathbf{R}^2$, on which two-dimensional Lebesgue measure satisfies the requirements of Haar measure.

Suppose now that $|\cdot|$ is an ultrametric absolute value function on k. Let N be the number of elements of the residue field $\overline{B}(0,1)/B(0,1)$, which is finite, because $\overline{B}(0,1)$ is compact and thus totally bounded, as in Section 2.11. We also have that $|\cdot|$ is discrete on k, so that there is a positive real number $\rho_1 < 1$ such that the positive values of $|\cdot|$ are the same as the integer powers of ρ_1, as before. In particular, $B(0,1) = \overline{B}(0,\rho_1)$ in this case.

Remember that $\overline{B}(0, \rho_1^j)$ is a subgroup of k, as a commutative group with respect to addition, for every $j \in \mathbf{Z}$. It is easy to see that

$$\overline{B}(0, \rho_1^j)/\overline{B}(0, \rho_1^{j+1}) \tag{4.10.1}$$

is isomorphic to $\overline{B}(0, 1)/\overline{B}(0, \rho_1)$ for each j, as a commutative group with respect to addition. Thus (4.10.1) has exactly N elements for every j. It follows that

$$\overline{B}(0, \rho_1^j)/\overline{B}(0, \rho_1^{j+l}) \tag{4.10.2}$$

has exactly N^l elements for every $j \in \mathbf{Z}$ and $l \in \mathbf{Z}_+$.

Let H be Haar measure on k, normalized so that

$$H(\overline{B}(0, 1)) = 1. \tag{4.10.3}$$

One can check that

$$H(\overline{B}(0, \rho_1^j)) = N^{-j} \tag{4.10.4}$$

for every $j \in \mathbf{Z}$, using the remarks in the preceding paragraph. Of course, this determines the Haar measure of all closed balls in k, because of invariance under translations.

4.11 Automorphisms of topological groups

An *automorphism* of a topological group G is an automorphism α of G as a group that is also a homeomorphism from G onto itself. It is easy to see that the automorphisms of G form a group with respect to composition of mappings. In particular, if $a \in G$, then conjugation by a defines an automorphism of G as a topological group.

Suppose that G is a locally compact topological group such that $\{e\}$ is a closed set, and let H_L be a left-invariant Haar measure on G. If α is an automorphism of G as a topological group, then one can check that $H_L(\alpha(E))$ satisfies the requirements of left-invariant Haar measure on G as well. It follows that there is a positive real number $\Lambda(\alpha)$ such that

$$H_L(\alpha(E)) = \Lambda(\alpha) = H_L(E) \tag{4.11.1}$$

for all Borel subsets E of G, by uniqueness of left-invariant Haar measure. Of course, there is an analogous statement for right-invariant Haar measure.

If E_0 is a Borel subset of G such that

$$0 < H_L(E_0) < +\infty \tag{4.11.2}$$

and

$$H_L(\alpha(E_0)) = H_L(E_0), \tag{4.11.3}$$

then

$$\Lambda(\alpha) = 1. \tag{4.11.4}$$

Of course, (4.11.3) holds automatically when

$$\alpha(E_0) = E_0. \tag{4.11.5}$$

In particular, (4.11.4) holds for every automorphism α of G as a topological group when G is compact.

Let $a \in G$ be given, and suppose for the moment that

$$\alpha(x) = a\,x\,a^{-1} \tag{4.11.6}$$

for every $x \in G$. If E is a Borel subset of G, then

$$H_L(\alpha(E)) = H_L(a\,E\,a^{-1}) = H_L(E\,a^{-1}). \tag{4.11.7}$$

This corresponds to some of the remarks in Section 4.6.

Let k be a field with an absolute value function $|\cdot|$. In particular, k may be considered as a commutative topological group with respect to addition. If $a \in k$ and $a \neq 0$, then

$$x \mapsto a\,x \tag{4.11.8}$$

defines an automorphism of k as a commutative topological group with respect to addition. Suppose that k is locally compact with respect to the topology determined by the metric associated to $|\cdot|$. Let H be Haar measure on k, as a locally compact commutative topological group with respect to addition. If $a \in k$ satisfies $|a| = 1$, then it is easy to see that H is invariant under (4.11.8). This uses the fact that open and closed balls in k centered at 0 are invariant under (4.11.8) when $|a| = 1$.

4.12 Some related examples

Let k be a field with an ultrametric absolute value function $|\cdot|$, and suppose that $|\cdot|$ is not the trivial absolute value function on k, and that k is locally compact with respect to the metric associated to $|\cdot|$. Also let n be a positive integer, and let $M_n(k)$ be the algebra of $n \times n$ matrices with entries in k. If $a = (a_{j,l}) \in M_n(k)$, then let $\|a\|$ be the maximum of $|a_{j,l}|$ over $1 \leq j, l \leq n$, which defines a submultiplicative ultranorm on $M_n(k)$, as before. It is easy to see that the topology determined on $M_n(k)$ by the ultrametric associated to $\|\cdot\|$ is the same as the product topology corresponding to the topology determined on k by the ultrametric associated to $|\cdot|$, where $M_n(k)$ is identified with the Cartesian product of n^2 copies of k in the obvious way. In particular, $M_n(k)$ may be considered as a commutative topological group with respect to addition, which corresponds to the product of n^2 copies of k as a commutative topological group with respect to addition.

Remember that the closed unit ball $\overline{B}(0,1)$ in k with respect to the ultrametric associated to $|\cdot|$ is a subring of k, so that $M_n(\overline{B}(0,1))$ is a subring of $M_n(k)$. Equivalently, $M_n(\overline{B}(0,1))$ is the closed unit ball in $M_n(k)$ with respect to the ultrametric associated to $\|\cdot\|$. Note that $M_n(\overline{B}(0,1))$ is compact and open in $M_n(k)$ with respect to the topology determined by this ultrametric, because $\overline{B}(0,1)$ is a compact open subset of k with respect to the topology determined by the ultrametric associated to $|\cdot|$. Let H_n be Haar measure on $M_n(k)$, as a commutative topological group with respect to addition.

If $a \in GL_n(\overline{B}(0,1))$, then left and right multiplication by a define one-to-one linear mappings from $M_n(k)$ onto itself that send $M_n(\overline{B}(0,1))$ onto itself. More precisely, left and right multiplication by a preserve $\|\cdot\|$ on $M_n(k)$, as in Section 1.12. In particular, left and right multiplication by a define automorphisms of $M_n(k)$, as a commutative topological group with respect to addition. One can use this to get that left and right multiplication by a preserve H_n on $M_n(k)$, as in the previous section.

We may consider $M_n(\overline{B}(0,1))$ as a commutative topological group with respect to addition as well, which is a compact open subgroup of $M_n(k)$. The restriction of H_n to $M_n(\overline{B}(0,1))$ satisfies the requirements of Haar measure on $M_n(\overline{B}(0,1))$.

Note that

$$\{x \in k : |x| = 1\} \tag{4.12.1}$$

is open and closed in k, with respect to the topology determined by the ultrametric associated to $|\cdot|$. This follows from the fact that $B(0,1)$ and $\overline{B}(0,1)$ are each open and closed in k.

Remember that $GL_n(\overline{B}(0,1))$ consists of $a \in M_n(\overline{B}(0,1))$ with $|\det a| = 1$. Observe that $GL_n(\overline{B}(0,1))$ is open and closed as a subset of $M_n(\overline{B}(0,1))$, because (4.12.1) is open and closed in k.

We may consider $GL_n(\overline{B}(0,1))$ as a compact topological group with respect to matrix multiplication, and the topology induced by the one on $M_n(\overline{B}(0,1))$. The restriction of H_n to $GL_n(\overline{B}(0,1))$ is invariant under left and right translations in $GL_n(\overline{B}(0,1))$, as before. It follows that the restriction of H_n to $GL_n(\overline{B}(0,1))$ satisfies the requirements of Haar measure on $GL_n(\overline{B}(0,1))$.

Of course, this is all much simpler when $n = 1$. Clearly (4.12.1) is a subgroup of $k \setminus \{0\}$, as a group with respect to multiplication. More precisely, (4.12.1) is a compact topological group, with respect to the topology induced by the topology determined on k by the ultrametric associated to $|\cdot|$. This is also an open set in k, as before. Let H be Haar measure on k, as a locally compact commutative topological group with respect to addition. The restriction of H to (4.12.1) satisfies the requirements of Haar measure on (4.12.1), as a topological group with respect to multiplication. This uses the fact that H is invariant under multiplication by $a \in k$ with $|a| = 1$, as in the previous section.

4.13 Quotient mappings

Let (X, τ_X) be a topological space, let Y be a set, and let f be a mapping from X onto Y. Under these conditions, the corresponding *quotient topology* on Y is defined by saying that $V \subseteq Y$ is an open set if and only if $f^{-1}(V)$ is an open set in X. It is easy to see that this defines a topology on Y, and that $E \subseteq Y$ is a closed set if and only if $f^{-1}(E)$ is a closed set in X. Note that f

is continuous with respect to the quotient topology on Y, and that Y satisfies the first separation condition with respect to the quotient topology if and only if for every $y \in Y$, $f^{-1}(\{y\})$ is a closed set in X.

Suppose now that (Y, τ_Y) is a topological space. A mapping f from X onto Y is said to be a *quotient mapping* as a mapping between topological spaces if τ_Y is the same as the quotient topology determined by τ_X and f. This means that f is continuous, and that $V \subseteq Y$ is an open set when $f^{-1}(V)$ is an open set in X. This is the same as saying that f is continuous, and that $E \subseteq Y$ is a closed set when $f^{-1}(E)$ is a closed set in X.

Let (Z, τ_Z) be another topological space, and let g be a mapping from Y into Z. If f is a quotient mapping from X onto Y, then it is easy to see that g is continuous as a mapping from Y into Z if and only if $g \circ f$ is continuous as a mapping from X into Z.

Let f be any mapping from X onto Y, and observe that

$$f(f^{-1}(V)) = V \tag{4.13.1}$$

for every $V \subseteq Y$. Suppose that f is also an *open mapping*, so that for each open set $U \subseteq X$, we have that $f(U)$ is an open set in Y. If $V \subseteq Y$ and $f^{-1}(V)$ is an open set in X, then (4.13.1) implies that V is an open set in Y. If f is continuous too, then it follows that f is a quotient mapping from X onto Y.

Similarly, suppose now that f maps closed subsets of X to closed subsets of Y. If $E \subseteq Y$ and $f^{-1}(E)$ is a closed set in X, then E is a closed set in Y, by (4.13.1). This implies that f is a quotient mapping from X onto Y when f is continuous as well.

If f is a continuous mapping from X onto Y, X is compact, and Y is Hausdorff, then f is a quotient mapping. Indeed, if $A \subseteq X$ is a closed set, then A is compact, so that $f(A)$ is compact, and thus closed in Y.

Let Y_0 be a subset of Y, and put $X_0 = f^{-1}(Y_0)$. Note that $f(X_0) = Y_0$, because $f(X) = Y$, by hypothesis. Suppose for the moment that f is a quotient mapping from X onto Y. If Y_0 is an open or closed set in Y, then X_0 has the same property in X, because f is continuous. In both cases, it is easy to see that the restriction of f to X_0 is a quotient mapping from X_0 onto Y_0, with respect to the topologies induced on X_0 and Y_0 by τ_X and τ_Y, respectively.

Observe that

$$f(A \cap X_0) = f(A) \cap Y_0 \tag{4.13.2}$$

for every $A \subseteq X$. If f is an open mapping from X onto Y, then it follows that the restriction of f to X_0 is an open mapping from X_0 onto Y_0, with respect to the induced topologies on X_0 and Y_0. Similarly, if f maps closed subsets of X to closed subsets of Y, then the restriction of f to X_0 has the same property as a mapping from X_0 onto Y_0, with respect to the induced topologies. Of course, if f is an open mapping, then the restriction of f to any open subset of X is an open mapping as well. If f maps closed subsets of X to closed subsets of Y, then the restriction of f to any closed set in X has the same property.

4.14 Quotient mappings and topological groups

Let G be a topological group, and let ϕ be a homomorphism from G onto a group H. If $U \subseteq G$, then

$$\phi^{-1}(\phi(U)) \tag{4.14.1}$$

is the same as the union of the translates of U, on the left or on the right, by elements of the kernel of ϕ. If U is an open set, then it follows that (4.14.1) is an open set as well. This implies that ϕ is an open mapping with respect to the corresponding quotient topology on H.

One can check that H is a topological group with respect to the corresponding quotient topology. Note that H satisfies the first separation condition with respect to the quotient topology exactly when the kernel of ϕ is a closed set. If $\mathcal{B}$ is a local base for the topology of G at the identity element, then it is easy to see that

$$\{\phi(U) : U \in \mathcal{B}\} \tag{4.14.2}$$

is a local base for the quotient topology on H at the identity element.

If the open subgroups of G form a local base for the topology at the identity element, then it follows that the open subgroups of H form a local base for the quotient topology at the identity element. Similarly, if the open normal subgroups of G form a local base for the topology at the identity element, then the open normal subgroups of H form a local base for the quotient topology at the identity element.

Suppose now that G is compact, and that H is a topological group with respect to some topology, where the set containing only the identity element in H is a closed set. If ϕ is a continuous homomorphism from G onto H, then ϕ is a quotient mapping, because H is Hausdorff, as in the previous section. It follows that ϕ is an open mapping under these conditions, as before. If G is profinite, then we get that H is profinite as well, using the remarks in the preceding paragraph.

Alternatively, let U be an open subgroup of G. Thus U is a closed set in G, which implies that U is compact, because G is compact. It follows that $\phi(U)$ is compact in H, so that $\phi(U)$ is closed in H, because H is Hausdorff. Note that U has finite index in G, because G is compact. This implies that $\phi(U)$ has finite index in H, because ϕ maps G onto H. This is another way to get that $\phi(U)$ is an open set in H, because $\phi(U)$ is a closed subgroup of H. If $\mathcal{B}$ is a local base for the topology of G at the identity element consisting of open subgroups, then it follows that (4.14.2) is a local base for the topology of H at the identity element.

Chapter 5

Commutative topological groups

5.1 The unit circle

Let

$$\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\} \tag{5.1.1}$$

be the unit circle in the complex plane $\mathbf{C}$, using the standard absolute value function on $\mathbf{C}$. This is a subgroup of the group $\mathbf{C} \setminus \{0\}$ of nonzero complex numbers, with respect to multiplication. More precisely, $\mathbf{T}$ is a compact topological group, with respect to the topology induced by the standard topology on $\mathbf{C}$.

Of course, the real line $\mathbf{R}$ is a commutative topological group with respect to addition and the standard topology. Using the complex exponential function, we get a continuous group homomorphism

$$x \mapsto \exp(2\,\pi\,i\,x) \tag{5.1.2}$$

from $\mathbf{R}$ onto $\mathbf{T}$. In fact, (5.1.2) is a local homeomorphism, and the kernel of (5.1.2) as a group homomorphism is $\mathbf{Z}$. Thus $\mathbf{T}$ may be identified with $\mathbf{R}/\mathbf{T}$, as a topological group. One can also use this to get Haar measure on $\mathbf{T}$ from one-dimensional Lebesgue measure.

We may consider $\mathbf{Q}$ as a subgroup of $\mathbf{R}$, so that $\mathbf{Q}/\mathbf{Z}$ is a subgroup of $\mathbf{R}/\mathbf{Z}$. This corresponds to the subgroup

$$\{z \in \mathbf{T} : z^n = 1 \text{ for some } n \in \mathbf{Z}_+\} \tag{5.1.3}$$

of $\mathbf{T}$.

Consider

$$\{z \in \mathbf{T} : \operatorname{Re} z > 0\}, \tag{5.1.4}$$

where $\operatorname{Re} z$ denotes the real part of a complex number z. Note that (5.1.4) is a relatively open subset of $\mathbf{T}$ that contains 1. If z is an element of (5.1.4) and

$z \neq 1$, then one can check that there is a positive integer n such that z^n is not contained in (5.1.4). This implies that $\{1\}$ is the only subgroup of $\mathbf{T}$ contained in (5.1.4).

Let A be a group, and let ϕ be a homomorphism from A into $\mathbf{T}$. Suppose that A_0 is a subgroup of A, and that

$$\operatorname{Re} \phi(a_0) > 0 \tag{5.1.5}$$

for every $a_0 \in A_0$. This means that $\phi(A_0)$ is contained in (5.1.4), which implies that

$$\phi(A_0) = \{1\}, \tag{5.1.6}$$

because $\phi(A_0)$ is a subgroup of $\mathbf{T}$. Equivalently,

$$\phi(a_0) = 1 \tag{5.1.7}$$

for every $a_0 \in A_0$.

Let A be a topological group, and suppose that the open subgroups of A form a local base for the topology of A at the identity element. If ϕ is a continuous homomorphism from A into $\mathbf{T}$, then there is an open subgroup A_0 of A such that $\phi(A_0)$ is contained in (5.1.4). This implies that (5.1.7) holds, as before.

If $\alpha \in \mathbf{R}$, then

$$\psi_\alpha(x) = \alpha\, x \tag{5.1.8}$$

is a continuous group homomorphism from $\mathbf{R}$ into itself. One can check that every continuous group homomorphism from $\mathbf{R}$ into itself is of this form for a unique $\alpha \in \mathbf{R}$.

Using the complex exponential function again, we get that

$$\rho_\alpha(x) = \exp(2\,\pi\, i\, \alpha\, x) \tag{5.1.9}$$

is a continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$ for every $\alpha \in \mathbf{R}$. If ρ is any continuous mapping from $\mathbf{R}$ into $\mathbf{T}$ such that

$$\rho(0) = 1, \tag{5.1.10}$$

then it is well known that there is a unique continuous mapping ψ from $\mathbf{R}$ into itself such that

$$\rho(x) = \exp(2\,\pi\, i\, \psi(x)) \tag{5.1.11}$$

for every $x \in \mathbf{R}$, and

$$\psi(0) = 0. \tag{5.1.12}$$

If ρ is a continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$, then it is not too difficult to show that ψ is a continuous group homomorphism from $\mathbf{R}$ into itself. This means that ρ is of the form (5.1.9) for a unique $\alpha \in \mathbf{R}$.

If $j \in \mathbf{Z}$, then

$$\chi_j(z) = z^j \tag{5.1.13}$$

defines a continuous homomorphism from $\mathbf{T}$ into itself. If χ is any continuous homomorphism from $\mathbf{T}$ into itself, then

$$\rho(x) = \chi(\exp(2\,\pi\,i\,x)) \tag{5.1.14}$$

defines a continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$. This is the same as (5.1.9) for a unique $\alpha \in \mathbf{R}$, as before. One can use this to verify that χ is of the form (5.1.13) for a unique $j \in \mathbf{Z}$.

5.2 Dual groups

Let C be a commutative group, and let X be a nonempty set. The set C^X of all mappings from X into C is a commutative group, where the group operations are defined pointwise.

If A is a commutative group too, then let $\mathrm{Hom}(A, C)$ be the collection of group homomorphisms from A into C. It is easy to see that this is a subgroup of C^A.

Let A be a commutative topological group, where the set containing only the identity element is a closed set. Consider the set $\widehat{A}$ of all continuous group homomorphisms from A into $\mathbf{T}$. This is a subgroup of $\mathrm{Hom}(A, \mathbf{T})$. This group is called the *dual group* of A, and may be denoted $\widehat{A}$. Of course, if A is equipped with the discrete topology, then $\widehat{A}$ consists of all group homomorphisms from A into $\mathbf{T}$.

Consider $A = \mathbf{Z}$, as a commutative group with respect to addition. If $c \in C$, then there is a unique group homomorphism ϕ_c from $\mathbf{Z}$ into C such that

$$\phi_c(1) = c. \tag{5.2.1}$$

Of course, if ϕ is any group homomorphism from $\mathbf{Z}$ into C, then $\phi = \phi_c$, with $c = \phi(1)$. This defines an isomorphism between C and $\mathrm{Hom}(\mathbf{Z}, C)$.

Remember that $\mathbf{R}$ is a commutative topological group with respect to addition and the standard topology, and that every continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$ can be expressed as ρ_α in (5.1.9) for a unique $\alpha \in \mathbf{R}$. Observe that

$$\rho_{\alpha+\beta} = \rho_\alpha\,\rho_\beta \tag{5.2.2}$$

on $\mathbf{R}$ for every $\alpha, \beta \in \mathbf{R}$. This means that

$$\alpha \mapsto \rho_\alpha \tag{5.2.3}$$

is a group isomorphism from $\mathbf{R}$ onto its dual group.

Remember that every element of $\widehat{\mathbf{T}}$ can be expressed as χ_j in (5.1.13) for a unique $j \in \mathbf{Z}$. Clearly

$$\chi_{j+l} = \chi_j\,\chi_l \tag{5.2.4}$$

on $\mathbf{T}$ for every $j, l \in \mathbf{Z}$, so that

$$j \mapsto \chi_j \tag{5.2.5}$$

defines a group isomorphism from $\mathbf{Z}$ onto the dual $\widehat{\mathbf{T}}$ of $\mathbf{T}$.

Let A be a commutative group again, where the group operations are expressed additively. As usual, $x \in A$ is said to be a *torsion element* of A if

$$n \cdot x = 0 \tag{5.2.6}$$

for some $n \in \mathbf{Z}_+$. Remember that $n \cdot x$ is the sum of n x's in A. In this case, if ϕ is a homomorphism from A into C, then $\phi(x)$ satisfies the analogous condition in C.

If every element of A is a torsion element, then A is said to be a *torsion commutative group*. If ϕ is a homomorphism from A into C, then it follows that $\phi(A)$ is a torsion subgroup of C. In particular, if ϕ is a homomorphism from A into $\mathbf{T} \cong \mathbf{R}/\mathbf{Z}$, then this implies that $\phi(A)$ is contained in the subgroup (5.1.3) of $\mathbf{T}$ corresponding to $\mathbf{Q}/\mathbf{Z}$.

Suppose now that A is a commutative topological group, where the set containing only the identity element is a closed set, and the open subgroups of A form a local base for the topology of A at the identity element. Let $\phi \in \widehat{A}$ be given, and let A_0 be an open subgroup of A such that (5.1.7) holds. If q_0 is the natural quotient mapping from A onto A/A_0, then it follows that there is a homomorphism ϕ_0 from A/A_0 into $\mathbf{T}$ such that

$$\phi = \phi_0 \circ q_0. \tag{5.2.7}$$

Suppose that A is also totally bounded as a topological group, so that A_0 has finite index in A. In particular, this means that $\phi(A)$ is a finite subgroup of $\mathbf{T}$. This implies that there is a positive integer n such that $\phi(x)^n = 1$ for every $x \in A$, so that

$$\phi^n = 1 \tag{5.2.8}$$

in $\widehat{A}$. Thus every element of $\widehat{A}$ is torsion under these conditions. This is related to Example 4 on p3 of [25].

5.3 Topology on dual groups

Let A be a commutative topological group again, where the set containing only the identity element is a closed set. One often considers the topology on the dual group $\widehat{A}$ that corresponds to uniform convergence on nonempty compact subsets of A. It is easy to see that $\widehat{A}$ is Hausdorff with respect to this topology, and one can check that $\widehat{A}$ is a commutative topoogical group with respect to this topology. Here we shall be primarily concerned with the cases where A is compact, or A is equipped with the discrete topology.

