Some topics in analysis related to Cartesian products, metrics, and ultrametrics

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Preface

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Chapter 1

Metrics and related notions

1.1 Some remarks about Cartesian products

Let I be a nonempty set, and let X_j be a set for each $j \in I$. The *Cartesian product*

$$X = \prod_{j \in I} X_j \tag{1.1.1}$$

can be defined as usual as the set of all functions f from I into $\bigcup_{j \in I} X_j$ such that $f(j) \in X_j$ for every $j \in I$. If n is a positive integer and $I = \{1, \ldots, n\}$ is the set of positive integers from 1 to n, then the Cartesian product may be denoted $\prod_{j=1}^n X_j$, and its elements identified with n-tuples $x = (x_1, \ldots, x_j)$ with $x_j \in I_j$ for each $j = 1, \ldots, n$. If I is the set $\mathbf{Z}_+$ of all positive integers, then the Cartesian product may be denoted $\prod_{j=1}^\infty X_j$, and its elements identified with infinite sequences $x = \{x_j\}_{j=1}^\infty$ such that $x_j \in I_j$ for every $j \geq 1$. Similar, if I is the set $\mathbf{Z}$ of all integers, then the Cartesian product may be denoted $\prod_{j=-\infty}^\infty X_j$, and its elements identified with doubly-infinite sequences $x = \{x_j\}_{j=-\infty}^\infty$ such that $x_j \in X_j$ for every j.

Often one is concerned with situations where X_j is equipped with some additional structure for each $j \in I$, and the Cartesian product X may be equipped with some related structure. If X_j is a group for each $j \in I$, for instance, then X is group too, where the group operations are defined coordinatewise. If k is a field and X_j is a vector space over k for every $j \in I$, then X is a vector space over k as well, with respect to coordinatewise addition and scalar multiplication.

If X_j is a topological space for each $j \in I$, then X is a topological space with respect to the product topology. If X_j is a metric space for each $j \in I$, and I has only finitely many elements, then it is easy to define metrics on X that are compatible with the corresponding product topology. One can also do this when I is countably infinite, and we shall discuss this further later.

If X_j is a σ-finite measure space for every $j \in I$, and I has only finitely many elements, then one can define a suitable product measure on X. If X_j is a probability space for every $j \in I$, then one can define a corresponding product probability measure on X, even when I has infinitely many elements.

If X_j is a metric space for each $j \in I$, and I is countably infinite, then one can reduce to the case where $I = \mathbf{Z}_+$, to define a compatible metric on X. Although the choice of the enumeration of I does not affect the product topology on X, it can be relevant for the resulting geometry on X.

Similarly, if X_j is a probability space for each $j \in I$, then the corresponding product probability measure on X does not depend on any particular ordering of the elements of I. However, this can be important in the consideration of certain filtrations of σ-subalgebras of measurable subsets of X.

Let X^0 be a set, and suppose that $X_j = X^0$ for every $j \in I$. In this case, a one-to-one mapping from I onto itself leads to a one-to-one mapping from X onto itself, by permuting the coordinates of elements of X. If X^0 is equipped with additional structure, and X is equipped with a corresponding product structure, then one may be concerned with the behavior of the mappings on X just mentioned. In particular, if $I = \mathbf{Z}$, then one may be concerned with shift mappings on X, corresponding to translations on $\mathbf{Z}$.

There is a well-known notion of a *uniform structure* on a set, which is less precise than a metric, and which determines a topology on the set. An arbitrary product of uniform spaces has a natural product uniform structure, and we shall say more about this later too.

1.2 Metrics and semimetrics

Let X be a set. A nonnegative real-valued function $d(x,y)$ defined for $x, y \in X$ is said to be a *semimetric* or *pseudometric* on X if it satisfies the following three conditions. First,

$$d(x,x) = 0 \quad \text{for every } x \in X. \tag{1.2.1}$$

Second,

$$d(x,y) = d(y,x) \quad \text{for every } x, y \in X. \tag{1.2.2}$$

Third,

$$d(x,z) \le d(x,y) + d(y,z) \quad \text{for every } x, y, z \in X, \tag{1.2.3}$$

which is known as the *triangle inequality*. If we also have that

$$d(x,y) > 0 \quad \text{for every } x, y \in X \text{ with } x \ne y, \tag{1.2.4}$$

then $d(\cdot,\cdot)$ is said to be a *metric* on X. The *discrete metric* on X is defined as usual by putting $d(x,y)$ equal to 0 when $x = y$, and to 1 when $x \ne y$, and it is easy to see that this defines a metric on X.

Let I be a nonempty set, let X_j be a set for every $j \in I$, and let $X = \prod_{j \in I} X_j$ be their Cartesian product. If $x \in X$ and $l \in I$, then let x_l denote the lth coordinate of x in X_l. Suppose that $d_l(\cdot,\cdot)$ is a semimetric on X_l, and put

$$\widehat{d}_l(x,y) = d_l(x_l, y_l) \tag{1.2.5}$$

for every $x, y \in X$. It is easy to see that this defines a semimetric on X.

Let $d(\cdot,\cdot)$ be a semimetric on a set X. If $x \in X$ and r is a positive real number, then *open ball* in X centered at x with radius r with respect to d is defined by

$$B(x,r) = B_d(x,r) = \{y \in X : d(x,y) < r\}. \tag{1.2.6}$$

Similarly, if $x \in X$ and r is a nonnegative real number, then the *closed ball* in X centered at x with radius r with respect to d is defined by

$$\overline{B}(x,r) = \overline{B}_d(x,r) = \{y \in X : d(x,y) \le r\}. \tag{1.2.7}$$

We may also use the notation $B_X(x,r) = B_{X,d}(x,r)$ and $\overline{B}_X(x,r) = \overline{B}_{X,d}(x,r)$, to indicate the role of X.

A subset U of X is said to be an *open set* with respect to d if for every $x \in U$ there is an $r > 0$ such that

$$B(x,r) \subseteq U. \tag{1.2.8}$$

Of course, this is the same as the definition that one normally uses for metric spaces. One can check that open balls in X with respect to d are open sets, and that the collection of open sets defines a topology on X, in the same way as for metric spaces. One can verify that closed balls in X with respect to d are closed sets with respect to this topology as well.

If $d(\cdot,\cdot)$ is a metric on X, then it is easy to see that X is Hausdorff with respect to the corresponding topology. Conversely, if X satisfies the first or even zeroth separation condition with respect to the topology determined by a semimetric $d(\cdot,\cdot)$, then $d(\cdot,\cdot)$ is a metric on X. In some situations, it can be helpful to consider collections of semimetrics on a set, instead of a single semimetric.

1.3 Ultrametrics and semi-ultrametrics

A semimetric $d(\cdot,\cdot)$ on a set X is said to be a *semi-ultrametric* on X if

$$d(x,z) \le \max(d(x,y), d(y,z)) \quad \text{for every } x, y, z \in X. \tag{1.3.1}$$

Note that (1.3.1) implies the ordinary triangle inequality. Similarly, a metric $d(\cdot,\cdot)$ on X is said to be an *ultrametric* on X if it satisfies (1.3.1). One can check that the discrete metric on X is an ultrametric.

Let I be a nonempty set, let X_j be a set for every $j \in I$, and let $X = \prod_{j \in I} X_j$ be their Cartesian product again. If $l \in I$ and d_l is a semi-ultrametric on X_l, then (1.2.5) defines a semi-ultrametric on X.

Suppose that $d(\cdot,\cdot)$ is a semi-ultrametric on a set X. If r is a positive real number, then one can verify that

$$d(x,y) < r \tag{1.3.2}$$

defines an equivalence relation on X. The corresponding equivalence classes in X are the same as the open balls in X of radius r with respect to d. In particular, the complement of an open ball of radius r in X can be expressed as the union of open balls of radius r in X. This implies that the complement of an open ball of radius r in X is an open set in X, with respect to the topology determined by d, so that open balls in X are closed sets.

Similarly, if r is a nonnegative real number, then

$$d(x,y) \le r \tag{1.3.3}$$

defines an equivalence relation on X. The equivalence classes in X corresponding to (1.3.3) are the same as the closed balls of radius r in X with respect to d. In particular, every closed ball of radius r in X contains the closed balls of radius r centered at each of its elements. If $r > 0$, then it follows that every closed ball of radius r in X is an open set, so that closed balls of radius r are open sets. If $r = 0$, then (1.3.3) defines an equivalence relation on X for any semimetric $d(\cdot,\cdot)$ on X.

Let $\mathcal{P}$ be a *partition* of X, which is to say a collection of pairwise-disjoint nonempty subsets of X whose union is X. This leads to an equivalence relation $\sim_{\mathcal{P}}$ on X, where $x \sim_{\mathcal{P}} y$ when $x, y \in X$ are elements of the same element of $\mathcal{P}$. In this case, the elements of $\mathcal{P}$ are the same as the equivalence classes in X with respect to $\sim_{\mathcal{P}}$, by construction. Conversely, any equivalence relation on X determines a partition of X, consisting of the corresponding equivalence classes in X.

If $x, y \in X$, then put

$$\begin{aligned} d_{\mathcal{P}}(x,y) &= 0 \quad \text{when } x \sim_{\mathcal{P}} y \\ &= 1 \quad \text{when } x \not\sim_{\mathcal{P}} y. \end{aligned} \tag{1.3.4}$$

It is easy to see that this defines a semi-ultrametric on X. If $0 < r \le 1$, then the open balls in X of radius r with respect to $d_{\mathcal{P}}(\cdot,\cdot)$ are the same as the elements of $\mathcal{P}$. Similarly, if $0 \le r < 1$, then the closed balls in X of radius r with respect to $d_{\mathcal{P}}(\cdot,\cdot)$ are the same as the elements of $\mathcal{P}$.

Let us say that a semimetric $d(\cdot,\cdot)$ on X is a *discrete semimetric* if for every $x, y \in X$, $d(x,y)$ is equal to either 0 or 1. One can check that this implies that $d(\cdot,\cdot)$ is a semi-ultrametric on X. In this case, the open balls in X of radius r, $0 < r \le 1$, with respect to d are the same, and they are also the same as the closed balls in X of radius r, $0 \le r < 1$. This defines a partition of X, and $d(\cdot,\cdot)$ is the same as the semi-ultrametric on X associated to this partition as in the preceding paragraph.

1.4 Absolute value functions

Let k be a field. A nonnegative real-valued function $|\cdot|$ on k is said to be an *absolute value function* on k if it satisfies the following three conditions. First, for each $x \in k$,

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{1.4.1}$$

Second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x, y \in k. \tag{1.4.2}$$

Third,

$$|x + y| \le |x| + |y| \quad \text{for every } x, y \in k. \tag{1.4.3}$$

It is well known that the standard absolute value functions on the real line $\mathbf{R}$ and the complex plane $\mathbf{C}$ are absolute value functions in this sense. The *trivial absolute value function* may be defined on any field k by putting $|x|$ equal to 0 when $x = 0$, and equal to 1 otherwise. It is easy to see that this defines an absolute value function on k in this sense.

If $|\cdot|$ is any absolute value function on a field k, then one can show that $|1| = 1$, where the first $1 = 1_k$ is the multiplicative identity element in k, and the second $1 = 1_{\mathbf{R}}$ is the multiplicative identity element in $\mathbf{R}$. This uses the fact that $1 \cdot 1 = 1$ in k. Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n, then $|x| = 1$. In particular, $|-1| = 1$, because $(-1)^2 = 1$ in k.

Using this, one can check that

$$d(x, y) = |x - y| \tag{1.4.4}$$

defines a metric on k. This is the same as the standard Euclidean metric on $\mathbf{R}$ or $\mathbf{C}$ when $|\cdot|$ is the standard absolute value function. If $|\cdot|$ is the trivial absolute value function on any field k, then (1.4.4) is the same as the discrete metric on k.

An absolute value function $|\cdot|$ on a field k is said to be an *ultrametric absolute value function* on k if

$$|x + y| \le \max(|x|, |y|) \quad \text{for every } x, y \in k. \tag{1.4.5}$$

Of course, (1.4.5) implies (1.4.3). It is easy to see that the trivial absolute value function on k is an ultrametric absolute value function. If $|\cdot|$ is an ultrametric absolute value function on k, then (1.4.4) is an ultrametric on k.

Let p be a prime number, and let x be a rational number. The *p-adic absolute value* $|x|_p$ of x is defined as follows. If $x = 0$, then $|x|_p = 0$. Otherwise, if $x \ne 0$, then x can be expressed as $p^j\,(a/b)$, where $a, b, j \in \mathbf{Z}$, $a, b \ne 0$, and neither a nor b is an integer multiple of p. In this case, we put

$$|x|_p = p^{-j}. \tag{1.4.6}$$

One can check that this defines an ultrametric absolute value function on the rational numbers $\mathbf{Q}$. The corresponding ultrametric

$$d_p(x, y) = |x - y|_p \tag{1.4.7}$$

is known as the *p-adic metric* on $\mathbf{Q}$.

Let k be any field again, and let $|\cdot|$ be an absolute value function on k. If $x \in k$ and $n \in \mathbf{Z}_+$, then let $n \cdot x$ be the sum of n 1's in k. If there are positive integers n such that $|n \cdot 1|$ can be arbitrarily large, then $|\cdot|$ is said to be *archimedian* on k. Otherwise, if there is a positive real number C such that

$$|n \cdot 1| \le C \tag{1.4.8}$$

for every $n \in \mathbf{Z}_+$, then $|\cdot|$ is said to be *non-archimedean* on k.

If $|\cdot|$ is an ultrametric absolute value function on k, then (1.4.8) holds with $C = 1$, so that $|\cdot|$ is non-archimedean on k. Conversely, it is well known that any non-archimedean absolute value function on k is an ultrametric absolute value function on k. Of course, the standard absolute value functions on $\mathbf{R}$ and $\mathbf{C}$ are archimedean.

1.5 Finitely many semimetrics

Let X be a set, and let $d_1, \ldots, d_n$ be finitely many semimetrics on X. If $x, y \in X$, then put

$$d(x, y) = \max_{1 \le j \le n} d_j(x, y) \tag{1.5.1}$$

and

$$d'(x, y) = \sum_{j=1}^{n} d_j(x, y). \tag{1.5.2}$$

One can check that these define semimetrics on X too. If $d_1, \ldots, d_n$ are semi-ultrametrics on X, then (1.5.1) is a semi-ultrametric on X as well.

If $x \in X$ and r is a positive real number, then

$$B_d(x, r) = \bigcap_{j=1}^{n} B_{d_j}(x, r). \tag{1.5.3}$$

Similarly, if r is a nonnegative real number, then

$$\overline{B}_d(x, r) = \bigcap_{j=1}^{n} \overline{B}_{d_j}(x, r). \tag{1.5.4}$$

Observe that

$$d(x, y) \le d'(x, y) \le n\, d(x, y) \tag{1.5.5}$$

for every $x, y \in X$. One can use this to compare open and closed balls in X with respect to d' with open and closed balls with respect to d.

One can also verify that

$$d''(x, y) = \Big(\sum_{j=1}^{n} d_j(x, y)^2\Big)^{1/2} \tag{1.5.6}$$

defines a semimetric on X, using the triangle inequality for the standard Euclidean norm on $\mathbf{R}^n$. It is easy to see that

$$d(x, y) \le d''(x, y) \le n^{1/2}\, d(x, y) \tag{1.5.7}$$

for every $x, y \in X$. As before, one can use this to compare open and closed balls in X with respect to d'' with open and closed balls with respect to d.

Now let $X_1, \ldots, X_n$ be finitely many sets, and put $X = \prod_{j=1}^n X_j$. Suppose that d_j is a semimetric on X_j for each $j = 1, \ldots, n$, and let $\widehat{d_j}$ be the corresponding semimetric on X, as in Section 1.2. If $x, y \in X$, then put

$$\widehat{d}(x,y) = \max_{1 \le j \le n} \widehat{d_j}(x,y) = \max_{1 \le j \le n} d_j(x_j, y_j), \tag{1.5.8}$$

$$\widehat{d}'(x,y) = \sum_{j=1}^n \widehat{d_j}(x,y) = \sum_{j=1}^n d_j(x_j, y_j), \tag{1.5.9}$$

$$\widehat{d}''(x,y) = \Big(\sum_{j=1}^n \widehat{d_j}(x,y)^2\Big)^{1/2} = \Big(\sum_{j=1}^n d_j(x_j, y_j)^2\Big)^{1/2}. \tag{1.5.10}$$

These define semimetrics on X, as before. If d_j is a metric on X_j for every $j = 1, \ldots, n$, then each of these defines a metric on X.

If d_j is a semi-ultrametric on X_j for each $j = 1, \ldots, n$, then one can check that $\widehat{d_j}$ is a semi-ultrametric on X for every $j = 1, \ldots, n$. This implies that (1.5.8) is a semi-ultrametric on X, as before.

If $x \in X$, then

$$B_{X,\widehat{d}}(x,r) = \prod_{j=1}^n B_{X_j,d_j}(x_j, r) \tag{1.5.11}$$

for every $r > 0$, and

$$\overline{B}_{X,\widehat{d}}(x,r) = \prod_{j=1}^n \overline{B}_{X_j,d_j}(x,r) \tag{1.5.12}$$

for every $r \ge 0$. Using (1.5.11), one can verify that the topology determined on X by $\widehat{d}$ is the same as the product topology, corresponding to the topology determined on X_j by d_j for each $j = 1, \ldots, n$.

Note that

$$\widehat{d}(x,y) \le \widehat{d}'(x,y) \le n\, \widehat{d}(x,y) \tag{1.5.13}$$

and

$$\widehat{d}(x,y) \le \widehat{d}''(x,y) \le n^{1/2}\, \widehat{d}(x,y) \tag{1.5.14}$$

for every $x, y \in X$, as in (1.5.5) and (1.5.7). In particular, this implies that the topologies determined on X by $\widehat{d}'$ and $\widehat{d}''$ are the same as the product topology too.

1.6 Snowflake semimetrics

If a is a positive real number with $a \le 1$, then it is well known that

$$(r+t)^a \le r^a + t^a \tag{1.6.1}$$

for all nonnegative real numbers r, t. To see this, observe first that

$$\max(r, t) \le (r^a + t^a)^{1/a} \tag{1.6.2}$$

for every $r, t \ge 0$. Using this, we get that

$$r + t \le \max(r, t)^{1-a}\,(r^a + t^a) \le (r^a + t^a)^{(1-a)/a+1} = (r^a + t^a)^{1/a} \tag{1.6.3}$$

for every $r, t \ge 0$. This implies (1.6.1), as desired.

Let X be a set, and let $d(\cdot, \cdot)$ be a semimetric on X. If $0 < a \le 1$, then it is easy to see that

$$d(x, y)^a \tag{1.6.4}$$

is a semimetric on X too, using (1.6.1). If $d(\cdot, \cdot)$ is a metric on X, then (1.6.4) is a metric on X as well. Similarly, if $d(\cdot, \cdot)$ is a semi-ultrametric on X, then one can check that (1.6.4) is a semi-ultrametric on X for every $a > 0$. If $d(\cdot, \cdot)$ is an ultrametric on X, then (1.6.4) is an ultrametric on X for every $a > 0$.

Let $d(\cdot, \cdot)$ be a semimetric on X again, and suppose that (1.6.4) is a semimetric on X for some $a > 0$. If $x \in X$, then it is easy to see that

$$B_{d^a}(x, r^a) = B_d(x, r) \tag{1.6.5}$$

for every $r > 0$, and that

$$\overline{B}_{d^a}(x, r^a) = \overline{B}_d(x, r) \tag{1.6.6}$$

for every $r \ge 0$. In particular, $d(\cdot, \cdot)$ and (1.6.4) determine the same topologies on X.

Let $X_1, \ldots, X_n$ be finitely many sets, and let $d_j(\cdot, \cdot)$ be a semimetric on X_j for each $j = 1, \ldots, n$. Also let $a_1, \ldots, a_n$ be positive real numbers, and suppose that

$$d_j(x_j, y_j)^{a_j} \tag{1.6.7}$$

is a semimetric on X_j for every $j = 1, \ldots, n$. Of course, we can get semimetrics on $X = \prod_{j=1}^n X_j$ using $d_1, \ldots, d_n$ as in the previous section. We can get semimetrics on X using $d_1^{a_1}, \ldots, d_n^{a_n}$ in the same way. Note that the topology determined on X by any of these semimetrics is the same as the product topology, corresponding to the topology determined on X_j by d_j for each $j = 1, \ldots, n$. In particular,

$$\widetilde{d}(x, y) = \max_{1 \le j \le n} d_j(x_j, y_j)^{a_j} \tag{1.6.8}$$

defines a semimetric on X under these conditions. If $x \in X$, then

$$B_{X, \widetilde{d}}(x, r) = \prod_{j=1}^n B_{X_j, d_j^{a_j}}(x, r) = \prod_{j=1}^n B_{X_j, d_j}(x, r^{1/a_j}) \tag{1.6.9}$$

for every $r > 0$, and

$$\overline{B}_{X, \widetilde{d}}(x, r) = \prod_{j=1}^n \overline{B}_{X_j, d_j^{a_j}}(x, r) = \prod_{j=1}^n \overline{B}_{X, d_j}(x, r^{1/a_j}) \tag{1.6.10}$$

for every $r \geq 0$.

Let X be any set again, and let $d(\cdot, \cdot)$ be a metric on X. Also let a be a positive real number, and suppose that (1.6.4) is a metric on X. It is easy to see that a sequence of elements of X is a Cauchy sequence with respect to $d(\cdot, \cdot)$ if and only if it is a Cauchy sequence with respect to (1.6.4). It follows that X is complete as a metric space with respect to $d(\cdot, \cdot)$ if and only if X is complete with respect to (1.6.4).

1.7 Snowflakes and absolute values

Let k be a field, and let $|\cdot|$ be an absolute value function on k. If $0 < a \leq 1$, then one can check that

$$|x|^a \tag{1.7.1}$$

is an absolute value function on k, using (1.6.1). If $|\cdot|$ is an ultrametric absolute value function on k, then (1.7.1) is an ultrametric absolute value function on k for every $a > 0$. If $k = \mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$ with the standard Euclidean absolute value function $|\cdot|$, then it is easy to see that (1.7.1) is not an absolute value function on k when $a > 1$.

Let k be any field again, and let $|\cdot|_1$ and $|\cdot|_2$ be absolute value functions on k. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{1.7.2}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* as absolute value functions on k. Of course,

$$|x - y|_2 = |x - y|_1^a \tag{1.7.3}$$

for every $x, y \in k$ in this case. This implies that the topologies determined on k by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same, as in the previous section. Conversely, if the topologies determined on k by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$, then it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent in the sense just defined.

A famous theorem of Ostrowski implies that any absolute value function on $\mathbf{Q}$ is either the trivial absolute value function, or that it is equivalent to the standard Euclidean absolute value function on $\mathbf{Q}$, or that it is equivalent to the p-adic absolute value function on $\mathbf{Q}$ for some prime number p.

Let k be a field with an archimedean absolute value function $|\cdot|$, and suppose that k is complete with respect to the metric associated to $|\cdot|$. Under these conditions, another famous theorem of Ostrowski implies that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to an absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard absolute value function.

Let k be any field with an absolute value function $|\cdot|$ again. If k is not already complete with respect to the metric associated to $|\cdot|$, then one can pass to a completion in a standard way. More precisely, the completion is also a field, and $|\cdot|$ extends to an absolute value function on the completion in a natural way. The completion is unique up to a suitable isomorphic equivalence.

Suppose that $|\cdot|^a$ is an absolute value function on k as well for some $a > 0$. If k is complete with respect to the metric associated to $|\cdot|$, then k is complete with respect to the metric associated to $|\cdot|^a$, as in the previous section. Otherwise, a completion of k with respect to $|\cdot|$ can also be used as a completion with respect to $|\cdot|$, by taking the ath power of the natural extension of $|\cdot|$ to the completion.

If p is a prime number, then the field $\mathbf{Q}_p$ of *p-adic numbers* is obtained by completing $\mathbf{Q}$ with respect to the p-adic absolute value function.

1.8 Lipschitz conditions

Let X, Y be sets with semimetrics d_X, d_Y, respectively. As usual, a mapping f from X into Y is said to be *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d_Y(f(x), f(w)) < \epsilon \tag{1.8.1}$$

for every $x, w \in X$ with $d_X(x, w) < \delta$. Of course, uniform continuity implies ordinary continuity.

Let α be a positive real number, and let C be a nonnegative real number. A mapping f from X into Y is said to be *Lipschitz* of order α with constant C if

$$d_Y(f(x), f(w)) \le C\, d_X(x, w)^\alpha \tag{1.8.2}$$

for every $x, w \in X$. Note that this implies that f is uniformly continuous. Constant mappings from X into Y are clearly Lipschitz of any order $\alpha > 0$, with constant $C = 0$. If d_Y is a metric on Y, and a mapping f from X into Y is Lipschitz of some order $\alpha > 0$ with constant $C = 0$, then f is constant on X.

Suppose for the moment that Y is the real line with the standard Euclidean metric. In this case, (1.8.2) is the same as saying that

$$|f(x) - f(w)| \le C\, d_X(x, w)^\alpha \tag{1.8.3}$$

for every $x, w \in X$, where $|\cdot|$ is the standard absolute value function on $\mathbf{R}$. One can check that this holds if and only if

$$f(x) \le f(w) + C\, d_X(x, w)^\alpha \tag{1.8.4}$$

for every $x, w \in X$. More precisely, in order to obtain (1.8.3) from (1.8.4), one can also use the analogue of (1.8.4) with the roles of x and w exchanged.

Let $x_0 \in X$ and $\alpha > 0$ be given, and put

$$f_{\alpha, x_0}(x) = d_X(x, x_0)^\alpha \tag{1.8.5}$$

for every $x \in X$. If $d_X(x, w)^\alpha$ is a semimetric on X, then f_{α, x_0} satisfies (1.8.4) with $C = 1$ for every $x, w \in X$. This implies that f_{α, x_0} is Lipschitz of order α with constant $C = 1$ as a real-valued function on X, as before.

Suppose now that f is a real-valued function on the real line, where $\mathbf{R}$ is equipped with the standard Euclidean metric. If f is Lipschitz of order $\alpha > 1$ with some constant C, then it is well known and not too difficult to show that f is constant on $\mathbf{R}$.

Let X, Y be arbitrary sets with semimetrics d_X, d_Y again. If a mapping f from X into Y is Lipschitz of order $\alpha = 1$ with constant $C \ge 0$, then one may simply say that f is Lipschitz with constant C.

A mapping f from X into Y is an *isometry* with respect to d_X and d_Y if

$$d_Y(f(x), f(w)) = d_X(x, w) \tag{1.8.6}$$

for every $x, w \in X$. Similarly, f is said to be *bilipschitz* with constant $C \ge 1$ if

$$C^{-1}\, d_X(x, w) \le d_Y(f(x), f(w)) \le C\, d_X(x, w) \tag{1.8.7}$$

for every $x, w \in X$. This implies that f is injective on X when d_X is a metric on X. Of course, (1.8.6) is the same as (1.8.7) with $C = 1$. If f is injective, then (1.8.7) is the same as saying that f is Lipschitz with constant C, and that the inverse mapping f^{-1} is Lipschitz with constant C, as a mapping from $f(X)$ into X, and using the restriction of d_Y to $f(X)$.

Let $X_1, \ldots, X_n$ be finitely many sets, and put $X = \prod_{j=1}^n X_j$, as usual. If $1 \le l \le n$, then let p_l be the natural coordinate projection from X into X_l, so that $p_l(x) = x_l$ is the lth coordinate of $x \in X$. Let d_j be a semimetric on X_j for each $j = 1, \ldots, n$, and let $\widehat{d_j}$ be the corresponding semimetric on X, as in Section 1.2. Thus

$$\widehat{d_l}(x, y) = d_l(p_l(x), p_l(y)) \tag{1.8.8}$$

for every $x, y \in X$ and $l = 1, \ldots, n$, so that p_l is an isometry from X into X_l with respect to $\widehat{d_l}$ and d_l. Similarly, for each $l = 1, \ldots, n$, p_l is Lipschitz of order $\alpha = 1$ with constant $C = 1$ with respect to any of the semimetrics $\widehat{d}$, $\widehat{d}'$, and $\widehat{d}''$ defined on X as in Section 1.5, and d_l on X_l.

1.9 Norms and seminorms

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *seminorm* or *pseudonorm* with respect to $|\cdot|$ on k if it satisfies the following two conditions. First,

$$N(t\, v) = |t|\, N(v) \tag{1.9.1}$$

for every $v \in V$ and $t \in k$. Second,

$$N(v + w) \le N(v) + N(w) \tag{1.9.2}$$

for every $v, w \in V$. Note that (1.9.1) implies that $N(v) = 0$ when $v = 0$. If we also have that

$$N(v) > 0 \quad \text{when } v \ne 0, \tag{1.9.3}$$

then N is said to be a *norm* on V. If N is any seminorm on V, then the set of $v \in V$ such that $N(v) = 0$ is a linear subspace of V.

A nonnegative real-valued function N on V is said to be a *semi-ultranorm* if it satisfies (1.9.1) and

$$N(v + w) \le \max(N(v), N(w)) \tag{1.9.4}$$

for every $v, w \in V$. Of course, (1.9.4) implies (1.9.2). If N also satisfies (1.9.3), then N is said to be an *ultranorm* on V. If N is a semi-ultranorm on V, and $N(v) > 0$ for some $v \in V$, then one can check that $|\cdot|$ is an ultrametric absolute value function on k.

If N is a seminorm on V, then it is easy to see that

$$d_N(v, w) = N(v - w) \tag{1.9.5}$$

is a semimetric on V. This is a metric on V when N is a norm on V. Similarly, this is a semi-ultrametric on V when N is a semi-ultranorm on V, and an ultrametric on V when N is an ultranorm on V.

If $|\cdot|$ is the trivial absolute value function on k, then the *trivial ultranorm* is defined on V by putting $N(v)$ equal to 0 when $v = 0$, and equal to 1 otherwise. One can verify that this defines an ultranorm on V, for which the corresponding ultrametric is the discrete metric.

If $|\cdot|$ is any absolute value function on k and $0 < a \le 1$, then $|\cdot|^a$ is an absolute value function on k too, as in Section 1.7. If N is a seminorm on V with respect to $|\cdot|$ on k, then one can check that

$$N(v)^a \tag{1.9.6}$$

is a seminorm on V with respect to $|\cdot|^a$. Similarly, if $|\cdot|$ is an ultrametric absolute value function on k, then $|\cdot|^a$ is an ultrametric absolute value function on k for every $a > 0$. If N is a semi-ultranorm on V with respect to $|\cdot|$ on k, then it is easy to see that (1.9.6) is a semi-ultranorm on V with respect to $|\cdot|^a$ on k. In both cases, we have that

$$d_{N^a}(v, w) = N(v - w)^a = d_N(v, w)^a \tag{1.9.7}$$

for every $v, w \in V$.

1.10 Some basic examples

Let X be a nonempty set, and let k be a field with an absolute value function $|\cdot|$. The space $c(X, k)$ of all k-valued functions on X is a vector space over k, with respect to pointwise addition and scalar multiplication of functions. If $x \in X$, then put

$$N_x(f) = |f(x)| \tag{1.10.1}$$

for every $f \in c(X, k)$. It is easy to see that this defines a seminorm on $c(X, k)$, with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then N_x is a semi-ultranorm on $c(X, k)$.

If $f \in c(X, k)$, then the *support* of f is defined to be the set of $x \in X$ such that $f(x) \neq 0$. Let $c_{00}(X, k)$ be the set of $f \in c(X, k)$ whose support has only finitely many elements. It is easy to see that $c_{00}(X, k)$ is a linear subspace of $c(X, k)$. Of course, if X has only finitely many elements, then $c_{00}(X, k) = c(X, k)$.

If $f \in c_{00}(X, k)$, then put

$$\|f\|_1 = \sum_{x \in X} |f(x)|, \tag{1.10.2}$$

$$\|f\|_2 = \Big(\sum_{x \in X} |f(x)|^2\Big)^{1/2}, \tag{1.10.3}$$

$$\|f\|_\infty = \max_{x \in X} |f(x)|. \tag{1.10.4}$$

More precisely, the sums over $x \in X$ on the right sides of the first two reduce to finite sums of nonnegative real numbers, by hypothesis. One can check that each of these is a norm on $c_{00}(X, k)$, with respect to $|\cdot|$ on k. This uses the triangle inequality for the standard Euclidean norm on $\mathbf{R}^n$ to get the triangle inequality for $\|\cdot\|_2$, as usual. If $|\cdot|$ is an ultrametric absolute value function on $|\cdot|$, then $\|\cdot\|_\infty$ is an ultranorm on $c_{00}(X, k)$.

Similarly, let a be a positive real-valued function on X. If $f \in c_{00}(X, k)$, then put

$$\|f\|_{1,a} = \sum_{x \in X} a(x)\,|f(x)|, \tag{1.10.5}$$

$$\|f\|_{2,a} = \Big(\sum_{x \in X} a(x)^2\,|f(x)|^2\Big)^{1/2}, \tag{1.10.6}$$

$$\|f\|_{\infty,a} = \max_{x \in X} (a(x)\,|f(x)|). \tag{1.10.7}$$

One can verify that each of these defines a norm on $c_{00}(X, k)$, with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then $\|\cdot\|_{\infty,a}$ is an ultranorm on $c_{00}(X, k)$. Of course, if $a(x) = 1$ for every $x \in X$, then these three norms are the same as the ones in the preceding paragraph.

Let $f \in c_{00}(X, k)$ be given, and observe that

$$\|f\|_{\infty,a} \le \|f\|_{1,a} \tag{1.10.8}$$

and

$$\|f\|_{\infty,a} \le \|f\|_{2,a}. \tag{1.10.9}$$

Using (1.10.8), we get that

$$\|f\|_{2,a}^2 = \sum_{x \in X} a(x)^2\,|f(x)|^2 \le \|f\|_{1,a}\,\|f\|_{\infty,a} \le \|f\|_{1,a}^2. \tag{1.10.10}$$

Thus

$$\|f\|_{2,a} \le \|f\|_{1,a}. \tag{1.10.11}$$

Let b be another positive real-valued function on X. Suppose for the moment that X has only finitely many elements. It is easy to see that

$$\|f\|_{1,a} \le \Big(\sum_{x\in X} a(x)/b(x)\Big)\,\|f\|_{\infty,b}. \tag{1.10.12}$$

Similarly,

$$\|f\|_{2,a} \le \Big(\sum_{x\in X} a(x)^2/b(x)^2\Big)^{1/2}\,\|f\|_{\infty,b}. \tag{1.10.13}$$

We also have that

$$\|f\|_{1,a} \le \Big(\sum_{x\in X} a(x)^2/b(x)^2\Big)^{1/2}\,\|f\|_{2,b}, \tag{1.10.14}$$

by the Cauchy–Schwarz inequality.

Suppose now that $X = \mathbf{Z}_+$. If $\sum_{j=1}^\infty a(j)/b(j)$ converges, as an infinite series of positive real numbers, then

$$\|f\|_{1,a} \le \Big(\sum_{j=1}^\infty a(j)/b(j)\Big)\,\|f\|_{\infty,b}. \tag{1.10.15}$$

Similarly, if $\sum_{j=1}^\infty a(j)^2/b(j)^2$ converges, then

$$\|f\|_{2,a} \le \Big(\sum_{j=1}^\infty a(j)^2/b(j)^2\Big)^{1/2}\,\|f\|_{\infty,b}. \tag{1.10.16}$$

In this case, we also get that

$$\|f\|_{1,a} \le \Big(\sum_{j=1}^\infty a(j)^2/b(j)^2\Big)^{1/2}\,\|f\|_{2,b}. \tag{1.10.17}$$

1.11 Regularity and normality

Let X be a topological space. Remember that X is said to satisfy the *first separation condition* if for every pair of distinct elements x, y of X, there is an open set $U \subseteq X$ such that $x \in U$ and $y \notin U$. This is symmetric in x and y, so that there is also an open set $V \subseteq X$ such that $y \in V$ and $x \notin U$. One may also say that X is a T_1 *space* in this case. It is well known that this holds if and only if every subset of X with only one element is a closed set.

Similarly, X satisfies the *second separation condition* if for every pair x, y of distinct elements of X, there are disjoint open subsets U, V of X such that $x \in U$ and $y \in V$. One may say that X is a T_2 *space* in this case, or that X is *Hausdorff.*. Of course, this implies that X satisfies the first separation condition.

We say that X satisfies the *zeroth separation condition* if for every pair x, y of distinct elements of X, there is an open set in X that contains one of the two points, and not the other. Equivalently, one may say that X is a T_0 *space* in this case. The first separation condition clearly implies this one.

Let us say that X is *regular in the strict sense* if for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there are disjoint open subsets U, V of X such that $x \in U$ and $E \subseteq V$. If X also satisfies the first or even zeroth separation condition, then X is said to be *regular in the strong sense.* In this case, one may say that X satisfies the *third separation condition*, or equivalently that X is a T_3 *space*, but sometimes these terms are used for regularity in the strict sense. Similarly, *regularity* is sometimes used to mean regularity in the strict sense, and sometimes it is used to mean regularity in the strong sense.

We say that X is *normal in the strict sense* if for every pair A, B of disjoint closed subsets of X, there are disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. If X satisfies the first separation condition as well, then X is said to be *normal in the strong sense.* If X is normal in the strong sense, then one may say that X satisfies the *fourth separation condition*, or equivalently that X is a T_4 *space*, but these terms are sometimes used for regularity in the strict sense. Sometimes *normality* is used to mean normality in the strict sense, and sometimes it is used to mean normality in the strong sense.

A pair of subsets A, B of X are said to be *separated* in X if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset, \tag{1.11.1}$$

where $\overline{A}$, $\overline{B}$ are the closures of A, B in X, respectively, as usual. If $Y \subseteq X$ and $A, B \subseteq Y$, then it is well known that A, B are separated in X if and only if A, B are separated in Y, with respect to the induced topology. This is because the closure of A in Y is the same as the intersection of Y with the closure of A in X.

We say that X is *completely normal in the strict sense* if for every pair A, B of separated subsets of X there are disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. If X satisfies the first separation condition too, then X is said to be *completely normal in the strong sense.* If X is completely normal in the strong sense, then one may say that X satisfies the *fifth separation condition*, or equivalently that X is a T_5 *space*, but these terms are sometimes used for complete normality in the strict sense. As before, *complete normality* is sometimes used to mean complete normality in the strict sense, and sometimes it is used to mean complete normality on the strong sense.

If X is regular in the strong sense, then it is easy to see that X is Hausdorff. Similarly, if X is normal in the strong sense, then X is regular in the strong sense. If X is completely normal in the strict sense, then X is normal in the strict sense, because disjoint closed subsets of X are separated in X. If X is completely normal in the strong sense, then it follows that X is normal in the strong sense.

Let Y be a subset of X again, equipped with the induced topology. If X satisfies the zeroth, first, or second separation condition, then it is well known that Y has the same property. Similarly, if X is regular in the strict sense, then Y is regular in the strict sense. If X is regular in the strong sense, then it follows that Y is regular in the strong sense.

If X is completely normal in the strict sense, then Y is completely normal in the strict sense. This uses the fact that separated subsets of Y are separated in X as well. If X is completely normal in the strong sense, then it follows that Y is completely normal in the strong sense.

It is well known that X is regular in the strict sense if and only if for every $x \in X$ and open set $W \subseteq X$ with $x \in W$, there is an open set $U \subseteq X$ such that $x \in U$ and $\overline{U} \subseteq W$. Similarly, X is normal in the strict sense if and only if for every pair of open subsets U, W of X with $\overline{U} \subseteq W$, there is an open set $V \subseteq X$ such that $\overline{U} \subseteq V$ and $\overline{V} \subseteq W$.

We say that X is *completely regular in the strict sense* if for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there is a continuous real-valued function f on X such that $f(x) > 0$ and $f(y) = 0$ for every $y \in E$. Of course, this uses the standard topology on $\mathbf{R}$, as the range of f. One may also ask that $f(x) = 1$, and that $0 \le f(z) \le 1$ for every $z \in X$.

If X also satisfies the first or even zeroth separation condition, then X s said to be *completely regular in the strong sense*. In this case, one may say that X satisfies *separation condition number three-and-a-half*, or equivalently that X is a $T_{3\frac{1}{2}}$ *space*, but these terms are sometimes used for complete regularity in the strict sense. As usual, *complete reguarity* is sometimes used to mean complete regularity in the strict sense, and sometimes it is used to mean complete regularity in the strong sense.

If X is completely regular in the strict sense, then it is easy to see that X is regular in the strict sense, because the real line is Hausdorff with respect to the standard topology. If X is completely regular in the strong sense, then it follows that X is regular in the strong sense. If X is normal in the strong sense, then it is well known that X is completely regular in the strong sense, by Urysohn's lemma.

If X is completely regular in the strict sense and $Y \subseteq X$, then it is easy to see that Y is completely regular in the strict sense, with respect to the induced topology. If X is completely regular in the strong sense, then it follows that Y is completely regular in the strong sense, with respect to the induced topology.

1.12 Related properties of semimetrics

Let X be a set, and let $d(\cdot,\cdot)$ be a semimetric on X. If $Y \subseteq X$, then the restriction of $d(\cdot,\cdot)$ to elements of Y defines a semimetric on Y. It is well known that the topology determined on Y by the restriction of $d(\cdot,\cdot)$ to Y is the same as the topology induced on Y by the topology determined on X by $d(\cdot,\cdot)$. To see this, let $y \in Y$ and $r > 0$ be given, and let $B_X(y,r)$, $B_Y(y,r)$ be the open balls in X, Y, respectively, centered at y, with radius r, and with respect to $d(\cdot,\cdot)$ or its restriction to Y, as appropriate. Observe that

$$B_Y(y,r) = B_X(y,r) \cap Y. \tag{1.12.1}$$

Using this, one can check that any open subset of Y with respect to the induced topology is also an open set with respect to the topology determined by the restriction of $d(\cdot,\cdot)$ to Y. One can also use (1.12.1) to get that $B_Y(y,r)$ is an open set in Y with respect to the induced topology. Of course, every open set in Y with respect to the topology determined by the restriction of $d(\cdot,\cdot)$ to Y can be expressed as a union of open balls in Y. It follows that every such subset of Y is an open set with respect to the induced topology on Y, because it is a union of open sets with respect to the induced topology.

As mentioned in Section 1.2, X is Hausdorff with respect to the topology determined by $d(\cdot,\cdot)$ when $d(\cdot,\cdot)$ is a metric on X, and it is necessary for $d(\cdot,\cdot)$ to be a metric on X in order for X to satisfy the first or even zeroth separation condition with respect to the topology determined by $d(\cdot,\cdot)$. One can check that X is regular in the strict sense with respect to the topology determined by $d(\cdot,\cdot)$, using the fact that closed balls in X with respect to $d(\cdot,\cdot)$ are closed sets. One can also verify that X is completely normal in the strict sense with respect to the topology determined by $d(\cdot,\cdot)$, in the same way as for metric spaces.

If $x_0 \in X$, then

$$f_{x_0}(x) = d(x, x_0) \tag{1.12.2}$$

is continuous as a real-valued function on X, as in Section 1.8. Using this, it is easy to see that X is completely regular in the strict sense with respect to the topology determined by $d(\cdot,\cdot)$.

A topological space Y is said to be *zero dimensional* if the collection of subsets of Y that are both open and closed is a base for the topology of Y. In this case, Y is regular in the strict sense, and in fact completely regular in the strict sense. This also implies that any subset of Y is zero dimensional, with respect to the induced topology. If $d(\cdot,\cdot)$ is a semi-ultrametric on X, then X is zero dimensional with respect to the topology determined by $d(\cdot,\cdot)$.

A topological space Y is said to be *totally separated* if for every pair of distinct elements y_1, y_2 of Y, there is an open set $U_1 \subseteq Y$ such that U_1 is a closed set, $y_1 \in U_1$, and $y_2 \in Y \setminus U_1$. Of course, this means that $U_2 = Y \setminus U_1$ is open and closed, $y_1 \in Y \setminus U_2$, and $y_2 \in U_2$. If Y is totally separated, then Y is Hausdorff, and every subset of Y is totally separated, with respect to the induced topology. If Y is zero dimensional and satisfies the zeroth separation condition, then Y is totally separated. If $d(\cdot,\cdot)$ is an ultrametric on X, then X is totally separated with respect to the topology determined by $d(\cdot,\cdot)$.

Let Y be a set, and let τ_1, τ_2 be topologies on Y, with $\tau_1 \subseteq \tau_2$. If Y satisfies the zeroth, first, or second separation condition with respect to τ_1, then it is easy to see that Y has the same property with respect to τ_2. Similarly, if Y is totally separated with respect to τ_1, then Y is totally separated with respect to τ_2.

Remember that a subset E of a topological space Y is said to be *totally disconnected* if E does not contain any connected sets with at least two elements. If Y is totally separated, then one can check that Y is totally disconnected.

1.13 Formal series

Let k be a field, and let T be an indeterminate. As in [3, 10], we normally use upper-case letters for indeterminates, and lower-case letters for elements of k. Let $k((T))$ be the space of formal sums of the form

$$f(T) = \sum_{j=j_0}^{\infty} f_j\, T^j, \tag{1.13.1}$$

where $j_0 \in \mathbf{Z}$ and $f_j \in k$ for each $j \geq j_0$. More precisely, we consider $f_j \in k$ to be defined for every $j \in \mathbf{Z}$, with $f_j = 0$ when $j < j_0$. We may also use the notation

$$f(T) = \sum_{j>>-\infty} f_j\, T^j \tag{1.13.2}$$

for an element of $k((T))$, as in [3].

Equivalently, $k((T))$ may be considered as the space of k-valued functions on $\mathbf{Z}$ that are equal to zero at all but at most finitely many negative integers. This is a linear subspace of the space $c(\mathbf{Z}, k)$ of all k-valued functions on $\mathbf{Z}$, as a vector space over k with respect to pointwise addition and scalar multiplication of functions. Of course, this corresponds to termwise addition and scalar multiplication of formal sums as in the preceding paragraph.

Let $f(T)$ be as in (1.13.1), and let $g(T) = \sum_{l=l_0} g_l\, T^l$ be another element of $k((T))$. If $n \in \mathbf{Z}$, then put

$$h_n = \sum_{j=-\infty}^{\infty} f_j\, g_{n-j} = \sum_{j+l=n} f_j\, g_l, \tag{1.13.3}$$

where the second sum is taken over all $j, l \in \mathbf{Z}$ with $j + l = n$. It is easy to see that all but finitely many terms in these sums are equal to 0, so that these sums reduce to finite sums in k. One can also check that $h_n = 0$ when $n < j_0 + l_0$, so that

$$h(T) = \sum_{n=j_0+l_0}^{\infty} h_n\, T^n \tag{1.13.4}$$

is an element of $k((T))$. Put

$$f(T)\, g(T) = h(T), \tag{1.13.5}$$

which is defined to be the product of $f(T)$ and $g(T)$ in $k((T))$.

It is well known not difficult to verify that $k((T))$ is a commutative associative algebra over k with respect to this definition of multiplication. If we identify elements of k with the corresponding multiple of T^0 in $k((T))$, then k corresponds to a subalgebra of $k((T))$. It is easy to see that the multiplicative identity element 1 in k corresponds to the multiplicative identity element of $k((T))$ in this way.

The algebra $k[T]$ of formal polynomials in T with coefficients in k may be identified with the subalgebra of $k((T))$ consisting of $f(T)$ as in (1.13.1) with

$j_0 = 0$ and $f_j = 0$ for all but finitely many $j \geq j_0$. Similarly, the algebra $k[[T]]$ of formal power series in T with coefficients in k may be identified with the subalgebra of $k((T))$ consisting of $f(T)$ as in (1.13.1) with $j_0 = 0$.

If $a(T) \in k[[T]]$ and n is a nonnegative integer, then

$$\sum_{l=0}^{n} a(T)^l \, T^l \tag{1.13.6}$$

defines an element of $k[[T]]$, where the $l = 0$ term is interpreted as being equal to 1, as usual. Note that

$$(1 - a(T)\,T) \sum_{l=0}^{n} a(T)^l \, T^l = 1 - a(T)^{n+1} \, T^{n+1}, \tag{1.13.7}$$

by a standard computation.

It is easy to define

$$\sum_{l=0}^{\infty} a(T)^l \, T^l \tag{1.13.8}$$

as an element of $k[[T]]$, by taking the coefficient of T^j in the sum to be the same as in (1.13.6) when $n \geq j$. One can check that

$$(1 - a(T)\,T) \sum_{l=0}^{\infty} a(T)^l \, T^l = 1, \tag{1.13.9}$$

using (1.13.7). This shows that $1 - a(T)T$ has multiplicative inverse equal to (1.13.8) in $k[[T]]$.

Let $f(T)$ be as in (1.13.1) again, and suppose that $f_{j_0} \neq 0$. In this case, $f(T)$ can be expressed as

$$f(T) = f_{j_0} \, T^{j_0} \, (1 - a(T)\,T) \tag{1.13.10}$$

for some $a(T) \in k[[T]]$. It follows that $f(T)$ has a multiplicative inverse in $k((T))$, by the remarks in the preceding paragraph. Thus $k((T))$ is a field.

If $f(T)$ is as in (1.13.1) and $f_{j_0} \neq 0$, then put

$$j_0(f(T)) = j_0. \tag{1.13.11}$$

This may be interpreted as being $+\infty$ when $f(T) = 0$. One can check that

$$j_0(f(T)\,g(T)) = j_0(f(T)) + j_0(g(T)) \tag{1.13.12}$$

and

$$j_0(f(T) + g(T)) \geq \min(j_0(f(T)), j_0(g(T))) \tag{1.13.13}$$

for every $f(T), g(T) \in k((T))$, with suitable interpretations when $f(T)$ or $g(T)$ is 0.

Let r be a positive real number less than or equal to 1. If $f(T) \in k((T))$ and $f(T) \neq 0$, then put

$$|f(T)|_r = r^{j_0(f(T))}, \tag{1.13.14}$$

and put $|f(T)|_r = 0$ when $f(T) = 0$. It is easy to see that $|\cdot|_r$ defines an ultrametric absolute value function on $k((T))$, using (1.13.12) and (1.13.13). If $r = 1$, then this is the trivial absolute value function on $k((T))$. If a is a positive real number, then $0 < r^a \leq 1$, and

$$|f(T)|_r^a = |f(T)|_{r^a} \tag{1.13.15}$$

for every $f(T) \in k((T))$.

1.14 Bounded sets

Let X be a set, and let $d(x, y)$ be a semimetric on X. One may say that a subset E of X is *bounded* with respect to d if E is contained in a ball in X. It may be helpful to consider the empty set as a bounded subset of X, even when $X = \emptyset$. Equivalently, one may say that E is bounded when the set of nonnegative real numbers of the form $d(x, y)$, with $x, y \in E$, has an upper bound in $\mathbf{R}$. Of course, this holds automatically when $E = \emptyset$.

If $x, y \in X$, then

$$B(x, r) \subseteq B(y, d(x, y) + r) \tag{1.14.1}$$

for every $r > 0$, and

$$\overline{B}(x, r) \subseteq \overline{B}(y, d(x, y) + r) \tag{1.14.2}$$

for every $r \geq 0$. If E is contained in a ball in X centered at a point $y \in X$, then this implies that E is contained in a ball centered at any $x \in X$, with a radius that depends on x. It follows that the union of finitely many bounded subsets of X is bounded as well.

If E is compact with respect to the topology determined on X by $d(\cdot, \cdot)$, and $x \in X$, then E is contained in an open ball in X centered at x with respect to $d(\cdot, \cdot)$. This can be seen using the family of open balls $B(x, r)$ with $r \in \mathbf{Z}_+$ as an open covering of E in X.

If E is a nonempty bounded subset of X, then the *diameter* of E with respect to $d(\cdot, \cdot)$ is defined as usual by

$$\operatorname{diam} E = \operatorname{diam}_d E = \sup\{d(x, y) : x, y \in E\}. \tag{1.14.3}$$

This may be interpreted as being 0 when $E = \emptyset$, and as being $+\infty$ when E is not bounded in X.

If E is a bounded subset of X, then one can check that the closure $\overline{E}$ of E in X with respect to the topology determined by $d(\cdot, \cdot)$ is bounded too, with

$$\operatorname{diam} \overline{E} = \operatorname{diam} E. \tag{1.14.4}$$

If $A \subseteq E$, then A is bounded as well, with

$$\operatorname{diam} A \leq \operatorname{diam} E. \tag{1.14.5}$$

If $E \subseteq X$ is bounded and $x \in E$, then

$$E \subseteq \overline{B}(x, \operatorname{diam} E). \tag{1.14.6}$$

If $E \subseteq \overline{B}(y, r)$ for some $y \in X$ and $r \geq 0$, then it is easy to see that

$$\operatorname{diam} E \leq 2\, r. \tag{1.14.7}$$

If $d(\cdot, \cdot)$ is a semi-ultrametric on X, then we have that

$$\operatorname{diam} E \leq r \tag{1.14.8}$$

in this case.

