Some Aspects of Harmonic Analysis on Commutative Topological Groups

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Preface

Some topics related to Fourier transforms of complex Borel measures on commutative topological groups are discussed. In particular, this includes integrable functions with respect to Haar measure on locally compact groups.

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Chapter 1

Topological groups and vector spaces

1.1 Commutative topological groups

Let A be a commutative group, with the group operations expressed additively, and suppose that A is also equipped with a topology. If the group operations on A are continuous, then A is said to be a *topological group*. More precisely, this means that addition on A is continuous as a mapping from $A \times A$ into itself, using the corresponding product topology on $A \times A$. Similarly, the mapping that sends $x \in A$ to its additive inverse $-x$ should be continuous. It is customary to require that $\{0\}$ be a closed set in A too.

Note that any commutative group A is a topological group with respect to the discrete topology on A. The real line $\mathbf{R}$ is a commutative topological group with respect to addition and the standard topology on $\mathbf{R}$. The unit circle

$$\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\} \tag{1.1}$$

in the complex plane $\mathbf{C}$ is a commutative topological group with respect to multiplication of complex numbers and the topology induced on $\mathbf{T}$ by the standard topology on $\mathbf{C}$. Here $|z|$ denotes the usual absolute value or modulus of $z \in \mathbf{C}$. If A is a commutative topological group and B is a subgroup of A, then B is a commutative topological group with respect to the topology induced on B by the topology on A.

Let A be a commutative topological group. Continuity of addition on A implies that for each $a \in A$, the translation mapping

$$x \mapsto x + a \tag{1.2}$$

is continuous as a mapping from A into itself. The inverse of this mapping is given by translation by $-a$, which is also continuous for the same reason, so that (1.2) is a homeomorphism from A onto itself for each $a \in A$. It follows in particular that subsets of A with only one element are closed sets, so that A satisfies the first separation condition. It is well known that A is Hausdorff and regular as a topological space, as we shall soon see.

If $a \in A$ and $E \subseteq A$, then we put

$$a + E = E + a = \{a + x : x \in E\} \tag{1.3}$$

and

$$-E = \{-x : x \in E\}, \tag{1.4}$$

which are the images of E under the translation mapping (1.2) and $x \mapsto -x$, respectively. We may also use $a - E$ for $a + (-E)$, and $E - a$ for $E + (-a)$. If E is open, closed, or compact in A, then each of these sets has the same property, because of the continuity of translations on A and of the mapping $x \mapsto -x$. If $E_1, E_2 \subseteq A$, then we put

$$E_1 + E_2 = \bigcup_{x \in E_1} (x + E_2) = \bigcup_{y \in E_2} (E_1 + y) = \{x + y : x \in E_1,\, y \in E_2\}, \tag{1.5}$$

which is the image of $E_1 \times E_2$ under addition as a mapping from $A \times A$ into A. If either E_1 or E_2 is an open set in A, then $E_1 + E_2$ is an open set in A, because it is a union of translates of open sets. If E_1 and E_2 are both compact subsets of A, then $E_1 \times E_2$ is compact in $A \times A$ with respect to the corresponding product topology, by Tychonoff's theorem. This implies that $E_1 + E_2$ is compact in A, by continuity of addition on A as a mapping from $A \times A$ into A. As before, we may use $E_1 - E_2$ instead of $E_1 + (-E_2)$.

If U is an open subset of A that contains 0, then there are open subsets U_1, U_2 of A that contain 0 and satisfy

$$U_1 + U_2 \subseteq U. \tag{1.6}$$

This corresponds exactly to continuity of addition on A as a mapping from $A \times A$ into A at $(0, 0)$. In order to verify that addition on A is continuous as a mapping from $A \times A$ into A at every point in $A \times A$, it suffices to check that this condition holds, and that translations are continuous on A.

If A is a commutative topological group, E is any subset of A, and $U \subseteq A$ is an open set that contains 0, then it is easy to see that

$$\overline{E} \subseteq E + U, \tag{1.7}$$

where $\overline{E}$ denotes the closure of E in A. More precisely, one can check that

$$\overline{E} = \bigcap \{E + U : U \subseteq A \text{ is an open set with } 0 \in U\}. \tag{1.8}$$

If U, U_1, U_2 are open subsets of A that contain 0 and satisfy (1.6), then we get that

$$\overline{U_1} \subseteq U_1 + U_2 \subseteq U, \tag{1.9}$$

using (1.7) in the first step.

Remember that a topological space X is said to be *regular* in the strict sense if for each point $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $E \subseteq V$. This is equivalent to asking that for each point $x \in X$ and open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$ and $\overline{U} \subseteq W$. If X is regular in the strict sense and X satisfies the first separation condition, then one may say that X is regular in the strong sense, or that X satisfies the third separation condition. This implies that X is Hausdorff, because subsets of X with only one element are closed sets.

If A is a commutative topological group, then A is regular as a topological space in the strong sense, and hence A is Hausdorff in particular. More precisely, the continuity of the group operations on A imply that A is regular in the strict sense. This uses (1.6) and (1.9) to get that for each open set $U \subseteq A$ with $0 \in U$ there is an open set $U_1 \subseteq A$ such that $0 \in U_1$ and $\overline{U_1} \subseteq U$. The analogous statement at any other point in A can be reduced to this one using continuity of translations. As before, the additional requirement that $\{0\}$ be a closed set in A implies that A satisfies the first separation condition, so that A is regular in the strong sense.

Let A be a commutative topological group again, let K be a compact subset of A, let W be an open set in A, and suppose that $K \subseteq W$. If $x \in K$, then continuity of addition on A implies that there is an open set $U(x) \subseteq A$ such that $0 \in U(x)$ and

$$x + U(x) + U(x) \subseteq W. \tag{1.10}$$

The sets $x + U(x)$ with $x \in K$ form an open covering of K in A, and so the compactness of K implies that there are finitely many elements $x_1, \ldots, x_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} (x_j + U(x_j)). \tag{1.11}$$

Put

$$U = \bigcap_{j=1}^{n} U(x_j), \tag{1.12}$$

which is an open set in A that contains 0. We also have that

$$K + U \subseteq \bigcup_{j=1}^{n} (x_j + U(x_j) + U) \subseteq \bigcup_{j=1}^{n} (x_j + U(x_j) + U(x_j)) \subseteq W. \tag{1.13}$$

If $K \subseteq A$ is compact and $E \subseteq A$ is a closed set, then it is well known that $K + E$ is a closed set in A too. To see this, let $a \in A \setminus (K + E)$ be given, so that

$$a - K \subseteq A \setminus E. \tag{1.14}$$

The argument in the preceding paragraph implies that there is an open set $U \subseteq A$ such that $0 \in U$ and

$$(a - K) + U \subseteq A \setminus E, \tag{1.15}$$

because $a - K$ is compact and $A \setminus E$ is an open set in A. Equivalently, this means that

$$a + U \subseteq A \setminus (K + E). \tag{1.16}$$

It follows that $A \setminus (K + E)$ is an open set in A, so that $K + E$ is a closed set in A, as desired.

1.2 Semimetrics

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function defined on $X \times X$. As usual, $d(x, y)$ is said to be a *semimetric* on X if

$$d(x, x) = 0 \tag{1.17}$$

for every $x \in X$,

$$d(x, y) = d(y, x) \tag{1.18}$$

for every $x, y \in X$, and

$$d(x, z) \leq d(x, y) + d(y, z) \tag{1.19}$$

for every $x, y, z \in X$. If we also have that

$$d(x, y) > 0 \tag{1.20}$$

for every $x, y \in X$ with $x \neq y$, then $d(\cdot, \cdot)$ is said to be a *metric* on X.

The *discrete metric* can be defined on any set X by putting $d(x, y)$ equal to 1 when $x \neq y$, and to 0 when $x = y$. It is easy to see that this defines a metric on X. The standard Euclidean metrics on $\mathbf{R}$ and $\mathbf{C}$ are defined by

$$d(x, y) = |x - y|, \tag{1.21}$$

where $|\cdot|$ is the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. In this case, we have that

$$d(a\,x, a\,y) = |a\,x - a\,y| = |a|\,|x - y| \tag{1.22}$$

for all real or complex numbers a, x, and y, as appropriate. If $d(x, y)$ is a semimetric on a set X and Y is a subset of X, then the restriction of $d(x, y)$ to $x, y \in Y$ defines a semimetric on Y, which is a metric on Y when $d(x, y)$ is a metric on X.

Let $d(\cdot, \cdot)$ be a semimetric on a set X. The *open ball* in X centered at a point $x \in X$ with radius $r > 0$ with respect to d is defined by

$$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}. \tag{1.23}$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \geq 0$ with respect to d is defined by

$$\overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}. \tag{1.24}$$

A subset U of X is said to be an *open set* with respect to d if for each $x \in U$ there is an $r > 0$ such that

$$B(x, r) \subseteq U. \tag{1.25}$$

This defines a topology on X, for which open balls are open sets, and closed balls are closed sets, by standard arguments. In particular, one can use this to verify that X is regular in the strict sense with respect to this topology. If d is a metric on X, then X is Hausdorff with respect to this topology.

Of course, the discrete metric on X determines the discrete topology on X, and the standard Euclidean metrics on $\mathbf{R}$ and $\mathbf{C}$ determine their standard topologies. Let d be a semimetric on a set X, let Y be a subset of X, and let d_Y be the restriction of d to elements of X, as before. The open ball in Y centered at a point $y \in Y$ with radius $r > 0$ with respect to d_Y is the same as the intersection of Y with the open ball in X centered at y with radius r with respect to d. Similarly, the closed ball in Y centered at $y \in Y$ with radius $r \geq 0$ with respect to d_Y is the same as the intersection of Y with the closed ball in X centered at y with radius r with respect to d. The topology determined on Y by d_Y is the same as the topology induced on Y by the topology determined on X by d, by a standard argument.