If A is compact, then the topology on $\widehat{A}$ mentioned in the preceding paragraph corresponds to uniform convergence on A, and can be defined using the supremum metric. In fact, this topology reduces to the discrete topology on $\widehat{A}$. More precisely, suppose that $\phi, \psi \in \widehat{A}$ satisfy

$$|\phi(a) - \psi(a)| < 1 \tag{5.3.1}$$

for every $a \in A$. This implies that $\rho = \phi/\psi$ satisfies

$$|\rho(a) - 1| < 1 \tag{5.3.2}$$

for every $a \in A$. In particular, this means that $\rho(A)$ is contained in (5.1.4). Observe that $\rho(A)$ is a subgroup of $\mathbf{T}$, because $\rho \in \widehat{A}$. It follows that $\rho(A) = \{1\}$, as before, so that $\phi = \psi$ on A.

If A is equipped with the discrete topology, then the only compact subsets of A are finite sets. This means that uniform convergence on compact subsets of A is the same as pointwise convergence on A. Let $\mathbf{T}^A$ be the space of all mappings from A into $\mathbf{T}$, as usual. This is the same as the Cartesian product of copies of $\mathbf{T}$ indexed by A. The topology on $\mathbf{T}^A$ that corresponds to pointwise convergence on A is the same as the product topology on this Cartesian product, using the standard topology on $\mathbf{T}$. Of course, $\mathbf{T}^A$ is compact with respect to the product topology, by Tychonoff's theorem. One can check that $\widehat{A}$ is a closed set in $\mathbf{T}^A$ with respect to the product topology, which implies that $\widehat{A}$ is compact in $\mathbf{T}^A$. The topology induced on $\widehat{A}$ by the product topology on $\mathbf{T}^A$ is the same as the topology on $\widehat{A}$ that corresponds to pointwise convergence on A.

Suppose that $A = \mathbf{Z}$, as a commutative group with respect to addition, and equipped with the discrete topology. If $z \in \mathbf{T}$, then

$$\phi_z(j) = z^j \tag{5.3.3}$$

defines a group homomorphism from $\mathbf{Z}$ into $\mathbf{T}$. If $w \in \mathbf{T}$ too, then

$$\phi_{z\,w} = \phi_z\,\phi_w \tag{5.3.4}$$

on $\mathbf{Z}$. It is easy to see that every group homomorphism from $\mathbf{Z}$ into $\mathbf{T}$ is of the form (5.3.3) for a unique $z \in \mathbf{T}$, so that

$$z \mapsto \phi_z \tag{5.3.5}$$

is a group isomorphism from $\mathbf{T}$ onto the dual of $\mathbf{Z}$, as in the previous section. One can check that (5.3.5) is a homeomorphism from $\mathbf{T}$ onto the dual of $\mathbf{Z}$ with respect to the topology on the dual of $\mathbf{Z}$ corresponding to pointwise convergence on $\mathbf{Z}$.

Suppose now that $A = \mathbf{R}$, as a commutative topological group with respect to addition and the standard topology. One can verify that the group isomorphism (5.2.3) from $\mathbf{R}$ onto its dual group is a homeomorphism, with respect to the topology on the dual group that corresponds to uniform convergence on compact subsets of $\mathbf{R}$.

Remember that (5.2.5) is a group isomorphism from $\mathbf{Z}$ onto the dual $\widehat{\mathbf{T}}$ of $\mathbf{T}$. This isomorphism is a homeomorphism with respect to the discrete topology on $\mathbf{Z}$, because the topology on $\widehat{\mathbf{T}}$ corresponding to uniform convergence on compact subsets of $\mathbf{T}$ is the discrete topology, as before.

Let A be a commutative topological group, where the group operations are expressed additively, and where $\{0\}$ is a closed set in A. Suppose for the moment that $x \in A$ is torsion, so that $n \cdot x = 0$ for some positive integer n. If $\phi \in \widehat{A}$, then it follows that

$$\phi(x)^n = 1. \tag{5.3.6}$$

Observe that

$$\{\psi \in \widehat{A} : \psi(x) = 1\} \tag{5.3.7}$$

is a subgroup of $\widehat{A}$. More precisely, this is an open subgroup of $\widehat{A}$ under these conditions, because of (5.3.6). We also get that (5.3.7) has finite index in $\widehat{A}$, by (5.3.6).

Suppose that A is equipped with the discrete topology, and that every element of A is torsion. In this case, the open subgroups of the form (5.3.7) form a local sub-base for the usual topology on $\widehat{A}$ at the identity element. This means that $\widehat{A}$ is profinite under these conditions, because $\widehat{A}$ is compact, as before. This is related to Example 4 on p3 of [25].

5.4 Direct products

Let I be a nonempty set, and let A_j be a commutative group for each $j \in I$. The Cartesian product

$$A = \prod_{j \in I} A_j \tag{5.4.1}$$

is a commutative group as well, where the group operations are defined coordinatewise. This is the *direct product* of the A_j's.

Let C be another commutative group. If $l \in I$ and ϕ_l is a homomorphism from A_l into C, then

$$\widetilde{\phi}_l(x) = \phi_l(x_l) \tag{5.4.2}$$

defines a homomorphism from A into C. Let $l_1, \ldots, l_n$ be finitely many elements of I, and for each $r = 1, \ldots, n$, let ϕ_{l_r} be a homomorphism from A_{l_r} into C. Thus $\widetilde{\phi}_{l_r}$ can be defined as in (5.4.2) for $r = 1, \ldots, n$, and is a homomorphism from A into C. Using the group operation on C, one can combine $\widetilde{\phi}_{l_1}, \ldots, \widetilde{\phi}_{l_n}$ to get another homomorphism from A into C.

If $l \in I$, then there is a natural injective homomorphism ι_l from A_l into A, which sends $x_l \in A_l$ to the element of A whose lth coordinate is equal to x_l, and whose jth coordinate for $j \in I$ with $j \neq l$ is equal to the identity element in A_j. Let ϕ be any homomorphism from A into C, and put

$$\phi_l = \phi \circ \iota_l \tag{5.4.3}$$

for each $l \in I$. This defines a homomorphism from A_l into C, so that $\widetilde{\phi}_l$ defined in (5.4.2) is a homomorphism from A into C. If I has only finitely many elements, then ϕ is the same as the combination of $\widetilde{\phi}_l$ over $l \in I$, using the group operation on C.

Suppose from now on in this section that A_j is a commutative topological group for each $j \in I$, and that the set containing only the identity element is a closed set in A_j for every $j \in I$. This implies that A is a commutative topological group with respect to the product topology, in which the set containing only the identity element is a closed set. We shall also take $C = \mathbf{T}$ in the previous remarks. If $l \in I$ and $\phi_l \in \widehat{A_l}$, then $\widetilde{\phi_l}$ defined in (5.4.2) is an element of $\widehat{A}$. If $l_1, \ldots, l_n$ are finitely many elements of I, and $\phi_{l_r} \in \widehat{A_{l_r}}$ for each $r = 1, \ldots, n$, then $\widetilde{\phi}_{l_r} \in \widehat{A}$ for every $r = 1, \ldots, n$, so that their product

$$\widetilde{\phi}_{l_1}(x) \cdots \widetilde{\phi}_{l_n}(x) \tag{5.4.4}$$

is an element of $\widehat{A}$ too.

Conversely, let $\phi \in \widehat{A}$ be given. If $l \in I$, then the homomorphism ι_l from A_l into A mentioned earlier is continuous. This implies that ϕ_l defined in (5.4.3) is an element of $\widehat{A_l}$ for every $l \in I$. It follows that $\widetilde{\phi_l}$ defined in (5.4.2) is an element of $\widehat{A}$ for every $l \in I$.

Because ϕ is continuous, there is an open subset U of A such that U contains the identity element and

$$\operatorname{Re} \phi(x) > 0 \tag{5.4.5}$$

for every $x \in U$. We may as well take U to be of the form

$$U = \prod_{j \in I} U_j, \tag{5.4.6}$$

where $U_j \subseteq A_j$ is an open set that contains the identity element for every $j \in I$, and $U_j = A_j$ for all but finitely many $j \in I$. If $j \in I$, then put

$$B_j = A_j \quad \text{when } U_j = A_j, \tag{5.4.7}$$

and take B_j to be the subset of A_j that contains only the identity element otherwise. Thus

$$B = \prod_{j \in I} B_j \tag{5.4.8}$$

is a subgroup of A contained in U.

By construction, $\phi(B)$ is contained in (5.1.4). This implies that

$$\phi(B) = \{1\}, \tag{5.4.9}$$

as before. This means that $\phi(x)$ depends on $x_j \in A_j$ for only finitely many $j \in I$. It follows that ϕ can be expressed as the product of $\widetilde{\phi_l}$ for finitely many $l \in I$.

5.5 Direct sums

Let I be a nonempty set again, and let A_j be a commutative group for each $j \in I$. Thus the direct product A of the A_j's can be defined as in the previous section. The *direct sum* of the A_j's is the subgroup

$$A^{\oplus} = \bigoplus_{j \in I} A_j \tag{5.5.1}$$

of A consisting of $x \in A$ such that x_j is the identity element in A_j for all but finitely many $j \in I$. Of course, this is the same as A when I has only finitely many elements.

Let C be a commutative group, and suppose that for each $l \in I$ we have a homomorphism ϕ_l from A_l into C. If $x \in A^{\oplus}$, then $\phi_l(x)$ is the identity element in C for all but finitely many $l \in I$. This permits us to define

$$\phi(x) \in C, \tag{5.5.2}$$

by combining $\phi_l(x)$, $l \in I$, using the group operation on C. It is easy to see that this defines a homomorphism from $A^{\oplus}$ into C.

Conversely, let ϕ be any homomorphism from $A^{\oplus}$ into C. If $l \in I$, then let ι_l be the natural inclusion of A_l into $A^{\oplus}$, as in the previous section, and put $\phi_l = \phi \circ \iota_l$, as before. Thus ϕ_l is a homomorphism from A_l into C for every $l \in I$, and one can check that for every $x \in A^{\oplus}$, $\phi(x)$ is the same as the element of C obtained by combining $\phi_l(x_l)$, $l \in I$.

Let us now consider A_j to be a commutative topological group with respect to the discrete topology for each $j \in I$. Let us take the direct sum $A^{\oplus}$ to be equipped with the discrete topology as well. The remarks in the previous paragraphs show that there is a natural group isomorphism between the dual $\widehat{A^{\oplus}}$ of $A^{\oplus}$ and the direct product

$$\prod_{j \in I} \widehat{A_j} \tag{5.5.3}$$

of the duals of the A_j's.

As in Section 5.3, we can take $\widehat{A^{\oplus}}$ to be equipped with the topology that corresponds to pointwise convergence of homomorphisms on $A^{\oplus}$. One can check that this is the same as the topology that corresponds to pointwise convergence on the subset

$$\bigcup_{l \in I} \iota_l(A_l) \tag{5.5.4}$$

of $A^{\oplus}$, because $A^{\oplus}$ is generated as a group by (5.5.4). Using this, one can verify that the isomorphism between $\widehat{A^{\oplus}}$ and (5.5.3) mentioned in the preceding paragraph is a homeomorphism. This uses the topology on $\widehat{A_j}$ corresponding to pointwise convergence on A_j for each $j \in I$, and the associated product topology on (5.5.3).

Let A_j be any commutative topological group for which the set containing only the identity element is a closed set for each $j \in I$, and let A be the direct product of the A_j's, equipped with the product topology. The remarks in the previous section show that there is a natural group isomorphism between the dual $\widehat{A}$ of A and the direct sum

$$\bigoplus_{j \in I} \widehat{A_j} \tag{5.5.5}$$

of the duals of the A_j's.

Suppose for the moment that A_j is compact for each $j \in I$, so that A is compact too. In this case, we use the discrete topology on $\widehat{A_j}$ for each $j \in I$, and the discrete topology on $\widehat{A}$, as in Section 5.3. Let us take (5.5.5) to be equipped with the discrete topology too, so that the isomorphism mentioned in the preceding paragraph is a homeomorphism.

Suppose now that I has only finitely many elements, so that (5.5.3) and (5.5.5) are the same. Let us take $\widehat{A_j}$ to be equipped with the topology corresponding to uniform convergence on compact subsets of A_j, and similarly for $\widehat{A}$, as before. One can check that this topology on $\widehat{A}$ corresponds exactly to the associated product topology on (5.5.3). This uses the fact that products of compact subsets of the A_j's are compact subsets of A, by Tychonoff's theorem. This also uses the fact that any compact subset of A is contained in the product of its projections in the A_j's, which are compact subsets of the A_j's.

5.6 Separating points

Let X be a set, and let $\mathcal{E}$ be a collection of functions on X. As usual, we say that $\mathcal{E}$ *separates points* in X if for every $x, y \in X$ with $x \neq y$ there is an $f \in \mathcal{E}$ such that $f(x) \neq f(y)$. If X is a group, and $\mathcal{E}$ is a collection of homomorphisms from X into some other groups, then it suffices to check that the identity element in X can be separated from other elements of X in this way. If A is $\mathbf{Z}$, $\mathbf{R}$, or $\mathbf{T}$, as a commutative topological group equipped with its usual topology, then one can check that $\widehat{A}$ separates points in A.

Let n be a positive integer, and consider $A = \mathbf{Z}/n\,\mathbf{Z}$, as a commutative group with respect to addition, and equipped with the discrete topology. Group homomorphisms from $\mathbf{Z}/n\,\mathbf{Z}$ into $\mathbf{T}$ correspond exactly to group homomorphisms from $\mathbf{Z}$ into $\mathbf{T}$ that send n to 1. If $z \in \mathbf{T}$, then $\phi_z(j) = z^j$ has this property exactly when

$$z^n = 1. \tag{5.6.1}$$

This leads to an isomorphism between the subgroup

$$\{z \in \mathbf{T} : z^n = 1\} \tag{5.6.2}$$

of $\mathbf{T}$ and the dual of $\mathbf{Z}/n\,\mathbf{Z}$. In particular, the dual of $\mathbf{Z}/n\,\mathbf{Z}$ separates points in $\mathbf{Z}/n\,\mathbf{Z}$.

Let I be a nonempty set, and let A_j be a commutative topological group for each $j \in I$, where the set containing only the identity element is a closed set. Thus $A = \prod_{j \in I} A_j$ is a commutative topological group with respect to the product topology, and where the set containing only the identity element is a closed set, as before. If $\widehat{A_j}$ separates points in A_j for every $j \in I$, then it is easy to see that $\widehat{A}$ separates points in A. This uses the elements of $\widehat{A}$ associated to elements of $\widehat{A_l}$, $l \in I$, as in (5.4.2).

Suppose for the moment that A is a commutative group with only finitely many elements, equipped with the discrete topology. It is well known that A

is isomorphic to the direct sum of finitely many cyclic groups of finite order. The dual of a cyclic group of finite order is a cyclic group of the same order, as before. The dual of A corresponds to the direct sum of the duals of the cyclic groups whose direct sum is A, as in the previous section. In particular, $\widehat{A}$ separates points in A, as in the preceding paragraph.

Let A be a commutative topological group, where the set containing only the identity element is a closed set. If A_0 is a subgroup of A, then A_0 is a topological group with respect to the induced topology. If $\phi \in \widehat{A}$, then the restriction of ϕ to A_0 is an element of the dual $\widehat{A_0}$ of A_0. In particular, if $\widehat{A}$ separates points in A, then $\widehat{A_0}$ separates points in A_0.

The space $C(A) = C(A, \mathbf{C})$ of all continuous complex-valued functions on A is a commutative algebra over $\mathbf{C}$ with respect to pointwise addition and multiplication of functions. Suppose that B is a subgroup of the dual $\widehat{A}$ of A. Let $\mathcal{E}(B)$ be the linear span of B in $C(A)$, as a vector space over $\mathbf{C}$. It is easy to see that $\mathcal{E}(B)$ is a subalgebra of $C(A)$. If $\phi \in B$, then the complex conjugate $\overline{\phi}$ of ϕ is the same as $1/\phi$, which is an element of B too. This implies that $\mathcal{E}(B)$ is invariant under complex conjugation as well. Of course, $\mathcal{E}(B)$ contains the constant functions on A, because B contains the constant function equal to 1 on A, which is the identity element in $\widehat{A}$.

Suppose that A is compact, so that the elements of $C(A)$ are bounded on A. The topology on $C(A)$ that corresponds to uniform convergence on A is the same as the topology determined by the supremum metric on $C(A)$, which is the metric associated to the supremum norm. If B separates points in A, then $\mathcal{E}(B)$ separates points in A. This implies that $\mathcal{E}(B)$ is dense in $C(A)$ with respect to the supremum metric, by the Stone–Weierstrass theorem.

It is well known that $\widehat{A}$ separates points in A when A is compact. It follows that $\mathcal{E}(\widehat{A})$ is dense in $C(A)$ with respect to the supremum metric, as in the preceding paragraph.

Let us verify that $\widehat{A}$ separates points in A when A is profinite. If $x \in A$ is not the identity element, then there is an open subset U of A that contains x and not the identity element, because A satisfies the first separation condition. We can take U to be an open subgroup of A, because A is profinite. Note that A/U is a finite commutative group in this case. Let q be the natural quotient mapping from A onto A/U, so that $q(x)$ is not the identity element in A/U, by construction. Thus there is a homomorphism ϕ from A/U into $\mathbf{T}$ such that

$$\phi(q(x)) \neq 1, \tag{5.6.3}$$

as before. One can check that $\phi \circ q$ is continuous as a mapping from A into $\mathbf{T}$, because U is an open subgroup of A. This means that $\phi \circ q \in \widehat{A}$, so that $\widehat{A}$ separates points in A.

5.7 Compact commutative groups

Let A be a compact commutative topological group, where the group operations are expressed additively, and $\{0\}$ is a closed set in A. Also let H be a Haar measure on A. If $a \in A$ and $\phi \in \widehat{A}$, then

$$\int_A \phi(x)\,dH(x) = \int_A \phi(x+a)\,dH(x) = \phi(a) \int_A \phi(x)\,dH(x), \tag{5.7.1}$$

using translation-invariance of Haar measure in the first step. It follows that

$$\int_A \phi\,dH = 0 \tag{5.7.2}$$

unless $\phi(a) = 1$ for every $a \in A$.

Let $L^2(A)$ be the usual space of square-integrable complex-valued functions on A with respect to H. If $f, g \in L^2(A)$, then $|f|\,|g|$ is integrable on A with respect to H, and we put

$$\langle f, g \rangle = \langle f, g \rangle_{L^2(A)} = \int_A f\,\overline{g}\,dH. \tag{5.7.3}$$

This defines an inner product on $L^2(A)$, for which the corresponding norm is the usual L^2 norm. It is well known that the space $C(A) = C(A, \mathbf{C})$ of continuous complex-valued functions on A is dense in $L^2(A)$, which is related to the regularity properties of Haar measure.

If $\phi, \psi \in \widehat{A}$, then $\phi\,\overline{\psi} = \phi/\psi \in \widehat{A}$, and

$$\langle \phi, \psi \rangle = 0 \tag{5.7.4}$$

when $\phi \neq \psi$, by (5.7.2). If H is normalized so that

$$H(A) = 1, \tag{5.7.5}$$

then it follows that the elements of $\widehat{A}$ are orthonormal with respect to (5.7.3). It is well known that $\widehat{A}$ separates points in A, so that the linear span of $\widehat{A}$ is dense in $C(A)$ with respect to the supremum metric, as in the previous section. This implies that the linear span of $\widehat{A}$ is dense in $L^2(A)$, because $C(A)$ is dense in $L^2(A)$, as in the preceding paragraph. This means that $\widehat{A}$ is an orthonormal basis for $L^2(A)$.

Let B be a subgroup of $\widehat{A}$, and suppose that B separates points in A. This implies that the linear span of B is dense in $C(A)$ with respect to the supremum metric, as before, and in particular that the linear span of B is dense in $L^2(A)$. If $\phi \in \widehat{A}$ is not in B, then ϕ is orthogonal to every element of B, as in (5.7.4). This would imply that

$$\langle \phi, \phi \rangle = 0, \tag{5.7.6}$$

by approximating ϕ by elements of the linear span of B. This is a contradiction, which means that

$$B = \widehat{A}. \tag{5.7.7}$$

Let A_0 be a closed subgroup of A, so that A_0 is a compact commutative topological group too. If $\phi \in \widehat{A}$, then the restriction of ϕ to A_0 is an element of $\widehat{A_0}$. Let B_0 be the set of restrictions to A_0 of the elements of $\widehat{A}$, which is a subgroup of $\widehat{A_0}$. Using the fact that $\widehat{A}$ separates points in A, we get that B_0 separates points in A_0. This implies that

$$B_0 = \widehat{A_0}, \tag{5.7.8}$$

as in (5.7.7).

5.8 An extension argument

Let C be a commutative group, where the group operations are expressed additively. Suppose that C is *divisible*, in the sense that for every $c \in C$ and positive integer n there is a $c' \in C$ such that

$$n \cdot c' = c. \tag{5.8.1}$$

Note that $\mathbf{Q}/\mathbf{Z}$ and $\mathbf{T}$ are divisible.

Let A be a commutative group, where the group operations are expressed additively, and let B be a subgroup of A. Also let ϕ be a homomorphism from B into C, let $x \in A$ be given, and let B_1 be the subgroup of A generated by B and x. Under these conditions, it is well known that ϕ can be extended to a homomorphism ϕ_1 from B_1 into C.

To see this, suppose first that $n \cdot x \in B$ for some $n \in \mathbf{Z}_+$, and let n_1 be the smallest positive integer with this property. Let us begin by choosing $\phi_1(x) \in C$ so that

$$n_1 \cdot \phi_1(x) = \phi(n_1 \cdot x). \tag{5.8.2}$$

Note that every element of B_1 can be expressed in a unique way as $b + r \cdot x$, where $b \in B$ and $r \in \{0, \ldots, n_1 - 1\}$. In this case, we put

$$\phi_1(b + r \cdot x) = \phi(b) + r \cdot \phi_1(x). \tag{5.8.3}$$

One can check that this holds for every $r \in \mathbf{Z}$, because of (5.8.2), and hence that ϕ_1 defines a homomorphism from B_1 into C.

Suppose now that $n \cdot x$ is not an element of B for any $n \in \mathbf{Z}_+$, so that every element of B_1 can be expressed in a unique way as $b + r \cdot x$ for some $r \in \mathbf{Z}$. In this case, we can take $\phi_1(x)$ to be any element of C, and define ϕ_1 on B_1 as in (5.8.3) for every $b \in B$ and $r \in \mathbf{Z}$. One can verify that this defines a homomorphism from B_1 into C, as desired.

If A is generated by B and only finitely or countably many elements of A, then one can repeat the process to extend ϕ to a homomorphism from A into C. Otherwise, one can get such an extension using Zorn's lemma or Hausdorff's maximality principle.

Let a_0 be an element of A, and let B_0 be the subgroup of A generated by a_0. We can get homomorphisms ϕ_0 from B_0 into C in the same way as before. More precisely, if $n \cdot a_0 = 0$ for some $n \in \mathbf{Z}_+$, then let n_0 be the smallest positive integer with this property. If we take $\phi_0(a_0)$ to be an element of C that satisfies

$$n_0 \cdot \phi_0(a_0) = 0, \tag{5.8.4}$$

then we can put

$$\phi_0(r \cdot a_0) = r \cdot \phi(a_0) \tag{5.8.5}$$

when $r = 0, \ldots, n_0 - 1$. Otherwise, if $n \cdot a_0 \neq 0$ for every $n \in \mathbf{Z}_+$, then we can take $\phi_0(a_0)$ to be any element of C, and define ϕ_0 on B_0 as in (5.8.5) for every $r \in \mathbf{Z}$.