Suppose that $d(x, y)^a$ is also a semimetric on X for some $a > 0$. Observe that $E \subseteq X$ is bounded with respect to $d(\cdot, \cdot)$ if and only if E is bounded with respect to $d(\cdot, \cdot)^a$, with

$$\operatorname{diam}_{d^a} E = (\operatorname{diam}_d E)^a. \tag{1.14.9}$$

Let $d_1, \ldots, d_n$ be finitely many semimetrics on X, and let d be their maximum, which is a semimetric on X too, as in Section 1.5. One can check that $E \subseteq X$ is bounded with respect to d if and only if E is bounded with respect to d_j for each $j = 1, \ldots, n$, with

$$\operatorname{diam}_d E = \max_{1 \leq j \leq n} (\operatorname{diam}_{d_j} E). \tag{1.14.10}$$

Let d' be the sum of $d_1, \ldots, d_n$, which is another semimetric on X, as in Section 1.5 again. It is easy to see that $E \subseteq X$ is bounded with respect to d' if and only if E is bounded with respect to d_j for each $j = 1, \ldots, n$, with

$$\max_{1 \leq j \leq n} (\operatorname{diam}_{d_j} E) \leq \operatorname{diam}_{d'} E \leq \sum_{j=1}^{n} \operatorname{diam}_{d_j} E. \tag{1.14.11}$$

Remember that the square root of the sum of the squares of $d_1, \ldots, d_n$ defines a semimetric d'' on X, as in Section 1.5. One can verify that $E \subseteq X$ is bounded with respect to d'' if and only if E if bounded with respect to d_j for each $j = 1, \ldots, n$, with

$$\max_{1 \leq j \leq n} (\operatorname{diam}_{d_j} E) \leq \operatorname{diam}_{d''} E \leq \Big(\sum_{j=1}^{n} (\operatorname{diam}_{d_j} E)^2 \Big)^{1/2}. \tag{1.14.12}$$

Let d_X be a semimetric on X, and let Y be a set with a semimetric d_Y. Suppose that f is a mapping from X into Y that is Lipschitz of order $\alpha > 0$ with constant $C \geq 0$, as in Section 1.8. If $E \subseteq X$ is bounded with respect to d_X, then $f(E)$ is bounded in Y with respect to d_Y, with

$$\operatorname{diam}_{d_Y} f(E) \leq C\, (\operatorname{diam}_{d_X} E)^\alpha. \tag{1.14.13}$$

Let $X_1, \ldots, X_n$ be finitely many sets, and put $X = \prod_{j=1}^n X_j$. Also let d_j be a semimetric on X_j for each $j = 1, \ldots, n$, and let $\widehat{d}_j$ be the corresponding semimetric on X, as in Section 1.2. This leads to semimetrics $\widehat{d}$, $\widehat{d}'$, and $\widehat{d}''$ on X, as in Section 1.5. If $E_j \subseteq X_j$ is bounded with respect to d_j for each $j = 1, \ldots, n$, then $E = \prod_{j=1}^n E_j$ is bounded with respect to these semimetrics on X.

More precisely, suppose that $E_j \neq \emptyset$ for each $j = 1, \ldots, n$. Under these conditions,

$$\operatorname{diam}_{\widehat{d}} E = \max_{1 \le j \le n} (\operatorname{diam}_{d_j} E_j). \tag{1.14.14}$$

Similarly,

$$\max_{1 \le j \le n} (\operatorname{diam}_{d_j} E_j) \le \operatorname{diam}_{\widehat{d}'} E \le \sum_{j=1}^n \operatorname{diam}_{d_j} E_j \tag{1.14.15}$$

and

$$\max_{1 \le j \le n} (\operatorname{diam}_{d_j} E_j) \le \operatorname{diam}_{\widehat{d}''} E \le \Big(\sum_{j=1}^n (\operatorname{diam}_{d_j} E_j)^2\Big)^{1/2}. \tag{1.14.16}$$

1.15 Totally bounded sets

Let X be a set, and let $d(x, y)$ be a semimetric on X again. A subset E of X is said to be *totally bounded* in X with respect to d if for every $r > 0$, E is contained in the union of finitely many open balls in X of radius r. This is equivalent to asking that for every $r > 0$, E can be covered by finitely many closed balls of radius r in X. Totally bounded sets are automatically bounded, because the union of finitely many bounded sets is bounded, as in the previous section. Note that the union of finitely many totally bounded subsets of X is totally bounded as well.

If E is compact with respect to the topology determined on X by $d(\cdot, \cdot)$, then it is easy to see that E is totally bounded with respect to d, by covering E with open balls of radius r for any $r > 0$. Of course, subsets of totally bounded sets are totally bounded too. If E is any totally bounded subset of X, then it is easy to see that the closure $\overline{E}$ of E in X with respect to the topology determined by $d(\cdot, \cdot)$ is totally bounded, using the characterization of totally bounded sets in terms of coverings by closed balls. It is well known that subsets of complete metric spaces that are both closed and totally bounded are compact.

One can check that $E \subseteq X$ is totally bounded if and only if for every $r > 0$, E is contained in the union of finitely many sets, each of which has diameter less than or equal to r. One may as well take these sets to be contained in E, by taking their intersections with E, if necessary. If E is totally bounded, then it follows that for each $r > 0$, E can be covered by finitely many open balls in X of radius r, centered at points in E.

Let X_0 be a subset of X, so that the restriction of $d(\cdot, \cdot)$ to X_0 defines a semimetric on X_0. If $E \subseteq X_0$, then one can check that E is totally bounded as a subset of X if and only if E is totally bounded as a subset of X_0. This can be seen using the characterization of total boundedness in terms of covering E

by finitely many subsets of itself of small diameter, or in terms of covering E by finitely many balls of small radius centered at elements of E.

Let Y be another set with a semimetric d_Y, and let f be a uniformly continuous mapping from X into Y. If $E \subseteq X$ is totally bounded, then one can verify that $f(E)$ is totally bounded in Y.

Let $d_1, \ldots, d_n$ be finitely many semimetrics on X, and let d be their maximum, so that d is a semimetric on X as well, as in Section 1.5. If $A_1, \ldots, A_n$ are subsets of X, then

$$\operatorname{diam}_d \Big(\bigcap_{j=1}^n A_j \Big) \le \max_{1 \le j \le n} (\operatorname{diam}_{d_j} A_j). \tag{1.15.1}$$

If $E \subseteq X$ is totally bounded with respect to d, then E is clearly totally bounded with respect to d_j for each $j = 1, \ldots, n$. Conversely, suppose that E is totally bounded with respect to d_j for each $j = 1, \ldots, n$, and let $r > 0$ be given. Thus, for each $j = 1, \ldots, n$, E is contained in the union of finitely many subsets of X, each of which has diameter less than or equal to r with respect to d_j. If one takes the intersection of sets from each of these coverings, then one gets a subset of X with diameter less than or equal to r with respect to d, as in (1.15.1). There are only finitely many subsets of X obtained by taking intersections in this way, and E is contained in the union of these intersections. This means that E is contained in the union of finitely many subsets of X with diameter less than or equal to r with respect to d. It follows that E is totally bounded in X with respect to d under these conditions.

Let n be a positive integer again, and let X_j be a set with a semimetric d_j for each $j = 1, \ldots, n$. Put $X = \prod_{j=1}^n X_j$, and let $\widehat{d_l}$ be the semimetric on X corresponding to d_l on X_l for each $l = 1, \ldots, n$. Remember that the maximum $\widehat{d}$ of $\widehat{d_l}$, $1 \le l \le n$, is a semimetric on X too, as in Section 1.5. If $E_j \subseteq X_j$ is totally bounded with respect to d_j for each $j = 1, \ldots, n$, then

$$E = \prod_{j=1}^n E_j \tag{1.15.2}$$

is totally bounded in X with respect to $\widehat{d}$.

To see this, let $r > 0$ be given, so that E_j can be covered by finitely many subsets of X_j with diameter less than or equal to r for each $j = 1, \ldots, n$. If $A_j \subseteq X_j$ has diameter less than or equal to r with respect to d_j for each $j = 1, \ldots, n$, then

$$A = \prod_{j=1}^n A_j \tag{1.15.3}$$

has diameter less than or equal to r in X with respect to $\widehat{d}$, as in the previous section. This permits one to cover E by finitely many subsets of X with diameter less than or equal to r with respect to $\widehat{d}$.

Alternatively, let p_l be the natural coordinate projection from X into X_l for each $l = 1, \ldots, n$. It is easy to see that

$$p_l^{-1}(E_l) \tag{1.15.4}$$

is totally bounded in X with respect to $\widehat{d}_l$ for each $l = 1, \ldots, n$, because E_l is totally bounded in X with respect to d_l. This implies that E is totally bounded in X with respect to $\widehat{d}_l$ for each $l = 1, \ldots, n$, because $E \subseteq p_l^{-1}(E_l)$. It follows that E is totally bounded in X with respect to $\widehat{d}$, because $\widehat{d}$ is the maximum of $\widehat{d}_1, \ldots, \widehat{d}_n$, as before.

Chapter 2

Collections of semimetrics

2.1 Collections and topologies

Let X be a set, and let $\mathcal{M}$ be a nonempty collection of semimetrics on X. Let us say that a subset U of X is an *open set* with respect to $\mathcal{M}$ if for every $x \in U$ there are finitely many elements $d_1, \ldots, d_n$ of $\mathcal{M}$ and positive real numbers $r_1, \ldots, r_n$ such that

$$\bigcap_{j=1}^{n} B_{d_j}(x, r_j) \subseteq U. \tag{2.1.1}$$

One can check that this defines a topology on X.

If $d \in \mathcal{M}$, then open sets in X with respect to d are open sets with respect to $\mathcal{M}$. In particular, open balls in X with respect to d are open sets with respect to $\mathcal{M}$. Similarly, closed balls in X with respect to d are closed sets with respect to $\mathcal{M}$. One can use this to get that X is regular in the strict sense with respect to the topology determined by $\mathcal{M}$.

Let us say that $\mathcal{M}$ is *nondegenerate* on X if for every $x, y \in X$ with $x \neq y$ there is a $d \in \mathcal{M}$ such that

$$d(x, y) > 0. \tag{2.1.2}$$

One can check that X is Hausdorff with respect to the topology determined by $\mathcal{M}$ in this case. One can also verify that nondegeneracy is necessary for X to satisfy the first or even zeroth separation conditions with respect to the topology determined by $\mathcal{M}$.

If $d \in \mathcal{M}$ and $x_0 \in X$, then

$$f_{x_0,d}(x) = d(x, x_0) \tag{2.1.3}$$

is continuous as a real-valued function on X with respect to the topology determined on X by d, as in Section 1.8. This implies that $f_{x_0,d}$ is continuous with respect to the topology determined on X by $\mathcal{M}$. One can use this to get that X is completely regular in the strict sense with respect to the topology determined by $\mathcal{M}$.

If the elements of $\mathcal{M}$ are semi-ultrametrics on X, then it is easy to see that X is zero dimensional with respect to the topology determined by $\mathcal{M}$. If $\mathcal{M}$ is also nondegenerate on X, then X is totally separated with respect to this topology.

If $\mathcal{M}$ has only finitely many elements, then one can get a single semimetric on X that determines the same topology, as in Section 1.5. If $\mathcal{M}$ is nondegenerate on X, then one gets a metric on X in this way.

Let Y be a subset of X, and let $\mathcal{M}_Y$ be the collection of semimetrics on Y obtained by restricting the elements of $\mathcal{M}$ to Y. One can check that the topology determined on Y by $\mathcal{M}_Y$ is the same as the topology induced on Y by the topology determined on X by $\mathcal{M}$.

It is easy to see that a sequence $\{x_j\}_{j=1}^\infty$ of elements of X converges to $x \in X$ with respect to the topology determined by $\mathcal{M}$ if and only if $\{x_j\}_{j=1}^\infty$ converges to x with respect to every $d \in \mathcal{M}$. More precisely, this works for convergence of nets in X too.

Let I be a nonempty set, let X_j be a set for each $j \in I$, and consider $X = \prod_{j \in I} X_j$. If $l \in I$ and d_l is a semimetric on X_l, then let $\widehat{d_l}(x, y) = d_l(x_l, y_l)$ be the corresponding semimetric on X, as in Section 1.2. Let $\mathcal{M}_l$ be a nonempty collection of semimetrics on X_l for each $l \in I$, and put

$$\widehat{\mathcal{M}_l} = \{\widehat{d_l} : d_l \in \mathcal{M}_l\} \tag{2.1.4}$$

for every $l \in I$. Thus

$$\mathcal{M} = \bigcup_{l \in I} \widehat{\mathcal{M}_l} \tag{2.1.5}$$

is a nonempty collection of semimetrics on X. If $\mathcal{M}_l$ is nondegenerate on X_l for each $l \in I$, then it is easy to see that $\mathcal{M}$ is nondegenerate on X.

Let p_l be the natural coordinate mapping from X into X_l for each $l \in I$, so that $p_l(x) = x_l$ is the lth coordinate of $x \in X$. If $x \in X$, $l \in I$, and $d_l \in \mathcal{M}_l$, then

$$p_l^{-1}(B_{X_l, d_l}(p_l(x), r)) = B_{X, \widehat{d_l}}(x, r) \tag{2.1.6}$$

for every $r > 0$, and

$$p_l^{-1}(\overline{B}_{X_l, d_l}(p_l(x), r)) = \overline{B}_{X, \widehat{d_l}}(x, r) \tag{2.1.7}$$

for every $r \geq 0$. Using this, one can check that the topology determined on X by $\mathcal{M}$ is the same as the product topology defined using the topology determined on X_l by $\mathcal{M}_l$ for each $l \in I$.

2.2 Collections and uniform continuity

Let X, Y be sets, let $\mathcal{M}_X$ be a nonempty collection of semimetrics on X, and let d_Y be a semimetric on Y. Let us say that a mapping f from X into Y is *uniformly continuous* with respect to $\mathcal{M}_X$ on X and d_Y on Y if for every $\epsilon > 0$

there are finitely many elements $d_1, \ldots, d_n$ of $\mathcal{M}_X$ and positive real numbers $\delta_1, \ldots, \delta_n$ such that

$$d_Y(f(x), f(w)) < \epsilon \tag{2.2.1}$$

for every $x, w \in X$ with

$$d_j(x, w) < \delta_j \quad \text{for each } j = 1, \ldots, n. \tag{2.2.2}$$

Of course, this reduces to the usual definition of uniform continuity when $\mathcal{M}_X$ has only one element. It is easy to see that uniform continuity in this sense implies ordinary continuity, with respect to the topology determined on X by $\mathcal{M}_X$, and the topology determined on Y by d_Y.

If $\mathcal{M}_X$ has only finitely many elements, then one can combine them to get a single semimetric on X as in Section 1.5. In this case, uniform continuity with respect to $\mathcal{M}_X$ on X is equivalent to uniform continuity with respect to a combined semimetric on X of this type.

Suppose now that $\mathcal{M}_Y$ is a nonempty collection of semimetrics on Y. It is easy to see that a mapping from a topological space into Y is continuous with respect to the topology determined on Y by $\mathcal{M}_Y$ if and only if for every $d_Y \in \mathcal{M}_Y$, the mapping is continuous with respect to the topology determined on Y by d_Y.

Let us say that a mapping f from X into Y is *uniformly continuous* with respect to $\mathcal{M}_X$, $\mathcal{M}_Y$ if for every $d_Y \in \mathcal{M}_Y$, f is uniformly continuous with respect to $\mathcal{M}_X$ on X and d_Y on Y, as before. This implies that f is continuous in the ordinary sense with respect to the topologies determined on X, Y by $\mathcal{M}_X$, $\mathcal{M}_Y$, respectively, by the remarks in the previous two paragraphs.

Let Z be another set with a nonempty collection $\mathcal{M}_Z$ of semimetrics. If a mapping f from X into Y is uniformly continuous with respect to $\mathcal{M}_X$, $\mathcal{M}_Y$, and if a mapping g from Y into Z is uniformly continuous with respect to $\mathcal{M}_Y$, $\mathcal{M}_Z$, then one can verify that their composition $g \circ f$ is uniformly continuous as a mapping from X into Z with respect to $\mathcal{M}_X$, $\mathcal{M}_Z$.

Let X_0 be a subset of X, and let $\mathcal{M}_{X_0}$ be the collection of semimetrics on X_0 obtained by restricting the elements of $\mathcal{M}_X$ to X_0. If f is a mapping from X into Y, and f is uniformly continuous with respect to $\mathcal{M}_X$ on X and a semimetric d_Y on Y, then the restriction of f to X_0 is uniformly continuous with respect to $\mathcal{M}_{X_0}$ on X_0 and d_Y on Y. If f is uniformly continuous with respect to $\mathcal{M}_X$ on X and $\mathcal{M}_Y$ on Y, then it follows that the restriction of f to X_0 is uniformly continuous with respect to $\mathcal{M}_{X_0}$ on X_0 and $\mathcal{M}_Y$ on Y. Note that the natural inclusion mapping from X_0 into X is uniformly continuous, with respect to $\mathcal{M}_{X_0}$ on X_0 and $\mathcal{M}_X$ on X.

It is sometimes helpful to consider uniform continuity along a subset X_0 of X. A mapping f from X into Y is said to be *uniformly continuous along* X_0 with respect to $\mathcal{M}_X$ on X and a semimetric d_Y of Y if for every $\epsilon > 0$ there are finitely many elements $d_1, \ldots, d_n$ of $\mathcal{M}_X$ and positive real numbers $\delta_1, \ldots, \delta_n$ such that (2.2.1) holds for every $x \in X_0$ and $w \in X$ that satisfy (2.2.2). Similarly, let us say that f is uniformly continuous along X_0 with respect to $\mathcal{M}_X$ on X and $\mathcal{M}_Y$ on Y if for every $d_Y \in \mathcal{M}_Y$, f is uniformly continuous along X_0 with respect to $\mathcal{M}_X$ on X and d_Y on Y. In both cases, uniform continuity of f along X_0 implies that the restriction of f to X_0 is uniformly continuous with respect to $\mathcal{M}_{X_0}$, as in the preceding paragraph. Uniform continuity of f along X_0 also implies that f is continuous at every point on X_0, with respect to the topology determined on X by $\mathcal{M}_X$, and the topology determined on Y by d_Y or $\mathcal{M}_Y$, as appropriate.

Suppose that a mapping f from X into Y is continuous at every point in X_0, with respect to the topology determined on X by $\mathcal{M}_X$, and the topology determined on Y by a semimetric d_Y. If X_0 is compact in X, then f is uniformly continuous along X_0. Indeed, let $\epsilon > 0$ be given, and observe that for every $x \in X_0$ there is a finite subset $\mathcal{M}_X(x)$ of $\mathcal{M}_X$ and a positive real number $\delta(x)$ such that

$$d_Y(f(x), f(w)) < \epsilon/2 \tag{2.2.3}$$

for every $w \in X$ with

$$d_X(x, w) < \delta(x) \quad \text{for each } d_X \in \mathcal{M}_X(x). \tag{2.2.4}$$

If $x \in X_0$, then put

$$B_X(x) = \bigcap_{d_X \in \mathcal{M}_X(x)} B_{X,d_X}(x, \delta(x)/2), \tag{2.2.5}$$

which is an open set in X that contains x. If X_0 is compact, then there are finitely many elements $x_1, \ldots, x_n$ of X_0 such that

$$X_0 \subseteq \bigcup_{j=1}^{n} B_X(x_j). \tag{2.2.6}$$

Put

$$\mathcal{M}_X(X_0, \epsilon) = \bigcup_{j=1}^{n} \mathcal{M}_X(x_j), \tag{2.2.7}$$

which is a finite subset of $\mathcal{M}_X$, and

$$\delta(X_0, \epsilon) = \min_{1 \le j \le n} (\delta(x_j)/2). \tag{2.2.8}$$

Let $x \in X_0$ and $w \in X$ be given, with

$$d_X(x, w) < \delta(X_0, \epsilon) \quad \text{for every } d_X \in \mathcal{M}_X(X_0, \epsilon). \tag{2.2.9}$$

Note that $x \in B_X(x_j)$ for some j, $1 \le j \le n$, by (2.2.6). Thus

$$d_X(x_j, x) < \delta(x_j)/2 \quad \text{for every } d_X \in \mathcal{M}_X(x_j), \tag{2.2.10}$$

so that

$$d_X(x_j, w) \le d_X(x_j, x) + d_X(x, w) < \delta(x_j)/2 + \delta(X_0, \epsilon) \le \delta(x_j) \tag{2.2.11}$$

for every $d_X \in \mathcal{M}_X(x_j)$.

It follows that

$$d_Y(f(x_j), f(x)),\ d_Y(f(x_j), f(w)) < \epsilon/2, \tag{2.2.12}$$

by the way that $\delta(x_j)$ was chosen. This implies that

$$\begin{aligned} d_Y(f(x), f(w)) &\le d_Y(f(x), f(x_j)) + d_Y(f(x_j), f(w)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned} \tag{2.2.13}$$

as desired. If f is continuous at every point in X_0 with respect to the topology determined on X by $\mathcal{M}_X$ and the topology determined on Y by $\mathcal{M}_Y$, and if X_0 is compact in X, then we get that f is uniformly continuous along X_0, with respect to $\mathcal{M}_X$ on X and $\mathcal{M}_Y$ on Y.

2.3 Uniform structures

Let X be a set, and put

$$\Delta = \Delta_X = \{(x, x) : x \in X\}. \tag{2.3.1}$$

If $A, B \subseteq X \times X$, then put

$$A^{-1} = \{(x, y) : (y, x) \in A\} \tag{2.3.2}$$

and

$$\begin{aligned} A \circ B = \{(x, z) &: \text{there is a } y \in X \text{ such that} \\ &(x, y) \in A \text{ and } (y, z) \in B\}. \end{aligned} \tag{2.3.3}$$

If

$$A^{-1} = A, \tag{2.3.4}$$

then A is said to be *symmetric* in $X \times X$.

A *uniformity* or *uniform structure* on X is a nonempty collection $\mathcal{U}$ of subsets of $X \times X$ that satisfies the following five conditions, as on p176 of [15]. First, if $U \in \mathcal{U}$, then

$$\Delta \subseteq U. \tag{2.3.5}$$

Second, if $U \in \mathcal{U}$, then

$$U^{-1} \in \mathcal{U}. \tag{2.3.6}$$

Third, if $U \in \mathcal{U}$, then there is a $V \in \mathcal{U}$ such that

$$V \circ V \subseteq U. \tag{2.3.7}$$

Fourth, if $U, V \in \mathcal{U}$, then

$$U \cap V \in \mathcal{U}. \tag{2.3.8}$$

Fifth, if $U \in \mathcal{U}$ and $U \subseteq W \subseteq X \times X$, then

$$W \in \mathcal{U}. \tag{2.3.9}$$

Under these conditions, $(X, \mathcal{U})$ is said to be a *uniform space*.

A collection $\mathcal{B}$ of subsets of $X \times X$ is said to be a *base* for a uniformity $\mathcal{U}$ on X if $\mathcal{U}$ is the same as the collection of subsets W of $X \times X$ for which there is a $U \in \mathcal{B}$ such that

$$U \subseteq W. \tag{2.3.10}$$

In particular, this means that $\mathcal{B} \subseteq \mathcal{U}$.

Let $\mathcal{B}$ be a nonempty collection of subsets of $X \times X$ that satisfies the first three conditions in the definition of a uniformity. If the intersection of any two elements of $\mathcal{B}$ contains another element of $\mathcal{B}$ as a subset, then one can check that $\mathcal{B}$ is a base for a uniformity on X. Of course, this holds when $\mathcal{B}$ satisfies the fourth condition in the definition of a uniformity.

Let $\mathcal{B}_0$ be a collection of subsets of $X \times X$, and let $\mathcal{B}_1$ be the collection of subsets of $X \times X$ obtained by taking the intersections of finitely many elements of $\mathcal{B}_0$. If $\mathcal{B}_1$ is a base for a uniformity $\mathcal{U}$ on X, then $\mathcal{B}_0$ is said to be a *sub-base* for $\mathcal{U}$.

Let $\mathcal{B}_0$ be a nonempty collection of subsets of $X \times X$ that satisfies the first three conditions in the definition of a uniformity. If $\mathcal{B}_1$ is as in the preceding paragraph, then one can verify that $\mathcal{B}_1$ satisfies the first four conditions in the definition of a uniformity. This implies that $\mathcal{B}_1$ is a base for a uniformity on X, as before.

Let $d(\cdot, \cdot)$ be a semimetric on X, and for each positive real number r, put

$$U_{d,r} = \{(x, y) \in X \times X : d(x, y) < r\}. \tag{2.3.11}$$

Clearly

$$\Delta \subseteq U_{d,r} \tag{2.3.12}$$

and

$$U_{d,r}^{-1} = U_{d,r} \tag{2.3.13}$$

for every $r > 0$. It is easy to see that

$$U_{d,r} \circ U_{d,t} \subseteq U_{d,r+t} \tag{2.3.14}$$

for every $r, t > 0$, using the triangle inequality. It follows that

$$\{U_{d,r} : r > 0\} \tag{2.3.15}$$

satisfies the first four conditions in the definition of a uniformity. This implies that (2.3.15) is a base for a uniformity on X, as before.

It is well known that a uniformity $\mathcal{U}$ on X corresponds to a semimetric on X in this way if and only there is a base for $\mathcal{U}$ with only finitely or countably many elements, as in the metrization theorem on p186 of [15]. Of course, the "only if" part can be verified directly.

Similarly, let $\mathcal{M}$ be a nonempty collection of semimetrics on X. It is easy to see that

$$\{U_{d,r} : d \in \mathcal{M},\, r > 0\} \tag{2.3.16}$$

satisfies the first three conditions in the definition of a uniformity. It follows that (2.3.16) is a sub-base for a uniformity on X, as before. It is well known that every uniformity corresponds to a collection of semimetrics in this way, as in Theorem 15 on p188 of [15].

2.4 More on uniform structures

Let $(X,\mathcal{U})$ be a uniform space. If $U \subseteq X \times X$ and $x \in X$, then put

$$U[x] = \{y \in X : (x,y) \in U\}. \tag{2.4.1}$$

A subset W of X is said to be an *open set* with respect to $\mathcal{U}$ if for every $x \in W$ there is a $U \in \mathcal{U}$ such that

$$U[x] \subseteq W, \tag{2.4.2}$$

as on p178 of [15]. One can check that this defines a topology on X. If $\mathcal{U}$ corresponds to a nonempty collection $\mathcal{M}$ of semimetrics on X, then the topology on X associated to $\mathcal{U}$ is the same as the topology determined on X by $\mathcal{M}$ as in Section 2.1.

Let A be a subset of X, and consider the interior of A with respect to the topology determined by $\mathcal{U}$. One can show that this is the same as the set of $x \in A$ for which there is a $U \in \mathcal{U}$ such that

$$U[x] \subseteq A, \tag{2.4.3}$$

as in Theorem 4 on p178 of [15]. In particular, if $x \in X$ and $V \in \mathcal{U}$, then x is an element of the interior of $V[x]$ with respect to the topology determined by $\mathcal{U}$.

If $U \subseteq X \times X$ and $A \subseteq X$, then put

$$U[A] = \{y \in X : (x,y) \in U \text{ for some } x \in A\} = \bigcup_{x \in A} U[x]. \tag{2.4.4}$$

One can show that the closure of A with respect to the topology determined by $\mathcal{U}$ can be expressed as

$$\overline{A} = \bigcap_{U \in \mathcal{U}} U[A], \tag{2.4.5}$$

as in the first part of Theorem 7 on p179 of [15]. In particular,

$$\overline{A} \subseteq U[A] \tag{2.4.6}$$

for every $U \in \mathcal{U}$. Using this, one can check that X is regular in the strict sense, with respect to the topology determined by $\mathcal{U}$.

A uniform space $(X,\mathcal{U})$ is said to be *separated* or *Hausdorff* if

$$\bigcap_{U \in \mathcal{U}} U = \Delta. \tag{2.4.7}$$

It is well known that X is Hausdorff with respect to the topology associated to $\mathcal{U}$ if and only if $(X,\mathcal{U})$ is Hausdorff as a uniform space, as on p180 of [15]. If $\mathcal{U}$ corresponds to a nonempty collection $\mathcal{M}$ of semimetrics on X, then this happens exactly when $\mathcal{M}$ is nondegenerate on X, as in Section 2.1.

Let $\mathcal{U}_X$ be a uniformity on X, and let Y be another set with a uniformity $\mathcal{U}_Y$. Under these conditions, one can define *uniform continuity* for mappings from X into Y with respect to $\mathcal{U}_X$, $\mathcal{U}_Y$ as on p180 of [15]. Uniformly continuous mappings in this sense are automatically continuous with respect to the topologies associated to the uniformities on X, Y.

Suppose that $\mathcal{U}_X, \mathcal{U}_Y$ are the uniformities associated to nonempty collections $\mathcal{M}_X$, $\mathcal{M}_Y$ of semimetrics on X, Y, respectively. In this case, uniform continuity of mappings from X to Y with respect to $\mathcal{U}_X$, $\mathcal{U}_Y$ is equivalent to uniform continuity with respect to $\mathcal{M}_X$, $\mathcal{M}_Y$, as defined in Section 2.2.

Let $\mathcal{U}_1, \mathcal{U}_2$ be uniformities on X. Note that

$$\mathcal{U}_2 \subseteq \mathcal{U}_1 \tag{2.4.8}$$

if and only if the identity mapping on X is uniformly continuous as a mapping from X equipped with $\mathcal{U}_1$ into X equipped with $\mathcal{U}_2$.

Let $\mathcal{M}_1$, $\mathcal{M}_2$ be nonempty collections of semimetrics on X, and let $\mathcal{U}_1$, $\mathcal{U}_2$ be the corresponding uniformities on X. It follows that (2.4.8) holds if and only if the identity mapping on X is uniformly continuous as a mapping from X equipped with $\mathcal{M}_1$ into X equipped with $\mathcal{M}_2$, in the sense of Section 2.2. Thus

$$\mathcal{U}_1 = \mathcal{U}_2 \tag{2.4.9}$$

if and only if the identity mapping on X is uniformly continuous as a mapping from X equipped with $\mathcal{M}_1$ into X equipped with $\mathcal{M}_2$, and as a mapping from X equipped with $\mathcal{M}_2$ into X equipped with $\mathcal{M}_1$.

Let I be a nonempty set, and let $(X_j,\mathcal{U}_j)$ be a uniform space for each $j \in I$. A natural *product uniformity* can be defined on $X = \prod_{j \in I} X_j$ as on p182 of [15]. The topology on X associated to the product uniformity is the same as the product topology obtained from the topology on X_j associated to $\mathcal{U}_j$ for each $j \in I$.

Suppose that for each $j \in I$, $\mathcal{U}_j$ is the uniformity associated to a nonempty collection $\mathcal{M}_j$ of semimetrics on X_j. If $l \in I$, then we get a corresponding collection $\widehat{\mathcal{M}_l}$ of semimetrics on X, as in Section 2.1. Put $\mathcal{M} = \bigcup_{l \in I} \widehat{\mathcal{M}_l}$, which is a nonempty collection of semimetrics on X. The product uniformity on X is the same as the uniformity associated to $\mathcal{M}$.

Let $(X,\mathcal{U})$ be a uniform space again, and let Y be a subset of X. One can check that

$$\mathcal{U}_Y = \{U \cap (Y \times Y) : U \in \mathcal{U}\} \tag{2.4.10}$$

defines a uniformity on Y, as on p182 of [15]. The topology on Y associated to $\mathcal{U}_Y$ is the same as the topology induced on Y by the topology on X associated to $\mathcal{U}$. If $\mathcal{U}$ is the uniformity associated to a nonempty collection $\mathcal{M}$ of semimetrics on X, then $\mathcal{U}_Y$ is the same as the uniformity associated to the collection $\mathcal{M}_Y$ of semimetrics on Y obtained by restricting the elements of $\mathcal{M}$ to Y.

2.5 Quasimetrics

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function on $X \times X$ that satisfies the first two conditions (1.2.1) and (1.2.2) in the definition of a semimetric. If there is a nonnegative real number C such that

$$d(x, z) \le C\,(d(x, y) + d(y, z)) \tag{2.5.1}$$

for every $x, y, z \in M$, then $d(\cdot, \cdot)$ may be called a *semi-quasimetric* on X. If we also have that $d(x, y) > 0$ for every $x, y \in X$ with $x \ne y$, then $d(\cdot, \cdot)$ is called a *quasimetric* on X.

Alternatively, one might ask that

$$d(x, z) \le C' \max(d(x, y), d(y, z)) \tag{2.5.2}$$

for every $x, y, z \in X$. This condition implies (2.5.1), with $C = C'$. Conversely, (2.5.1) implies (2.5.2), with $C' = 2\,C$.

Let $U_{d,r} \subseteq X \times X$ be as in (2.3.11) for each $r > 0$. Of course, (1.2.1) and (1.2.2) imply that (2.3.12) and (2.3.13) hold for every $r > 0$, respectively. If (2.5.1) holds, then we get that

$$U_{d,r} \circ U_{d,t} \subseteq U_{d,C\,(r+t)} \tag{2.5.3}$$

for every $r, t > 0$. Similarly, if (2.5.2) holds, then

$$U_{d,r} \circ U_{d,t} \subseteq U_{d,C' \max(r,t)} \tag{2.5.4}$$

for every $r, t > 0$.

Using either (2.5.3) or (2.5.4), one can check that (2.3.16) is a base for a uniformity on X when $d(\cdot, \cdot)$ is a semi-quasimetric on X. If $d(\cdot, \cdot)$ is a quasimetric on X, then this uniformity is Hausdorff.

If $d(\cdot, \cdot)$ is a semi-quasimetric on X and a is a positive real number, then one can verify that $d(x, y)^a$ defines a semi-quasimetric on X too. Note that

$$U_{d^a, r^a} = U_{d,r} \tag{2.5.5}$$

for every $r > 0$, so that the uniformities associated to $d(\cdot, \cdot)$ and $d(\cdot, \cdot)^a$ on X are the same.

If $d(\cdot, \cdot)$ is a quasimetric on X, then there is metric $\rho(\cdot, \cdot)$ on X and positive real numbers C_0, α such that

$$C_0^{-1}\,\rho(x, y) \le d(x, y)^\alpha \le C_0\,\rho(x, y) \tag{2.5.6}$$

for every $x, y \in X$, as in [18]. Of course, there is an analogous statement for semi-quasimetrics.

Let k be a field, and let $|\cdot|$ be a nonnegative real-valued function on k that satisfies the first two conditions (1.4.1) and (1.4.2) in the definition of an absolute value function. If there is a positive real number C such that

$$|x + y| \le C\,(|x| + |y|) \tag{2.5.7}$$

for every $x, y \in k$, then $|\cdot|$ may be called a *quasimetric absolute value function* on k. Equivalently, one can ask that there be a positive real number C' such that

$$|x + y| \le C'\,(|x| + |y|) \tag{2.5.8}$$

for every $x, y \in k$, as before.

One can check that (2.5.8) holds if and only if

$$|1 + z| \le C' \tag{2.5.9}$$

for every $z \in k$ with $|z| \le 1$. This corresponds to the condition (iii) on p12 of [3].

If $|\cdot|$ is a quasimetric absolute value function on k and a is a positive real number, then it is easy to see that $|x|^a$ defines a quasimetric absolute value function on k as well. In fact, $|x|^\alpha$ is an absolute value function on k for some $\alpha > 0$, as in the corollary on p14 of [3].

2.6 Boundedness and total boundedness

Let X be a set with a nonempty collection $\mathcal{M}_X$ of semimetrics on X. Let us say that $E \subseteq X$ is *bounded* with respect to $\mathcal{M}_X$ if for each $d_X \in \mathcal{M}_X$, E is bounded with respect to d_X. This implies that E is bounded with respect to the maximum of any nonempty finite set of semimetrics in $\mathcal{M}_X$, as in Section 1.14.

If E is compact with respect to the topology determined on X by $\mathcal{M}_X$, then it is easy to see that E is bounded with respect to $\mathcal{M}_X$, because of the analogous statement for boundedness with respect to a single semimetric. Similarly, the union of finitely many subsets of X that are bounded with respect to $\mathcal{M}_X$ is bounded with respect to $\mathcal{M}_X$ as well. If $E \subseteq X$ is bounded with respect to $\mathcal{M}_X$, then the closure $\overline{E}$ of E with respect to the topology determined by $\mathcal{M}_X$ is bounded with respect to $\mathcal{M}_X$ too. Of course, every subset of E is bounded with respect to $\mathcal{M}_X$ in this case.

Let us say that $E \subseteq X$ is *totally bounded* with respect to $\mathcal{M}_X$ if for each $d_X \in \mathcal{M}_X$, E is totally bounded with respect to d_X. This implies that E is bounded with respect to every $d_X \in \mathcal{M}_X$, so that E is bounded with respect to $\mathcal{M}_X$. If E is totally bounded with respect to $\mathcal{M}_X$, then it follows that E is totally bounded with respect to the maximum of any finite set of semimetrics in $\mathcal{M}_X$, as in Section 1.15.

If E is compact with respect to the topology determined on X by $\mathcal{M}_X$, then E is totally bounded with respect to $\mathcal{M}_X$, because of the analogous statement for total boundedness with respect to a single semimetric. One can check that the union of finitely many totally bounded sets is totally bounded, the closure of a totally bounded set is totally bounded, and that subsets of totally bounded sets are totally bounded.

Let Y be another set with a nonempty collection of semimetrics $\mathcal{M}_Y$, and let f be a mapping from X to Y that is uniformly continuous with respect to $\mathcal{M}_X$, $\mathcal{M}_Y$, as in Section 2.2. If $E \subseteq X$ is totally bounded with respect to $\mathcal{M}_X$, then one can check that $f(E)$ is totally bounded in Y with respect to $\mathcal{M}_Y$. This uses the fact that E is totally bounded with respect to the maximum of any nonempty finite set of semimetrics in $\mathcal{M}_X$, as before.

Let I be a nonempty set, let X_j be a nonempty set for each $j \in I$, and let E_j be a nonempty subset of X_j for every $j \in I$. Put $X = \prod_{j \in I} X_j$ and

$$E = \prod_{j \in I} E_j, \tag{2.6.1}$$

so that $E \subseteq X$. Suppose that d_l is a semimetric on X_l for some $l \in I$, and let $\widehat{d_l}$ be the corresponding semimetric on X, as in Section 1.2. It is easy to see that E is bounded in X with respect to $\widehat{d_l}$ if and only if E_l is bounded in X_l with respect to d_l, and in fact

$$\mathrm{diam}_{\widehat{d_l}} E = \mathrm{diam}_{d_l} E_l. \tag{2.6.2}$$

Similarly, one can check that E is totally bounded in X with respect to $\widehat{d_l}$ if and only if E_l is totally bounded in X_l with respect to d_l.

Suppose that $\mathcal{M}_l$ is a nonempty collection of semimetrics on X_l for each l in I, and let $\widehat{\mathcal{M}_l}$ be the corresponding collection of semimetrics on X, as in Section 2.1. It is easy to see that E is bounded in X with respect to $\mathcal{M} = \bigcup_{l \in I} \widehat{\mathcal{M}_l}$ if and only if E_l is bounded in X_l with respect to $\mathcal{M}_l$ for every $l \in I$, using the remarks in the preceding paragraph. Similarly, E is totally bounded in X with respect to $\mathcal{M}$ if and only if E_l is totally bounded in X_l with respect to $\mathcal{M}_l$ for every $l \in I$.

Let $(X, \mathcal{U})$ be a uniform space. A subset E of X is said to be *totally bounded* if for every $U \in \mathcal{U}$ there is a finite set $A \subseteq X$ such that

$$E \subseteq U[A], \tag{2.6.3}$$

as on p198 of [15]. Here $U[A]$ is as in Section 2.4. If $\mathcal{U}$ corresponds to a nonempty collection $\mathcal{M}$ of semimetrics on X, then this is equivalent to total boundedness with respect to $\mathcal{M}$. If E is compact with respect to the topology determined on X by $\mathcal{U}$, then one can check that E is totally bounded with respect to $\mathcal{U}$.

If $U \in \mathcal{U}$, then let us say that $B \subseteq X$ is *U-small* when

$$B \times B \subseteq U \tag{2.6.4}$$

One can show that $E \subseteq X$ is totally bounded with respect to $\mathcal{U}$ if and only if for every $U \in \mathcal{U}$, E is contained in the union of finitely many U-small subsets of X, as on p198 of [15]. If $Y \subseteq X$ and $E \subseteq Y$, then one can check that E is totally bounded in X with respect to $\mathcal{U}$ if and only if E is totally bounded in Y with respect to the uniformity $\mathcal{U}_Y$ induced on Y by $\mathcal{U}$ as in Section 2.4.

2.7 Truncating semimetrics

Let X be a set, let $d(x, y)$ be a semimetric on X, and let t be a positive real number. Put

$$d_t(x, y) = \min(d(x, y), t) \tag{2.7.1}$$

for every $x, y \in X$. One can check that this defines a semimetric on X, which is a metric when $d(\cdot, \cdot)$ is a metric on X. If $d(\cdot, \cdot)$ is a semi-ultrametric on X, then one can verify that (2.7.1) is a semi-ultrametric on X, and thus an ultrametric on X when $d(\cdot, \cdot)$ is an ultrametric on X.

If $x \in X$ and r is a positive real number, then

$$\begin{aligned} B_{d_t}(x, r) &= B_d(x, r) \quad \text{when } r \le t \\ &= X \quad \text{when } r > t. \end{aligned} \tag{2.7.2}$$

Similarly, if r is a nonnegative real number, then

$$\begin{aligned} \overline{B}_{d_t}(x, r) &= \overline{B}_d(x, r) \quad \text{when } r < t \\ &= X \quad \text{when } r \ge t. \end{aligned} \tag{2.7.3}$$

In particular, the topology determined on X by d_t is the same as the topology determined by d. One can check that $E \subseteq X$ is totally bounded with respect to d_t if and only if E is totally bounded with respect to d.

Let $U_{d,r} \subseteq X \times X$ be as in (2.3.11) for each $r > 0$, and similarly for d_t. Observe that

$$\begin{aligned} U_{d_t,r} &= U_{d,r} \quad \text{when } r \le t \\ &= X \times X \quad \text{when } r > t. \end{aligned} \tag{2.7.4}$$

Using this, one can check that the uniform structures on X corresponding to d_t and d are the same.

Let k be a field with an absolute value function $|\cdot|$, let V be a vector space over k, and let N be a seminorm on V with respect to $|\cdot|$ on k. If $v \in V$, then put

$$N_t(v) = \min(N(v), t). \tag{2.7.5}$$

Observe that for each $a \in k$ with $|a| = 1$, we have that

$$N_t(a\,v) = N_t(v), \tag{2.7.6}$$

because $N(a\,v) = |a|\,N(v) = N(v)$. Similarly, if $a \in k$ and $|a| \le 1$, then

$$N_t(a\,v) \le N_t(v), \tag{2.7.7}$$

because $N(a\,v) = |a|\,N(v) \le N(v)$. Of course, $N_t(0) = 0$, because $N(0) = 0$. If $N(v)$ is a norm on V, then $N_t(v) > 0$ for every $v \in V$ with $v \ne 0$. If $|\cdot|$ is the trivial absolute value function on k, then N_t satisfies the homogeneity condition (1.9.1) of a seminorm on V with respect to $|\cdot|$.

One can check that

$$N_t(v + w) \le N_t(v) + N_t(w) \tag{2.7.8}$$

for every $v, w \in V$. Similarly, if N is a semi-ultranorm on V with respect to $|\cdot|$, then one can verify that

$$N_t(v + w) \le \max(N_t(v), N_t(w)) \tag{2.7.9}$$

for every $v, w \in V$. If $|\cdot|$ is the trivial absolute value function on k, then it follows that N_t is a seminorm or semi-ultranorm on V, as appropriate. In this case, if N is a norm or ultranorm on V, then N_t has the same property.

2.8 Sequences of semimetrics

Let X be a set, and let $d_1, d_2, d_3, \ldots$ be an infinite sequence of semimetrics on X. Suppose that

$$\sup_{x,y \in X} d_j(x, y) \to 0 \quad \text{as } j \to \infty. \tag{2.8.1}$$

Of course, this can always be arranged using suitable truncations of semimetrics, as in the previous section.

Put

$$d(x, y) = \max_{j \ge 1} d_j(x, y) \tag{2.8.2}$$

for every $x, y \in X$. More precisely, this is equal to 0 when $d_j(x, y) = 0$ for every $j \ge 1$. Otherwise, if $d_l(x, y) > 0$ for some $l \ge 1$, then $d_j(x, y) < d_l(x, y)$ for all sufficiently large j, by (2.8.1). This implies that the right side of (2.8.2) reduces to the maximum of finitely many terms, and is thus attained.

One can check that (2.8.2) defines a semimetric on X. If d_j is a semi-ultrametric on X for each $j \ge 1$, then (2.8.2) is a semi-ultrametric on X too. If the collection of semimetrics d_j, $j \ge 1$, is nondegenerate on X, then (2.8.2) is a metric on X.

Observe that

$$B_d(x, r) = \bigcap_{j=1}^{\infty} B_{d_j}(x, r) \tag{2.8.3}$$

for every $x \in X$ and $r > 0$. In fact,

$$B_d(x, r) = \bigcap_{j=1}^{l} B_{d_j}(x, r) \tag{2.8.4}$$

when l is sufficiently large, depending only on r, by (2.8.1). This implies that open balls in X are open sets with respect to the topology determined on X by the collection of d_j's, $j \ge 1$. It follows that open sets in X with respect to the topology determined by d are open sets with respect to the topology determined by the collection of d_j's. It is easy to see that open sets in X with respect to the topology determined by the collection of d_j's, $j \ge 1$, are open sets with respect to d, so that the two topologies are the same.

Let $U_{d,r}$ be as in Section 2.3, and similarly for d_j, $j \ge 1$. One can check that

$$U_{d,r} = \bigcap_{j=1}^{\infty} U_{d_j,r} \tag{2.8.5}$$

for every $r > 0$. More precisely,

$$U_{d,r} = \bigcap_{j=1}^{l} U_{d_j,r} \tag{2.8.6}$$

when l is sufficiently large, by (2.8.1). One can use this to get that the uniformity on X corresponding to d is the same as the one associated to the collections of d_j's, $j \ge 1$. Equivalently, the identity mapping on X is uniformly continuous as a mapping from X equipped with the collection of d_j's, $j \ge 1$, into X equipped with d, and as a mapping from X equipped with d into X equipped with the collection of d_j's, $j \ge 1$.

If $E \subseteq X$ is totally bounded with respect to d, then E is clearly totally bounded with respect to d_j for each $j \ge 1$. This means that E is totally bounded with respect to the collection of d_j's, $j \ge 1$, as before. Conversely, if E is totally bounded with respect to the collection of d_j's, $j \ge 1$, then E is totally bounded with respect to d. This can be obtained from the previous remarks, and one can argue more directly, as follows.

Note that

$$\max_{1 \le j \le l} d_j(x, y) \tag{2.8.7}$$

is a semimetric on X for each $l \ge 1$. If E is totally bounded with respect to d_j for each $j \ge 1$, then E is totally bounded with respect to (2.8.7) for every $l \ge 1$, as in Section 1.15. Let $r > 0$ be given, and let l be large enough so that (2.8.4) holds for every $x \in X$. Because E is totally bounded with respect to (2.8.7), E can be covered by finitely many open balls of radius r with respect to (2.8.7). This implies that E can be covered by finitely many open balls with respect to d, by (2.8.4), and because the right side of (2.8.4) is the same as the open ball in X centered at x with radius r with respect to (2.8.7).

2.9 Equicontinuity

Let X be a topological space, and let Y be a set with a semimetric d_Y. A collection $\mathcal{E}$ of mappings from X into Y is said to be *equicontinuous* at a point $x \in X$ with respect to d_X if for every $\epsilon > 0$ there is an open set $U \subseteq X$ such that $x \in U$ and

$$d_Y(f(x), f(w)) < \epsilon \tag{2.9.1}$$

for every $w \in U$. Of course, this implies that every element of $\mathcal{E}$ is continuous at x, with respect to the topology determined on Y by d_Y. If $\mathcal{E}$ has only finitely many elements, each of which is continuous at x, then it is easy to see that $\mathcal{E}$ is equicontinuous at x.

Let $\mathcal{M}_Y$ be a nonempty collection of semimetrics on Y. Let us say that $\mathcal{E}$ is *equicontinuous* at x with respect to $\mathcal{M}_Y$ if $\mathcal{E}$ is equicontinuous at x with respect to every $d_Y \in \mathcal{M}_Y$. If $\mathcal{U}_Y$ is a uniformity on Y, then equicontinuity of $\mathcal{E}$ at x with respect to $\mathcal{U}_Y$ can be defined as on p232 of [15]. If $\mathcal{U}_Y$ is the uniformity associated to $\mathcal{M}_Y$, then this is equivalent to equicontinuity with respect to $\mathcal{M}_Y$.

Let $\mathcal{M}_X$ be a nonempty collection of semimetrics on X, and let d_Y be a semimetric on Y again. Let us say that $\mathcal{E}$ is *uniformly equicontinuous* with respect to $\mathcal{M}_X$ on X and d_Y on Y if for every $\epsilon > 0$ there are finitely many semimetrics $d_1, \ldots, d_n \in \mathcal{M}_X$ and positive real numbers $\delta_1, \ldots, \delta_n$ such that (2.9.1) holds for every $x, w \in X$ with

$$d_j(x, w) < \delta_j \quad \text{for each } j = 1, \ldots, n. \tag{2.9.2}$$

If $\mathcal{M}_Y$ is a nonempty collection of semimetrics on Y, then let us say that $\mathcal{E}$ is *uniformly equicontinuous* with respect to $\mathcal{M}_X$ on X and $\mathcal{M}_Y$ on Y if $\mathcal{E}$ is uniformly equicontinuous on X with respect to every $d_Y \in \mathcal{M}_Y$. This implies that every element of $\mathcal{E}$ is uniformly continuous with respect to $\mathcal{M}_X$ on X and $\mathcal{M}_Y$ on Y. If $\mathcal{E}$ has only finitely many elements, each of which is uniformly continuous on X, then $\mathcal{E}$ is uniformly equicontinuous on X.

If $\mathcal{U}_X, \mathcal{U}_Y$ are uniformities on X, Y, respectively, then uniform equicontinuity with respect to $\mathcal{U}_X, \mathcal{U}_Y$ can be defined as on p239 of [15]. If $\mathcal{U}_X, \mathcal{U}_Y$ are the uniformities corresponding to $\mathcal{M}_X$, $\mathcal{M}_Y$, respectively, then uniform equicontinuity with respect to $\mathcal{U}_X$, $\mathcal{U}_Y$ is equivalent to uniform equicontinuity with respect to $\mathcal{M}_X$, $\mathcal{M}_Y$.

Let $\mathcal{M}_X$ be a nonempty collection of semimetrics on X again, let d_Y be a semimetric on Y, and let X_0 be a subset of X. Let us say that $\mathcal{E}$ is *uniformly equicontinuous along* X_0 with respect to $\mathcal{M}_X$ on X and d_Y on Y if for every $\epsilon > 0$ there are finitely many semimetrics $d_1, \ldots, d_n \in \mathcal{M}_X$ and $\delta_1, \ldots, \delta_n > 0$ such that (2.9.1) holds for every $x \in X_0$ and $w \in X$ that satisfy (2.9.2). If $\mathcal{M}_Y$ is a nonempty collection of semimetrics on Y, then let us say that $\mathcal{E}$ is *uniformly equicontinuous along* X_0 with respect to $\mathcal{M}_X$ on X and $\mathcal{M}_Y$ on Y if $\mathcal{E}$ is uniformly equicontinuous along X_0 with respect to every $d_Y \in \mathcal{Y}$. This implies that the elements of $\mathcal{E}$ are uniformly continuous along X_0 with respect to $\mathcal{M}_X$ on X and $\mathcal{M}_Y$ on Y, as usual. If $\mathcal{E}$ has only finitely many elements, each of which is uniformly continuous along X_0, then $\mathcal{E}$ is uniformly equicontinuous along X_0.

Suppose that $\mathcal{E}$ is uniformly equicontinuous along X_0. This implies that the restrictions of the elements of $\mathcal{E}$ to X_0 are uniformly equicontinuous on X_0, with respect to the collection of semimetrics on X_0 obtained by restricting the elements of $\mathcal{M}_X$ to X_0. This also implies that the elements of $\mathcal{E}$ are equicontinuous at every element of X_0, with respect to the topology determined on X by $\mathcal{M}_X$.

Suppose now that X_0 is a compact subset of X, and that $\mathcal{E}$ is equicontinuous at every element of X_0, with respect to the topology determined on X by $\mathcal{M}_X$. Under these conditions, one can use an argument like the one in Section 2.2 to get that $\mathcal{E}$ is uniformly equicontinuous along X_0.

2.10 q-Metrics and q-semimetrics

If q_1, q_2 are positive real numbers with $q_1 \le q_2$, then one can check that

$$(r^{q_2} + t^{q_2})^{1/q_2} \le (r^{q_1} + t^{q_1})^{1/q_1} \tag{2.10.1}$$

for all nonnegative real numbers r, t, using (1.6.1). Observe that

$$\max(r, t) \le (r^q + t^q)^{1/q} \le 2^{1/q} \max(r, t) \tag{2.10.2}$$

for every $q > 0$ and $r, t \ge 0$. This implies that

$$(r^q + t^q)^{1/q} \to \max(r, t) \quad \text{as } q \to \infty, \tag{2.10.3}$$

because of the well-known fact that $2^{1/q} \to 1$ as $q \to \infty$.

Let X be a set, let q be a positive real number, and let $d(x, y)$ be a nonnegative real-valued function on $X \times X$ that satisfies the first two conditions (1.2.1), (1.2.2) in the definition of a semimetric. Let us say that $d(\cdot, \cdot)$ is a *q-semimetric* on X if

$$d(x, z)^q \le d(x, y)^q + d(y, z)^q \tag{2.10.4}$$

for every $x, y, z \in X$. If we also have that $d(x, y) > 0$ when $x \ne y$, then $d(\cdot, \cdot)$ is a *q-metric* on X. Thus q-metrics and q-semimetrics are the same as ordinary metrics and semimetrics, respectively, when $q = 1$. Note that $d(\cdot, \cdot)$ is a q-metric or q-semimetric on X exactly when $d(x, y)^q$ is an ordinary metric or semimetric on X, respectively.

Of course, (2.10.4) is the same as saying that

$$d(x, z) \le (d(x, y)^q + d(y, z)^q)^{1/q} \tag{2.10.5}$$

for every $x, y, z \in X$. The right side of (2.10.5) is monotonically decreasing in q, as in (2.10.1). This means that the property of being a q-metric or q-semimetric becomes more restrictive as q increases.

If $d(\cdot, \cdot)$ is a semi-ultrametric on X, then $d(\cdot, \cdot)$ is a q-semimetric on X for every $q > 0$, because of the first inequality in (2.10.2). It is convenient to consider ultrametrics and semi-ultrametrics as q-metrics and q-semimetrics with $q = \infty$, respectively.

Suppose that $d(\cdot, \cdot)$ is a q-semimetric on X for some $q > 0$. If a is any positive real number, then it is easy to see that $d(x, y)^a$ is a (q/a)-semimetric on X.

If $d(\cdot, \cdot)$ is a q-semimetric on X for some $q > 0$, then one can check that $d(\cdot, \cdot)$ is a semi-quasimetric on X. The result from [18] mentioned in Section 2.5 can be reformulated as saying that every quasimetric on X is comparable to a q-metric on X for some $q > 0$. There is an analogous statement for semi-quasimetrics, as before.