Let A be a commutative group, and let $d(x, y)$ be a semimetric on A. If

$$d(x + a, y + a) = d(x, y) \tag{1.26}$$

for every $a, x, y \in A$, then d is said to be *invariant under translations* on A. This implies that translations on A are homeomorphisms with respect to the topology determined on A by d. If we take $a = -x - y$ in (1.26), then we get that

$$d(-x, -y) = d(x, y) \tag{1.27}$$

for every $x, y \in X$. This implies that $x \mapsto -x$ is a homeomorphism from A onto itself, with respect to the topology determined by d. Using the triangle inequality (1.19) and translation-invariance (1.26), we obtain that

$$\begin{aligned} d(x_0 + y_0, x + y) &\leq d(x_0 + y_0, x + y_0) + d(x + y_0, x + y) \\ &= d(x_0, x) + d(y_0, y) \end{aligned} \tag{1.28}$$

for every $x_0, x, y_0, y \in A$. It follows that

$$B(x_0, r) + B(y_0, t) \subseteq B(x_0 + y_0, r + t) \tag{1.29}$$

for every $x_0, y_0 \in A$ and $r, t > 0$ in this situation. This implies that addition is continuous as a mapping from $A \times A$ into A, using the topology determined on A by d, and the corresponding product topology on $A \times A$. If d is a translation-invariant metric on A, then A is a topological group with respect to the topology corresponding to d.

The discrete metric on any commutative group A is obviously invariant under translations on A. It is easy to see that the standard Euclidean metrics on $\mathbf{R}$ and $\mathbf{C}$ are invariant under translations too. These metrics are also invariant under multiplication by real or complex numbers with absolute value equal to 1, by (1.22). This implies that the restriction of the standard Euclidean metric on $\mathbf{C}$ to the unit circle $\mathbf{T}$ is invariant under translations in $\mathbf{T}$ as a commutative group with respect to multiplication. If A is a commutative group, B is a subgroup of A, and d is a semimetric on A that is invariant under translations on A, then the restriction d_B of d to B is invariant under translations on B.

If d is a semimetric on any set X, then there is a local base for the topology determined on X by d at any point $x \in X$ with only finitely or countably many elements. It suffices to use a sequence of open balls in X centered at x with radii converging to 0. If A is a commutative topological group, then a local base for the topology of A at 0 leads to local bases at every point in A, using translations. If there is a local base for the topology of A at 0 with only finitely or countably many elements, then it is well known that there is a translation-invariant metric on A that determines the same topology.

1.3 Collections of semimetrics

Let X be a set, and let $\mathcal{M}$ be a nonempty collection of semimetrics on X. Let us say that a subset U of X is an *open set* with respect to $\mathcal{M}$ if for each $x \in U$ there are finitely many semimetrics $d_1, \ldots, d_l \in \mathcal{M}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\bigcap_{j=1}^{l} B_{d_j}(x, r_j) \subseteq U, \tag{1.30}$$

where $B_{d_j}(x, r_j)$ is as in (1.23) in the previous section. It is easy to see that this defines a topology on X. If $d \in \mathcal{M}$, then every open set in X with respect to d is an open set with respect to $\mathcal{M}$, and hence every closed set in X with respect to d is a closed set with respect to $\mathcal{M}$. In particular, open balls in X with respect to d are open sets with respect to $\mathcal{M}$, and closed balls in X with respect to d are closed sets with respect to $\mathcal{M}$. It follows that the collection of all open balls in X with respect to elements of $\mathcal{M}$ is a sub-base for the topology determined on X by $\mathcal{M}$. One can check that the topology determined on X by $\mathcal{M}$ is regular in the strict sense. If for each $x, y \in X$ with $x \neq y$ there is a $d \in \mathcal{M}$ such that $d(x, y) > 0$, then $\mathcal{M}$ is said to be *nondegenerate* on X. This implies that X is Hausdorff with respect to the topology determined by $\mathcal{M}$.

Let Y be a subset of X. If d is a semimetric on X, then the restriction d_Y of d to elements of Y defines a semimetric on Y, as in the previous section. Let $\mathcal{M}$ be a nonempty collection of semimetrics on X, and let

$$\mathcal{M}_Y = \{d_Y : d \in \mathcal{M}\} \tag{1.31}$$

be the collection of semimetrics on Y obtained by restricting the elements of $\mathcal{M}$ to Y. One can check that the topology determined on Y by $\mathcal{M}_Y$ is the same as the topology induced on Y by the topology determined on X by $\mathcal{M}$, as in the case of a single semimetric. More precisely, if $E \subseteq Y$ is an open set with respect to the topology induced on Y by the topology determined on X by $\mathcal{M}$, then $E = U \cap Y$ for some open set $U \subseteq X$ with respect to $\mathcal{M}$, and one can verify that E is an open set with respect to $\mathcal{M}_Y$ on Y. In the other direction, if $d \in \mathcal{M}$, $x \in Y$, and $r > 0$, then the open ball in Y centered at x with radius r with respect to d_Y is the same as the intersection of Y with the open ball in X centered at x with radius r with respect to d. This implies that these open balls in Y are open sets with respect to the topology induced on Y by the topology determined on X by $\mathcal{M}$. It follows that every open set in Y with respect to the topology determined by $\mathcal{M}_Y$ is an open set with respect to the topology induced on Y by the topology determined on X by $\mathcal{M}$, because these open balls in Y form a sub-base for the topology determined on Y by $\mathcal{M}_Y$. Of course, if $\mathcal{M}$ is nondegenerate on X, then $\mathcal{M}_Y$ is nondegenerate on Y.

Let A be a commutative group. If $\mathcal{M}$ is a nonempty collection of translation-invariant semimetrics on A, then one can verify that the group operations on A are continuous with respect to the topology determined on A by $\mathcal{M}$. This is analogous to the case of a single translation-invariant semimetric, as in the previous section. If $\mathcal{M}$ is also nondegenerate, then A is a commutative topological group with respect to the topology determined by $\mathcal{M}$. Conversely, if A is any commutative topological group, then it is well known that there is a nondegenerate collection $\mathcal{M}$ of translation-invariant semimetrics on A that determines the same topology on A.

Let X be any set again, and let $d_1, \ldots, d_l$ be finitely many semimetrics on X. Under these conditions, one can check that

$$d(x, y) = \max_{1 \le j \le l} d_j(x, y) \tag{1.32}$$

defines a semimetric on X. We also have that

$$B_d(x, r) = \bigcap_{j=1}^{l} B_{d_j}(x, r) \tag{1.33}$$

for every $x \in X$ and $r > 0$, where the corresponding open balls are as defined in (1.23) in Section 1.2. Using this, one can verify that the topology determined on X by d as in Section 1.2 is the same as the topology determined on X by the collection $\{d_1, \ldots, d_l\}$ as before. Note that d is a metric on X exactly when this collection of semimetrics is nondegenerate on X. One can also use other combinations of $d_1, \ldots, d_l$, such as their sum, with somewhat different relationships between the corresponding balls. If $d_1, \ldots, d_l$ are translation-invariant semimetrics on a commutative group A, then (1.32) is invariant under translations on A too, which would work for the sum of the d_j's as well.

1.4 Sequences of semimetrics

Let d be a semimetric on a set X, and let t be a positive real number. It is easy to see that

$$d_t(x, y) = \min(d(x, y), t) \tag{1.34}$$

also defines a semimetric on X. By construction,

$$B_{d_t}(x,r) = B_d(x,r) \tag{1.35}$$

for every $x \in X$ when $0 \le r \le t$, where the corresponding open balls are as defined in (1.23) in Section 1.2. Similarly,

$$B_{d_t}(x,r) = X \tag{1.36}$$

for every $x \in X$ when $r > t$. It follows that the topology determined on X by d_t as in Section 1.2 is the same as the topology determined by d for every $t > 0$. Note that $d_t(x,y) = 0$ exactly when $d(x,y) = 0$. In particular, if d is a metric on X, then d_t is a metric on X.

Now let $d_1, d_2, d_3, \ldots$ be an infinite sequence of semimetrics on X, and put

$$d_j'(x,y) = \min(d_j(x,y), 1/j) \tag{1.37}$$

for every $j \ge 1$ and $x,y \in X$. Thus, for each $j \ge 1$, d_j' is a semimetric on X that determines the same topology on X as d_j, as in the preceding paragraph. Put

$$d'(x,y) = \max_{j \ge 1} d_j'(x,y) \tag{1.38}$$

for every $x,y \in X$. More precisely, this is equal to 0 when $d_j'(x,y) = 0$ for every j. Otherwise, if $d_l'(x,y) > 0$ for some $l \ge 1$, then (1.38) reduces to the maximum of $d_j'(x,y)$ over finitely many j, since $d_j'(x,y) \le 1/j$ for each j, by construction. As in the case of finitely many semimetrics in the previous section, one can check that (1.38) defines a semimetric on X. If the collection of semimetrics d_j, $j \ge 1$, is nondegenerate on X, then the collection of semimetrics d_j', $j \ge 1$, is nondegenerate on X too, and d' is a metric on X.

Because the maximum on the right side of (1.38) is always attained, we have that

$$B_{d'}(x,r) = \bigcap_{j=1}^{\infty} B_{d_j'}(x,r) \tag{1.39}$$

for every $x \in X$ and $r > 0$, where these open balls are as defined in (1.23) in Section 1.2 again. If $r > 1$, then this reduces to

$$B_{d'}(x,r) = X, \tag{1.40}$$

as in (1.36). If $0 < r \le 1$, and $j(r)$ is the largest positive integer that is less than or equal to $1/r$, then we get that

$$B_{d'}(x,r) = \bigcap_{j=1}^{j(r)} B_{d_j}(x,r) \tag{1.41}$$

for every $x \in X$, using (1.35) and (1.36). One can use this to verify that the topology determined on X by d' is the same as the topology determined on X

by the collection of semimetrics d_j, $j \ge 1$. This is the same as the topology determined on X by the collection of semimetrics d'_j, $j \ge 1$.

Let A be a commutative group. If d is a translation-invariant semimetric on A, then (1.34) is invariant under translations on A for every $t > 0$. Similarly, if $d_1, d_2, d_3, \ldots$ is an infinite sequence of translation-invariant semimetrics on A, then (1.37) is invariant under translations on A for each $j \ge 1$. It follows that (1.38) is invariant under translations on A as well.

1.5 Seminorms

Let V be a vector space over the real or complex numbers. A nonnegative real-valued function N on V is said to be a *seminorm* on V if

$$N(t\,v) = |t|\,N(v) \tag{1.42}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$N(v + w) \le N(v) + N(w) \tag{1.43}$$

for every $v, w \in V$. In (1.42), $|t|$ is the standard absolute function on $\mathbf{R}$ or $\mathbf{C}$, and we get that $N(0) = 0$ by taking $t = 0$. If we also have that

$$N(v) > 0 \tag{1.44}$$

for every $v \in V$ with $v \ne 0$, then N is said to be a *norm* on V. Note that the standard absolute value functions on $\mathbf{R}$ and $\mathbf{C}$ may be considered as norms on $\mathbf{R}$ or $\mathbf{C}$ as one-dimensional vector spaces over themselves.