If $C \neq \{0\}$, then we can take $\phi_0(a_0) \neq 0$ in the second case. In the first case, if $a_0 \neq 0$, then we can choose $\phi_0(a_0) \neq 0$ when $C = \mathbf{Q}/\mathbf{Z}$ or $\mathbf{T}$. In both cases, we can extend ϕ_0 to a homomorphism from A into C, as before.

5.9 Extensions and open subgroups

Let A be a commutative topological group, and suppose for the moment that B is an open subgroup of A. Let ϕ be a continuous homomorphism from B into $\mathbf{T}$, with respect to the induced topology on B. Under these conditions, any extension of ϕ to a homomorphism from a subgroup of A that contains B into $\mathbf{T}$ is continuous, with respect to the induced topology on the domain.

Let A_0 be an open subgroup of A, and suppose that $a \in A$ is not an element of A_0. If q_0 is the natural quotient homomorphism from A onto A/A_0, then $q_0(a) \neq 0$. It follows that there is a homomorphism ψ_0 from A/A_0 into $\mathbf{T}$ such that

$$\psi_0(q_0(a)) \neq 1, \tag{5.9.1}$$

as in the previous section. Put

$$\psi = \psi_0 \circ q_0, \tag{5.9.2}$$

which defines a homomorphism from A into $\mathbf{T}$. Clearly ψ is continuous, because A_0 is an open subgroup of A.

Suppose now that the set containing only the identity element in A is a closed set, and that the open subgroups of A form a local base for the topology of A at the identity element. In this case, the argument in the preceding paragraph implies that $\widehat{A}$ separates points in A. This was mentioned earlier when A is profinite.

Let B_1 be a subgroup of A, considered as a topological group with respect to the induced topology, and let $\phi_1 \in \widehat{B_1}$ be given. We would like to show that ϕ_1 can be extended to an element of $\widehat{A}$. Observe first that there is an open subgroup A_1 of A such that

$$\phi_1(A_1 \cap B_1) = \{1\}, \tag{5.9.3}$$

as in Section 5.1. This uses the fact that the intersections of B_1 with open subgroups of A form a local base for the induced topology on B_1 at the identity element.

Put

$$B_2 = A_1 + B_1 \tag{5.9.4}$$

which is an open subgroup of A. We would like to extend ϕ_1 to a homomorphism ϕ_2 from B_2 into $\mathbf{T}$, by putting

$$\phi_2(a_1 + b_1) = \phi_1(b_1) \tag{5.9.5}$$

for every $a_1 \in A_1$ and $b_1 \in B_1$. It is easy to see that this is well defined, using (5.9.3). By construction, $\phi_2(A_1) = \{1\}$, which implies that ϕ_2 is continuous on B_2, with respect to the induced topology. We can extend ϕ_2 to a homomorphism from A into $\mathbf{T}$ as in the previous section, and it follows that this extension is continuous as well.

5.10 Dual homomorphisms

Let A and B be commutative topological groups, where the subsets of A, B containing only the identity element are closed sets, and let h be a continuous homomorphism from A into B. If ϕ is a continuous homomorphism from B into $\mathbf{T}$, then

$$\widehat{h}(\phi) = \phi \circ h \tag{5.10.1}$$

is a continuous homomorphism from A into $\mathbf{T}$. This defines $\widehat{h}$ as a mapping from $\widehat{B}$ into $\widehat{A}$, and it is easy to see that $\widehat{h}$ is a group homomorphism. This is the *dual homomorphism* associated to h. One can check that $\widehat{h}$ is continuous with respect to the usual topologies on $\widehat{A}$, $\widehat{B}$ corresponding to uniform convergence on compact subsets of A, B, respectively, because h maps compact subsets of A to compact subsets of B.

Let C be another commutative topological group, where the set containing only the identity element is a closed set, and let h' be a continuous homomorphism from B into C. Thus the composition $h' \circ h$ of h and h' is a continuous homomorphism from A into C. If $\psi \in \widehat{C}$, then

$$(\widehat{h' \circ h})(\psi) = \psi \circ h' \circ h = \widehat{h}((\widehat{h'})(\psi)). \tag{5.10.2}$$

This means that

$$(\widehat{h' \circ h}) = \widehat{h} \circ \widehat{h'}, \tag{5.10.3}$$

as mappings from $\widehat{C}$ into $\widehat{A}$. If h is a group isomorphism from A onto B that is also a homeomorphism, then $\widehat{h}$ is a group isomorphism from $\widehat{B}$ onto $\widehat{A}$ that is a homeomorphism too, with

$$(\widehat{h})^{-1} = (\widehat{h^{-1}}) \tag{5.10.4}$$

as mappings from $\widehat{A}$ onto $\widehat{B}$.

Suppose for the moment that A and B are equipped with the discrete topology, and that h is injective. If $\psi \in \widehat{A}$, then

$$\phi_0(h(x)) = \psi(x) \tag{5.10.5}$$

defines a homomorphism from $h(A)$ into $\mathbf{T}$. This can be extended to a group homomorphism ϕ from B into $\mathbf{T}$, as in Section 5.8. By construction,

$$\widehat{h}(\phi) = \phi \circ h = \psi. \tag{5.10.6}$$

Thus $\widehat{h}$ maps $\widehat{B}$ onto $\widehat{A}$ under these conditions.

Suppose for the moment now that A and B are compact, and that h is injective. In this case, h defines a homeomorphism from A onto $h(A)$ with respect to the topology induced on $h(A)$ by the topology on B, because A is compact, B is Hausdorff, and thus $h(A)$ is Hausdorff with respect to the induced topology. If $\psi \in \widehat{A}$, then (5.10.5) defines ϕ_0 as a continuous homomorphism from $h(A)$ into $\mathbf{T}$, with respect to the induced topology on $h(A)$. It follows that ϕ_0 can be extended to a continuous group homomorphism ϕ from B into $\mathbf{T}$, as in Section 5.7. If B is profinite, then this can be obtained as in the previous section. Of course, (5.10.6) holds by construction, as before. This shows that $\widehat{h}$ maps $\widehat{B}$ onto $\widehat{A}$ under these conditions as well.

If h is any continuous homomorphism from A into B, then the kernel of $\widehat{h}$ consists of the $\phi \in \widehat{B}$ such that $h(A)$ is contained in the kernel of ϕ. Of course, if $\phi \in \widehat{B}$, then the kernel of ϕ is a closed set in B, because ϕ is continuous. This means that the kernel of $\widehat{h}$ consists of the $\phi \in \widehat{B}$ whose kernel contains the closure $\overline{h(A)}$ of $h(A)$ in B. If $h(A)$ is a dense subset of B, then $\widehat{h}$ is injective.

If h is any continuous homomorphism from A into B and $\phi \in \widehat{B}$, then the kernel of $\widehat{h}(\phi)$ contains the kernel of $\widehat{h}$ in A. Suppose that h maps A onto B, and that the kernel of $\psi \in \widehat{A}$ contains the kernel of h. This permits one to define ϕ_0 as a homomorphism from B into $\mathbf{T}$ as in (5.10.5). Of course, if B is equipped with the discrete topology, then ϕ_0 is automatically continuous.

Suppose that A is compact, and let us verify that ϕ_0 is continuous on B. If E is a closed subset of $\mathbf{T}$, then it suffices to check that $\phi_0^{-1}(E)$ is a closed set in B. Observe that

$$\phi_0^{-1}(E) = h(\psi^{-1}(E)), \tag{5.10.7}$$

by (5.10.5). Because ψ is continuous, $\psi^{-1}(E)$ is a closed set in A. This implies that $\psi^{-1}(E)$ is compact, because A is compact. It follows that $h(\psi^{-1}(E))$ is compact, because h is continuous. This means that $h(\psi^{-1}(E))$ is a closed set in B, because B is Hausdorff. The continuity of ϕ_0 could also be obtained from the remarks in Section 4.13.

5.11 Second duals

Let A be a commutative topological group, where the set containing only the identity element is a closed set. Remember that the dual group $\widehat{A}$ is a Hausdorff commutative topological group, with respect to the topology that corresponds to uniform convergence on compact subsets of A, as in Section 5.3. Thus the dual $\widehat{\widehat{A}}$ of $\widehat{A}$ can be defined in the same way as before.

If $a \in A$, then let E_a be the mapping from $\widehat{A}$ into $\mathbf{T}$ defined by

$$E_a(\phi) = \phi(a) \tag{5.11.1}$$

for every $\phi \in \widehat{A}$. This defines a group homomorphism from $\widehat{A}$ into $\mathbf{T}$. Clearly E_a is continuous on $\widehat{A}$ with respect to the topology that corresponds to uniform convergence on compact subsets of A, because $\{a\}$ is a compact subset of A. This means that

$$E_a \in \widehat{\widehat{A}} \tag{5.11.2}$$

for every $a \in A$.

If $\phi \in \widehat{A}$, then (5.11.1) is a homomorphism from A into $\mathbf{T}$, as a function of a. This means that

$$a \mapsto E_a \tag{5.11.3}$$

defines a group homomorphism from A into $\widehat{\widehat{A}}$. This mapping is continuous under suitable conditions, with respect to the same type of topology on $\widehat{\widehat{A}}$ as before. In particular, if A is equipped with the discrete topology, then $\widehat{A}$ is compact, and we get the discrete topology on $\widehat{\widehat{A}}$.

If A is compact, then we get the discrete topology on $\widehat{A}$, as in Section 5.3. This means that the topology on $\widehat{\widehat{A}}$ that corresponds to uniform convergence on nonempty compact subsets of $\widehat{A}$ is the same as the topology that corresponds to pointwise convergence on $\widehat{A}$. Of course, if $\phi \in \widehat{A}$, then (5.11.1) is continuous as a function of $a \in A$ with values in $\mathbf{T}$. This implies that (5.11.3) is continuous on A with respect to the topology on $\widehat{\widehat{A}}$ that corresponds to pointwise convergence on $\widehat{A}$.

Note that (5.11.3) is injective as a mapping from A into $\widehat{\widehat{A}}$ exactly when $\widehat{A}$ separates points in A. It is easy to see that

$$\{E_a : a \in A\} \tag{5.11.4}$$

automatically separates points in $\widehat{A}$. If A is equipped with the discrete topology, then $\widehat{A}$ is compact, and it follows that (5.11.4) is $\widehat{\widehat{A}}$, as in Section 5.7. In this case, $\widehat{A}$ separates points in A, as in Section 5.8. Thus (5.11.3) is a group isomorphism from A onto $\widehat{\widehat{A}}$ when A is equipped with the discrete topology.

If A is compact, then (5.11.4) is a compact subset of $\widehat{\widehat{A}}$, because (5.11.3) is continuous, as before. Remember that $\widehat{A}$ separates points in A in this case, so that (5.11.3) is injective. It follows that (5.11.3) is a homeomorphism from A onto (5.11.4) under these conditions, with respect to the topology induced on (5.11.4) by the usual topology on $\widehat{\widehat{A}}$. This uses the fact that $\widehat{\widehat{A}}$ is Hausdorff with respect to this topology, so that (5.11.4) is Hausdorff with respect to the induced topology. More precisely, this also uses the well-known fact that a one-to-one continuous mapping from a compact topological space onto a Hausdorff space is a homeomorphism.

It is well known that (5.11.4) is equal to $\widehat{\widehat{A}}$ when A is compact. We can show this more directly when A is profinite, as in the next section.

5.12 Another dual

Let A be a commutative topological group again, where the set containing only the identity element is a closed set. If $\phi \in \widehat{A}$ and $\Psi \in \widehat{\widehat{A}}$, then put

$$\mathcal{E}_\phi(\Psi) = \Psi(\phi). \tag{5.12.1}$$

This defines a continuous group homomorphism from $\widehat{\widehat{A}}$ into $\mathbf{T}$, so that

$$\mathcal{E}_\phi \in \widehat{\widehat{\widehat{A}}} \tag{5.12.2}$$

for every $\phi \in \widehat{A}$. We also have that

$$\phi \mapsto \mathcal{E}_\phi \tag{5.12.3}$$

is a group homomorphism from $\widehat{A}$ into $\widehat{\widehat{\widehat{A}}}$, as before.

If A is compact, then $\widehat{A}$ is equipped with the discrete topology, as in Section 5.3. This implies that (5.12.3) is a group isomorphism from $\widehat{A}$ onto $\widehat{\widehat{\widehat{A}}}$, as in the previous section. In particular, every element of $\widehat{\widehat{\widehat{A}}}$ is of the form (5.12.1) for some $\phi \in \widehat{A}$ in this case.

Suppose that A is profinite, so that $\widehat{A}$ is torsion, as in Section 5.2. This implies that $\widehat{\widehat{A}}$ is profinite, as in Section 5.3.

Let E be the subgroup (5.11.4) of $\widehat{\widehat{A}}$, which we would like to show is $\widehat{\widehat{A}}$. Note that E is a closed set in $\widehat{\widehat{A}}$, because E is compact, as in the previous section, and $\widehat{\widehat{A}}$ is Hausdorff. Thus it suffices to show that E is dense in $\widehat{\widehat{A}}$.

Let U be an open subset of $\widehat{\widehat{A}}$ that contains the identity element. We would like to verify that

$$E + U = \widehat{\widehat{A}}. \tag{5.12.4}$$

This will imply that E is dense in $\widehat{\widehat{A}}$, by the characterization of the closure of E in Section 2.4.

We may as well take U to be an open subgroup of $\widehat{\widehat{A}}$, because $\widehat{\widehat{A}}$ is profinite, as before. This means that $E + U$ is an open subgroup of $\widehat{\widehat{A}}$ too.

We would like to show that the quotient of $\widehat{\widehat{A}}$ by $E+U$ has only one element. Otherwise, there is a nontrivial homomorphism from the quotient into $\mathbf{T}$, as in Section 5.8.

The composition of this homomorphism with the natural quotient mapping from $\widehat{\widehat{A}}$ onto its quotient by $E+U$ is a nontrivial homomorphism from $\widehat{\widehat{A}}$ into $\mathbf{T}$ whose kernel contains $E + U$. In particular, this homomorphism is continuous, because $E + U$ is an open subgroup of $\widehat{\widehat{A}}$.

Thus this homomorphism from $\widehat{\widehat{A}}$ into $\mathbf{T}$ is an element of $\widehat{\widehat{\widehat{A}}}$. This means that this homomorphism can be expressed as (5.12.1) for some $\phi \in \widehat{A}$, as before.

If $a \in A$, then E_a defined as in (5.11.1) is an element of $\widehat{\widehat{A}}$, and

$$\mathcal{E}_\phi(E_a) = E_a(\phi) = \phi(a). \tag{5.12.5}$$

This is equal to 1 for every $a \in A$, because E is contained in the kernel of $\mathcal{E}_\phi$. This means that ϕ is the identity element in $\widehat{A}$. It follows that $\mathcal{E}_\phi$ is the trivial homomorphism from $\widehat{\widehat{A}}$ into $\mathbf{T}$. This is a contradiction, because this homomorphism is supposed to be nontrivial on $\widehat{\widehat{A}}$.

5.13 Torsion-free groups

Let A be a commutative group, where the group operations are expressed additively. If n is a positive integer, then

$$\nu_n(x) = n \cdot x \tag{5.13.1}$$

is a homomorphism from A into itself. If for every $x \in A$ with $x \neq 0$ and every $n \in \mathbf{Z}_+$ we have that $n \cdot x \neq 0$, then A is said to be *torsion free*. Equivalently, this means that ν_n is injective on A for every $n \in \mathbf{Z}_+$. Note that A is divisible exactly when ν_n maps A onto itself for every $n \in \mathbf{Z}_+$.

Suppose now that A is a commutative topological group, and that $\{0\}$ is a closed set in A. If $n \in \mathbf{Z}_+$, then ν_n is continuous as a mapping from A into itself, by continuity of addition on A. Thus the dual homomorphism $\widehat{\nu_n}$ is defined from $\widehat{A}$ into itself as in Section 5.10. More precisely, if $\phi \in \widehat{A}$ and $x \in A$, then

$$(\widehat{\nu_n}(\phi))(x) = \phi(\nu_n(x)) = \phi(n \cdot x) = \phi(x)^n. \tag{5.13.2}$$

Equivalently,

$$\widehat{\nu_n}(\phi) = \phi^n \tag{5.13.3}$$

for every $\phi \in \widehat{A}$.

If $\nu_n(A)$ is dense in A, then $\widehat{\nu_n}$ is injective on $\widehat{A}$, as in Section 5.10. This means that if $\phi \in \widehat{A}$ is not the identity element, then ϕ^n is not the identity element in $\widehat{A}$. In particular, if A is divisible, then $\widehat{A}$ is torsion free.

Suppose that ν_n is injective on A for some $n \in \mathbf{Z}_+$. If A is equipped with the discrete topology, or A is compact, then $\widehat{\nu_n}$ maps $\widehat{A}$ onto itself, as in Section 5.10. In particular, if A is torsion free, and A is either equipped with the discrete topology or A is compact, then $\widehat{A}$ is divisible.

5.14 Second duals of discrete groups

Let A be a commutative group, with the group operations expressed additively, and equipped with the discrete topology. Also let Ψ be an element of the dual

$\widehat{\widehat{A}}$ of the dual $\widehat{A}$ of A. Thus Ψ is a continuous homomorphism from $\widehat{A}$ into $\mathbf{T}$, with respect to the topology on $\widehat{A}$ that corresponds to pointwise convergence of homomorphisms from A into $\mathbf{T}$.

In particular, the set of $\phi \in \widehat{A}$ such that

$$\operatorname{Re}\Psi(\phi) > 0 \tag{5.14.1}$$

is an open subset of $\widehat{A}$ that contains the identity element. This implies that there are finitely many elements $x_1, \ldots, x_n$ of A such that (5.14.1) holds when $\phi(x_j)$ is sufficiently close to 1 for each $j = 1, \ldots, n$.

Put

$$B = \{\phi \in \widehat{A} : \phi(x_j) = 1 \text{ for each } j = 1, \ldots, n\}, \tag{5.14.2}$$

which is a closed subgroup of $\widehat{A}$. If $\phi \in B$, then (5.14.1) holds, as before. It follows that

$$\Psi(B) = \{1\}, \tag{5.14.3}$$

because $\Psi(B)$ is a subgroup of $\mathbf{T}$, as in Section 5.1.

Let A_1 be the subgroup of A generated by $x_1, \ldots, x_n$. This may be considered as a commutative topological group with respect to the discrete topology, which is the same as the topology induced on A_1 by the discrete topology on A. Let ι_1 be the natural inclusion mapping from A_1 into A, which is a continuous group homomorphism. The dual homomorphism $\widehat{\iota_1}$ from $\widehat{A}$ into $\widehat{A_1}$ is defined by

$$\widehat{\iota_1}(\phi) = \phi \circ \iota_1 \tag{5.14.4}$$

for every $\phi \in \widehat{A}$, as in Section 5.10. Observe that

$$B = \{\phi \in \widehat{A} : \phi(x) = 1 \text{ for every } x \in A_1\}, \tag{5.14.5}$$

which is the same as the kernel of $\widehat{\iota_1}$.

If $\phi_1 \in \widehat{A_1}$, then there is a $\widetilde{\phi}_1 \in \widehat{A}$ such that $\widetilde{\phi}_1 = \phi_1$ on A_1, as in Section 5.8. We would like to put

$$\Psi_1(\phi_1) = \Psi(\widetilde{\phi}_1). \tag{5.14.6}$$

This does not depend on the choice of $\widetilde{\phi}_1$, because of (5.14.3). It is easy to see that this defines Ψ_1 as a homomorphism from $\widehat{A_1}$ into $\mathbf{T}$. Equivalently, $\widehat{\iota_1}$ maps $\widehat{A}$ onto $\widehat{A_1}$, and

$$\Psi_1(\widehat{\iota_1}(\phi)) = \Psi(\phi) \tag{5.14.7}$$

for every $\phi \in \widehat{A}$.

In fact, Ψ_1 is continuous on $\widehat{A_1}$, so that

$$\Psi_1 \in \widehat{\widehat{A_1}}. \tag{5.14.8}$$

This can be seen as in Section 5.10, using the compactness of $\widehat{A}$.

Of course, A_1 is finitely generated as a commutative group, by construction. This implies that A_1 is isomorphic to the direct sum of finitely many cyclic groups. It follows that the dual of A_1 is isomorphic to the direct sum of the duals of the corresponding cyclic groups. More precisely, this isomorphism is also a homeomorphism with respect to the appropriate topologies, as in Section 5.5. Similarly, the dual of $\widehat{A_1}$ is isomorphic to the direct sum of the duals of the duals of the cyclic groups that make up A_1.

In this case, one can check directly that there is an $a \in A_1$ such that

$$\Psi_1(\phi_1) = \phi_1(a) \tag{5.14.9}$$

for every $\phi_1 \in \widehat{A_1}$. This reduces to the analogous statement for cyclic groups. It follows that

$$\Psi(\phi) = \phi(a) \tag{5.14.10}$$

for every $\phi \in \widehat{A}$. This was mentioned in Section 5.11, using another argument.

5.15 Topological vector spaces

A vector space V over the real numbers is said to be a *topological vector space* is V is equipped with a topology for which the vector space operations are continuous. More precisely, this means that addition on V is continuous as a mapping from $V \times V$ into V, using the associated product topology on $V \times V$. Similarly, scalar multiplication on V should be continuous as a mapping from $\mathbf{R} \times V$ into V, using the product topology on $\mathbf{R} \times V$ corresponding to the standard topology on $\mathbf{R}$ and the given topology on V. One may also ask that $\{0\}$ be a closed set in V, as usual.

Of course, one can consider analogous notions for vector spaces over other fields. We would like to consider vector spaces over $\mathbf{R}$ for the moment, in connection with duals of commutative topological groups.

If V is a topological vector space over $\mathbf{R}$, then continuity of scalar multiplication implies in particular that for every $t \in \mathbf{R}$,

$$v \mapsto t\,v \tag{5.15.1}$$

is continuous as a mapping from V into itself. If $t \neq 0$, then it follows that (5.15.1) is a homeomorphism on V. In particular, continuity of addition on V and continuity of (5.15.1) with $t = -1$ implies that V is a topological group with respect to addition. If $\{0\}$ is a closed set in V, then V is Hausdorff, as before.

Let V' be the *dual space* of continuous linear functionals on V, which is to say continuous linear mappings from V into $\mathbf{R}$. It is easy to see that V' is a vector space over the real numbers, with respect to pointwise addition and scalar multiplication of linear functionals. If $\lambda \in V'$, then

$$\phi_\lambda(v) = \exp(2\,\pi\,i\,\lambda(v)) \tag{5.15.2}$$

is a continuous mapping from V into $\mathbf{T}$ that is a group homomorphism with respect to addition on V. In fact,

$$\lambda \mapsto \phi_\lambda \tag{5.15.3}$$

is a group homomorphism from V' into the dual group of continuous group homomorphisms from V into $\mathbf{T}$, with respect to addition on V'.

Observe that (5.15.3) is injective on V'. One can show that every continuous group homomorphism from V into $\mathbf{T}$ is of the form (5.15.2) for some $\lambda \in V'$. This uses the characterization of continuous group homomorphisms from $\mathbf{R}$ into $\mathbf{T}$ mentioned in Section 5.1.

A set $E \subseteq V$ with $0 \in E$ is said to be *starlike about* 0 if for every $v \in E$ and $t \in \mathbf{R}$ with $0 \leq t \leq 1$, we have that

$$t\,v \in E. \tag{5.15.4}$$

It is well known that the open subsets of V that contain 0 and are starlike about 0 form a local base for the topology of V at 0. One can use this to show that if (5.15.2) is continuous on V, then λ is continuous on V.