Let $d(\cdot, \cdot)$ be a q-semimetric on X for some $q > 0$. Of course, one can define open and closed balls in X with respect to d in the same way as for ordinary semimetrics, as well as the topology determined on X by d. It is easy to see that many of the same properties hold as for ordinary semimetrics, and one can also reduce to that case using $d(x, y)^q$ when $q < 1$. In particular, open and closed balls with respect to d are open and closed sets, respectively, with respect to the topology determined by d.

Similarly, let $\mathcal{M}$ be a nonempty collection of q-semimetrics on X, where more precisely each $d \in \mathcal{M}$ is a q_d-semimetric for some $q_d > 0$ that may depend on d. As before, it is easy to see that many of the same properties of collections of semimetrics can be extended to collections of q-semimetrics. One can also often reduce to the previous case, using $d(x, y)^{q_d}$ when $d \in \mathcal{M}$ and $q_d < 1$.

2.11 Finitely many q-semimetrics

Let n be a positive integer, and let $a_1, \ldots, a_n$ be nonnegative real numbers. If r is a positive real number, then

$$\max_{1 \le j \le n} a_j \le \Big(\sum_{j=1}^n a_j^r\Big)^{1/r} \le n^{1/r} \Big(\max_{1 \le j \le n} a_j\Big). \tag{2.11.1}$$

It follows that

$$\Big(\sum_{j=1}^n a_j^r\Big)^{1/r} \to \max_{1 \le j \le n} a_j \quad \text{as } r \to \infty, \tag{2.11.2}$$

because $n^{1/r} \to 1$ as $r \to \infty$. If r_1, r_2 are positive real numbers with $r_1 \le r_2$, then

$$\begin{aligned} \sum_{j=1}^n a_j^{r_2} &\le \Big(\max_{1 \le j \le n} a_j\Big)^{r_2 - r_1} \sum_{j=1}^n a_j^{r_1} \\ &\le \Big(\sum_{j=1}^n a_j^{r_1}\Big)^{(r_2 - r_1)/r_1 + 1} = \Big(\sum_{j=1}^n a_j^{r_1}\Big)^{r_2/r_1}, \end{aligned} \tag{2.11.3}$$

using the first inequality in (2.11.1) with $r = r_1$ in the second step. This implies that

$$\Big(\sum_{j=1}^n a_j^{r_2}\Big)^{1/r_2} \le \Big(\sum_{j=1}^n a_j^{r_1}\Big)^{1/r_1}, \tag{2.11.4}$$

which is the same as (2.10.1) when $n = 2$.

If $1 < r < \infty$ and $b_1, \ldots, b_n$ are nonnegative real numbers too, then it is well known that

$$\Big(\sum_{j=1}^n (a_j + b_j)^r\Big)^{1/r} \le \Big(\sum_{j=1}^n a_j^r\Big)^{1/r} + \Big(\sum_{j=1}^n b_j^r\Big)^{1/r}. \tag{2.11.5}$$

This is *Minkowski's inequality* for finite sums. Of course, equality holds trivially when $r = 1$. If $0 < r \le 1$, then

$$\sum_{j=1}^{n} (a_j + b_j)^r \le \sum_{j=1}^{n} a_j^r + \sum_{j=1}^{n} b_j^r, \tag{2.11.6}$$

by (1.6.1), which could also be obtained from (2.11.4), with $n = 2$. One can check directly that

$$\max_{1 \le j \le n} (a_j + b_j) \le \Big(\max_{1 \le j \le n} a_j \Big) + \Big(\max_{1 \le j \le n} b_j \Big), \tag{2.11.7}$$

which is the analogue of (2.11.5) for $r = \infty$.

Let X be a set, and let $d_j(\cdot, \cdot)$ be a q_j-semimetric on X for some $q_j > 0$, $j = 1, \ldots, n$. Put

$$q_0 = \min(q_1, \ldots, q_n), \tag{2.11.8}$$

so that $d_j(\cdot, \cdot)$ may be considered as a q_0-semimetric on X for each $j = 1, \ldots, n$. One can check that

$$d(x, y) = \max_{1 \le j \le n} d_j(x, y) \tag{2.11.9}$$

defines a q_0-semimetric on X as well.

If r is a positive real number, then put

$$\rho_r(x, y) = \Big(\sum_{j=1}^{n} d_j(x, y)^r \Big)^{1/r}. \tag{2.11.10}$$

One may consider (2.11.9) as the analogue of (2.11.10) for $r = \infty$. If $r \le q_0$, then one can verify that (2.11.10) is an r-semimetric on X. More precisely, this can be obtained from (2.11.6), using the exponent $r/q_0 \le 1$ when $q_0 < \infty$. Alternatively, this can be obtained from the fact that d_j is an r-semimetric on X when $r \le q_0$.

If $q_0 \le r$, then one can verify that (2.11.10) is a q_0-semimetric on X. This uses Minkowski's inequality with exponent $r/q_0 \ge 1$.

2.12 Uniform convergence and supremum semimetrics

Let X, Y be nonempty sets, and let d be a q_d-semimetric on Y for some $q_d > 0$. A sequence $\{f_j\}_{j=1}^{\infty}$ of mappings from X into Y is said to converge *uniformly* to a mapping f from X into Y with respect to d if for every $\epsilon > 0$ there is a positive integer L such that

$$d(f_j(x), f(x)) < \epsilon \tag{2.12.1}$$

for every $j \ge L$ and $x \in X$. If $\mathcal{M}_Y$ is a nonempty collection of q-semimetrics on Y, then we say that $\{f_j\}_{j=1}^{\infty}$ converges to f *uniformly* with respect to $\mathcal{M}_Y$

if $\{f_j\}_{j=1}^\infty$ converges to f uniformly with respect to every $d \in \mathcal{M}$. Of course, this implies that $\{f_j\}_{j=1}^\infty$ converges to f pointwise on X, with respect to the topology determined on Y by $\mathcal{M}_Y$.

If $\mathcal{U}$ is a uniformity on Y, then one can define uniform convergence of sequences or nets of mappings from X into Y with respect to $\mathcal{U}$, as on p226 of [15]. If $\mathcal{U}$ is the uniformity associated to a nonempty collection $\mathcal{M}_Y$ of q-semimetrics on Y, then uniform convergence of a sequence of mappings from X into Y with respect to $\mathcal{U}$ is equivalent to uniform convergence with respect to $\mathcal{M}_Y$, as in the preceding paragraph. Of course, one can define uniform convergence of nets of mappings from X into Y with respect to $\mathcal{M}_Y$ analogously, so that the previous statement can be extended to nets.

Let d be a q_d-semimetric on Y for some $q_d > 0$ again. A mapping f from X into Y is said to be *bounded* with respect to d if $f(X)$ is a bounded set in Y with respect to d. Let $\mathcal{B}(X, Y) = \mathcal{B}_d(X, Y)$ be the space of bounded mappings from X into Y with respect to d. Of course, if Y is bounded with respect to d, then $\mathcal{B}(X, Y)$ consists of all mappings from X into Y.

If $f, g \in \mathcal{B}_d(X, Y)$, then it is easy to see that $d(f(x), g(x))$ is bounded as a real-valued function on X. In this case, let us put

$$\theta(f, g) = \theta_d(f, g) = \sup_{x \in X} d(f(x), g(x)). \tag{2.12.2}$$

One can check that this defines a q_d-semimetric on $\mathcal{B}_d(X, Y)$, which is the *supremum q_d-semimetric* associated to d. If d is a q_d-metric on Y, then (2.12.2) is a q_d-metric on $\mathcal{B}_d(X, Y)$.

One can verify that a sequence $\{f_j\}_{j=1}^\infty$ of elements of $\mathcal{B}_d(X, Y)$ converges to $f \in \mathcal{B}_d(X, Y)$ with respect to (2.12.2) if and only if $\{f_j\}_{j=1}^\infty$ converges to f uniformly with respect to d. More precisely, if $\{f_j\}_{j=1}^\infty$ is a sequence of elements of $\mathcal{B}_d(X, Y)$ that converges to a mapping f from X into Y uniformly with respect to d, then $f \in \mathcal{B}_d(X, Y)$ too. There are analogous statements for nets of elements of $\mathcal{B}_d(X, Y)$, as before.

Let t be a positive real number, and let $d_t(y, z)$ be the minimum of $d(y, z)$ and t for every $y, z \in Y$, as in Section 2.7. This defines a q_d-semimetric on Y, as before. If $f, g \in \mathcal{B}_d(X, Y)$, then it is easy to see that

$$\theta_{d_t}(f, g) = \min(\theta_d(f, g), t). \tag{2.12.3}$$

Of course, Y is automatically bounded with respect to d_t, so that $\mathcal{B}_{d_t}(X, Y)$ contains all mappings from X into Y. Note that a sequence or net of mappings from X into Y converges uniformly with respect to d_t if and only if it converges uniformly with respect to d.

Let $\mathcal{M}_Y$ be a nonempty collection of q-semimetrics on Y again. Let us say that a mapping f from X into Y is *bounded* with respect to $\mathcal{M}_Y$ if f is bounded with respect to every $d \in \mathcal{M}_Y$. Let $\mathcal{B}(X, Y)$ be the collection of mappings from X into Y that are bounded with respect to $\mathcal{M}$. Thus

$$\{\theta_d : d \in \mathcal{M}_Y\} \tag{2.12.4}$$

is a nonempty collection of q-semimetrics on $\mathcal{B}(X, Y)$.

A sequence or net of elements of $\mathcal{B}(X, Y)$ converges to an element of $\mathcal{B}(X, Y)$ with respect to the topology determined on $\mathcal{B}(X, Y)$ by (2.12.4) if and only if it converges with respect to θ_d for every $d \in \mathcal{M}_Y$, as in Section 2.1. This happens if and only if the sequence or net converges uniformly with respect to every $d \in \mathcal{M}_Y$, as before, which means that it converges uniformly with respect to $\mathcal{M}_Y$.

If $\mathcal{U}$ is a uniformity on Y, then one can define the corresponding uniformity of uniform convergence on any collection of mappings from X into Y, as on p226 of [15]. Uniform convergence of a sequence or net of mappings from X into Y is the same as convergence with respect to the topology determined by the uniformity of uniform convergence. If $\mathcal{U}$ is the uniformity on Y associated to a nonempty collection of q-semimetrics $\mathcal{M}_Y$ on Y, then the uniformity of uniform convergence on the space $\mathcal{B}(X, Y)$ of bounded mappings associated to $\mathcal{U}$ is the same as the uniformity on $\mathcal{B}(X, Y)$ associated to (2.12.4).

If Y is bounded with respect to every $d \in \mathcal{M}_Y$, then $\mathcal{B}(X, Y)$ is the set of all mappings from X into Y. One can always reduce to this case, as before.

2.13 Supremum semimetrics and compact sets

Let X be a nonempty topological space, and let Y be a nonempty set. If Y is equipped with a topology, then we let $C(X, Y)$ be the space of continuous mappings from X into Y.

Let d be a q_d-semimetric on Y for some $q_d > 0$, and let us take Y to be equipped with the topology determined by d for the moment. Also let K be a nonempty compact subset of X. If $f \in C(X, Y)$, then $f(K)$ is a compact subset of Y, which implies that $f(K)$ is bounded with respect to f. Thus the restrictions of elements of $C(X, Y)$ to K are bounded as mappings from K into Y, with respect to d.

If $f, g \in C(X, Y)$, then put

$$\theta_K(f, g) = \theta_{K,d}(f, g) = \sup_{x \in K} d(f(x), g(x)). \tag{2.13.1}$$

This defines a q_d-semimetric on $C(X, Y)$, as in the previous section, which is the *supremum q_d-semimetric* associated to K and d.

Thus

$$\{\theta_{K,d} : K \subseteq X,\ K \neq \emptyset,\ K \text{ compact}\} \tag{2.13.2}$$

is a collection of q_d-semimetrics on $C(X, Y)$, which is nonempty because finite subsets of X are compact. If d is a q_d-metric on Y, then (2.13.2) is nondegenerate on $C(X, Y)$.

Let $\mathcal{M}_Y$ be a nonempty collection of semimetrics on Y, and let us now take Y to be equipped with the topology determined by $\mathcal{M}_Y$. Consider the collection

$$\{\theta_{K,d} : K \subseteq X,\ K \neq \emptyset,\ K \text{ compact},\ d \in \mathcal{M}_Y\} \tag{2.13.3}$$

of q-semimetrics on $C(X,Y)$, which is the same as the union of (2.13.2) over $d \in \mathcal{M}_Y$. If $\mathcal{M}_Y$ is nondegenerate on Y, then (2.13.3) is nondegenerate on $C(X,Y)$.

A sequence $\{f_j\}_{j=1}^\infty$ of elements of $C(X,Y)$ is said to converge to an element f of $C(X,Y)$ *uniformly on compact sets* with respect to $\mathcal{M}_Y$ if for every nonempty compact subset K of X, the restrictions of the f_j's to K converge uniformly to the restriction of f to K with respect to $\mathcal{M}_Y$. This is equivalent to $\{f_j\}_{j=1}^\infty$ converging to f with respect to (2.13.3) on $C(X,Y)$. Of course, one can consider nets of elements of $C(X,Y)$ too.

Suppose that $\mathcal{U}$ is a uniformity on Y. Given any collection of subsets of X, one can define a uniformity on any collection of mappings from X into Y, which corresponds to uniform convergence on the given subsets of X with respect to $\mathcal{U}$, as on p228 of [15]. In particular, one can do this for the collection of compact subsets of X, as on p229 of [15].

Suppose that $\mathcal{U}$ is the uniformity on Y associated to a nonempty collection $\mathcal{M}_Y$ of q-semimetrics on Y. In this case, the uniformity on $C(X,Y)$ corresponding to uniform convergence on compact subsets of X is the same as the uniformity associated to the collection (2.13.3) of q-semimetrics on $C(X,Y)$.

2.14 Uniform convergence and continuity

Let X be a nonempty topological space, and let Y be a nonempty set with a q_d-semimetric for some $q_d > 0$, which determines a topology on Y in particular. If a sequence or net of continuous mappings from X into Y converges uniformly to a mapping f from X into Y with respect to d, then it is well known and not difficult to show that f is continuous as well.

Similarly, let $\mathcal{M}_Y$ be a nonempty collection of q-semimetrics on Y, which determines a topology on Y. If a sequence or net of continuous mappings from X into Y converges uniformly to a mapping f from X into Y with respect to $\mathcal{M}_Y$, then f is continuous too.

Let $\mathcal{U}$ be a uniformity on Y, which determines a topology on Y. If a sequence or net of continuous mappings from X into Y converges uniformly to a mapping f from X into Y with respect to $\mathcal{U}$, then f is continuous. Alternatively, the set $C(X,Y)$ of continuous mappings from X into Y is a closed set in the space of all mappings from X into Y, with respect to the topology determined by the uniformity of uniform convergence.

Let $\mathcal{M}_Y$ be a nonempty collection of q-metrics on Y again, and let $\mathcal{B}(X,Y)$ be the space of mappings from X into Y that are bounded with respect to $\mathcal{M}_Y$, as in Section 2.12. This leads to the corresponding collection of supremum q-semimetrics on $\mathcal{B}(X,Y)$, as in (2.12.4). Under these conditions, the space

$$\mathcal{B}(X,Y) \cap C(X,Y) \tag{2.14.1}$$

is a closed set in $\mathcal{B}(X,Y)$, with respect to the topology determined by (2.12.4). Remember that $\mathcal{B}(X,Y)$ contains all mappings from X into Y when Y is bounded with respect to every $d \in \mathcal{M}_Y$, and that we can always reduce to that case.

If Y is equipped with a topology, then a mapping f from X into Y is said to be *continuous on compact sets* if for every compact subset K of X, the restriction of f to K is continuous with respect to the induced topology on K. Suppose that Y is equipped with a nonempty collection $\mathcal{M}_Y$ of q-semimetrics or a uniformity $\mathcal{U}$, and thus a topology. If a sequence or net of continuous mappings from X into Y converges to a mapping f from X into Y uniformly on compact subsets of X, then f is continuous on compact subsets of X.

Suppose for the moment that X is *locally compact*, so that every element of X is contained in an open subset of X that is contained in a compact subset of X. If Y is equipped with a topology, and a mapping f from X into Y is continuous on compact subsets of X, then f is continuous on X.

Let X be any topological space again, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of X that converges to a point $x \in X$. One can check that

$$\{x_j : j \in \mathbf{Z}_+\} \cup \{x\} \tag{2.14.2}$$

is a compact subset of X.

Suppose that Y is equipped with a topology, and let $x \in X$ be given. A mapping f from X into Y is said to be *sequentially continuous at x* if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of X that converges to x, $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y. If f is continuous at x, then f is sequentially continuous at x. If f is not continuous at x, and if there is a local base for the topology of X at x with only finitely or countably many elements, then f is not sequentially continuous at x.

If a mapping f from X into Y is continuous on compact sets, then it follows that f is sequentially continuous at every point in X. This implies that f is continuous when X satisfies the *first countability condition*, which means that there is a local base for the topology of X at every point with only finitely or countably many elements.

2.15 Cauchy sequences

Let X be a set, and let d be a q_d-semimetric on X for some $q_d > 0$. A sequence $\{x_j\}_{j=1}^\infty$ of elements of X is said to be a *Cauchy sequence* with respect to d if

$$d(x_j, x_l) \to 0 \quad \text{as } j, l \to \infty, \tag{2.15.1}$$

as usual. It is well known and easy to see that if $\{x_j\}_{j=1}^\infty$ converges to an element of X with respect to d, then $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to d.

Let n be a positive integer, and for each $r = 1, \ldots, n$, let d_r be a q_r-semimetric on X for some $q_r > 0$. If we take q_0 to be the minimum of $q_1, \ldots, q_n$, then the maximum of $d_1, \ldots, d_n$ is a q_0-semimetric on X, as in Section 1.5. It is easy to see that a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is a Cauchy sequence with respect to the maximum of $d_1, \ldots, d_n$ if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to d_r for each $r = 1, \ldots, n$.

Let $\mathcal{M}_X$ be a nonempty collection of q-semimetrics on X. Let us say that a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is a *Cauchy sequence* with respect to $\mathcal{M}_X$ if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to every $d \in \mathcal{M}_X$. If $\{x_j\}_{j=1}^\infty$ converges to an element of X with respect to the topology determined by $\mathcal{M}_X$, then it follows that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $\mathcal{M}_X$.

If $\mathcal{U}$ is a uniformity on X, then one can define Cauchy nets in X with respect to $\mathcal{U}$ as on p190 of [15]. If $\mathcal{U}$ is the uniformity associated to a nonempty collection $\mathcal{M}_X$ of q-semimetrics on X, then a Cauchy sequence in X with respect to $\mathcal{U}$ is the same as a Cauchy sequence with respect to $\mathcal{M}_X$. Of course, one can define Cauchy nets in X with respect to $\mathcal{M}_X$ analogously, so that the previous statement can be extended to nets.

Let $\{x_j\}_{j=1}^\infty$ be a Cauchy sequence of elements of X with respect to a nonempty collection $\mathcal{M}_X$ of q-semimetrics on X. Under these conditions, one can check that

$$\{x_j : j \in \mathbf{Z}_+\} \tag{2.15.2}$$

is totally bounded in X with respect to $\mathcal{M}_X$. Similarly, if $\mathcal{U}$ is a uniformity on X, and $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in X with respect to $\mathcal{U}$, then (2.15.2) is totally bounded in X with respect to $\mathcal{U}$.

Suppose that for each positive integer r, d_r is a q_r-semimetric on X for some $q_r > 0$. Suppose for the moment that there is a $q_0 > 0$ such that $q_r \ge q_0$ for every $r \ge 1$, which means that d_r may be considered as a q_0-semimetric on X for each $r \ge 1$. If the d_r's converge to 0 uniformly on $X \times X$ as $r \to \infty$, then the maximum of the d_r's is also a q_0-semimetric on X, as in Section 2.8. Under these conditions, one can check that a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is a Cauchy sequence with respect to the maximum of the d_r's if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to d_r for each $r \ge 1$.

Of course, one can simply reduce to the case where d_r is an ordinary semimetric on X for every $r \ge 1$, by replacing d_r with $d_r(\cdot,\cdot)^{q_r}$ when $q_r \le 1$. However, this could affect the condition that the d_r's converge to 0 uniformly on $X \times X$ as $r \to \infty$. One could deal with this by truncating or rescaling the d_r's, if necessary.

Let $\mathcal{M}_X$ be a nonempty collection of q-semimetrics on X again, and let Y be a set with a nonempty collection $\mathcal{M}_Y$ of q-semimetrics. Suppose that a mapping f from X into Y is uniformly continuous with respect to $\mathcal{M}_X$, $\mathcal{M}_Y$, respectively. If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of X that is a Cauchy sequence with respect to $\mathcal{M}_X$, then one can check that $\{f(x_j)\}_{j=1}^\infty$ is a Cauchy sequence in Y with respect to $\mathcal{M}_Y$.

Chapter 3

Topological groups

3.1 Definitions and basic properties

A *topological group* is a group G equipped with a topology such that the group operations are continuous. This means that multiplication in the group is continuous, as a mapping from $G \times G$ into G, using the corresponding product topology on $G \times G$, and that

$$x \mapsto x^{-1} \tag{3.1.1}$$

is continuous as a mapping from G into itself. It follows that (3.1.1) is a homeomorphism from G onto itself, because this mapping is its own inverse. Sometimes the condition that the set containing only the identity element e is a closed set is included in the definition of a topological group, and sometimes it is considered as an additional condition.

Of course, any group is a topological group with respect to the discrete topology. If k is a field with an absolute value function $|\cdot|$, then it is easy to see that k is a topological group with respect to addition and the topology determined by the metric associated to $|\cdot|$. One can also check that $k \setminus \{0\}$ is a topological group with respect to multiplication, and the topology determined by the restriction to $k \setminus \{0\}$ of the metric associated to $|\cdot|$. A subgroup of a topological group is a topological group with respect to the induced topology. Cartesian products of topological groups are topological groups with respect to the product topology, and where the group operations are defined coordinatewise.

Let G be a topological group. If $a \in G$, then continuity of multiplication on G implies that the corresponding left and right translation mappings

$$x \mapsto a\,x \tag{3.1.2}$$

and

$$x \mapsto x\,a \tag{3.1.3}$$

are continuous on G. This means that these mappings are homeomorphisms on G, because their inverses are given by left and right translations by a^{-1}, respectively. If E is any subset of G, then we put

$$a\,E = \{a\,x : x \in E\} \tag{3.1.4}$$

and

$$E\,a = \{x\,a : x \in E\}. \tag{3.1.5}$$

If E is open, closed, or compact, for instance, then it follows that $a\,E$ and $E\,a$ have the same property.

Similarly, put

$$E^{-1} = \{x^{-1} : x \in E\}. \tag{3.1.6}$$

If E is open, closed, or compact, then continuity of (3.1.1) implies that E^{-1} has the same property. If $E^{-1} = E$, then E is said to be *symmetric*. Note that

$$(a\,E)^{-1} = E^{-1}\,a^{-1}, \quad (E\,a)^{-1} = a^{-1}\,E^{-1}. \tag{3.1.7}$$

If A, B are subsets of G, then put

$$A\,B = \{a\,b : a \in A,\, b \in B\} = \bigcup_{a \in A} a\,B = \bigcup_{b \in B} A\,b. \tag{3.1.8}$$

If either A or B is an open set, then $A\,B$ is a union of open sets, by continuity of translations, and thus an open set. If A and B are compact, then $A \times B$ is compact, by Tychonoff's theorem. This implies that $A\,B$ is compact, by continuity of multiplication. Clearly

$$(A\,B)^{-1} = B^{-1}\,A^{-1} \tag{3.1.9}$$

for any A, B.

If W is an open subset of G that contains e, then there are open subsets U, V of G that contain e and satisfy

$$U\,V \subseteq W. \tag{3.1.10}$$

This corresponds exactly to continuity of multiplication at e.

It is well known that topological groups are regular in the strict sense, and we shall say more about that later. If $\{e\}$ is a closed set, then G satisfies the first separation condition, because of continuity of translations. This implies that G is regular in the strong sense, and Hausdorff in particular.

3.2 Two associated uniformities

Let G be a topological group, with multiplicative identity element e. If U is an open subset of G that contains e, then put

$$U_L = \{(x, y) \in G \times G : x^{-1}\,y \in U\} \tag{3.2.1}$$

and

$$U_R = \{(x, y) \in G \times G : x\,y^{-1} \in U\}. \tag{3.2.2}$$

Equivalently,

$$(x, y) \in U_L \quad \text{if and only if} \quad y \in x\, U \tag{3.2.3}$$

and

$$(x, y) \in U_R \quad \text{if and only if} \quad x \in U\, y. \tag{3.2.4}$$

Note that (3.2.4) is the same as saying that

$$(x, y) \in U_R \quad \text{if and only if} \quad y \in U^{-1}\, x. \tag{3.2.5}$$

If $a \in G$, then consider the mappings

$$(x, y) \mapsto (a\, x, a\, y) \tag{3.2.6}$$

and

$$(x, y) \mapsto (x\, a, y\, a) \tag{3.2.7}$$

from $G \times G$ onto itself. Of course, these correspond to left and right translation by a in both coordinates, respectively. It is easy to see that (3.2.6) maps (3.2.1) onto itself, and that (3.2.7) maps (3.2.2) onto itself.

Let U be an open set in G that contains e, so that U^{-1} is an open set that contains e too. Observe that

$$\begin{aligned} (U^{-1})_L &= \{(x, y) \in G \times G : x^{-1}\, y \in U^{-1}\} \\ &= \{(x, y) \in G \times G : y^{-1}\, x \in U\} \end{aligned} \tag{3.2.8}$$

and

$$\begin{aligned} (U^{-1})_R &= \{(x, y) \in G \times G : x\, y^{-1} \in U^{-1}\} \\ &= \{(x, y) \in G \times G : y\, x^{-1} \in U\}. \end{aligned} \tag{3.2.9}$$

These also correspond to exchanging the x and y coordinates in U_L, U_R, respectively.

Let V be another open set in G that contains e, so that $U \cap V$ is an open set that contains e as well. Clearly

$$U_L \cap V_L = (U \cap V)_L \tag{3.2.10}$$

and

$$U_R \cap V_R = (U \cap V)_R. \tag{3.2.11}$$

One can check that

$$U_L \circ V_L = (U\, V)_L \tag{3.2.12}$$

and

$$U_R \circ V_R = (U\, V)_R, \tag{3.2.13}$$

where the left sides are as in Section 2.3.

Let $\mathcal{B}_L$, $\mathcal{B}_R$ be the collections of subsets of $G \times G$ of the form (3.2.1), (3.2.2), respectively, where U is an open subset of G that contains e. One can verify that $\mathcal{B}_L$ and $\mathcal{B}_R$ satisfy the first four conditions in the definition of a uniformity in

Section 2.3, using the prevous remarks. Thus $\mathcal{B}_L$, $\mathcal{B}_R$ are bases for uniformities $\mathcal{U}_L, \mathcal{U}_R$ on G, respectively, as before.

Let U be an open subset of G that contains e again. If $x \in G$, then

$$U_L[x] = x\,U \tag{3.2.14}$$

and

$$U_R[x] = U^{-1}\,x, \tag{3.2.15}$$

where the left sides are as in Section 2.4. Similarly, if A is any subset of G, then

$$U_L[A] = A\,U \tag{3.2.16}$$

and

$$U_R[A] = U^{-1}\,A, \tag{3.2.17}$$

where the left sides are as in Section 2.4 too.

One can verify that the topology on G is the same as those associated to $\mathcal{U}_L$ and $\mathcal{U}_R$, as on p210 of [15]. This implies that G is regular in the strict sense, because the topology associated to any uniformity has this property. Of course, this can also be seen more directly.

If $\{e\}$ is a closed set in G, then one can check that G is Hausdorff with respect to $\mathcal{U}_L$ and $\mathcal{U}_R$, as in Section 2.4. This means that G is Hausdorff as a topological space, as before.

If G is commutative, then it is easy to see that

$$(U^{-1})_L = U_R \tag{3.2.18}$$

for every open set U in G that contains e. This implies that $\mathcal{B}_L = \mathcal{B}_R$, so that

$$\mathcal{U}_L = \mathcal{U}_R. \tag{3.2.19}$$

3.3 Translation-invariant semimetrics

Let G be a group. A semimetric $d(\cdot, \cdot)$ on G is said to be *invariant under left translations* if

$$d(a\,x, a\,y) = d(x, y) \tag{3.3.1}$$

for every $a, x, y \in G$. Similarly, $d(\cdot, \cdot)$ is said to be *invariant under right translations* if

$$d(x\,a, y\,a) = d(x, y) \tag{3.3.2}$$

for every $a, x, y \in G$. Of course, one can define translation invariance of q-semimetrics in the same way. If G is commutative, then invariance under left and right translations are the same.

Suppose that G is a topological group, and let $\mathcal{U}_L, \mathcal{U}_R$ be the uniformities on G defined in the previous section. It is well known that there is a collection $\mathcal{M}_L$ of left-invariant semimetrics on G for which $\mathcal{U}_L$ is the associated uniformity, as on p210 of [15]. Similarly, there is a collection $\mathcal{M}_R$ of right-invariant semimetrics on G for which $\mathcal{U}_R$ is the associated uniformity. In particular, the topology on G is the same as the topologies determined by $\mathcal{M}_L$, $\mathcal{M}_R$.

If $\mathcal{M}_L$ is any nonempty collection of left-invariant semimetrics on G that determines the same topology on G, then it is easy to see that $\mathcal{U}_L$ is the same as the uniformity on G associated to $\mathcal{M}_L$. Similarly, if $\mathcal{M}_R$ is a nonempty collection of right-invariant semimetrics on G that determines the same topology on G, then $\mathcal{U}_R$ is the same as the uniformity on G associated to $\mathcal{M}_R$.

Let $\mathcal{B}_0$ be a local base for the topology of G at e. One can check that

$$\{U_L : U \in \mathcal{B}_0\} \tag{3.3.3}$$

is a base for $\mathcal{U}_L$, where U_L is as in (3.2.1). Similarly,

$$\{U_R : U \in \mathcal{B}_0\} \tag{3.3.4}$$

is a base for $\mathcal{U}_R$, where U_R is as in (3.2.2).

If $\mathcal{B}_0$ has only finitely many elements, then (3.3.3) and (3.3.4) have only finitely many elements as well. In this case, a famous theorem states that there are left and right-invariant semimetrics that determine the same topology on G. This means that the uniformities on G associated to these semimetrics are the same as $\mathcal{U}_L$, $\mathcal{U}_R$, respectively.

If $d(\cdot, \cdot)$ is any semimetric on G, then

$$\widetilde{d}(x, y) = d(x^{-1}, y^{-1}) \tag{3.3.5}$$

defines a semimetric on G as well. It is easy to see that $d(\cdot, \cdot)$ is invariant under left translations if and only if (3.3.5) is invariant under right translations.

Let $\mathcal{M}_L$ be a nonempty collection of left-invariant semimetrics on G, so that

$$\{\widetilde{d} : d \in \mathcal{M}_L\} \tag{3.3.6}$$

consists of right-invariant semimetrics. One can check that $x \mapsto x^{-1}$ sends the topology determined by $\mathcal{M}_L$ to the topology determined by (3.3.6).

If G is a topological group, then the topology on G is the same as the one determined by $\mathcal{M}$ if and only if it is the same as the one determined by (3.3.6). Note that the uniformities $\mathcal{U}_L$ and $\mathcal{U}_R$ correspond to each other under the mapping $x \mapsto x^{-1}$.

If a semimetric $d(\cdot, \cdot)$ on G is invariant under left or right translations, then it is easy to see that

$$d(e, x) = d(x^{-1}, e) \tag{3.3.7}$$

for every $x \in G$. If $d(\cdot, \cdot)$ is invariant under both left and right translations, then one can verify that

$$d(x, y) = d(x^{-1}, y^{-1}) \tag{3.3.8}$$

for every $x, y \in G$. If $d(\cdot, \cdot)$ is invariant under left or right translations and satisfies (3.3.8), then it follows that $d(\cdot, \cdot)$ is invariant under both left and right translations.

3.4 Translations and uniform continuity

Let G_1, G_2 be topological groups, with identity elements e_1, e_2, respectively, and let ϕ be a mapping from G_1 into G_2. Let us say that ϕ is *left-invariant uniformly continuous* if for every open subset U_2 of G_2 that contains e_2 there is an open subset U_1 of G_1 that contains e_1 such that

$$\phi(x_1\, U_1) \subseteq \phi(x_1)\, U_2 \tag{3.4.1}$$

for every $x_1 \in G_1$. Of course, this implies that ϕ is continuous. More precisely, this is equivalent to the uniform continuity of ϕ with respect to the uniformities $\mathcal{U}_{1,L}$, $\mathcal{U}_{2,L}$ on G_1, G_2, respectively, as in Section 3.2.

Similarly, let us say that ϕ is *right-invariant uniformly continuous* if for every open subset U_2 of G_2 that contains e_2 there is an open subset U_1 of G_1 that contains e_1 such that

$$\phi(U_1\, x_1) \subseteq U_2\, \phi(x_1) \tag{3.4.2}$$

for every $x_1 \in G_1$. This implies that ϕ is continuous, and is equivalent to the uniform continuity of ϕ with respect to the uniformities $\mathcal{U}_{1,R}$, $\mathcal{U}_{2,R}$ on G_1, G_2, respectively, as in Section 3.2. Of course, left and right-invariant uniform continuity are the same when G_1 and G_2 are commutative. If ϕ is a group homomorphism from G_1 into G_2 that is continuous at e_1, then it is easy to see that ϕ is both left and right-invariant uniformly continuous.

If the topologies on G_1, G_2 are determined by collections $\mathcal{M}_{1,L}$, $\mathcal{M}_{2,L}$ of left-invariant semimetrics, respectively, then left-invariant uniform continuity of ϕ is equivalent to uniform continuity with respect to $\mathcal{M}_{1,L}$, $\mathcal{M}_{2,L}$. Similarly, if the topologies on G_1, G_2 are determined by collections $\mathcal{M}_{1,R}$, $\mathcal{M}_{2,R}$ of right-invariant semimetrics, respectively, then right-invariant uniform continuity of ϕ is equivalent to uniform continuity with respect to $\mathcal{M}_{1,R}$, $\mathcal{M}_{2,R}$.

Let G be a topological group, let Y be a set, and let ϕ be a mapping from G into Y. Let us say that ϕ is *left-invariant uniformly continuous* with respect to a semimetric d_Y on Y if for every $\epsilon > 0$ there is an open subset U of G that contains e such that

$$d_Y(\phi(x), \phi(w)) < \epsilon \tag{3.4.3}$$

for every $x, w \in G$ with $w \in x\, U$. This implies that ϕ is continuous with respect to the topology determined on Y by d_Y. In fact, this is equivalent to the uniform continuity of ϕ with respect to the uniformity $\mathcal{U}_L$ on G, as in Section 3.2, and the uniformity on Y associated to d_Y.

Similarly, let us say that ϕ is *right-invariant uniformly continuous* with respect to d_Y on Y if for every $\epsilon > 0$ there is an open subset U of G that contains e such that (3.4.3) holds for every $x, w \in G$ with $w \in U\, x$. This implies that ϕ is continuous, and is equivalent to the uniform continuity of ϕ with respect to the uniformity $\mathcal{U}_R$ on G as in Section 3.2. If G is commutative, then left and right-invariant uniform continuity with respect to d_Y are the same.

Let $\mathcal{M}_Y$ be a nonempty collection of semimetrics on Y. Let us say that ϕ is left or right-invariant uniformly continuous with respect to $\mathcal{M}_Y$ on Y if ϕ is left or right-invariant uniformly continuous, as appropriate, with respect to every $d_Y \in \mathcal{M}_Y$. This is equivalent to the uniform continuity of ϕ with respect to $\mathcal{U}_L$ or $\mathcal{U}_R$ on G, as appropriate, and the uniformity on Y associated to $\mathcal{M}_Y$.

If the topology on G is determined by a collection $\mathcal{M}_L$ of left-invariant semimetrics, then left-invariant uniform continuity of ϕ is equivalent to uniform continuity with respect to $\mathcal{M}_L$. Similarly, if the topology on G is determined by a collection $\mathcal{M}_R$ of right-invariant semimetrics, then right-invariant uniform continuity of ϕ is equivalent to uniform continuity with respect to $\mathcal{M}_R$.

3.5 Translations and equicontinuity

Let X and Y be topological spaces, and let $x_0 \in X$ and $y_0 \in Y$ be given. Also let $\mathcal{E}$ be a collection of mappings from X into Y such that

$$\phi(x_0) = y_0 \tag{3.5.1}$$

for every $\phi \in \mathcal{E}$. Let us say that $\mathcal{E}$ is *equicontinuous* at x_0 if for every open set $V \subseteq Y$ with $y_0 \in V$ there is an open set $U \subseteq X$ such that $x_0 \in U$ and

$$\phi(U) \subseteq V \tag{3.5.2}$$

for every $\phi \in \mathcal{E}$. Of course, this implies that every element of $\mathcal{E}$ is continuous at x_0. If $\mathcal{E}$ has only finitely many elements, each of which is continuous at x_0, then it is easy to see that $\mathcal{E}$ is equicontinuous at x_0.

Let G be a topological group, and let $\mathcal{E}$ be a collection of mappings from X into G. Let us say that $\mathcal{E}$ is *left-invariant equicontinuous* at x_0 if for every open subset V of G that contains e there is an open set $U \subseteq X$ such that $x_0 \in U$ and

$$\phi(U) \subseteq \phi(x_0)\, V \tag{3.5.3}$$

for every $\phi \in \mathcal{E}$. Similarly, let us say that $\mathcal{E}$ is *right-invariant equicontinuous* at x_0 if for every open subset V of G that contains e there is an open set $U \subseteq X$ such that $x_0 \in U$ and

$$\phi(U) \subseteq V\, \phi(x_0) \tag{3.5.4}$$

for every $\phi \in \mathcal{E}$. Each of these conditions implies that every element of $\mathcal{E}$ is continuous at x_0, and both conditions hold when $\mathcal{E}$ has only finitely many elements, each of which is continuous at x_0. If there is a $y_0 \in G$ such that (3.5.1) holds for every $\phi \in \mathcal{E}$, then each of these conditions is equivalent to equicontinuity at x_0 as in the preceding paragraph.

Left and right-invariant equicontinuity at x_0 are equivalent to equicontinuity with respect to the uniformities $\mathcal{U}_L$, $\mathcal{U}_R$ on G, respectively, as in Section 3.2. If the topology on G is determined by a collection $\mathcal{M}_L$ of left-invariant semimetrics, or a collection $\mathcal{M}_R$ of right-invariant semimetrics, then left or right-invariant equicontinuity at x_0 is equivalent to equicontinuity at x_0 with respect to $\mathcal{M}_L$ or $\mathcal{M}_R$, as appropriate. If G is commutative, then left and right-invariant invariant equicontinuity at x_0 are the same.

Let G_1, G_2 be topological groups with identity elements e_1, e_2, respectively, and let $\mathcal{E}$ be a collection of mappings from G_1 into G_2. Let us say that $\mathcal{E}$ is *left-invariant uniformly equicontinuous* if for every open subset U_2 of G_2 that contains e_2 there is an open subset U_1 of G_1 that contains e_1 such that (3.4.1) holds for every $\phi \in \mathcal{E}$ and $x_1 \in G$. Similarly, let us say that $\mathcal{E}$ is *right-invariant uniformly equicontinuous* if for every open subset U_2 of G_2 that contains e_2 there is an open subset U_1 of G_1 that contains e_1 such that (3.4.2) holds for every $\phi \in \mathcal{E}$ and $x_1 \in G_1$. If $\mathcal{E}$ is left or right-invariant uniformly equicontinuous, then every element of $\mathcal{E}$ is left or right-invariant uniformly continuous, as appropriate. The converse holds in both cases when $\mathcal{E}$ has only finitely many elements, as usual.

Left-invariant uniform equicontinuity is equivalent to uniform equicontinuity with respect to the uniformities $\mathcal{U}_{1,L}$, $\mathcal{U}_{2,L}$ on G_1, G_2, respectively, as in Section 3.2. If the topologies on G_1, G_2 are determined by collections $\mathcal{M}_{1,L}$, $\mathcal{M}_{2,L}$ of left-invariant semimetrics, then left-invariant uniform equicontinuity is equivalent to uniform equicontinuity with respect to $\mathcal{M}_{1,L}$, $\mathcal{M}_{2,L}$. Of course, there are analogous statements for right-invariant uniform equicontinuity. If G_1 and G_2 are commutative, then left and right-invariant uniform equicontinuity are the same.

If $\mathcal{E}$ is a collection of group homomorphisms from G_1 into G_2, then $\phi(e_1) = e_2$ for every $\phi \in \mathcal{E}$. If $\mathcal{E}$ is equicontinuous at e_1, then $\mathcal{E}$ is both left and right-invariant uniformly equicontinuous.

Let G be a topological group, let Y be a set, and let $\mathcal{E}$ be a collection of mappings from G into Y. Let us say that $\mathcal{E}$ is *left-invariant uniformly equicontinuous* with respect to a semimetric d_Y on Y if for every $\epsilon > 0$ there is an open subset U of G that contains e such that (3.4.3) holds for every $\phi \in \mathcal{E}$ and $x, w \in G$ with $w \in x\,U$. Similarly, let us say that $\mathcal{E}$ is *right-invariant uniformly equicontinuous* with respect to d_Y if for every $\epsilon > 0$ there is an open subset U of G that contains e such that (3.4.3) holds for every $\phi \in \mathcal{E}$ and $x, w \in G$ with $w \in U\,x$. In each case, every element of $\mathcal{E}$ is left or right-invariant uniformly continuous with respect to d_Y, as appropriate. If $\mathcal{E}$ has only finitely many elements, then the converse holds, as before.

Left and right-invariant uniform equicontinuity with respect to d_Y are equivalent to uniform equicontinuity with respect to the uniformities $\mathcal{U}_L$, $\mathcal{U}_R$ on G, respectively, as in Section 3.2, and the uniformity on Y associated to d_Y. If G is commutative, then left and right-invariant uniform equicontinuity are the same.

If $\mathcal{M}_Y$ is a nonempty collection of semimetrics on Y, then we say that $\mathcal{E}$ is left or right-invariant uniformly equicontinuous with respect to $\mathcal{M}_Y$ on Y when $\mathcal{E}$ is left or right-invariant uniformly equicontinuous, as appropriate, with respect to every $d_Y \in \mathcal{M}_Y$. This is equivalent to the uniform equicontinuity of $\mathcal{E}$ with respect to $\mathcal{U}_L$ or $\mathcal{U}_R$ on G, as appropriate, and the uniformity on Y associated to $\mathcal{M}_Y$. If the topology on G is determined by a collection $\mathcal{M}_L$ of left-invariant semimetrics, or a collection $\mathcal{M}_R$ of right-invariant semimetrics, then left or right-invariant uniform equicontinuity of $\mathcal{E}$ is equivalent to uniform equicontinuity with respect to $\mathcal{M}_L$ or $\mathcal{M}_R$ on G, as appropriate.

3.6 Compatible semimetrics

Let X be a topological space, and let $d(\cdot, \cdot)$ be a semimetric on X. Let us say that $d(\cdot, \cdot)$ is *compatible* with the topology on X at a point $x_0 \in X$ if the identity mapping on X is continuous at x_0 as a mapping from X with its given topology into X with the topology determined by $d(\cdot, \cdot)$. Equivalently, this means that for each $r > 0$, x_0 is in the interior of $B_d(x_0, r)$ with respect to the given topology on X. This is the same as saying that

$$f_{x_0}(x) = d(x, x_0) \tag{3.6.1}$$

is continuous at x_0, as a real-valued function of x on X.

Let us say that $d(\cdot, \cdot)$ is *compatible* with the topology on X if $d(\cdot, \cdot)$ is compatible with the topology on X at every $x_0 \in X$, which means that the given topology on X is at least as strong as the topology determined by $d(\cdot, \cdot)$. This is equivalent to the continuity of (3.6.1) as a real-valued function of x on X for every x_0, because (3.6.1) is continuous with respect to $d(\cdot, \cdot)$, as in Section 1.8. Alternatively, $d(\cdot, \cdot)$ is compatible with the topology on X if and only if every open ball in X with respect to $d(\cdot, \cdot)$ is an open set with respect to the given topology on X.

Let us say that $d(\cdot, \cdot)$ is *compatible* with a uniformity $\mathcal{U}$ on X if the identity mapping on X is uniformly continuous as a mapping from X equipped with $\mathcal{U}$ into X equipped with $d(\cdot, \cdot)$. Of course, this implies that $d(\cdot, \cdot)$ is compatible with the topology on X associated to $\mathcal{U}$.

Note that (3.6.1) is uniformly continuous as a real-valued function on X with respect to $d(\cdot, \cdot)$ for every $x_0 \in X$, as in Section 1.8. In fact, the collection of these functions is uniformly equicontinuous on X with respect to $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is compatible with a uniformity $\mathcal{U}$ on X, then it follows that the collection of these functions is uniformly equicontinuous with respect to $\mathcal{U}$ on X.

Let G be a topological group, and let $d(\cdot, \cdot)$ be a semimetric on G. Suppose that $d(\cdot, \cdot)$ is compatible with the topology of G at e. If $d(\cdot, \cdot)$ is invariant under left translations, then it follows that $d(\cdot, \cdot)$ is compatible with the topology of G at every point. More precisely, one can check that $d(\cdot, \cdot)$ is compatible with the uniformity $\mathcal{U}_L$, as in Section 3.2. Equivalently, this means that the identity mapping on G is left-invariant uniformly continuous as a mapping from G as a topological group into G with respect to $d(\cdot, \cdot)$, as in the previous section.

Similarly, if $d(\cdot, \cdot)$ is invariant under right translations, then $d(\cdot, \cdot)$ is compatible with the topology of G at every point. In this case, one can verify that $d(\cdot, \cdot)$ is compatible with the uniformity $\mathcal{U}_R$, as in Section 3.2. This means that the identity mapping on G is right-invariant uniformly continuous as a mapping from G as a topological group into G with respect to $d(\cdot, \cdot)$, as before.

If $d(\cdot, \cdot)$ is invariant under left translations, then the collection of functions of the form (3.6.1) with $x_0 \in G$ is left-invariant uniformly equicontinuous as a collection of real-valued functions on G. Similarly, if $d(\cdot, \cdot)$ is invariant under right translations, then this collection is right-invariant uniformly equicontinuous as a collection of real-valued functions on G.

3.7 More on translation invariance

Let G be a group, and let $d(\cdot,\cdot)$ be a semimetric on G. If $d(\cdot,\cdot)$ is invariant under left or right translations, then $B_d(e,r)$ is a symmetric set for every $r > 0$, by (3.3.7). Similarly, $\overline{B}_d(e,r)$ is symmetric for every $r \geq 0$.

Let $x, y \in G$ be given. If $d(\cdot,\cdot)$ is invariant under left translations on G, then

$$d(e, x\,y) \leq d(e,x) + d(x, x\,y) = d(e,x) + d(e,y). \tag{3.7.1}$$

Similarly, if $d(\cdot,\cdot)$ is invariant under right translations, then

$$d(e, x\,y) \leq d(e,y) + d(y, x\,y) = d(e,x) + d(e,y). \tag{3.7.2}$$

In both cases, we get that

$$B_d(e,r)\,B_d(e,t) \subseteq B_d(e, r+t) \tag{3.7.3}$$

for every $r, t > 0$, and

$$\overline{B}_d(e,r)\,\overline{B}_d(e,t) \subseteq \overline{B}_d(e, r+t) \tag{3.7.4}$$

for every $r, t \geq 0$.

Suppose for the moment that $d(\cdot,\cdot)$ is a semi-ultrametric on G. If $d(\cdot,\cdot)$ is invariant under left translations, then

$$d(e, x\,y) \leq \max(d(e,x), d(x, x\,y)) = \max(d(e,x), d(e,y)) \tag{3.7.5}$$

for every $x, y \in G$. If $d(\cdot,\cdot)$ is invariant under right translations, then

$$d(e, x\,y) \leq \max(d(e,y), d(y, x\,y)) = \max(d(e,x), d(e,y)). \tag{3.7.6}$$

It follows that open and closed balls centered at e with respect to $d(\cdot,\cdot)$ are subgroups of G in both cases. If $d(\cdot,\cdot)$ is invariant under both left and right translations, then one can check that open and closed balls centered at e with respect to $d(\cdot,\cdot)$ are normal subgroups of G.

If $d(\cdot,\cdot)$ is any semimetric on G that is invariant under left or right translations, then $\overline{B}_d(e,0)$ is a subgroup of G. If $d(\cdot,\cdot)$ is invariant under both left and right translations, then $\overline{B}_d(e,0)$ is a normal subgroup of G.

Let A be a subgroup of G. If $x, y \in G$, then put

$$\begin{aligned} d_{A,L}(x,y) &= 0 \quad \text{when } x\,A = y\,A \\ &= 1 \quad \text{when } x\,A \neq y\,A. \end{aligned} \tag{3.7.7}$$

One can check that this is a semi-ultrametric on G that is invariant under left translations. More precisely, this is the same as the discrete semimetric associated to the partition of G into left cosets of A, as in Section 1.3.

Similarly, if we put

$$\begin{aligned} d_{A,R}(x,y) &= 0 \quad \text{when } A\,x = A\,y \\ &= 1 \quad \text{when } A\,x \neq A\,y, \end{aligned} \tag{3.7.8}$$

then we get a right-invariant semi-ultrametric on G. This is the same as the discrete semimetric associated to the partition of G into right cosets of A. If A is a normal subgroup, then (3.7.7) and (3.7.8) are the same. Thus we get a semi-ultrametric that is invariant under both left and right translations in this case.

Suppose now that G is a topological group, and that $d(\cdot, \cdot)$ is a semimetric on G that is compatible with the topology of G at e. If $d(\cdot, \cdot)$ is invariant under left or right translations, then $d(\cdot, \cdot)$ is compatible with the topology of G at every point, as in the previous section. If $d(\cdot, \cdot)$ is also a semi-ultrametric on G, then it follows that open balls centered at e with respect to $d(\cdot, \cdot)$ are open subgroups. Similarly, closed balls centered at e of positive radius are open subgroups in this case.

If A is an open subgroup of G, then it is easy to see that A is a closed set too. More precisely, the complement of A is a union of cosets of A, and thus an open set. Note that (3.7.7) and (3.7.8) are compatible with the topology of G in this case.

3.8 Translations and total boundedness

Let G be a topological group, and let E be a subset of G. Let us say that E is *left-invariant totally bounded* in G if for every open subset U of G that contains e there are finitely many elements $x_1, \ldots, x_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} x_j\, U. \tag{3.8.1}$$

Similarly, let us say that E is *right-invariant totally bounded* in G if for every open subset U of G that contains e there are finitely many elements $x_1, \ldots, x_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} U\, x_j. \tag{3.8.2}$$

Of course, these two properties are the same when G is commutative. If E is compact, then it is easy to see that E is both left and right-invariant totally bounded in G.

If E is left or right-invariant totally bounded, then it is easy to see that every subset of E has the same property. The union of finitely many left-invariant totally bounded sets is left-invariant totally bounded, and similarly for right-invariant total boundedness. If E is left or right-invariant totally bounded, then one can check that the closure $\overline{E}$ of E has the same property, because G is regular in the strict sense as a topological space.

Remember that total boundedness with respect to a uniformity can be defined as in Section 2.6, and that $\mathcal{U}_L, \mathcal{U}_R$ are as in Section 3.2. One can verify that E is left or right-invariant totally bounded if and only if E is totally bounded with respect to $\mathcal{U}_L$ or $\mathcal{U}_R$, as appropriate. If the topology on G is determined by a collection $\mathcal{M}_L$ of left-invariant semimetrics, or a collection $\mathcal{M}_R$ of right-invariant semimetrics, then E is left or right-invariant totally bounded if and only if E is totally bounded with respect to $\mathcal{M}_L$ or $\mathcal{M}_R$, as appropriate.

Let U be an open subset of G that contains e. Let us say that a subset A of G is *left-invariant U-small* if

$$A^{-1}\,A \subseteq U. \tag{3.8.3}$$

Equivalently, this means that

$$A \times A \subseteq U_L, \tag{3.8.4}$$

so that A is U_L-small in the sense of Section 2.6. Similarly, let us say that A is *right-invariant U-small* if

$$A\,A^{-1} \subseteq U. \tag{3.8.5}$$

This is the same as saying that

$$A \times A \subseteq U_R, \tag{3.8.6}$$

so that A is U_R-small in the sense of Section 2.6.

Alternatively, A is left-invariant U-small if and only if

$$A \subseteq x\,U \tag{3.8.7}$$

for every $x \in A$, and A is right-invariant U-small if and only if

$$A \subseteq U\,y \tag{3.8.8}$$

for every $y \in A$. Let V be another open subset of G that contains e. If A is contained in a left translate of V, then

$$A^{-1}\,A \subseteq V^{-1}\,V, \tag{3.8.9}$$

so that A is $(V^{-1}\,V)$-small. If A is contained in a right translate of V, then

$$A\,A^{-1} \subseteq V\,V^{-1}, \tag{3.8.10}$$

so that A is $(V\,V^{-1})$-small.

One can check that E is left or right-invariant totally bounded if and only if for every open subset U of G that contains e, E is contained in the union of finitely many left or right-invariant U-small sets, as appropriate. If E is contained in a subgroup of G, then E is left or right-invariant totally bounded in G if and only if E is left or right-invariant totally bounded in the subgroup, as appropriate, as a topological group with respect to the induced topology.

Let G_1, G_2 be topological groups, let ϕ be a mapping from G_1 into G_2, and let E_1 be a subset of G_1. If E_1 is left-invariant totally bounded in G_1, and ϕ is left-invariant uniformly continuous, then $\phi(E_1)$ is left-invariant totally bounded in G_2. Similarly, if E_1 is right-invariant totally bounded in G_1, and ϕ is right-invariant uniformly continuous, then $\phi(E_1)$ is right-invariant totally bounded in G_2.