If N is a seminorm on a real or complex vector space V, then

$$d(v, w) = d_N(v, w) = N(v - w) \tag{1.45}$$

defines a translation-invariant semimetric on V as a commutative group with respect to addition. This uses (1.42) with $t = -1$ to get that (1.45) is symmetric in v and w. We also get that

$$d(t\,v, t\,w) = |t|\,d(v, w) \tag{1.46}$$

for every $v, w \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, in this case, using (1.42) again. Of course, if N is a norm on V, then (1.45) defines a metric on V. The standard metrics on $\mathbf{R}$ and $\mathbf{C}$ correspond to the standard absolute value functions on $\mathbf{R}$ and $\mathbf{C}$ in this way.

Let $\mathcal{N}$ be a nonempty collection of seminorms on a real or complex vector space V. Thus

$$\mathcal{M} = \mathcal{M}(\mathcal{N}) = \{d_N : N \in \mathcal{N}\} \tag{1.47}$$

defines a nonempty collection of translation-invariant semimetrics on V, where d_N corresponds to $N \in \mathcal{N}$ as in (1.45). This leads to a topology on V determined by $\mathcal{N}$, as in Section 1.3. Let us say that $\mathcal{N}$ is *nondegenerate* on V if for each

$v \in V$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$. This implies that (1.47) is nondegenerate as a collection of semimetrics on V, as in Section 1.3, so that the corresponding topology on V is Hausdorff.

If $N_1, \ldots, N_l$ are finitely many seminorms on V, then it is easy to see that

$$N(v) = \max_{1 \le j \le l} N_j(v) \tag{1.48}$$

defines a seminorm on V. Let d_j be the semimetric on V corresponding to N_j as in (1.45) for each $j = 1, \ldots, l$, and let d correspond to (1.48) in the same way. Thus d is the same as the maximum of $d_1, \ldots, d_l$, as in (1.32) in Section 1.3. If $\{N_1, \ldots, N_l\}$ is nondegenerate on V, then N is a norm on V. As before, one can also consider other combinations of finitely many seminorms on V, such as sums.

1.6 Topological vector spaces

Let V be a vector space over the real or complex numbers, and suppose that V is also equipped with a topology. In order for V to be a topological vector space, the vector space operations on V should be continuous. As for commutative topological groups, continuity of addition on V means that addition defines a continuous mapping from $V \times V$ into V, using the associated product topology on $V \times V$. Similarly, continuity of scalar multiplication on V means that scalar multiplication should be continuous as a mapping from $\mathbf{R} \times V$ or $\mathbf{C} \times V$, as appropriate, into V. This uses the product topology on $\mathbf{R} \times V$ or $\mathbf{C} \times V$ associated to the standard topology on $\mathbf{R}$ or $\mathbf{C}$, as appropriate.

In particular, continuity of scalar multiplication implies that

$$v \mapsto t\,v \tag{1.49}$$

is continuous as a mapping from V into itself for each $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. If $t \neq 0$, then it follows that (1.49) is a homeomorphism on V, because the inverse mapping corresponds to multiplication by $1/t$. Of course, (1.49) is the same as $v \mapsto -v$ when $t = -1$. As before, we also ask that $\{0\}$ be a closed set in V, so that a topological vector space is a commutative topological group with respect to addition. Note that the real and complex numbers may be considered as one-dimensional real and complex topological vector spaces, with respect to their standard topologies.

Let $\mathcal{N}$ be a nonempty collection of seminorms on a real or complex vector space V, which leads to a topology on V, as in the previous section. One can check that the vector space operations on V are continuous with respect to this topology. Continuity of addition on V has already been mentioned for collections of translation-invariant semimetrics, so that one really only needs to consider continuity of scalar multiplication. If $\mathcal{N}$ is nondegenerate on V, then it follows that V is a topological vector space with respect to this topology. In particular, if N is a norm on V, then V is a topological vector space with respect to the topology determined by the metric associated to N as in (1.45) in the previous section.

If N is a seminorm on a real or complex vector space V, then the *open ball* in V centered at $v \in V$ and with radius $r > 0$ with respect to N is defined by

$$B_N(v, r) = \{w \in V : N(v - w) < r\}. \tag{1.50}$$

This is the same as the open ball $B_{d_N}(v, r)$ in V centered at v with radius r with respect to the semimetric d_N on V associated to N as in (1.45) in the previous section, where $B_{d_N}(v, r)$ is as in (1.23) in Section 1.2. It is easy to see that (1.50) is a convex subset of V, because N is a seminorm on V.

A topological vector space V is said to be *locally convex* if there is a local base for the topology of V at 0 consisting of convex open sets. If the topology on V is determined by a nondegenerate collection $\mathcal{N}$ of seminorms, then V is locally convex. This uses the fact that open balls in V centered at 0 with respect to elements of $\mathcal{N}$ form a local sub-base for the topology determined on V by $\mathcal{N}$ at 0. Conversely, if V is a locally convex topological vector space over $\mathbf{R}$ or $\mathbf{C}$, then it is well known that there is a nondegenerate collection $\mathcal{N}$ of seminorms on V that determines the same topology on V.

Let V be a real or complex vector space again, and put

$$t\,E = \{t\,v : v \in E\} \tag{1.51}$$

for each $E \subseteq V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. Let us say that E is *balanced* in V if

$$t\,E \subseteq E \tag{1.52}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t| \le 1$. If E is balanced and nonempty, then $0 \in E$, since we can take $t = 0$ in (1.52). Open and closed balls in V centered at 0 with respect to a seminorm N on V are balanced, because of the homogeneity property (1.42) of seminorms.

Suppose that V is a topological vector space over $\mathbf{R}$ or $\mathbf{C}$, and let W be an open subset of V that contains 0. Continuity of scalar multiplication on V at $(0, 0)$ implies that there is a $\delta > 0$ and an open set $U \subseteq V$ such that $0 \in U$ and

$$t\,U \subseteq W \tag{1.53}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $|t| < \delta$. Put

$$U_1 = \bigcup_{0 < |t| < \delta} t\,U, \tag{1.54}$$

where more precisely the union is taken over all real or complex numbers t, as appropriate, with $0 < |t| < \delta$. Thus

$$U_1 \subseteq W, \tag{1.55}$$

by (1.53). Observe that $t\,U$ is an open set in V when $t \neq 0$, since (1.49) is a homeomorphism from V onto itself. This implies that U_1 is an open set in V

too, because it is a union of open sets. Of course, $0 \in U_1$, because $0 \in U$, and U_1 is balanced in V, by construction. This shows that the balanced open subsets of V containing 0 form a local base for the topology of V at 0.

Let E be a subset of a real or complex vector space V, and put

$$\widetilde{E} = \bigcup_{|t| \le 1} t\, E, \tag{1.56}$$

where more precisely the union is taken over all real or complex numbers t, as appropriate, with $|t| \le 1$. By construction, $\widetilde{E}$ is a balanced subset of V that contains E. Alternatively, put

$$E_1 = \{|t| \le 1\} \times E, \tag{1.57}$$

where $\{|t| \le 1\}$ is the set of real or complex numbers t, as appropriate, with $|t| \le 1$. Thus $\widetilde{E}$ is the same as the image of E_1 in V under scalar multiplication as a mapping defined on $\mathbf{R} \times V$ or $\mathbf{C} \times V$, as appropriate.

Remember that $\{|t| \le 1\}$ is compact with respect to the standard topology on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. If E is a compact subset of a real or complex topological vector space V, then $\widetilde{E}$ is compact in V too. This uses the fact that E_1 is compact in $\mathbf{R} \times V$ or $\mathbf{C} \times V$, as appropriate, by Tychonoff's theorem, and continuity of scalar multiplication on V.

1.7 Continuous functions

Let $C(X, Y)$ denote the space of continuous mappings from a topological space X into another topological space Y. If X is a nonempty topological space, then $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ are vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, with respect to pointwise addition and scalar multiplication of functions. In fact, $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ are commutative algebras with respect to pointwise multiplication of functions. If f is a continuous real or complex-valued function on X and K is a nonempty compact subset of X, then put

$$N_K(f) = \sup_{x \in K} |f(x)|. \tag{1.58}$$

This defines a seminorm on each of $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$, which are the *supremum seminorms* associated to K. The collections of these seminorms, corresponding to all nonempty compact subsets of X, determine topologies on $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$, as in Section 1.5. These collections of seminorms are nondegenerate, because finite subsets of X are compact. Thus $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ are topological vector spaces with respect to these topologies. If f, g are continuous real or complex-valued functions on X, then it is easy to see that

$$N_K(f\, g) \le N_K(f)\, N_K(g) \tag{1.59}$$

for every nonempty compact set $K \subseteq X$. Using this, one can check that multiplication of functions on X defines continuous mappings from $C(X, \mathbf{R}) \times C(X, \mathbf{R})$

into $C(X, \mathbf{R})$ and from $C(X, \mathbf{C}) \times C(X, \mathbf{C})$ into $C(X, \mathbf{C})$, where the domains of these mappings are equipped with the associated product topologies.

As before, $C(X, \mathbf{T})$ denotes the space of continuous mappings from X into the unit circle $\mathbf{T}$. This is a subset of $C(X, \mathbf{C})$, and $C(X, \mathbf{T})$ is also a commutative group with respect to pointwise multiplication of functions. More precisely, $C(X, \mathbf{T})$ is a commutative topological group, with respect to the topology induced by the one defined on $C(X, \mathbf{C})$ in the preceding paragraph. Remember that

$$N_K(f - g) \tag{1.60}$$

defines a semimetric on $C(X, \mathbf{C})$ for every nonempty compact set $K \subseteq X$, as in Section 1.5. The restriction of (1.60) to $f, g \in C(X, \mathbf{T})$ defines a semimetric on $C(X, \mathbf{T})$, and the collection of these semimetrics determines a topology on $C(X, \mathbf{T})$, as in Section 1.3. The topology determined on $C(X, \mathbf{T})$ by this collection of semimetrics is the same as the topology induced on $C(X, \mathbf{T})$ by the topology determined on $C(X, \mathbf{C})$ by the analogous collection of semimetrics, as before. If f, g are continuous complex-valued functions on X and a is a continuous mapping from X in $\mathbf{T}$, then

$$N_K(a\, f - a\, g) = N_K(a\, (f - g)) = N_K(f - g) \tag{1.61}$$

for every nonempty compact set $K \subseteq X$. This implies that the restriction of (1.60) to $f, g \in C(X, \mathbf{T})$ is invariant under translations on $C(X, \mathbf{T})$, as a commutative group with respect to pointwise multiplication of functions.