If the convex open subsets of V form a base for the topology of V, then V is said to be *locally convex*. It is well known that this implies that V' separates points in V, because of the Hahn–Banach theorem.

Chapter 6

Direct and inverse limits

6.1 Direct systems of commutative groups

Let I be a nonempty set with a partial ordering $\preceq$. Suppose that $(I, \preceq)$ is a *directed set* or *directed system*, so that for every $j, l \in I$ there is an $r \in I$ with $j, l \preceq r$.

Let A_j be a commutative group for every $j \in I$, where the group operations are expressed additively. Suppose that for every $j, l \in I$ with $j \preceq l$ we have a homomorphism $\alpha_{j,l}$ from A_j into A_l that satisfies the following two properties. First, $\alpha_{j,j}$ is the identity mapping on A_j for every $j \in I$. Second, if $j, l, r \in I$ satisfy $j \preceq l \preceq r$, then

$$\alpha_{l,r} \circ \alpha_{j,l} = \alpha_{j,r}. \tag{6.1.1}$$

Under these conditions, the family of commutative groups A_j and homomorphisms $\alpha_{j,l}$ is said to form a *direct* or *inductive system* over $(I, \preceq)$.

Remember that the direct sum $\bigoplus_{j \in I} A_j$ of the A_j's is a commutative group, as in Section 5.5. If $l \in I$, then let ι_l be the natural injection from A_l into $\bigoplus_{j \in I} A_j$, so that for each $x_l \in A_l$ the jth coordinate of $\iota_l(x_l)$ is equal to x_l when $j = l$, and to 0 when $j \neq l$, as before. Let B be the subset of $\bigoplus_{j \in I} A_j$ consisting of finite sums of elements of the form

$$\iota_l(x_l) - \iota_r(\alpha_{l,r}(x_l)), \tag{6.1.2}$$

where $l, r \in I$, $l \preceq r$, and $x_l \in A_l$. Thus B is a subgroup of $\bigoplus_{j \in I} A_j$, and we put

$$\varinjlim A_j = \Big(\bigoplus_{j \in I} A_j\Big)/B, \tag{6.1.3}$$

which is a commutative group too. This is the *direct* or *inductive limit* of the direct system of A_j's, $j \in I$.

Let q be the natural quotient mapping from $\bigoplus_{j \in I} A_j$ onto (6.1.3). If $l \in I$, then put

$$\beta_l = q \circ \iota_l, \tag{6.1.4}$$

which is a homomorphism from A_l into (6.1.3). If $r \in I$ and $l \preceq r$, then

$$\beta_l = \beta_r \circ \alpha_{l,r}, \tag{6.1.5}$$

by construction. More precisely, the direct limit consists of the group (6.1.3) together with the homomorphisms β_l, as in Exercise 14 on p32f of [1].

One can check that every element of (6.1.3) may be expressed as $\beta_l(x_l)$ for some $l \in I$ and $x_l \in A_l$, using the fact that $(I, \preceq)$ is a directed system. If $\beta_l(x_l) = 0$, then one can verify that

$$\alpha_{l,r}(x_l) = 0 \tag{6.1.6}$$

for some $r \in I$ with $l \preceq r$. This corresponds to Exercise 15 on p33 of [1].

Let C be another commutative group, and suppose that for each $l \in I$ we have a homomorphism γ_l from A_l into C. If $r \in I$ and $l \preceq r$, then suppose also that

$$\gamma_l = \gamma_r \circ \alpha_{l,r}. \tag{6.1.7}$$

Under these conditions, one can check that there is a unique homomorphism γ from (6.1.3) into C such that

$$\gamma \circ \beta_l = \gamma_l \tag{6.1.8}$$

for every $l \in I$. One can verify that the direct limit is uniquely determined up to isomorphism by this property as well. This corresponds to Exercise 16 on p33 of [1].

Let A be a commutative group, and suppose for the moment that A_j is a subgroup of A for every $j \in I$. If $l, r \in I$ and $l \preceq r$, then suppose that

$$A_l \subseteq A_r. \tag{6.1.9}$$

It is easy to see that this implies that

$$\bigcup_{j \in I} A_j \tag{6.1.10}$$

is a subgroup of A, because $(I, \preceq)$ is a directed set. In this case, we can get a direct system by taking $\alpha_{j,l}$ to be the natural inclusion mapping from A_j into A_l when $j, l \in I$ and $j \preceq l$. One can check that (6.1.10) is isomorphic to the direct limit, as in Exercise 17 on p33 of [1].

6.2 Direct systems of sets

Let $(I, \preceq)$ be a nonempty directed set again, and let A_j be a set for each $j \in I$. Suppose that for every $j, l \in I$ with $j \preceq l$ we have a mapping $\alpha_{j,l}$ from A_j into A_l. As before, we ask that $\alpha_{j,j}$ be the identity mapping on A_j for every $j \in I$, and that

$$\alpha_{l,r} \circ \alpha_{j,l} = \alpha_{j,r} \tag{6.2.1}$$

for every $j, l, r \in I$ such that $j \preceq l \preceq r$. The family of sets A_j and mappings $\alpha_{j,l}$ may be called a *direct* or *inductive system* over $(I, \preceq)$.

Let $\widetilde{A}_j$ be a nonempty set that contains A_j for each $j \in I$. If $x, y \in \prod_{j \in I} \widetilde{A}_j$, then put $x \sim y$ when there is an $l \in I$ such that

$$x_j = y_j \tag{6.2.2}$$

for every $j \in I$ with $l \preceq j$. It is easy to see that this defines an equivalence relation on $\prod_{j \in I} \widetilde{A}_j$, because $(I, \preceq)$ is a directed set. Thus the quotient

$$\Big(\prod_{j \in I} \widetilde{A}_j\Big) / \sim \tag{6.2.3}$$

may be defined as usual as the set of equivalence classes in $\prod_{j \in I} \widetilde{A}_j$ associated to this equivalence relation.

Let $l \in I$ and $a_l \in A_l$ be given. Observe that there are $x \in \prod_{j \in I} \widetilde{A}_j$ such that

$$x_j = \alpha_{l,j}(a_l) \tag{6.2.4}$$

for every $j \in I$ with $l \preceq j$, and that any two such elements of $\prod_{j \in I} \widetilde{A}_j$ are equivalent with respect to $\sim$. This defines a mapping β_l from A_l into (6.2.3). If $r \in I$ and $l \preceq r$, then

$$\beta_l = \beta_r \circ \alpha_{l,r}, \tag{6.2.5}$$

by construction.

Put

$$\lim_{\longrightarrow} A_j = \bigcup_{l \in I} \beta_l(A_l), \tag{6.2.6}$$

which is a subset of (6.2.3). This may be considered as the *direct* or *inductive limit* of the direct system of A_j's, $j \in I$. More precisely, the direct limit consists of this set together with the mappings β_l from A_l into this set for each $l \in I$. Note that

$$\beta_l(A_l) \subseteq \beta_r(A_r) \tag{6.2.7}$$

when $l, r \in I$ and $l \preceq r$, by (6.2.5).

Let $l_1, l_2 \in I$, $a_{1,l_1} \in A_{l_1}$, and $a_{2,l_2} \in A_{l_2}$ be given. If

$$\beta_{l_1}(a_{1,l_1}) = \beta_{l_2}(a_{2,l_2}), \tag{6.2.8}$$

then there is an $r \in I$ such that

$$l_1, l_2 \preceq r \tag{6.2.9}$$

and

$$\alpha_{l_1,r}(a_{1,l_1}) = \alpha_{l_2,r}(a_{2,l_2}), \tag{6.2.10}$$

because $(I, \preceq)$ is a directed set. Conversely, if there is an $r \in I$ such that (6.2.9) and (6.2.10) hold, then

$$\alpha_{l_1,t}(a_{1,l_1}) = \alpha_{l_2,t}(a_{2,l_2}) \tag{6.2.11}$$

for every $t \in I$ with $r \preceq t$, by (6.2.1). This implies that (6.2.8) holds.

Let C be another set, and suppose that for every $l \in I$ we have a mapping γ_l from A_l into C. Suppose also that for every $l, r \in I$ with $l \preceq r$ we have that

$$\gamma_l = \gamma_r \circ \alpha_{l,r}. \tag{6.2.12}$$

It is easy to see that there is a unique mapping γ from (6.2.6) into C such that

$$\gamma \circ \beta_l = \gamma_l \tag{6.2.13}$$

for every $l \in I$, because $(I, \preceq)$ is a directed set. The direct limit is uniquely determined up to a suitable equivalence by this property.

Suppose for the moment that A is a set, that A_j is a subset of A for every $j \in I$, and that

$$A_j \subseteq A_l \tag{6.2.14}$$

for every $j, l \in I$ with $j \preceq l$. If $j, l \in I$ and $j \preceq l$, then let $\alpha_{j,l}$ be the obvious inclusion mapping from A_j into A_l. This clearly satisfies the requirements of a direct system of sets over $(I, \preceq)$. In this case, one can check that the direct limit is equivalent to

$$\bigcup_{j \in I} A_j, \tag{6.2.15}$$

where β_l corresponds to the natural inclusion mapping from A_l into (6.2.15) for each $l \in I$.

6.3 Direct systems of groups

Let $(I, \preceq)$ be a nonempty directed set, and let A_j be a group for every $j \in I$. Also let $\alpha_{j,l}$ be a homomorphism from A_j into A_l for every $j, l \in I$ with $j \preceq l$. As usual, we ask that $\alpha_{j,j}$ be the identity mapping on A_j for every $j \in I$, and that (6.2.1) hold for every $j, l, r \in I$ with $j \preceq l \preceq r$. The family of groups A_j and homomorphisms $\alpha_{j,l}$ form a *direct* or *inductive system* over $(I, \preceq)$.

In particular, A_j is a nonempty set for every $j \in I$, and we may continue as in the previous section, with $\widetilde{A}_j = A_j$ for each $j \in I$. The *direct* or *inductive limit* of the A_j's may be defined as a set as in (6.2.6), with the corresponding mappings β_l from A_l into (6.2.6) for every $l \in I$. One can check that there is a unique group structure on (6.2.6) such that β_l is a homomorphism from A_l into (6.2.6) for every $l \in I$. This uses the fact that any two elements of (6.2.6) are contained in $\beta_r(A_r)$ for some $r \in I$, because of (6.2.7) and the fact that $(I, \preceq)$ is a directed set.

Let C be another group, and suppose that for every $l \in I$ we have a homomorphism γ_l from A_l into C. Suppose also that for every $l, r \in I$ with $l \preceq r$ we have that (6.2.12) holds. This implies that there is a unique mapping γ from (6.2.6) into C such that (6.2.13) holds for every $l \in I$, as before. It is easy to see that γ is a homomorphism from C into (6.2.6) in this case. As usual, the direct limit is uniquely determined up to isomorphism by this property.

This description of the direct limit of a direct system of groups is equivalent to the one on p132 of [14]. If the A_j's are commutative groups, then this is equivalent to the direct limit defined in Section 6.1.

Let A be a group, and suppose that A_j is a subgroup of A for every $j \in I$. Suppose also that if $j, l \in I$ and $j \preceq l$, then $A_j \subseteq A_l$. If $j, l \in I$ and $j \preceq l$, then let $\alpha_{j,l}$ be the natural inclusion mapping from A_j into A_l, which is a group homomorphism. Note that the A_j's and $\alpha_{j,l}$'s satisfy the requirements of a direct system of groups over $(I, \preceq)$. One can check that $\bigcup_{j \in I} A_j$ is a subgroup of A, because $(I, \preceq)$ is a directed set, as before. Remember that the direct limit of the A_j's is equivalent to $\bigcup_{j \in I} A_j$ as a set, as in the previous section. One can verify that the direct limit of the A_j's is isomorphic to $\bigcup_{j \in I} A_j$ as a group in this case too.

6.4 Direct systems of topological spaces

Let $(I, \preceq)$ be a nonempty set, and let A_j be a topological space for every $j \in I$. Suppose that for every $j, l \in I$ with $j \preceq l$, we have a continuous mapping $\alpha_{j,l}$ from A_j into A_l. We ask that $\alpha_{j,j}$ be the identity mapping on A_j for every $j \in I$, and that (6.2.1) hold for every $j, l, r \in I$ with $j \preceq l \preceq r$, as before. Thus we can define the direct limit of the A_j's as a set as in Section 6.2. Remember that for each $l \in I$, we have a mapping β_l from A_l into the direct limit, as before.

Let τ_0 be a topology on the direct limit of the A_j's, $j \in I$. A basic compatibility condition with the topologies on the A_j's is that

$$\beta_l \text{ is continuous for every } l \in I \tag{6.4.1}$$

with respect to τ_0. There is a strongest topology on the direct limit with this property, by standard arguments. More precisely, a subset W of the direct limit is an open set with respect to this topology if and only if $\beta_l^{-1}(W)$ is an open set in A_l for every $l \in I$. Equivalently, a subset E of the direct limit is a closed set with respect to this topology if and only if $\beta_l^{-1}(E)$ is a closed set in A_l for every $l \in I$.

Let C be another topological space, and suppose that for every $l \in I$ we have a continuous mapping γ_l from A_l into C. Suppose also that for every $l, r \in I$ with $l \preceq r$ we have that $\gamma_l = \gamma_r \circ \alpha_{l,r}$, as in Section 6.2. This implies that there is a unique mapping γ from the direct limit of the A_j's into C such that $\gamma \circ \beta_l = \gamma_l$ for every $l \in I$, as before. If the direct limit is equipped with the strongest topology that satisfies (6.4.1), then γ is continuous. It is easy to see that this is the only topology on the direct limit such that (6.4.1) and this property hold.

If $l, r \in I$ and $l \preceq r$, then $\beta_l = \beta_r \circ \alpha_{l,r}$, as in Section 6.2. If E is a subset of the direct limit, then it follows that

$$\beta_l^{-1}(E) = \alpha_{l,r}^{-1}(\beta_r^{-1}(E)). \tag{6.4.2}$$

If $\beta_r^{-1}(E)$ is open or closed in A_r, then we get that $\beta_l^{-1}(E)$ has the same property in A_l, because $\alpha_{l,r}$ is continuous.

Suppose now that A is a set, $A_j \subseteq A$ for every $j \in I$, and that $A_j \subseteq A_l$ for every $j, l \in I$ with $j \preceq l$. If $j, l \in I$ and $j \preceq l$, then we ask that $\alpha_{j,l}$ be the obvious inclusion mapping from A_j into A_l. Suppose that

$$A = \bigcup_{j \in I} A_j, \tag{6.4.3}$$

which is equivalent to the direct limit of the A_j's as a set in this case, as in Section 6.2. Remember that β_l corresponds to the obvious inclusion of A_l into A for every $l \in I$, as before.

Let τ_1 be a topology on A. The compatibility condition (6.4.1) corresponds in this case to saying that

(6.4.4) for each $l \in I$, the topology on A_l is at least as strong as the topology induced on A_l by τ_1.

The strongest topology on A with this property can be defined by saying that $W \subseteq A$ is an open set if and only if $W \cap A_l$ is an open set in A_l for every $l \in I$. Equivalently, this means that $E \subseteq A$ is a closed set if and only if $E \cap A_l$ is a closed set in A_l for every $l \in I$.

If $l, r \in I$ and $l \preceq r$, then

$$E \cap A_l = (E \cap A_r) \cap A_l \tag{6.4.5}$$

for every $E \subseteq A$. If $E \cap A_r$ is open or closed in A_r, then $E \cap A_l$ has the same property in A_l, by the continuity of $\alpha_{l,r}$, as before.

Let $l_0 \in I$ and $E \subseteq A_{l_0}$ be given. In order to check that E is an open set in A with respect to the strongest topology that satisfies (6.4.4), it suffices to verify that for every $r \in I$ with $l_0 \preceq r$ we have that E is an open set in A_r. Similarly, in order to check that E is a closed set in A with respect to the strongest topology that satisfies (6.4.4), it suffices to verify that for every $r \in I$ with $l_0 \preceq r$ we have that E is a closed set in A_r.

Suppose for the moment that

(6.4.6) if $j, l \in I$ and $j \preceq l$, then the topology on A_j is the same as the topology induced by the topology on A_r.

Of course, this implies that the inclusion mapping from A_j into A_l is continuous. Let $l_0, r \in I$ and $E \subseteq A_{l_0}$ be given, with $l_0 \preceq r$. If A_{l_0} is an open set in A_r and E is an open set in A_{l_0}, then E is an open set in A_r. Similarly, if A_{l_0} is a closed set in A_r and E is a closed set in A_{l_0}, then E is a closed set in A_r.

Let τ be a topology on A, and suppose that

(6.4.7) for each $j \in I$, A_j is equipped with the topology induced by τ.

Note that (6.4.6) holds in this case, and that (6.4.4) holds with $\tau_1 = \tau$. This means that τ is contained in the strongest topology on A that satisfies (6.4.4).

If A_l is an open set in A with respect to τ for every $l \in I$, then τ is the same as the strongest topology on A that satisfies (6.4.4).

If one is working with locally convex topological vector spaces over the real numbers, then one normally considers topologies on direct limits that are locally convex as well. This is compatible with looking at continuity of linear mappings into other locally convex topological vector spaces over $\mathbf{R}$.

6.5 Inverse systems of sets

Let $(I, \preceq)$ be a nonempty directed set, and let Y_j be a nonempty set for each $j \in I$. Suppose that for every $j, l \in I$ with $j \preceq l$, we have a mapping $\phi_{j,l}$ from Y_l into Y_j with the following two properties. First, $\phi_{j,j}$ is the identity mapping on Y_j for every $j \in I$. Second, if $j, l, r \in I$ and $j \preceq l \preceq r$, then

$$\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}. \tag{6.5.1}$$

Under these conditions, the family of sets Y_j, $j \in I$, with the associated mappings $\phi_{j,l}$ is said to be an *inverse* or *projective system*.

Sometimes we may wish to ask also that

$$\phi_{j,l}(Y_l) = Y_j \tag{6.5.2}$$

for every $j, l \in I$ with $j \preceq l$. In this case, we may say that the inverse system is *surjective*, as on p103 of [1].

If y is an element of the Cartesian product $\prod_{j \in I} Y_j$ and $l \in I$, then we let y_l be the lth coordinate of y in Y_l, as usual. Put

$$Y = \Big\{ y \in \prod_{j \in I} Y_j : \phi_{l,r}(y_r) = y_l \text{ for every } l, r \in I \text{ with } l \preceq r \Big\}. \tag{6.5.3}$$

This is the *inverse* or *projective limit* of the Y_j's, which may be denoted

$$\varprojlim Y_j. \tag{6.5.4}$$

If $l \in I$, then let π_l be the standard coordinate projection from $\prod_{j \in I} Y_j$ onto Y_l, so that $\pi_l(y) = y_l$ for every $y \in \prod_{j \in I} Y_j$. Also let ρ_l be the restriction of π_l to Y, which is a mapping from Y into Y_l. If $l, r \in I$ and $l \preceq r$, then

$$\phi_{l,r} \circ \rho_r = \rho_l, \tag{6.5.5}$$

by construction. More precisely, the inverse limit consists of the set Y together with these mappings ρ_l.

Let X be another set, and suppose that for each $j \in I$ we have a mapping θ_j from X into Y_j. This leads to a mapping θ fom X into $\prod_{j \in I} Y_j$, with

$$\pi_l \circ \theta = \theta_l \tag{6.5.6}$$

for every $l \in I$. If

$$\phi_{l,r} \circ \theta_r = \theta_l \quad \text{for every } l, r \in I \text{ with } l \preceq r, \tag{6.5.7}$$

then

$$\theta(X) \subseteq Y. \tag{6.5.8}$$

In this case, (6.5.6) is the same as saying that

$$\rho_l \circ \theta = \theta_l \tag{6.5.9}$$

for every $l \in I$, and θ is uniquely determined as a mapping from X into Y by (6.5.9). The inverse limit is uniquely determined up to suitable equivalence by this property.

If $l, r \in I$ and $l \preceq r$, then put

$$E_{l,r} = \Big\{ y \in \prod_{j \in I} Y_j : \phi_{l,r}(y_r) = y_l \Big\}. \tag{6.5.10}$$

Observe that

$$Y = \bigcap \{E_{l,r} : l, r \in I,\, l \preceq r\} \tag{6.5.11}$$

by construction.

Let $l_1, \ldots, l_n$, $r_1, \ldots, r_n$ be finitely many elements of I, with $l_m \preceq r_m$ for $m = 1, \ldots, n$. We would like to check that

$$\bigcap_{m=1}^{n} E_{l_m, r_m} \neq \emptyset. \tag{6.5.12}$$

Because I is a directed system, there is a $u \in I$ such that $r_m \preceq u$ for every $m = 1, \ldots, n$. Remember that $Y_j \neq \emptyset$ for every $j \in I$, by hypothesis, and let y_u be an element of Y_u. Let y be an element of $\prod_{j \in I} Y_j$ such that

$$y_j = \phi_{j,u}(y_u) \tag{6.5.13}$$

for every $j \in I$ with $j \preceq u$. If $l, r \in I$ and $l \preceq r \preceq u$, then

$$\phi_{l,r}(y_r) = \phi_{l,r}(\phi_{r,u}(y_u)) = \phi_{l,u}(y_u) = y_l. \tag{6.5.14}$$

In particular, this implies that y is an element of the intersection on the left side of (6.5.12).

Suppose for the moment that the inverse system of Y_j's, $j \in I$, is surjective, and let $t \in I$ and $y_t \in Y_t$ be given. In the preceding paragraph, we can take $u \in I$ so that $t \preceq u$ as well, and we can take $y_u \in Y_u$ so that

$$\phi_{t,u}(y_u) = y_t. \tag{6.5.15}$$

Using the same argument as before, we get that

$$\Big(\bigcap_{m=1}^{n} E_{l_m, r_m} \Big) \cap \pi_t^{-1}(\{y_t\}) \neq \emptyset. \tag{6.5.16}$$

6.6 Inverse systems of topological spaces

Let us continue with the same notation and hypotheses as in the previous section. In this section, we suppose also that

$$Y_j \text{ is a topological space for every } j \in I, \tag{6.6.1}$$

and that

$$\phi_{j,l} \text{ is continuous as a mapping from } Y_l \text{ into } Y_j \text{ for every } j, l \in I \text{ with } j \preceq l. \tag{6.6.2}$$

Let us take $\prod_{j \in I} Y_j$ to be equipped with the associated product topology, and Y to be equipped with the induced topology. In this case,

$$\rho_l \text{ is continuous as a mapping from } Y \text{ into } Y_l \text{ for every } l \in I. \tag{6.6.3}$$

This should be considered as part of the inverse limit of the Y_j's as a topological space.

If U_l is an open subset of Y_l for some $l \in I$, then $\pi_l^{-1}(U_l)$ is an open subset of $\prod_{j \in I} Y_j$ with respect to the product topology, and

$$\rho_l^{-1}(U_l) = \pi_l^{-1}(U_l) \cap Y \tag{6.6.4}$$

is a relatively open subset of Y. Of course, open subsets of $\prod_{j \in I} Y_j$ of the form $\pi_l^{-1}(U_l)$ with $l \in I$ and $U_l \subseteq Y_l$ an open set form a sub-base for the product topology. One can check that the subsets of Y of the form (6.6.4) with $l \in I$ and $U_l \subseteq Y_l$ an open set form a base for the induced topology on Y, using the fact that $(I, \preceq)$ is a directed set.