Let G be a topological group again, and let Y be a set with a nonempty collection $\mathcal{M}_Y$ of semimetrics. Also let ϕ be a mapping from G into Y, and let E be a subset of G. If E is left-invariant totally bounded and ϕ is left-invariant uniformly continuous, then $\phi(E)$ is totally bounded in Y with respect to $\mathcal{M}_Y$. Similarly, if E is right-invariant totally bounded and ϕ is right-invariant uniformly continuous, then $\phi(E)$ is totally bounded in Y with respect to $\mathcal{M}_Y$.

3.9 More on small sets

Let G be a topological group, and let E be a subset of G again. One can check that E is left-invariant totally bounded if and only if E^{-1} is right-invariant totally bounded. In particular, left and right-invariant total boundedness are the same for symmetric sets.

Let A be a subset of G, and let U be an open subset of G that contains e. Thus U^{-1} and $U \cap U^{-1}$ are open sets that contain e as well. It is easy to see that A is left or right-invariant U-small if and only if A is left or right-invariant U^{-1}-small, as appropriate. In this case, A is left or right-invariant $(U \cap U^{-1})$-small, as appropriate. We also have that A is left-invariant U-small if and only if A^{-1} is right-invariant U-small.

Let $U_1, \ldots, U_n$ be finitely many open subsets of G, each of which contains e. Thus

$$U = \bigcap_{j=1}^{n} U_j \tag{3.9.1}$$

is an open set that contains e too. Suppose that for each $j = 1, \ldots, n$, A_j is a subset of G that is left-invariant U_j-small. Under these conditions, it is easy to see that

$$\bigcap_{j=1}^{n} A_j \tag{3.9.2}$$

is left-invariant U-small. Similarly, if A_j is right-invariant U_j-small for each $j = 1, \ldots, n$, then (3.9.2) is right-invariant U-small.

Let E be a subset of G, and suppose that for each $j = 1, \ldots, n$, E can be covered by finitely many subsets of G that are left-invariant U_j-small. One can check that E can be covered by finitely many subsets of G that are left-invariant U-small, using the remarks in the preceding paragraph. More precisely, there are finitely many sets obtained by taking intersections of sets from the n coverings of E, and E is covered by these sets. Similarly, if for each $j = 1, \ldots, n$, E can be covered by finitely many subsets of G that are right-invariant U_j-small, then E can be covered by finitely many sets that are right-invariant U-small.

Let I be a nonempty set, and let G_j be a topological group for each $j \in I$. Thus $G = \prod_{j \in I} G_j$ is a topological group as well, where the group operations are defined coordinatewise, and with respect to the product topology. If $l \in I$, then let p_l be the usual coordinate projection from G onto G_l. Note that p_l is a continuous group homomorphism from G onto G_l. In particular, p_l maps left and right-invariant totally bounded subsets of G to left and right-invariant totally bounded subsets of G_l, respectively.

Suppose that for each $j \in I$, E_j is a left-invariant totally bounded subset of G, and put

$$E = \prod_{j \in I} E_j. \tag{3.9.3}$$

Let $l \in I$ be given, and let U_l be an open subset of G_l that contains the identity element e_l. Thus E_l can be covered by finitely many subsets of G_l that are left-invariant U_l-small. Note that $p_l^{-1}(U_l)$ is an open subset of G that contains the identity element. One can check that $E \subseteq p_l^{-1}(E_l)$ is contained in the union of finitely many subsets of G that are left-invariant $p_l^{-1}(U_l)$-small.

Let $l_1, \ldots, l_n$ be finitely many elements of I, and let U_{l_r} be an open subset of G_{l_r} that contains e_{l_r} for each $r = 1, \ldots, n$. It follows that for each $r = 1, \ldots, n$, E can be covered by finitely many subsets of G that are left-invariant $p_{l_r}^{-1}(U_{l_r})$-small, as in the previous paragraph. Put

$$U = \bigcap_{r=1}^{n} p_{l_r}^{-1}(U_{l_r}), \tag{3.9.4}$$

which is an open subset of G that contains the identity element. Under these conditions, we get that E can be covered by finitely many left-invariant U-small sets, as before. This implies that E is left-invariant totally bounded in G, and there is an analogous statement for right-invariant total boundedness.

3.10 Equicontinuity and conjugations

Let G be a group. If $a \in G$, then put

$$C_a(x) = a\,x\,a^{-1} \tag{3.10.1}$$

for every $x \in G$. Of course, C_a is an *inner automorphism* of G. Note that

$$C_a(C_b(x)) = C_{a\,b}(x) \tag{3.10.2}$$

for every $a, b, x \in G$.

Suppose now that G is a topological group. In this case, C_a is continuous for every $a \in G$, and $(C_a)^{-1} = C_{a^{-1}}$ is continuous too. If A is a subset of G, then we may be interested in the equicontinuity of

$$\{C_a : a \in A\} \tag{3.10.3}$$

at e, as in Section 3.5. This means that for every open subset V of G that contains e, there is an open subset U of G that contains e such that

$$C_a(U) = a\,U\,a^{-1} \subseteq V \tag{3.10.4}$$

for every $a \in A$.

Suppose that A is right-invariant totally bounded, and let us check that (3.10.3) is equicontinuous at e. Let V be an open subset of G that contains e. The continuity of the group operations at e implies that there is a symmetric open subset U_0 of G that contains e and satisfies

$$U_0\, U_0\, U_0 \subseteq V. \tag{3.10.5}$$

Because A is right-invariant totally bounded, there are finitely many elements $a_1, \ldots, a_n$ of G such that

$$A \subseteq \bigcup_{j=1}^{n} U_0\, a_j. \tag{3.10.6}$$

This means that

$$A^{-1} \subseteq \bigcup_{j=1}^{n} a_j^{-1}\, U_0, \tag{3.10.7}$$

because U_0 is symmetric.

Observe that $a_j\, U\, a_j^{-1}$ is an open set that contains e for each $j = 1, \ldots, n$, so that

$$U = \bigcap_{j=1}^{n} a_j^{-1}\, U_0\, a_j, \tag{3.10.8}$$

is an open set that contains e as well. By construction,

$$a_j\, U\, a_j^{-1} \subseteq a_j\, (a_j^{-1}\, U_0\, a_j)\, a_j^{-1} = U_0 \tag{3.10.9}$$

for every $j = 1, \ldots, n$. This implies that

$$U_0\, (a_j\, U\, a_j^{-1})\, U_0 \subseteq U_0\, U_0\, U_0 \subseteq V \tag{3.10.10}$$

for every $j = 1, \ldots, n$. Equivalently,

$$(U_0\, a_j)\, U\, (a_j^{-1}\, U_0) \subseteq V \tag{3.10.11}$$

for every $j = 1, \ldots, n$. If $a \in A$, then $a \in U_0\, a_j$ for some j, and it follows that

$$a\, U\, a^{-1} \subseteq (U_0\, a_j)\, U\, (a_j^{-1}\, U_0) \subseteq V, \tag{3.10.12}$$

as desired.

Of course, if G is commutative, then C_a is the identity mapping for every $a \in G$, and the equicontinuity of the C_a's is trivial. If G is any group equipped with the discrete topology, then one can simply take $U = \{e\}$ in (3.10.4). Note that left or right-invariant totally bounded sets have only finitely many elements in this case.

3.11 Subgroups and conjugations

Let G be a topological group, and let A be a subgroup of G. Suppose that (3.10.3) is equicontinuous at e, and let V be an open subset of G that contains e. Thus there is an open subset U of G that contains e and for which (3.10.4) holds for every $a \in A$. It follows that

$$U_1 = \bigcup_{a \in A} a\, U\, a^{-1} \tag{3.11.1}$$

is an open set that contains e and satisfies

$$U_1 \subseteq V. \tag{3.11.2}$$

We also have that

$$C_{a_1}(U_1) = a_1\, U_1\, a_1^{-1} = U_1 \tag{3.11.3}$$

for every $a_1 \in A$, by construction.

Let $\mathcal{B}_A$ be the collection of open subsets of G that contain e and are invariant under conjugation by elements of A. If (3.10.3) is equicontinuous at e, then $\mathcal{B}_A$ is a local base for the topology of G at e, as in the preceding paragraph. Conversely, if $\mathcal{B}_A$ is a local base for the topology of G at e, then it is easy to see that (3.10.3) is equicontinuous at e.

Let d be a semimetric on G, and suppose that d is invariant under conjugations by elements of A. This implies that open and closed balls centered at e with respect to d are invariant under conjugations by elements of A. If the topology on G is determined by a collection of semimetrics with this property, then it follows that $\mathcal{B}_A$ is a local base for the topology of G at e.

Let d be a semimetric on G again, and suppose for the moment that d is invariant under left translations. In this case, d is invariant under conjugations by elements of A if and only if d is invariant under right translations by elements of A. Similarly, if d is invariant under right translations, then d is invariant under conjugations by elements of A if and only if d is invariant under left translations by elements of A.

If d is a semimetric on G, then one can try to get a semimetric that is invariant under conjugations by elements of A, as follows. Put

$$d_A(x, y) = \sup_{a \in A} d(C_a(x), C_a(y)) \tag{3.11.4}$$

for every $x, y \in G$. Let us suppose that the supremum on the right is finite for each x, y. In particular, this holds when $d(\cdot, \cdot)$ is bounded. Remember that this can always be arranged, as in Section 2.7.

Note that $d(C_a(x), C_a(y))$ is a semimetric for every $a \in A$. Using this, one can check that d_A is a semimetric on G. If d is a semi-ultrametric on G, then d_A is a semi-ultrametric on G as well. It is easy to see that d_A is invariant under conjugations by elements of A, by construction. Of course,

$$d(x, y) \le d_A(x, y) \tag{3.11.5}$$

for every $x, y \in G$.

If d is invariant under left translations, then $d(C_a(x), C_a(y))$ is invariant under left translations for every $a \in A$. This implies that d_A is invariant under left translations too. It follows that d_A is invariant under right translations by elements of A, because it is invariant under cnjugations by elements of A. Similarly, if d is invariant under right translations, then $d(C_a(x), C_a(y))$ is invariant under right translations for every $a \in A$, and d_A is invariant under right translations. This means that d_A is invariant under left translations by elements of A, as before.

Suppose that (3.10.3) is equicontinuous at e. If d is compatible with the topology of G at e, then one can check that d_A is compatible with the topology of G at e too.

3.12 Subgroups and topologies

Let G be a group, and let A be a subgroup of G. Suppose for the moment that G is a topological group, and that A is an open set. Of course, this implies that the left and right cosets of A are open sets too. It follows that a subset U of G is an open set if and only if the intersection of U with each of the left cosets of A is an open set, and if and only if the intersection of U with each of the right cosets of A is an open set.

If $x \in G$, then

$$U \cap (x\,A) = x\,((x^{-1}\,U) \cap A), \tag{3.12.1}$$

which is an open set exactly when $(x^{-1}\,U) \cap A$ is an open set. Similarly,

$$U \cap (A\,x) = ((U\,x^{-1}) \cap A)\,x, \tag{3.12.2}$$

which is an open set exactly when $(U\,x^{-1}) \cap A$ is an open set.

If $w \in G$, then $w^{-1}\,A\,w$ is an open subgroup of G, and thus

$$(w^{-1}\,A\,w) \cap A \tag{3.12.3}$$

is an open subgroup too. Of course, $C_w(x) = w\,x\,w^{-1}$ maps (3.12.3) onto

$$(w\,A\,w^{-1}) \cap A. \tag{3.12.4}$$

Suppose now that A is a topological group, which is a subgroup of a group G. If $w \in G$, then $w^{-1}\,A\,w$ is a subgroup of G, so that (3.12.3) is a subgroup of A. Suppose that for every $w \in G$,

$$(w^{-1}\,a\,w) \cap A \text{ is an open subgroup of } A, \tag{3.12.5}$$

and that

$$\text{the restriction of } C_w \text{ to } (w^{-1}\,A\,w) \cap A \text{ is continuous as a mapping into } A, \tag{3.12.6}$$

with respect to the topology induced on (3.12.3) by the topology on A. This means that

(3.12.7) the restriction of C_w to $(w^{-1}\,A\,w) \cap A$ is a homeomorphism onto $(w\,A\,w^{-1}) \cap A$

for every $w \in G$, with respect to the topologies induced by the topology on A, because the inverse mapping corresponds to w^{-1} in the same way.

If $x \in G$, then there is a unique topology on $x\,A$ such that $a \mapsto x\,a$ is a homeomorphism from A onto $x\,A$. If $x \in A$, so that $x\,A = A$, then this is the same as the initial topology on A, by continuity of left translations on A. Similarly, if $y \in G$ and $x\,A = y\,A$, then we get the same topology on this left coset of A using x or y, because of continuity of left translations on A. Note that left translation by $w \in G$ defines a homeomorphism from $x\,A$ onto $w\,x\,A$, with respect to these topologies of $x\,A$ and $w\,x\,A$.

In the same way, there is a unique topology on $A\,x$ such that $a \mapsto a\,x$ is a homeomorphism from A onto $A\,x$. If $x \in A$, then this is the same as the initial topology on A, by continuity of right translations on A. If $y \in G$ and $A\,x = A\,y$, then we get the same topology on this right coset of A using x or y, because of continuity of right translations on A. As before, right translation by $w \in G$ defines a homeomorphism from $A\,x$ onto $A\,x\,w$, with respect to these topologies on $A\,x$ and $A\,x\,w$.

Let $x, y \in G$ be given, and let us check that

(3.12.8) $(x\,A) \cap (A\,y)$ is an open subset of $x\,A$ and $A\,y$,

with respect to the topologies defined on $x\,A$ and $A\,y$ earlier. Of course, this is trivial when $(x\,A) \cap (A\,y) = \emptyset$, and so we may suppose that

$$(x\,A) \cap (A\,y) \neq \emptyset. \tag{3.12.9}$$

Let w be an element of $(x\,A) \cap (A\,y)$, so that

$$x\,A = w\,A, \quad A\,y = A\,w. \tag{3.12.10}$$

Thus

$$(x\,A) \cap (A\,y) = (w\,A) \cap (A\,w) = w\,(A \cap (w^{-1}\,A\,w)) \tag{3.12.11}$$

and

$$(x\,A) \cap (A\,y) = (w\,A) \cap (A\,w) = ((w\,A\,w^{-1}) \cap A)\,w. \tag{3.12.12}$$

It follows that (3.12.8) holds, because of (3.12.5).

We also have that

(3.12.13) the topologies induced on $(x\,A) \cap (A\,y)$ by the topologies on $x\,A$ and $A\,y$ are the same.

This is trivial when $(x\,A) \cap (A\,y) = \emptyset$, and so we may suppose that (3.12.9) holds again. If $w \in (x\,A) \cap (A\,y)$, then (3.12.13) is the same as saying that

(3.12.14) the topologies induced on $(w\,A) \cap (A\,w)$ by the topologies on $w\,A$ and $A\,w$ are the same.

One can check that (3.12.14) follows from (3.12.7).

If U is a subset of G, then we would like to say that U is an open set when

(3.12.15) $U \cap (x\,A)$ is an open set in $x\,A$ for every $x \in G$.

We would also like to say that U is an open set when

(3.12.16) $U \cap (A\,y)$ is an open set in $A\,y$ for every $y \in G$.

One can verify that (3.12.15) and (3.12.16) are equivalent, using (3.12.8) and (3.12.13). It is easy to see that this defines a topology on G, for which the left and right cosets of A are open sets. The topologies induced on the left and right cosets of A by this topology are the topologies defined earlier on the cosets of A, by construction.

Left and right translations are continuous on G, because of the continuity of left and right translations as mappings between left and right cosets of A, respectively. One can check that G is a topological group with respect to this topology, using the continuity of the group operations on A.

3.13 Subgroups and semimetrics

Let G be a group, and let A be a subgroup of G. Also let $d_0(\cdot,\cdot)$ be a semimetric on A, and let r_0, r_1 be nonnegative real numbers. Suppose that

(3.13.1) $$d_0(x,y) \le r_0$$

for every $x, y \in A$, which can always be arranged as in Section 2.7.

Suppose for the moment that $d_0(\cdot,\cdot)$ is invariant under left translations on A. Let $x, y \in G$ be given, and put

(3.13.2) $$d_L(x,y) = r_1 \quad \text{when } x\,A \ne y\,A.$$

Otherwise, if

(3.13.3) $$x\,A = y\,A,$$

then

(3.13.4) $$A\,x^{-1} = A\,y^{-1}.$$

If w is an element of this right coset of A, then $w\,x, w\,y \in A$, and we would like to put

(3.13.5) $$d_L(x,y) = d_0(w\,x, w\,y).$$

This does not depend on the choice of w, because d_0 is invariant under left translations on A.

If

$$r_0 \le 2\, r_1, \tag{3.13.6}$$

then one can check that d_L is a semimetric on G. If d_0 is a semi-ultrametric on A, and

$$r_0 \le r_1, \tag{3.13.7}$$

then d_L is a semi-ultrametric on G. In both cases, d_L is invariant under left translations on G. If d_0 is also invariant under right translations on A, then d_L is invariant under right translations by elements of A too. Note that d_L is the same as d_0 on A, and that A is an open subset of G with respect to d_L when $r_1 > 0$.

Suppose now that d_0 is invariant under right translations on A. Let $x, y \in G$ be given again, and put

$$d_R(x, y) = r_1 \quad \text{when } A\, x \ne A\, y. \tag{3.13.8}$$

If instead

$$A\, x = A\, y, \tag{3.13.9}$$

then

$$x^{-1}\, A = y^{-1}\, A, \tag{3.13.10}$$

and we let w be an element of this left coset of A. Thus $x\, w, y\, w \in A$, and we would like to put

$$d_R(x, y) = d_0(x\, w, y\, w). \tag{3.13.11}$$

It is easy to see that this does not depend on the choice of w, because d_0 is invariant under right translations on A.

If (3.13.6) holds, then one can verify that d_R is a semimetric on G, as before. If d_0 is a semi-ultrametric on A and (3.13.7) holds, then d_R is a semi-ultrametric on G. One can check that d_R is invariant under right translations on G in both cases. If d_0 is also invariant under left translations on A, then d_R is invariant under left translations by elements of A as well. By construction, d_R is the same as d_0 on A, and A is an open set in G with respect to d_R when $r_1 > 0$.

3.14 Translations and continuity conditions

Let G be a group, and suppose that G is equipped with a topology. Consider the condition that

(3.14.1) left and right translations are continuous on G.

Of course, this implies that

(3.14.2) conjugations are continuous on G.

If either left or right translations are continuous, then it is easy to see that (3.14.2) implies (3.14.1). More precisely, one check that (3.14.2) holds when conjugations are continuous at e in this case.

If (3.14.1) holds, and if $x \mapsto x^{-1}$ is continuous at e, then one can verify that $x \mapsto x^{-1}$ is continuous at every point. Of course, if $x \mapsto x^{-1}$ is continuous at every point, then continuity of left and right translations are equivalent.

Suppose that (3.14.1) holds again. If multiplication in the group is continuous as a mapping from $G \times G$ into G at (e, e), then one can check that multiplication is continuous at every point in $G \times G$.

Let $\mathcal{M}_L$ be a nonempty collection of semimetrics on G that are invariant under left translations, and suppose that G is equipped with the topology determined by $\mathcal{M}_L$. Of course, left translations are continuous on G in this case. It is easy to see that $x \mapsto x^{-1}$ is continuous at e with respect to this topology, using (3.3.7). We also have that multiplication on G is continuous as a mapping from $G \times G$ into G at (e, e), by (3.7.3).

Similarly, let $\mathcal{M}_R$ be a nonempty collection of semimetrics on G that are invariant under right translations. If G is equipped with the topology determined by $\mathcal{M}_R$, then right translations are continuous. As before, $x \mapsto x^{-1}$ is continuous at e with respect to this topology, and multiplication is continuous as a mapping from $G \times G$ into G at (e, e).

If $\mathcal{M}$ is a nonempty collection of semimetrics on G that are invariant under both left and right translations, then left and right translations are continuous with respect to the corresponding topology on G. It follows that the group operations are continuous on G, by the earlier remarks. Note that the elements of $\mathcal{M}$ are invariant under conjugations. This implies that conjugations are equicontinuous at e, as in Section 3.11.

Suppose now that G is a topological group, and let $\mathcal{B}_0$ be a local base for the topology of G at e. If U is an open subset of G that contains e, then let $U_L, U_R \subseteq G \times G$ be as in Section 3.2. One can check that

$$\mathcal{B}_{0,L} = \{U_L : U \in \mathcal{B}_0\} \tag{3.14.3}$$

and

$$\mathcal{B}_{0,R} = \{U_R : U \in \mathcal{B}_0\} \tag{3.14.4}$$

are bases for the uniformities $\mathcal{U}_L$, $\mathcal{U}_R$ defined in Section 3.2, respectively.

Let $\mathcal{B}_1$ be the collection of open subsets of G that contain e and are invariant under conjugations. This is a local base for the topology of G at e if and only if conjugations on G are equicontinuous at e, as in Section 3.11.

Equivalently, $\mathcal{B}_1$ consists of the open subsets U of G that contain e and satisfy

$$a\,U = U\,a \tag{3.14.5}$$

for every $a \in G$. Observe that $U \in \mathcal{B}_1$ if and only if $U^{-1} \in \mathcal{B}_1$.

If $U \in \mathcal{B}_1$, then it is easy to see that

$$(U^{-1})_L = U_R. \tag{3.14.6}$$

If $\mathcal{B}_1$ is a local base for the topology of G at e, then one can use this to get that

$$\mathcal{U}_L = \mathcal{U}_R. \tag{3.14.7}$$

More precisely, if $\mathcal{B}_{1,L}$, $\mathcal{B}_{1,R}$ are as in (3.14.3), (3.14.4), respectively, then

$$\mathcal{B}_{1,L} = \mathcal{B}_{1,R}. \tag{3.14.8}$$

Conversely, suppose that (3.14.7) holds, and let us check that conjugations are equicontinuous at e. Let V be an open subset of G that contains e, so that V^{-1} has the same properties, and thus $(V^{-1})_R \in \mathcal{U}_R$. Using (3.14.7), we get that $(V^{-1})_R \in \mathcal{U}_L$, so that there is an open subset U of G that contains e and satisfies

$$U_L \subseteq (V^{-1})_R. \tag{3.14.9}$$

If $x \in G$, then it follows that

$$x\,U = U_L[x] \subseteq (V^{-1})_R[x] = V\,x, \tag{3.14.10}$$

where the two equalities are as in Section 3.2. This is the same as saying that

$$x\,U\,x^{-1} \subseteq V \tag{3.14.11}$$

for every $x \in G$, as desired.

3.15 Cauchy sequences in topological groups

Let G be a topological group, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of G. Let us say that $\{x_j\}_{j=1}^\infty$ is a *left-invariant Cauchy sequence* if for every open subset U of G that contains e, there is a positive integer L such that

$$x_j^{-1}\,x_l \in U \tag{3.15.1}$$

for every $j, l \geq L$. Equivalently, this means that

$$x_l \in x_j\,U \tag{3.15.2}$$

for every $j, l \geq L$. Similarly, $\{x_j\}_{j=1}^\infty$ is said to be a *right-invariant Cauchy sequence* if for every open subset U of G that contains e there is a positive integer L such that

$$x_j\,x_l^{-1} \in U \tag{3.15.3}$$

for every $j, l \geq L$. This is the same as saying that

$$x_j \in U\,x_l \tag{3.15.4}$$

for every $j, l \geq L$.

Left and right-invariant Cauchy sequences are the same as Cauchy sequences with respect to the uniformities $\mathcal{U}_L$ and $\mathcal{U}_R$ defined in Section 3.2, respectively.

If the topology on G is determined by a collection $\mathcal{M}_L$ of left-invariant semimetrics, then $\{x_j\}_{j=1}^{\infty}$ is a left-invariant Cauchy sequence if and only if $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to $\mathcal{M}_L$, as in Section 2.15. Similarly, if the topology on G is determined by a collection $\mathcal{M}_R$ of right-invariant semimetrics, then $\{x_j\}_{j=1}^{\infty}$ is a right-invariant Cauchy sequence if and only if $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to $\mathcal{M}_R$. Of course, there are analogous conditions and statements for nets.

If $\{x_j\}_{j=1}^{\infty}$ converges to an element of G, then one can check that $\{x_j\}_{j=1}^{\infty}$ is both a left and right-invariant Cauchy sequence. It is easy to see that $\{x_j\}_{j=1}^{\infty}$ is a left-invariant Cauchy sequence if and only if $\{x_j^{-1}\}_{j=1}^{\infty}$ is a right-invariant Cauchy sequence. If G is commutative, then left and right-invariant Cauchy sequences are the same. More precisely, this also works when conjugations are equicontinuous at e.

If $\{x_j\}_{j=1}^{\infty}$ is a left-invariant Cauchy sequence, then

$$\{x_j : j \geq 1\} \tag{3.15.5}$$

is left-invariant totally bounded. Similarly, if $\{x_j\}_{j=1}^{\infty}$ is a right-invariant Cauchy sequence, then (3.15.5) is right-invariant totally bounded.

Let Y be a set with a nonempty collection $\mathcal{M}_Y$ of semimetrics, and let ϕ be a mapping from G into Y. If $\{x_j\}_{j=1}^{\infty}$ is a left-invariant Cauchy sequence, and ϕ is left-invariant uniformly continuous, then $\{\phi(x_j)\}_{j=1}^{\infty}$ is a Cauchy sequence in Y with respect to $\mathcal{M}_Y$. Similarly, if $\{x_j\}_{j=1}^{\infty}$ is a right-invariant Cauchy sequence, and ϕ is right-invariant uniformly continuous, then $\{\phi(x_j)\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to $\mathcal{M}_Y$. A left-invariant uniformly continuous mapping between topological groups sends left-invariant Cauchy sequences to sequences with the same property, and similarly for right-invariant uniformly continuous mappings. In particular, continuous group homomorphisms send left and right-invariant Cauchy sequences to sequences with the same property.

Chapter 4

Absolute values and matrices

4.1 Some norms on k^n

Let k be a field, and let $|\cdot|$ be an absolute value function on k. Also let n be a positive integer, and let k^n be the space of n-tuples of elements of k, as usual. This is a vector space over k, with respect to coordinatewise addition and scalar multiplication.

If $v = (v_1, \ldots, v_n) \in k^n$, then put

$$\|v\|_1 = \sum_{j=1}^{n} |v_j|, \tag{4.1.1}$$

$$\|v\|_2 = \Big(\sum_{j=1}^{n} |v_j|^2\Big)^{1/2}, \tag{4.1.2}$$

$$\|v\|_\infty = \max_{1 \le j \le n} |v_j|. \tag{4.1.3}$$

One can check that these define norms on k^n with respect to $|\cdot|$ on k. More precisely, these norms correspond to some of the examples in Section 1.10, with $X = \{1, \ldots, n\}$. If $|\cdot|$ is an ultrametric absolute value function on k, then (4.1.3) is an ultranorm on k^n, as before.

It is easy to see that

$$\|v\|_\infty \le \|v\|_1,\ \|v\|_2 \tag{4.1.4}$$

for every $v \in k^n$. We also have that

$$\|v\|_2 \le \|v\|_1 \tag{4.1.5}$$

for every $v \in k^n$, as in Section 1.10. Observe that

$$\|v\|_1 \le n\,\|v\|_\infty \tag{4.1.6}$$

and

(4.1.7) $$\|v\|_2 \le n^{1/2}\,\|v\|_\infty$$

for every $v \in k^n$. One can use the Cauchy–Schwarz inequality to get that

(4.1.8) $$\|v\|_1 \le n^{1/2}\,\|v\|_2$$

for every $v \in k^n$, as before.

If $v, w \in k^n$, then put

(4.1.9) $$d_1(v,w) = \|v-w\|_1,$$

(4.1.10) $$d_2(v,w) = \|v-w\|_2,$$

(4.1.11) $$d_\infty(v,w) = \|v-w\|_\infty,$$

which define metrics on k^n, as in Section 1.9. Note that (4.1.11) is an ultrametric on k^n when $|\cdot|$ is an ultrametric absolute value function on k. As in the preceding paragraph, we have that

(4.1.12) $$d_\infty(v,w) \le d_2(v,w) \le d_1(v,w)$$

for every $v, w \in k^n$. We also get that

(4.1.13) $$d_1(v,w) \le n^{1/2}\,d_2(v,w) \le n\,d_\infty(v,w)$$

for every $v, w \in k^n$.

In particular, these three metrics determine the same topology on k^n. Of course, k^n is the same as the Cartesian product of n copies of k. These three metrics on k^n are the same as those considered in Section 1.5, using the metric associated to $|\cdot|$ on each copy of k. The topology determined on k^n by these metrics is the same as the product topology on k^n, using the topology determined on k by the metric associated to $|\cdot|$ on each factor, as before. These metrics also determine the same uniform structure on k^n, and the same collection of bounded subsets of k^n.

4.2 Absolute values and metrics

Let k be a field with an absolute value function $|\cdot|$, and let $d(x,y) = |x-y|$ be the corresponding metric on k. We can also define metrics on $k^2 = k \times k$ as in the previous section. It is easy to see that addition on k is continuous as a mapping from $k \times k$ into k, with respect to the topology determined on k by $d(\cdot,\cdot)$, and the corresponding product topology on $k \times k$. More precisely, addition on k is uniformly continuous as a mapping from $k \times k$ into k, with respect to $d(\cdot,\cdot)$ on k, and any of the metrics on $k \times k$ considered in the previous section.

One can check that multiplication on k is continuous as a mapping from $k \times k$ into k as well, using standard arguments. In fact, the restriction of this mapping to any bounded subset of $k \times k$ is uniformly continuous.

One can also verify that $x \mapsto 1/x$ is continuous as a mapping from $k \setminus \{0\}$ into itself, with respect to the topology induced by the topology determined on k by $d(\cdot, \cdot)$. If r is a positive real number, then the restriction of this mapping to the set of $x \in k$ with $|x| \geq r$ is uniformly continuous. It follows that k is a topological group with respect to addition and the topology determined on k by $d(\cdot, \cdot)$, and that $k \setminus \{0\}$ is a topological group with respect to multiplication and the induced topology.

Of course, $d(x, y)$ is invariant under translations on k. Observe that

$$d(a\,x, a\,y) = |a\,x - a\,y| = |a|\,|x - y| = |a|\,d(x, y) \tag{4.2.1}$$

for every $a, x, y \in k$.

Clearly

$$\{x \in k : |x| = 1\} \tag{4.2.2}$$

is a subgroup of $k \setminus \{0\}$, as a group with respect to multiplication. Using (4.2.1), we get that the restriction of $d(\cdot, \cdot)$ to (4.2.2) is invariant under translations, as a group with respect to multiplication. Note that (4.2.2) is a closed set in k, with respect to the topology determined by $d(\cdot, \cdot)$. If $|\cdot|$ is an ultrametric absolute value function on k, then (4.2.2) is an open set in k.

The set $\mathbf{R}_+$ of positive real numbers is an open subset of the real line with respect to the standard topology, and a subgroup of $\mathbf{R} \setminus \{0\}$, as a group with respect to multiplication. The exponential function defines a homeomorphism from $\mathbf{R}$ onto $\mathbf{R}_+$, with respect to the standard topology on $\mathbf{R}$ and the induced topology on $\mathbf{R}_+$. The exponential function is also a group isomorphism from $\mathbf{R}$, as a group with respect to addition, onto $\mathbf{R}_+$, as a group with respect to multiplication. The logarithm is the inverse mapping, and

$$|\log x - \log y| \tag{4.2.3}$$

defines a metric on $\mathbf{R}_+$ that is invariant under translations, with respect to multiplication on $\mathbf{R}_+$. This corresponds to the standard Euclidean metric on $\mathbf{R}$, using the isomorphism between $\mathbf{R}$ and $\mathbf{R}_+$ given by the exponential function, and the topology determined on $\mathbf{R}_+$ by (4.2.3) is the same as the topology induced by the standard topology on $\mathbf{R}$.

If $k = \mathbf{R}$ with the standard absolute value function, then (4.2.2) is the subgroup $\{1, -1\}$ of $\mathbf{R} \setminus \{0\}$, as a group with respect to multiplication. There is an obvious group isomorphism from $\{1, -1\} \times \mathbf{R}_+$ onto $\mathbf{R} \setminus \{0\}$, which is defined by

$$(a, b) \mapsto a\,b \tag{4.2.4}$$

for $a \in \{1, -1\}$ and $b \in \mathbf{R}_+$, so that $a\,b \in \mathbf{R} \setminus \{0\}$. The topology induced on $\{1, -1\}$ by the standard topology on $\mathbf{R}$ is the discrete topology, and the isomorphism from $\{1, -1\} \times \mathbf{R}_+$ onto $\mathbf{R} \setminus \{0\}$ defined by (4.2.4) is a homeomorphism with respect to the corresponding product topology. One can use this and (4.2.3) to get a metric on $\mathbf{R} \setminus \{0\}$ that is invariant under translations, with respect to multiplication, and for which the corresponding topology on $\mathbf{R} \setminus \{0\}$ is the same as the topology induced by the standard topology on $\mathbf{R}$.

If $k = \mathbf{C}$ with the standard absolute value function, then (4.2.2) is the same as the unit circle

$$\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}. \tag{4.2.5}$$

There is an obvious group isomorphism from $\mathbf{T} \times \mathbf{R}_+$ onto $\mathbf{C} \setminus \{0\}$, defined by (4.2.4) again, now with $a \in \mathbf{T}$, $b \in \mathbf{R}_+$, and $a\, b \in \mathbf{C} \setminus \{0\}$. This isomorphism is a homeomorphism, with respect to the topology determined on $\mathbf{T}$ by the standard topology on $\mathbf{C}$, and the corresponding product topology on $\mathbf{T} \times \mathbf{R}_+$. One can use this and the previous translation-invariant metrics on $\mathbf{T}$ and $\mathbf{R}_+$ to get a translation-invariant metric on $\mathbf{C} \setminus \{0\}$, as a group with respect to multiplication. More precisely, one can get a translation-invariant metric on $\mathbf{C} \setminus \{0\}$ in this way for which the corresponding topology is the same as the topology induced by the standard topology on $\mathbf{C}$.

4.3 Absolute values and ultrametrics

Let k be a field with an ultrametric absolute value function $|\cdot|$. If $u, v \in k$ satisfy

$$|u - v| < |v|, \tag{4.3.1}$$

then

$$|u| = |v|. \tag{4.3.2}$$

Indeed,

$$|u| \le \max(|v|, |u - v|) \le |v| \tag{4.3.3}$$

when $|u - v| \le |v|$. We also have that

$$|v| \le \max(|u|, |u - v|), \tag{4.3.4}$$

which implies that $|v| \le |u|$ when (4.3.1) holds.

If $x, y \in k \setminus \{0\}$, then put

$$\begin{aligned} \delta(x, y) &= |x|^{-1}\,|x - y| = |y|^{-1}\,|x - y| \quad \text{when } |x| = |y| \\ &= 1 \quad \text{when } |x| \ne |y|. \end{aligned} \tag{4.3.5}$$

Of course,

$$|x - y| \le \max(|x|, |y|) \tag{4.3.6}$$

for every $x, y \in k$. This implies that

$$\delta(x, y) \le 1 \tag{4.3.7}$$

when $|x| = |y|$, and thus for all $x, y \in k \setminus \{0\}$. If

$$|x - y| < \max(|x|, |y|), \tag{4.3.8}$$

then $|x| = |y|$, as in (4.3.2). It follows that

$$\delta(x, y) < 1 \tag{4.3.9}$$

if and only if (4.3.8) holds.

It is easy to see that (4.3.5) is symmetric in x and y, and is equal to 0 exactly when $x = y$. Let $x, y, z \in k \setminus \{0\}$ be given, and let us verify that

$$\delta(x, z) \le \max(\delta(x, y), \delta(y, z)). \tag{4.3.10}$$

This holds automatically when the right side is equal to one, because of (4.3.7). Otherwise, if the right side is less than one, then $|x| = |y| = |z|$, by definition of $\delta(\cdot, \cdot)$. In this case, (4.3.10) follows from the ultrametric version of the triangle inequality for $|\cdot|$ on k.

If $a, x, y \in k \setminus \{0\}$, then

$$\delta(a\,x, a\,y) = \delta(x, y), \tag{4.3.11}$$

by construction. This means that $\delta(\cdot, \cdot)$ is invariant under translations on $k \setminus \{0\}$, as a group with respect to multiplication. Note that $\delta(\cdot, \cdot)$ is the same as $d(\cdot, \cdot)$ on (4.2.2).

Let $d(y, z) = |y - z|$ be the usual metric on k associated to $|\cdot|$, and let $B_d(x, r)$, $\overline{B}_d(x, r)$ be the open and closed balls in k centered at $x \in k$ with radius $r > 0$ with respect to $d(\cdot, \cdot)$. Suppose that $x \in k \setminus \{0\}$, and let $B_\delta(x, r)$, $\overline{B}_\delta(x, r)$ be the open and closed balls of radius $r > 0$ centered at x in $k \setminus \{0\}$ with respect to $\delta(\cdot, \cdot)$. Observe that

$$B_\delta(x, r) = B_d(x, r\,|x|) \tag{4.3.12}$$

when $0 < r \le 1$, and that

$$\overline{B}_\delta(x, r) = \overline{B}_d(x, r\,|x|) \tag{4.3.13}$$

when $0 < r < 1$. In particular, this implies that the topology determined on $k \setminus \{0\}$ by $\delta(\cdot, \cdot)$ is the same as the one induced by the topology determined on k by $d(\cdot, \cdot)$.

4.4 Total boundedness in $k \setminus \{0\}$

Let k be a field with an absolute value function $|\cdot|$, and let $d(x, y) = |x - y|$ be the corresponding metric on k. If $t \in k$ and $E \subseteq k$, then put

$$t\,E = \{t\,x : x \in E\}. \tag{4.4.1}$$

Let $x \in k$ and $r > 0$ be given, and let $B_d(x, r)$, $\overline{B}_d(x, r)$ be the open and closed balls in k centered at x with radius r with respect to $d(\cdot, \cdot)$, as in the previous section. If $t \in k$ and $t \ne 0$, then

$$t\,B_d(x, r) = B_d(t\,x, |t|\,r) \tag{4.4.2}$$

and

$$t\,\overline{B}_d(x, r) = \overline{B}_d(t\,x, |t|\,r). \tag{4.4.3}$$

Let $x \in k \setminus \{0\}$ be given. If $y \in k$ and $d(x, y) \le r < |x|$, then it is easy to see that

$$|x| - r \le |y| \le |x| + r. \tag{4.4.4}$$

Note that

$$x\, B_d(e, r) = B_d(x, r\,|x|) \tag{4.4.5}$$

and

$$x\, \overline{B}_d(e, r) = \overline{B}_d(x, r\,|x|) \tag{4.4.6}$$

for every $r > 0$, as in (4.4.2), (4.4.3). If $0 < r < 1$ and y is an element of (4.4.6), then

$$(1 - r)\,|x| \le |y| \le (1 + r)\,|x|, \tag{4.4.7}$$

by (4.4.4).

Remember that $k \setminus \{0\}$ is a topological group with respect to multiplication, and the topology induced by the topology determined on k by $d(\cdot, \cdot)$. Of course, left and right-invariant total boundedness in $k \setminus \{0\}$ are the same, as in Section 3.8. It is easy to see that $E \subseteq k \setminus \{0\}$ is totally bounded in $k \setminus \{0\}$ as a topological group with respect to multiplication if and only if for every $r > 0$ there are finitely many elements $x_1, \ldots, x_n$ of $k \setminus \{0\}$ such that

$$E \subseteq \bigcup_{j=1}^{n} x_j\, B_d(e, r). \tag{4.4.8}$$

Equivalently, this means that

$$E \subseteq \bigcup_{j=1}^{n} B_d(x_j, r\,|x_j|), \tag{4.4.9}$$

by (4.4.6). In particular, we can take $r = 1/2$ to get that

$$\{|x| : x \in E\} \tag{4.4.10}$$

has an upper bound in $\mathbf{R}$ and a positive lower bound, because of (4.4.7).

In (4.4.8) and (4.4.9), we may as well ask that

$$(x_j\, B_d(e, r)) \cap E = B_d(x_j, r\,|x_j|) \cap E \ne \emptyset, \tag{4.4.11}$$

because otherwise the set corresponding to x_j is not needed to cover E. We may also restrict our attention to $r \le 1/2$. Under these conditions, the upper and lower bounds for (4.4.10) mentioned in the preceding paragraph lead to an upper bound and positive lower bound for $|x_j|$, $1 \le j \le n$, depending only on E, because of (4.4.7).

Left and right-invariant total boundedness in k, as a topological group with respect to addition, are the same too, and they are the same as total boundedness with respect to $d(\cdot, \cdot)$, as in Section 3.8. If $E \subseteq k \setminus \{0\}$ is totally bounded in $k \setminus \{0\}$, as a topological group with respect to multiplication, then one can use the upper bound for $|x_j|$ mentioned in the previous paragraph to get that E is totally bounded with respect to $d(\cdot, \cdot)$.

If $E \subseteq k$ is totally bounded with respect to k, then for every $r > 0$ there are finitely many elements $x_1, \ldots, x_n$ of k such that

$$E \subseteq \bigcup_{j=1}^{n} B_d(x_j, r), \tag{4.4.12}$$

and we may as well ask that $B_d(x_j, r) \cap E \neq \emptyset$ for each $j = 1, \ldots, n$, as before. If $E \subseteq k \setminus \{0\}$, and there is a positive lower bound for (4.4.10), then we can get a positive lower bound for $|x_j|$ when r is sufficiently small, using (4.4.4). Under these conditions, one can check that E is totally bounded in $k \setminus \{0\}$, as a topological group with respect to multiplication.

Suppose now that $|\cdot|$ is an ultrametric absolute value function on k, and let $\delta(\cdot, \cdot)$ be the ultrametric defined on $k \setminus \{0\}$ in (4.3.5). Note that $E \subseteq k \setminus \{0\}$ is totally bounded in $k \setminus \{0\}$ as a topological group with respect to multiplication if and only if E is totally bounded with respect to $\delta(\cdot, \cdot)$, as in Section 3.8. In this case, one can use (4.4.8) or (4.4.9) with $r = 1$ to get that (4.4.10) has only finitely many elements.

4.5 Discrete absolute value functions

Let k be a field, and let $|\cdot|$ be an absolute value function on k. Observe that

$$\{|x| : x \in k \setminus \{0\}\} \tag{4.5.1}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. If 1 is not a limit point of (4.5.1), with respect to the standard topology on $\mathbf{R}$, then $|\cdot|$ is said to be *discrete* as an absolute value function on k.

Put

$$\rho_1 = \sup\{|x| : x \in k,\ |x| < 1\}, \tag{4.5.2}$$

so that $0 \leq \rho_1 \leq 1$. It is easy to see that $\rho_1 = 0$ if and only if $|\cdot|$ is the trivial absolute value function on k. One can also check that $\rho_1 < 1$ if and only if $|\cdot|$ is discrete on k.

Suppose that $|\cdot|$ is nontrivial and discrete on k, so that $0 < \rho_1 < 1$. One can verify that the supremum in (4.5.2) is attained in this case, so that there is an $x_1 \in k$ such that

$$|x_1| = \rho_1. \tag{4.5.3}$$

In fact, (4.5.1) consists exactly of the integer powers of ρ_1 under these conditions.

If k has positive characteristic, then $|\cdot|$ is non-archimedean on k, and thus an ultrametric absolute value function. If $|\cdot|$ is archimedean on k, then it follows that k has characteristic 0, so that there is a natural embedding of $\mathbf{Q}$ into k. This leads to an absolute value function on $\mathbf{Q}$, which is archimedean as well.

Ostrowski's theorem implies that an archimedean absolute value function on $\mathbf{Q}$ is equivalent to the standard Euclidean absolute value function. In particular, this means that an archimedean absolute value function on $\mathbf{Q}$ is not discrete. Combining this with the remarks in the preceding paragraph, we get that any archimdean absolute value function on k is not discrete. Equivalently, if $|\cdot|$ is a discrete absolute value function on k, then $|\cdot|$ is non-archimedean on k.

Suppose that $|\cdot|$ is nontrivial and discrete on k again, so that $|\cdot|$ is an ultrametric absolute value function on k, as in the previous paragraph. Remember that

$$\{x \in k : |x| = 1\} \tag{4.5.4}$$

is a subgroup of $k \setminus \{0\}$, as a group with respect to multiplication. If $x_1 \in k$ is as in (4.5.3), then every element of $k \setminus \{0\}$ can be expressed in a unique way as

$$x\, x_1^j, \tag{4.5.5}$$

where $x \in k$, $|x| = 1$, and $j \in \mathbf{Z}$. This leads to a group isomorphism between $k \setminus \{0\}$ and the product of (4.5.4) and $\mathbf{Z}$, as a group with respect to addition.

Of course, (4.5.4) is a topological group, with respect to the topology induced by the topology determined on k by the ultrametric associated to $|\cdot|$. We also have that (4.5.4) is an open subset of k, because $|\cdot|$ is an ultrametric absolute value function on k. We may consider $\mathbf{Z}$ as a topological group too, using the discrete topology. The group isomorphism between $k \setminus \{0\}$ and the product of (4.5.4) and $\mathbf{Z}$ mentioned in the previous paragraph is a homeomorphism, with respect to the topology induced on $k \setminus \{0\}$ by the topology determined on k by the ultrametric associated to $|\cdot|$, and the product topology on the product of (4.5.4) and $\mathbf{Z}$.

4.6 Local total boundedness

Let X be a set with a uniformity $\mathcal{U}$. We say that X is *locally totally bounded* with respect to $\mathcal{U}$ if for every $x \in X$ there is an open set U with respect to the topology determined by $\mathcal{U}$ such that $x \in U$ and $\mathcal{U}$ is totally bounded with respect to $\mathcal{U}$. If X is locally compact with respect to the topology determined by $\mathcal{U}$, then X is locally totally bounded with respect to $\mathcal{U}$, because compact subsets of X are totally bounded.

Let G be a topological group, and note that G is locally compact when there is an open subset of G that contains e and is contained in a compact set, because of continuity of translations. Let us say that G is *locally totally bounded* if there is an open subset U of G such that $e \in U$ and U is either left or right-invariant totally bounded in G. We may as well ask that U be symmetric, by replacing it with $U \cap U^{-1}$. In this case, left and right-invariant total boundedness of U are the same.

Let $\mathcal{U}_L$, $\mathcal{U}_R$ be the uniformities on G defined in Section 3.2, and remember that left and right-invariant total boundedness are equivalent to total boundedness with respect to $\mathcal{U}_L$ and $\mathcal{U}_R$, respectively, as in Section 3.8. If G is locally totally bounded with respect to $\mathcal{U}_L$ or $\mathcal{U}_R$, then it is easy to see that G is locally totally bounded as a topological group, as in the preceding paragraph.

Conversely, suppose that G is locally totally bounded as a topological group, and let U be a symmetric open subset of G that contains e and is left and right-invariant totally bounded. Note that $a\,U\,a^{-1}$ has the same properties for every $a \in G$, because conjugation by a is continuous. One can check that $a\,U\,b$ is left and right-invariant totally bounded for every $a, b \in G$, which implies in particular that G is locally totally bounded with respect to both $\mathcal{U}_L$ and $\mathcal{U}_R$.

In this case, if a subset E of G is left or right-invariant totally bounded, then E is contained in the union of finitely many left or right translates of U. It follows that E is both left and right-invariant totally bounded, because left and right translates of U are both left and right-invariant totally bounded.

Let k be a field with an absolute value function $|\cdot|$. Let us refer to a subset E of k as being *totally bounded* if E is totally bounded with respect to the metric associated to $|\cdot|$, which is the same as saying that E is totally bounded in k as a topological group with respect to addition and the topology determined by the metric associated to $|\cdot|$. In this case, it is easy to see that $t\,E$ is totally bounded in k for every $t \in k$.

Let us say that k is *locally totally bounded* with respect to $|\cdot|$ if k is locally totally bounded with respect to the metric associated to $|\cdot|$, which is the same as saying that k is locally totally bounded as a topological group with respect to addition. Equivalently, this means that there is a positive real number r such that the open ball in k centered at 0 with radius r with respect to the metric associated to $|\cdot|$ is totally bounded. Note that this holds when $|\cdot|$ is the trivial absolute value function on k.

If $|\cdot|$ is nontrivial on k, then there is an $x \in k$ such that $x \neq 0$ and $|x| \neq 1$. This implies that there are $y, z \in k$ such that $0 < |y| < 1$ and $|z| > 1$, using x and $1/x$. Of course, it follows that

$$|y^j| = |y|^j \to 0 \quad \text{and} \quad |z^j| = |z|^j \to +\infty \quad \text{as } j \to \infty. \tag{4.6.1}$$

If k is locally totally bounded, then one can use this to get that there are open balls in k centered at 0 of arbitrarily large radius that are totally bounded. This means that all bounded subsets of k are totally bounded.

Similarly, if k is locally compact, and $|\cdot|$ is nontrivial on k, then closed and bounded sets in k are compact. One can check that k is complete with respect to the metric associated to $|\cdot|$ in this case. If $|\cdot|$ is the trivial absolute value function on k, then k is locally compact and complete.

If k is complete with respect to the metric associated to $|\cdot|$, then subsets of k that are both closed and totally bounded are compact. If k is also locally totally bounded, and $|\cdot|$ is nontrivial on k, then it follows that closed and bounded subsets of k are compact.

4.7 Some subgroups of k

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let $B(x, r)$, $\overline{B}(x, r)$ be the open and closed balls in k centered at $x \in k$ with radius $r > 0$ with respect to the ultrametric associated to $|\cdot$. It is easy to see that open and closed balls centered at 0 are subgroups of k, as a group with respect to addition, as in Section 3.7.

In fact, $\overline{B}(0,1)$ is a subring of k that contains the multiplicative identity element, and $B(0,r)$, $\overline{B}(0,r)$ are ideals in $\overline{B}(0,1)$ when $0 < r \le 1$. One can check that the quotient ring

$$\overline{B}(0,1)/B(0,1) \tag{4.7.1}$$

is a field, which is called the *residue field* associated to $|\cdot|$ on k. If $|\cdot|$ is the trivial absolute value function on k, then $\overline{B}(0,1) = k$, $B(0,1) = \{0\}$, and the residue field reduces to k.

If $\overline{B}(0,1)$ is totally bounded, then one can verify that the residue field is finite. If $B(0,1)$ is totally bounded, then $|\cdot|$ is discrete on k. More precisely, this holds when $B(0,1)$ is contained in the union of finitely many balls of radius strictly less than one.

Suppose now that the residue field (4.7.1) is finite, and that $|\cdot|$ is nontrivial and discrete on k. Let ρ_1 be as in (4.5.2), so that $0 < \rho_1 < 1$, and the positive values of $|\cdot|$ on k are the same as the integer powers of ρ_1. In particular, $B(0,1) = \overline{B}(0,\rho_1)$, so that $\overline{B}(0,1)$ can be expressed as the union of finitely many pairwise-disjoint closed balls of radius ρ_1.

If j is any integer, then one can use translations and dilations to get that any closed ball in k of radius ρ_1^j can be expressed as the union of finitely many closed balls of radius ρ_1^{j+1}. If l is any positive integer, then one can repeat the process to get that closed balls in k of radius ρ_1^j can be expressed as the union of finitely many closed balls of radius ρ_1^{j+l}. This means that closed balls in k are totally bounded.

4.8 p-Adic integers

Let k be a field, and let x be an element of k. If n is a nonnegative integer, then it is well known and easy to see that

$$(1-x)\sum_{j=0}^{n} x^j = 1 - x^{n+1}, \tag{4.8.1}$$

where x^j is interpreted as being equal to 1 when $j = 0$. If $x \ne 1$, then it follows that

$$\sum_{j=0}^{n} x^j = (1 - x^{n+1})\,(1-x)^{-1}. \tag{4.8.2}$$

Let $|\cdot|$ be an absolute value function on k. If $|x| < 1$, then $|x^{n+1}| = |x|^{n+1}$ tends to 0 as $n \to \infty$, and thus

$$\sum_{j=0}^{n} x^j \to (1-x)^{-1} \quad \text{as } n \to \infty, \tag{4.8.3}$$

with respect to the metric on k associated to $|\cdot|$.

Let p be a prime number, and let $|\cdot|_p$ be the p-adic absolute value on $\mathbf{Q}_p$. If $x \in \mathbf{Q}$ and $|x|_p \le 1$, then x can be expressed as a/b, where $a, b \in \mathbf{Z}$, $b \ne 0$, and b is not a multiple of p. It follows that there is a $c \in \mathbf{Z}$ such that $b\,c = 1 - p\,y$ for some $y \in \mathbf{Z}$. The argument in the preceding paragraph implies that $x = a\,c\,(1 - p\,y)^{-1}$ can be approximated by integers with respect to the p-adic metric.

Put

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \le 1\}. \tag{4.8.4}$$

It is easy to see that $\mathbf{Q} \cap \mathbf{Z}_p$ is dense in $\mathbf{Z}_p$ with respect to the p-adic metric, because $\mathbf{Q}$ is dense in $\mathbf{Q}_p$, by construction. One can check that $\mathbf{Z}$ is dense in $\mathbf{Z}_p$, using the remarks in the previous paragraph. More precisely, $\mathbf{Z}_p$ is the same as the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$.

If $j \in \mathbf{Z}$, then $p^j\,\mathbf{Z}_p$ is the same as the closed ball in $\mathbf{Q}_p$ centered at 0 with radius p^{-j} with respect to the p-adic metric. This is a subgroup of $\mathbf{Q}_p$, as a group with respect to addition, as in the previous section. We also have that $\mathbf{Z}_p$ is a subring of $\mathbf{Q}_p$, and that $p^j\,\mathbf{Z}_p$ is an ideal in $\mathbf{Z}_p$ when $j \ge 0$, as before. Thus the quotient $\mathbf{Z}_p/(p^j\,\mathbf{Z}_p)$ is a commutative ring when $j \ge 0$.