Let Y be another nonempty topological space, and let h be a continuous mapping from X into Y. If f is a continuous real or complex-valued function on Y, then the composition $f \circ h$ defines a continuous function on X. It is easy to see that

$$T_h(f) = f \circ h \tag{1.62}$$

defines linear mappings from $C(Y, \mathbf{R})$ into $C(X, \mathbf{R})$ and from $C(Y, \mathbf{C})$ into $C(X, \mathbf{C})$. These mappings are also algebra homomorphisms with respect to pointwise multiplication of functions. Similarly, (1.62) defines a group homomorphism from $C(Y, \mathbf{T})$ into $C(X, \mathbf{T})$, as commutative groups with respect to pointwise multiplication of functions.

Let K be a nonempty compact subset of X, so that $h(K)$ is a nonempty compact subset of Y. If f is a continuous real or complex-valued function on Y, then

$$N_{K,X}(T_h(f)) = \sup_{x \in K} |f(h(x))| = \sup_{y \in h(K)} |f(y)| = N_{h(K),Y}(f), \tag{1.63}$$

where the additional subscripts X, Y in the supremum seminorms are used to indicate the topological space on which the functions are defined. It follows that

$$N_{K,X}(T_h(f) - T_h(g)) = N_{K,X}(T_h(f - g)) = N_{h(K),Y}(f - g) \tag{1.64}$$

for all continuous real or complex-valued functions f, g on Y, using the linearity of T_h in the first step. Using this, one can check that T_h is continuous as a mapping from $C(Y, \mathbf{R})$ into $C(X, \mathbf{R})$, and as a mapping from $C(Y, \mathbf{C})$ into $C(X, \mathbf{C})$, with respect to the topologies defined on these spaces as before. Similarly, T_h is continuous as a mapping from $C(Y, \mathbf{T})$ into $C(X, \mathbf{T})$, with respect to the topologies defined on these spaces as before.

1.8 Cartesian products

Let I be a nonempty set, let X_j be a set for each $j \in I$, and let

$$X = \prod_{j \in I} X_j \tag{1.65}$$

be the corresponding Cartesian product. If $x \in X$ and $j \in I$, then we let x_j denote the jth coordinate of x in X_j. If $l \in I$ and d_l is a semi-metric on X_l, then it is easy to see that

$$\widetilde{d}_l(x, y) = d_l(x_l, y_l) \tag{1.66}$$

defines a semimetric on X. Let $\mathcal{M}_l$ be a nonempty collection of semimetrics on X_l for each $l \in I$, and let

$$\widetilde{\mathcal{M}}_l = \{\widetilde{d}_l : d_l \in \mathcal{M}_l\} \tag{1.67}$$

be the collection of semimetrics on X that correspond to elements of $\mathcal{M}_l$ as in (1.66). Thus

$$\widetilde{\mathcal{M}} = \bigcup_{l \in I} \widetilde{\mathcal{M}}_l \tag{1.68}$$

is a nonempty collection of semimetrics on X, which leads to a topology on X, as in Section 1.3. Of course, $\mathcal{M}_l$ determines a topology on X_l for each $l \in I$, as in Section 1.3 again. One can check that the topology determined on X by (1.68) is the same as the product topology on X associated to the topology on X_l determined by $\mathcal{M}_l$ for each $l \in I$. Note that (1.68) is nondegenerate on X when $\mathcal{M}_l$ is nondegenerate on X_l for each $l \in I$.

If A_j is a commutative group for each $j \in I$, then the Cartesian product

$$A = \prod_{j \in I} A_j \tag{1.69}$$

is a commutative group as well, where the group operations are defined coordinatewise. Similarly, if A_j is a commutative topological group for each $j \in I$, then one can check that A is a commutative topological group, with respect to the associated product topology. Note that a translation-invariant semimetric d_l on A_l for some $l \in I$ leads to a translation-invariant semimetric $\widetilde{d}_l$ on A as in (1.66).

Suppose now that V_j, $j \in I$, is either a family of real vector spaces, or a family of complex vector spaces. As before, the Cartesian product

$$V = \prod_{j \in I} V_j \tag{1.70}$$

is a vector space over the real or complex numbers too, as appropriate, where the vector space operations are defined coordinatewise. If V_j is a topological vector space for each $j \in I$, then one can verify that V is a topological vector space too, with respect to the associated product topology. If $l \in I$ and N_l is a seminorm on V_l, then

$$\widetilde{N}_l(v) = N_l(v_l) \tag{1.71}$$

defines a seminorm on V. Observe that

$$\widetilde{N}_l(v - w) = N_l(v_l - w_l) \tag{1.72}$$

for every $v, w \in V$. Remember that N_l and $\widetilde{N}_l$ determine semimetrics on V_l and V as in (1.45) in Section 1.5, respectively. The simple identity (1.72) says exactly that these semimetrics on V_l and V correspond to each other as in (1.66).

Let $\mathcal{N}_l$ be a nonempty collection of seminorms on V_l for each $l \in I$, and let

$$\widetilde{\mathcal{N}}_l = \{\widetilde{N}_l : N_l \in \mathcal{N}_l\} \tag{1.73}$$

be the corresponding collection of seminorms on V, as in (1.71). Thus

$$\widetilde{\mathcal{N}} = \bigcup_{l \in I} \widetilde{\mathcal{N}}_l \tag{1.74}$$

is a nonempty collection of seminorms on V, which leads to a topology on V, as in Section 1.5. This is the same as the product topology on V associated to the topologies on the V_l's determined by the $\mathcal{N}_l$'s. This can be obtained from the analogous statement for semimetrics mentioned earlier, since the semimetrics on V associated to elements of $\widetilde{\mathcal{N}}$ correspond to the semimetrics on the V_l's associated to elements of the $\mathcal{N}_l$'s as in (1.66). As before, (1.74) is nondegenerate on V when $\mathcal{N}_l$ is nondegenerate on V_l for each $l \in I$.

1.9 Functions on discrete sets

Let X be any nonempty set, and let $c(X, Y)$ be the space of all mappings from X into a set Y. This is the same as the Cartesian product of copies of Y indexed by X. In particular, $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ are vector spaces over the real and complex numbers, respectively, with respect to pointwise addition and scalar multiplication. More precisely, $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ are commutative algebras with respect to pointwise multiplication. Similarly, $c(X, \mathbf{T})$ is a commutative group with respect to pointwise multiplication.

If $x \in X$, then

$$N_x(f) = |f(x)| \tag{1.75}$$

defines a seminorm on each of $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, for which the corresponding semimetrics are given by

$$N_x(f - g) = |f(x) - g(x)|. \tag{1.76}$$

The collections of these seminorms (1.75) with $x \in X$ are clearly nondegenerate on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, and determine topologies on these spaces, as in Section 1.5. These are the same as the topologies determined on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ by the collections of semimetrics (1.76) with $x \in X$, as in Section 1.3. If we identify $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ with Cartesian copies of $\mathbf{R}$ and $\mathbf{C}$ indexed by X, respectively, then these topologies correspond exactly to the product topologies associated to the standard topologies on $\mathbf{R}$ and $\mathbf{C}$, as in the previous section. Note that multiplication of functions on X defines continuous mappings from $c(X, \mathbf{R}) \times c(X, \mathbf{R})$ into $c(X, \mathbf{R})$ and from $c(X, \mathbf{C}) \times c(X, \mathbf{C})$ into $c(X, \mathbf{C})$, using the topologies on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ just mentioned, and the corresponding product topologies on the domains of these mappings.

If $x \in X$, then (1.76) defines a semimetric on $c(X, \mathbf{T})$, which is invariant under translations on $c(X, \mathbf{T})$. As in the preceding paragraph, the collection of these semimetrics is clearly nondegerate on $c(X, \mathbf{T})$, and determines a topology on $c(X, \mathbf{T})$, as in Section 1.3. This topology corresponds exactly to the product topology on $c(X, \mathbf{T})$ as a Cartesian product of copies of $\mathbf{T}$, using the topology induced on $\mathbf{T}$ by the standard topology on $\mathbf{C}$, as before. In particular, $c(X, \mathbf{T})$ is a commutative topological group with respect to this topology. Note that $c(X, \mathbf{T})$ is compact with respect to this topology, by Tychonoff's theorem, since $\mathbf{T}$ is compact.

If X is equipped with the discrete topology, and Y is any topological space, then $c(X, Y)$ is the same as the space $C(X, Y)$ of all continuous mappings from X into Y. Remember that compact subsets of X have only finitely many elements in this case. This implies that the topologies determined on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ by the collections of seminorms (1.75) with $x \in X$ are the same as the topologies defined on $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ in Section 1.7. Similarly, the topology determined on $c(X, \mathbf{T})$ by the collection of semimetrics (1.76) with $x \in X$ is the same as the topology on $C(X, \mathbf{T})$ discussed in Section 1.7 in this situation. Of course, (1.75) corresponds to (1.58) in Section 1.7 with $K = \{x\}$, and (1.76) corresponds to (1.60) in the same way.

1.10 Dual spaces

As usual, a *linear functional* on a real or complex vector space V is a linear mapping from V into the real or complex numbers, as appropriate, where $\mathbf{R}$ and $\mathbf{C}$ are considered as one-dimensional vector spaces over themselves. If V is a topological vector space, then a linear functional λ on V is said to be *continuous* if λ is continuous with respect to the standard topology on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, as the range of λ. The *dual space* of continuous linear functionals on V may be denoted V'. This is also a vector space over $\mathbf{R}$ or $\mathbf{C}$, as appropriate, with respect to pointwise addition and scalar multiplication of functions. If a linear functional λ on V is continuous at 0, then it is easy to see that λ is continuous everywhere on V, by continuity of translations.

Let λ be a linear functional on V, and suppose that there is an open set

$U \subseteq V$ such that $0 \in U$ and

$$|\lambda(v)| < 1 \tag{1.77}$$

for every $v \in U$. If $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and $t \neq 0$, then $t\,U$ is an open set in V that contains 0, and

$$|\lambda(v)| < |t| \tag{1.78}$$

for every $v \in t\,U$. This implies that λ is continuous at 0, so that λ is continuous on V, as before. Conversely, if λ is continuous at 0, then there is an open set $U \subseteq V$ that contains 0 and satisfies (1.77).