If Y_j is Hausdorff for every $j \in I$, then $\prod_{j \in I} Y_j$ is Hausdorff with respect to the product topology, and Y is Hausdorff with respect to the induced topology. If $l, r \in I$ and $l \preceq r$, then one can verify that the set $E_{l,r}$ defined in (6.5.10) is a closed set in $\prod_{j \in I} Y_j$ in this case. This implies that Y is a closed set in $\prod_{j \in I} Y_j$ too, by (6.5.11).

Suppose for the moment that Y_j is compact and Hausdorff for every $j \in I$. This implies that $\prod_{j \in I} Y_j$ is compact with respect to the product topology, by Tychonoff's theorem, and that Y is compact in $\prod_{j \in I} Y_j$, because it is a closed set with respect to the product topology. We also get that $Y \neq \emptyset$ under these conditions, because Y is the intersection of a nonempty family of nonempty closed subsets of $\prod_{j \in I} Y_j$ with the finite intersection property, as in the previous section. If the inverse system of Y_j's, $j \in I$, is surjective, then we can use (6.5.16) to get that ρ_t maps Y onto Y_t for every $t \in I$.

Suppose that X is a topological space, and that θ_j is a continuous mapping from X into Y_j for every $j \in I$. This means that the corresponding mapping θ from X into $\prod_{j \in I} Y_j$ is continuous, with respect to the product topology on the range. If (6.5.7) holds, then θ may be considered as a mapping from X into Y, which is continuous with respect to the induced topology on Y. The inverse limit is uniquely determined up to homeomorphism by this property.

If X is compact, then $\theta(X)$ is a compact subset of Y. It follows that $\theta(X)$ is a closed set in Y when Y_j is Hausdorff for every $j \in I$, so that Y is Hausdorff.

Suppose that $\theta_j(X)$ is dense in Y_j for every $j \in I$. If (6.5.7) holds, then one can check that $\theta(X)$ is dense in Y.

6.7 Inverse systems of groups

Let $(I, \preceq)$ be a nonempty directed set, and let Y_j be a group for every $j \in I$. If $j, l \in I$ and $j \preceq l$, then let $\phi_{j,l}$ be a homomorphism from Y_l into Y_j. As before, we ask that $\phi_{j,j}$ be the identity mapping on Y_j for every $j \in I$, and that

$$\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r} \tag{6.7.1}$$

for every $j, l, r \in I$ with $j \preceq l \preceq r$.

Of course, $\prod_{j \in I} Y_j$ may be considered as a group as well, where the group operations are defined coordinatewise. It is easy to see that the set Y defined in (6.5.3) is a subgroup of $\prod_{j \in I} Y_j$ under these conditions. Note that the mapping ρ_l from Y into Y_l defined in Section 6.5 is a group homomorphism for every $l \in I$ in this case.

Suppose that X is a group, and that θ_j is a group homomorphism from X into Y_j for every $j \in I$. This implies that the corresponding mapping θ from X into $\prod_{j \in I} Y_j$ is a group homomorphism too. If (6.5.7) holds, then θ may be considered as a group homomorphism from X into Y. Remember that θ satisfies (6.5.9) in this case, and is uniquely determined by it. The inverse limit is uniquely determined up to isomorphism by this property, as usual.

If Y_j is a profinite group for every $j \in I$, then $\prod_{j \in I} Y_j$ is a profinite group with respect to the corresponding product topology, as in Section 2.14. Note that Y is a closed set in $\prod_{j \in I} Y_j$ in this case, as in the previous section. It follows that Y is a profinite group with respect to the induced topology, as indicated on p3 of [25]. If the inverse system of Y_j's, $j \in I$, is surjective, then we also have that ρ_t maps Y onto Y_t for every $t \in I$, as before.

Let G be a group, and let $\mathcal{B}$ be a nonempty collection of normal subgroups of G of finite index. Suppose that for every $A_1, A_2 \in \mathcal{B}$ there is an $A_3 \in \mathcal{B}$ such that

$$A_3 \subseteq A_1 \cap A_2. \tag{6.7.2}$$

Let $\preceq$ be the partial order defined on $\mathcal{B}$ by saying that for each $A, B \in \mathcal{B}$,

$$A \preceq B \quad \text{if and only if} \quad B \subseteq A. \tag{6.7.3}$$

Observe that $(\mathcal{B}, \preceq)$ is a directed system, because of (6.7.2).

If $A \in \mathcal{B}$, then let q_A be the natural quotient mapping from G onto G/A, which is a finite group. If $A, B \in \mathcal{B}$ and $A \preceq B$, so that $B \subseteq A$, then there is a unique homomorphism $\Phi_{A,B}$ from G/B onto G/A such that

$$\Phi_{A,B} \circ q_B = q_A. \tag{6.7.4}$$

Note that $\Phi_{A,A}$ is the identity mapping on G/A for every $A \in \mathcal{B}$. If $A, B, C \in \mathcal{B}$ satisfy $A \preceq B \preceq C$, which is to say that $C \subseteq B \subseteq A$, then

$$\Phi_{A,B} \circ \Phi_{B,C} = \Phi_{A,C}. \tag{6.7.5}$$

Thus the inverse limit

$$\lim_{\longleftarrow} G/A \tag{6.7.6}$$

of the quotients G/A, $A \in \mathcal{B}$, can be defined as a group as before.

More precisely, the inverse limit (6.7.6) is a subgroup of

$$\prod_{A \in \mathcal{B}} (G/A), \tag{6.7.7}$$

where the group operations are defined coordinatewise, as usual. Using the quotient mappings q_A, we get a homomorphism Θ from G into (6.7.7). This homomorphism maps G into the inverse limit (6.7.6), because of (6.7.4). Note that the kernel of Θ is

$$\bigcap_{A \in \mathcal{B}} A. \tag{6.7.8}$$

Let us take G/A to be equipped with the discrete topology for every $A \in \mathcal{B}$, so that (6.7.7) is profinite with respect to the product topology. This implies that the inverse limit (6.7.6) is a closed subgroup of (6.7.7), and thus a profinite group with respect to the induced topology, as before. We also have that Θ maps G onto a dense subset of the inverse limit (6.7.6), because q_A is surjective for every $A \in \mathcal{B}$, by construction.

Suppose now that G is a totally bounded topological group, for which the open normal subgroups form a local base for the topology at the identity element. Let $\mathcal{B}$ be any local base for the topology at the identity element consisting of open normal subgroups. If $A_1, A_2 \in \mathcal{B}$, then $A_1 \cap A_2$ is an open set that contains the identity element, which implies that there is an $A_3 \in \mathcal{B}$ that satisfies (6.7.2). Thus the inverse limit (6.7.6) can be defined as before.

If $A \in \mathcal{B}$, then the natural quotient mapping q_A from G onto G/A is continuous with respect to the discrete topology on G/A, because A is an open set. This implies that the corresponding mapping Θ from G into (6.7.7) is continuous with respect to the product topology on the range. Note that (6.7.8) contains only the identity element when the set containing only the identity element is a closed set. In this case, one can check that Θ is a homeomorphism from G onto its image in (6.7.7), with respect to the topology induced by the product topology, because $\mathcal{B}$ is a local base for the topology on G at the identity element.

If G is compact, then Θ maps G onto a compact subset of (6.7.7). This implies that Θ maps G onto the inverse limit (6.7.6), because Θ maps G onto a dense subset of (6.7.6). One can define profinite groups to be inverse limits of finite groups with the discrete topology, as on p3 of [25].

6.8 Inverse systems along sequences

Let Y_j be a nonempty set for every positive integer j, and suppose that ψ_j is a mapping from Y_{j+1} into Y_j for each j. If $j, l \in \mathbf{Z}_+$ and $j < l$, then put

$$\phi_{j,l} = \psi_j \circ \cdots \circ \psi_{l-1}, \tag{6.8.1}$$

which maps Y_l into Y_j. This may be interpreted as the identity mapping on Y_j when $j = l$. If $j, l, r \in \mathbf{Z}_+$ and $j \leq l \leq r$, then $\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}$, by construction. If ψ_j maps Y_{j+1} onto Y_j for every j, then $\phi_{j,l}$ maps Y_l onto Y_j when $j \leq l$.

If we take $I = \mathbf{Z}_+$ with the standard ordering, then (6.5.3) reduces to

$$Y = \Big\{ y \in \prod_{j=1}^{\infty} Y_j : \psi_j(y_{j+1}) = y_j \text{ for every } j \in \mathbf{Z}_+ \Big\}. \tag{6.8.2}$$

This may be used as the definition of the inverse or projective limit in this case. If ψ_j maps Y_{j+1} onto Y_j for every j, then it is easy to see that for each $l \in \mathbf{Z}_+$ there is a $y \in Y$ such that y_l is any element of Y_l, by choosing suitable $y_j \in Y_j$ recursively when $j > l$.

Let X be another set, and let θ_j be a mapping from X into Y_j for every $j \in \mathbf{Z}_+$. This leads to a mapping θ from X into $\prod_{j=1}^{\infty} Y_j$, as before. If

$$\psi_j \circ \theta_{j+1} = \theta_j \quad \text{for every } j \geq 1, \tag{6.8.3}$$

then $\theta(X) \subseteq Y$.

If $l \in \mathbf{Z}_+$, then put

$$E_l = \Big\{ y \in \prod_{j=1}^{\infty} Y_j : \psi_l(y_{l+1}) = y_l \Big\}, \tag{6.8.4}$$

so that

$$Y = \bigcap_{l=1}^{\infty} E_l. \tag{6.8.5}$$

If Y_j is a Hausdorff topological space for every $j \geq 1$, and ψ_j is continuous as a mapping from Y_{j+1} onto Y_j for every j, then E_l is a closed set in $\prod_{j=1}^{\infty} Y_j$ with respect to the product topology for every $l \geq 1$. This implies that Y is a closed set in $\prod_{j=1}^{\infty} Y_j$ too, as before.

Suppose that Y_j is a group for every $j \geq 1$, and that ψ_j is a homomorphism from Y_{j+1} into Y_j for each j. This implies that $\phi_{j,l}$ is a homomorphism from Y_l into Y_j when $j \leq l$. As in the previous section, $\prod_{j=1}^{\infty} Y_j$ is a group in this case, where the group operations are defined coordinatewise, and it is easy to see that Y is a subgroup of $\prod_{j=1}^{\infty} Y_j$. This corresponds to the definition of the inverse limit on p103 of [1].

Let G be a group, and let $A_1, A_2, A_3, \ldots$ be a sequence of normal subgroups of G such that

$$A_{j+1} \subseteq A_j \tag{6.8.6}$$

for every j. Also let q_j be the natural quotient homomorphism from G onto G/A_j for each j. Observe that for each j there is a unique homomorphism Ψ_j from G/A_{j+1} onto G/A_j such that

$$\Psi_j \circ q_{j+1} = q_j. \tag{6.8.7}$$

This permits one to define the inverse limit

$$\lim_{\longleftarrow} G/A_j \tag{6.8.8}$$

as a subgroup of $\prod_{j=1}^{\infty}(G/A_j)$ as in (6.8.2).

Using the quotient mappings q_j, we get a homomorphism Θ from G into $\prod_{j=1}^{\infty}(G/A_j)$, as before. More precisely, Θ maps G into the inverse limit (6.8.8), because of (6.8.7). The kernel of Θ is $\bigcap_{j=1}^{\infty} A_j$, as before.

Let G be a topological group, for which the open normal subgroups form a local base for the topology at the identity element. Suppose that there is also a countable local base for the topology at the identity element. This implies that there is a sequence $A_1, A_2, A_3, \ldots$ of open normal subgroups of G such that the collection $\mathcal{B}$ of the A_j's is a local base for the topology at the identity element. We may as well ask that (6.8.6) hold for every j, by replacing A_j with $A_1 \cap \cdots \cap A_j$ for each j, if necessary. Thus the inverse limit (6.8.8) can be defined as before.

Let us take G/A_j to be equipped with the discrete topology for each j, and $\prod_{j=1}^{\infty}(G/A_j)$ to be equipped with the corresponding product topology, as usual. If the set containing only the identity element is a closed set in G, then Θ is a homeomorphism onto its image, as in the previous section. This is related to Exercise 2 on p6 of [25].

6.9 Inverse systems of subsets

Let $(I, \preceq)$ be a nonempty directed set, and let Y_j be a nonempty set for each $j \in I$. Suppose that for every $j, l \in I$ with $j \le l$, we have a mapping $\phi_{j,l}$ from Y_l into Y_j, where $\phi_{j,j}$ is the identity mapping on Y_j for each $j \in I$, and $\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}$ when $j, l, r \in I$ and $j \preceq l \preceq r$. Remember that the inverse limit of the Y_j's is defined to be the set Y of $y \in \prod_{j \in I} Y_j$ such that $\phi_{l,r}(y_r) = y_l$ for every $l, r \in I$ with $l \preceq r$, as in Section 6.5.

Let W_j be a nonempty subset of Y_j for every $j \in I$, and suppose that

$$\phi_{j,l}(W_l) \subseteq W_j \tag{6.9.1}$$

for every $j, l \in I$ with $j \preceq l$. Under these conditions, the family of W_j's, $j \in I$, with the restrictions of the mappings $\phi_{j,l}$ to W_l, is an inverse system. The inverse limit of the W_j's is given by

$$W = \Big\{w \in \prod_{j \in I} W_j : \phi_{l,r}(w_r) = w_l \text{ for every } l, r \in I \text{ with } l \preceq r\Big\}, \tag{6.9.2}$$

as before. Equivalently,

$$W = \Big(\prod_{j \in I} W_j\Big) \cap Y. \tag{6.9.3}$$

Suppose for the moment that Y_j is a topological space for every $j \in I$, and that $\phi_{j,l}$ is continuous as a mapping from Y_l onto Y_j when $j, l \in I$ and $j \preceq l$. In this case, we take $\prod_{j \in I} Y_j$ to be equipped with the corresponding product topology, and Y to be equipped with the induced topology. Let us take W_j to be equipped with the topology induced by the one on Y_j for every $j \in I$, so that

(6.9.4) the restriction of $\phi_{j,l}$ to W_l is continuous

as a mapping from W_l into W_j when $j, l \in I$ and $j \preceq l$. Note that the corresponding product topology on $\prod_{j \in I} W_j$ is the same as the topology induced on $\prod_{j \in I} W_j$ by the product topology on $\prod_{j \in I} Y_j$. It is easy to see that the topology induced on W by the product topology on $\prod_{j \in I} W_j$ is the same as the topology induced by the topology on Y just mentioned.

If W_j is a closed set in Y_j for every $j \in I$, then $\prod_{j \in I} W_j$ is a closed set in $\prod_{j \in I} Y_j$ with respect to the product topology. In this case, W is a closed set in Y with respect to the topology induced by the product topology on $\prod_{j \in I} Y_j$, because of (6.9.3).

Suppose now that Y_j is a group for every $j \in I$, and that $\phi_{j,l}$ is a homomorphism from Y_l onto Y_j when $j, l \in I$ and $j \preceq l$. This implies that $\prod_{j \in I} Y_j$ is a group, where the group operations are defined coordinatewise, and that Y is a subgroup of $\prod_{j \in I} Y_j$, as in Section 6.7. If W_j is a subgroup of Y_j for every $j \in I$, then $\prod_{j \in I} W_j$ is a subgroup of $\prod_{j \in I} Y_j$, and W is a subgroup of Y.

Suppose that Y_j is a finite group equipped with the discrete topology for every $j \in I$, which means that the homomorphisms $\phi_{j,l}$ are automatically continuous. Remember that Y is a profinite group in this case, as in Section 6.7. If $l \in I$, then let π_l be the standard coordinate projection from $\prod_{j \in I} Y_j$ onto Y_l, and let ρ_l be the restriction of π_l to Y, as in Section 6.5. Thus ρ_l is a continuous group homomorphism from Y into Y_l for each $l \in I$, as before. In particular, the kernel of ρ_l is an open normal subgroup of Y for every $l \in I$.

One can check that the kernels of the ρ_l's form a local base for the topology of Y at the identity element, using the fact that $(I, \preceq)$ is a directed system. This is analogous to an earlier remark about (6.6.4).

If $l \in I$, then the restriction of π_l to $\prod_{j \in I} W_j$ is the same as the standard coordinate projection from $\prod_{j \in I} W_j$ onto W_l. Similarly, the restriction of ρ_l to W is the analogue of ρ_l for W, which maps W into W_l.

6.10 Direct systems and homomorphisms

Let $(I, \preceq)$ be a nonempty directed set, and suppose that we have a direct system of groups A_j, $j \in I$, as in Section 6.3. Thus for each $j, l \in I$ with $j \preceq l$ we have a homomorphism $\alpha_{j,l}$ from A_j into A_l with the properties mentioned earlier. If

$l \in I$, then we also get a homomorphism β_l into the direct limit of the A_j's, as before.

Let C be another group. If γ is a homomorphism from the direct limit of the A_j's into C, then

$$\gamma_l = \gamma \circ \beta_l \tag{6.10.1}$$

is a homomorphism from A_l into C for every $l \in I$. If $l, r \in I$ and $l \preceq r$, then

$$\gamma_r \circ \alpha_{l,r} = \gamma_l, \tag{6.10.2}$$

because of the analogous property of β_l, β_r, as in Section 6.2. Conversely, if γ_l is a homomorphism from A_l into C for each $l \in I$, and if (6.10.2) holds for every $l, r \in I$ with $l \preceq r$, then there is a unique homomorphism γ from the direct limit of the A_j's into C such that (6.10.1) holds for every $l \in I$, as in Section 6.3.

If B is any group, then let $\mathrm{Hom}(B, C)$ be the set of all group homomorphisms from B into C. This is a commutative group when C is commutative, where the group operations for mappings from B into C are defined pointwise. Let B_1, B_2 be groups, and let h_1 be a homomorphism from B_1 into B_2. If $\zeta \in \mathrm{Hom}(B_2, C)$, then

$$\widetilde{h}_1(\zeta) = \zeta \circ h_1 \tag{6.10.3}$$

is an element of $\mathrm{Hom}(B_1, C)$. This defines a mapping from $\mathrm{Hom}(B_2, C)$ into $\mathrm{Hom}(B_1, C)$, which is a group homomorphism when C is commutative. Let B_3 be another group, and let h_2 be a homomorphism from B_2 into B_3. Thus $h_2 \circ h_1$ is a homomorphism from B_1 into B_3, and it is easy to see that

$$(\widetilde{h_2 \circ h_1}) = \widetilde{h}_1 \circ \widetilde{h}_2. \tag{6.10.4}$$

If $B_1 = B_2$ and h_1 is the identity mapping on B_1, then $\widetilde{h}_1$ is the identity mapping on $\mathrm{Hom}(B_1, C)$.

If $j, l \in I$ and $j \preceq l$, then we can define $\widetilde{\alpha}_{j,l}$ as a mapping from $\mathrm{Hom}(A_l, C)$ into $\mathrm{Hom}(A_j, C)$ as in the preceding paragraph. If $j, l, r \in I$ and $j \preceq l \preceq r$, then

$$\widetilde{\alpha}_{j,r} = \widetilde{\alpha}_{j,l} \circ \widetilde{\alpha}_{l,r}, \tag{6.10.5}$$

by (6.10.4), and because $\alpha_{j,r} = \alpha_{l,r} \circ \alpha_{j,l}$. Note that $\widetilde{\alpha}_{j,j}$ is the identity mapping on $\mathrm{Hom}(A_j, C)$ for every $j \in I$, because $\alpha_{j,j}$ is the identity mapping on A_j. This means that the family of sets

$$\mathrm{Hom}(A_j, C),\ j \in I, \tag{6.10.6}$$

is an inverse system, with respect to the corresponding family of maps $\widetilde{\alpha}_{j,l}$. If C is commutative, then this may be considered as an inverse system of commutative groups.

Of course, (6.10.2) is the same as saying that

$$\widetilde{\alpha}_{l,r}(\gamma_r) = \gamma_l. \tag{6.10.7}$$

This leads to a one-to-one correspondence between homomorphisms from the direct limit of the A_j's into C and elements of the inverse limit of (6.10.6). If C is commutative, then this is an isomorphism between these commutative groups.

Let us now take A_j to be a commutative group equipped with the discrete topology for every $j \in I$, and take the direct limit of the A_j's to be equipped with the discrete topology as well. Let us also take $C = \mathbf{T}$, so that $\mathrm{Hom}(A_j, C)$ is the same as the dual $\widehat{A_j}$ of A_j for each $j \in I$, and similarly for the direct limit of the A_j's. Thus we get an isomorphism between the dual of the direct limit of the A_j's and the inverse limit of $\widehat{A_j}$, $j \in I$, as in the preceding paragraph. Remember that the topology on $\widehat{A_j}$ defined in Section 5.3 is the same as the topology that corresponds to pointwise convergence on A_j for each $j \in I$, because A_j is equipped with the discrete topology. Similarly, the topology on the dual of the direct limit of the A_j's is the same as the topology that corresponds to pointwise convergence on the direct limit.

Using this topology on $\widehat{A_j}$ for each $j \in I$, we get a topology on the inverse limit of the $\widehat{A_j}$'s, as in Section 6.6. One can check that this corresponds exactly to the topology on the dual of the direct limit of the A_j's mentioned in the preceding paragraph. This uses the fact that the direct limit of the A_j's is the union of $\beta_l(A_l)$ over $l \in I$, as in Sections 6.1, 6.2.

6.11 Inverse systems and homomorphisms

Let $(I, \preceq)$ be a nonempty directed set, and let Y_j be a group for every $j \in I$. If $j, l \in I$ and $j \preceq l$, then let $\phi_{j,l}$ be a group homomorphism from Y_l into Y_j. As usual, we ask that $\phi_{j,j}$ be the identity mapping on Y_j for every $j \in I$, and that $\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}$ for every $j, l, r \in I$ with $j \preceq l \preceq r$. Thus the inverse limit of the Y_j's, $j \in I$, may be defined as the subset Y of $\prod_{j \in I} Y_j$ considered in Section 6.5. More precisely, $\prod_{j \in I} Y_j$ is a group, where the group operations are defined coordinatewise, and we have seen that Y is a subgroup of $\prod_{j \in I} Y_j$. If $l \in I$, then let π_l be the standard coordinate projection from $\prod_{j \in I} Y_j$ onto Y_l, and let ρ_l be the restriction of π_l to Y, as before. Of course, these are group homomorphisms into Y_l.

Let C be another group. If $j, l \in I$ and $j \preceq l$, then we get a mapping $\widetilde{\phi}_{j,l}$ from $\mathrm{Hom}(Y_j, C)$ into $\mathrm{Hom}(Y_l, C)$ associated to $\phi_{j,l}$ as in the previous section. If C is commutative, then $\widetilde{\phi}_{j,l}$ is a homomorphism between commutative groups, as before. If $j, l, r \in I$ and $j \preceq l \preceq r$, then

$$\widetilde{\phi}_{j,r} = \widetilde{\phi}_{l,r} \circ \widetilde{\phi}_{j,l}, \tag{6.11.1}$$

because of (6.10.4). Of course, $\widetilde{\phi}_{j,j}$ is the identity mapping on $\mathrm{Hom}(Y_j, C)$ for every $j \in I$.

This shows that

$$\mathrm{Hom}(Y_j, C),\ j \in I, \tag{6.11.2}$$

is a direct system, with respect to the corresponding family of maps $\widetilde{\phi}_{j,l}$. If C is commutative, then this may be considered as a direct system of commutative groups. If $l \in I$, then there is a natural mapping β_l from $\mathrm{Hom}(Y_l, C)$ into the direct limit of (6.11.2), as in Sections 6.1, 6.2. This is a homomorphism between commutative groups when C is commutatitive, as before.