There is a natural ring homomorphism from $\mathbf{Z}$ into $\mathbf{Z}_p/(p^j\,\mathbf{Z}_p)$ for each $j \ge 0$, obtained by composing the obvious inclusion of $\mathbf{Z}$ into $\mathbf{Z}_p$ with the quotient mapping from $\mathbf{Z}_p$ onto $\mathbf{Z}_p/(p^j\,\mathbf{Z}_p)$. This homomorphism maps $\mathbf{Z}$ onto $\mathbf{Z}_p/(p^j\,\mathbf{Z}_p)$, because $\mathbf{Z}$ is dense in $\mathbf{Z}_p$, as before. The kernel of this homomorphism is $\mathbf{Z} \cap (p^j\,\mathbf{Z}_p)$, which is the same as $p^j\,\mathbf{Z}$. This leads to a natural ring isomorphism from $\mathbf{Z}/(p^j\,\mathbf{Z})$ onto $\mathbf{Z}_p/(p^j\,\mathbf{Z}_p)$ for every $j \ge 0$.

In particular, this implies that $\mathbf{Z}_p$ is totally bounded. Of course, $\mathbf{Z}_p$ is a closed set in $\mathbf{Q}_p$, with respect to the p-adic metric. It follows that $\mathbf{Z}_p$ is compact, because $\mathbf{Q}_p$ is complete with respect to the p-adic metric.

4.9 Seminorms on k^n

Let k be a field with an absolute value function $|\cdot|$, and let n be a positive integer. Also let N be a seminorm on k^n, with respect to $|\cdot|$ on k. The standard basis vectors $e_1, \ldots, e_n$ in k^n can be defined as usual by taking the jth coordinate of e_l to be 1 when $j = l$, and 0 when $j \ne l$. If $v \in k^n$, then we get that

$$N(v) = N\Big(\sum_{l=1}^n v_l\, e_l\Big) \le \sum_{l=1}^n N(e_l)\,|v_l|. \tag{4.9.1}$$

Let $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ be the norms on k^n defined in Section 4.1. If $v \in k^n$, then it is easy to see that

$$N(v) \le \Big(\max_{1 \le l \le n} N(e_l)\Big)\,\|v\|_1 \tag{4.9.2}$$

and

$$N(v) \le \Big(\sum_{l=1}^n N(e_l)\Big)\,\|v\|_\infty, \tag{4.9.3}$$

using (4.9.1). Similarly,

$$N(v) \le \Big(\sum_{l=1}^{n} N(e_l)^2\Big)^{1/2} \|v\|_2, \tag{4.9.4}$$

by the Cauchy–Schwarz inequality.

If N is a semi-ultranorm on k^n, then

$$N(v) \le \max_{1 \le l \le n} (N(e_l)\,|v_l|) \tag{4.9.5}$$

for every $v \in k^n$. This implies that

$$N(v) \le \Big(\max_{1 \le l \le n} N(e_l)\Big) \|v\|_\infty \tag{4.9.6}$$

for every $v \in k^n$.

Suppose for the moment that there is a positive real number c such that

$$c\,\|v\|_\infty \le N(v) \tag{4.9.7}$$

for every $v \in k^n$. Of course, this implies that N is a norm on k^n. In this case, it is easy to see that the topology determined on k^n by the metric associated to N is the same as the topology determined by the metric associated to $\|\cdot\|_\infty$. This is the same as the product topology on k^n, using the topology determined on k by the metric associated to $|\cdot|$, as before.

Note that a sequence of elements of k^n corresponds to n sequences of elements of k, by taking the coordinates of the terms of the sequence in k^n. One can check that a sequence of elements of k^n is a Cauchy sequence with respect to the metric associated to $\|\cdot\|_\infty$ if and only if the corresponding n sequences in k are Cauchy sequences with respect to the metric associated to $|\cdot|$. If k is complete with respect to the metric associated to $|\cdot|$, then it follows that k^n is complete with respect to the metric associated to $\|\cdot\|_\infty$.

If (4.9.7) holds, then a Cauchy sequence of elements of k^n with respect to the metric associated to N is also a Cauchy sequence with respect to the metric associated to $\|\cdot\|_\infty$. If k is complete with respect to the metric associated to $|\cdot|$, then one can use this to get that k^n is complete with respect to the metric associated to N.

Let $(X, d(x,y))$ be a metric space, and let E be a subset of X. If E is complete as a metric space with respect to the restriction of $d(\cdot,\cdot)$ to E, then it is well known that E is a closed set in X, with respect to the topology determined by $d(\cdot,\cdot)$. To see this, let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of E that converges to an element x of X. Under these conditions, $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in E, which converges to an element of E, by hypothesis. This implies that $x \in E$, by the uniqueness of the limit of a convergent sequence in a metric space.

4.10 Norms and completeness

Let k be a field with an absolute value function $|\cdot|$ again, let n be a positive integer, and let N be a norm on k^n with respect to $|\cdot|$ on k. If k is complete with respect to the metric associated to $|\cdot|$, then it is well known that there is a positive real number c such that (4.9.7) holds for every $v \in k^n$.

To show this, we use induction on n. The $n = 1$ case is clear, because any norm on k with respect to $|\cdot|$ is a positive constant multiple of $|\cdot|$.

Suppose that $n \geq 2$, and consider

$$\{v \in k^n : v_n = 0\}, \tag{4.10.1}$$

which is a linear subspace of k^n. This can be identified with k^{n-1} in an obvious way, so that the restriction of N to (4.10.1) corresponds to a norm on k^{n-1}. Using our induction hypothesis, we get that there is a positive real number c_0 such that

$$c_0 \, \|v\|_\infty \leq N(v) \tag{4.10.2}$$

for every $v \in k^n$ with $v_n = 0$.

It follows from this and the completeness of k that (4.10.1) is complete as a metric space with respect to the metric associated to the restriction of N to (4.10.1), as in the previous section. This implies that (4.10.1) is a closed set in k^n with respect to the metric associated to N, as before.

Let e_n be the nth standard basis vector in k^n, with jth coordinate equal to 1 when $j = n$, and to 0 otherwise. Of course, e_n is not an element of (4.10.1). It follows that there is a positive real number c_1 such that

$$N(v - e_n) \geq c_1 \tag{4.10.3}$$

for every $v \in k^n$ with $v_n = 0$, because (4.10.1) is a closed set in k^n with respect to the metric associated to N.

Equivalently, (4.10.3) says that

$$N(v) \geq c_1 \, |v_n| \tag{4.10.4}$$

for every $v \in k^n$. More precisely, this is trivial when $v_n = 0$, and otherwise one can reduce to the case where $v_n = 1$ using the homogeneity of N on k^n.

If v is any element of k^n, then

$$c_0 \, \|v - v_n \, e_n\|_\infty \leq N(v - v_n \, e_n), \tag{4.10.5}$$

by (4.10.2). We also have that

$$N(v - v_n \, e_n) \leq N(v) + |v_n| \, N(e_n) \leq (1 + c_1 \, N(e_n)) \, N(v), \tag{4.10.6}$$

by (4.10.4). Thus

$$c_0 \, \|v - v_n \, e_n\|_\infty \leq (1 + c_1 \, N(e_n)) \, N(v). \tag{4.10.7}$$

Note that $\|v - v_n \, e_n\|_\infty = \max_{1 \leq j \leq n-1} |v_j|$. One can get (4.9.7) for a suitable $c > 0$ using (4.10.4) and (4.10.7).

If k is locally compact and $|\cdot|$ is nontrivial on k, then one can use another argument, as follows. In this case, closed and bounded subsets of k are compact, as in Section 4.6. This implies that closed and bounded subsets of k^n are compact, with respect to the metric associated to $\|\cdot\|_\infty$, by Tychonoff's theorem. In particular, the set of $v \in k^n$ with $\|v\|_\infty = 1$ is compact.

Note that N is continuous as a real-valued function on k^n with respect to the metric on k^n associated to N, as in Section 1.8. This implies that N is continuous as a real-valued function on k^n, with respect to the metric on k^n associated to $\|\cdot\|_\infty$, because of (4.9.3). It follows that N attains its minumum on the set of $v \in k^n$ with $\|v\|_\infty = 1$, because this set is compact with respect to the metric on k^n associated to $\|\cdot\|_\infty$. Of course, the minimum of N on this set is positive, because N is a norm on k^n, by hypothesis. It is easy to see that (4.9.7) holds with c equal to this minimum, using homogeneity of the norm.

4.11 Some remarks about matrices

Let n be a positive integer, and let R be a ring. The space $M_n(R)$ of $n \times n$ matrices with entries in R is a ring too, with respect to entrywise addition of matrices, and matrix multiplication.

Suppose that R has a multiplicative identity element e. The corresponding *identity matrix* $I = I_n$ in $M_n(R)$ is the matrix with diagonal entries equal to e, and all other entries equal to 0. This is the multiplicative identity element in $M_n(R)$.

Let $GL_n(R)$ be the set of elements of $M_n(R)$ with a multiplicative inverse in $M_n(R)$. Of course, this is a group with respect to matrix multiplication.

Suppose now that multiplication on R is commutative. If $a \in M_n(R)$, then the determinant $\det a$ of a can be defined as an element of R in the usual way. It is well known that a has a multiplicative inverse in $M_n(R)$ if and only if $\det a$ has a multiplicative inverse in R.

Let k be a field, and note that $M_n(k)$ is a vector space over k, with respect to entrywise addition and scalar multiplication. In fact, $M_n(k)$ is an associative algebra over k, with respect to matrix multiplication. In this case, $a \in M_n(k)$ has a multiplicative inverse in $M_n(k)$ if and only if $\det a \neq 0$.

Let $|\cdot|$ be an absolute value function on k, and let us now consider k to be equipped with the topology determined by the metric associated to $|\cdot|$. If we consider $M_n(k)$ to be the Cartesian product of n^2 copies of k, then we get a corresponding product topology on $M_n(k)$. One can check that matrix addition and multiplication are continuous on $M_n(k)$, as mappings from $M_n(k) \times M_n(k)$ into $M_n(k)$, and using the corresponding product topology on $M_n(k) \times M_n(k)$. One can also verify that the determinant defines a continuous function from $M_n(k)$ into k.

In particular, $GL_n(k)$ is an open subset of $M_n(k)$. One can check that $a \mapsto a^{-1}$ is continuous on $GL_n(k)$, with respect to the topology induced by the one on $M_n(k)$, using Cramer's rule. It follows that $GL_n(k)$ is a topological group with respect to this topology.

Suppose now that $|\cdot|$ is an ultrametric absolute value function on k. Thus the closed unit ball $\overline{B}(0,1) = \overline{B}_k(0,1)$ in k with respect to the ultrametric associated to $|\cdot|$ is both open and closed in k, and a subring of k that contains the multiplicative identity element, as before. This means that $M_n(\overline{B}(0,1))$ is both open and closed as a subset of $M_n(k)$, and that $M_n(\overline{B}(0,1))$ is a subring of $M_n(k)$ that contains the identity matrix I.

Observe that $x \in \overline{B}(0,1)$ has a multiplicative inverse in $\overline{B}(0,1)$ if and only if $|x| = 1$. This implies that

$$GL_n(\overline{B}(0,1)) = \{a \in M_n(\overline{B}(0,1)) : |\det a| = 1\}. \tag{4.11.1}$$

Remember that the set of $x \in k$ with $|x| = 1$ is both open and closed in k, because $|\cdot|$ is an ultrametric absolute value function on k. This implies that $GL_n(\overline{B}(0,1))$ is both open and closed in $M_n(k)$, because the determinant is continuous on $M_n(k)$.

4.12 Supremum and Lipschitz seminorms

Let k be a field with an absolute value function $|\cdot|$, let X be a nonempty set, and let V be a linear subspace of the space of all k-valued functions on X. Also let E be a nonempty subset of X, and suppose that the elements of V are bounded on E with respect to $|\cdot|$ on k. If $f \in V$, then put

$$\|f\|_{sup,E} = \sup_{x \in E} |f(x)|. \tag{4.12.1}$$

One can check that this defines a seminorm on V, as a vector space over k, with respect to $|\cdot|$ on k. This is called the *supremum seminorm* on V corresponding to E.

If $E = X$, then this defines a norm on V, which may be denoted $\|f\|_{sup}$. If $|\cdot|$ is an ultrametric absolute value function on k, then (4.12.1) is a semi-ultranorm on V.

Suppose that V is a subalgebra of the algebra of all k-valued functions on X, with respect to pointwise multiplication of functions. If $f, g \in V$, then we have that

$$\|f\,g\|_{sup,E} \le \|f\|_{sup,E}\,\|g\|_{sup,E}. \tag{4.12.2}$$

Let $d(\cdot,\cdot)$ be a semimetric on X, and let α be a positive real number. Remember that a k-valued function f on X is said to be *Lipschitz* of order α with constant $C \ge 0$ on X with respect to the metric associated to $|\cdot|$ on k if

$$|f(x) - f(w)| \le C\, d(x,w)^\alpha \tag{4.12.3}$$

for every $x, w \in X$, as in Section 1.8. It is easy to see that the space $\mathrm{Lip}_\alpha(X,k)$ of all k-valued functions on X that are Lipschitz of order α with some constant C is a linear subspace of the space of all k-valued functions on X.

If $f \in \mathrm{Lip}_\alpha(X,k)$, then we would like to put

$$\|f\|_{\mathrm{Lip}_\alpha(X,k)} = \sup\left\{\frac{|f(x) - f(w)|}{d(x,w)^\alpha} : x, w \in X,\ d(x,w) > 0\right\}. \tag{4.12.4}$$

Let us interpret this as being equal to 0 when $d(x, w) = 0$ for every x, w in X. Note that $f(x) = f(w)$ when $d(x, w) = 0$, by hypothesis. Equivalently, $\|f\|_{\mathrm{Lip}_\alpha(X,k)}$ is the smallest $C \geq 0$ such that (4.12.3) holds.

One can check that this defines a seminorm on $\mathrm{Lip}_a(X, k)$, as a vector space over k, with respect to $|\cdot|$ on k. More precisely, $\|f\|_{\mathrm{Lip}_a(X,k)} = 0$ exactly when f is constant on X. If $|\cdot|$ is an ultrametric absolute value function on k, then this defines a semi-ultranorm on $\mathrm{Lip}_a(X, k)$.

If f, g are k-valued functions on X, then

$$|f(x)\,g(x) - f(w)\,g(w)| \leq |f(x)|\,|g(x) - g(w)| + |f(x) - f(w)|\,|g(w)| \tag{4.12.5}$$

for every $x, w \in X$. If $|\cdot|$ is an ultrametric absolute value function on X, then

$$|f(x)\,g(x) - f(w)\,g(w)| \leq \max(|f(x)|\,|g(x) - g(w)|, |f(x) - f(w)|\,|g(w)|) \tag{4.12.6}$$

for every $x, w \in X$. If f, g are bounded on X, and $f, g \in \mathrm{Lip}_\alpha(X, k)$, then it is easy to see that $f\,g \in \mathrm{Lip}_\alpha(X, k)$, with

$$\|f\,g\|_{\mathrm{Lip}_\alpha(X,k)} \leq \|f\|_{sup}\,\|g\|_{\mathrm{Lip}_\alpha(X,k)} + \|f\|_{\mathrm{Lip}_\alpha(X,k)}\,\|g\|_{sup}. \tag{4.12.7}$$

If $|\cdot|$ is an ultrametric absolute value function on k, then we get that

$$\|f\,g\|_{\mathrm{Lip}_\alpha(X,k)} \leq \max(\|f\|_{sup}\,\|g\|_{\mathrm{Lip}_\alpha(X,k)}, \|f\|_{\mathrm{Lip}_\alpha(X,k)}\,\|g\|_{sup}). \tag{4.12.8}$$

4.13 Some norms on matrices

Let k be a field with an absolute value function $|\cdot|$, and let n be a positive integer. Remember that the space $M_n(k)$ of $n \times n$ matrices with entries in k is an associative algebra over k, with respect to matrix multiplication. If $a = (a_{j,l}) \in M_n(k)$, then put

$$N_{1,\infty}(a) = \max_{1 \leq l \leq n} \Big(\sum_{j=1}^{n} |a_{j,l}|\Big), \tag{4.13.1}$$

$$N_{\infty,1}(a) = \max_{1 \leq j \leq n} \Big(\sum_{l=1}^{n} |a_{j,l}|\Big), \tag{4.13.2}$$

$$N_{\infty,\infty}(a) = \max_{1 \leq j,l \leq n} |a_{j,l}|. \tag{4.13.3}$$

It is easy to see that these define norms on $M_n(k)$, as a vector space over k, and with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then (4.13.3) is an ultranorm on $M_n(k)$.

Let $a^t = (a^t_{j,l}) \in M_n(k)$ be the *transpose* of a, so that $a^t_{j,l} = a_{l,j}$ for $j, l = 1, \ldots, n$. Observe that

$$N_{1,\infty}(a^t) = N_{\infty,1}(a) \tag{4.13.4}$$

and

$$N_{\infty,\infty}(a^t) = N_{\infty,\infty}(a). \tag{4.13.5}$$

It is easy to see that

$$N_{\infty,\infty}(a) \le N_{1,\infty}(a),\ N_{\infty,1}(a) \tag{4.13.6}$$

and

$$N_{1,\infty}(a),\ N_{\infty,1}(a) \le n\, N_{\infty,\infty}(a). \tag{4.13.7}$$

We also have that

$$N_{1,\infty}(I) = N_{\infty,1}(I) = N_{\infty,\infty}(I) = 1, \tag{4.13.8}$$

where I is the identity matrix in $M_n(k)$, as before. Note that the topologies determined on $M_n(k)$ by the metrics associated to these norms are the same as the product topology corresponding to the topology determined on k by the metric associated to $|\cdot|$, where $M_n(k)$ is considered as the Cartesian product of n^2 copies of k.

Let $b = (b_{j,l})$ be another element of $M_n(k)$, and let $c = (c_{j,l}) \in M_n(k)$ be the product of a and b, so that

$$c_{j,r} = \sum_{l=1}^{n} a_{j,l}\, b_{l,r} \tag{4.13.9}$$

for $j, r = 1, \ldots, n$. Thus

$$|c_{j,r}| \le \sum_{l=1}^{n} |a_{j,l}|\, |b_{l,r}| \tag{4.13.10}$$

for $j, r = 1, \ldots, n$. Using this, one can check that

$$N_{1,\infty}(c) \le N_{1,\infty}(a)\, N_{1,\infty}(b). \tag{4.13.11}$$

Similarly, one can verify that

$$N_{\infty,1}(c) \le N_{\infty,1}(a)\, N_{\infty,1}(b). \tag{4.13.12}$$

Let us suppose from now on in thia section that $|\cdot|$ is an ultrametric absolute value function on k. In this case, we get that

$$|c_{j,r}| \le \max_{1 \le l \le n} (|a_{j,l}|\, |b_{l,r}|) \tag{4.13.13}$$

for $j, r = 1, \ldots, n$. This implies that

$$N_{\infty,\infty}(c) \le N_{\infty,\infty}(a)\, N_{\infty,\infty}(b). \tag{4.13.14}$$

Let $\overline{B}_k(0,1)$ be the closed unit ball in k with respect to the ultrametric associated to $|\cdot|$, as before. Thus $M_n(\overline{B}_k(0,1))$ is the same as the closed unit ball in $M_n(k)$ with respect to the ultrametric associated to $N_{\infty,\infty}$.

If $a, b \in M_n(\overline{B}_k(0,1))$, then

$$|\det a - \det b| \le N_{\infty,\infty}(a - b). \tag{4.13.15}$$

This can be verified directly, and one can also use the remarks about products of bounded Lipschitz functions in the previous section.

4.14 Some subgroups of $GL_n(k)$

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let n be a positive integer again. Remember that $GL_n(k)$ is the group of invertible elements in $M_n(k)$, and let $N_{\infty,\infty}$ be the ultranorm defined on $M_n(k)$ in the previous section. Let us use $B_k(\cdot,\cdot)$, $\overline{B}_k(\cdot,\cdot)$ for open and closed balls in k with respect to the ultrametric associated to $|\cdot|$, and $B(\cdot,\cdot)$, $\overline{B}(\cdot,\cdot)$ for open and closed balls in $M_n(k)$ with respect to the ultrametric associated to $N_{\infty,\infty}$. Thus

$$\overline{B}(0,1) = M_n(\overline{B}_k(0,1)), \tag{4.14.1}$$

as in the previous section. Remember that (4.14.1) is a subring of $M_n(k)$ that contains the identity matrix I, and that $GL_n(\overline{B}_k(0,1))$ is the group of invertible elements of this subring.

If $a \in M_n(\overline{B}_k(0,1))$ and $b \in M_n(k)$, then

$$N_{\infty,\infty}(a\,b),\ N_{\infty,\infty}(b\,a) \le N_{\infty,\infty}(a)\,N_{\infty,\infty}(b) \le N_{\infty,\infty}(b), \tag{4.14.2}$$

by (4.13.14). Similarly, if $a \in GL_n(\overline{B}_k(0,1))$ and $c \in M_n(k)$, then

$$N_{\infty,\infty}(a^{-1}\,c),\ N_{\infty,\infty}(c\,a^{-1}) \le N_{\infty,\infty}(c). \tag{4.14.3}$$

This implies that

$$N_{\infty,\infty}(b) \le N_{\infty,\infty}(a\,b),\ N_{\infty,\infty}(b\,a) \tag{4.14.4}$$

for every $b \in M_n(k)$. It follows that

$$N_{\infty,\infty}(a\,b) = N_{\infty,\infty}(b\,a) = N_{\infty,\infty}(b) \tag{4.14.5}$$

for every $a \in GL_n(\overline{B}_k(0,1))$ and $b \in M_n(k)$. In particular, $N_{\infty,\infty}(a) = 1$ when $a \in GL_n(\overline{B}_k(0,1))$.

Of course,

$$d_{N_{\infty,\infty}}(b,c) = N_{\infty,\infty}(b-c) \tag{4.14.6}$$

is the ultrametric on $M_k(k)$ associated to $N_{\infty,\infty}$. Using (4.14.5), we get that this ultrametric is invariant under left and right multiplication by elements of $GL_n(\overline{B}_k(0,1))$. This means that the restriction of (4.14.6) to $GL_n(\overline{B}_k(0,1))$ is invariant under left and right translations in $GL_n(\overline{B}_k(0,1))$, as a group with respect to matrix multiplication. Note that

$$N_{\infty,\infty}(a-I) = N_{\infty,\infty}(a^{-1}-I) \tag{4.14.7}$$

when $a \in GL_n(\overline{B}_k(0,1))$. If $a, b \in GL_n(\overline{B}_k(0,1))$, then we have that

$$\begin{aligned} N_{\infty,\infty}(a\,b-I) &\le \max(N_{\infty,\infty}(a\,b-b), N_{\infty,\infty}(b-I)) \\ &= \max(N_{\infty,\infty}(a-I), N_{\infty,\infty}(b-I)). \end{aligned} \tag{4.14.8}$$

More precisely, this works when $a \in M_n(k)$ and $b \in GL_n(\overline{B}_k(0,1))$. The same conclusion can be obtained analogously when $a \in GL_n(\overline{B}_k(0,1))$ and $b \in M_n(k)$.

Remember that $GL_n(\overline{B}_k(0,1))$ consists of all $a \in M_n(\overline{B}_k(0,1))$ such that $|\det a| = 1$, as in (4.11.1). If $a \in GL_n(\overline{B}_k(0,1))$, $b \in M_n(\overline{B}_k(0,1))$, and

$$N_{\infty,\infty}(a - b) < 1, \tag{4.14.9}$$

then

$$|\det a - \det b| < 1, \tag{4.14.10}$$

by (4.13.15). This implies that

$$|\det b| = 1, \tag{4.14.11}$$

as in Section 4.3, so that $b \in GL_n(\overline{B}_k(0,1))$ too.

Observe that

$$B(I,1) \subseteq \overline{B}(I,1) = \overline{B}(0,1). \tag{4.14.12}$$

In fact,

$$B(I,1) \subseteq GL_n(\overline{B}_k(0,1)), \tag{4.14.13}$$

by the remarks in the preceding paragraph. More precisely, $B(I,1)$ is a normal subgroup of $GL_n(\overline{B}(0,1))$, as in Section 3.7. This could also be obtained from some of the properties of $N_{\infty,\infty}$ mentioned earlier. Similarly, $B(I,r)$ and $\overline{B}(I,r)$ are normal subgroups of $GL_n(\overline{B}_k(0,1))$ when $0 < r < 1$.

4.15 Two ultrametrics on $GL_n(k)$

Let us continue with the same notation and hypotheses as in the previous section. Let $a, b \in GL_n(k)$ be given, and put

$$\begin{aligned} \delta_L(a,b) &= N_{\infty,\infty}(a^{-1}\,b - I) \quad \text{when } a^{-1}\,b \in GL_n(\overline{B}_k(0,1)) \\ &= 1 \quad \text{otherwise.} \end{aligned} \tag{4.15.1}$$

Of course, $a^{-1}\,b$ is an element of $GL_n(\overline{B}_k(0,1))$ if and only if $b^{-1}\,a = (a^{-1}\,b)^{-1}$ has this property. In this case,

$$N_{\infty,\infty}(a^{-1}\,b - I) = N_{\infty,\infty}(b^{-1}\,a - I), \tag{4.15.2}$$

by (4.14.7). This implies that (4.15.1) is symmetric in a, b.

It is easy to see that

$$\delta_L(a,b) \le 1 \tag{4.15.3}$$

when $a^{-1}\,b \in GL_n(\overline{B}_k(0,1))$, and thus for all $a, b \in GL_n(k)$. If $a, b \in GL_n(k)$ and

$$N_{\infty,\infty}(a^{-1}\,b - I) < 1, \tag{4.15.4}$$

then $a^{-1}\,b \in GL_n(\overline{B}_k(0,1))$, as in (4.14.13). It follows that $a, b \in GL_n(k)$ satisfy

$$\delta_L(a,b) < 1 \tag{4.15.5}$$

if and only if (4.15.4) holds. Note that $\delta_L(a,b) = 0$ if and only if $a^{-1}\,b = I$, which means that $a = b$.

Let $a, b, c \in GL_n(k)$ be given, and let us check that

$$\delta_L(a,c) \le \max(\delta_L(a,b), \delta_L(b,c)). \tag{4.15.6}$$

This is trivial when the right side is equal to one, by (4.15.3). If the right side is less than one, then $a^{-1}\,b, b^{-1}\,c \in GL_n(\overline{B}_k(0,1))$, which implies that $a^{-1}\,c$ is an element of $GL_n(\overline{B}_k(0,1))$ too. Under these conditions, we have that

$$\begin{aligned} N_{\infty,\infty}(a^{-1}\,c - I) &= N_{\infty,\infty}((a^{-1}\,b)\,(b^{-1}\,c) - I) \\ &\le \max(N_{\infty,\infty}(a^{-1}\,b - I), N_{\infty,\infty}(b^{-1}\,c - I)), \end{aligned} \tag{4.15.7}$$

as in (4.14.8). This implies (4.15.6), as desired.

It follows that (4.15.1) defines an ultrametric on $GL_n(k)$. This ultrametric is invariant under left translations on $GL_n(k)$, as a group with respect to matrix multiplication, by construction. If $a, b \in GL_n(\overline{B}_k(0,1))$, then one can check that

$$\delta_L(a,b) = N_{\infty,\infty}(a - b). \tag{4.15.8}$$

More precisely, this uses the fact that $N_{\infty,\infty}$ is invariant under left multiplication by elements of $GL_n(\overline{B}_k(0,1))$, as in the previous section.

Let $a, b \in GL_n(k)$ be given again, and put

$$\begin{aligned} \delta_R(a,b) &= N_{\infty,\infty}(a\,b^{-1} - I) \quad \text{when } a\,b^{-1} \in GL_n(\overline{B}_k(0,1)) \\ &= 1 \quad \text{otherwise.} \end{aligned} \tag{4.15.9}$$

Observe that

$$\delta_R(a,b) = \delta_L(a^{-1}, b^{-1}). \tag{4.15.10}$$

It follows that (4.15.9) defines an ultrametric on $GL_n(k)$. This ultrametric is clearly invariant under right translations on $GL_n(k)$, as a group with respect to matrix multiplication. One can verify that

$$\delta_R(a,b) = N_{\infty,\infty}(a - b) \tag{4.15.11}$$

for every $a, b \in GL_n(\overline{B}_k(0,1))$, because $N_{\infty,\infty}$ is invariant under right multiplication by elements of $GL_n(\overline{B}_k(0,1))$, as before.

Chapter 5

ℓ^r Spaces and operator seminorms

5.1 q-Norms and q-seminorms

Let k be a field, and let $|\cdot|$ be a nonnegative real-valued function on k that satisfies the first two conditions (1.4.1) and (1.4.2) in the definition of an absolute value function. Let us say that $|\cdot|$ is a *q-absolute value function* on k for some positive real number q if

$$|x+y|^q \le |x|^q + |y|^q \tag{5.1.1}$$

for every $x, y \in k$. Equivalently, this means that

$$|x+y| \le (|x|^q + |y|^q)^{1/q} \tag{5.1.2}$$

for every $x, y \in k$. The right side decreases monotonically in q, by (2.10.1), so that this condition becomes more restrictive as q increases. A q-absolute value function is the same as an ordinary absolute value function when $q = 1$, and it is convenient to consider an ultrametric absolute value function as a q-absolute value function with $q = \infty$.

If $|\cdot|$ is a q-absolute value function on k, then $|x-y|$ defines a q-metric on k. In this case, if a is a positive real number, then $|x|^a$ is a (q/a)-absolute value function on k. It is easy to see that a q-absolute value function for any $q > 0$ is a quasimetric absolute value function. Conversely, a quasimetric absolute value function is a q-absolute value function for some $q > 0$, by the corollary on p14 of [3].

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let V be a vector space over k. Also let N be a nonnegative real-valued function on V that satisfies the homogeneity condition (1.9.1) with respect to $|\cdot|$, and let q_N be a positive real number. Let us say that N is a *q_N-seminorm* on V with respect to $|\cdot|$ on k if

$$N(v+w)^{q_N} \le N(v)^{q_N} + N(w)^{q_N} \tag{5.1.3}$$

for every $v, w \in V$. If we also have that $N(v) > 0$ when $v \neq 0$, then we say that N is a q_N-*norm* on V with respect to $|\cdot|$ on k. Thus q_N-norms and q_N-seminorms are the same as ordinary norms and seminorms when $q_N = 1$.

As before, (5.1.3) is the same as asking that

$$N(v + w) \leq (N(v)^{q_N} + N(w)^{q_N})^{1/q_N} \tag{5.1.4}$$

for every $v, w \in V$. The right side is monotonically decreasing in q_n, by (2.10.1), so that this condition becomes more restrictive as q_N increases. We may consider an ultranorm or semi-ultranorm as a q_N-norm or q_N-seminorm with $q_N = \infty$, respectively. If N is a q_N-norm or q_N-seminorm on V, then $d_N(v, w) = N(v-w)$ is a q_N-metric or q_N-semimetric on V, as appropriate. If N is a q_N-seminorm on V and $N(v) > 0$ for some $v \in V$, then one can check that $|\cdot|$ is a q_N-absolute value function on V.

Let a be a positive real number, so that $|\cdot|^a$ is a (q_k/a)-absolute value function on k. If N is a q_N-norm or q_N-seminorm on V with respect to $|\cdot|$, then it is easy to see that $N(v)^a$ is a (q_N/a)-norm or (q_N/a)-seminorm on V, as appropriate, with respect to $|\cdot|^a$ on k.

Let $\mathcal{N}$ be a nonempty collection of q-seminorms on V. More precisely, every $N \in \mathcal{N}$ should be a q_N-seminorm on V with respect to $|\cdot|$ on k, for some $q_N > 0$ that may depend on N. Thus

$$\mathcal{M} = \mathcal{M}(\mathcal{N}) = \{d_N : N \in \mathcal{N}\} \tag{5.1.5}$$

is a collection of q-semimetrics on V, which can be used to define a topology or uniform structure on V, as in Section 2.10.

Let us say that $\mathcal{N}$ is *nondegenerate* on V if for every $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$. This means that (5.1.5) is nondegenerate as a collection of q-semimetrics on V.

5.2 Examples of q-seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. Also let l be a positive integer, and for each $j = 1, \ldots, l$, let N_j be a q_{N_j}-seminorm on V with respect to $|\cdot|$ on k for some $q_{N_j} > 0$. If we put

$$q_0 = \min(q_{N_1}, \ldots, q_{N_l}), \tag{5.2.1}$$

then N_j may be considered as a q_0-seminorm on V for each $j = 1, \ldots, l$. Under these conditions, one can verify that

$$\max_{1 \leq j \leq l} N_j(v) \tag{5.2.2}$$

is a q_0-seminorm on V with respect to $|\cdot|$ on k too.

Let r be a positive real number, and consider

$$\Big(\sum_{j=1}^{l} N_j(v)^r\Big)^{1/r}, \tag{5.2.3}$$

which is a nonnegative real-valued function on V. Of course, (5.2.2) is the analogue of this for $r = \infty$. If $r \le q_0$, then one can check that (5.2.3) is an r-seminorm on V with respect to $|\cdot|$ on k. As in Section 2.11, one can do this using (2.11.6) with the exponent $r/q_0 \le 1$ when $q_0 < \infty$, or using the fact that N_j is an r-seminorm on V for each $j = 1, \ldots, l$ when $r \le q_0$. If $q_0 \le r$, then one can check that (5.2.3) is a q_0-seminorm on V, using Minkowski's inequality with exponent $r/q_0 \ge 1$.

Let X be a nonempty set, and remember that $c_{00}(X, k)$ is the space of k-valued functions on X with finite support, as in Section 1.10. Also let a be a positive real-valued function on X, and let r be a positive real number. If $f \in c_{00}(X, k)$, then put

$$\|f\|_{r,a} = \Big(\sum_{x \in X} (a(x)\,|f(x)|)^r\Big)^{1/r}. \tag{5.2.4}$$

Note that this is the same as in Section 1.10 when $r = 1, 2$. The analogue of this for $r = \infty$ is

$$\|f\|_{\infty,a} = \max_{x \in X}(a(x)\,|f(x)|), \tag{5.2.5}$$

as usual.

One can verify that (5.2.5) is a q_k-norm on $c_{00}(X, k)$ with respect to $|\cdot|$ on k. If $r \le q_k$, then (5.2.4) is an r-norm on $c_{00}(X, k)$ with respect to $|\cdot|$ on k. As before, this can be obtained using (2.11.6) with the exponent $r/q_k \le 1$ when $q_k < \infty$, or from the fact that $|\cdot|$ is an r-absolute value function on k when $r \le q_k$. If $q_k \le r$, then (5.2.4) is a q_k-norm on $c_{00}(X, k)$ with respect to $|\cdot|$ on k. This can be seen using Minkowski's inequality, with exponent $r/q_k \ge 1$.

If $0 < r_1 \le r_2 \le \infty$, then

$$\|f\|_{r_2,a} \le \|f\|_{r_1,a} \tag{5.2.6}$$

for every $f \in c_{00}(X, k)$, as in (2.11.1) and (2.11.4). If $a(x) = 1$ for every $x \in X$, then we may use $\|f\|_r$ to denote $\|f\|_{r,a}$, as in Section 1.10. In particular, $\|f\|_\infty$ is the same as the supremum norm $\|f\|_{sup}$ on $c_{00}(X, k)$, as in Section 4.12.

Let b be another positive real-valued function on X, and suppose for the moment that X has only finitely many elements. Observe that

$$\|f\|_{r,a} \le \Big(\sum_{x \in X} (a(x)/b(x))^r\Big)^{1/r} \|f\|_{\infty,b} \tag{5.2.7}$$

for every $f \in c_{00}(X, k)$.

Suppose that $0 < r_1 < r_2 < \infty$, and that $0 < r_3 < \infty$ satisfies

$$1/r_1 = 1/r_2 + 1/r_3. \tag{5.2.8}$$

This means that $r_1/r_2 + r_1/r_3 = 1$, so that one can use Hölder's inequality to get that

$$\|f\|_{r_1,a}^{r_1} = \sum_{x \in X} (a(x)/b(x))^{r_1}\,(b(x)\,|f(x)|)^{r_1} \tag{5.2.9}$$

$$\leq \Big(\sum_{x \in X} (a(x)/b(x))^{r_3}\Big)^{r_1/r_3} \|f\|_{r_2,b}^{r_1}$$

for every $f \in c_{00}(X,k)$. It follows that

$$\|f\|_{r_1,a} \leq \Big(\sum_{x \in X} (a(x)/b(x))^{r_3}\Big)^{1/r_3} \|f\|_{r_2,b} \tag{5.2.10}$$

for every $f \in c_{00}(X,k)$.

Similarly, suppose now that $X = \mathbf{Z}_+$. If $\sum_{j=1}^\infty (a(j)/b(j))^r$ converges, as an infinite series of positive real numbers, then

$$\|f\|_{r,a} \leq \Big(\sum_{j=1}^\infty (a(j)/b(j))^r\Big)^{1/r} \|f\|_{\infty,b} \tag{5.2.11}$$

for every $f \in c_{00}(\mathbf{Z}_+,k)$. Suppose that $0 < r_1 < r_2 < \infty$ and $0 < r_3 < \infty$ satisfy (5.2.8) again. If $\sum_{j=1}^\infty (a(j)/b(j))^{r_3}$ converges as an infinite series of positive real numbers, then one can use Hölder's inequality to get that

$$\|f\|_{r_1,a} \leq \Big(\sum_{j=1}^\infty (a(j)/b(j))^{r_3}\Big)^{1/r_3} \|f\|_{r_2,b} \tag{5.2.12}$$

for every $f \in c_{00}(\mathbf{Z}_+,k)$, as before.

5.3 ℓ^∞ Spaces

Let X be a nonempty set, let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Consider the space of $\ell^\infty(X,k)$ of all k-valued functions on X that are bounded with respect to $|\cdot|$ on k. This is the same as $\mathcal{B}(X,k)$, as in Section 2.12, where k is equipped with the q_k-metric associated to $|\cdot|$. It is easy to see that $\ell^\infty(X,k)$ is a linear subspace of the space $c(X,k)$ of all k-valued functions on X.

If $f \in \ell^\infty(X,k)$, then put

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,k)} = \sup_{x \in X} |f(x)|. \tag{5.3.1}$$

This is the same as the supremum norm $\|f\|_{sup}$ on $\ell^\infty(X,k)$, as in Section 4.12. More precisely, one can check that (5.3.1) is a q_k-norm on $\ell^\infty(X,k)$, with respect to $|\cdot|$ on k. The q_k-metric on $\ell^\infty(X,k)$ associated to (5.3.1) is the same as the supremum q_k-metric corresponding to the q_k-metric on k associated to $|\cdot|$, as in Section 2.12.

Similarly, let a be a positive real-valued function on X, and let $\ell_a^\infty(X,k)$ be the space of k-valued functions f on X such that

$$a(x)\,|f(x)| \text{ is bounded on } X. \tag{5.3.2}$$

This is a linear subspace of $c(X,k)$, as before. If $f \in \ell^\infty_a(X,k)$, then put

$$\|f\|_{\infty,a} = \|f\|_{\ell^\infty_a(X,k)} = \sup_{x\in X}(a(x)\,|f(x)|). \tag{5.3.3}$$

One can check that this defines a q_k-norm on $\ell^\infty_a(X,k)$ with respect to $|\cdot|$ on k, as before. Of course, $\ell^\infty_a(X,k)$ and $\|f\|_{\infty,a}$ are the same as $\ell^\infty(X,k)$ and $\|f\|_\infty$ when $a(x)=1$ for every $x \in X$.

Let Y be a nonempty set, and let $d(\cdot,\cdot)$ be a q_d-metric on Y for some $q_d > 0$. Remember that one can define the notion of Cauchy sequences in Y with respect to $d(\cdot,\cdot)$ in the usual way, as in Section 2.15. As usual, we say that Y is *complete* with respect to $d(\cdot,\cdot)$ if every Cauchy sequence in Y converges to an element of Y.

Let $\mathcal{B}(X,Y)$ be the space of bounded mappings from X into Y with respect to $d(\cdot,\cdot)$, and let $\theta(\cdot,\cdot)$ be the corresponding supremum q_d-metric on $\mathcal{B}(X,Y)$, as in Section 2.12. If Y is complete with respect to $d(\cdot,\cdot)$, then $\mathcal{B}(X,Y)$ is complete with respect to $\theta(\cdot,\cdot)$, by standard arguments. More precisely, let $\{f_j\}_{j=1}^\infty$ be a sequence of bounded mappings from X into Y that is a Cauchy sequence with respect to $\theta(\cdot,\cdot)$. If $x \in X$, then it is easy to see that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in Y with respect to $d(\cdot,\cdot)$. This implies that $\{f_j(x)\}_{j=1}^\infty$ converges to an element $f(x)$ of Y, because Y is complete.

Thus $\{f_j\}_{j=1}^\infty$ converges pointwise to a mapping f from X into Y. Using this and the Cauchy condition for $\{f_j\}_{j=1}^\infty$ with respect to $\theta(\cdot,\cdot)$, one can verify that $\{f_j\}_{j=1}^\infty$ converges to f uniformly on X. One can use this and the boundedness of the f_j's to get that f is bounded on X too. This means that $f \in \mathcal{B}(X,Y)$, and that $\{f_j\}_{j=1}^\infty$ converges to f with respect to $\theta(\cdot,\cdot)$, as desired.

Suppose now that k is complete, with respect to the q_k-metric associated to $|\cdot|$. This implies that $\ell^\infty(X,k)$ is complete with respect to the corresponding supremum q_k-metric, as in the previous paragraphs. An analogous argument shows that $\ell^\infty_a(X,k)$ is complete with respect to the q_k-metric associated to (5.3.3), as follows.

Let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $\ell^\infty_a(X,k)$, with respect to the q_k-metric associated to (5.3.3). If $x \in X$, then it is easy to see that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in k with respect to the q_k-metric associated to $|\cdot|$, as before. This implies that $\{f_j(x)\}_{j=1}^\infty$ converges to an element $f(x)$ of k, because k is complete, by hypothesis. This means that $\{f_j\}_{j=1}^\infty$ converges pointwise to a k-valued function f on X.

One can use the Cauchy condition for $\{f_j\}_{j=1}^\infty$ with respect to the q_k-metric associated to (5.3.3) to get that

$$a(x)\,|f_j(x) - f(x)| \to 0 \quad \text{as } j \to \infty \tag{5.3.4}$$

uniformly on X. One can use this to get that $f \in \ell^\infty_a(X,k)$, because f_j is an element of $\ell^\infty_a(X,k)$ for each j. This uniform convergence condition also implies that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the q_k-metric associated to (5.3.3), as desired.

5.4 Vanishing at infinity

Let X be a nonempty set, and let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$. A k-valued function f on X is said to *vanish at infinity* with respect to $|\cdot|$ on k if for every $\epsilon > 0$, there are only finitely many $x \in X$ such that

$$|f(x)| \geq \epsilon. \tag{5.4.1}$$

Of course, this holds automatically when X has only finitely many elements. If $X = \mathbf{Z}_+$, then f vanishes at infinity if and only if $|f(j)| \to 0$ as $j \to \infty$.

Let $c_0(X, k)$ be the space of k-valued functions on X that vanish at infinity with respect to $|\cdot|$ on k. If $f \in c_0(X, k)$, then it is easy to see that f is bounded on X with respect to $|\cdot|$ on k, so that

$$c_0(X, k) \subseteq \ell^\infty(X, k). \tag{5.4.2}$$

More precisely, $c_0(X, k)$ is a linear subspace of $\ell^\infty(X, k)$. One can check that $c_0(X, k)$ is a closed set in $\ell^\infty(X, k)$, with respect to the supremum q_k-metric. If k is complete with respect to the q_k-metric associated to $|\cdot|$, then it follows that $c_0(X, k)$ is complete with respect to the supremum q_k-metric, because $\ell^\infty(X, k)$ is complete, as in the previous section.

Remember that the support of a k-valued function f on X is defined to be the set of $x \in X$ such that $f(x) \neq 0$, and that $c_{00}(X, k)$ is the space of k-valued functions on X with finite support. Clearly

$$c_{00}(X, k) \subseteq c_0(X, k), \tag{5.4.3}$$

and one can check that $c_0(X, k)$ is the same as the closure of $c_{00}(X, k)$ in $\ell^\infty(X, k)$, with respect to the supremum q_k-metric. If $f \in c_0(X, k)$, then one can verify that the support of f has only finitely or countably many elements.

Let a be a positive real-valued function on X, and let $c_{0,a}(X, k)$ be the space of k-valued functions f on X such that

$$a(x)\,|f(x)| \text{ vanishes at infinity,} \tag{5.4.4}$$

as a real-valued function on X. As before,

$$c_{0,a}(X, k) \subseteq \ell^\infty_a(X, k), \tag{5.4.5}$$

and in fact $c_{0,a}(X, k)$ is a linear subspace of $\ell^\infty_a(X, k)$. One can check that $c_{0,a}(X, k)$ is a closed set in $\ell^\infty_a(X, k)$, with respect to the q_k-metric associated to $\|\cdot\|_{\infty,a}$. If k is complete with respect to the q_k-metric associated to $|\cdot|$, then it follows that $c_{0,a}(X, k)$ is complete with respect to the q_k-metric associated to $\|\cdot\|_{\infty,a}$.

Of course,

$$c_{00}(X, k) \subseteq c_{0,a}(X, k), \tag{5.4.6}$$

and one can verify that $c_{0,a}(X, k)$ is the same as the closure of $c_{00}(X, k)$ in $\ell^\infty_a(X, k)$, with respect to the q_k-metric associated to $\|\cdot\|_{\infty,a}$. If $f \in c_{a,0}(X, k)$, then the support of f has only finitely or countably many elements, because the support of $a\,|f|$ has only finitrly or countably many elements, as before.

Let b be another positive real-valued function on X, and suppose that

$$a(x)/b(x) \text{ is bounded on } X. \tag{5.4.7}$$

It is easy to see that

$$\ell^\infty_b(X, k) \subseteq \ell^\infty_a(X, k) \tag{5.4.8}$$

and

$$c_{0,b}(X, k) \subseteq c_{0,a}(X, k). \tag{5.4.9}$$

If $f \in \ell^\infty_b(X, k)$, then we have that

$$\|f\|_{\infty,a} \le \Big(\sup_{x\in X}(a(x)/b(x))\Big)\,\|f\|_{\infty,b}. \tag{5.4.10}$$

Suppose now that

$$a(x)/b(x) \text{ vanishes at infinity on } X, \tag{5.4.11}$$

as a real-valued function on X. Observe that

$$\ell^\infty_b(X, k) \subseteq c_{0,a}(X, k). \tag{5.4.12}$$

5.5 Nonnegative sums

Let X be a nonempty set, and let f be a nonnegative real-valued function on X. If E is a nonempty subset of X, then

$$\sum_{x\in E} f(x) \tag{5.5.1}$$

can be defined as a nonnegative extended real number, by taking the supremum of

$$\sum_{x\in A} f(x) \tag{5.5.2}$$

over all nonempty finite subsets A of E.

It is easy to see that

$$\sum_{x\in E} t\,f(x) = t\sum_{x\in E} f(x) \tag{5.5.3}$$

for every positive real number t. If g is another nonnegative real-valued function on X, then one can verify that

$$\sum_{x\in E}(f(x)+g(x)) = \sum_{x\in E} f(x) + \sum_{x\in E} g(x). \tag{5.5.4}$$

If E_1, E_2 are disjoint nonempty subsets of X, then it follows that

$$\sum_{x \in E_1 \cup E_2} f(x) = \sum_{x \in E_1} f(x) + \sum_{x \in E_2} f(x). \tag{5.5.5}$$

Let us say that f is *summable* on E if (5.5.1) is finite. If $E = \mathbf{Z}_+$, then the summability of f on E is equivalent to the convergence of

$$\sum_{j=1}^{\infty} f(j), \tag{5.5.6}$$

as an infinite series of nonnegative real numbers. In this case, (5.5.1) and (5.5.6) are the same.

Suppose that f is summable on $E \subseteq X$. If E_0 is a nonempty subset of E, then f is summable on E_0 too. If t is a nonnegative real number, then $t\,f$ is summable on E, and satisfies (5.5.3). If g is also summable on E, then $f + g$ is summable on E, by (5.5.4).

If f is summable on E, then for each $\epsilon > 0$ there is a nonempty finite subset $A(\epsilon)$ of E such that

$$\sum_{x \in E} f(x) < \sum_{x \in A(\epsilon)} f(x) + \epsilon. \tag{5.5.7}$$

This implies that

$$\sum_{x \in E \setminus A(\epsilon)} f(x) < \epsilon, \tag{5.5.8}$$

by (5.5.5). It follows in particular that f vanishes at infinity on E.

Let r be a positive real number, and let us say that f is *r-summable* on E if $f(x)^r$ is summable on E. In this case, $t\,f$ is r-summable on E for every nonnegative real number t. If g is r-summable on E as well, then one can check that $f + g$ is r-summable on E.

If $0 < r \le 1$, then

$$(f(x) + g(x))^r \le f(x)^r + g(x)^r \tag{5.5.9}$$

for every $x \in X$, as in (1.6.1). This implies that

$$\sum_{x \in E} (f(x) + g(x))^r \le \sum_{x \in E} f(x)^r + \sum_{x \in E} g(x)^r. \tag{5.5.10}$$

Suppose now that $1 \le r < \infty$, and that f and g are r-summable on E. Under these conditions, Minkowski's inequality for sums implies that $f + g$ is r-summable on E, with

$$\Big(\sum_{x \in E} (f(x) + g(x))^r\Big)^{1/r} \le \Big(\sum_{x \in E} f(x)^r\Big)^{1/r} + \Big(\sum_{x \in E} g(x)^r\Big)^{1/r}. \tag{5.5.11}$$

If f is r-summable on E for any $r > 0$, then $f(x)^r$ vanishes at infinity on E, and thus f vanishes at infinity on E. If r_0 is another positive real number with

$r_0 \geq r$, then it is easy to see that f is r_0-summable on E, because f is bounded on E. More precisely,

$$\Big(\sum_{x \in E} f(x)^{r_0}\Big)^{1/r_0} \leq \Big(\sum_{x \in E} f(x)^r\Big)^{1/r}, \tag{5.5.12}$$

as in (2.11.4).

5.6 ℓ^r Spaces

Let X be a nonempty set, let r be a positive real number, and let a be a positive real-valued function on X. Also let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Consider the space $\ell^r_a(X,k)$ of k-valued functions f on X such that

$$a(x)\,|f(x)| \text{ is } r\text{-summable on } X. \tag{5.6.1}$$

If $f \in \ell^r_a(X,k)$, then put

$$\|f\|_{r,a} = \|f\|_{\ell^r_a(X,k)} = \Big(\sum_{x \in X} (a(x)\,|f(x)|)^r\Big)^{1/r}. \tag{5.6.2}$$

If $a(x) = 1$ for every $x \in X$, then we may use the notation $\ell^r(X,k)$ and $\|f\|_r = \|f\|_{\ell^r(X,k)}$ for $\ell^r_a(X,k)$ and $\|f\|_{r,a}$, respectively, as usual.

One can check that $\ell^r_a(X,k)$ is a linear subspace of $c(X,k)$. Clearly

$$c_{00}(X,k) \subseteq \ell^r_a(X,k), \tag{5.6.3}$$

and (5.6.2) is the same as in Section 5.2 when $f \in c_{00}(X,k)$. As before, (5.6.2) is an r-norm on $\ell^r_a(X,k)$ with respect to $|\cdot|$ on k when $r \leq q_k$. If $q_k \leq r$, then (5.6.2) is a q_k-norm on $\ell^r_a(X,k)$ with respect to $|\cdot|$ on k. In both cases, one can verify that $c_{00}(X,k)$ is dense in $\ell^r_a(X,k)$ with respect to the q_k or r-metric associated to (5.6.2), as appropriate.

If $0 < r_1 \leq r_2 \leq \infty$, then

$$\ell^{r_1}_a(X,k) \subseteq \ell^{r_2}_a(X,k), \tag{5.6.4}$$

and

$$\|f\|_{r_2,a} \leq \|f\|_{r_1,a} \tag{5.6.5}$$

for every $f \in \ell^{r_1}_a(X,k)$. If $r_1 < \infty$, then

$$\ell^{r_1}_a(X,k) \subseteq c_{0,a}(X,k). \tag{5.6.6}$$

If k is complete with respect to the q_k-metric associated to $|\cdot|$, then $\ell^r_a(X,k)$ is complete with respect to the q_k or r-metric associated to (5.6.2), as appropriate.

Indeed, suppose that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\ell_a^r(X,k)$ with respect to the q_k or r-metric associated to (5.6.2), so that

$$\|f_j - f_l\|_{r,a} \to 0 \quad \text{as } j, l \to \infty. \tag{5.6.7}$$

Using this, it is easy to see that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in k for every $x \in X$, with respect to the q_k-metric associated to $|\cdot|$. This implies that $\{f_j\}_{j=1}^\infty$ converges pointwise to a k-valued function f on X, because k is complete. Under these conditions, one can check that $f \in \ell_a^r(X,k)$, and that

$$\|f_j - f\|_{r,a} \to 0 \quad \text{as } j \to \infty, \tag{5.6.8}$$

as desired.

Let b be another positive real-valued function on X. If

$$a(x)/b(x) \text{ is } r\text{-summable on } X, \tag{5.6.9}$$

then it is easy to see that

$$\ell_b^\infty(X,k) \subseteq \ell_a^r(X,k). \tag{5.6.10}$$

More precisely, if $f \in \ell_b^\infty(X,k)$, then

$$\|f\|_{r,a} \le \Big(\sum_{x\in X} (a(x)/b(x))^r\Big)^{1/r} \|f\|_{\infty,b}. \tag{5.6.11}$$

If $a(x)/b(x)$ is bounded on X, then

$$\ell_b^r(X,k) \subseteq \ell_a^r(X,k). \tag{5.6.12}$$

In this case,

$$\|f\|_{r,a} \le \Big(\sup_{x\in X} (a(x)/b(x))\Big) \|f\|_{r,b} \tag{5.6.13}$$

for every $f \in \ell_b^\infty(X,k)$.

Suppose that r_1, r_2, r_3 are positive real numbers with $1/r_1 = 1/r_2 + 1/r_3$. If $a(x)/b(x)$ is r_3-summable on X, then

$$\ell_b^{r_2}(X,k) \subseteq \ell_a^{r_1}(X,k). \tag{5.6.14}$$

Indeed, if $f \in \ell_b^{r_2}(X,k)$, then

$$\|f\|_{r_1,a} \le \Big(\sum_{x\in X} (a(x)/b(x))^{r_3}\Big)^{1/r_3} \|f\|_{r_2,b}, \tag{5.6.15}$$

by Hölder's inequality, as in Section 5.2.