Suppose for the moment that the topology on V is determined by a nonempty collection $\mathcal{N}$ of seminorms on V, and let λ be a linear functional on V again. If there are finitely many seminorms $N_1, \ldots, N_l$ in $\mathcal{N}$ and a nonnegative real number C such that

$$|\lambda(v)| \le C \max_{1 \le j \le l} N_j(v) \tag{1.79}$$

for every $v \in V$, then it is easy to see that λ is continuous on V. Conversely, if λ is continuous on V, then there are finitely many elements of $\mathcal{N}$ such that (1.79) holds for some $C \geq 0$ and every $v \in V$. More precisely, if λ is continuous at 0, then there is an open set $U \subseteq V$ that contains 0 and satisfies (1.77), as before. If the topology on V is determined by $\mathcal{N}$, then there are finitely many seminorms $N_1, \ldots, N_l$ in $\mathcal{N}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\bigcap_{j=1}^{l} B_{N_j}(0, r_j) \subseteq U, \tag{1.80}$$

where $B_{N_j}(0, r_j)$ is as in (1.50) in Section 1.6. This corresponds to (1.30) in Section 1.3 in this situation. One can get (1.79) from (1.77) and (1.80) using standard arguments.

Let I be a nonempty set, and let V_j, $j \in I$, be a family of topological vector spaces, all real or all complex. Also let V be their Cartesian product, as in (1.70) in Section 1.8, equipped with the corresponding product topology. If $l \in I$ and λ_l is a continuous linear functional on V_l, then

$$\widetilde{\lambda}_l(v) = \lambda_l(v_l) \tag{1.81}$$

defines a continuous linear functional on V. It follows that finite sums of linear functionals on V of this type are continuous as well. Conversely, let us check that any continuous linear functional λ on V can be expressed as a finite sum of linear functionals of the form (1.81). Of course, this is very easy to do when I has only finitely many elements. If I has infinitely many elements, then we want to show that continuity of λ implies that $\lambda(v)$ only depends on finitely many coordinates v_j of v.

As before, continuity of λ at 0 implies that there is an open set $U \subseteq V$ that contains 0 and satisfies (1.77). In this situation, we can take U to be of the form

$$U = \prod_{j \in I} U_j, \tag{1.82}$$

where $U_j \subseteq V_j$ is an open set that contains 0 for each $j \in I$, and $U_j = V_j$ for all but finitely many $j \in I$. Put $W_j = V_j$ when $U_j = V_j$, $W_j = \{0\}$ otherwise, and

$$W = \prod_{j \in I} W_j, \tag{1.83}$$

which is a linear subspace of V. Observe that

$$W \subseteq t\,U \tag{1.84}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, with $t \neq 0$. This implies that $\lambda(v) = 0$ for every $v \in W$, because (1.78) holds when $t \neq 0$. Equivalently, this means that $\lambda(v) = 0$ when $v_j = 0$ for each of the finitely many $j \in I$ such that $U_j \neq V_j$. It follows that $\lambda(v)$ depends only on v_j for the finitely many $j \in I$ such that $U_j \neq V_j$, as desired.

If V is any vector space over the complex numbers, then V may be considered as a vector space over the real numbers too. If λ is a complex linear functional on V, then the real part of λ defines a linear functional on V as a real vector space. One can check that every real linear functional on V is the real part of a unique complex linear functional on V. Similarly, if V is a topological vector space over the complex numbers, then V is also a topological vector space the real numbers. As before, continuous real linear functionals on V correspond exactly to real parts of continuous complex linear functionals on V.

If V is any real or complex topological vector space, then V' may be considered as a linear subspace of the space of all continuous real or complex-valued functions on V, as appropriate. In particular, if K is a nonempty compact subset of V, then

$$N_K(\lambda) = \sup_{v \in K} |\lambda(v)| \tag{1.85}$$

defines a seminorm on V', as in (1.58) in Section 1.7. The collection of these seminorms is nondegenerate on V', as before, so that V' becomes a topological vector space with respect to the corresponding topology, as in Section 1.6. Remember that every compact subset of V is contained in a balanced compact set, as in Section 1.6. This implies that we can restrict our attention to supremum seminorms associated to nonempty balanced compact subsets of V, and get the same topology on V'.

Let V, W be topological vector spaces, both real or both complex, and let T be a linear mapping from V into W. If λ is a continuous linear functional on W, then

$$T'(\lambda) = \lambda \circ T \tag{1.86}$$

is a continuous linear functional on V. This defines a linear mapping T' from W' into V', which is the *dual linear mapping* associated to T. If K is a nonempty compact subset of V, then $T(K)$ is a nonempty compact subset of W, and

$$N_{K,V'}(T'(\lambda)) = N_{K,V'}(\lambda \circ T) = N_{T(K),W'}(\lambda) \tag{1.87}$$

for every $\lambda \in W'$. The additional subscripts V', W' in the supremum seminorms here are used to indicate the spaces on which the seminorms are defined. Of course, (1.87) is essentially the same as (1.63) in Section 1.7, with slightly different notation. It follows that T' is continuous as a mapping from W' into V' with respect to the topologies corresponding to supremum seminorms associated to nonempty compact sets, as before.

1.11 Dual groups

Let A, B be commutative groups, with the group operations expressed additively. The collection of all group homomorphisms from A into B is a commutative group as well, with respect to pointwise addition of B-valued functions on A. If A and B are commutative topological groups, then the collection of all continuous group homomorphisms from A into B is a commutative group with respect to pointwise addition too. If a group homomorphism ϕ from A into B is continuous at 0, then it is easy to see that ϕ is continuous everywhere on A, by continuity of translations.

The *dual group* $\widehat{A}$ associated to a commutative topological group A is defined to be the group of continuous homomorphisms from A into the unit circle $\mathbf{T}$. This is a commutative group with respect to pointwise multiplication of functions on A, as before. Of course, if A is equipped with the discrete topology, then $\widehat{A}$ consists of all group homomorphisms from A into $\mathbf{T}$.

Let us consider some basic examples, starting with group $\mathbf{Z}$ of integers with respect to addition, equipped with the discrete topology. If $z \in \mathbf{T}$, then

$$\phi_z(j) = z^j \tag{1.88}$$

defines a homomorphism from $\mathbf{Z}$ into $\mathbf{T}$. If ϕ is any group homomorphism from $\mathbf{Z}$ into $\mathbf{T}$, and if $z = \phi(1)$, then it is easy to see that $\phi = \phi_z$ on $\mathbf{Z}$. This shows that the dual of $\mathbf{Z}$ as a commutative topological group with respect to the discrete topology is isomorphic to $\mathbf{T}$ in a simple way.

If $a \in \mathbf{R}$, then

$$\phi_a(x) = a\,x \tag{1.89}$$

defines a continuous homomorphism from $\mathbf{R}$ into itself, where $\mathbf{R}$ is considered as a commutative topological group with respect to addition and the standard topology. If ϕ is any continuous group homomorphism from $\mathbf{R}$ into itself, and if $a = \phi(1)$, then one can check that

$$\phi(x) = a\,x \tag{1.90}$$

for every $x \in \mathbf{R}$. More precisely, (1.90) is clear when $x \in \mathbf{Z}$, and one can verify (1.90) for rational numbers x using the fact that ϕ is a group homomorphism. This implies that (1.90) holds for every $x \in \mathbf{R}$, by continuity.

The complex exponential function $\exp z$ defines a continuous homomorphism from $\mathbf{C}$ as a commutative topological group with respect to addition and the standard topology into the multiplicative group $\mathbf{C} \setminus \{0\}$ of nonzero complex numbers, with respect to the topology induced on $\mathbf{C} \setminus \{0\}$ by the standard topology on $\mathbf{C}$. If $a \in \mathbf{R}$, then

$$\exp(i\,a\,x) \tag{1.91}$$

defines a continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$, and every continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$ is of this form. More precisely, if ϕ is any continuous mapping from $\mathbf{R}$ into $\mathbf{T}$ such that $\phi(0) = 1$, then it is well known that there is a unique continuous mapping ψ from $\mathbf{R}$ into itself such that $\psi(0) = 0$ and

$$\phi(x) = \exp(i\,\psi(x)) \tag{1.92}$$

for every $x \in \mathbf{R}$. Of course, this implies that

$$\phi(x+t)/\phi(t) = \exp(i\,\psi(x+t))/\exp(i\,\psi(t)) = \exp(i\,(\psi(x+t) - \psi(t)) \tag{1.93}$$

for every $x, t \in \mathbf{R}$. If ϕ is also a group homomorphism from $\mathbf{R}$ into $\mathbf{T}$, then the left side of (1.93) reduces to $\phi(x)$. Observe that

$$\psi_t(x) = \psi(x+t) - \psi(t) \tag{1.94}$$

is a continuous mapping from $\mathbf{R}$ into itself that satisfies $\psi_t(0) = 0$ for every t in $\mathbf{R}$, by construction. Under these conditions, the uniqueness of the representation (1.92) implies that $\psi_t(x) = \psi(x)$ for every $x, t \in \mathbf{R}$, which is the same as saying that ψ is a group homomorphism from $\mathbf{R}$ into itself with respect to addition. Because ψ is continuous, there is an $a \in \mathbf{R}$ such that $\psi(x) = a\,x$ for every $x \in \mathbf{R}$, as in the previous paragraph. This shows that ϕ can be expressed as in (1.91), and it is easy to see that a is uniquely determined by ϕ.

If $j \in \mathbf{Z}$, then

$$\phi_j(z) = z^j \tag{1.95}$$

defines a continuous group homomorphism from $\mathbf{T}$ into itself, and every continuous group homomorphism from $\mathbf{T}$ into itself is of this form. Indeed, if ϕ is any continuous homomorphism from $\mathbf{T}$ into itself, then

$$\phi(\exp(i\,x)) \tag{1.96}$$

defines a continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$. Hence there is an $a \in \mathbf{R}$ such that (1.96) is equal to (1.91) for every $x \in \mathbf{R}$, as in the preceding paragraph. One can check that $a \in \mathbf{Z}$ in this situation, because

$$\exp(2\,\pi\,i\,t) = 1 \tag{1.97}$$

exactly when $t \in \mathbf{Z}$. This implies that ϕ can be expressed as in (1.95), as desired.