Similarly, if $l \in I$, then we get a mapping $\widetilde{\rho}_l$ from $\mathrm{Hom}(Y_l, C)$ into $\mathrm{Hom}(Y, C)$ associated to ρ_l as in the previous section. If C is commutative, then $\widetilde{\rho}_l$ is a homomorphism between commutative groups, as usual. If $l, r \in I$ and $l \preceq r$, then

$$\widetilde{\rho}_l = \widetilde{\rho}_r \circ \widetilde{\phi}_{l,r}, \tag{6.11.3}$$

because of (6.10.4) and the fact that $\rho_l = \phi_{l,r} \circ \rho_r$, as in Section 6.5. This leads to a unique mapping γ from the direct limit of (6.11.2) into $\mathrm{Hom}(Y, C)$ such that

$$\gamma \circ \beta_l = \widetilde{\rho}_l \tag{6.11.4}$$

for every $l \in I$, as in Sections 6.1, 6.2. If C is commutative, then γ is a homomorphism between commutative groups, as before.

If $\widetilde{\rho}_l$ is injective as a mapping from $\mathrm{Hom}(Y_l, C)$ into $\mathrm{Hom}(Y, C)$ for every $l \in I$, then one can check that γ is injective on the direct limit of (6.11.2). If ρ_l maps Y onto Y_l for some $l \in I$, then it is easy to see that $\widetilde{\rho}_l$ is injective on $\mathrm{Hom}(Y_l, C)$.

6.12 Inverse systems and dual groups

Let $(I, \preceq)$ be a nonempty directed set again, and let Y_j be a commutative topological group for every $j \in I$. If $j, l \in I$ and $j \preceq l$, then let $\phi_{j,l}$ be a continuous group homomorphism from Y_l into Y_j. As before, we ask that $\phi_{j,j}$ be the identity mapping on Y_j for every $j \in I$, and that $\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}$ for every $j, l, r \in I$ with $j \preceq l \preceq r$. Remember that $\prod_{j \in I} Y_j$ is a commutative topological group with respect to the associated product topology. Thus the inverse limit Y of the Y_j's, $j \in I$, is a commutative topological group with respect to the induced topology.

Let us consider analogues of the remarks in the previous section with $C = \mathbf{T}$ and continuous homomorphisms. If A is a commutative topological group, then $\widehat{A}$ denotes the dual group of continuous homomorphisms from A into $\mathbf{T}$, as in Section 5.2. If $j, l \in I$ and $j \preceq l$, then let $\widehat{\phi}_{j,l}$ be the dual homomorphism from $\widehat{Y_j}$ into $\widehat{Y_l}$, as in Section 5.10. Note that $\widehat{\phi}_{j,j}$ is the identity mapping on $\widehat{A_j}$ for every $j \in I$, because $\phi_{j,j}$ is the identity mapping on A_j. If $j, l, r \in I$ and $j \preceq l \preceq r$, then

$$\widehat{\phi}_{j,r} = \widehat{\phi}_{l,r} \circ \widehat{\phi}_{j,l}, \tag{6.12.1}$$

by the remarks about duals of compositions of continuous homomorphisms in Section 5.10.

It follows that

$$\widehat{Y_j},\ j \in I, \tag{6.12.2}$$

is a direct system of commutative groups, with respect to the corresponding family of homomorphisms $\widehat{\phi}_{j,l}$. If $l \in I$, then there is a natural homomorphism β_l from $\widehat{A_l}$ into the direct limit of (6.12.2), as in Sections 6.1, 6.2.

If $l \in I$, then let ρ_l be the usual mapping from Y into Y_l, which is a continuous group homomorphism in this case. This leads to a dual homomorphism $\widehat{\rho_l}$ from $\widehat{Y_l}$ into $\widehat{Y}$, as before. If $l, r \in I$ and $l \preceq r$, then

$$\widehat{\rho_l} = \widehat{\rho_r} \circ \widehat{\phi_{l,r}}, \tag{6.12.3}$$

because $\rho_l = \phi_{l,r} \circ \rho_r$, and the usual properties of dual homomorphisms. Using this, we get a unique homomorphism γ from the direct limit of (6.12.2) into $\widehat{Y}$ such that

$$\gamma \circ \beta_l = \widehat{\rho_l} \tag{6.12.4}$$

for every $l \in I$, as in Sections 6.1, 6.2.

Suppose for the moment that for every $j \in I$, the set containing only the identity element is a closed set in Y_j, so that Y_j is Hausdorff. Suppose that Y_j is also compact for every $j \in I$, and that $\phi_{j,l}$ maps Y_l onto Y_j for every $j, l \in I$ with $j \preceq l$. Under these conditions, ρ_l maps Y onto Y_l for every $l \in I$, as in Section 6.6. This implies that $\widehat{\rho_l}$ is injective on $\widehat{Y_l}$ for every $l \in I$. One can use this to get that γ is injective on the direct limit of (6.12.2), as before.

6.13 Duals of inverse limits

Let us continue with the same notation and hypotheses as at the beginning of the previous section. Let η be a continuous homomorphism from Y into $\mathbf{T}$, which is to say an element of $\widehat{Y}$. Observe that

$$\mathrm{Re}\, \eta > 0 \tag{6.13.1}$$

on an open set $U \subseteq Y$ that contains the identity element.

As in Section 6.6, there is an $l_0 \in I$ and an open set $U_{l_0} \subseteq Y_{l_0}$ such that U_{l_0} contains the identity element and

$$\rho_{l_0}^{-1}(U_{l_0}) \subseteq U. \tag{6.13.2}$$

In particular, this means that the kernel of ρ_{l_0} is contained in U, so that (6.13.1) holds on the kernel of ρ_{l_0}. It follows that

$$\eta \equiv 1 \text{ on the kernel of } \rho_{l_0}, \tag{6.13.3}$$

as in Section 5.1.

Suppose for the rest of the section that for each $j \in I$, the set containing only the identity element is a closed set in Y_j, and that Y_j is compact. Suppose also that for every $j, l \in I$ with $j \preceq l$, we have that $\phi_{j,l}(Y_l) = Y_j$. This implies that $\rho_l(Y) = Y_l$ for every $l \in I$, as in the previous section. Combining this with (6.13.3), we get that there is a homomorphism η_{l_0} from Y_{l_0} into $\mathbf{T}$ such that

$$\eta = \eta_{l_0} \circ \rho_{l_0}. \tag{6.13.4}$$

Remember that Y is compact under these conditions, as in Section 6.6. One can use the continuity of ρ_{l_0} and η, and the fact that $\rho_{l_0}(Y) = Y_{l_0}$, to get that η_{l_0} is continuous on Y_{l_0}, as in Section 5.10. This means that $\eta_{l_0} \in \widehat{Y_{l_0}}$, and that

$$\eta = \widehat{\rho}_{l_0}(\eta_{l_0}), \tag{6.13.5}$$

by (6.13.4). This shows that the mapping γ from the direct limit of (6.12.2) into $\widehat{Y}$ obtained from the $\widehat{\rho}_l$'s, $l \in I$, as in the previous section is surjective, and thus an isomorphism.

6.14 Subsets of inverse limits

Let $(I, \preceq)$ be a nonempty directed set, and let Y_j, $j \in I$, be an inverse system of nonempty sets over $(I, \preceq)$, as in Section 6.5. Thus Y_j is a nonempty set for every $j \in I$, and for each $j, l \in I$ with $j \preceq l$, we have a mapping $\phi_{j,l}$ from Y_l into Y_j. As before, $\phi_{j,j}$ should be the identity mapping on Y_j for every $j \in I$, and $\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}$ for every $j, l, r \in I$ with $j \preceq l \preceq r$. Remember that the inverse limit of the Y_j's is the set Y of $y \in \prod_{j \in I} Y_j$ such that $\phi_{l,r}(y_r) = y_j$ for every $l, r \in I$ with $l \preceq r$. If $l \in I$, then we let π_l be the standard coordinate projection from $\prod_{j \in I} Y_j$ onto Y_l, and we let ρ_l be the restriction of π_l to Y, as usual.

Let Z be a nonempty subset of Y, and put

$$W_l = \rho_l(Z) \tag{6.14.1}$$

for every $l \in I$, which is a nonempty subset of Y_l. If $l, r \in I$ and $l \preceq r$, then

$$\phi_{l,r}(W_r) = W_l, \tag{6.14.2}$$

because $\phi_{l,r} \circ \rho_r = \rho_l$, as in Section 6.5. This implies that the family of sets W_l, $l \in I$, together with the restriction of the mapping $\phi_{l,r}$ to W_r for $l, r \in I$ with $l \preceq r$, is a surjective inverse system. The inverse limit W of the W_l's, $l \in I$, is the same as the intersection of Y with $\prod_{l \in I} W_l$, as in Section 6.9. In particular,

$$Z \subseteq W, \tag{6.14.3}$$

by (6.14.1).

If $w \in W$ and $r \in I$, then $w_r \in W_r = \rho_r(Z)$, so that there is a $z \in Z$ such that $w_r = z_r$. This implies that

$$w_l = z_l \tag{6.14.4}$$

for every $l \in I$ with $l \preceq r$, because $w, z \in Y$. If $l_1, \ldots, l_n$ are finitely many elements of L, then there is an $r \in I$ such that $l_m \preceq r$ for every $m = 1, \ldots, n$, because $(I, \preceq)$ is a directed set. In particular, if $w \in W$ then there is a $z \in Z$ such that (6.14.4) holds for $l = l_1, \ldots, l_n$.

Suppose now that Y_j is a topological space for every $j \in I$, and that $\phi_{j,l}$ is continuous as a mapping from Y_l into Y_j for every $j, l \in I$ with $j \preceq l$. As before, we take Y to be equipped with the topology induced by the product topology on $\prod_{j \in I} Y_j$. In this case, the remarks in the preceding paragraph imply that

$$W \subseteq \overline{Z}, \tag{6.14.5}$$

where $\overline{Z}$ is the closure of Z in Y. If Z is a closed set in Y, then we get that $W = Z$.

If Z is compact in Y, then W_l is compact in Y_l for every $l \in I$, because ρ_l is continuous. Suppose that Y_j is Hausdorff for every $j \in I$, so that $\prod_{j \in I} Y_j$ is Hausdorff with respect to the product topology, and Y is Hausdorff with respect to the induced topology. This implies that Z is a closed set in Y when Z is compact.

6.15 Mappings between inverse limits

Let $(I, \preceq)$ be a nonempty directed system, and let Y_j^1 and Y_j^2, $j \in I$, be inverse systems of nonempty sets over $(I, \preceq)$, as in Section 6.5. If $i = 1, 2$, then Y_j^i is a nonempty set for every $j \in I$, and for each $j, l \in I$ with $j \preceq l$, we have a mapping $\phi_{j,l}^i$ from Y_l^i into Y_j^i. As usual, $\phi_{j,j}^i$ should be the identity mapping on Y_j^i for every $j \in I$, and

$$\phi_{j,r}^i = \phi_{j,l}^i \circ \phi_{l,r}^i \tag{6.15.1}$$

for every $j, l, r \in I$ with $j \preceq l \preceq r$. The inverse limit of Y_j^i, $j \in I$, is the set Y^i of $y^i \in \prod_{j \in I} Y_j^i$ such that

$$\phi_{l,r}^i(y_r^i) = y_l^i \tag{6.15.2}$$

for every $l, r \in I$ with $l \preceq r$. If $l \in I$, then let π_l^i be the standard coordinate projection from $\prod_{j \in I} Y_j^i$ onto Y_l^i, and let ρ_l^i be the restriction of π_l^i to Y^i, as before.

Let f_j be a mapping from Y_j^1 into Y_j^2 for each $j \in I$. This leads to a mapping F from $\prod_{j \in I} Y_j^1$ into $\prod_{j \in I} Y_j^2$, using f_j in the jth coordinate for each $j \in I$, so that

$$\pi_l^2 \circ F = f_l \circ \pi_l^1 \tag{6.15.3}$$

for every $l \in I$. Equivalently, if $y^1 \in \prod_{j \in I} Y_j^1$ and

$$y_j^2 = f_j(y_j^1) \tag{6.15.4}$$

for every $j \in I$, then the corresponding element y^2 of $\prod_{j \in I} Y_j^2$ is equal to $F(y^1)$.

If $l, r \in I$ and $l \preceq r$, then suppose that

$$f_l \circ \phi_{l,r}^1 = \phi_{l,r}^2 \circ f_r. \tag{6.15.5}$$

If $y^1 \in Y^1$ and $y^2 = F(y^1)$, then

$$\phi_{l,r}^2(y_r^2) = \phi_{l,r}^2(f_r(y_r^1)) = f_l(\phi_{l,r}^1(y_r^1)) = f_l(y_l^1) = y_l^2, \tag{6.15.6}$$

so that $y^2 \in Y^2$. Let f be the restriction of F to Y^1, which maps Y^1 into Y^2. Note that

$$\rho_l^2 \circ f = f_l \circ \rho_l^1 \tag{6.15.7}$$

for every $l \in I$, by (6.15.3).

Suppose for the moment that Y_j^i is a topological space for every $j \in I$ and $i = 1, 2$, and that f_j is continuous as a mapping from Y_j^1 into Y_j^2 for every $j \in I$. This implies that F is continuous as a mapping from $\prod_{j\in I} Y_j^1$ into $\prod_{j\in I} Y_j^2$, with respect to the corresponding product topologies. If (6.15.5) holds for every $l, r \in I$ with $l \preceq r$, then it follows that f is continuous as a mapping from Y^1 into Y_2, with respect to the topology induced on Y^i by the product topology on $\prod_{j\in I} Y_j^i$ for $i = 1, 2$. Of course, one normally asks that $\phi_{j,l}^i$ be continuous as a mapping from Y_l^i into Y_j^i for every $j, l \in I$ with $j \preceq l$ and $i = 1, 2$ in this case, as before.

Suppose now that Y_j^i is a group for every $j \in I$ and $i = 1, 2$, and that f_j is a homomorphism from Y_j^1 into Y_j^2 for every $j \in I$. This means that F is a homomorphism from $\prod_{j\in I} Y_j^1$ into $\prod_{j\in I} Y_j^2$, where the group operations are defined on the products coordinatewise. Suppose that $\phi_{j,l}^i$ is a homomorphism from Y_l^i into Y_j^i for every $j, l \in I$ with $j \preceq l$ and $i = 1, 2$, so that Y^i is a subgroup of $\prod_{j\in I} Y_j^i$ for $i = 1, 2$. If (6.15.5) holds for every $l, r \in I$ with $l \preceq r$, then f is a homomorphism from Y^1 into Y^2.

Chapter 7

Indices and Sylow subgroups

7.1 Counting functions

Let us call a function $c(p)$ defined for prime numbers p with values in

$$\mathbf{Z}_+ \cup \{0, +\infty\} \tag{7.1.1}$$

a *counting function*. In this case, the formal product

$$\prod_p p^{c(p)} \tag{7.1.2}$$

over all prime numbers p may be considered as a type of extension of a positive integer, as on p5 of [25]. Of course, this product defines a positive integer when $c(p) < +\infty$ for every p, and $c(p) = 0$ for all but finitely many p. Every positive integer corresponds to a unique counting function with these properties in this way.

Addition of nonnegative integers can be extended to (7.1.1) in the usual way, where the sum of $+\infty$ and any element of (7.1.1) is $+\infty$. The sum of two counting functions can be defined pointwise, and is a counting function as well. Equivalently, the product of two formal products as in (7.1.2) is the formal product associated to the sum of the corresponding counting functions, as on p5 of [25]. A formal product (7.1.2) is considered to be a power of a prime number p_1 if the corresponding counting function $c(p)$ is equal to 0 for every prime number $p \neq p_1$.

Note that (7.1.1) is well ordered by the standard ordering. This means that any nonempty collection of counting functions has a pointwise minimum which is also a counting function. This corresponds to taking the greatest common divisor of the associated formal products, as on p5 of [25]. In particular, a pair of formal products as in (7.1.2) are considered to be relatively prime when the minimum of the corresponding counting functions is equal to 0.

Similarly, every nonempty subset of (7.1.1) has a least upper bound in (7.1.1). More precisely, the maximum is attained except for infinite subsets of $\mathbf{Z}_+ \cup \{0\}$, for which the supremum is $+\infty$. Thus any nonempty collection of counting functions has a pointwise supremum that is a counting function too. This corresponds to taking the least common multiple of the associated formal products, as on p5 of [25].

Let I be a nonempty set, and let $c_j(p)$ be a counting function for each $j \in I$. If I has only finitely many elements, then

$$\sum_{j \in I} c_j(p) \tag{7.1.3}$$

can be defined as an element of (7.1.1) for every prime number p as before. Otherwise, the sum can be defined by taking the supremum of the sum of $c_j(p)$ over all nonempty finite subsets of I. The product of the formal products associated to $c_j(p)$, $j \in I$, can be defined as the formal product associated to (7.1.3).

7.2 Orders of profinite groups

Let G be a profinite group, and let U be an open normal subgroup of G. Thus G/U is a finite group of order $n_U \in \mathbf{Z}_+$. Let c_U be the counting function associated to n_U, so that $c_U(p)$ is the number of factors of p in n_U for every prime number p. Put

$$c(p) = \sup_U c_U(p) \tag{7.2.1}$$

for every prime number p, where the supremum is taken over all open normal subgroups U of G. This defines a counting function, and the *order* of G is considered to be the associated formal product, as on p6 of [25].

Let U, V be open normal subgroups of G such that $U \subseteq V$. Thus V/U corresponds to a normal subgroup of G/U, and the quotient of G/U by V/U is isomorphic to G/V. In particular, n_U is equal to n_V times the order of V/U. This implies that

$$c_V(p) \le c_U(p) \tag{7.2.2}$$

for every prime number p. It follows that $c(p)$ can be obtained by taking the supremum of $c_U(p)$ over any collection of open normal subgroups U of G that form a local base for the topology at the identity element.

Suppose for the moment that G has only finitely many elements, and let U_0 be the subgroup of G consisting of only the identity element. Thus U_0 is an open normal subgroup of G, and $c(p) = c_{U_0}(p)$ for every prime number p, by (7.2.2). This means that the usual definition of the order of G agrees with this one in this case.

The previous definition of $c(p)$ and the order of G works as well for totally bounded topological groups G for which the open normal subgroups form a local base for the topology at the identity element. In this case, one could get a profinite group from G as in Section 2.14. One can verify that one would get the same counting function for G as for the associated profinite group, basically because one would get the same quotients by open normal subgroups.

If U is any open subgroup of G, then U has finite index in G, and one can take $n_U \in \mathbf{Z}_+$ to be the usual index of U in G. Let V be another open subgroup of G with $U \subseteq V$, and let c_U, c_V be the counting functions associated to n_U, n_V, respectively. It is well known that n_U is equal to n_V times the index of U in V, so that (7.2.2) holds for every prime number p in particular.

A finite group is said to be a *p-group* for some prime number p if the order of the group is a power of p. Similarly, a profinite group G is said to be a *pro-p-group* if its order in the sense defined before is a power of p, as on p6 of [25]. Equivalently, this means that G/U is a p-group for every open normal subgroup U of G.

Let I be a nonempty set, and let G_j be a profinite group for every $j \in I$. Remember that the product $\prod_{j \in I} G_j$ is a profinite group too, with respect to the product topology. One can check that the counting funtion associated to $\prod_{j \in I} G_j$ is the same as the sum over $j \in I$ of the counting function associated to G_j. Equivalently, the order of $\prod_{j \in I} G_j$ is the product of the orders of the G_j's, $j \in I$.

7.3 Indices of subgroups

Let G be a profinite group, and let H be a subgroup of G. If U is an open normal subgroup of G, then let q_U be the natural quotient mapping from G onto G/U. Thus $q_U(H)$ is a subgroup of G/U, which is isomorphic to $H/(H \cap U)$. Put

$$n^{U,H} = [G/U : q_U(H)], \tag{7.3.1}$$

where the right side is the usual index of $q_U(H)$ in G/U. This is a positive integer, and we let $c^{U,H} = c_G^{U,H}$ be the counting function associated to $n^{U,H}$.

Put

$$c^H(p) = c_G^H(p) = \sup_U c^{U,H}(p) \tag{7.3.2}$$

for every prime number p, where the supremum is taken over all open normal subgroups U of G. This is a counting function, and the *index* of H in G as a profinite group is defined to be the associated formal product,

$$(G : H) = \prod_p p^{c^H(p)}, \tag{7.3.3}$$

as on p5 of [25]. If H is the subgroup of G consisting of only the identity element, then (7.3.3) is the same as the order of G as defined in the previous section, as on p6 of [25].

We use parentheses to express this version of the index, rather than the usual index $[G : H]$, which may be considered as a cardinal number. As in the previous section, one could also consider totally bounded topological groups G

for which the open normal subgroups form a local base for the topology at the identity element.

If H is any subgroup of G, then the closure $\overline{H}$ of H in G is a subgroup of G as well, and it is easy to see that

$$q_U(\overline{H}) = q_U(H) \tag{7.3.4}$$

for every open normal subgroup U of G. This implies that

$$(G : \overline{H}) = (G : H). \tag{7.3.5}$$

One may wish to restrict one's attention to closed subgroups of G, as in [25].

If U is a normal subgroup of G, then

$$W = U\,H = H\,U \tag{7.3.6}$$

is a subgroup of G, as in Section 3.8. Suppose that U is an open normal subgroup of G, so that W is an open subgroup of G. Equivalently,

$$W = q_U^{-1}(q_U(H)), \tag{7.3.7}$$

and $q_U(W) = q_U(H)$ in particular. Of course, $U \subseteq W$, so that $q_U(W) = W/U$, and

$$n^{U,H} = [G/U : W/U]. \tag{7.3.8}$$

It follows that

$$n^{U,H} = [G : W]. \tag{7.3.9}$$

Let U' be another open normal subgroup of G with

$$U \subseteq U', \tag{7.3.10}$$

and put

$$W' = U'\,H = H\,U' \tag{7.3.11}$$

as before. Thus $W \subseteq W'$, so that

$$[G : W] = [G : W'] \cdot [W' : W]. \tag{7.3.12}$$

In particular, this means that

$$c^{U',H}(p) \le c^{U,H}(p) \tag{7.3.13}$$

for every prime number p. This implies that $c^H(p)$ can be obtained by taking the supremum of $c^{U,H}(p)$ over any collection of open normal subgroups U of G that form a local base for the topology at the identity element.

Let V be an open subgroup of G with $H \subseteq V$. Because the open normal subgroups of G form a local base for the topology at the identity element, there is an open normal subgroup U of G such that

$$U \subseteq V. \tag{7.3.14}$$

Let W be as in (7.3.6), so that

$$W \subseteq V. \tag{7.3.15}$$

Also let $n_V, n_W \in \mathbf{Z}_+$ be the indices of V, W in G, respectively, and let c_V, c_W be their associated counting functions, as in the previous section. Observe that

$$c_V(p) \le c_W(p) = c^{U,H}(p) \tag{7.3.16}$$

for every prime number p, using (7.2.2) in the first step, and (7.3.9) in the second step.