5.7 Operator q-seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-seminorms on V, W for some $q_V, q_W > 0$, respectively, and with respect to $|\cdot|$ on k. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \le C\, N_V(v) \tag{5.7.1}$$

for every $v \in V$. This implies that

$$N_W(T(u) - T(v)) = N_W(T(u - v)) \le C\, N_V(u - v) \tag{5.7.2}$$

for every $u, v \in V$.

Thus (5.7.1) implies that T is Lipschitz of order 1 with constant C with respect to the q_V, q_W-semimetrics associated to N_V, N_W, respectively. In particular, this means that T is uniformly continuous with respect to these semimetrics. If $|\cdot|$ is not the trivial absolute value function on k, and if T is continuous at 0 with respect to these semimetrics on V, W, then one can check that T is bounded. More precisely, it suffices to ask that $N_W(T(v))$ be bounded as a real-valued function on a ball in V centered at 0 with positive radius with respect to the q_V-semimetric associated to N_V in this case.

Let $\mathcal{L}(V, W)$ be the space of all linear mappings from V into W. Of course, this is a vector space over k, with respect to pointwise addition and scalar multiplication. Similarly, let $\mathcal{BL}(V, W)$ be the space of linear mappings from V into W that are bounded with respect to N_V, N_W. One can check that this is a linear subspace of $\mathcal{L}(V, W)$.

If $T \in \mathcal{BL}(V, W)$, then we would like to put

$$\|T\|_{op} = \|T\|_{op,VW} = \sup\left\{\frac{N_W(T(v))}{N_V(v)} : v \in V,\ N_V(v) > 0\right\}. \tag{5.7.3}$$

Let us interpret this as being equal to 0 when $N_V(v) = 0$ for every $v \in V$. Note that $N_W(T(v)) = 0$ for every $v \in V$ such that $N_V(v) = 0$, because T is bounded. Equivalently, $\|T\|_{op}$ is the smallest $C \ge 0$ such that (5.7.1) holds for every $v \in V$.

One can check that $\|\cdot\|_{op}$ defines a q_W-seminorm on $\mathcal{BL}(V, W)$, with respect to $|\cdot|$ on k. If N_W is a q_W-norm on W, then $\|\cdot\|_{op}$ is a q_W-norm on $\mathcal{BL}(V, W)$.

Let Z be another vector space over k, and let N_Z be a q_Z-seminorm on Z for some $q_Z > 0$, with respect to $|\cdot|$ on k. Suppose that T_1 is a bounded linear mapping from V into W, and that T_2 is a bounded linear mapping from W into Z. If $v \in V$, then

$$\begin{aligned} N_Z((T_2 \circ T_1)(v)) = N_Z(T_2(T_1(v))) &\le \|T_2\|_{op,WZ}\, N_W(T_1(v)) \\ &\le \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}\, N_V(v). \end{aligned} \tag{5.7.4}$$

This means that $T_2 \circ T_1$ is bounded as a linear mapping from V into Z, with

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW} \, \|T_2\|_{op,WZ}. \tag{5.7.5}$$

Suppose now that N_W is a q_W-norm on W, and that W is complete with respect to the associated q_W-metric. Under these conditions,

$$\begin{aligned} &\mathcal{BL}(V, W) \text{ is complete with respect to} \\ &\text{the } q_W\text{-metric associated to } \|\cdot\|_{op}. \end{aligned} \tag{5.7.6}$$

Indeed, let $\{T_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathcal{BL}(V, W)$ with respect to this q_W-metric. This means that for each $\epsilon > 0$ there is a positive integer $L(\epsilon)$ such that

$$\|T_j - T_l\|_{op} < \epsilon \tag{5.7.7}$$

for every $j, l \ge L(\epsilon)$. If $v \in V$, then it follows that

$$\begin{aligned} N_W(T_j(v) - T_l(v)) &= N_W((T_j - T_l)(v)) \\ &\le \|T_j - T_l\|_{op} \, N_V(v) \le \epsilon \, N_V(v) \end{aligned} \tag{5.7.8}$$

for every $j, l \ge L(\epsilon)$.

This implies that $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in W, with respect to the q_W-metric associated to N_W. It follows that this sequence converges to a unique element $T(v)$ of W, because W is complete, by hypothesis. One can check that T is linear as a mapping from V into W. We also get that

$$N_W(T(v) - T_l(v)) \le \epsilon \, N_V(v) \tag{5.7.9}$$

for every $l \ge L(\epsilon)$ and $v \in V$, by (5.7.8). Using this, one can verify that T is a bounded linear mapping from V into W, with

$$\|T - T_l\|_{op} \le \epsilon \tag{5.7.10}$$

for every $l \ge L(\epsilon)$.

Let us continue to ask that N_W be a q_W-norm on W, and that W be complete with respect to the associated q_W-metric. Let V_0 be a linear subspace of V, and suppose that V_0 is dense in V, with respect to the q_V-semimetric associated to N_V. Also let T_0 be a linear mapping from V_0 into W, and suppose that T_0 is bounded, with respect to the restriction of N_V to V_0. Under these conditions,

$$\begin{aligned} &\text{there is a unique extension of } T_0 \text{ to a} \\ &\text{bounded linear mapping } T \text{ from } V \text{ into } W, \end{aligned} \tag{5.7.11}$$

by standard arguments. More precisely, uniqueness follows easily from the continuity of bounded linear mappings.

To get the existence of such an extension, let $v \in V$ be given, and let $\{v_j\}_{j=1}^\infty$ be a sequence of elements of V_0 that converges to v, with respect to the q_V-semimetric associated to N_V. In particular, $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the q_V-semimetric associated to N_V. One can use this to check that

$$\{T_0(v_j)\}_{j=1}^{\infty} \text{ is a Cauchy sequence in } W, \tag{5.7.12}$$

with respect to the q_W-metric associated to N_W, because T_0 is bounded on V_0. It follows that

$$\{T_0(v_j)\}_{j=1}^{\infty} \text{ converges in } W, \tag{5.7.13}$$

because W is complete. We would like to use the limit of this sequence to define $T(v)$ as an element of W.

One can verify that the limit of $\{T_0(v_j)\}_{j=1}^{\infty}$ does not depend on the particular sequence $\{v_j\}_{j=1}^{\infty}$ of elements of V_0 that converges to v, using the boundedness of T_0 on V_0. Thus we can define T as a mapping from V into W in this way, and it is easy to see that T is linear, because T_0 is linear on V_0. One can also get the boundedness of T on V from the boundedness of T_0 on V_0. More precisely, one can verify that

$$\|T\|_{op,VW} = \|T_0\|_{op,V_0W}. \tag{5.7.14}$$

5.8 Bounded vector-valued functions

Let X be a nonempty set, let k be a field, and let V be a vector space over k. The space $c(X,V)$ of all V-valued functions on X is a vector space over k too, with respect to pointwise addition and scalar multiplication of functions. As before, the *support* of $f \in c(X,V)$ is defined to be the set of $x \in X$ such that $f(x) \neq 0$. The space $c_{00}(X,V)$ of $f \in c(X,V)$ whose support has only finitely many elements is a linear subspace of $c(X,k)$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q_N-seminorm on V for some $q_N > 0$, with respect to $|\cdot|$ on k. Consider the space $\ell^\infty(X,V)$ of all V-valued functions f on X that are bounded with respect to N on V, so that $N(f(x))$ is bounded as a nonnegative real-valued function on X. This is the same as $\mathcal{B}(X,V)$, as in Section 2.12, where V is equipped with the q_N-semimetric associated to N. It is easy to see that $\ell^\infty(X,V)$ is a linear subspace of $c(X,V)$.

Similarly, let a be a positive real-valued function on X, and let $\ell^\infty_a(X,V)$ be the space of V-valued functions f on X such that

$$a(x)\,N(f(x)) \text{ is bounded on } X. \tag{5.8.1}$$

One can check that this is a linear subspace of $c(X,V)$, and that

$$\|f\|_{\infty,a} = \|f\|_{\ell^\infty_a(X,V)} = \sup_{x\in X}(a(x)\,N(f(x))) \tag{5.8.2}$$

defines a q_N-seminorm on $\ell^\infty_a(X,V)$, with respect to $|\cdot|$ on k. If $a(x) = 1$ for every $x \in X$, then $\ell^\infty_a(X,V)$ is the same as $\ell^\infty(X,V)$, and $\|f\|_{\infty,a}$ may be denoted $\|f\|_\infty = \|f\|_{\ell^\infty(X,V)}$. In this case, the q_N-semimetric on $\ell^\infty(X,k)$

associated to $\|\cdot\|_{\infty,a}$ is the same as the supremum semimetric corresponding to the q_N-semimetric on V associated to N, as in Section 2.12.

If N is a q_N-norm on V, then (5.8.2) defines a q_N-norm on $\ell^\infty_a(X,k)$. If V is also complete with respect to the q_N-metric associated to N, then one can check that $\ell^\infty_a(X,V)$ is complete with respect to the q_N-metric associated to (5.8.2), as in Section 5.3. More precisely, if $a(x)=1$ for every $x\in X$, then this follows from the analogous statement for $\mathcal{B}(X,Y)$ mentioned earlier. Note that $f\in c(X,k)$ has finite support in X when $N(f(x))$ has finite support in X and N is a q_N-norm on V.

Let us say that $f\in c(X,V)$ *vanishes at infinity* on X with respect to N if

$$a(x)\,N(f(x)) \text{ vanishes at infinity} \tag{5.8.3}$$

as a real-valued function on X. Let $c_{0,a}(X,V)$ be the set of these functions, and note that

$$c_{00}(X,V)\subseteq c_{0,a}(X,V)\subseteq \ell^\infty_a(X,V). \tag{5.8.4}$$

One can check that $c_{0,a}(X,V)$ is a closed linear subspace of $\ell^\infty_a(X,V)$, with respect to the q_N-semimetric associated to (5.8.2), and in fact that it is the closure of $c_{00}(X,V)$ in $\ell^\infty_a(X,V)$. If $a(x)=1$ for every $x\in X$, then $c_{0,a}(X,V)$ may be denoted $c_0(X,V)$. If $f\in c_{0,a}(X,V)$ and N is a q_N-norm on V, then the support of f in X has only finitely or countably many elements.

5.9 r-Summable vector-valued functions

Let us continue with the same notation and hypotheses as in the previous section. Let r be a positive real number, and let $\ell^r_a(X,V)$ be the space of V-valued functions f on X such that

$$a(x)\,N(f(x)) \text{ is } r\text{-summable on } X. \tag{5.9.1}$$

In this case, we put

$$\|f\|_{r,a}=\|f\|_{\ell^r_a(X,V)}=\Big(\sum_{x\in X}(a(x)\,N(f(x)))^r\Big)^{1/r}. \tag{5.9.2}$$

As usual, we may use the notation $\ell^r(X,V)$ and $\|f\|_r=\|f\|_{\ell^r(X,V)}$ when $a(x)=1$ for every $x\in X$.

One can check that $\ell^r_a(X,V)$ is a linear subspace of $c(X,k)$, with

$$c_{00}(X,V)\subseteq \ell^r(X,V)\subseteq c_{0,a}(X,V). \tag{5.9.3}$$

One can also verify that (5.9.2) is an r-seminorm on $\ell^r_a(X,V)$ with respect to $|\cdot|$ on k when $r\le q_N$, and that it is a q_N-seminorm when $q_N\le r$, as in Section 5.2. It is easy to see that $c_{00}(X,V)$ is dense in $\ell^r_a(X,V)$ with respect to the q_N or r-semimetric associated to (5.9.2), as appropriate. If $0<r_1\le r_2\le\infty$, then

$$\ell^{r_1}_a(X,V)\subseteq \ell^{r_2}_a(X,V), \tag{5.9.4}$$

with

(5.9.5)
$$\|f\|_{r_2,a} \le \|f\|_{r_1,a}$$

for every $f \in \ell^{r_1}_a(X, V)$, as before.

Suppose for the moment that N is a q_N-norm on V, so that (5.9.2) is a q_N or r-norm on $\ell^r_a(X, V)$, as appropriate. If V is complete with respect to the q_N-metric associated to N, then one can check that $\ell^r_a(X, V)$ is complete with respect to the q_N or r-metric associated to (5.9.2), as in Section 5.6. If $f \in \ell^r_a(X, V)$, then the support of f in X has only finitely or countably many elements.

Let b be another positive real-valued function on X. If $a(x)/b(x)$ is bounded on X, then

(5.9.6)
$$\ell^r_b(X, V) \subseteq \ell^r_a(X, V)$$

for $0 < r \le \infty$. If $f \in \ell^r_b(X, V)$, then

(5.9.7)
$$\|f\|_{r,a} \le \Big(\sup_{x\in X}(a(x)/b(x))\Big)\, \|f\|_{r,b},$$

as before. Similarly,

(5.9.8)
$$c_{0,b}(X, V) \subseteq c_{0,a}(X, V)$$

in this case. If $a(x)/b(x)$ vanishes at infinity on X, then

(5.9.9)
$$\ell^\infty_b(X, V) \subseteq c_{0,a}(X, V).$$

Let $r_1, r_2, r_3 > 0$ be given, with $1/r_1 = 1/r_2 + 1/r_3$ and $r_3 < \infty$. If

(5.9.10)
$$a(x)/b(x) \text{ is } r_3\text{-summable on } X,$$

then

(5.9.11)
$$\ell^{r_2}_b(X, V) \subseteq \ell^{r_1}_a(X, V),$$

as in Section 5.6. More precisely, if $f \in \ell^{r_2}_b(X, V)$, then

(5.9.12)
$$\|f\|_{r_1,a} \le \Big(\sum_{x\in X}(a(x)/b(x))^{r_3}\Big)^{1/r_3} \|f\|_{r_2,b},$$

as in Section 5.6.

5.10 Some bounded linear mappings

Let X be a nonempty set, let k be a field, and let V be a vector space over k. If $f \in c_{00}(X, V)$, then

(5.10.1)
$$\sum_{x\in X} f(x)$$

reduces to a finite sum in V, and thus defines an element of V. This defines a linear mapping from $c_{00}(X, V)$ into V.

If $x \in X$, then let δ_x be the k-valued function on X with $\delta_x(y) = 1$ when $x = y$, and $\delta_x(y) = 0$ when $x \neq y$. It is easy to see that δ_x, $x \in X$, form a basis for $c_{00}(X, k)$, as a vector space over k. More precisely, if $f \in c_{00}(X, k)$, then

$$f = \sum_{x \in X} f(x)\, \delta_x, \tag{5.10.2}$$

where the right side reduces to a finite sum in $c_{00}(X, k)$, as before.

If T is a linear mapping from $c_{00}(X, k)$ into V, then

$$T(f) = \sum_{x \in X} f(x)\, T(\delta_x) \tag{5.10.3}$$

for every $f \in c_{00}(X, k)$, by (5.10.2). Note that any V-valued function on X corresponds to $T(\delta_x)$ for a unique such linear mapping T.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q_N-seminorm on V with respect to $|\cdot|$ on k for some $q_N > 0$. Also let a be a positive real-valued function on X, and let $0 < r \leq \infty$ be given. Thus $\|\cdot\|_{r,a} = \|\cdot\|_{\ell^r_a(X,k)}$ can be defined on $\ell^r_a(X, k)$ as in Sections 5.3 and 5.6. Note that

$$\|\delta_x\|_{\ell^r_a(X,k)} = a(x) \tag{5.10.4}$$

for every $x \in X$, by construction.

Let T be a linear mapping from $c_{00}(X, k)$ into V, and suppose for the moment that T is bounded with respect to the restriction of $\|\cdot\|_{\ell^r_a(X,k)}$ to $c_{00}(X, k)$, and using N on V. This implies that

$$N(T(\delta_x)) \leq \|T\|_{op}\, \|\delta_x\|_{\ell^r_a(X,k)} = \|T\|_{op}\, a(x) \tag{5.10.5}$$

for every $x \in X$, where $\|T\|_{op}$ is the corresponding operator q_N-seminorm of T. Equivalently,

$$N(T(\delta_x))/a(x) \leq \|T\|_{op} \tag{5.10.6}$$

for every $x \in X$.

Let T be any linear mapping from $c_{00}(X, k)$ into V, and let $f \in c_{00}(X, k)$ be given. If $q_N < \infty$, then we can use (5.10.3) to get that

$$\begin{aligned} N(T(f))^{q_N} &\leq \sum_{x \in X} |f(x)|^{q_N}\, N(T(\delta_x))^{q_N} \\ &= \sum_{x \in X} (a(x)\, |f(x))^{q_N}\, (N(T(\delta_x))/a(x))^{q_N}. \end{aligned} \tag{5.10.7}$$

Similarly, if $q_N = \infty$, then we get that

$$\begin{aligned} N(T(f)) &\leq \max_{x \in X}(|f(x)|\, N(T(\delta_x))) \\ &= \max_{x \in X}((a(x)\, |f(x)|)\, (N(T(\delta_x))/a(x))). \end{aligned} \tag{5.10.8}$$

Suppose that

$$N(T(\delta_x))/a(x) \text{ is bounded on } X, \tag{5.10.9}$$

so that $T(\delta_x)$ is an element of $\ell^\infty_{1/a}(X, V)$, as a V-valued function on X. If $f \in c_{00}(X, k)$, then

$$N(T(f)) \le \Big(\sup_{x \in X} (N(T(\delta_x))/a(x))\Big) \, \|f\|_{\ell^{q_N}_a(X,k)} \tag{5.10.10}$$

by (5.10.7) or (5.10.8), as appropriate. This means that T is bounded with respect to the restriction of $\|\cdot\|_{q_N,a} = \|\cdot\|_{\ell^{q_N}_a(X,k)}$ to $c_{00}(X, k)$, and using N on V. More precisely, the corresponding operator q_N-seminorm $\|T\|_{op}$ of T satisfies

$$\|T\|_{op} \le \sup_{x \in X} (N(T(\delta_x))/a(x)). \tag{5.10.11}$$

In fact,

$$\|T\|_{op} = \sup_{x \in X} (N(T(\delta_x))/a(x)) \tag{5.10.12}$$

under these conditions, because of (5.10.6), with $r = q_N$.

Suppose now that N is a q_N-norm on V, and that V is complete with respect to the associated q_N-metric. Let $0 < r \le \infty$ be given again, and let T be a bounded linear mapping from $c_{00}(X, k)$ into V, with respect to the restriction of $\|\cdot\|_{r,a} = \|\cdot\|_{\ell^r_a(X,k)}$ to $c_{00}(X, k)$, and using N on V. If $r < \infty$, then T has a unique extension to a bounded linear mapping from $\ell^r_a(X, k)$ into V, as in Section 5.7. Similarly, if $r = \infty$, then T has a unique extension to a bounded linear mapping from $c_{0,a}(X, k)$ into V, using the restriction of $\|\cdot\|_{\infty,a} = \|\cdot\|_{\ell^\infty_a(X,k)}$ to $c_{0,a}(X, k)$. This uses the density of $c_{00}(X, k)$ in $\ell^r_a(X, k)$ when $r < \infty$, and in $c_{0,a}(X, k)$ when $r = \infty$, as in Sections 5.4 and 5.6.

5.11 Vector-valued Lipschitz mappings

Let X be a nonempty set, and let $d(\cdot, \cdot)$ be a q_d-semimetric on X for some $q_d > 0$. Also let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q_N-seminorm N with respect to $|\cdot|$ on k for some $q_N > 0$. As in Section 1.8, a V-valued function f on X is said to be *Lipschitz* of order $\alpha > 0$ with constant $C \ge 0$ with respect to the q_N-semimetric on V associated to N if

$$N(f(x) - f(w)) \le C \, d(x, w)^\alpha \tag{5.11.1}$$

for every $x, w \in X$. One can check that the space $\mathrm{Lip}_\alpha(X, V)$ of all functions f that satisfy this condition for some $C \ge 0$ is a vector space over k, with respect to pointwise addition and scalar multiplication of functions.

If $f \in \mathrm{Lip}_\alpha(X, V)$, then we would like to put

$$\|f\|_{\mathrm{Lip}_\alpha(X,V)} = \sup\Big\{\frac{N(f(x) - f(w))}{d(x, w)^\alpha} : x, w \in X, \, d(x, w) > 0\Big\}, \tag{5.11.2}$$

as in Section 4.12. We interpret this as being equal to 0 when $d(x, w) = 0$ for every $x, w \in X$, as before. Of course, if $x, w \in X$ and $d(x, w) = 0$, then

$N(f(x) - f(w)) = 0$, by hypothesis. Alternatively, $\|f\|_{\mathrm{Lip}_\alpha(X,V)}$ is the smallest $C \geq 0$ such that (5.11.1) holds.

One can check that this defines a q_N-seminorm on $\mathrm{Lip}_\alpha(X, V)$, with respect to $|\cdot|$ on k. Note that $\|f\|_{\mathrm{Lip}_\alpha(X,V)} = 0$ when f is constant on X. If N is a q_N-norm on V, and $\|f\|_{\mathrm{Lip}_\alpha(X,V)} = 0$, then f is constant on X.

Let a be a k-valued function on X, and let f be a V-valued function on X, so that $a\, f$ is a V-valued function on X too. If $q_N < \infty$, then

$$\begin{aligned} & N(a(x)\, f(x) - a(w)\, f(w))^{q_N} \\ (5.11.3) \quad \leq \quad & |a(x)|^{q_N}\, N(f(x) - f(w))^{q_N} + |a(x) - a(w)|^{q_N}\, N(f(w))^{q_N} \end{aligned}$$

for every $x, w \in X$. Similarly, if $q_N = \infty$, then

$$\begin{aligned} (5.11.4) \quad & N(a(x)\, f(x) - a(w)\, f(w)) \\ & \leq \max(|a(x)|\, N(f(x) - f(w)), |a(x) - a(w)|\, N(f(w))) \end{aligned}$$

for every $x, w \in X$. If a and f are each both bounded and Lipschitz of order α on X, then it follows that $a\, f \in \mathrm{Lip}_\alpha(X, V)$, with

$$(5.11.5) \quad \|a\, f\|_{\mathrm{Lip}_\alpha(X,V)}^{q_N} \leq \|a\|_{\ell^\infty(X,k)}^{q_N}\, \|f\|_{\mathrm{Lip}_\alpha(X,V)}^{q_N} + \|a\|_{\mathrm{Lip}_\alpha(X,k)}^{q_N}\, \|f\|_{\ell^\infty(X,V)}^{q_N}$$

when $q_N < \infty$, and

$$\begin{aligned} (5.11.6) \quad & \|a\, f\|_{\mathrm{Lip}_\alpha(X,V)} \\ & \leq \max(\|a\|_{\ell^\infty(X,k)}\, \|f\|_{\mathrm{Lip}_\alpha(X,V)}, \|a\|_{\mathrm{Lip}_\alpha(X,k)}\, \|f\|_{\ell^\infty(X,V)}) \end{aligned}$$

when $q_N = \infty$.

If $\alpha = 1$, then we may simply say that the elements of $\mathrm{Lip}_\alpha(X, V)$ are Lipschitz on X, as in Section 1.8. We may also use the notation $\mathrm{Lip}(X, V)$ and $\|f\|_{\mathrm{Lip}(X,V)}$ in this case.

Suppose for the moment that X is a vector space over k too, and that $d(\cdot, \cdot)$ is the q_d-semimetric associated to a q_d-seminorm on X with respect to $|\cdot|$ on k. Remember that a bounded linear mapping T from X into V is Lipschitz with respect to the associated q-semimetrics, as in Section 5.7. In this case, the corresponding operator q_N-seminorm $\|T\|_{op}$ is the same as $\|T\|_{\mathrm{Lip}(X,V)}$.

Let Y be a set with a q_Y-semimetric d_Y for some $q_Y > 0$, and let Z be a set with a q_Z-semimetric d_Z for some $q_Z > 0$. Suppose that a mapping f from X into Y is Lipschitz of order $\alpha > 0$ with constant $C_1 \geq 0$ with respect to d and d_Y, and that a mapping g from Y into Z is Lipschitz of order $\beta > 0$ with constant $C_2 \geq 0$ with respect to d_Y and d_Z. If $x, w \in X$, then

$$\begin{aligned} (5.11.7) \qquad d_Z((g \circ f)(x), (g \circ f)(w)) & \leq C_2\, d_Y(f(x), f(w))^\beta \\ & \leq C_1^\beta\, C_2\, d(x, w)^{\alpha\, \beta}. \end{aligned}$$

This means that $g \circ f$ is Lipschitz of order $\alpha\, \beta$, as a mapping from X into Z, with constant $C_1^\beta\, C_2$, and with respect to d and d_Z.

5.12 Bilipschitz conditions and isometries

Let us continue with the same notation and hypotheses as at the beginning of the previous section. Suppose that f is a V-valued function on X such that

$$N(f(x) - f(w)) \ge c\, d(x, w) \tag{5.12.1}$$

for some $c > 0$ and every $x, w \in X$. This implies that f is injective when $d(\cdot, \cdot)$ is a q_d-metric on X, as in Section 1.8. If f is injective on X, then (5.12.1) is the same as saying that f^{-1} is Lipschitz of order 1 with constant $1/c$ as a mapping from $f(X)$ onto X, with respect to the restriction to $f(X)$ of the q_N-semimetric on V associated to N, as before.

Let g be another mapping from X into V, and suppose that $f - g$ is Lipschitz of order one on X. If $q_N < \infty$, then

$$\begin{aligned} c^{q_N}\, d(x, w)^{q_N} &\le N(f(x) - f(w))^{q_N} \\ &\le N((f(x) - g(x)) - (f(w) - g(w)))^{q_N} + N(g(x) - g(w))^{q_N} \\ &\le \|f - g\|_{\mathrm{Lip}(X,V)}^{q_N}\, d(x, w)^{q_N} + N(g(x) - g(w))^{q_N} \end{aligned} \tag{5.12.2}$$

for every $x, w \in X$. This means that

$$(c^{q_N} - \|f - g\|_{\mathrm{Lip}(X,V)}^{q_N})\, d(x, w)^{q_N} \le N(g(x) - g(w))^{q_N} \tag{5.12.3}$$

for every $x, w \in X$. Of course, this is interesting only when

$$\|f - g\|_{\mathrm{Lip}(X,V)} < c. \tag{5.12.4}$$

In this case, we get that

$$(c^{q_N} - \|f - g\|_{\mathrm{Lip}(X,V)}^{q_N})^{1/q_N}\, d(x, w) \le N(g(x) - g(w)) \tag{5.12.5}$$

for every $x, w \in X$.

If $q_N = \infty$, then

$$\begin{aligned} c\, d(x, w) &\le \max(N((f(x) - g(x)) - (f(w) - g(w))), N(g(x) - g(w))) \\ &\le \max(\|f - g\|_{\mathrm{Lip}(X,V)}\, d(x, w), N(g(x) - g(w))) \end{aligned} \tag{5.12.6}$$

for every $x, w \in X$. If (5.12.4) holds, then we get that

$$c\, d(x, w) \le N(g(x) - g(w)) \tag{5.12.7}$$

for every $x, w \in X$. More precisely, this holds automatically when $d(x, w) = 0$, and otherwise it can be obtained from (5.12.6).

Remember that f is bilipschitz as a mapping from X into V, with respect to the q_N-semimetric on V associated to N, if f is Lipschitz, and (5.12.1) holds for some $c > 0$, as in Section 1.8. If (5.12.4) holds, then it follows that g is bilipschitz as well.

Similarly, f is an isometry as a mapping from X into V if

$$N(f(x) - f(w)) = d(x, w) \tag{5.12.8}$$

for every $x, w \in X$. Of course, this is the same as saying that f is Lipschitz, with $\|f\|_{\mathrm{Lip}(X,V)} \le 1$, and that (5.12.1) holds with $c = 1$. In this case, if $q_N = \infty$, and

$$\|f - g\|_{\mathrm{Lip}(X,V)} < 1, \tag{5.12.9}$$

then (5.12.7) holds with $c = 1$, and $\|g\|_{\mathrm{Lip}(X,V)} \le 1$, because $\|\cdot\|_{\mathrm{Lip}(X,V)}$ is a semi-ultranorm on $\mathrm{Lip}(X,V)$. This means that g is an isometry as a mapping from X into V, as before.

Let Y be a set with a q_Y-semimetric d_Y for some $q_Y > 0$ again, and let Z be a set with a q_Z-semimetric d_Z for some $q_Z > 0$. If f is a bilipschitz mapping from X into Y, and g is a bilipschitz mapping from Y into Z, then it is easy to see that $g \circ f$ is bilipschitz as a mapping from X into Z. Similarly, if f and g are isometries, then $g \circ f$ is an isometry from X into Z.

Suppose now that f is a one-to-one mapping from X onto Y. If f is bilipschitz, then f^{-1} is bilipschitz as a mapping from Y onto X. In particular, if f is an isometry, then f^{-1} is an isometry as a mapping from Y onto X. Of course, f has to be injective when it is bilipschitz and $d(\cdot,\cdot)$ is a q_d-metric on X.

5.13 Some conditions on linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-seminorms on V, W for some $q_V, q_W > 0$, respectively, and with respect to $|\cdot|$ on k. Suppose that T is a linear mapping from V into W such that

$$N_W(T(v)) \ge c\, N_V(v) \tag{5.13.1}$$

for some $c > 0$ and every $v \in V$. If T is injective on V, then this is the same as saying that T^{-1} is bounded as a linear mapping from $T(V)$ onto V, with respect to the restriction of N_W to $T(V)$, and with operator q_V-seminorm less than or equal to $1/c$. Of course, (5.13.1) implies that T is injective on V when N_V is a q_V-norm on V.

Note that

$$c\, N_V(u - v) \le N_W(T(u - v)) = N_W(T(u) - T(v)) \tag{5.13.2}$$

for every $u, v \in V$ in this case. Although the remarks in the previous section could be used here, analogous statements for linear mappings can be obtained a bit more directly, as follows.

Let R be another linear mapping from V into W, and suppose that $R - T$ is bounded as a linear mapping from V into W. If $q_W < \infty$, then

$$\begin{aligned} c^{q_W}\, N_V(v)^{q_W} &\le N_W(T(v))^{q_W} \\ &\le N_W(R(v) - T(V))^{q_W} + N_W(R(v))^{q_W} \\ &\le \|R - T\|_{op,VW}^{q_W}\, N_V(v)^{q_W} + N_W(R(v))^{q_W} \end{aligned} \tag{5.13.3}$$

for every $v \in V$. Thus

$$(c^{q_W} - \|R - T\|_{op,VW}^{q_W})\, N_V(v)^{q_W} \le N_W(R(v))^{q_W} \tag{5.13.4}$$

for every $v \in V$, which is interesting only when

$$\|R - T\|_{op,VW} < c. \tag{5.13.5}$$

Under these conditions, we obtain that

$$(c^{q_W} - \|R - T\|_{op,VW}^{q_W})^{1/q_W}\, N_V(v) \le N_W(R(v)) \tag{5.13.6}$$

for every $v \in V$.

If $q_W = \infty$, then

$$\begin{aligned} c\, N_V(v) &\le \max(N_W(R(v) - T(v)), N_W(R(v))) \\ &\le \max(\|R - T\|_{op,VW}\, N_V(v), N_W(R(v))) \end{aligned} \tag{5.13.7}$$

for every $v \in V$. If (5.13.5) holds, then we get that

$$c\, N_V(v) \le N_W(R(v)) \tag{5.13.8}$$

for every $v \in V$. As before, this holds automatically when $N_V(v) = 0$, and it follows from (5.13.7) when $N_V(v) > 0$.

We say that T is an *isometric linear mapping* from V into W with respect to N_V, N_W if

$$N_W(T(v)) = N_V(v) \tag{5.13.9}$$

for every $v \in V$. This implies that

$$N_W(T(u) - T(v)) = N_W(T(u - v)) = N_V(u - v) \tag{5.13.10}$$

for every $u, v \in V$, so that T is an isometry from V into W with respect to the q_V, q_W-semimetrics associated to N_V, N_W, respectively. Equivalently, T is an isometric linear mapping if and only if T is a bounded linear mapping, with $\|T\|_{op,VW} \le 1$, and (5.13.1) holds with $c = 1$. If we also have that $q_N = \infty$, and

$$\|R - T\|_{op,VW} < 1, \tag{5.13.11}$$

then R is an isometric linear mapping too. This is because (5.13.8) holds with $c = 1$, as before, and $\|R\|_{op,VW} \le 1$, since $\|\cdot\|_{op,VW}$ is a semi-ultranorm on $\mathcal{BL}(V, W)$.

A one-to-one bounded linear mapping T from V onto W is said to be *invertible* as a bounded linear mapping if T^{-1} is bounded as a linear mapping from W onto V. This means that (5.13.1) holds for some $c > 0$, as before.

Let Z be another vector space over k, with a q_Z-seminorm N_Z with respect to $|\cdot|$ on k for some $q_Z > 0$. Suppose that T is a linear mapping from V into W that satisfies (5.13.1) for some $c > 0$ again, and that T_0 is a linear mapping from W into Z such that

$$c_0\, N_W(w) \le N_Z(T_0(w)) \tag{5.13.12}$$

for some $c_0 > 0$ and every $w \in W$. Under these conditions,

$$N_Z((T_0 \circ T)(v)) \geq c_0 \, N_W(T(v)) \geq c \, c_0 \, N_V(v) \tag{5.13.13}$$

for every $v \in V$. Similarly, if T is an isometric linear mapping from V into W, and T_0 is an isometric linear mapping from W into Z, then $T_0 \circ T$ is an isometric linear mapping from V into Z.

5.14 Dense linear subspaces

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let W be a vector space over k, and let N_W be a q_W-seminorm on W with respect to $|\cdot|$ on k for some $q_W > 0$. Also let W_0 be a linear subspace of W, and let c be a positive real number with $c < 1$. Suppose that for each $w \in W$ there is a $w_0 \in W_0$ such that

$$N_W(w - w_0) \leq c \, N_W(w). \tag{5.14.1}$$

We would like to check that W_0 is dense in W, with respect to the q_W-semimetric associated to N_W.

Let $w \in W$ be given, and suppose that $w_1, \ldots, w_l \in W_0$ have been chosen for some $l \in \mathbf{Z}_+$, in such a way that

$$N_W\Big(w - \sum_{j=1}^{l} w_j\Big) \leq c^l \, N_W(w). \tag{5.14.2}$$

Of course, we can do this when $l = 1$, by hypothesis. Applying our hypothesis to $w - \sum_{j=1}^{l} w_j$, we get $w_{l+1} \in W_0$ such that

$$N_W\Big(w - \sum_{j=1}^{l+1} w_j\Big) \leq c \, N_W\Big(w - \sum_{j=1}^{l} w_j\Big) \leq c^{l+1} \, N_W(w). \tag{5.14.3}$$

Thus we can do this for every $l \geq 1$, which implies that W_0 is dense in W, because $c < 1$.

Let W_1 be a linear subspace of W that is dense in W, with respect to the q_W-semimetric associated to N_W, and let c_1 be a positive real number with $c_1 < 1$. Suppose that for every $w \in W_1$ there is a $w_0 \in W_0$ such that

$$N_W(w - w_0) \leq c_1 \, N_W(w). \tag{5.14.4}$$

Let c_0 be a real number with

$$c_1 < c_0 < 1. \tag{5.14.5}$$

If $w \in W$, then we would like to find a $w_0 \in W_0$ such that

$$N_W(w - w_0) \leq c_0 \, N_W(w). \tag{5.14.6}$$

If $N_W(w) = 0$, then we can simply take $w_0 = 0$.

Otherwise, if $N_W(w) > 0$, then we can first choose $w_1 \in W_1$ such that $N_W(w - w_1)$ is as small as we like. In particular, this means that $N_W(w_1)$ is as close to $N_W(w)$ as we like. By hypothesis, there is a $w_0 \in W_0$ such that

$$N_W(w_1 - w_0) \le c_1 \, N_W(w_1). \tag{5.14.7}$$

One can use this to get (5.14.6), because of (5.14.5). It follows that W_0 is dense in W with respect to the q_W-semimetric associated to N_W, as before.

Suppose now that N_W is a q_W-norm on W, and let W_0 be a subset of W. If W_0 is complete with respect to the restriction of the q_W-metric associated to N_W to W_0, then W_0 is a closed set in W, as in Section 4.9. If W_0 is dense in W as well, then it follows that $W_0 = W$.

5.15 Surjectivity of linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k again. Also let N_V, N_W be q_V, q_W-seminorms on V, W for some $q_V, q_W > 0$, respectively, and with respect to $|\cdot|$ on k. Suppose that T is a linear mapping from V into W such that

$$c \, N_V(v) \le N_W(T(v)) \tag{5.15.1}$$

for some $c > 0$ and every $v \in V$. Let R be a linear mapping from V into W such that $R - T$ is bounded as a linear mapping from V into W, with respect to N_V and N_W, respectively.

If $v \in V$, then

$$\begin{aligned} N_W(T(v) - R(v)) = N_W((R - T)(v)) &\le \|R - T\|_{op,VW} \, N_V(v) \\ &\le c^{-1} \, \|R - T\|_{op,VW} \, N_W(T(v)). \end{aligned} \tag{5.15.2}$$

Suppose that

$$\|R - T\|_{op,VW} < c, \tag{5.15.3}$$

so that $c^{-1} \, \|R - T\|_{op,VW} < 1$. If $T(V)$ is dense in W, with respect to the q_W-semimetric associated to N_W, then it follows that $R(V)$ is dense in W as well, as in the previous section.

Suppose from now on in this section that N_V, N_W are q_V, q_W-norms on V, W, respectively. In particular, (5.15.1) implies that T is injective. Remember that R satisfies an analogous condition, because of (5.15.3), as in Section 5.13. Thus R is injective too.

Suppose for the moment that V and W are finite-dimensional as vector spaces over k, with the same dimension. In this case, the injectivity of T implies that T maps V onto W. Similarly, the injectivity of R implies that it maps V onto W.

Suppose now that T is also bounded, as a linear mapping from V into W. If V is complete with respect to the q_V-metric associated to N_V, then one can check that $T(V)$ is complete with respect to the restriction of the q_W-metric associated to N_W to $T(V)$. This implies that $T(V)$ is a closed set in W with respect to the q_W-metric associated to N_W, as in Section 4.9. If $T(V)$ is dense in W, then it follows that

$$T(V) = W, \tag{5.15.4}$$

as mentioned in the previous section.

Under these conditions, R is bounded as a linear mapping from V into W, because $R - T$ is bounded, by hypothesis. If V is complete, then $R(V)$ is a closed set in W, as in the preceding paragraph. This uses (5.15.3) to get that R satisfies the same type of property as T, as before. We also have that $R(V)$ is dense in W, because of (5.15.3) and (5.15.4), as before. This means that

$$R(V) = W, \tag{5.15.5}$$

as in the previous paragraph.

This shows that the set of invertible bounded linear mappings from V onto W is an open set in $\mathcal{BL}(V, W)$ with respect to the q_W-metric associated to the operator q_W-norm when V is complete. Note that W is complete with respect to the q_W-metric associated to N_W when V is complete and there is an invertible bounded linear mapping from V onto W. Similarly, the set of invertible bounded linear mappings from V onto W is an open subset of $\mathcal{BL}(V, W)$ when V has finite dimension. More precisely, if there is an invertible linear mapping from V onto W, then W has the same dimension as V.

Chapter 6

Subadditivity and sub-invariance

6.1 Subadditivity on semigroups

Let Σ be a semigroup. If $A \subseteq \Sigma$ and $x \in \Sigma$, then put $x\,A = \{x\,a : a \in A\}$ and $A\,x = \{a\,x : a \in A\}$, as usual. If $B \subseteq \Sigma$ too, then let $A\,B$ be the set of products $a\,b$, with $a \in A$ and $b \in B$.

Let N be a nonnegative real-valued function on Σ, and let q be a positive real number. Let us say that N is *q-subadditive* on Σ if

$$N(x\,y)^q \le N(x)^q + N(y)^q \tag{6.1.1}$$

for every $x, y \in \Sigma$. If this holds with $q = 1$, then we may simply say that N is *subadditive* on Σ.

As usual, (6.1.1) is the same as saying that

$$N(x\,y) \le (N(x)^q + N(y)^q)^{1/q} \tag{6.1.2}$$

for every $x, y \in \Sigma$. The right side of (6.1.2) decreases monotonically in q, as in (2.10.1). This implies that q-subadditivity becomes more restrictive as q increases.

Similarly, let us say that N is *ultrasubadditive* on Σ if

$$N(x\,y) \le \max(N(x), N(y)) \tag{6.1.3}$$

for every $x, y \in \Sigma$. This implies that N is q-subadditive on Σ when $0 < q < \infty$. We shall consider ultrasubadditivity to be the same as q-subadditivity with $q = +\infty$, as before.

Suppose that Σ has an identity element e. If

$$N(e) = 0, \tag{6.1.4}$$

then we say that N is *normalized* on Σ. If

$$N(x) > 0 \tag{6.1.5}$$

for every $x \in \Sigma$ with $x \neq e$, then we say that N is *nondegenerate* on Σ. If Σ is a group, and if

$$N(x^{-1}) = N(x) \tag{6.1.6}$$

for every $x \in \Sigma$, then we say that N is *symmetric* on Σ.

Let a be a positive real number. If N is q-subadditive on Σ for some $q > 0$, then

$$N(x)^a \tag{6.1.7}$$

is (q/a)-subadditive on Σ. If Σ has an identity element e and N is normalized or nondegenerate on Σ, then (6.1.7) has the same property. If Σ is a group, and N is symmetric on Σ, then (6.1.7) is symmetric on Σ as well.

If Σ is a commutative semigroup, then we may use additive notation for the semigroup operation. In this case, q-subadditivity of a nonnegative real-valued function N on Σ means that

$$N(x + y)^q \le N(x)^q + N(y)^q \tag{6.1.8}$$

for every $x, y \in \Sigma$ when $q < \infty$, and that

$$N(x + y) \le \max(N(x), N(y)) \tag{6.1.9}$$

for every $x, y \in \Sigma$ when $q = \infty$. If Σ has an identity element, then it may be denoted 0 under these conditions. If Σ is a commutative group, then a nonnegative real-valued function N on Σ is *symmetric* when

$$N(-x) = N(x) \tag{6.1.10}$$

for every $x \in \Sigma$.

6.2 Some examples and additional properties

Let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Observe that $|\cdot|$ is q_k-subadditive, normalized, nondegenerate, and symmetric on k, as a commutative group with respect to addition.

In particular, the standard absolute value function is subadditive on the real line, as a commutative group with respect to addition. One can check that

$$\max(x, 0), \quad \max(-x, 0) \tag{6.2.1}$$

are subadditive on $\mathbf{R}$ as well.

The sets $\mathbf{R}_+$ and $\mathbf{R}_+ \cup \{0\}$ of positive and nonnegative real numbers, respectively, are sub-semigroups of $\mathbf{R}$, as a commutative group with respect to addition. If $0 < a \le 1$, then

$$x^a \text{ is subadditive on } \mathbf{R}_+ \cup \{0\}, \tag{6.2.2}$$

as in Section 1.6. If t is a positive real number, then one can verify that

$$\min(x, t) \tag{6.2.3}$$

is subadditive on $\mathbf{R}_+ \cup \{0\}$.

Let Σ be a semigroup, and let N be a nonnegative real-valued function on Σ. Put

$$B_N(r) = \{x \in \Sigma : N(x) < r\} \tag{6.2.4}$$

for every positive real number r, and

$$\overline{B}_N(r) = \{x \in \Sigma : N(x) \le r\} \tag{6.2.5}$$

for every nonnegative real number r. If N is q-subadditive on Σ for some positive real number q, then

$$B_N(r_1)\, B_N(r_2) \subseteq B_N((r_1^q + r_2^q)^{1/q}) \tag{6.2.6}$$

for every $r_1, r_2 > 0$, and

$$\overline{B}_N(r_1)\, \overline{B}_N(r_2) \subseteq \overline{B}_N((r_1^q + r_2^q)^{1/q}) \tag{6.2.7}$$

for every $r_1, r_2 \ge 0$.

Similarly, if N is ultrasubadditive on Σ, then

$$B_N(r_1)\, B_N(r_2) \subseteq B_N(\max(r_1, r_2)) \tag{6.2.8}$$

for every $r_1, r_2 > 0$, and

$$\overline{B}_N(r_1)\, \overline{B}_N(r_2) \subseteq \overline{B}_N(\max(r_1, r_2)) \tag{6.2.9}$$

for every $r_1, r_2 \ge 0$. This means that $B_N(r)$ is a sub-semigroup of Σ for every $r > 0$, and that $\overline{B}_N(r)$ is a sub-semigroup of Σ for every $r \ge 0$. Note that $\overline{B}_N(0)$ is a sub-semigroup of Σ when N is q-subadditive on Σ for any $q > 0$. If Σ has an identity element e, and N is normalized on Σ, then $e \in B_N(r)$ for every $r > 0$, and $e \in \overline{B}_N(r)$ for every $r \ge 0$.

Suppose for the moment that Σ is a group, and that N is symmetric on Σ. In this case, $B_N(r)$ is symmetric in Σ for every $r > 0$, and $\overline{B}_N(r)$ is symmetric for every $r \ge 0$. If N is ultrasubadditive and normalized on Σ, then $B_N(r)$ is a subgroup of Σ for every $r > 0$, and $\overline{B}_N(r)$ is a subgroup of Σ for every $r \ge 0$. If N is normalized and q-subadditive on Σ for any $q > 0$, then $\overline{B}_N(0)$ is a subgroup of Σ.

Let a be a positive real number, so that N^a is another nonnegative real-valued function on Σ. Note that

$$B_{N^a}(r^a) = B_N(r) \tag{6.2.10}$$

for every $r > 0$, and that

$$\overline{B}_{N^a}(r^a) = \overline{B}_N(r) \tag{6.2.11}$$

for every $r \ge 0$.

6.3 Subadditivity and compositions

Let Σ_1, Σ_2 be semigroups, and let ϕ be a semigroup homomorphism from Σ_1 into Σ_2. If a nonnegative real-valued function N_2 on Σ_2 is q_2-subadditive for some $q_2 > 0$, then

$$N_2 \circ \phi \text{ is } q_2\text{-subadditive on } \Sigma_1. \tag{6.3.1}$$

Suppose that Σ_1, Σ_2 have identity elements e_1, e_2, respectively, and that

$$\phi(e_1) = e_2. \tag{6.3.2}$$

Note that this holds automatically when $\phi(\Sigma_1) = \Sigma_2$. If N_2 is normalized on Σ_2, then $N_2 \circ \phi$ is normalized on Σ_1. If N_2 is nondegenerate on Σ_2, and $x_1 \in \Sigma_1$ satisfies $\phi(x_1) = e_2$ only when $x_1 = e_1$, then $N_2 \circ \phi$ is nondegenerate on Σ_1. If Σ_1, Σ_2 are groups, and N_2 is symmetric on Σ_2, then $N_2 \circ \phi$ is symmetric on Σ_1.

Let Σ be a semigroup, and let N be a nonnegative real-valued function on Σ. Also let α be a monotonically increasing nonnegative real-valued function on $\mathbf{R}_+ \cup \{0\}$. If N is ultrasubadditive on Σ, then

$$\alpha \circ N \text{ is ultrasubadditive on } \Sigma. \tag{6.3.3}$$

If N is subadditive on Σ, and α is subadditive on $\mathbf{R}_+ \cup \{0\}$, then it is easy to see that

$$\alpha \circ N \text{ is subadditive on } \Sigma. \tag{6.3.4}$$

If Σ is a group, and N is symmetric on Σ, then $\alpha \circ N$ is symmetric on Σ as well.

Suppose that Σ has an identity element. If N is normalized on Σ, and $\alpha(0) = 0$, then $\alpha \circ N$ is normalized on Σ. If N is nondegenerate on Σ, and $\alpha(r) > 0$ when $r > 0$, then $\alpha \circ N$ is nondegenerate on Σ.

Let q be a positive real number, and put

$$\alpha_q(x) = \alpha(x^{1/q})^q \tag{6.3.5}$$

for every $x \geq 0$. It is easy to see that α_q is monotonically increasing as a nonnegative real-valued function on $\mathbf{R}_+ \cup \{0\}$, because α is monotonically increasing, by hypothesis. If α_q is aubadditive on $\mathbf{R}_+ \cup \{0\}$, and N is q-subadditive on Σ, then one can check that

$$\alpha \circ N \text{ is } q\text{-subadditive on } \Sigma. \tag{6.3.6}$$

More precisely, this is the same as saying that

$$(\alpha(N(u)))^q = \alpha_q(N(u)^q) \tag{6.3.7}$$

is subadditive on Σ. This can be obtained from the subadditivity of α_q and N^q, as before.

Let t be a positive real number, and put

$$\alpha(x) = \min(x, t) \tag{6.3.8}$$

for every $x \geq 0$. This is a monotonically increasing nonnegative real-valued function on $\mathbf{R}_+ \cup \{0\}$, with

$$\alpha_q(x) = \min(x, t^q) \tag{6.3.9}$$

for every $x \geq 0$. Thus α_q is subadditive on $\mathbf{R}_+ \cup \{0\}$, as in the previous section.

If N is q-subadditive on Σ, then it follows that

$$N_t(u) = \min(N(u), t) \tag{6.3.10}$$

is q-subadditive on Σ as well.

6.4 Sub-invariance under translations

Let Σ be a semigroup, and let $d(\cdot, \cdot)$ be a q-semimetric on Σ for some $q > 0$. Let us say that $d(\cdot, \cdot)$ is *sub-invariant under left translations* on Σ if

$$d(a\,x, a\,y) \leq d(x, y) \tag{6.4.1}$$

for every $a, y, x \in \Sigma$. Similarly, let us say that $d(\cdot, \cdot)$ is *sub-invariant under right translations* if

$$d(x\,a, y\,a) \leq d(x, y) \tag{6.4.2}$$

for every $a, x, y \in \Sigma$.

If Σ is a group, then sub-invariance under left or right translations implies invariance under left or right translations, as appropriate. More precisely, suppose that Σ has an identity element, and that $a \in \Sigma$ has an inverse $a^{-1} \in \Sigma$. In this case, sub-invariance under left or right translations by a and a^{-1} implies invariance under left or right translations by a, as appropriate.

Let Σ be a semigroup with an identity element e, and let $d(\cdot, \cdot)$ be a q-semimetric on Σ for some $q > 0$ again. Thus

$$N_d(x) = d(e, x) \tag{6.4.3}$$

is a normalized nonnegative real-valued function on Σ, which is nondegenerate when $d(\cdot, \cdot)$ is a q-metric on Σ. If $d(\cdot, \cdot)$ is sub-invariant under left or right translations on Σ, then one can check that N_d is q-subadditive on G, as in Section 3.7.

If Σ is a group, and $d(\cdot, \cdot)$ is sub-invariant under left or right translations, then $d(\cdot, \cdot)$ is invariant under left or right translations, as appropriate. In this case, N_d is symmetric on Σ, as in Section 3.3.

Suppose now that Σ is a group, and that N is a nonnegative real-valued function on Σ that is q_N-subadditive for some $q_N > 0$, normalized, and symmetric. Under these conditions, one can check that

$$d_{N,L}(x, y) = N(x^{-1}\,y) = N(y^{-1}\,x) \tag{6.4.4}$$

and

$$d_{N,R}(x, y) = N(x\,y^{-1}) = N(y\,x^{-1}) \tag{6.4.5}$$

define q_N-semimetrics on Σ. More precisely, (6.4.4) is invariant under left translations on Σ, and (6.4.5) is invariant under right translations. If N is also nondegenerate on Σ, then these are q_N-metrics on Σ. Note that

$$d_{N,R}(x,y) = d_{N,L}(x^{-1}, y^{-1}) \tag{6.4.6}$$

for every $x, y \in \Sigma$, and that

$$N(x) = d_{N,L}(e,x) = d_{N,R}(e,x) \tag{6.4.7}$$

for every $x \in \Sigma$.

Let q_0 be a positive real number with $q_0 \le q_N$, so that N is q_0-subadditive on Σ, as in Section 6.1. Similarly, $d_{N,L}$ and $d_{N,R}$ may be considered as q_0-semimetrics on Σ, so that $d_{N,L}(\cdot,\cdot)^{q_0}$ and $d_{N,R}(\cdot,\cdot)^{q_0}$ are semimetrics on Σ. It follows that $N(x)^{q_0}$ is Lipschitz of order q_0 with constant $C = 1$ on Σ with respect to $d_{N,L}$ and $d_{N,R}$, as in Section 1.8. In particular, this implies that N is continuous with respect to $d_{N,L}$ and $d_{N,R}$.

Suppose for the moment that Σ is a topological group. If N is continuous at e as a real-valued function on Σ, then $d_{N,L}$ and $d_{N,R}$ are compatible with the topology on Σ, as in Section 3.6. In this case, N is continuous on Σ, as before. More precisely, N^{q_0} is both left and right-invariant uniformly continuous on Σ.

Let t be a positive real number, and let N_t be as in (6.3.10). Also let $d_{N_t,L}$ and $d_{N_t,R}$ be the q_N-semimetrics corresponding to N_t as before. Observe that

$$d_{N_t,L}(x,y) = \min(d_{N,L}(x,y), t) \tag{6.4.8}$$

and

$$d_{N_t,R}(x,y) = \min(d_{N,R}(x,y), t) \tag{6.4.9}$$

for every $x, y \in \Sigma$. Thus $d_{N_t,L}$ and $d_{N_t,R}$ have the properties mentioned in Section 2.7, in relation to $d_{N,L}$ and $d_{N,R}$, respectively.

Let $d(\cdot,\cdot)$ be a q-semimetric on Σ for some $q > 0$, and let N_d be as in (6.4.3). If $d(\cdot,\cdot)$ is invariant under left translations on Σ, then it is easy to see that $d(\cdot,\cdot)$ is the same as the left-invariant q-semimetric associated to N_d as in (6.4.4). Similarly, if $d(\cdot,\cdot)$ is invariant under right translations, then $d(\cdot,\cdot)$ is the same as the right-invariant q-semimetric associated to N_d as in (6.4.6).