Let V be a topological vector space over the real numbers, which may be considered as a commutative topological group with respect to addition. If λ is a continuous linear functional on V, then

$$\exp(i\,\lambda(v)) \tag{1.98}$$

defines a continuous group homomorphism from V into $\mathbf{T}$, and every continuous group homomorphism from V into $\mathbf{T}$ is of this form. To see this, let ϕ be any continuous group homomorphism from V into $\mathbf{T}$, and put

$$\phi_v(t) = \phi(t\,v) \tag{1.99}$$

for every $t \in \mathbf{R}$ and $v \in V$. This defines a continuous group homomorphism from $\mathbf{R}$ into $\mathbf{T}$ for each $v \in V$. It follows that for each $v \in V$ there is a unique $\lambda(v) \in \mathbf{R}$ such that

$$\phi_v(t) = \exp(i\,t\,\lambda(v)) \tag{1.100}$$

for every $t \in \mathbf{R}$, as before. One can check that $\lambda(v)$ is a linear functional on V, using uniqueness. The condition that ϕ be continuous at 0 in V means that there are neighborhoods of 0 in V on which ϕ is close to 1 in $\mathbf{T}$. This implies that there are neighborhoods of 0 in V on which λ is close to 0 in $\mathbf{R}$ modulo integer multiples of $2\,\pi$. As in Section 1.6, we can also take these neighborhoods of 0 in V to be balanced. Using this, one can verify that λ is close to 0 on these neighborhoods of 0 in V, so that λ is continuous at 0 on V, as desired.

Let I be a nonempty set, let A_j be a commutative topological group for each $j \in I$, and let A be their Cartesian product, as in (1.69) in Section 1.8. Thus A is a commutative topological group as well, where the group operations are defined coordinatewise, and using the corresponding product topology. If $l \in I$ and $\phi_l \in \widehat{A_l}$, then

$$\widetilde{\phi_l}(a) = \phi_l(a_l) \tag{1.101}$$

defines an element of $\widehat{A}$. It follows that products of finitely many elements of $\widehat{A}$ of this form are in $\widehat{A}$ too. Conversely, let us show that any element ϕ of $\widehat{A}$ can be expressed as the product of finitely many elements of $\widehat{A}$ of the form (1.101). This is easy to do when I has only finitely many elements. Otherwise, we would like to use continuity of ϕ to show that $\phi(a)$ depends on only finitely many coordinates a_j of a.

Observe that

$$\{z \in \mathbf{T} : \operatorname{Re} z > 0\} \tag{1.102}$$

is a relatively open subset of $\mathbf{T}$ that contains 1, where $\operatorname{Re} z$ denotes the real part of $z \in \mathbf{C}$. If $\phi \in \widehat{A}$, then there is an open set $U \subseteq A$ such that $0 \in U$ and $\phi(U)$ is contained in (1.102), because ϕ is continuous at 0 on A. We may as well take U to be of the form

$$U = \prod_{j \in I} U_j, \tag{1.103}$$

where $U_j \subseteq A_j$ is an open set for each $j \in I$, $0 \in U_j$ for each $j \in I$, and $U_j = A_j$ for all but finitely many $j \in I$. Put $B_j = A_j$ when $U_j = A_j$, $B_j = \{0\}$ otherwise, and

$$B = \prod_{j \in I} B_j, \tag{1.104}$$

so that B is a subgroup of A and $B \subseteq U$. Thus $\phi(B)$ is a subgroup of $\mathbf{T}$ which is contained in $\phi(U)$, and hence $\phi(B)$ is contained in (1.102). This implies that

$$\phi(B) = \{1\}, \tag{1.105}$$

because $\{1\}$ is the only subgroup of $\mathbf{T}$ contained in (1.102). It follows that $\phi(a)$ only depends on a_j for the finitely many $j \in I$ such that $U_j \neq A_j$, as desired.

1.12 The dual topology

Let A be a commutative topological group, whose dual group $\widehat{A}$ may be considered as a subgroup of the group $C(A, \mathbf{T})$ of all continuous $\mathbf{T}$-valued functions on A, with respect to pointwise multiplication of functions. If K is a nonempty compact subset of A, then

$$\sup_{x \in K} |\phi(x) - \psi(x)| \tag{1.106}$$

defines a semimetric on $\widehat{A}$ that is invariant under translations, as in (1.60) in Section 1.7. The collection of these semimetrics is nondegenerate on $\widehat{A}$, as before, and $\widehat{A}$ is a commutative topological group with respect to the topology determined by this collection, as in Section 1.3. Equivalently, this topology on $\widehat{A}$ is the same as those induced by the analogous topologies on $C(A, \mathbf{T})$ and $C(A, \mathbf{C})$, as in Section 1.7.

Suppose for the moment that A is equipped with the discrete topology, so that $\widehat{A}$ reduces to the group of all group homomorphisms from A into $\mathbf{T}$, and $C(A, \mathbf{T})$ reduces to the group $c(A, \mathbf{T})$ of all $\mathbf{T}$-valued functions on A, as in Section 1.9. Thus $c(A, \mathbf{T})$ can be identified with the Cartesian product of copies of $\mathbf{T}$ indexed by A, as before, equipped with the corresponding product topology. In particular, $c(A, \mathbf{T})$ is compact with respect to this topology, by Tychonoff's theorem. It is easy to see that $\widehat{A}$ is a closed set in $c(A, \mathbf{T})$ with respect to this topology. More precisely, for each $a, b \in A$,

$$\{\phi \in c(A, \mathbf{T}) : \phi(a + b) = \phi(a)\, \phi(a)\} \tag{1.107}$$

is a closed set in $c(A, \mathbf{T})$ with respect to this topology, because of continuity of multiplication on $\mathbf{T}$. This implies that $\widehat{A}$ is a closed set in $c(A, \mathbf{T})$, because $\widehat{A}$ is the same as the intersection of the sets (1.107) over all $a, b \in A$. If follows that $\widehat{A}$ is a compact subset of $c(A, \mathbf{T})$, so that $\widehat{A}$ is compact with respect to the topology described in the preceding paragraph. If $A = \mathbf{Z}$ with the discrete topology, for instance, then we have seen in the previous section that $\widehat{A}$ is isomorphic to $\mathbf{T}$ in a simple way, and one can check that this isomorphism is a homeomorphism with respect to the standard topology on $\mathbf{T}$.

Suppose now that A is compact, so that one can take $K = A$ in (1.106). In this case, the corresponding topology on $\widehat{A}$ is the discrete topology. More precisely, if $\phi \in \widehat{A}$ is uniformly close to the constant function equal to 1 on A, then $\phi(A)$ is a subgroup of $\mathbf{T}$ contained in (1.102). This implies that

$$\phi(A) = \{1\}, \tag{1.108}$$

so that the subset of $\widehat{A}$ consisting of the constant function equal to 1 on A is an open set in $\widehat{A}$. It follows that the topology on $\widehat{A}$ described earlier is the discrete topology, using translations in $\widehat{A}$.

Suppose that $A = \mathbf{R}$, as a commutative group with respect to addition, and equipped with the standard topology. Remember that there is a simple group isomorphism between $\widehat{A}$ and $\mathbf{R}$, as in the previous section. In this case, one can check directly that this isomorphism is a homeomorphism with respect to the topology on $\widehat{A}$ described earlier and the standard topology on $\mathbf{R}$.

Let V be a topological vector space over the real numbers, and let A be V considered as a commutative topological group with respect to addition. As in the previous section, there is a simple group isomorphism between $\widehat{A}$ and the dual space V', where V' is considered as a commutative group with respect to addition. One can verify that this isomorphism is a homeomorphism with the respect to the topology on $\widehat{A}$ described earlier and the analogous topology on V', discussed in Section 1.10. More precisely, it is easy to see that the mapping from V' into $\widehat{A}$ is continuous. In order to get the continuity of the mapping from $\widehat{A}$ into V', we use the fact that every compact subset of V is contained in a balanced compact set, as in Section 1.6. This permits us to restrict our attention to supremum semimetrics associated to nonempty balanced compact subsets K of V in the earlier definition of the topology on $\widehat{A}$. If two elements of $\widehat{A}$ are uniformly close on a nonempty balanced compact set $K \subseteq V$, then the corresponding elements of V' are uniformly close on K as well. This implies that the mapping from $\widehat{A}$ into V' is continuous, as desired.

Let n be a positive integer, and suppose that A_j is a commutative topological group for each $j = 1, \ldots, n$. Thus

$$A = \prod_{j=1}^{n} A_j \tag{1.109}$$

is a commutative topological group, where the group operations are defined coordinatewise, and using the corresponding product topology, as in Section 1.8. If $\phi_j \in \widehat{A}_j$ for each $j = 1, \ldots, n$, then

$$\phi(a) = \prod_{j=1}^{n} \phi_j(a_j) \tag{1.110}$$

defines an element of $\widehat{A}$, and every element of $\widehat{A}$ can be expressed as in (1.110) in a unique way. This defines a group isomorphism from

$$\prod_{j=1}^{n} \widehat{A}_j \tag{1.111}$$

onto $\widehat{A}$. One can check that this isomorphism is a homeomorphism with respect to the topologies on $\widehat{A}_j$ and $\widehat{A}$ described earlier, and the corresponding product topology on (1.111). More precisely, if K_j is a compact subset of A_j for each $j = 1, \ldots, n$, then

$$\prod_{j=1}^{n} K_j \tag{1.112}$$

is a compact subset of A, by Tychonoff's theorem. If K is any compact subset of A, then the image of K in A_j under the coordinate projection mapping from A onto A_j is compact for each $j = 1, \ldots, n$, because these coordinate projection mappings are continuous. Of course, K is contained in the product of its coordinate projections in $A_1, \ldots, A_n$. This permits us to restrict our attention to supremum semimetrics associated to compact subsets of A of the form (1.112) in the definition of the topology on $\widehat{A}$. The supremum semimetric on $\widehat{A}$ associated to a compact subset of A of the form (1.112) can be estimated in terms of the supremum semimetrics on $\widehat{A}_j$ associated to the compact sets $K_j \subseteq A_j$ for $j = 1, \ldots, n$. Using this, one can verify that the isomorphism from (1.111) onto $\widehat{A}$ mentioned earlier is a homeomorphism, as desired.

Let A, B be commutative topological groups, and let h be a continuous group homomorphism from A into B. If $\phi \in \widehat{B}$, then

$$\widehat{h}(\phi) = \phi \circ h \tag{1.113}$$

is in $\widehat{A}$. This defines a group homomorphism $\widehat{h}$ from $\widehat{B}$ into $\widehat{A}$, which is the *dual homomorphism* associated to h. Let K be a nonempty compact subset of A, so that $h(K)$ is a nonempty compact subset of B. If $\phi, \psi \in \widehat{B}$, then

$$\begin{aligned} \sup_{x \in K} |(\widehat{h}(\phi))(x) - (\widehat{h}(\psi))(x)| &= \sup_{x \in K} |\phi(h(x)) - \psi(h(x))| \\ &= \sup_{y \in h(K)} |\phi(y) - \psi(y)|. \end{aligned} \tag{1.114}$$

This is essentially the same as (1.64) in Section 1.7, with slightly different notation. It follows from (1.114) that $\widehat{h}$ is continuous with respect to the topologies on $\widehat{A}$, $\widehat{B}$ described earlier.