Alternatively,

$$c^H(p) = \sup_{H \subseteq V} c_V(p) \tag{7.3.17}$$

for every prime number p, where the supremum is taken over all open subgroups V of G with $H \subseteq V$. This corresponds to the second characterization of the index on p5 of [25]. More precisely, the fact that the supremum is less than or equal to $c^H(p)$ follows from (7.3.16) and the previous definition (7.3.2) of $c^H(p)$. To get equality, one can use the fact that if U is an open normal subgroup of G, then W defined in (7.3.6) is an open subgroup of G that contains H.

If H is an open subgroup of G, then the definition (7.3.3) of the index of H in G is equivalent to the usual definition of the index. This follows by taking $V = H$ in (7.3.17), and using (7.2.2). This corresponds to part of part (iii) of Proposition 2 on p5 of [25].

7.4 Indices of sub-subgroups

Let G be a profinite group, and let H, K be subgroups of G, with

$$K \subseteq H. \tag{7.4.1}$$

We would like to check that

$$(G : K) = (G : H) \cdot (H : K), \tag{7.4.2}$$

where the indices are as defined in the previous section. This corresponds to part (i) of Proposition 2 on p5 of [25].

More precisely, one can take G to be a totally bounded topological group for which the open normal subgroups form a local base for the topology at the identity element, as before. This implies that H has the analogous properties with respect to the induced topology, so that $(H : K)$ can be defined as before too. If one takes G to be profinite, then one can take H, K to be closed subgroups, so that they are profinite with respect to the induced topology.

Let c_G^H, c_G^K be the counting functions used to define the indices of H, K in G, as in (7.3.2). Similarly, let c_H^K be the counting function used to define the index of K in H. Thus (7.4.2) is the same as saying that

$$c_G^K(p) = c_G^H(p) + c_H^K(p) \tag{7.4.3}$$

for every prime number p.

Let U be an open normal subgroup of G, and let $q_{G,U}$ be the natural quotient mapping from G onto G/U. Observe that

$$q_{G,U}(K) \subseteq q_{G,U}(H), \tag{7.4.4}$$

so that

$$[G/U : q_{G,U}(K)] = [G/U : q_{G,U}(H)] \cdot [q_{G,U}(H) : q_{G,U}(K)]. \tag{7.4.5}$$

Put

$$U_H = U \cap H, \tag{7.4.6}$$

which is a normal subgroup of H that is relatively open in H. Let q_{H,U_H} be the natural quotient mapping from H onto H/U_H, which is isomorphic to $q_{G,U}(H)$. Using this isomorphism, $q_{H,U_H}(K)$ corresponds to $q_{G,U}(K)$, so that

$$[H/U_H : q_{H,U_H}(K)] = [q_{G,U}(H) : q_{G,U}(K)]. \tag{7.4.7}$$

Combining this with (7.4.5), we get that

$$[G/U : q_{G,U}(K)] = [G/U : q_{G,U}(H)] \cdot [H/U_H : q_{H,U_H}(K)]. \tag{7.4.8}$$

Let $c_G^{U,K}$ be the counting function associated to the left side of (7.4.8), and let $c_G^{U,H}$, $c_H^{U_H,K}$ be the counting functions associated to the two indices on the right side of (7.4.8), respectively, as in the previous section. Thus

$$c_G^{U,K}(p) = c_G^{U,H}(p) + c_H^{U_H,K}(p) \tag{7.4.9}$$

for all prime numbers p. It follows that

$$c_G^{U,K}(p) \le c_G^H(p) + c_H^K(p) \tag{7.4.10}$$

for all prime numbers p, by the definition of $c_G^H(p)$, $c_H^K(p)$. This implies that

$$c_G^K(p) \le c_G^H(p) + c_H^K(p) \tag{7.4.11}$$

for every prime number p.

Similarly, (7.4.9) implies that

$$c_G^{U,H}(p) + c_H^{U_H,K}(p) \le c_G^K(p) \tag{7.4.12}$$

for every prime number p, by the definition of c_G^K. In order to get (7.4.3), one can use the fact that the relatively open normal subgroups of H of the form (7.4.6), where U is an open normal subgroup of G, form a local base for the induced topology on H at the identity element. One can also use the fact that $c_G^{U,H}(p)$ and $c_H^{U_H,K}(p)$ can only get larger as U gets smaller, as in (7.3.13).

7.5 Finiteness of the index

Let G be a profinite group again, and let H be a subgroup of G. Note that

$$(G : H) = 1, \tag{7.5.1}$$

where the index is defined as in Section 7.3, if and only if $c^H(p) = 0$ for every prime number p, where $c^H(p)$ is as in (7.3.2). Clearly this happens if and only if $c^{U,H}(p) = 0$ for every open normal subgroup U of G, where $c^{U,H}$ is the counting function associated to the usual index (7.3.1) of $q_U(H)$ in G/U. Here q_U is the natural quotient mapping from G onto G/U, as before.

Thus (7.5.1) holds if and only if

$$[G/U : q_U(H)] = 1 \tag{7.5.2}$$

for every open normal subgroup U of G. This is the same as saying that

$$q_U(H) = G/U \tag{7.5.3}$$

for every open normal subgroup U in G. One can check that this happens if and only if H is dense in G. Of course, if H is a closed subgroup of G, then this means that H is the whole group. This also works when G is a totally bounded topological group, for which the open normal subgroups form a local base for the topology at the identity element.

Suppose now that the index $(G : H)$ of H in G defined in Section 7.3 corresponds to a positive integer. This means that

$$c^H(p) < +\infty \tag{7.5.4}$$

for every prime number p, and that

$$c^H(p) = 0 \tag{7.5.5}$$

for all but finitely many prime numbers p.

If V is an open subgroup of G, then let $n_V \in \mathbf{Z}_+$ be the usual index of V in G, and let c_V be the counting function associated to n_V, as in Section 7.2. If $H \subseteq V$, then

$$c_V(p) = 0 \tag{7.5.6}$$

for every prime number p for which (7.5.5) holds, by the characterization (7.3.17) of $c^H(p)$. If p is any prime number, then there is an open subgroup V_p of G such that

$$H \subseteq V_p \tag{7.5.7}$$

and

$$c_{V_p}(p) = c^H(p), \tag{7.5.8}$$

because of (7.3.17) and (7.5.4).

Using this, we can find an open subgroup V of G such that $H \subseteq V$ and

$$c_V(p) = c^H(p) \tag{7.5.9}$$

for every prime number p. More precisely, we can take V to be the intersection of open subgroups V_p as in the preceding paragraph, over the finitely many prime numbers p such that $c^H(p) > 0$. This also uses the fact that $c_V(p)$ can only get larger when V gets smaller, as in (7.2.2).

Remember that $(V : H)$ can be defined as in Section 7.3 too, and let c_V^H be the corresponding counting function. In fact,

$$c_V^H(p) = 0 \tag{7.5.10}$$

for every prime number p, because of (7.4.3) and (7.5.9). Equivalently, this means that

$$(V : H) = 1. \tag{7.5.11}$$

It follows that H is dense in V, as before. This is the same as saying that the closure $\overline{H}$ of H in G is equal to V. If H is a closed subgroup of G, then we get that $H = V$, so that H is an open subgroup of G under these conditions. This corresponds to part of part (iii) of Proposition 2 on p5 of [25]. This works as well when G is a totally bounded topological group, and the open normal subgroups of G form a local base for the topology at the identity element.

7.6 Chains of subgroups

Let G be a profinite group, and if H is a subgroup of G, then let c^H be the counting function used to define the index of H, as in (7.3.2). Also let $\mathcal{C}$ be a nonempty collection of subgroups of G, and put

$$H_{\mathcal{C}} = \bigcap_{H \in \mathcal{C}} H, \tag{7.6.1}$$

which is a subgroup of G as well. If $H \in \mathcal{C}$, then $H_{\mathcal{C}} \subseteq H$, and thus

$$c^H(p) \le c^{H_{\mathcal{C}}}(p) \tag{7.6.2}$$

for every prime number p, by (7.4.3).

Suppose from now on in this section that the elements of $\mathcal{C}$ are closed subgroups of G, so that $H_{\mathcal{C}}$ is a closed subgroup too. Suppose in addition that $\mathcal{C}$ is linearly ordered by inclusion, which means that for any two elements of $\mathcal{C}$, one is contained in the other.

Let V be an open subgroup of G such that

$$H_{\mathcal{C}} \subseteq V. \tag{7.6.3}$$

Observe that V, together with the complements of the elements of $\mathcal{C}$, form an open covering of G. Because G is compact, there is an open subcovering consisting of V together with the complements of finitely many elements of $\mathcal{C}$. It follows that there is an $H_0 \in \mathcal{C}$ such that

$$H_0 \subseteq V, \tag{7.6.4}$$

because $\mathcal{C}$ is linearly ordered by inclusion.

Let c_V be the counting function associated to the index of V in G, as in Section 7.2. Observe that

$$c_V(p) \le c^{H_0}(p) \tag{7.6.5}$$

for all prime numbers p, by (7.3.17) and (7.6.4). This implies that

$$c_V(p) \le \sup_{H \in \mathcal{C}} c^H(p) \tag{7.6.6}$$

for all prime numbers p. It follows that

$$c^{H_\mathcal{C}}(p) \le \sup_{H \in \mathcal{C}} c^H(p) \tag{7.6.7}$$

for all prime numbers p, by (7.3.17). Combining this with (7.6.2), we obtain that

$$c^{H_\mathcal{C}}(p) = \sup_{H \in \mathcal{C}} c^H(p) \tag{7.6.8}$$

for all prime numbers p, as in part (ii) of Proposition 2 on p5 of [25].

7.7 Orders of subgroups

Let G be a profinite group, and let A be a subgroup of G. More precisely, it suffices to ask for the moment that G be a totally bounded topological group for which the open normal subgroups form a local base for the topology at the identity element. One can check that A is totally bounded as a topological group with respect to the induced topology, using the characterization of total boundedness in terms of small sets mentioned in Section 2.8. If $\mathcal{B}$ is a local base for the topology of G at the identity element consisting of open normal subgroups of G, then

$$\mathcal{B}_A = \{A \cap U : U \in \mathcal{B}\} \tag{7.7.1}$$

is a local base for the induced topology on A that consists of relatively open normal subgroups of A.

If V is a normal subgroup of A that is an open set with respect to the induced topology on A, then A/V has only finitely many elements, and we let $c_{A,V}$ be the counting function associated to the number $n_{A,V}$ of elements of A/V. The counting function used to define the order of A is given by

$$c_A(p) = \sup_V c_{A,V}(p) \tag{7.7.2}$$

for every prime number p, where the supremum is taken over all relatively open normal subgroups V of A, as in Section 7.2. In fact, it suffices to take the supremum over any collection of relatively open normal subgroups V of A that form a local base for the induced topology on A at the identity element, as before.

If U is an open normal subgroup of G, then $A \cap U$ is a relatively open normal subgroup of A. Observe that

$$c_A(p) = \sup_U c_{A, A \cap U}(p) \tag{7.7.3}$$

for every prime number p, where the supremum is taken over all open normal subgroups U of G. This uses the fact that the collection of relatively open normal subgroups of A of the form $A \cap U$, where U is an open normal subgroup of G, form a local base for the induced topology on A at the identity element, because the open normal subgroups of G form a local base for the topology of G at the identity element. One could also take the supremum over any collection $\mathcal{B}$ of open normal subgroups U of G that form a local base for the topology at the identity element, because the corresponding collection (7.7.1) would form a local base for the induced topology on A at the identity element.

Let U be an open normal subgroup of G, and let q_U be the natural quotient mapping from G onto G/U. The kernel of the restriction of q_U to A is $A \cap U$, so that $q_U(A)$ is isomorphic to $A/(A \cap U)$. Thus $n_{A, A \cap U}$ is the same as the number of elements of $q_U(A)$. It is easy to see that

$$q_U(\overline{A}) = q_U(A), \tag{7.7.4}$$

where $\overline{A}$ is the closure of A in G. This implies that

$$n_{\overline{A}, \overline{A} \cap U} = n_{A, A \cap U}, \tag{7.7.5}$$

so that

$$c_{\overline{A}, \overline{A} \cap U} = c_{A, A \cap U}, \tag{7.7.6}$$

and thus

$$c_{\overline{A}} = c_A. \tag{7.7.7}$$

Suppose that A is profinite, which happens in particular when G is profinite and A is a closed subgroup of G. It follows from (7.7.3) that A is a pro-p-group for some prime number p if and only if

$$c_{A, A \cap U}(p') = 0 \tag{7.7.8}$$

for every open normal subgroup U of G and prime number $p' \neq p$. Equivalently, this means that $A/(A \cap U)$ is a p-group for every open normal subgroup U of G. Of course, this is the same as saying that $q_U(A)$ is a p-group for every open normal subgroup U of G. More precisely, it suffices to consider any collection of open normal subgroups U of G that form a local base for the topology at the identity element.

7.8 Sylow p-subgroups

Let p be a prime number, and let G be a finite group. Remember that a subgroup A of G is said to be a *Sylow p-subgroup* if A is a p-subgroup of G, which is to say that it is a p-group, and the index $[G : A]$ is not a multiple of p. Equivalently, this means that the order of A is a power of p, and the largest power of p of which the order of G is a multiple. The first Sylow theorem states that G has a Sylow p-subgroup.

Suppose that ϕ is a homomorphism from G onto another group H. If A is a p-subgroup of G, then it is easy to see that $\phi(A)$ is a p-subgroup of H. If A is a Sylow p-subgroup of G, then $\phi(A)$ is a Sylow p-subgroup of H. Indeed, put

(7.8.1) $$A_1 = \phi^{-1}(\phi(A)),$$

which is a subgroup of G that contains A. Note that

(7.8.2) $$[G : A] = [G : A_1] \cdot [A_1 : A]$$

and

(7.8.3) $$[G : A_1] = [H : \phi(A)].$$

Using (7.8.2), we get that $[G : A_1]$ is not a multiple of p. This means that $[H : \phi(A)]$ is not a multiple of p, as desired.

Let B be a Sylow p-subgroup of H, and let us check that B corresponds to a Sylow p-subgroup of G in this way. Put

(7.8.4) $$B_1 = \phi^{-1}(B),$$

which is a subgroup of G, and let B_0 be a Sylow p-subgroup of B_1. Observe that

(7.8.5) $$[G : B_0] = [G : B_1] \cdot [B_1 : B_0]$$

and

(7.8.6) $$[G : B_1] = [H : B].$$

It follows that the index of B_0 in G is not a multiple of p, so that B_0 is a Sylow p-subgroup of G.

We would like to verify that

(7.8.7) $$\phi(B_0) = B.$$

Of course,

(7.8.8) $$\phi(B_0) \subseteq \phi(B_1) = B,$$

by construction. We also have that $\phi(B_0)$ is a Sylow p-subgroup of H, because B_0 is a Sylow p-subgroup of G, as before. This implies that $\phi(B_0)$ has the same number of elements as B, so that (7.8.7) holds.

Suppose now that G is a profinite group, and that A is a closed subgroup of G, so that A is profinite with respect to the induced topology. If A is a pro-p-group, as in Section 7.2, and if the index $(G : A)$ of A in G as a profinite group is not a multiple of p, then A is said to be a *Sylow p-subgroup* of G as a profinite group, as on p6 of [25]. If c^A is the counting function used to define $(G : A)$ as in Section 7.3, then the second condition means that $c^A(p) = 0$.

If U is an open normal subgroup of G, then let q_U be the natural quotient mapping from G onto G/U, as before. Remember that c^A is defined to be the supremum over all open normal subgroups U of G of the counting functions $c^{U,A}$ associated to the index of $q_U(A)$ in G/U, as in Section 7.3. Thus $c^A(p) = 0$ if and only if

(7.8.9) $$c^{U,A}(p) = 0$$

for every open normal subgroup U of G. As usual, it suffices to consider any collection of open normal subgroups U of G that form a local base for the topology at the identity element.

The first part of Proposition 3 on p7 of [25] states that G has a Sylow p-subgroup. This will be discussed further in the next section.

7.9 Sylow subgroups and inverse systems

Let G be a profinite group, and let $\mathcal{B}$ be a local base for the topology of G at the identity element consisting of open normal subgroups. If $U, V \in \mathcal{B}$, then put $U \preceq V$ when $V \subseteq U$, as in Section 6.7. Thus $(\mathcal{B}, \preceq)$ is a directed system, as before.

If $U \in \mathcal{B}$, then let q_U be the natural quotient mapping from G onto the finite group G/U. If $V \in \mathcal{B}$ satisfies $U \preceq V$, so that $V \subseteq U$, then there is a unique homomorphism $\Phi_{U,V}$ from G/V onto G/U such that

(7.9.1) $$\Phi_{U,V} \circ q_V = q_U,$$

as in Section 6.7. If $W \in \mathcal{B}$ and $V \preceq W$, so that $W \subseteq V$, then we get that

(7.9.2) $$\Phi_{U,V} \circ \Phi_{V,W} = \Phi_{U,W},$$

as before. In fact, G corresponds to the inverse limit of the quotients G/U, $U \in \mathcal{B}$, under these conditions. Here we take G/U to be equipped with the discrete topology for every $U \in \mathcal{B}$.

If $U \in \mathcal{B}$, then let $P(U)$ be the collection of Sylow p-subgroups of G/U, which is a nonempty finite set. If $V \in \mathcal{B}$, $U \preceq V$, and $A \in P(V)$, then

(7.9.3) $$\Phi_{U,V}(A) \in P(U),$$

as in the previous section. This defines a mapping $\widetilde{\Phi}_{U,V}$ from $P(V)$ into $P(U)$, and in fact this mapping is surjective, as mentioned earlier. If $W \in \mathcal{B}$ and $V \preceq W$, then it is easy to see that

(7.9.4) $$\widetilde{\Phi}_{U,V} \circ \widetilde{\Phi}_{V,W} = \widetilde{\Phi}_{U,W},$$

because of (7.9.2).

Thus the family of sets $P(U)$, $U \in \mathcal{B}$, with the associated mappings $\widetilde{\Phi}_{U,V}$, is a surjective inverse system. The corresponding inverse limit

(7.9.5) $$\varprojlim P(U)$$

can be defined as in Section 6.5. Note that (7.9.5) is nonempty, as in Section 6.6. More precisely, this can be seen by taking $P(U)$ equipped with the discrete topology for every $U \in \mathcal{B}$, so that $P(U)$ is compact and Hausdorff. This can be obtained more directly when there is a countable local base for the topology of G at the identity element, so that one can take $\mathcal{B}$ to consist of a nested sequence of open normal subgroups of G, as in Section 6.8.

Let us take an element of (7.9.5), which assigns to each $U \in \mathcal{B}$ an element A_U of $P(U)$. If $U, V \in \mathcal{B}$ and $U \preceq V$, then

$$\widetilde{\Phi}_{U,V}(A_V) = A_U, \tag{7.9.6}$$

by definition of (7.9.5). Equivalently, this means that $\Phi_{U,V}$ maps A_V onto A_U.

It follows that the family of subgroups A_U of G/U, $U \in \mathcal{B}$, is a surjective inverse system with respect to the restrictions of the mappings $\Phi_{U,V}$ to A_V. The inverse limit

$$\lim_{\longleftarrow} A_U \tag{7.9.7}$$

is a closed subgroup of

$$\lim_{\longleftarrow} G/U, \tag{7.9.8}$$

as in Section 6.9. Note that A_U is a p-group for every $U \in \mathcal{B}$, because A_U is an element of $P(U)$. This implies that (7.9.7) is a pro-p-group.

Remember that (7.9.8) is contained in the Cartesian product $\prod_{U \in \mathcal{B}}(G/U)$, by construction. If $V \in \mathcal{B}$, then let π_V be the standard coordinate projection from $\prod_{U \in \mathcal{B}}(G/U)$ onto G/V, and let ρ_V be the restriction of π_V to (7.9.8). Remember that ρ_V maps (7.9.8) onto G/V under these conditions, as in Section 6.6. Similarly, ρ_V maps (7.9.7) onto A_V.

The index of (7.9.7) in (7.9.8) as a profinite group, as in Section 7.3, can be obtained from the index of A_V in G/V, $V \in \mathcal{B}$, as before. In particular, the index of (7.9.7) in (7.9.8) is not a multiple of p, because $A_V \in P(V)$ for every $V \in \mathcal{B}$. This means that (7.9.7) is a Sylow p-subgroup of (7.9.8), as a profinite group. Thus (7.9.7) corresponds to a Sylow p-subgroup of G, because G is isomorphic to (7.9.8) as a profinite group, as in Section 6.7.

7.10 Conjugates of Sylow subgroups

Let p be a prime number, and let G be a finite group again. Part of the second Sylow theorem is that the Sylow p-subgroups of G are conjugate in G.

Now let G be a profinite group, and let A, A' be closed subgroups of G that are Sylow p-subgroups, as in Section 7.8. The second part of Proposition 3 on p7 of [25] states that A and A' are conjugate in G.

To see this, let $\mathcal{B}$ be a local base for the topology of G at the identity element consisting of open normal subgroups. If $U, V \in \mathcal{B}$ and $V \subseteq U$, then put $U \preceq V$, as before, so that $(\mathcal{B}, \preceq)$ is a directed system. Let q_U be the natural quotient mapping from G onto G/U for every $U \in \mathcal{B}$, and if $V \in \mathcal{B}$ satisfies $U \preceq V$, then let $\Phi_{U,V}$ be the unique homomorphism from G/V onto G/U that satisfies (7.9.1).

Remember that A, A' are pro-p-groups, whose indices in G are not multiples of p. If $U \in \mathcal{B}$, then $q_U(A)$, $q_U(A')$ are isomorphic to $A/(A \cap U)$, $A'/(A' \cap U)$, respectively. This implies that $q_U(A)$, $q_U(A')$ are p-groups for every $U \in \mathcal{B}$, because A, A' are pro-p-groups.

Similarly, the second condition means that the indices of $q_U(A)$, $q_U(A')$ in G/U are not multiples of p for any $U \in \mathcal{B}$. It follows that $q_U(A)$, $q_U(A')$ are Sylow p-subgroups of G/U for each $U \in \mathcal{B}$. Thus $q_U(A)$ and $q_U(A')$ are conjugate in G/U for every $U \in \mathcal{B}$, by the second Sylow theorem.

If $U \in \mathcal{B}$, then let $Q(U)$ be the set of elements of G/U that can be used to conjugate $q_U(A)$ onto $q_U(A')$. Suppose that $U, V \in \mathcal{B}$ satisfy $U \preceq V$, and observe that

$$\Phi_{U,V}(q_V(A)) = q_U(A), \quad \Phi_{U,V}(q_V(A')) = q_U(A'), \tag{7.10.1}$$

by (7.9.1). This implies that

$$\Phi_{U,V}(Q(V)) \subseteq Q(U). \tag{7.10.2}$$

It follows that the family of sets $Q(U)$, $U \in \mathcal{B}$, is an inverse system, with respect to the restriction of $\Phi_{U,V}$ to $Q(V)$ for every $U, V \in \mathcal{B}$ with $U \preceq V$.

Of course, $Q(U)$ has only finitely many elements for each $U \in \mathcal{B}$, because G/U has only finitely many elements. We also have that $Q(U) \neq \emptyset$ for every $U \in \mathcal{B}$, by the second Sylow theorem, as before. Let us take $Q(U)$ to be equipped with the discrete topology for every $U \in \mathcal{B}$, so that $Q(U)$ is compact and Hausdorff. This implies that the inverse limit

$$\varprojlim Q(U) \tag{7.10.3}$$

of the family of $Q(U)$'s, $U \in \mathcal{B}$, is nonempty, as in Section 6.6.