6.5 Combining subadditive functions

Let Σ be a semigroup, and let l be a positive integer. Also let N_j be a nonnegative real-valued function on Σ for each $j = 1, \ldots, l$, and suppose that N_j is q_j-subadditive on Σ for some $q_j > 0$. Put

$$q_0 = \min(q_1, \ldots, q_l), \tag{6.5.1}$$

so that N_j is q_0-subadditive on Σ for each $j = 1, \ldots, l$. One can check that

$$N(x) = \max_{1 \le j \le l} N_j(x) \tag{6.5.2}$$

is q_0-subadditive on Σ as well under these conditions. Note that

$$B_N(r) = \bigcap_{j=1}^{l} B_{N_j}(r) \tag{6.5.3}$$

for every $r > 0$, and

$$\overline{B}_N(r) = \bigcap_{j=1}^{l} \overline{B}_{N_j}(r) \tag{6.5.4}$$

for every $r \geq 0$, using the notation in (6.2.4) and (6.2.5), respectively.

Let r be a positive real number, and consider

$$\Big(\sum_{j=1}^{l} N_j(x)^r\Big)^{1/r}, \tag{6.5.5}$$

which is a nonnegative real-valued function of $x \in \Sigma$. If $r \leq q_0$, then (6.5.5) is r-subadditive on Σ. One can check this using (2.11.6) with the exponent $r/q_0 \leq 1$ when $q_0 < \infty$, or using the fact that N_j is r-subadditive on Σ for each $j = 1, \ldots, l$ when $r \leq q_0$, as in Sections 2.11 and 5.2. If $q_0 \leq r$, then one can verify that (6.5.5) is q_0-subadditive on Σ, using Minkowski's inequality with exponent $r/q_0 \geq 1$, as before.

Suppose for the moment that Σ has an identity element e. If N_j is normalized for each $j = 1, \ldots, l$, then (6.5.2) and (6.5.5) are normalized. It is easy to see that (6.5.2) and (6.5.5) are nondegenerate exactly when for every $x \in \Sigma$ with $x \neq e$ we have that $N_j(x) > 0$ for some j. If Σ is a group, and N_j is symmetric on Σ for every $j = 1, \ldots, l$, then (6.5.2) and (6.5.5) are symmetric on Σ as well.

Now let $N_1, N_2, N_3, \ldots$ be an infinite sequence of nonnegative real-valued functions on Σ, and suppose that for each $j \geq 1$, N_j is q_j-subadditive on Σ for some $q_j > 0$. Let us also ask that there be a $q_0 > 0$ such that

$$q_0 \leq q_j \tag{6.5.6}$$

for every $j \geq 1$. This can always be arranged with $q_0 = 1$, for instance, by replacing N_j with $N_j^{q_j}$ when $q_j < 1$. It follows that N_j is q_0-subadditive on Σ for every $j \geq 1$.

Let us suppose too that

$$\sup_{x \in \Sigma} N_j(x) \to 0 \quad \text{as } j \to \infty. \tag{6.5.7}$$

This can be arranged by replacing N_j by the minimum of N_j and a positive real number that tends to 0 as $j \to \infty$, as in Section 6.3. If $x \in \Sigma$, then we put

$$N(x) = \max_{j \geq 1} N_j(x). \tag{6.5.8}$$

More precisely, it is easy to see that the maximum is attained under these conditions. One can check that N is q_0-subadditive on Σ.

Suppose that Σ is a group, and that N_j is normalized and symmetric on Σ for every $j \geq 1$, so that N is normalized and symmetric too. Observe that N is nondegenerate on Σ exactly when for every $x \in \Sigma$ with $x \neq 0$ there is a $j \in \mathbf{Z}_+$ such that $N_j(x) > 0$.

Let $d_{N_j,L}$, $d_{N_j,R}$ be as in the previous section for each $j \geq 1$, and similarly for $d_{N,L}$, $d_{N,R}$. Thus

$$d_{N,L}(x,y) = \max_{j \geq 1} d_{N_j,L}(x,y) \tag{6.5.9}$$

and

$$d_{N,R}(x,y) = \max_{j \geq 1} d_{N_j,R}(x,y) \tag{6.5.10}$$

for every $x, y \in \Sigma$. The topologies determined on Σ by $d_{N,L}$ and $d_{N,R}$ are the same as the topologies determined by the collections of $d_{N_j,L}$ and $d_{N_j,R}$, $j \geq 1$, respectively, as in Section 2.8.

6.6 Subadditivity and sub-semigroups

Let Σ be a semigroup, and let A be a sub-semigroup of Σ. If $x \in \Sigma$, then put

$$\begin{aligned} N_A(x) &= 0 \quad \text{when } x \in A \\ &= 1 \quad \text{when } x \notin A. \end{aligned} \tag{6.6.1}$$

One can check that N_A is ultrasubadditive on Σ. If Σ has an identity element e, and $e \in A$, then N_A is normalized on Σ.

Suppose now that Σ is a group, and that A is a subgroup of Σ. Thus N_A is normalized and symmetric on Σ. This leads to semi-ultrametrics $d_{N_A,L}$ and $d_{N_A,R}$ on Σ, as in Section 6.4.

We also have left and right-invariant semi-ultrametrics $d_{A,L}$ and $d_{A,R}$ on Σ corresponding to A as in Section 3.7. It is easy to see that

$$d_{N_A,L} = d_{A,L}, \quad d_{N_A,R} = d_{A,R}. \tag{6.6.2}$$

Alternatively,

$$N_A(x) = d_{A,L}(e,x) = d_{A,R}(e,x) \tag{6.6.3}$$

for every $x \in \Sigma$.

Let Σ be a semigroup again, and let A be a sub-semigroup of Σ. Also let N_0 be a nonnegative real-valued function on A, and let r_0, r_1 be nonnegative real numbers. Suppose that

$$N_0(x) \leq r_0 \tag{6.6.4}$$

for every $x \in A$, which can always be arranged by taking the minimum of N_0 and r_0. If $x \in \Sigma$, then put

$$\begin{aligned} N(x) &= N_0(x) \quad \text{when } x \in A \\ &= r_1 \quad \text{when } x \notin A. \end{aligned} \tag{6.6.5}$$

Suppose that N_0 is q-subadditive on A for some $q > 0$. If

$$r_0 \le r_1, \tag{6.6.6}$$

then one can check that N is q-subadditive on Σ. More precisely, suppose that $q < \infty$, and that

$$r_0 \le 2^{1/q}\, r_1. \tag{6.6.7}$$

Suppose also that if $x, y \in \Sigma$ have the property that $x\,y \in A$, and if at least one of x and y is an element of A, then x and y are both in A. Under these conditions, one can verify that N is q-subadditive on G.

Suppose that Σ has an identity element e. Note that the condition on A mentioned in the preceding paragraph holds when $e \in A$, and A is a group. If $e \in A$ and N_0 is normalized on A, then N is normalized on Σ. If N_0 is nondegenerate on A and $r_1 > 0$, then N is nondegenerate on Σ. If Σ is a group, A is a subgroup of Σ, and N_0 is symmetric on A, then N is symmetric on Σ.

Suppose that Σ is a group, A is a subgroup of G, and that N_0 is normalized, symmetric, and q-subadditive on A for some $q > 0$. Suppose also that (6.6.6) or (6.6.7) holds, as appropriate, so that N is normalized, symmetric, and q-subadditive on Σ, as before. Using N_0, we get q-semimetrics $d_{N_0,L}$ and $d_{N_0,R}$ on A, as in Section 6.4. Similarly, we have q-semimetrics $d_{N,L}$ and $d_{N,R}$ on Σ associated to N, as before. One can check that $d_{N,L}$, $d_{N,R}$ correspond to $d_{N_0,L}$, $d_{N_0,R}$ as in Section 3.13, respectively.

6.7 Balanced functions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. Suppose for the moment that N is a q_N-seminorm on V with respect to $|\cdot|$ on k for some $q_N > 0$. This implies that N is q_N-subadditive, normalized, and symmetric on V, as a commutative group with respect to addition. If N is a q_N-norm on V, then N is nondegenerate on V.

Let us say that a nonnegative real-valued function N on V is *balanced* with respect to $|\cdot|$ on k if

$$N(t\,v) \le N(v) \tag{6.7.1}$$

for every $v \in V$ and $t \in k$ with $|t| \le 1$. This implies that

$$N(t\,v) = N(v) \tag{6.7.2}$$

for every $v \in V$ and $t \in k$ with $|t| = 1$. In particular, this means that N is symmetric on V. If N satisfies the homogeneity condition in the definition of a seminorm, then N is balanced on V.

Suppose that N is balanced on V. If α is a monotonically increasing nonnegative real-valued function on $\mathbf{R}_+ \cup \{0\}$, then it is easy to see that $\alpha \circ N$ is balanced on V. Thus

$$N_\tau(v) = \min(N(v), \tau) \tag{6.7.3}$$

is balanced on V for every positive real number τ.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k. In this case, N satisfies the homogeneity condition in the definition of a seminorm if and only if N is balanced and normalized on V.

Let $|\cdot|$ be any q_k-absolute value function on V again, and suppose that N is balanced on V with respect to $|\cdot|$ on k. If a is a positive real number, then $N(x)^a$ is balanced on V with respect to $|\cdot|$ on k as well.

Remember that $|\cdot|^a$ is a (q_k/a)-absolute value function on k, as in Section 5.1. Observe that N is balanced on V with respect to $|\cdot|$ if and only if N is balanced with respect to $|\cdot|^a$.

Let l be a positive integer, and suppose that N_j is a balanced nonnegative real-valued function on V for each $j = 1, \ldots, l$. Under these conditions, the maximum of $N_1, \ldots, N_l$ is balanced on V as well. If r is a positive real number, then (6.5.5) is balanced on V too.

Similarly, let $N_1, N_2, N_3, \ldots$ be an infinite sequence of nonnegative real-valued balanced functions on V. If the N_j's are bounded pointwise on V, then their supremum is balanced on V.

6.8 Subadditivity and conjugations

Let Σ be a semigroup with an identity element e, and let A be a sub-semigroup of Σ such that $e \in A$, and every element of A has an inverse in A, so that A is a group. Under these conditions, a nonnegative real-valued function N on Σ is invariant under conjugations by elements of A when

$$N(a\,x\,a^{-1}) = N(x) \tag{6.8.1}$$

for every $a \in A$ and $x \in \Sigma$. Equivalently, this means that

$$N(a\,x) = N(x\,a) \tag{6.8.2}$$

for every $a \in A$ and $x \in \Sigma$. If N is invariant under conjugations by elements of A, then $B_N(r)$ is invariant under conjugations by elements of A for every $r > 0$, and $\overline{B}_N(r)$ is invariant under conjugations by elements of A for every $r \geq 0$, where $B_N(r)$, $\overline{B}_N(r)$ are as in Section 6.1.

Let $d(\cdot,\cdot)$ be a q-semimetric on Σ for some $q > 0$. Of course, $d(\cdot,\cdot)$ is invariant under conjugations by elements of A when

$$d(a\,x\,a^{-1}, a\,y\,a^{-1}) = d(x, y) \tag{6.8.3}$$

for every $a \in A$ and $x, y \in \Sigma$. This is equivalent to asking that

$$d(a\,x, a\,y) = d(x\,a, y\,a) \tag{6.8.4}$$

for every $a \in A$ and $x, y \in \Sigma$. In this case, $d(e, x)$ is invariant under conjugations by elements of A, as a function of $x \in \Sigma$.

Suppose for the moment that Σ is a group, and that N is normalized, symmetric, and q_N-subadditive on Σ for some $q_N > 0$. Let $d_{N,L}$, $d_{N,R}$ be the corresponding q_N-semimetrics on Σ, as in Section 6.4. It is easy to see that the invariance of N, $d_{N,L}$, and $d_{N,R}$ under conjugations by elements of A are all equivalent. Note that $d_{N,L}$ is invariant under conjugations by elements of A if and only if $d_{N,L}$ is invariant under right translations by elements of A, because $d_{N,L}$ is invariant under left translations, as in Section 3.11. Similarly, $d_{N,R}$ is invariant under conjugations by elements of A if and only if $d_{N,R}$ is invariant under left translations by elements of A.

If N is invariant under conjugations by arbitrary elements of Σ, then we may simply say that N is invariant under conjugations on Σ. One can check that this happens if and only if $d_{N,L} = d_{N,R}$. In this case, one may denote this q_N-semimetric as d_N. If $q_N = \infty$, then it follows that $B_N(r)$ is a normal subgroup of Σ for every $r > 0$, and that $\overline{B}_N(r)$ is a normal subgroup of Σ for every $r \geq 0$.

If N is any nonnegative real-valued function on Σ, then one can get a function that is invariant under conjugations by elements of A by putting

$$N_A(x) = \sup_{a \in A} N(a\,x\,a^{-1}) \tag{6.8.5}$$

for every $x \in \Sigma$. More precisely, let us suppose that the supremum on the right is finite for every $x \in \Sigma$, which holds in particular when N is bounded on Σ. Of course, this can always be arranged by taking the minimum of N and some positive real number.

Suppose that N is normalized and q_N-subadditive on Σ for some $q_N > 0$ again, so that $N(a\,x\,a^{-1})$ has these properties as a function of $x \in \Sigma$ for every $a \in A$. One can use this to check that N_A has the same properties on Σ. Suppose now that Σ is a group, and that N is also symmetric on Σ. This implies that $N(a\,x\,a^{-1})$ is symmetric as a function of $x \in \Sigma$ for every $a \in A$, so that N_A is symmetric on Σ as well.

Let $d_{N,L}$, $d_{N,R}$ be as in Section 6.4 again, and let $d_{N_A,L}$, $d_{N_A,R}$ be the analogous q_N-semimetrics on Σ corresponding to N_A. One can verify that $d_{N_A,L}$, $d_{N_A,R}$ can be obtained from $d_{N,L}$, $d_{N,R}$, respectively, by taking the appropriate supremum over conjugations by elements of A, as in Section 3.11.

Suppose that Σ is equipped with a topology, and that conjugations by elements of A are equicontinuous at e as mappings on Σ, as in Section 3.5. If N is continuous at e, as a real-valued function on Σ, then it is easy to see that N_A is continuous at e as well.

6.9 Subadditivity and boundedness

Let Σ be a semigroup, and let N be a nonnegative real-valued function on Σ that is q-subadditive for some $q > 0$. Let us say that a subset E of Σ is *bounded* with respect to N if N is bounded on E. If A, B are subsets of Σ that are bounded with respect to N, then $A\,B$ is bounded with respect to N too.

Equivalently, E is bounded with respect to N if there is a nonnegative real number r such that

$$E \subseteq \overline{B}_N(r), \tag{6.9.1}$$

where the right side is as in Section 6.1. If E is also nonempty, then put

$$\rho_N(E) = \sup_{x \in E} N(x), \tag{6.9.2}$$

which may be interpreted as being 0 when $E = \emptyset$. This is the same as the smallest $r \geq 0$ such that (6.9.1) holds. If A, B are bounded subsets of Σ with respect to N, then

$$\rho_N(A\,B)^q \leq \rho_N(A)^q + \rho_N(B)^q \tag{6.9.3}$$

when $q < \infty$, and

$$\rho_N(A\,B) \leq \max(\rho_N(A), \rho_N(B)) \tag{6.9.4}$$

when $q = \infty$.

Let X be a nonempty set, and let us say that a mapping f from X into Σ is *bounded* with respect to N if $f(X)$ is a bounded set with respect to N. Let $\mathcal{B}(X, \Sigma) = \mathcal{B}_N(X, \Sigma)$ be the space of mappings from X into Σ that are bounded with respect to N. Note that this is a semigroup with respect to pointwise multiplication of functions on X.

If $f \in \mathcal{B}(X, \Sigma)$, then put

$$N_{sup}(f) = N_{sup,X}(f) = \sup_{x \in X} N(f(x)). \tag{6.9.5}$$

Equivalently,

$$N_{sup}(f) = \rho_N(f(X)). \tag{6.9.6}$$

One can check that this is q-subadditive on $\mathcal{B}(X, \Sigma)$.

Suppose that Σ has an identity element e, so that the constant function on X equal to e at every point is the identity element in $\mathcal{B}(X, \Sigma)$. If N is normalized on Σ, then (6.9.5) is normalized on $\mathcal{B}(X, \Sigma)$. Similarly, if N is nondegenerate on Σ, then (6.9.5) is nondegenerate on $\mathcal{B}(X, \Sigma)$.

Suppose now that Σ is a group, and that N is normalized and symmetric on Σ. If a subset E of Σ is bounded with respect to N, then E^{-1} is bounded with respect to N too. More precisely,

$$\rho_N(E^{-1}) = \rho_N(E). \tag{6.9.7}$$

Let $d_{N,L}$ and $d_{N,R}$ be the q-semimetrics on Σ associated to N as in Section 6.4. If E is bounded with respect to N, then it is easy to see that E is bounded with respect to both $d_{N,L}$ and $d_{N,R}$. Conversely, if E is bounded with respect to either $d_{N,L}$ or $d_{N,R}$, then E is bounded with respect to N.

Similarly, if $f \in \mathcal{B}(X, \Sigma)$, then $f(x)^{-1}$ is bounded on X with respect to N too, with

$$N_{sup}(f(\cdot)^{-1}) = N_{sup}(f). \tag{6.9.8}$$

This implies that $\mathcal{B}(X, \Sigma)$ is a group, with respect to pointwise multiplication of functions. Note that the boundedness of a mapping f from X into Σ with respect to N is equivalent to boundedness of f with respect to $d_{N,L}$, $d_{N,R}$.

Let $d_{N_{sup},L}$ and $d_{N_{sup},R}$ be the q-semimetrics on $\mathcal{B}(X,\Sigma)$ associated to (6.9.5) as in Section 6.4. One can check that these are the same as the supremum q-semimetrics on $\mathcal{B}(X,\Sigma)$ associated to $d_{N,L}$ and $d_{N,R}$, respectively, as in Section 2.12.

If N is invariant under conjugations on Σ, then (6.9.5) is invariant under conjugations on $\mathcal{B}(X,\Sigma)$. In this case, $d_{N,L} = d_{N,R}$, as in the previous section, and similarly $d_{N_{sup},L} = d_{N_{sup},R}$.

6.10 Some examples of sub-invariance

Let X be a nonempty set, let d be a q-semimetric on X for some $q > 0$, and suppose that X is bounded with respect to d. The space $\mathcal{M}(X)$ of all mappings from X into itself is a semigroup with respect to composition, with the identity mapping on X as the identity element. Let θ be the corresponding supremum q-semimetric on $\mathcal{M}(X)$, as in Section 2.12. It is easy to see that

$$\theta(f \circ h, g \circ h) \le \theta(f,g) \tag{6.10.1}$$

for every $f,g,h \in \mathcal{M}(X)$, so that θ is sub-invariant under right translations on $\mathcal{M}(X)$. Note that

$$\theta(f \circ h, g \circ h) = \theta(f,g) \tag{6.10.2}$$

when $h(X) = X$.

Let $C(X) = C(X,X)$ be the space of continuous mappings from X into itself, with respect to d. This is a sub-semigroup of $\mathcal{M}(X)$ that contains the identity mapping. One can check that (6.10.2) holds when $f,g \in C(X)$, $h \in \mathcal{M}(X)$, and $h(X)$ is dense in X with respect to d.

If h is an isometry from X into itself with respect to d, then

$$\theta(h \circ f, h \circ g) = \theta(f,g) \tag{6.10.3}$$

for every $f,g \in \mathcal{M}(X)$. If h is Lipschitz of order one with constant $C = 1$ with respect to d, then

$$\theta(h \circ f, \theta \circ g) \le \theta(f,g) \tag{6.10.4}$$

for every $f,g \in \mathcal{M}(X)$. If h is a uniformly continuous mapping from X into itself with respect to d, then one can verify that

$$f \mapsto h \circ f \tag{6.10.5}$$

is uniformly continuous as a mapping from $\mathcal{M}(X)$ into itself, with respect to θ.

Let $f_0, f, g_0, g \in \mathcal{M}(X)$ be given. If $q < \infty$, then

$$\begin{aligned} \theta(g \circ f, g_0 \circ f_0)^q &\le \theta(g \circ f, g_0 \circ f)^q + \theta(g_0 \circ f, g_0 \circ f_0)^q \\ &\le \theta(g, g_0)^q + \theta(g_0 \circ f, g_0 \circ f_0)^q. \end{aligned} \tag{6.10.6}$$

Similarly, if $q = \infty$, then

$$\begin{aligned} \theta(g \circ f, g_0 \circ f_0) &\le \max(\theta(g \circ f, g_0 \circ f), \theta(g_0 \circ f, g_0 \circ f_0)) \\ &\le \max(\theta(g, g_0), \theta(g_0 \circ f, g_0 \circ f_0)). \end{aligned} \tag{6.10.7}$$

If g_0 is Lipschitz of order one with constant $C = 1$, then

$$\theta(g \circ f, g_0 \circ f_0)^q \le \theta(g, g_0)^q + \theta(f, f_0)^q \tag{6.10.8}$$

when $q < \infty$, and

$$\theta(g \circ f, g_0 \circ f_0) \le \max(\theta(g, g_0), \theta(f, f_0)) \tag{6.10.9}$$

when $q = \infty$.

If g_0 is uniformly continuous on X, then one can check that

$$(f, g) \mapsto g \circ f \tag{6.10.10}$$

is continuous at (f_0, g_0), as a mapping from $\mathcal{M}(X) \times \mathcal{M}(X)$ into $\mathcal{M}(X)$. This uses the topology determined on $\mathcal{M}(X)$ by θ, and the corresponding product topology on $\mathcal{M}(X) \times \mathcal{M}(X)$. Note that the collection of uniformly continuous mappings from X into itself is a sub-semigroup of $C(X)$ that contains the identity mapping. The collection of Lipschitz mappings of order one with constant $C = 1$ from X into itself is a sub-semigroup of the semigroup of uniformly continuous mappings. The collection of isometric mappings from X into itself is a sub-semigroup of this semigroup.

6.11 Mappings into topological groups

Let X be a topological space, and let Σ be a semigroup. Suppose that Σ is equipped with a topology, and that the semigroup operation on Σ is continuous as a mapping from $\Sigma \times \Sigma$ into Σ, using the corresponding product topology on $\Sigma \times \Sigma$. It is easy to see that the space $C(X, \Sigma)$ of continuous mappings from X into Σ is a semigroup, with respect to pointwise multiplication of functions. If Σ has an identity element e, then the constant function on X equal to e at every point is the identity element in $C(X, \Sigma)$.

Suppose from now on in this section that Σ is a topological group. If f is a continuous mapping from X into Σ, then $f(x)^{-1}$ is continuous on X too. This means that $C(X, \Sigma)$ is a group, with respect to pointwise multiplication of functions.

Let $d(\cdot, \cdot)$ be a q_d-semimetric on X for some $q_d > 0$. Let us say that a mapping f from X into Σ is *left-invariant uniformly continuous* with respect to $d(\cdot, \cdot)$ if for every open subset U of Σ that contains e there is a $\delta > 0$ such that

$$f(w) \in f(x)\, U \tag{6.11.1}$$

for every $x, w \in X$ with $d(x, w) < \delta$. Similarly, let us say that f is *right-invariant uniformly continuous* with respect to $d(\cdot, \cdot)$ on X if for every open subset U of Σ that contains e there is a $\delta > 0$ such that

$$f(w) \in U\, f(x) \tag{6.11.2}$$

for every $x, w \in X$ with $d(x, w) < \delta$. Each of these conditions implies that f is continuous with respect to the topology determined on X by $d(\cdot, \cdot)$.

Let $\mathcal{U}_L$, $\mathcal{U}_R$ be the uniformities defined on Σ as in Section 3.2. It is easy to see that left and right-invariant uniform continuity of mappings from X into Σ, as in the preceding paragraph, are equivalent to uniform continuity with respect to the uniformity on X associated to $d(\cdot, \cdot)$ and $\mathcal{U}_L$, $\mathcal{U}_R$, respectively. Of course, one can also consider left and right-invariant uniform continuity conditions with respect to a collection of q-semimetrics on X, or a uniformity on X.

Let f be a mapping from X into Σ again. Observe that

(6.11.3) f is right-invariant uniformly continuous on X if and only if $f(x)^{-1}$ is left-invariant uniformly continuous on X.

Suppose now that conjugations on Σ are equicontinuous at e. This is equivalent to the condition that $\mathcal{U}_L = \mathcal{U}_R$, as in Section 3.14. It follows that left and right-invariant uniform continuity of mappings from X into Σ are the same in this case. Of course, this can also be verified directly, in terms of (6.11.1) and (6.11.2). This is a bit simpler if one uses the fact that the collection $\mathcal{B}_1$ of open subsets of Σ that contain e and are invariant under conjugations is a local base for the topology of Σ at e, as in Section 3.11.

Thus, under these conditions, we may simply say that a mapping f from X into Σ is uniformly continuous with respect to $d(\cdot, \cdot)$ on X when f is left or equivalently right-invariant uniformly continuous. If f and g are uniformly continuous mappings from X into Σ, then one can check that

(6.11.4) $$f(x)\, g(x) \text{ is uniformly continuous on } X.$$

This is a bit simpler using the fact that $\mathcal{B}_1$ is a local base for the topology of Σ at e, as before. This implies that the uniformly continuous mappings from X into Σ form a subgroup of $C(X, \Sigma)$.

6.12 Left and right sub-invariance

Let Σ be a semigroup, and let $d(\cdot, \cdot)$ be a q_d-semimetric on Σ for some $q_d > 0$. Suppose that $d(\cdot, \cdot)$ is sub-invariant under both left and right translations on Σ, and let $u, v, y, z \in \Sigma$ be given. If $q_d < \infty$, then

(6.12.1) $$d(u\,v, y\,z)^{q_d} \le d(u\,v, y\,v)^{q_d} + d(y\,v, y\,z)^{q_d} \le d(u, y)^{q_d} + d(v, z)^{q_d}.$$

Similarly, if $q_d = \infty$, then

(6.12.2) $$d(u\,v, y\,z) \le \max(d(u\,v, y\,v), d(y\,v, y\,z)) \le \max(d(u, y), d(v, z)).$$

In both cases, it follows in particular that the semigroup operation on Σ is continuous as a mapping from $\Sigma \times \Sigma$ into Σ, with respect to the topology determined on Σ by $d(\cdot, \cdot)$, and the corresponding product topology on $\Sigma \times \Sigma$.

Note that $\Sigma \times \Sigma$ may be considered as a semigroup, where the semigroup operation is defined coordinatewise. The remarks in the preceding paragraph imply that $d(\cdot,\cdot)$ is q_d-subadditive on $\Sigma \times \Sigma$. Conversely, if $d(\cdot,\cdot)$ is q-subadditive on $\Sigma \times \Sigma$ for any $q > 0$, then it is easy to see that $d(\cdot,\cdot)$ is sub-invariant under left and right translations on Σ.

Let X be a nonempty set, and let $d_X(\cdot,\cdot)$ be a q_X-semimetric on X for some $q_X > 0$. The space $C(X,\Sigma)$ of continuous mappings from X into Σ is a semigroup with respect to pointwise multiplication of functions, as in the previous section. One can check that the collection of uniformly continuous mappings from X into Σ is a sub-semigroup of $C(X,\Sigma)$, using the q_d-subadditivity of $d(\cdot,\cdot)$ on $\Sigma \times \Sigma$.

Let $\mathrm{Lip}_\alpha(X,\Sigma)$ be the space of mappings from X into Σ that are Lipschitz of order $\alpha > 0$ with some constant $C \geq 0$ with respect to d_N on Σ. If $\alpha = 1$, then this space may be denoted $\mathrm{Lip}(X,\Sigma)$, and we may simply say that its elements are Lipschitz on X, as before. One can verify that $\mathrm{Lip}_\alpha(X,\Sigma)$ is a sub-semigroup of $C(X,\Sigma)$ as well, using the q_d-subadditivity of $d(\cdot,\cdot)$ on $\Sigma \times \Sigma$.

If $f \in \mathrm{Lip}_\alpha(X,\Sigma)$, then we would like to put

$$\|f\|_{\mathrm{Lip}_\alpha(X,\Sigma)} = \sup\left\{\frac{d(f(x),f(w))}{d_X(x,w)^\alpha} : x,w \in X,\ d_X(x,w) > 0\right\}, \tag{6.12.3}$$

as in Sections 4.12 and 5.11. This should be interpreted as being equal to 0 when $d_X(x,w) = 0$ for every $x,w \in X$, as before. If $x,w \in X$ and $d_X(x,w) = 0$, then $d(f(x),f(w)) = 0$, by hypothesis. It is easy to see that (6.12.3) is the smallest $C \geq 0$ such that f is Lipschitz of order α with constant C on X. If $\alpha = 1$, then (6.12.3) may be denoted $\|f\|_{\mathrm{Lip}(X,\Sigma)}$.

One can check that (6.12.3) is q_d-subadditive on $\mathrm{Lip}_\alpha(X,\Sigma)$, using the q_d-subadditivity of $d(\cdot,\cdot)$ on $\Sigma \times \Sigma$ again. Of course, (6.12.3) is equal to 0 when f is constant on X. If $d(\cdot,\cdot)$ is a q_d-metric on Σ and (6.12.3) is equal to 0, then f is constant on X. If Σ has an identity element e, then the constant function on X equal to e at every point is the identity element in $C(X,\Sigma)$, and thus in $\mathrm{Lip}_\alpha(X,\Sigma)$. In particular, (6.12.3) is normalized on $\mathrm{Lip}_\alpha(X,\Sigma)$ in this case.

Suppose now that Σ is a group, so that $d(\cdot,\cdot)$ is invariant under left and right translations, as in Section 6.4. This implies that $d(\cdot,\cdot)$ is invariant under $x \mapsto x^{-1}$, as in Section 3.3. Remember that Σ is a topological group with respect to the topology determined by $d(\cdot,\cdot)$ under these conditions, as in Section 3.14. Thus $C(X,\Sigma)$ is a group with respect to pointwise multiplication of functions, as in the previous section. The collection of uniformly continuous mappings from X into Σ is a subgroup of $C(X,\Sigma)$, as before.

Similarly, $\mathrm{Lip}_\alpha(X,\Sigma)$ is a subgroup of $C(X,\Sigma)$ in this case. More precisely, if $f \in \mathrm{Lip}_\alpha(X,\Sigma)$, then

$$\|f(\cdot)^{-1}\|_{\mathrm{Lip}_\alpha(X,\Sigma)} = \|f\|_{\mathrm{Lip}_\alpha(X,\Sigma)}, \tag{6.12.4}$$

so that (6.12.3) is symmetric on $\mathrm{Lip}_\alpha(X,\Sigma)$.

6.13 Bilipschitz mappings into groups

Let us continue with the same notation and hypotheses as in the previous section. In particular, we suppose that Σ is a group, so that $d(\cdot,\cdot)$ is invariant under left and right translations on Σ.

Let f and g be mappings from X into Σ, and let $x, w \in X$ be given. If $q_d < \infty$, then

$$d(f(x), f(w))^{q_d} \le d(f(x)\, g(x)^{-1}, f(w)\, g(w)^{-1})^{q_d} + d(g(x), g(w))^{q_d}, \tag{6.13.1}$$

as in (6.12.1). Similarly,

$$d(f(x), f(w))^{q_d} \le d(g(x)^{-1}\, f(x), g(w)^{-1}\, f(w))^{q_d} + d(g(x), g(w))^{q_d}. \tag{6.13.2}$$

If $q_d = \infty$, then

$$d(f(x), f(w)) \le \max(d(f(x)\, g(x)^{-1}, f(w)\, g(w)^{-1}), d(g(x), g(w))), \tag{6.13.3}$$

as in (6.12.2). We also have that

$$d(f(x), f(w)) \le \max(d(g(x)^{-1}\, f(x), g(w)^{-1}\, f(w)), d(g(x), g(w))). \tag{6.13.4}$$

Suppose that

$$d(f(x), f(w)) \ge c\, d_X(x, w) \tag{6.13.5}$$

for some $c > 0$ and all $x, w \in X$. This implies that f is injective on X when $d_X(\cdot,\cdot)$ is a q_X-metric on X, as usual. If f is injective on X, then (6.13.5) is the same as saying that f^{-1} is Lipschitz of order 1 with constant $1/c$ as a mapping from $f(X)$ onto X, with respect to the restriction of $d(\cdot,\cdot)$ to $f(X)$, as in Section 1.8.

If $f(x)\, g(x)^{-1}$ is Lipschitz of order one on X, and $q_d < \infty$, then we get that

$$c^{q_d}\, d_X(x, w)^{q_d} \le \|f(\cdot)\, g(\cdot)^{-1}\|_{\mathrm{Lip}(X,\Sigma)}^{q_d}\, d_X(x, w)^{q_d} + d(g(x), g(w))^{q_d} \tag{6.13.6}$$

for every $x, w \in X$. Equivalently,

$$(c^{q_d} - \|f(\cdot)\, g(\cdot)^{-1}\|_{\mathrm{Lip}(X,\Sigma)}^{q_d})\, d_X(x, w)^{q_d} \le d(g(x), g(w))^{q_d} \tag{6.13.7}$$

for every $x, w \in X$. This is interesting only when

$$\|f(\cdot)\, g(\cdot)^{-1}\|_{\mathrm{Lip}(X,\Sigma)} < c, \tag{6.13.8}$$

in which case we obtain that

$$(c^{q_d} - \|f(\cdot)\, g(\cdot)^{-1}\|_{\mathrm{Lip}(X,\Sigma)}^{q_d})^{1/q_d}\, d_X(x, w) \le d(g(x), g(w)) \tag{6.13.9}$$

for every $x, w \in X$.

If $q_d = \infty$, then we obtain that

$$c\, d_X(x, w) \le \max(\|f(\cdot)\, g(\cdot)^{-1}\|_{\mathrm{Lip}(X,\Sigma)}\, d_X(x, w), d(g(x), g(w))) \tag{6.13.10}$$

for every $x, w \in X$. If (6.13.8) holds, then we obtain that

$$c\, d_X(x, w) \le d(g(x), g(w)) \tag{6.13.11}$$

for every $x, w \in X$. Of course, this holds automatically when $d_X(x, w) = 0$.

Suppose now that $g(x)^{-1}\, f(x)$ is Lipschitz of order one on X, with

$$\|g(\cdot)^{-1}\, f(\cdot)\|_{\mathrm{Lip}(X,\Sigma)} < c. \tag{6.13.12}$$

If $q_d < \infty$, then

$$(c^{q_d} - \|g(\cdot)^{-1}\, f(\cdot)\|_{\mathrm{Lip}(X,\Sigma)}^{q_d})^{1/q_d}\, d_X(x, w) \le d(g(x), g(w)) \tag{6.13.13}$$

for every $x, w \in X$, as before. If $q_d = \infty$, then (6.13.11) holds for every $x, w \in X$, by the same type of argument as before.

If f is Lipschitz of order one on X, and either $f(x)\, g(x)^{-1}$ or $g(x)^{-1}\, f(x)$ is Lipschitz of order one, then g is Lipschitz of order one on X as well, as in the previous section. If (6.13.8) or (6.13.12) holds too, then it follows that g is bilipschitz on X.

Suppose now that $q_d = \infty$, and that f is an isometry from X into Σ, so that f is Lipschitz of order one with constant $C = 1$, and (6.13.5) holds with $c = 1$. If $f(x)\, g(x)^{-1}$ is Lipschitz of order one, and (6.13.8) holds with $c = 1$, then we get that g is Lipschitz of order one with constant $C = 1$, and that (6.13.11) holds with $c = 1$, so that g is an isometry from X into Σ as well. Similarly, if $g(x)^{-1}\, f(x)$ is Lipschitz of order one, and (6.13.12) holds with $c = 1$, then g is an isometry from X into Σ.

6.14 Lipschitz homomorphisms

Let Σ_1, Σ_2 be groups, and let N_1, N_2 be nonnegative real-valued functions on Σ_1, Σ_2 that are normalized, symmetric, and q_1, q_2-subadditive for some $q_1, q_2 > 0$, respectively. This leads to semimetrics $d_{N_1,L}$, $d_{N_1,R}$ and $d_{N_2,L}$, $d_{N_2,R}$ on Σ_1 and Σ_2, respectively, as in Section 6.4. Remember that $d_{N_j,L}$ and $d_{N_j,R}$ correspond to each other under the mapping $x \mapsto x^{-1}$ on Σ_j, $j = 1, 2$. If e_1, e_2 are the identity elements in Σ_1, Σ_2, respectively, then

$$d_{N_j,L}(e_j, x) = d_{N_j,R}(e_j, x) = N_j(x) \tag{6.14.1}$$

for every $x \in \Sigma_j$, $j = 1, 2$, as before.

Let ϕ be a homomorphism from Σ_1 into Σ_2. If $x, w \in \Sigma_1$, then

$$d_{N_2,L}(\phi(x), \phi(w)) = N_2(\phi(x)^{-1}\, \phi(w)) = N_2(\phi(x^{-1}\, w)). \tag{6.14.2}$$

Similarly,

$$d_{N_2,R}(\phi(x), \phi(w)) = N_2(\phi(x)\, \phi(w)^{-1}) = N_2(\phi(x\, w^{-1})). \tag{6.14.3}$$

Let us say that ϕ is continuous at e_1 with respect to N_1, N_2 if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$N_2(\phi(x)) < \epsilon \tag{6.14.4}$$

for every $x \in \Sigma_1$ with $N_1(x) < \delta$. This is equivalent to the continuity of ϕ at e_1 with respect to $d_{N_1,L}$ or $d_{N_1,R}$ on Σ_1, and $d_{N_2,L}$ or $d_{N_2,R}$ on Σ_2. In this case, it is easy to see that ϕ is uniformly continuous with respect to $d_{N_1,L}$ on Σ_1 and $d_{N_2,L}$ on Σ_2, using (6.14.2). Similarly, ϕ is uniformly continuous with respect to $d_{N_1,R}$ on Σ_1 and $d_{N_2,R}$ on Σ_2, because of (6.14.3).

Let us say that ϕ is *Lipschitz* of order $\alpha > 0$ with constant $C \geq 0$ at e_1 with respect to N_1 on Σ_1 and N_2 on Σ_2 if

$$N_2(\phi(x)) \leq C\, N_1(x)^\alpha \tag{6.14.5}$$

for every $x \in \Sigma_1$. Of course, this implies that ϕ is continuous at e with respect to N_1, N_2. If N_2 is nondegenerate on Σ_2, then this condition holds with $C = 0$ only when $\phi(x) = e_2$ for every $x \in \Sigma_1$. If this condition holds for some $C \geq 0$, then ϕ is Lipschitz of order α with constant C with respect to $d_{N_1,L}$ on Σ_1 and $d_{N_2,L}$ on Σ_2, because of (6.14.2). Similarly, this condition implies that ϕ is Lipschitz of order α with constant C with respect to $d_{N_1,R}$ on Σ_1 and $d_{N_2,R}$ on Σ_2, because of (6.14.3).

Suppose that

$$N_2(\phi(x)) \geq c\, N_1(x) \tag{6.14.6}$$

for some $c > 0$ and every $x \in \Sigma_1$. This implies that the kernel of ϕ is trivial when N_1 is nondegenerate on Σ_1. If ϕ is injective on Σ_1, then (6.14.6) is the same as saying that ϕ^{-1} is Lipschitz of order one with constant $1/c$ at e_2 as a homomorphism from $\phi(\Sigma_1)$ onto Σ_1, with respect to N_1 on Σ_1 and the restriction of N_2 to $\phi(\Sigma_1)$. One can use (6.14.2) and (6.14.6) to get that

$$d_{N_2,L}(\phi(x), \phi(w)) \geq c\, d_{N_1,L}(x, w) \tag{6.14.7}$$

for every $x, w \in \Sigma_1$. Similarly, one can use (6.14.3) and (6.14.6) to get that

$$d_{N_2,R}(\phi(x), \phi(w)) \geq c\, d_{N_1,R}(x, w) \tag{6.14.8}$$

for every $x, w \in \Sigma_1$.

Let us say that ϕ is an *isometric homomorphism* with respect to N_1 on Σ_1 and N_2 on Σ_2 if

$$N_2(\phi(x)) = N_1(x) \tag{6.14.9}$$

for every $x \in \Sigma_1$. This implies that ϕ is an isometry with respect to $d_{N_1,L}$ on Σ_1 and $d_{N_2,L}$ on Σ_2, because of (6.14.2). Similarly, this implies that ϕ is an isometry with respect to $d_{N_1,R}$ on Σ_1 and $d_{N_2,R}$ on Σ_2, because of (6.14.3).

Of course, if Σ_2 is a commutative group, then $d_{N_2,L} = d_{N_2,R}$, which defines a q_2-semimetric d_{N_2} on Σ_2 that is invariant under left and right translations.

6.15 Dense subgroups

Let Σ be a group, and let N be a nonnegative real-valued function on Σ that is normalized, symmetric, and q-subadditive for some $q > 0$. This leads to q-semimetrics $d_{N,L}$ and $d_{N,R}$ on Σ, as in Section 6.4. If E is a symmetric subset of Σ, then it is easy to see that E is dense in Σ with respect to $d_{N,L}$ if and only if E is dense in Σ with respect to $d_{N,R}$. This uses the fact that $d_{N,L}$ and $d_{N,R}$ correspond to each other under the mapping $x \mapsto x^{-1}$ on Σ, as before.

Let Σ_0 be a subgroup of Σ, and let c be a positive real number with $c < 1$. Suppose that for each $w \in \Sigma$ there is a $w_0 \in \Sigma_0$ such that

$$d_{N,R}(w, w_0) = N(w\, w_0^{-1}) \le c\, N(w). \tag{6.15.1}$$

We would like to check that Σ_0 is dense in Σ with respect to $d_{N,R}$, as in Section 5.14. Let $w \in \Sigma$ be given, and suppose that $w_1, \ldots, w_r \in \Sigma_0$ have been chosen for some positive integer r, with

$$N(w\, w_1^{-1} \cdots w_r^{-1}) \le c^r\, N(w). \tag{6.15.2}$$

Under these conditions, there is a $w_{r+1} \in \Sigma_0$ such that

$$N(w\, w_1^{-1} \cdots w_r^{-1}\, w_{r+1}^{-1}) \le c\, N(w\, w_1^{-1} \cdots w_r^{-1}) \le c^{r+1}\, N(w), \tag{6.15.3}$$

by hypothesis. Thus we can continue the process, and get an infinite sequence $w_1, w_2, w_3, \ldots$ of elements of Σ_0 such that

$$d_{N,R}(w, w_r \cdots w_1) \le c^r\, N(w) \tag{6.15.4}$$

for every $r \ge 1$. This implies that Σ_0 is dense in Σ with respect to $d_{N,R}$, because $w_r \cdots w_1 \in \Sigma_0$.

Similarly, suppose that for every $w \in \Sigma$ there is a $w_0 \in \Sigma_0$ such that

$$d_{N,L}(w, w_0) = N(w_0^{-1}\, w) \le c\, N(w). \tag{6.15.5}$$

One can use the same type of argument as in the preceding paragraph to get that Σ_0 is dense in Σ with respect to $d_{N,L}$. In fact, this condition is equivalent to the previous one, with w, w_0 replaced with w^{-1}, w_0^{-1}, respectively.

Let Σ_1 be a subgroup of Σ that is dense with respect to $d_{N,R}$, or equivalently $d_{N,L}$, and let c_1 be a positive real number with $c_1 < 1$. Suppose that for every $w \in \Sigma_1$ there is a $w_0 \in \Sigma_0$ such that

$$d_{N,R}(w, w_0) = N(w\, w_0^{-1}) \le c_1\, N(w). \tag{6.15.6}$$

Let c_0 be a real number with $c_1 < c_0 < 1$. If $w \in \Sigma$, then we would like to find a $w_0 \in \Sigma_0$ such that

$$d_{N,R}(w, w_0) = N(w\, w_0^{-1}) \le c_0\, N(w), \tag{6.15.7}$$

as in Section 5.14. Of course, if $N(w) = 0$, then we can take w_0 to be the identity element e. Otherwise, if $N(w) > 0$, then we can first choose $w_1 \in \Sigma_1$ such that

$$d_{N,R}(w, w_1) = N(w\, w_1^{-1}) \tag{6.15.8}$$

is as small as we like. This implies in particular that $N(w_1)$ is as close as we like to $N(w)$. Because $w_1 \in \Sigma_1$, there is a $w_0 \in \Sigma_0$ such that

$$d_{N,R}(w_1, w_0) = N(w_1\, w_0^{-1}) \le c_1\, N(w_1). \tag{6.15.9}$$

One can use this to get (6.15.7) when (6.15.8) is small enough. This means that Σ_0 is dense in Σ_1 with respect to $d_{N,R}$, as before.

Similarly, suppose that for every $w \in \Sigma_1$ there is a $w_0 \in \Sigma_0$ such that

$$d_{N,L}(w, w_0) = N(w_0^{-1}\, w) \le c_1\, N(w). \tag{6.15.10}$$

If $w \in \Sigma$, then we can use the same type of argument to find a $w_0 \in \Sigma_0$ such that

$$d_{N,L}(w, w_0) = N(w_0^{-1}\, w) = c_0\, N(w). \tag{6.15.11}$$

Alternatively, one can reduce to the previous version, using the mapping $x \mapsto x^{-1}$ on Σ.

Chapter 7

Bilinear mappings and submultiplicativity

7.1 Bounded linear mappings

Let V, W be commutative groups, where the group operations are expressed additively. It will sometimes be convenient to refer to a group homomorphism from V into W as being *linear*. The space of linear mappings from V into W may be denoted $\mathcal{L}(V, W)$. This is a commutative group with respect to pointwise addition of mappings.

If V and W are vector spaces over a field k, then a linear mapping from V into W as vector spaces over k may be described as being *linear over* k. The space of linear mappings from V into W as vector spaces over k may be denoted $\mathcal{L}_k(V, W)$, to indicate the role of k.

Let V and W be commutative groups again, and let N_V, N_W be nonnegative real-valued functions on V, W, respectively. Suppose that N_V, N_W are normalized, symmetric, and q_V, q_W-subadditive on V, W for some $q_V, q_W > 0$, respectively. Let us say that a linear mapping T from V into W is *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \le C\, N_V(v) \tag{7.1.1}$$

for every $v \in V$. If $u, v \in V$, then it follows that

$$N_W(T(u) - T(v)) = N_W(T(u - v)) \le C\, N_V(u - v), \tag{7.1.2}$$

so that T is Lipschitz of order 1 with constant C with respect to the q_V, q_W-semimetrics on V, W associated to N_V, N_W, respectively. Conversely, this Lipschitz condition clearly implies that T is bounded in the sense of (7.1.1).

Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W with respect to N_V, N_W, in the sense of (7.1.1). One can check that this is a subgroup of $\mathcal{L}(V, W)$. If $T \in \mathcal{BL}(V, W)$, then we would like to put

$$\|T\|_{op} = \|T\|_{op,VW} = \sup\left\{\frac{N_W(T(v))}{N_V(v)} : v \in V,\, N_V(v) > 0\right\}, \tag{7.1.3}$$

as before. This may be interpreted as being equal to 0 when $N_V(v) = 0$ for every $v \in V$, as usual. Of course, if $v \in V$ and $N_V(v) = 0$, then $N_W(T(v)) = 0$, as in (7.1.1).

Equivalently, $\|T\|_{op}$ is the smallest $C \geq 0$ such that (7.1.1) holds for every $v \in V$. One can check that $\|\cdot\|_{op}$ is normalized, symmetric, and q_W-subadditive on $\mathcal{BL}(V, W)$. If N_W is nondegenerate on W, then $\|\cdot\|_{op}$ is nondegenerate on $\mathcal{BL}(V, W)$. Note that $\|T\|_{op}$ is also the same as the smallest $C \geq 0$ such that (7.1.2) holds for every $u, v \in V$. This corresponds to $\|T\|_{\mathrm{Lip}(V,W)}$ in Section 6.12 as well, using the q_V, q_W-semimetrics on V, W associated to N_V, N_W, respectively.

Let Z be another commutative group, and let N_Z be a nonnegative real-valued function on Z that is normalized, symmetric, and q_Z-subadditive for some $q_Z > 0$. Suppose that T_1 is a bounded linear mapping from V into W, and that T_2 is a bounded linear mapping from W into Z. Under these conditions, it is easy to see that $T_2 \circ T_1$ is a bounded linear mapping from V into Z, with

$$\|T_2 \circ T_1\|_{op,VZ} \leq \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}. \tag{7.1.4}$$

This was mentioned in Section 5.7 for linear mappings between vector spaces with q-seminorms. One could also look at this in terms of compositions of Lipschitz mappings, as in Section 5.11.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and suppose that V, W are vector spaces over k. If N_V, N_W are q_V, q_W-seminorms on V, W, respectively, then N_V, N_W are normalized, symmetric, and q_V, q_W-subadditive, on V, W, respectively, as commutative groups with respect to addition, as in Section 6.7. If T is a linear mapping from V into W as vector spaces over k, then the boundedness of T with respect to N_V, N_W considered here is the same as in Section 5.7. Similarly, (7.1.3) is the same as before. The space of bounded linear mappings from V into W as vector spaces over k may be denoted $\mathcal{BL}_k(V, W)$, to indicate the role of k.

7.2 Bounded bilinear mappings

Let V, W, and Z be commutative groups, where the group operations are expressed additively. A mapping b from $V \times W$ into Z is said to be *bilinear* if it is linear in each variable separately. This means that

$$b_{1,w}(v) = b(v, w) \tag{7.2.1}$$

is a linear mapping from V into Z for each $w \in W$, and that

$$b_{2,v}(w) = b(v, w) \tag{7.2.2}$$

is a linear mapping from W into Z for every $v \in V$. In this case,

$$w \mapsto b_{1,w} \tag{7.2.3}$$

is a linear mapping from W into $\mathcal{L}(V, Z)$, and

$$v \mapsto b_{2,v} \tag{7.2.4}$$

is a linear mapping from V into $\mathcal{L}(W, Z)$. Conversely, a linear mapping from W into $\mathcal{L}(V, Z)$, or from V into $\mathcal{L}(W, Z)$, leads to a bilinear mapping from $V \times W$ into Z in this way.

Suppose for the moment that V, W, and Z are vector spaces over a field k. A mapping b from $V \times W$ into Z is considered to be *bilinear* with V, W, and Z considered as vector spaces over k if it is linear in each variable separately, as a mapping between vector spaces over k. This means that (7.2.1) and (7.2.2) are linear over k, and we may say that b is *bilinear over k*. Under these conditions, (7.2.3) is linear over k as a mapping from W into $\mathcal{L}_k(V, Z)$, and (7.2.4) is linear over k as a mapping from V into $\mathcal{L}_k(W, Z)$. Conversely, a mapping from W into $\mathcal{L}_k(V, Z)$ that is linear over k, or a mapping from V into $\mathcal{L}_k(W, Z)$ that is linear over k, leads to a mapping from $V \times W$ into Z that is bilinear over k, as before.

Let V, W, and Z be commutative groups again, and let N_V, N_W, and N_Z be nonnegative real-valued functions on V, W, and Z, respectively. Suppose that N_V, N_W, and N_Z are normalized, symmetric, and q_V, q_W, q_Z-subadditive on V, W, Z for some $q_V, q_W, q_Z > 0$, respectively. A bilinear mapping b from $V \times W$ into Z is said to be *bounded* with respect to N_V, N_W, and N_Z if there is a nonnegative real number C such that

$$N_Z(b(v, w)) \le C\, N_V(v)\, N_W(w) \tag{7.2.5}$$

for every $v \in V$ and $w \in W$. Note that this implies that b is continuous at $(0, 0)$ in $V \times W$, with respect to the q_Z-semimetric on Z associated to N_Z, and a suitable product q-semimetric on $V \times W$, obtained from the q_V, q_W-semimetrics on V, W associated to N_V, N_W, respectively. One can check that b is continuous on all of $V \times W$ with respect to these q-semimetrics under these conditions, using standard arguments.

If $w \in W$, then (7.2.5) implies that (7.2.1) is a bounded linear mapping from V into Z, with

$$\|b_{1,w}\|_{op,VZ} \le C\, N_W(w). \tag{7.2.6}$$

Similarly, if $v \in V$, then (7.2.5) implies that (7.2.2) is a bounded linear mapping from W into Z, with

$$\|b_{2,v}\|_{op,WZ} \le C\, N_V(v). \tag{7.2.7}$$

In particular, this means that (7.2.3) maps W into $\mathcal{BL}(V, Z)$, and that (7.2.4) maps V into $\mathcal{BL}(W, Z)$. More precisely, these are bounded linear mappings, with respect to $\|\cdot\|_{op,VZ}$, $\|\cdot\|_{op,WZ}$ on $\mathcal{BL}(V, Z)$, $\mathcal{BL}(W, Z)$, respectively. Conversely, a bounded linear mapping from W into $\mathcal{BL}(V, Z)$, or from V into $\mathcal{BL}(W, Z)$, leads to a bounded bilinear mapping from $V \times W$ into Z in the same way as before.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and suppose that V, W, and Z are vector spaces over k again. Let N_V, N_W, and N_Z be q_V, q_W, and q_Z-seminorms on V, W, and Z with respect to $|\cdot|$ on k for some $q_V, q_W, q_Z > 0$, respectively, and let b be a mapping from $V \times W$ into Z that is bilinear over k. If $|\cdot|$ is not the trivial absolute value function on k, and if b is continuous at $(0,0)$ with respect to the q-semimetrics obtained from N_V, N_W, and N_Z as before, then one can verify that b is bounded with respect to N_V, N_W, and N_Z. More precisely, it suffices to ask that $N_Z(b(v,w))$ be bounded on the product of balls in V, W of positive radius centered at 0 with respect to N_V, N_W, respectively. This is analogous to the corresponding statement for linear mappings mentioned in Section 5.7.

7.3 Continuity properties and completeness

Let V, W, and Z be commutative groups, where the group operations are expressed additively, and let b be a bilinear mapping from $V \times W$ into Z. If $v_1, v_2 \in V$ and $w_1, w_2 \in W$, then

$$b(v_1, w_1) - b(v_2, w_2) = b(v_1 - v_2, w_1) + b(v_2, w_1 - w_2). \tag{7.3.1}$$

Let N_V, N_W, and N_Z be nonnegative real-valued functions on V, W, Z that are normalized, symmetric, and q_V, q_W, and q_Z-subadditive for some $q_V, q_W, q_Z > 0$, respectively. If b is bounded with respect to N_V, N_W, and N_Z, then one can use (7.3.1) to check that b is Lipschitz of order one on bounded subsets of $V \times W$, with respect to the q_Z-semimetric on Z associated to N_Z, and a suitable product q-semimetric on $V \times W$ obtained from the q_V, q_W-semimetrics on V, W associated to N_V, N_W, respectively. In particular, this implies that b is continuous on $V \times W$, as in the previous section.