1.13 Weak topologies

Let V be a vector space over the real or complex numbers. If λ is a linear functional on V, then

$$N_\lambda(v) = |\lambda(v)| \tag{1.115}$$

defines a seminorm on V. Let Λ be a nonempty collection of linear functionals on V, so that

$$\mathcal{N}(\Lambda) = \{N_\lambda : \lambda \in \Lambda\} \tag{1.116}$$

is a nonempty collection of seminorms on V. Let us say that Λ is *nondegenerate* on V if for every $v \in V$ with $v \neq 0$ there is a $\lambda \in \Lambda$ such that $\lambda(v) \neq 0$, which is equivalent to asking that Λ separate points in V. This implies that (1.116) is nondegenerate as a collection of seminorms on V, as in Section 1.5. The topology determined on V by (1.116) as in Section 1.5 is known as the *weak topology* on V associated to Λ. This is the same as the weakest topology on V such that every element of Λ is continuous on V. It follows that finite linear combinations of elements of Λ are also continuous on V with respect to this topology.

Conversely, if μ is a linear functional on V that is continuous with respect to this topology on V, then μ can be expressed as a linear combination of finitely many elements of Λ. To see this, remember that the continuity of μ implies that there are finitely many elements $\lambda_1, \ldots, \lambda_l$ of Λ and a nonnegative real number C such that

$$|\mu(v)| \le C \max_{1 \le j \le l} N_{\lambda_j}(v) \tag{1.117}$$

for every $v \in V$, as in (1.79) in Section 1.10. In this situation, this means that

$$|\mu(v)| \le C \max_{1 \le j \le l} |\lambda_j(v)| \tag{1.118}$$

for every $v \in V$. In particular, it follows that the intersection

$$\bigcap_{j=1}^{l} \{v \in V : \lambda_j(v) = 0\} \tag{1.119}$$

of the kernels of the λ_j's is contained in the kernel

$$\{v \in V : \mu(v) = 0\} \tag{1.120}$$

of μ. One can use this and basic linear algebra to get that μ can be expressed as a linear combination of $\lambda_1, \ldots, \lambda_l$, as desired.

If X is a nonempty set and $x \in X$, then

$$\lambda_x(f) = f(x) \tag{1.121}$$

defines a linear functional on each of $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$. The corresponding seminorms

$$N_{\lambda_x}(f) = |\lambda_x(f)| = |f(x)| \tag{1.122}$$

on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ are the same as (1.75) in Section 1.9. Of course,

$$\{\lambda_x : x \in X\} \tag{1.123}$$

is nondegenerate on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, essentially by construction. The weak topologies on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ determined by (1.123) are the same as the topologies discussed in Section 1.9.

Now let V be any real or complex topological vector space, and let V' be the corresponding dual space, as in Section 1.10. The corresponding *weak topology* on V is the topology obtained by taking Λ to be V' in the earlier discussion. The given topology on V is always at least as strong as this weak topology, because the elements of V' are continuous on V with respect to the given topology on V, by hypothesis. It is well known that V' separates points in V when V is locally convex, by the Hahn–Banach theorem.

If $v \in V$, then

$$L_v(\lambda) = \lambda(v) \tag{1.124}$$

defines a linear functional on V'. This leads to a seminorm

$$N_{v,V'}(\lambda) = |L_v(\lambda)| = |\lambda(v)| \tag{1.125}$$

on V' for each $v \in V$, as before. The collection

$$\{L_v : v \in V\} \tag{1.126}$$

of these linear functionals on V' is automatically nondegenerate, because $\lambda \in V'$ is nonzero exactly when there is a $v \in V$ such that $\lambda(v) \neq 0$. The topology on V' determined by (1.126) is known as the *weak* topology*. The topology on V' discussed in Section 1.10 is always at least as strong as the weak* topology, because subsets of V with only one element are compact.

1.14 Uniform continuity

Let $(X, d(x, y))$ be a metric space, and let E be a subset of X. Let us say that a real or complex-valued function f on X is *uniformly continuous along* E if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x \in E$ and $y \in X$ with $d(x, y) < \delta$, we have that

$$|f(x) - f(y)| < \epsilon. \tag{1.127}$$

If this holds with $E = X$, then f is said to be *uniformly continuous* on X. If f is uniformly continuous along $E \subseteq X$, then the restriction of f to E is uniformly continuous on E, where E is considered as a metric space, using the restriction of $d(x, y)$ to $x, y \in E$. This condition also implies that f is continuous as a function on X at every point in E. If f is a real or complex-valued function on X that is continuous at every point in a compact set $E \subseteq X$, then f is uniformly continuous along E. This is often considered in the case where $E = X$, but essentially the same argument works for compact subsets of X.

Now let A be a commutative topological group, and let E be a subset of A. A real or complex-valued function f on A is said to be *uniformly continuous along* E if for each $\epsilon > 0$ there is an open set $U \subseteq A$ such that $0 \in U$ and for every $x \in E$ and $y \in A$ with $y - x \in U$ we have that (1.127) holds. If this condition is satisfies with $E = A$, then f is said to be *uniformly continuous* on A. If f is uniformly continuous along $E \subseteq A$, then f is continuous as a function on A at every point in E. If a real or complex-valued function f on A is continuous at every point in a compact set $E \subseteq A$, then one can show that f is uniformly continuous along E, using an argument like the one for metric spaces. If the topology on A is determined by a translation-invariant metric $d(x, y)$ on A, then these uniform continuity conditions on A as a commutative topological group are equivalent to their analogues for A as a metric space with respect to $d(x, y)$. If A is any commutative topological group, then it is easy to see that every $\phi \in \widehat{A}$ is uniformly continuous on A. Similarly, if V is a real or complex topological vector space, then continuous linear functionals on V are uniformly continuous on V as a commutative topological group with respect to addition.

If A is a metric space or a commutative topological group, then linear combinations of real or complex-valued uniformly continuous functions on A are also uniformly continuous. One can check that products of bounded uniformly continuous functions on A are uniformly continuous as well. If a real or complex-valued function f on A can be approximated uniformly by uniformly continuous functions on A, then f is uniformly continuous on A too, by a standard argument.

Let X be a nonempty topological space, and let $C_b(X, \mathbf{R})$, $C_b(X, \mathbf{C})$ denote the spaces of real or complex-valued functions on X, respectively, that are bounded and continuous. These are linear subspaces of the corresponding spaces $C(X, \mathbf{R})$, $C(X, \mathbf{C})$ of all continuous functions on X, and more precisely they are subalgebras with respect to pointwise multiplication of functions. Of course, if X is compact, then every continuous real or complex-valued function on X is bounded. If f is a bounded real or complex-valued function on X, then the *supremum norm* of f is defined as usual by

$$\|f\|_{sup} = \sup_{x \in X} |f(x)|. \tag{1.128}$$

This defines a norm on each of $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$, which also satisfies

$$\|f\, g\|_{sup} \le \|f\|_{sup}\, \|g\|_{sup} \tag{1.129}$$

for all bounded real or complex-valued functions f, g on X. If A is a nonempty metric space or a commutative topological group, then we let

$$UC_b(A, \mathbf{R}),\ UC_b(A, \mathbf{C}) \tag{1.130}$$

be the spaces of real or complex-valued functions on A, respectively, that are bounded and uniformly continuous. These are closed subalgebras of $C_b(A, \mathbf{R})$, $C_b(A, \mathbf{C})$, respectively, by the remarks in the preceding paragraph.

1.15 Some simple extension arguments

Let A be a commutative group, with the group operations expressed additively. If $a \in A$ and j is a positive integer, then we let $j \cdot a$ be the sum of j a's in A. This can be extended to every integer j, by putting $0 \cdot a$ equal to 0 in A, and $(-j) \cdot a = -(j \cdot a)$ when $j \ge 1$. Of course, this is basically the same as considering A as a module over $\mathbf{Z}$.

Let B be a subgroup of A, and let ϕ be a group homomorphism from B into $\mathbf{T}$. Also let a be an element of A not in B, and let B_1 be the subgroup of A generated by a and B. We would like to show that there is an extension of ϕ to a group homomorphism ϕ_1 from B_1 into $\mathbf{T}$. Note that every element of B_1 can be expressed as

$$j \cdot a + b \tag{1.131}$$

for some $j \in \mathbf{Z}$ and $b \in B$. Suppose first that $j \cdot a \notin B$ for every positive integer j, which implies that this holds when j is a negative integer as well. In this case, the expression of an element of B_1 as in (1.131) is unique. In order to extend ϕ to B_1, we can put

$$\phi_1(j \cdot a + b) = \phi(b) \tag{1.132}$$

for every $j \in \mathbf{Z}$ and $b \in B$. This amounts to taking $\phi_1(a) = 1$, and one could also extend ϕ to B_1 by taking $\phi_1(a)$ to be any element of $\mathbf{T}$. Otherwise, let j_0 be the smallest positive integer such that $j_0 \cdot a \in B$, and let α be an element of $\mathbf{T}$ such that

$$\alpha^{j_0} = \phi(j_0 \cdot a). \tag{1.133}$$

In this case, one can check that

$$\phi_1(j \cdot a + b) = \alpha^j \, \phi(b) \tag{1.134}$$

defines a group homomorphism from B_1 into $\mathbf{T}$ that extends ϕ.

Repeating the process, one can extend ϕ to a homomorphism from any subgroup of A generated by B and finitely or countably many elements of A into $\mathbf{T}$. This may include A itself, and otherwise one can extend ϕ to a group homomorphism from A into $\mathbf{T}$ using Zorn's lemma or the Hausdorff maximality principle.

Let $x \in A$ with $x \neq 0$ be given, and let B be the subgroup of A generated by x. It is easy to see that there is a group homomorphism from B into $\mathbf{T}$ such that $\phi(x) \neq 1$. The previous extension argument implies that ϕ can be extended to a group homomorphism from A into $\mathbf{T}$ with the same property. It follows that group homomorphisms from A into $\mathbf{T}$ separate points in A.