Remember that the family of quotients G/U, $U \in \mathcal{B}$, is an inverse system with respect to the mappings $\Phi_{U,V}$. Similarly, the families of subgroups $q_U(A)$, $q_U(A')$ of G/U, $U \in \mathcal{B}$, are inverse systems with respect to the restrictions of $\Phi_{U,V}$ to $q_V(A)$, $q_V(A')$, because of (7.10.1). Their inverse limits

$$\varprojlim q_U(A) \tag{7.10.4}$$

and

$$\varprojlim q_U(A') \tag{7.10.5}$$

are subgroups of the inverse limit of G/U, $U \in \mathcal{B}$, as in Section 6.9. Of course, (7.10.3) is a subset of the inverse limit of G/U, $U \in \mathcal{B}$, too. One can check that (7.10.4) and (7.10.5) are conjugate in the inverse limit of G/U, $U \in \mathcal{B}$, using the elements of (7.10.3).

Remember that G is isomorphic to the inverse limit of G/U, $U \in \mathcal{B}$, as a profinite group, as in Section 6.7. Similarly, this isomorphism maps A, A' onto (7.10.4), (7.10.5), respectively. This implies that A and A' are conjugate in G, as desired.

7.11 Subgroups of Sylow subgroups

Let p be a prime number, and let G be a finite group. If a subgroup A of G is a p-group, then the third Sylow theorem states that A is contained in a Sylow p-subgroup of G.

Let ϕ be a homomorphism from G onto another group H. Also let A be a subgroup of G that is a p-group, and let C be a Sylow p-subgroup of G with

$$A \subseteq C. \tag{7.11.1}$$

Note that $\phi(A)$ is a p-group, and remember that $\phi(C)$ is a Sylow p-subgroup of H, as in Section 7.8. Of course,

$$\phi(A) \subseteq \phi(C). \tag{7.11.2}$$

Suppose that B is a Sylow p-subgroup of H such that

$$\phi(A) \subseteq B. \tag{7.11.3}$$

Thus $B_1 = \phi^{-1}(B)$ is a subgroup of G that contains A. The third Sylow theorem implies that there is a Sylow p-subgroup B_0 of B_1 such that

$$A \subseteq B_0. \tag{7.11.4}$$

We also have that $\phi(B_0) = B$ under these conditions, as in Section 7.8.

Suppose now that G is a profinite group, and let A be a closed subgroup of G. Thus A is a profinite group with respect to the induced topology, and we suppose also that A is a pro-p-group. Part (a) of Proposition 4 of [25] states that A is contained in a Sylow p-subgroup of G.

Let $\mathcal{B}$ be a local base for the topology of G at the identity element, which is a directed system with respect to the partial order $\preceq$ defined by putting $U \preceq V$ when $U, V \in \mathcal{B}$ and $V \subseteq U$, as before. If $U \in \mathcal{B}$, then let q_U be the natural quotient mapping from G onto G/U, and let $\Phi_{U,V}$ be the unique homomorphism from G/V onto G/U that satisfies (7.9.1) when $V \in \mathcal{B}$ and $U \preceq V$.

If $U \in \mathcal{B}$, then $q_U(A)$ is isomorphic to $A/(A \cap U)$. Note that $q_U(A)$ is a p-group, because A is a pro-p-group. Let $P^A(U)$ be the collection of Sylow p-subgroups of G/U that contain $q_U(A)$. The third Sylow theorem implies that $P^A(U) \neq \emptyset$.

If $U, V \in \mathcal{B}$, $U \preceq V$, and $C \in P^A(V)$, then

$$\Phi_{U,V}(C) \in P^A(U), \tag{7.11.5}$$

by the remarks at the beginning of the section. This defines a mapping $\widetilde{\Phi}^A_{U,V}$ from $P^A(V)$ into $P^A(U)$, which maps $P^A(V)$ onto $P^A(U)$, as before. If $W \in \mathcal{B}$ and $V \preceq W$, then

$$\widetilde{\Phi}^A_{U,V} \circ \widetilde{\Phi}^A_{V,W} = \widetilde{\Phi}^A_{U,W}, \tag{7.11.6}$$

by (7.9.2).

This shows that the family of sets $P^A(U)$, $U \in \mathcal{B}$, is a surjective inverse system with respect to the mappings $\widetilde{\Phi}^A_{U,V}$. Thus the inverse limit

$$\lim_{\longleftarrow} P^A(U) \tag{7.11.7}$$

can be defined as in Section 6.5. We also have that (7.11.7) is nonempty, as in Section 7.9.

Let us consider an element of (7.11.7), which assigns to each $U \in \mathcal{B}$ an element C_U of $P^A(U)$. If $U, V \in \mathcal{B}$ and $U \preceq V$, then

$$\widetilde{\Phi}^A_{U,V}(C_V) = C_U, \tag{7.11.8}$$

by definition of the inverse limit. This is the same as saying that $\Phi_{U,V}$ maps C_V onto C_U, by definition of $\widetilde{\Phi}^A_{U,V}$.

This means that the family of subgroups C_U of G/U, $U \in \mathcal{B}$, is a surjective inverse system with respect to the restrictions of the mappings $\Phi_{U,V}$ to C_V. The inverse limit

$$\lim_{\longleftarrow} C_U \tag{7.11.9}$$

is a closed subgroup of

$$\lim_{\longleftarrow} G/U, \tag{7.11.10}$$

as in Section 7.9. More precisely, (7.11.9) is a Sylow p-subgroup of (7.11.10), as a profinite group, as before.

Remember that (7.11.10) is isomorphic to G as a profinite group, so that (7.11.9) corresponds to a Sylow p-subgroup of G, as before. One can check that this Sylow p-subgroup of G contains A, because $C_U \in P^A(U)$ for every $U \in \mathcal{B}$.

7.12 Sylow subgroups and homomorphisms

Let G be a profinite group, and let H be a topological group, where the set containing only the identity element in H is a closed set. Also let $\mathcal{B}$ be a local base for the topology of G at the identity element consisting of open normal subgroups, and let ϕ be a continuous homomorphism from G onto H. If $U \in \mathcal{B}$, then $\phi(U)$ is an open set in H, as in Section 4.14. This implies that the collection of $\phi(U)$, $U \in \mathcal{B}$, is a local base for the topology of H at the identity element, as before. In particular, H is profinite too under these conditions.

If $U \in \mathcal{B}$, then U is a normal subgroup of G, and we let $q_{G,U}$ be the natural quotient mapping from G onto G/U. Similarly, $\phi(U)$ is a normal subgroup of H, and we let $q_{H,\phi(U)}$ be the natural quotient mapping from H onto $H/\phi(U)$. Observe that there is a unique homomorphism ϕ_U from G/U onto $H/\phi(U)$ such that

$$\phi_U \circ q_{G,U} = q_{H,\phi(U)} \circ \phi, \tag{7.12.1}$$

because U is contained in the kernel of the right side.

Let A be a closed subgroup of G, so that A is compact. This implies that $\phi(A)$ is compact, and thus closed in H. Let p be a prime number, and suppose for the moment that A is a pro-p-group. If $U \in \mathcal{B}$, then it follows that $q_{G,U}(A)$ is a p-group, as in Section 7.7. This implies that $\phi_U(q_{G,U}(A))$ is a p-group. This means that $q_{H,\phi(U)}(\phi(A))$ is a p-group, by (7.12.1). It follows that $\phi(A)$ is a pro-p-group, as before.

Suppose now that the index $(G : A)$ of A in G as a profinite group is not a multiple of p. If $U \in \mathcal{B}$, then we get that the index of $q_{G,U}(A)$ in G/U is not a multiple of p. This implies that the index of $\phi_U(q_{G,U}(A))$ in $H/\phi(U)$ is not a multiple of p. Equivalently, this means that the index of $q_{H,U}(\phi(A))$ in $H/\phi(U)$ is not a multiple of p, by (7.12.1). It follows that the index $(H : \phi(A))$ of $\phi(A)$ in H as a profinite group is not a multiple of p under these conditions.

If A is a Sylow p-subgroup of G, as a profinite group, then we obtain that $\phi(A)$ is a Sylow p-subgroup of H. This corresponds to part (b) of Proposition 4 on p7 of [25].

Chapter 8

8.1 Inverse systems and injections

Let $(I, \preceq)$ be a nonempty directed set, and let Y_j be a nonempty set for every $j \in I$, as in Section 6.5. Suppose that for every $j, l \in I$ with $j \preceq l$, we have a mapping $\phi_{j,l}$ from Y_l into Y_j such that $\phi_{j,j}$ is the identity mapping on Y_j for every $j \in I$, and $\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}$ when $j, l, r \in I$ satisfy $j \preceq l \preceq r$, as before. Remember that the inverse limit of the Y_j's is defined to be the set Y consisting of $y \in \prod_{j \in I} Y_j$ such that $\phi_{l,r}(y_r) = y_l$ for every $l, r \in I$ with $l \preceq r$.

If $l \in I$, then we let π_l be the standard coordinate projection from $\prod_{j \in I} Y_j$ onto Y_l, and we let ρ_l be the restriction of π_l to Y, as before. Thus ρ_l is a mapping from Y into Y_l, and $\phi_{l,r} \circ \rho_r = \rho_l$ when $l, r \in I$ and $l \preceq r$, by construction.

Suppose that

(8.1.1) $\qquad \phi_{j,l}$ is injective as a mapping from Y_l into Y_j

for every $j, l \in I$ with $j \preceq l$. Let $l \in I$ be given, and let us check that

(8.1.2) $\qquad \rho_l$ is injective as a mapping from Y into Y_l.

Equivalently, this means that $y \in Y$ is uniquely determined by y_l.

If $r \in I$ and $l \preceq r$, then y_r is uniquely determined by $y_l = \phi_{l,r}(y_r)$, because $\phi_{l,r}$ is injective. If $j \in I$ and $j \preceq r$, then $y_j = \phi_{j,r}(y_r)$ is uniquely determined by y_r, by the definition of Y. This means that y_j is uniquely determined by y_l when $j, l \preceq r$. Of course, for every $j \in I$ there is an $r \in I$ such that $j, l \preceq r$, because $(I, \preceq)$ is a directed set.

If Y_j is a topological space for every $j \in I$, and $\phi_{j,l}$ is continuous as a mapping from Y_l into Y_j for every $j, l \in I$ with $j \preceq l$, then we take Y to be equipped with the topology induced by the corresponding product topology on $\prod_{j \in I} Y_j$, as in Section 6.6. In this case, ρ_l is continuous as a mapping from Y into Y_l for every $l \in I$, by construction. Suppose that for every $j, l \in I$ with $j \preceq l$, we have that

(8.1.3) $\qquad \phi_{j,l}$ is a homeomorphism from Y_l onto its image in Y_j, with respect to the induced topology. Under these conditions, one can check that

(8.1.4) ρ_l is a homeomorphism from Y onto its image in Y_l,

with respect to the induced topology, for every $l \in I$.

Let Z be a set, and suppose now that

$$Y_j \subseteq Z \tag{8.1.5}$$

for every $j \in I$. If $j, l \in I$ and $j \preceq l$, then suppose that

$$Y_l \subseteq Y_j, \tag{8.1.6}$$

and let us take $\phi_{j,l}$ to be the natural inclusion mapping from Y_l into Y_j. This satisfies the requirments of an inverse system, so that Y and ρ_l, $l \in I$, can be defined as before.

Put

$$X = \bigcap_{j \in I} Y_j, \tag{8.1.7}$$

and for each $l \in I$, let θ_l be the natural inclusion mapping from X into Y_l. This leads to a mapping θ from X into $\prod_{j \in I} Y_j$, with $\pi_l \circ \theta = \theta_l$ for every $l \in I$, as in Section 6.5. If $l, r \in I$ and $l \preceq r$, then $\phi_{l,r} \circ \theta_r = \theta_l$ holds automatically, so that $\theta(X) \subseteq Y$, as before. In fact, it is easy to see that

$$\theta(X) = Y. \tag{8.1.8}$$

Note that $\rho_l \circ \theta = \theta_l$ for every $l \in I$, by construction, as before.

If Z is a topological space, then we may take

(8.1.9) Y_j to be equipped with the induced topology

for each $j \in I$. If $j, l \in I$ and $j \preceq l$, then it follows that $\phi_{j,l}$ is a homeomorphism from Y_l onto its image in Y_j, with respect to the induced topology. Let us take X to be equipped with the topology induced by Z, so that θ_l is a homeomorphism from X onto its image in Y_l for every $l \in I$. One can check that θ is a homeomorphism from X onto Y.

Suppose that Z is a Hausdorff topological space, and that Y_j is a compact subset of Z for every $j \in I$. This implies that Y_j is a closed set in Z for every $j \in I$, so that X is a closed set in Z, and in fact X is compact. One can verify that $X \neq \emptyset$, because $Y_j \neq \emptyset$ for every $j \in I$, by hypothesis. More precisely, one can consider X as the intersection of a compact set with a nonempty family of closed sets with the finite intersection property with respect to that closed set.

Let A be a nonempty set, and let Z_α be a subset of Z for every $\alpha \in A$. If $\alpha_1, \ldots, \alpha_n$ are finitely many elements of A, then suppose that

$$Z_{\alpha_1} \cap \cdots \cap Z_{\alpha_n} \tag{8.1.10}$$

is nonempty. Of course, the collection of nonempty finite subsets of A is a directed set, with respect to inclusion. Using this directed set, the corresponding family of sets of the form (8.1.10) satisfies the conditions mentioned earlier. The intersection of the sets of the form (8.1.10) is the same as $\bigcap_{\alpha \in A} Z_\alpha$.

8.2 Semimetrics and partitions

Let X be a set, and let $d(x, y)$ be a semimetric on X. It is easy to see that

$$d(x, y) = 0 \tag{8.2.1}$$

defines an equivalence relation on X. The corresponding equivalence classes in X are the same as the closed balls in X of radius 0 with respect to $d(\cdot, \cdot)$.

Suppose now that $d(\cdot, \cdot)$ is a semi-ultrametric on X. If r is a positive real number, then

$$d(x, y) < r \tag{8.2.2}$$

defines an equivalence relation on X. The corresponding equivalence classes in X are the open balls in X of radius r with respect to $d(\cdot, \cdot)$. Similarly, if r is a nonnegative real number, then

$$d(x, y) \le r \tag{8.2.3}$$

defines an equivalence relation on X. The equivalence classes in X corresponding to (8.2.3) are the closed balls in X of radius r with respect to $d(\cdot, \cdot)$.

Of course, if $r = 0$, then (8.2.3) is equivalent to (8.2.1). Let us say that a semimetric $d(\cdot, \cdot)$ is a *discrete semimetric* on X if for every $x, y \in X$,

$$d(x, y) = 0 \text{ or } 1. \tag{8.2.4}$$

The discrete metric on X is a discrete semimetric in this sense, and it is the only metric on X that is a discrete semimetric. It is easy to see that any discrete semimetric on X is a semi-ultrametric on X.

Let $\mathcal{P}$ be a *partition* of X, which is to say a collection of nonempty pairwise-disjoint subsets of X whose union is equal to X. This determines an equivalence relation $\sim_{\mathcal{P}}$ on X, where

$$x \sim_{\mathcal{P}} y \tag{8.2.5}$$

if and only if x and y are elements of the same subset of X in $\mathcal{P}$. In this case, the elements of $\mathcal{P}$ are the same as the equivalence classes in X associated to $\sim_{\mathcal{P}}$. Conversely, every equivalence relation on X leads to a partition of X into equivalence classes, which determines the same equivalence relation on X in this way.

If $\mathcal{P}$ is a partition of X and $x, y \in X$, then put

$$\begin{aligned} d_{\mathcal{P}}(x, y) &= 0 \quad \text{when } x \sim_{\mathcal{P}} y \\ &= 1 \quad \text{otherwise.} \end{aligned} \tag{8.2.6}$$

One can check that this defines a semi-ultrametric on X, which is the discrete semimetric on X associated to $\mathcal{P}$. Note that $\sim_{\mathcal{P}}$ is the same as the equivalence relation associated to $d_{\mathcal{P}}(\cdot, \cdot)$ as in (8.2.2) when $0 < r \le 1$, and as the equivalence relation associated to $d_{\mathcal{P}}(\cdot, \cdot)$ as in (8.2.3) when $0 \le r < 1$. Conversely, if $d(\cdot, \cdot)$ is any discrete semimetric on X, then $d(\cdot, \cdot)$ is the same as $d_{\mathcal{P}}(\cdot, \cdot)$ for some partition $\mathcal{P}$ of X. More precisely, one can take $\mathcal{P}$ to be the partition of X into equivalence classes using the equivalence relation (8.2.2) when $0 < r \le 1$, or the equivalence relation (8.2.3) when $0 \le r < 1$.

Note that a collection $\mathcal{P}$ of nonempty subsets of X is a partition of X exactly when every element of X is contained in a unique element of $\mathcal{P}$. If $\mathcal{P}_1$, $\mathcal{P}_2$ are partitions of X, then we say that $\mathcal{P}_2$ is a *refinement* of $\mathcal{P}_1$ if

(8.2.7) every element of $\mathcal{P}_2$ is a subset of an element of $\mathcal{P}_1$.

In terms of the corresponding equivalence relations $\sim_{\mathcal{P}_1}$, $\sim_{\mathcal{P}_2}$, this means that for every $x, y \in X$,

$$x \sim_{\mathcal{P}_2} y \text{ implies } x \sim_{\mathcal{P}_1} y. \tag{8.2.8}$$

Equivalently, if $d_{\mathcal{P}_1}(\cdot,\cdot)$, $d_{\mathcal{P}_2}(\cdot,\cdot)$ are as in (8.2.6), then $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ if and only if

$$d_{\mathcal{P}_1}(x, y) \le d_{\mathcal{P}_2}(x, y) \tag{8.2.9}$$

for every $x, y \in X$.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be finitely many partitions of X, and let $\mathcal{P}$ be the collection of nonempty subsets of X of the form

$$A_1 \cap \cdots \cap A_n, \tag{8.2.10}$$

where $A_j \in \mathcal{P}_j$ for each $j = 1, \ldots, n$. It is easy to see that $\mathcal{P}$ is a partition of X, which is a refinement of each of $\mathcal{P}_1, \ldots, \mathcal{P}_n$. If $\sim_{\mathcal{P}_1}, \ldots, \sim_{\mathcal{P}_n}$ and $\sim_{\mathcal{P}}$ are the equivalence relations corresponding to $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and $\mathcal{P}$ as before, respectively, then for each $x, y \in X$, we have that

(8.2.11) $x \sim_{\mathcal{P}} y$ if and only if $x \sim_{\mathcal{P}_j} y$ for every $j = 1, \ldots, n$.

If $d_{\mathcal{P}_j}(\cdot,\cdot)$, $1 \le j \le n$, and $d_{\mathcal{P}}(\cdot,\cdot)$ are as in (8.2.6), then

$$d_{\mathcal{P}}(x, y) = \max(d_{\mathcal{P}_1}(x, y), \ldots, d_{\mathcal{P}_n}(x, y)) \tag{8.2.12}$$

for every $x, y \in X$.

8.3 Inverse systems and semimetrics

Let I be a nonempty set, let X_j be a set for each $j \in I$, and let $X = \prod_{j \in I} X_j$ be their Cartesian product. Suppose that for each $j \in I$, we have a nonempty collection $\mathcal{M}_j$ of semimetrics on X_j. If $l \in I$ and $d_l \in \mathcal{M}_l$, then let $\widetilde{d_l}$ be the semimetric defined on X by $\widetilde{d_l}(x, y) = d_l(x_l, y_l)$ for every $x, y \in X$, as in Section 1.1. Put

$$\widetilde{\mathcal{M}_l} = \{\widetilde{d_l} : d_l \in \mathcal{M}_l\} \tag{8.3.1}$$

for every $l \in I$, and

$$\widetilde{\mathcal{M}} = \bigcup_{l \in I} \widetilde{\mathcal{M}_l}. \tag{8.3.2}$$

If $l \in I$, $d_l \in \mathcal{M}_l$, $x \in X$, and r is a positive real number, then it is easy to see that

$$B_{X,\widetilde{d_l}}(x,r) = \pi_l^{-1}(B_{X_l,d_l}(x_l,r)), \tag{8.3.3}$$

where π_l is the natural coordinate projection from X into X_l. Using this, one can check that

(8.3.4) the topology determined on X by $\widetilde{\mathcal{M}}$ is the same as the product topology,

where X_j is equipped with the topology determined by $\mathcal{M}_j$ for every $j \in I$. Note that

(8.3.5) $\widetilde{\mathcal{M}}$ is nondegenerate on X when $\mathcal{M}_j$ is nondegenerate on X_j for each $j \in I$,

where nondegeneracy is as defined in Section 2.1.

Now let $(I, \preceq)$ be a nonempty directed set, and let Y_j be a nonempty set for every $j \in I$, as in Section 6.5. Suppose as before that for every $j, l \in I$ with $j \preceq l$ we have a mapping $\phi_{j,l}$ from Y_l into Y_j such that $\phi_{j,j}$ is the identity mapping on Y_j for every $j \in I$, and $\phi_{j,r} = \phi_{j,l} \circ \phi_{l,r}$ when $j, l, r \in I$ and $j \preceq l \preceq r$. The inverse limit of the Y_j's is defined as usual to be the set Y of $y \in \prod_{j\in I} Y_j$ such that $\phi_{l,r}(y_r) = y_l$ for every $l, r \in I$ with $l \preceq r$. Let π_l be the standard coordinate projection from $\prod_{j\in I} Y_j$ onto Y_l for every $l \in I$, and let ρ_l be the restriction of π_l to Y, as before.

Suppose that for each $j \in I$ we have a nonempty collection $\mathcal{M}_j$ of semimetrics on Y_j, and let us take Y_j to be equipped with the topology determined by $\mathcal{M}_j$, as in Section 2.1. When considering Y as the inverse limit of the Y_j's as topological spaces, remember that one asks that $\phi_{j,l}$ be continuous as a mapping from Y_l into Y_j for every $j, l \in I$ with $j \preceq l$, as in Section 6.6. If $l \in I$ and $d_l \in \mathcal{M}_l$, then let $\widetilde{d_l}$ be the semimetric on $\prod_{j\in I} Y_j$ corresponding to d_l as before. Also let $\widetilde{\mathcal{M}_l}$ be as in (8.3.1) again, and let $\widetilde{\mathcal{M}}$ be as in (8.3.2). Thus the topology determined on $\prod_{j\in I} Y_j$ by $\widetilde{\mathcal{M}}$ is the corresponding product topology, as before.

Let $\widetilde{\mathcal{M}}_Y$ be the collection of the restrictions of the elements of $\widetilde{\mathcal{M}}$ to Y. The topology determined on Y by $\widetilde{\mathcal{M}}_Y$ is the same as the topology induced on Y by the topology determined on $\prod_{j\in I} Y_j$ by $\widetilde{\mathcal{M}}$, as in Section 2.1. In this case, this is

(8.3.6) the topology induced on Y by the product topology on $\prod_{j\in I} Y_j$,

which is the topology on Y considered in Section 6.6. If $l \in I$, $d_l \in \mathcal{M}_l$, $y \in Y$, and $r > 0$, then

$$B_{Y,\widetilde{d_l}}(y,r) = \pi_l^{-1}(B_{Y_l,d_l}(y_l,r)) \cap Y, \tag{8.3.7}$$

where more precisely the left side is the open ball in Y centered at y with radius r with respect to the restriction of $\widetilde{d_l}$ to Y. This means that

$$B_{Y,\widetilde{d_l}}(y,r) = \rho_l^{-1}(B_{Y_l,d_l}(y_l,r)). \tag{8.3.8}$$

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