Suppose now that N_Z is nondegenerate on Z, and that Z is complete with respect to the associated q_Z-metric. In this case, $\|\cdot\|_{op,VZ}$ is nondegenerate on $\mathcal{BL}(V,Z)$, as in Section 7.1. One can verify that

(7.3.2) $\mathcal{BL}(V,Z)$ is complete with respect to the q_Z-metric associated to $\|\cdot\|_{op,VZ}$,

as in Section 5.7.

Let V_0 be a subgroup of V, and suppose that V_0 is dense in V, with respect to the q_V-semimetric associated to N_V. Also let T_0 be a linear mapping from V_0 into Z that is bounded, with respect to the restriction of N_V to V_0. Under these conditions,

(7.3.3) there is a unique extension of T_0 to a bounded linear mapping T from V into Z,

as in Section 5.7 again. More precisely,

$$\|T\|_{op,VZ} = \|T_0\|_{op,V_0Z}, \tag{7.3.4}$$

as before.

Similarly, let W_0 be a subgroup of W that is dense in W with respect to the q_W-semimetric associated to N_W. Suppose that b_0 is a bilinear mapping from $V_0 \times W_0$ into Z that is bounded with respect to the restrictions of N_V, N_W to V_0, W_0, respectively. Under these conditions,

(7.3.5) there is a unique extension of b_0 to a bounded bilinear mapping b from $V \times W$ into Z.

The uniqueness of the extension can be obtained from the continuity of a bounded bilinear mapping on $V \times W$.

To get the existence of such an extension, one can use Cauchy sequences, as in Section 5.7. More precisely, if $v \in V$ and $w \in W$, then there are sequences $\{v_j\}_{j=1}^{\infty}$ and $\{w_j\}_{j=1}^{\infty}$ of elements of V_0, W_0 that converge to v, w with respect to the q_V, q_W-semimetrics associated to N_V, N_W, respectively. In particular, $\{v_j\}_{j=1}^{\infty}$ and $\{w_j\}_{j=1}^{\infty}$ may be considered as Cauchy sequences in V_0, W_0 with respect to the q_V, q_W-semimetrics associated to N_V, N_W, respectively. One can use this to check that

(7.3.6) $\{b_0(v_j, w_j)\}_{j=1}^{\infty}$ is a Cauchy sequence in Z,

with respect to the q_Z-metric associated to N_Z. This also uses the boundedness of b_0 on $V_0 \times W_0$, and the fact that Cauchy sequences with respect to a q-semimetric are bounded.

It follows that

(7.3.7) $\{b_0(v_j, w_j)\}_{j=1}^{\infty}$ converges in Z,

because Z is complete, by hypothesis. We would like to use the limit of this sequence to define $b(v, w)$ as an element of Z, as before. One can verify that this does not depend on the particular choices of sequences $\{v_j\}_{j=1}^{\infty}$ and $\{w_j\}_{j=1}^{\infty}$, using the boundedness of b_0 on $V_0 \times W_0$ again. It is easy to see that this extension is a bounded bilinear mapping on $V \times W$, because of the corresponding properties of b_0 on $V_0 \times W_0$.

Alternatively, if $w \in W_0$, then $b_0(v, w)$ may be considered as a bounded linear mapping from V_0 into Z, as a function of v. This has a unique extension to a bounded linear mapping from V into Z, as before. One can use this to get a unique extension of b_0 to a bounded bilinear mapping from $V \times W_0$ into Z. Similarly, this has a unique extension to a bounded bilinear mapping from $V \times W$ into Z.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and suppose that V, W, Z are vector spaces over k. Suppose also that N_V, N_W are q_V, q_W-seminorms on V, W, respectively, and that N_Z is a q_Z-norm on Z, with respect to $|\cdot|$ on k. Let V_0, W_0 be dense linear subspaces of V, W, with respect to the q_V, q_W-semimetrics associated to N_V, N_W, respectively. If b_0 is a bounded bilinear mapping from $V_0 \times W_0$ into Z, and Z is complete with respect to the q_Z-metric associated to N_Z, then b_0 has a unique extension to a bounded bilinear mapping b from $V \times W$ into Z. More precisely, b_0 is asked to be bilinear over k, and one can use this to get that b is bilinear over k too.

7.4 Related conditions on linear mappings

Let V, W be commutative groups, where the group operations are expressed additively, and let N_V, N_W be nonnegative real-valued functions on V, W that are normalized, symmetric, and q_V, q_W-subadditive for some $q_V, q_W > 0$, respectively. Suppose that T is a linear mapping from V into W such that

$$N_W(T(v)) \geq c\, N_V(v) \tag{7.4.1}$$

for some $c > 0$ and all $v \in V$. This type of condition was considered in Section 5.13 for linear mappings between vector spaces, and in Section 6.14 for homomorphisms between groups that need not be commutative. If T is injective on V, then (7.4.1) is the same as saying that T^{-1} is bounded as a linear mapping from $T(V)$ onto V, with respect to the restriction of N_W to $T(V)$, and with constant $1/c$, as before. If N_V is nondegenerate on V, then (7.4.1) implies that the kernel of T is trivial.

Of course, (7.4.1) implies that

$$c\, N_V(u - v) \leq N_W(T(u - v)) = N_W(T(u) - T(v)) \tag{7.4.2}$$

for every $u, v \in V$, as in Sections 5.13 and 6.14. Let us say that T is an *isometric linear mapping* from V into W with respect to N_V, N_W if

$$N_W(T(v)) = N_V(v) \tag{7.4.3}$$

for every $v \in V$. This means that

$$N_W(T(u) - T(v)) = N_W(T(u - v)) = N_V(u - v) \tag{7.4.4}$$

for every $u, v \in V$, so that T is an isometry from V into W with respect to the q_V, q_W-semimetrics associated to N_V, N_W, respectively, as before. Note that this holds exactly when T is a bounded linear mapping from V into W, with $\|T\|_{op,VW} \leq 1$, and (7.4.1) holds with $c = 1$, as in Section 5.13.

Suppose that (7.4.1) holds for some $c > 0$ again. Let R be a linear mapping from V into W such that $R - T$ is bounded, with

$$\|R - T\|_{op,VW} < c. \tag{7.4.5}$$

If $q_W < \infty$, then

$$(c^{q_W} - \|R - T\|_{op,VW}^{q_W})^{1/q_W}\, N_V(v) \leq N_W(R(v)) \tag{7.4.6}$$

for every $v \in V$, as in Section 5.13. If $q_W = \infty$, then

$$c\, N_V(v) \leq N_W(R(v)) \tag{7.4.7}$$

for every $v \in V$, as before. This corresponds to some of the remarks in Section 6.13 as well.

Suppose that T is an isometric linear mapping from V into W, and that $q_W = \infty$. Let R be a linear mapping from V into W again, where $R - T$ is bounded, with

$$\|R - T\|_{op,VW} < 1. \tag{7.4.8}$$

Under these conditions, R is an isometric linear mapping from V into W too, as in Sections 5.13 and 6.13.

Let $q_W > 0$ be arbitrary again. A one-to-one bounded linear mapping from V onto W is said to be *invertible* as a bounded linear mapping if the inverse is bounded as a linear mapping from W onto V, as in Section 5.13.

Suppose that (7.4.1) holds for some $c > 0$, and let R be a linear mapping from V into W such that $R - T$ is bounded and satisfies (7.4.5), as before. If $v \in V$, then

$$N_W(T(v) - R(v)) \le c^{-1} \|R - T\|_{op,VW} \, N_W(T(v)), \tag{7.4.9}$$

as in Section 5.15. If $T(V)$ is dense in W, with respect to the q_W-semimetric associated to N_W, then $R(V)$ is dense in W too, as in Section 6.15. This is essentially the same as for linear mappings between vector spaces, as in Sections 5.14 and 5.15.

Suppose from now on in this section that N_V, N_W are nondegenerate. In particular, this means that T is injective, because of (7.4.1). We have seen that R satisfies an analogous condition, by (7.4.5), so that R is injective as well.

Suppose that T is also bounded as a linear mapping from V into W, and that V is complete with respect to the q_V-metric associated to N_V. This implies that $T(V)$ is complete with respect to the restriction of the q_W-metric associated to N_W to $T(V)$, so that $T(V)$ is a closed set in W, as in Section 5.15. If $T(V)$ is dense in W, then it follows that $T(V) = W$, as before.

Similarly, R is bounded as a linear mapping from V into W, because $R - T$ is bounded. Under these conditions, we get that $R(V) = W$, as in the preceding paragraph. This shows that the set of invertible bounded linear mappings from V onto W is an open set in $\mathcal{BL}(V, W)$ with respect to the q_W-metric associated to $\| \cdot \|_{op,VW}$ when V is complete, as in Section 5.15.

7.5 Submultiplicative subadditive functions

Let R be a ring, and let N be a nonnegative real-valued function on R. Suppose that N is normalized, symmetric, and q-subadditive for some $q > 0$, as a function on R as a commutative group with respect to addition. If

$$N(x\,y) \le N(x)\,N(y) \tag{7.5.1}$$

for every $x, y \in R$, then N is said to be *submultiplicative* on R. Similarly, if

$$N(x\,y) = N(x)\,N(y) \tag{7.5.2}$$

for every $x, y \in R$, then N is said to be *multiplicative* on R.

If a is a positive real number, then N^a is (q/a)-subadditive on R, as in Section 6.1. If N is submultiplicative or multiplicative on R, then N^a has the same property.

If N is submultiplicative on R, then multiplication on R is continuous as a mapping from $R \times R$ into R, with respect to the q-semimetric on R associated to N, and a suitable product q-semimetric on $R \times R$, as in Section 7.2. This means that multiplication on R is continuous with respect to the corresponding product topology on $R \times R$.

Suppose that R has a multiplicative identity element e. If N is submultiplicative on R, then $N(e) \le N(e)^2$. If $N(e) = 0$, then it is easy to see that $N(x) = 0$ for every $x \in R$. Otherwise, if $N(e) > 0$, then we get that $N(e) \ge 1$. If N is multiplicative on R, then we have that $N(e) = 1$.

Let V be a commutative group, with the group operations expressed additively. Consider the space $\mathcal{L}(V) = \mathcal{L}(V, V)$ of linear mappings from V into itself, as in Section 7.1. This is a ring with respect to composition of linear mappings. Of course, the identity mapping $I = I_V$ on V is the mutliplicative identity element in $\mathcal{L}(V)$.

Let N_V be a nonnegative real-valued function on V that is normalized, symmetric, and q_V-subadditive for some $q_V > 0$. The space $\mathcal{BL}(V) = \mathcal{BL}(V, V)$ of bounded linear mappings from V into itself with respect to N_V is a subring of $\mathcal{L}(V)$, and $\| \cdot \|_{op}$ is submultiplicative on $\mathcal{BL}(V)$, as in Section 7.1. Note that the identity operator on V is bounded with respect to N_V. If $N_V(v) > 0$ for some $v \in V$, then $\|I\|_{op} = 1$.

Let k be a field, and let $\mathcal{A}$ be an associative algebra over k. This means that $\mathcal{A}$ is a vector space over k equipped with a binary operation that is associative and bilinear over k. In particular, $\mathcal{A}$ may be considered as a ring. Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. In this case, we may be interested in q-seminorms on $\mathcal{A}$ with respect to $| \cdot |$ on k that are submultiplicative or multiplicative.

Let X be a nonempty set, and remember that $\ell^\infty(X, k)$ is the space of k-valued functions on X that are bounded with respect to $|\cdot|$ on k, as in Section 5.3. This is a commutative algebra over k with respect to pointwise multiplication of functions, and it is easy to see that the corresponding supremum q_k-norm $\| \cdot \|_\infty$ is submultiplicative. The k-valued function on X equal to 1 at every point is the multiplicative identity element in $\ell^\infty(X, k)$, and has supremum norm equal to one.

Let V be a vector space over k. The space $\mathcal{L}_k(V) = \mathcal{L}_k(V, V)$ of linear mappings from V into itself, as a vector over k, is an associative algebra over k with respect to composition of linear mappings. Suppose that N_V is a q_V-seminorm on V with respect to $| \cdot |$ on k for some $q_V > 0$. The space $\mathcal{BL}_k(V) = \mathcal{BL}_k(V, V)$, of mappings from V into itself that are linear over k and bounded as linear mappings with respect to N_V is a subalgebra of $\mathcal{L}_k(V)$. The corresponding operator q_V-seminorm $\| \cdot \|_{op}$ is submultiplicative on $\mathcal{BL}(V)$, as before.

7.6 Multiplication operators

Let $\mathcal{A}$ be a ring. If $a \in \mathcal{A}$, then the corresponding left and right *multiplication operators* are defined by

$$M_a(x) = M_a^L(x) = a\,x \tag{7.6.1}$$

and

$$M_a^R(x) = x\,a \tag{7.6.2}$$

for every $x \in \mathcal{A}$. These define linear mappings from $\mathcal{A}$ into itself, as a commutative group with respect to addition. Note that $a \mapsto M_a$ and $a \mapsto M_a^R$ are linear as mappings from $\mathcal{A}$ into the space $\mathcal{L}(\mathcal{A})$ of linear mappings from $\mathcal{A}$ into itself, as a commutative group with respect to addition. If $\mathcal{A}$ is an algebra over a field k, then M_a and M_a^R are linear over k as mappings from $\mathcal{A}$ into itself, and $a \mapsto M_a$, $a \mapsto M_a^R$ are linear over k as mappings from $\mathcal{A}$ into $\mathcal{L}_k(\mathcal{A})$.

Suppose for the moment that $\mathcal{A}$ has a multiplicative identity element e. Note that M_e, M_e^R are the same as the identity operator on $\mathcal{A}$. If $a \in \mathcal{A}$, then

$$M_a(e) = M_a^R(e) = a. \tag{7.6.3}$$

In particular, this means that $a \mapsto M_a$, $a \mapsto M_a^R$ are injective on $\mathcal{A}$.

If $a, b, x \in \mathcal{A}$, then

$$M_a(M_b(x)) = M_a(b\,x) = a\,b\,x = M_{a\,b}(x) \tag{7.6.4}$$

and

$$M_a^R(M_b^R(x)) = M_a^R(x\,b) = x\,b\,a = M_{b\,a}^R(x). \tag{7.6.5}$$

Thus

$$M_a \circ M_b = M_{a\,b} \tag{7.6.6}$$

and

$$M_a^R \circ M_b^R = M_{b\,a}^R. \tag{7.6.7}$$

Of course, (7.6.6) implies that $a \mapsto M_a$ is a ring homomorphism from $\mathcal{A}$ into $\mathcal{L}(\mathcal{A})$.

Let N be a nonnegative real-valued function on $\mathcal{A}$ that is normalized, symmetric, and q-subadditive for some $q > 0$, as a function on $\mathcal{A}$ as a commutative group with respect to addition. Suppose that

$$N(x\,y) \le C\,N(x)\,N(y) \tag{7.6.8}$$

for some $C \ge 0$ and every $x, y \in \mathcal{A}$. If $a \in \mathcal{A}$, then

$$N(M_a(x)) \le C\,N(a)\,N(x) \tag{7.6.9}$$

and

$$N(M_a^R(x)) \le C\,N(a)\,N(x) \tag{7.6.10}$$

for every $x \in \mathcal{A}$. This means that M_a, M_a^R are bounded with respect to N as linear mappings from $\mathcal{A}$ into itself, with

$$\|M_a\|_{op},\ \|M_a^R\|_{op} \le C\, N(a). \tag{7.6.11}$$

If $\mathcal{A}$ has a multiplicative identity element e, then

$$N(a) \le \|M_a\|_{op}\, N(e),\ \|M_a^R\|_{op}\, N(e), \tag{7.6.12}$$

by (7.6.3).

Thus $a \mapsto M_a$, $a \mapsto M_a^R$ may be considered now as linear mappings from $\mathcal{A}$ into $\mathcal{BL}(\mathcal{A})$. More precisely, these are bounded linear mappings with respect to $\|\cdot\|_{op}$ on $\mathcal{BL}(\mathcal{A})$, by (7.6.11). Suppose that N is submultiplicative on $\mathcal{A}$, so that we can take $C = 1$. If $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, then we get that

$$\|M_a\|_{op} = \|M_a^R\|_{op} = N(a) \tag{7.6.13}$$

for every $a \in \mathcal{A}$.

7.7 Matrices and submultiplicativity

Let $\mathcal{A}$ be a ring, let n be a positive integer, and let $M_n(\mathcal{A})$ be the ring of $n \times n$ matrices with entries in $\mathcal{A}$. Also let N be a nonnegative real-valued function on $\mathcal{A}$ that is normalized, symmetric, and q_N-subadditive on $\mathcal{A}$ for some $q_N > 0$, as a function on $\mathcal{A}$ as a commutative group with respect to addition. We would like to consider subadditivity properties of related functions on $M_n(\mathcal{A})$. If N is submultiplicative on $\mathcal{A}$, then we would like to consider submultiplicativity on $M_n(\mathcal{A})$ as well.

If $1 \le j, l \le n$, then

$$N(a_{j,l}) \tag{7.7.1}$$

defines a nonnegative real-valued function of $a \in M_n(\mathcal{A})$. This function is normalized, symmetric, and q_N-subadditive on $M_n(\mathcal{A})$, as a commutative group with respect to addition.

Let r be a positive real number. If $1 \le l \le n$, then

$$\Big(\sum_{j=1}^{n} N(a_{j,l})^r\Big)^{1/r} \tag{7.7.2}$$

is normalized and symmetric on $M_n(\mathcal{A})$, as a commutative group with respect to addition. Similarly, if $1 \le j \le n$, then

$$\Big(\sum_{l=1}^{n} N(a_{j,l})^r\Big)^{1/r} \tag{7.7.3}$$

is normalized and symmetric on $M_n(\mathcal{A})$ too. If $r \le q_N$, then (7.7.2) and (7.7.3) are r-subadditive on $M_n(\mathcal{A})$, as in Section 6.5. If $q_N \le r$, then (7.7.2) and (7.7.3) are q_N-subadditive on $M_n(\mathcal{A})$, as before.

If $a = (a_{j,l}) \in M_n(\mathcal{A})$, then put

$$N_{r,\infty}(a) = \max_{1 \le l \le n} \Big(\sum_{j=1}^{n} N(a_{j,l})^r \Big)^{1/r}, \tag{7.7.4}$$

$$N_{\infty,r}(a) = \max_{1 \le j \le n} \Big(\sum_{l=1}^{n} N(a_{j,l})^r \Big)^{1/r}, \tag{7.7.5}$$

$$N_{\infty,\infty}(a) = \max_{1 \le j,l \le n} N(a_{j,l}). \tag{7.7.6}$$

These are nonnegative real-valued functions on $M_n(\mathcal{A})$ that are normalized and symmetric, as functions on $M_n(\mathcal{A})$ as a commutative group with respect to addition. Note that $N_{\infty,\infty}$ is q_N-subadditive on $M_n(\mathcal{A})$, as in Section 6.5. If $r \le q_N$, then $N_{r,\infty}$ and $N_{\infty,r}$ are r-subadditive on $M_n(\mathcal{A})$. If $r \ge q_N$, then $N_{r,\infty}$ and $N_{\infty,r}$ are q_N-subadditive on $M_n(\mathcal{A})$.

It is easy to see that

$$N_{\infty,\infty}(a) \le N_{r,\infty}(a),\ N_{\infty,r}(a) \le n^{1/r}\, N_{\infty,\infty}(a) \tag{7.7.7}$$

for every $a \in M_n(\mathcal{A})$. We also have that (7.7.2) and (7.7.3) are monotonically decreasing in r, as in Section 2.11. This implies that (7.7.4) and (7.7.5) are monotonically decreasing in r as well. If N is nondegenerate on $\mathcal{A}$, then $N_{r,\infty}$, $N_{\infty,r}$, $N_{\infty,\infty}$ are nondegenerate on $M_n(\mathcal{A})$.

Observe that

$$N_{r,\infty}(a^t) = N_{\infty,r}(a), \quad N_{\infty,\infty}(a^t) = N_{\infty,\infty}(a) \tag{7.7.8}$$

for every $a \in M_n(\mathcal{A})$, where a^t is the transpose of a. If $\mathcal{A}$ has a multiplicative identity element e, then the corresponding identity matrix $I = I_n$ in $M_n(\mathcal{A})$ can be defined as in Section 4.11. In this case, we have that

$$N_{r,\infty}(I) = N_{\infty,r}(I) = N_{\infty,\infty}(I) = N(e). \tag{7.7.9}$$

Suppose that $r \le q_N$, so that N is r-subadditive on $\mathcal{A}$, as in Section 6.1. Let $a, b \in M_n(\mathcal{A})$ be given, and put $c = a\, b$, so that

$$c_{j,m} = \sum_{l=1}^{n} a_{j,l}\, b_{l,m} \tag{7.7.10}$$

for $j, m = 1, \ldots, n$. If N is submultiplicative on $\mathcal{A}$, then it follows that

$$N(c_{j,m})^r \le \sum_{l=1}^{n} N(a_{j,l})^r\, N(b_{l,m})^r \tag{7.7.11}$$

for $j, m = 1, \ldots, n$. One can use this to check that

$$N_{r,\infty}(c) \le N_{r,\infty}(a)\, N_{r,\infty}(b) \tag{7.7.12}$$

and

$$N_{\infty,r}(c) \le N_{\infty,r}(a)\, N_{\infty,r}(b). \tag{7.7.13}$$

This shows that $N_{r,\infty}$ and $N_{\infty,r}$ are submultiplicative on $M_n(\mathcal{A})$ when $r \le q_N$.

If $q_N = \infty$, and N is submultiplicative on $\mathcal{A}$, then

$$N(c_{j,m}) \le \max_{1 \le l \le n} (N(a_{j,l})\, N(b_{l,m})) \tag{7.7.14}$$

for $j, m = 1, \ldots, n$. This implies that

$$N_{\infty,\infty}(c) \le N_{\infty,\infty}(a)\, N_{\infty,\infty}(b), \tag{7.7.15}$$

so that $N_{\infty,\infty}$ is submultiplicative on $M_n(\mathcal{A})$.

Let k be a field, and suppose now that $\mathcal{A}$ is an associative algebra over k. Note that $M_n(\mathcal{A})$ is an associative algebra over k as well, where scalar multiplication is defined entrywise. Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and suppose that N is a q_N-seminorm on $\mathcal{A}$ for some $q_N > 0$ with respect to $|\cdot|$ on k. This means that (7.7.1) is a q_N-seminorm on $M_n(\mathcal{A})$ for every $1 \le j, l \le n$. It follows that $N_{\infty,\infty}$ is a q_N-seminorm on $M_n(\mathcal{A})$.

If $r \le q_N$, then (7.7.2) and (7.7.3) are r-seminorms on $M_n(\mathcal{A})$. This implies that $N_{r,\infty}$ and $N_{\infty,r}$ are r-seminorms on $M_n(\mathcal{A})$, as before. If $r \ge q_N$, then (7.7.2) and (7.7.3) are q_N-seminorms on $M_n(\mathcal{A})$. This implies that $N_{r,\infty}$ and $N_{\infty,r}$ are q_N-seminorms on $M_n(\mathcal{A})$.

7.8 Continuity of inverses

Let $\mathcal{A}$ be a ring with a multiplicative identity element e, and let N be a nonnegative real-valued function on $\mathcal{A}$ that is normalized, symmetric, and q-subadditive for some $q > 0$, as a function on $\mathcal{A}$ as a commutative group with respect to addition. Suppose that N is also submultiplicative on $\mathcal{A}$, and that $N(e) > 0$. If x is an invertible element of $\mathcal{A}$, then

$$N(e) \le N(x)\, N(x^{-1}), \tag{7.8.1}$$

so that $N(x), N(x^{-1}) > 0$ in particular. If y is another invertible element of $\mathcal{A}$, then

$$x^{-1} - y^{-1} = x^{-1}\,(y - x)\, y^{-1}. \tag{7.8.2}$$

This implies that

$$N(x^{-1} - y^{-1}) \le N(x^{-1})\, N(y^{-1})\, N(x - y). \tag{7.8.3}$$

If $q < \infty$, then it follows that

$$N(y^{-1})^q \le N(x^{-1})^q + N(x^{-1})^q\, N(y^{-1})^q\, N(x - y)^q. \tag{7.8.4}$$

This implies that

$$(1 - N(x^{-1})^q\, N(x - y)^q)\, N(y^{-1})^q \le N(x^{-1})^q. \tag{7.8.5}$$

Suppose that

(7.8.6) $$N(x^{-1})\,N(x-y) < 1.$$

Under these conditions, we get that

(7.8.7) $$N(y^{-1}) \le (1 - N(x^{-1})^q\,N(x-y)^q)^{-1/q}\,N(x^{-1}).$$

Combining this with (7.8.3), we obtain that

(7.8.8) $$N(x^{-1} - y^{-1}) \le (1 - N(x^{-1})^q\,N(x-y)^q)^{-1/q}\,N(x^{-1})^2\,N(x-y).$$

If $q = \infty$, then we can use (7.8.3) to get that

(7.8.9) $$N(y^{-1}) \le \max(N(x^{-1}), N(x^{-1})\,N(y^{-1})\,N(x-y)).$$

If (7.8.6) holds, then it follows that

(7.8.10) $$N(y^{-1}) \le N(x^{-1}).$$

In this case, we obtain that

(7.8.11) $$N(x^{-1} - y^{-1}) \le N(x^{-1})^2\,N(x-y),$$

by (7.8.3).

Let $G(\mathcal{A})$ be the group of invertible elements of $\mathcal{A}$. The remarks in the previous paragraphs show that $x \mapsto x^{-1}$ is continuous on $G(\mathcal{A})$, with respect to the restriction to $G(\mathcal{A})$ of the q-semimetric on $\mathcal{A}$ associated to N. Note that multiplication on $\mathcal{A}$ is continuous as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, with respect to the q-semimetric on $\mathcal{A}$ associated to N and the corresponding product topology on $\mathcal{A} \times \mathcal{A}$, as in Sections 7.2 and 7.3. This implies that multiplication on $G(\mathcal{A})$ is continuous as a mapping from $G(\mathcal{A}) \times G(\mathcal{A})$ into $G(\mathcal{A})$, with respect to the induced topology on $G(\mathcal{A})$, and the corresponding product topology on $G(\mathcal{A}) \times G(\mathcal{A})$. Thus $G(\mathcal{A})$ is a topological group with respect to the induced topology.

Suppose for the moment that $N(e) = 1$, and consider

(7.8.12) $$U(\mathcal{A}) = \{u \in G(\mathcal{A}) : N(u),\ N(u^{-1}) \le 1\}.$$

This is a subgroup of $G(\mathcal{A})$, and a closed set in $G(\mathcal{A})$ with respect to the induced topology. If $u \in U(\mathcal{A})$ and $x \in \mathcal{A}$, then one can check that

(7.8.13) $$N(u\,x) = N(x\,u) = N(x).$$

Note that $N(u) = N(u^{-1}) = 1$ when $u \in U(\mathcal{A})$, because of (7.8.1).

Suppose now that N is nondegenerate on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to $\mathcal{A}$. Let $\{x_j\}_{j=1}^\infty$ be a sequence of invertible elements of $\mathcal{A}$ that converges to an element x of $\mathcal{A}$, with respect to the q-metric associated to N. In particular, this implies that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence. If $\{x_j^{-1}\}_{j=1}^\infty$ is bounded with respect to N, then it is easy to see that $\{x_j^{-1}\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{A}$ too, using (7.8.3). This implies that $\{x_j^{-1}\}_{j=1}^\infty$ converges in $\mathcal{A}$, and one can check that the limit is the multiplicative inverse of x in $\mathcal{A}$.

If $N(e) = 1$, then one can use the remarks in the preceding paragraph to get that $U(\mathcal{A})$ is a closed set in $\mathcal{A}$, with respect to the q-metric associated to N, under the same conditions on N and $\mathcal{A}$.

7.9 Infinite series and invertibility

Let $\mathcal{A}$ be a commutative group, where the group operations are expressed additively, and let N be a nonnegative real-valued function on $\mathcal{A}$ that is normalized, symmetric, nondegenerate, and q-subadditive for some $q > 0$. An infinite series $\sum_{j=1}^{\infty} a_j$ with terms in $\mathcal{A}$ is said to *converge* if the corresponding sequence of partial sums $\sum_{j=1}^{n} a_j$ converges in $\mathcal{A}$, with respect to the q-metric associated to N. In this case, the value of the sum is defined to be the limit of the sequence of partial sums, as usual.

One can check that the sequence of partial sums is a Cauchy sequence with respect to the q-metric associated to N if and only if for each $\epsilon > 0$ there is a positive integer L such that

$$N\Big(\sum_{j=l}^{n} a_j\Big) < \epsilon \tag{7.9.1}$$

for every $n \geq l \geq L$. This implies that

$$\lim_{l \to \infty} N(a_l) = 0, \tag{7.9.2}$$

by taking $l = n$ in (7.9.1).

Suppose for the moment that $q < \infty$, so that

$$N\Big(\sum_{j=l}^{n} a_j\Big)^q \leq \sum_{j=l}^{n} N(a_j)^q \tag{7.9.3}$$

for every $n \geq l \geq 1$. Let us say that $\sum_{j=1}^{\infty} a_j$ *converges q-absolutely* if

$$\sum_{j=1}^{\infty} N(a_j)^q \tag{7.9.4}$$

converges as an infinite series of nonnegative real numbers. One can check that this implies that the sequence of partial sums of $\sum_{j=1}^{\infty} a_j$ is a Cauchy sequence with respect to q-metric associated to N in this case. If $\mathcal{A}$ is complete with respect to the q-metric associated to N, then it follows that $\sum_{j=1}^{\infty} a_j$ converges in $\mathcal{A}$. Under these conditions, we also have that

$$N\Big(\sum_{j=1}^{\infty} a_j\Big)^q \leq \sum_{j=1}^{\infty} N(a_j)^q. \tag{7.9.5}$$

Suppose now that $q = \infty$, so that

$$N\Big(\sum_{j=l}^{n} a_j\Big) \leq \max_{l \leq j \leq n} N(a_j) \tag{7.9.6}$$

for every $n \geq l \geq 1$. If (7.9.2) holds, then it follows that the partial sums of $\sum_{j=1}^{\infty} a_j$ form a Cauchy sequence with respect to the ultrametric associated to

N. If $\mathcal{A}$ is complete with respect to this ultrametric, then $\sum_{j=1}^{\infty} a_j$ converges in $\mathcal{A}$. In this case, we get that

$$N\Big(\sum_{j=1}^{\infty} a_j\Big) \le \max_{j \ge 1} N(a_j). \tag{7.9.7}$$

More precisely, one can verify that the maximum on the right side is attained, because of (7.9.2).

Now let $\mathcal{A}$ be a ring with a multiplicative identity element e. If $x \in \mathcal{A}$ and n is a nonnegative integer, then

$$(e - x) \sum_{j=0}^{n} x^j = \Big(\sum_{j=0}^{n} x^j\Big)(e - x) = e - x^{n+1}, \tag{7.9.8}$$

where x^j is interpreted as being equal to e when $j = 0$.

Let N be a nonnegative real-valued function on $\mathcal{A}$ that is normalized, symmetric, nondegenerate, and q-subadditive for some $q > 0$, as a function on $\mathcal{A}$ as a commutative group with respect to addition. Suppose that N is also submultiplicative on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Note that

$$N(x^j) \le N(x)^j \tag{7.9.9}$$

for every positive integer j. If $N(x) < 1$, then $N(x^j) \to 0$ as $j \to \infty$, and $\sum_{j=0}^{\infty} N(x^j)^q$ converges when $q < \infty$. This implies that $\sum_{j=0}^{\infty} x^j$ converges in $\mathcal{A}$, as before. In this case, we get that

$$(e - x) \sum_{j=0}^{\infty} x_j = \Big(\sum_{j=0}^{\infty} x^j\Big)(e - x) = e, \tag{7.9.10}$$

by taking the limit as $n \to \infty$ in (7.9.8). This means that $e - x$ is invertible in $\mathcal{A}$, with

$$(e - x)^{-1} = \sum_{j=0}^{\infty} x^j. \tag{7.9.11}$$

Let y be an invertible element of $\mathcal{A}$, and suppose that $z \in \mathcal{A}$ satisfies

$$N(y^{-1})\, N(y - z) < 1. \tag{7.9.12}$$

Observe that

$$z = y - (y - z) = y\,(e - y^{-1}\,(y - z)), \tag{7.9.13}$$

and that $e - y^{-1}\,(y - z)$ is invertible in $\mathcal{A}$, because of (7.9.12). Thus z is invertible in $\mathcal{A}$ too. This shows that the group $G(\mathcal{A})$ of invertible elements of $\mathcal{A}$ is an open subset of $\mathcal{A}$ under these conditions.

7.10 Some subgroups of $U(\mathcal{A})$

Let $\mathcal{A}$ be a ring with a multiplicative identity element e, and let N be a nonnegative real-valued function on $\mathcal{A}$ that is normalized, symmetric, and ultrasubadditive, as a function on $\mathcal{A}$ as a commutative group with respect to addition. Suppose that N is also submultiplicative on $\mathcal{A}$, and that $N(e) = 1$. Let $B(x, r) = B_N(x, r)$ and $\overline{B}(x, r) = \overline{B}_N(x, r)$ be the usual open and closed balls in $\mathcal{A}$ with respect to the semi-ultrametric associated to N. Note that $\overline{B}(0, 1)$ is a subring of $\mathcal{A}$ that contains e under these conditions. The subgroup $U(\mathcal{A})$ of the group $G(\mathcal{A})$ of invertible elements of $\mathcal{A}$ defined in (7.8.12) is the same as the group $G(\overline{B}(0, 1))$ of invertible elements of $\overline{B}(0, 1)$ in this case.

If $x \in U(\mathcal{A})$, $y \in G(\mathcal{A})$, and $N(x - y) < 1$, then

$$y \in U(\mathcal{A}). \tag{7.10.1}$$

Indeed, $N(y) \le 1$ because $N(x) \le 1$ and N is ultrasubadditive on $\mathcal{A}$. We also have that

$$N(y^{-1}) \le N(x^{-1}) \le 1, \tag{7.10.2}$$

as in (7.8.10). If N is nondegenerate on $\mathcal{A}$, and $\mathcal{A}$ is complete with respect to the associated ultrametric, then $y \in G(\mathcal{A})$ when $y \in \mathcal{A}$ satisfies $N(x - y) < 1$ for some $x \in U(\mathcal{A})$, as in the previous section. In particular, $U(\mathcal{A})$ is an open subset of $\mathcal{A}$ in this case.

We can take $x = e$ in the preceding paragraph, to get that

$$B(e, 1) \cap G(\mathcal{A}) \subseteq U(\mathcal{A}). \tag{7.10.3}$$

If N is nondegenerate on $\mathcal{A}$, and $\mathcal{A}$ is complete with respect to the associated ultrametric, then $B(e, 1) \subseteq G(\mathcal{A})$, as in the previous section, so that

$$B(e, 1) \subseteq U(\mathcal{A}). \tag{7.10.4}$$

The semi-ultrametric $d_N(x, y) = N(x - y)$ on $\mathcal{A}$ associated to N is invariant under left and right multiplication by elements of $U(\mathcal{A})$, by (7.8.13). In particular, the restriction of $d_N(x, y)$ to $x, y \in U(\mathcal{A})$ is invariant under left and right translations on $U(\mathcal{A})$, as a group with respect to multiplication. This implies that open and closed balls in $U(\mathcal{A})$ centered at e with respect to the restriction of d_N to $U(\mathcal{A})$ are normal subgroups of $U(\mathcal{A})$, as in Section 3.7. This means that

$$B(e, r) \cap U(\mathcal{A}) \tag{7.10.5}$$

is a normal subgroup of $U(\mathcal{A})$ for every $r > 0$, and that

$$\overline{B}(e, r) \cap U(\mathcal{A}) \tag{7.10.6}$$

is a normal subgroup of $U(\mathcal{A})$ for every $r \ge 0$.

Of course,

$$U(\mathcal{A}) \subseteq \overline{B}(0, 1) = \overline{B}(e, 1), \tag{7.10.7}$$

by construction. Thus (7.10.5) is equal to $U(\mathcal{A})$ when $r > 1$, and (7.10.6) is equal to $U(\mathcal{A})$ when $r \geq 1$. Using (7.10.3), we get that

$$B(e,r) \cap U(\mathcal{A}) = B(e,r) \cap G(\mathcal{A}) \tag{7.10.8}$$

when $0 < r \leq 1$, and

$$\overline{B}(e,r) \cap U(\mathcal{A}) = \overline{B}(e,r) \cap G(\mathcal{A}) \tag{7.10.9}$$

when $0 \leq r < 1$. If N is nondegenerate on $\mathcal{A}$, and $\mathcal{A}$ is complete with respect to d_N, then (7.10.4) implies that

$$B(e,r) \cap U(\mathcal{A}) = B(e,r) \tag{7.10.10}$$

when $0 < r \leq 1$, and

$$\overline{B}(e,r) \cap U(\mathcal{A}) = \overline{B}(e,r) \tag{7.10.11}$$

when $0 \leq r < 1$.

7.11 Two semi-ultrametrics on $G(\mathcal{A})$

Let us continue with the same notation and hypotheses as at the beginning of the previous section. Note that

$$N(u^{-1} - e) = N(u - e) \tag{7.11.1}$$

for every $u \in U(\mathcal{A})$, because N is invariant under left or right multiplication by elements of $U(\mathcal{A})$, as in Section 7.8. We also have that

$$N(u) \leq 1 \tag{7.11.2}$$

for every $u \in U(\mathcal{A})$, as in (7.10.7).

If $x, y \in G(\mathcal{A})$, then put

$$\begin{aligned} \delta_L(x,y) &= N(x^{-1}\,y - e) \quad \text{when } x^{-1}\,y \in U(\mathcal{A}) \\ &= 1 \quad \text{otherwise.} \end{aligned} \tag{7.11.3}$$

It is easy to see that this is symmetric in x and y, using (7.11.1). Of course,

$$\delta_L(x,y) \leq 1 \tag{7.11.4}$$

for every $x, y \in G(\mathcal{A})$, by (7.11.2). If $x, y \in G(\mathcal{A})$ and

$$N(x^{-1}\,y - e) < 1, \tag{7.11.5}$$

then $x^{-1}\,y \in U(\mathcal{A})$, by (7.10.3). Thus $x, y \in G(\mathcal{A})$ satisfy

$$\delta_L(x,y) < 1 \tag{7.11.6}$$

if and only if (7.11.5) holds.

Clearly

$$\delta_L(x, x) = N(0) = 0 \tag{7.11.7}$$

for every $x \in G(\mathcal{A})$. If N is nondegenerate on $\mathcal{A}$, then $\delta_L(x, y) = 0$ only when $x = y$. If $x, y \in U(\mathcal{A})$, then

$$\delta_L(x, y) = N(x - y), \tag{7.11.8}$$

because N is invariant under left multiplication by elements of $U(\mathcal{A})$. Observe that (7.11.3) is invariant under left translations on $G(\mathcal{A})$, as a group with respect to multiplication, by construction.

If $x, y, z \in G(\mathcal{A})$, then we would like to verify that

$$\delta_L(x, z) \le \max(\delta_L(x, y), \delta_L(y, z)). \tag{7.11.9}$$

If the right side is equal to one, then this follows from (7.11.4). If the right side is less than one, then $x^{-1}\,y, y^{-1}\,z \in U(\mathcal{A})$, so that $x^{-1}\,z \in U(\mathcal{A})$ too. In this case, (7.11.9) follows from (7.11.8) and the ultrasubadditivity of N on $\mathcal{A}$. This shows that (7.11.3) defines a semi-ultrametric on $G(\mathcal{A})$.

Similarly, if $x, y \in G(\mathcal{A})$, then put

$$\begin{aligned} \delta_R(x, y) &= N(x\,y^{-1} - e) \quad \text{when } x\,y^{-1} \in U(\mathcal{A}) \\ &= 1 \quad\quad\quad\quad\quad\ \ \text{otherwise.} \end{aligned} \tag{7.11.10}$$

It is easy to see that

$$\delta_R(x, y) = \delta_L(x^{-1}, y^{-1}) \tag{7.11.11}$$

for every $x, y \in G(\mathcal{A})$. In particular, (7.11.10) defines a semi-ultrametric on $G(\mathcal{A})$, which is an ultrametric when N is nondegenerate on $\mathcal{A}$, as before. This semi-ultrametric is invariant under right translations on $G(\mathcal{A})$, as a group with respect to multiplication. If $x, y \in U(\mathcal{A})$, then

$$\delta_R(x, y) = N(x - y), \tag{7.11.12}$$

because N is invariant under right multiplication by elements of $U(\mathcal{A})$.

7.12 Total boundedness in $G(\mathcal{A})$

Let $\mathcal{A}$ be a ring with a multiplicative identity element e, and let N be a nonnegative real-valued function on $\mathcal{A}$ that is normalized, symmetric, and q-subadditive for some $q > 0$, as a function on $\mathcal{A}$ as a commutative group with respect to addition. Suppose also that N is submultiplicative on $\mathcal{A}$, and that $N(e) > 0$. Remember that the group $G(\mathcal{A})$ of invertible elements of $\mathcal{A}$ is a topological group, with respect to the topology induced by the topology determined on $\mathcal{A}$ by the q-semimetric associated to N, as in Section 7.8. Of course, this is the same as the topology determined on $G(\mathcal{A})$ by the restriction of $d_N(x, y) = N(x - y)$ to $x, y \in G(\mathcal{A})$.

Let $B(x,r)$ and $\overline{B}(x,r)$ be the usual open and closed balls in $\mathcal{A}$ with respect to the q-semimetric associated to N, as before. If $x, y \in \mathcal{A}$, then it is easy to see that

$$x\,B(y,r) \subseteq B(x\,y, N(x)\,r), \quad B(y,r)\,x \subseteq B(y\,x, N(x)\,r) \tag{7.12.1}$$

for every $r > 0$ when $N(x) > 0$. If $N(x) = 0$, then $x\,\mathcal{A}, \mathcal{A}\,x \subseteq \overline{B}(0,0)$. Similarly,

$$x\,\overline{B}(y,r) \subseteq \overline{B}(x\,y, N(x)\,r), \quad \overline{B}(y,r)\,x \subseteq \overline{B}(y\,x, N(x)\,r) \tag{7.12.2}$$

for every $r \geq 0$.

Suppose that $E \subseteq G(\mathcal{A})$ is left or right-invariant totally bounded in $G(\mathcal{A})$, as a topological group with respect to multiplication, as in Section 3.8. Thus, for each $r > 0$, E can be covered by finitely many left or right translates of $B(e,r) \cap G(\mathcal{A})$ in $G(\mathcal{A})$, as approriate. This implies that E is bounded with respect to N, by the remarks in the preceding paragraph.

More precisely, E can be covered by finitely many left or right translates of $B(e,r) \cap G(\mathcal{A})$ in $G(\mathcal{A})$ by elements of E, as appropriate. This follows from the fact that E can be covered by finitely many left or right-invariant U-small sets, with $U = B(e,r) \cap G(\mathcal{A})$, as in Section 3.8. One can use this and the boundedness of N on E to get that E is totally bounded with respect to d_N in $\mathcal{A}$. This also uses (7.12.1), with $y = e$. Equivalently, this means that E is totally bounded in $\mathcal{A}$ as a commutative topological group with respect to addition, and the topology determined by d_N.

If $r < 1/N(e)$, then $N(x^{-1})$ is bounded on $B(e,r) \cap G(\mathcal{A})$, as in Section 7.8. One can use this to get that $N(x^{-1})$ is bounded on E too.

Let $x \in G(\mathcal{A})$ be given, and note that $N(x^{-1}) > 0$. If $y \in \mathcal{A}$, then

$$B(y,r) \subseteq x\,B(x^{-1}\,y, N(x^{-1})\,r),\ B(y\,x^{-1}, N(x^{-1})\,r)\,x \tag{7.12.3}$$

for every $r > 0$, by (7.12.1). Similarly,

$$\overline{B}(y,r) \subseteq x\,\overline{B}(x^{-1}\,y, N(x^{-1})\,r),\ \overline{B}(y\,x^{-1}, N(x^{-1})\,r)\,x \tag{7.12.4}$$

for every $r \geq 0$, by (7.12.2).

Suppose now that $E \subseteq G(\mathcal{A})$ is totally bounded as a subset of $\mathcal{A}$ with respect to d_N, or equivalently as a subset of $\mathcal{A}$ as a commutative topological group with respect to addition. If $N(x^{-1})$ is bounded on E, then E is left and right-invariant totally bounded in $G(\mathcal{A})$ as well, as a topological group with respect to multiplication. Indeed, the total boundedness of E in $\mathcal{A}$ implies that E can be covered by finitely many balls of arbitrarily small radius, and we may take these balls to be centered at elements of E, as in Section 1.15. One can use this to cover E by finitely many left or right translates in $G(\mathcal{A})$ of balls centered at e with arbitrarily small radius, because $N(x^{-1})$ is bounded on E. More precisely, this uses (7.12.3) too, with $y = x$.

Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of $G(\mathcal{A})$. If $\{x_j\}_{j=1}^\infty$ is a left or right-invariant Cauchy sequence in $G(\mathcal{A})$, as a topological group with respect to multiplication, then the set of x_j's, $j \geq 1$, is left or right-invariant totally bounded in $G(\mathcal{A})$, as in Section 3.15. This implies that $N(x_j)$ and $N(x_j^{-1})$, $j \geq 1$, are bounded, as before. Under these conditions, one can check that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{A}$, with respect to d_N. Equivalently, $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{A}$, as a commutative topological group with respect to addition.

Conversely, suppose that $N(x_j^{-1})$, $j \geq 1$, is bounded, and that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to d_N, or equivalently as a sequence in $\mathcal{A}$, as a commutative topological group with respect to addition. Under these conditions, one can verify that $\{x_j\}_{j=1}^\infty$ is a left and right-invariant Cauchy sequence in $G(\mathcal{A})$, as a topological group with respect to multiplication.

Part II

Chapter 8

Some doubling and related conditions

8.1 Doubling conditions for semimetrics

Let X be a set, and let $d(x, y)$ be a q-semimetric on X for some $q > 0$, or simply a semi-quasimetric. Let us say that X is *doubling* with respect to $d(\cdot, \cdot)$ if there is a positive real number C such that for each $x \in X$ and $r > 0$, $\overline{B}_d(x, r)$ can be covered by at most C closed balls in X with respect to $d(\cdot, \cdot)$ of radius $r/2$. This implies that for each $x \in X$, $r > 0$, and positive integer j, $\overline{B}_d(x, r)$ can be covered by at most C^j closed balls in X with respect to $d(\cdot, \cdot)$ of radius $2^{-j}\, r$.

This type of doubling condition is often formulated in terms of open balls instead of closed balls. It is easy to see that the two formulations are equivalent, although perhaps with a different constant.

It will sometimes be more convenient to use another formulation of the doubling condition, in terms of diameters of sets. This formulation asks that there be a positive real number C_0 such that if E is a bounded subset of X with respect to $d(\cdot, \cdot)$, then E can be covered by at most C_0 bounded subsets of X, each of which has diameter less than or equal to one-half the diameter of E. Of course, this implies that for each $j \geq 1$, E can be covered by C_0^j bounded sets, each of which has diameter less than or equal to $2^{-j} \operatorname{diam}_d E$.

As before, one can check that this formulation is equivalent to the previous ones, perhaps with a different constant. This uses the fact that the diameter of a ball in X can be estimated in terms of its radius. Note that this depends on q when $d(\cdot, \cdot)$ is a q-semimetric, and otherwise on the constant in the semi-quasimetric condition.

Let X_0 be a subset of X, equipped with the restriction of $d(x, y)$ to $x, y \in X_0$. If X is doubling with respect to $d(\cdot, \cdot)$, then X_0 is doubling with respect to the restriction of $d(\cdot, \cdot)$ to X_0. If one uses the formulation of the doubling condition in terms of diameters, then one can use the same doubling constant for X_0 as for X.

Let $d_1(\cdot,\cdot)$ be a q_1-semimetric on X for some $q_1 > 0$, or a semi-quasimetric on X. Suppose that $d(\cdot,\cdot)$ and $d_1(\cdot,\cdot)$ are each bounded by a constant multiple of the other. Under these conditions, one can check that X is doubling with respect to $d(\cdot,\cdot)$ if and only if X is doubling with respect to $d_1(\cdot,\cdot)$. More precisely, the doubling constants for $d(\cdot,\cdot)$ and $d_1(\cdot,\cdot)$ can be estimated in terms of each other.

Let a be a positive real number. Remember that if $d(\cdot,\cdot)$ is a q-semimetric on X, then $d(\cdot,\cdot)^a$ is a (q/a)-semimetric on X. Otherwise, if $d(\cdot,\cdot)$ is a semi-quasimetric on X, then $d(\cdot,\cdot)^a$ is a semi-quasimetric on X too, with a suitable constant. One can verify that X is doubling with respect to $d(\cdot,\cdot)$ if and only if X is doubling with respect to $d(\cdot,\cdot)^a$. In this case, the doubling constants for $d(\cdot,\cdot)$ and $d(\cdot,\cdot)^a$ can be estimated in terms of each other, as usual.

8.2 Doubling conditions and Cartesian products

Let X be a set, and let n be a positive integer. Suppose that for each $j = 1,\ldots,n$, $d_j(\cdot,\cdot)$ is a q_j-semimetric on X for some $q_j > 0$, or a semi-quasimetric on X. Put

$$d(x,y) = \max_{1\le j\le n} d_j(x,y) \tag{8.2.1}$$

for each $x, y \in X$. This is a q-semimetric on X for a suitable $q > 0$, or a semi-quasimetric on X, with a suitable semi-quasimetric constant.

Suppose that X is doubling with respect to d_j for each $j = 1,\ldots,n$. More precisely, suppose that X is doubling with respect to d_j with constant $C_j > 0$ for each $j = 1,\ldots,n$, using the formulation of the doubling condition in terms of diameters of bounded sets. Let us check that X is doubling with respect to d, with constant

$$C_1\, C_2 \cdots C_n, \tag{8.2.2}$$

again usng the formulation of the doubling condition in terms of diameters of bounded sets.

Let E be a bounded subset of X with respect to d. This implies that E is bounded with respect to d_j for each $j = 1,\ldots,n$, with

$$\mathrm{diam}_{d_j} E \le \mathrm{diam}_d E. \tag{8.2.3}$$

Thus, for each $j = 1,\ldots,n$, E can be covered by at most C_j bounded subsets of X with respect to d_j, where each of these sets has diameter with respect to d_j less than or equal to one-half $\mathrm{diam}_{d_j} E$. Consider the family of subsets of X obtained by taking the interesection of n sets, where for each $j = 1,\ldots,n$, the jth set is one of the sets used in the jth covering of E just mentioned. It is easy to see that this family consists of at most (8.2.2) sets. Each set in the family has diameter with respect to d_j less than or equal to $\mathrm{diam}_d E$ for each $j = 1,\ldots,n$. It follows that each set in the family is bounded with respect to d, with diameter less than or equal to one-half $\mathrm{diam}_d E$. We also have that E is covered by the sets in this family, as desired.

Now let $X_1, \ldots, X_n$ be n sets, and let $X = \prod_{j=1}^n X_j$ be their Cartesian product. Suppose that for each $j = 1, \ldots, n$, $d_{X_j}(x_j, y_j)$ is a q_j-semimetric on X_j for some $q_j > 0$, or a semi-quasimetric on X_j. Put

$$d_j(x, y) = d_{X_j}(x_j, y_j) \tag{8.2.4}$$

for each $j = 1, \ldots, n$ and $x, y \in X$, as in Section 1.2, but with different notation. This is a q_j-semimetric or semi-quasimetric on X, as appropriate.

Suppose that X_j is doubling with respect to d_{X_j} for each $j = 1, \ldots, n$. One can check that this implies that X satisfies the analogous condition with respect to d_j for each $j = 1, \ldots, n$. It follows that X is doubling with respect to (8.2.1), as before.

8.3 Some examples and related properties

Let X be a set, and let $d(\cdot, \cdot)$ be a q-semimetric on X for some $q > 0$, or a semi-quasimetric on X. If X is doubling with respect to $d(\cdot, \cdot)$, then every bounded set E in X with respect to $d(\cdot, \cdot)$ is totally bounded with respect to $d(\cdot, \cdot)$. In particular, this implies that E has a subset with only finitely or countably many elements that is dense in E. It follows that X is separable with respect to $d(\cdot, \cdot)$, because X can be expressed as the union of a sequence of bounded sets.

It is easy to see that the real line is doubling with respect to the standard Euclidean metric. Similarly, the complex plane is doubling with respect to the standard metric.

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let $d(\cdot, \cdot)$ be the corresponding ultrametric on k. If k is doubling with respect to $d(\cdot, \cdot)$, then the closed unit ball in k is totally bounded, as before. This implies that the residue field associated to $|\cdot|$ is finite, and that $|\cdot|$ is discrete on k, as in Section 4.7.

Conversely, suppose that the residue field associated to $|\cdot|$ is finite, and that $|\cdot|$ is discrete on k. If $|\cdot|$ is the trivial absolute value function on k, then k is the same as the residue field, and k is doubling with respect to $d(\cdot, \cdot)$.

Otherwise, if $|\cdot|$ is not the trivial absolute value function on k, then there is a positive real number ρ_1 such that $\rho_1 < 1$ and the positive values of $|\cdot|$ on k are the same as the integer powers of ρ_1, as in Section 4.5. Let N be the number of elements of the residue field associated to $|\cdot|$. Under these conditions, the closed unit ball in k is the union of N pairwise-disjoint open balls of radius 1, which are the same as closed balls of radius ρ_1. If j is any integer, then one can use translations and dilations to get that any closed ball in k of radius ρ_1^j is the union of N pairwise-disjoint closed balls of radius ρ_1^{j+1}, as in Section 4.7. It is easy to see that this implies that k is doubling with respect to $d(\cdot, \cdot)$.

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