1.16 The dual of $C(X)$

Let X be a nonempty topological space, and let λ be a continuous linear functional on $C(X, \mathbf{R})$ or $C(X, \mathbf{C})$, with respect to the topology defined in Section 1.7. As in (1.79) in Section 1.10, this means that there is a nonnegative real number C and finitely many nonempty compact subsets $K_1, \ldots, K_l$ of X such that

$$|\lambda(f)| \leq C \max_{1 \leq j \leq l} N_{K_j}(f) \tag{1.135}$$

for every continuous real or complex-valued function f on X, as appropriate. Here N_K is the supremum seminorm associated to a nonempty compact subset K of X, as in (1.58) in Section 1.7. In this situation, we can take $K = \bigcup_{j=1}^{l} K_l$, to get that

$$|\lambda(f)| \leq C \, N_K(f) \tag{1.136}$$

for every continuous real or complex-valued function f on X, as appropriate. If X is compact, then we may as well take $K = X$, so that $N_K(f)$ is the same as the supremum norm $\|f\|_{sup}$ of f, as in (1.128) in Section 1.14.

If a continuous real or complex-valued function f on X satisfies $f(x) = 0$ for every $x \in K$, then (1.136) implies that $\lambda(f) = 0$. If f_1, f_2 are continuous real or complex-valued functions on X that satisfy

$$f_1(x) = f_2(x) \tag{1.137}$$

for every $x \in K$, then it follows that

$$\lambda(f_1) = \lambda(f_2). \tag{1.138}$$

Of course, K can also be considered as a topological space, with respect to the topology induced by the one on X. If f is a continuous real or complex-valued function on X, then the restriction of f to K defines a continuous real or complex-valued function on K, as appropriate. Using (1.138), we get that λ determines a linear functional λ_0 on the space of real or complex-valued functions on K that are restrictions to K of continuous real or complex-valued functions on X, as appropriate. If every continuous real or complex-valued function on K is the restriction to K of a continuous real or complex-valued function on X, then λ_0 defines a linear functional on $C(K, \mathbf{R})$ or $C(K, \mathbf{C})$, as appropriate. In this case, λ_0 satisfies an estimate like (1.136), and hence is continuous with respect to the topology determined on $C(K, \mathbf{R})$ or $C(K, \mathbf{C})$ by the supremum norm. Otherwise, one can use the Hahn–Banach theorem to extend λ_0 to a linear functional on $C(K, \mathbf{R})$ or $C(K, \mathbf{C})$, as appropriate, that satisfies the same type of estimate.

Let Y be another topological space, and let h be a continuous mapping from X into Y. If f is a continuous real or complex-valued function on Y, then

$$T_h(f) = f \circ h \tag{1.139}$$

is a continuous function on X. As in Section 1.7, (1.139) defines continuous linear mappings from $C(Y, \mathbf{R})$, $C(Y, \mathbf{C})$ into $C(X, \mathbf{R})$, $C(X, \mathbf{C})$, respectively, with respect to the topologies defined on these spaces as before. This leads to corresponding dual linear mappings T_h' from $C(X, \mathbf{R})'$, $C(X, \mathbf{C})'$ into $C(Y, \mathbf{R})'$, $C(Y, \mathbf{C})'$, respectively, as in Section 1.10. More precisely, if λ is a continuous linear functional on $C(X, \mathbf{R})$ or $C(X, \mathbf{C})$, then $T_h'(\lambda)$ is the continuous linear functional on $C(Y, \mathbf{R})$ or $C(Y, \mathbf{C})$, as appropriate, defined by

$$(T_h'(\lambda))(f) = \lambda(T_h(f)) = \lambda(f \circ h) \tag{1.140}$$

for every continuous real or complex-valued function f on Y, as appropriate. If λ satisfies (1.136) for some $C \ge 0$ and nonempty compact set $K \subseteq X$, then we have that

$$|\lambda(f \circ h)| \le C \sup_{x \in K} |f(h(x))| = C \sup_{y \in h(K)} |f(y)| \tag{1.141}$$

for every continuous real or complex-valued function f on Y, as appropriate. Of course, $h(K)$ is a compact subset of Y under these conditions, because h is continuous.

Suppose that continuous real-valued functions on X separate points in X. In particular, this implies that X is Hausdorff, so that compact subsets of X are closed sets. If $x \in X$, $K_1 \subseteq X$ is compact, and $x \in X \setminus K_1$, then there is a continuous mapping from X into $[0,1]$ that is equal to 0 at x and equal to 1 on K_1, by standard arguments. One can also choose this mapping to be equal to 0 on a neighborhood of x. Similarly, if K_0, K_1 are disjoint compact subsets of X, then there is a continuous mapping from X into $[0,1]$ that is equal to 0 on K_0 and to 1 on K_1. If $K \subseteq X$ is compact, then every continuous real or complex-valued function on K extends to a continuous real or complex-valued function on X, as appropriate. This can be shown using an argument like the one used to prove Tietze's extension theorem.

1.17 Bounded sets

Let V be a topological vector space over the real or complex numbers. A subset E of V is said to be *bounded* if for every open set $U \subseteq V$ with $0 \in U$ there is a $t_0 \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that

$$E \subseteq t_0\, U. \tag{1.142}$$

If U is balanced in V, then (1.142) implies that

$$E \subseteq t\, U \tag{1.143}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t| \geq |t_0|$.

If $U \subseteq V$ is any open set that contains 0, then we have seen that there is a balanced open set $U_1 \subseteq V$ such that $0 \in U_1$ and $U_1 \subseteq U$, as in Section 1.6. If $E \subseteq V$ is a bounded set, then $E \subseteq t\, U_1$ for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t|$ is sufficiently large. This implies that (1.143) holds for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t|$ is sufficiently large.

In order to verify that a subset E of V is bounded, it suffices to consider open sets $U \subseteq V$ in a local base for the topology of V at 0. In particular, it is enough to consider balanced open subsets of V that contain 0. Note that subsets of bounded sets in V are bounded too. It is easy to see that the union of finitely many bounded subsets of V is bounded as well.

Let $v \in V$ be given, and observe that

$$r \mapsto r\, v \tag{1.144}$$

defines a continuous mapping from $\mathbf{R}$ or $\mathbf{C}$, as appropriate, into V, because of continuity of scalar multiplication on V. If $U \subseteq V$ is an open set that contains 0, then it follows that $r\, v \in U$ when $|r|$ is sufficiently small, by continuity of (1.144) at 0. This implies that $\{v\}$ is a bounded set in V. Hence finite subsets of V are bounded in V, because the union of finitely many bounded sets is bounded.

Remember that V is regular as a topological space, as in Section 1.1. If $U \subseteq V$ is an open set that contains 0, then it follows that there is an open set

$U_0 \subseteq V$ such that $0 \in U_0$ and $\overline{U_0} \subseteq U$. Using this, one can check that the closure of a bounded subset of V is bounded in V too.

Suppose that $E_1, E_2 \subseteq V$ are bounded sets, and let $U \subseteq V$ be an open set with $0 \in U$. As in Section 1.1, continuity of addition on V at 0 implies that there are open sets $U_1, U_2 \subseteq V$ such that $0 \in U_1, U_2$ and $U_1 + U_2 \subseteq U$. Because E_1 and E_2 are bounded in V, we have that $E_1 \subseteq t\,U_1$ and $E_2 \subseteq t\,U_2$ for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t|$ is sufficiently large. This implies that

$$E_1 + E_2 \subseteq t\,U_1 + t\,U_2 \subseteq t\,U \tag{1.145}$$

when $|t|$ is sufficiently large, so that $E_1 + E_2$ is bounded in V as well.

If the topology on V is determined by a nonempty collection $\mathcal{N}$ of seminorms on V, then one can check that $E \subseteq V$ is bounded if and only if each element of $\mathcal{N}$ is bounded on V. If V is locally convex and $E \subseteq V$ is a bounded set, then one can verify that the convex hull of E is bounded in V too.

Let $U \subseteq V$ be an open set that contains 0 again. Observe that

$$\bigcup_{j=1}^{\infty} j\,U = V, \tag{1.146}$$

because subsets of V with only one element are bounded, as before. If U is also balanced, then we have that $j\,U \subseteq l\,U$ when $j \le l$. If $K \subseteq V$ is compact, then it follows that $K \subseteq j\,U$ for some positive integer j. This implies that K is bounded in V, because balanced open subsets of V that contain 0 form a local base for the topology of V at 0.

Let E be a subset of V, and put $\widetilde{E} = \bigcup_{|t| \le 1} t\,E$, as in (1.56) in Section 1.6. As before, the union is taken over all $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t| \le 1$. If E is bounded in V, then $\widetilde{E}$ is bounded in V too. This uses the fact that one can restrict one's attention to balanced open sets $U \subseteq V$ that contain 0 in the definition of a bounded set.

A sequence $\{v_j\}_{j=1}^{\infty}$ of elements of V is said to be *bounded* if the set of v_j's is bounded in V. Let $\{t_j\}_{j=1}^{\infty}$ be a sequence of real or complex numbers, as appropriate, that converges to 0. If $\{v_j\}_{j=1}^{\infty}$ is a bounded sequence in V, then it is easy to see that $\{t_j\,v_j\}_{j=1}^{\infty}$ converges to 0 in V.

If a sequence $\{v_j\}_{j=1}^{\infty}$ of elements of V converges to some $v \in V$, then $\{v_j\}_{j=1}^{\infty}$ is bounded in V. This follows from the fact that the set of v_j's together with v is compact in V. One can also check this more directly, using the fact that subsets of V with one element are bounded.

If $E \subseteq V$ is a bounded set, then every sequence of elements of E is bounded in V. If E is not bounded in V, then there is an open set $U \subseteq V$ such that $0 \in U$ and E is not contained in $j\,U$ for any positive integer j. Thus, for each positive integer j, there is a $v_j \in E$ such that $(1/j)\,v_j$ is not in U. This implies that $\{(1/j)\,v_j\}_{j=1}^{\infty}$ does not converge to 0 in V.

1.18 Bounded linear mappings

Let V, W be topological vector spaces, both real or both complex. A linear mapping T from V into W is said to be *bounded* if for every bounded set $E \subseteq V$ we have that $T(E)$ is bounded in W. It is easy to see that continuous linear mappings are bounded, directly from the definitions.

Let X, Y be topological spaces, and let f be a mapping from X into Y. As usual, f is said to be *sequentially continuous* at a point $x \in X$ if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of X that converges to x, we have that $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y. If f is continuous at x, then f is sequentially continuous at x. If f is sequentially continuous at x, and if there is a local base for the topology of X at x with only finitely or countably many elements, then it is well known that f is continuous at x.

Let V, W be as before, and let T be a linear mapping from V into W that is sequentially continuous at 0. Also let $\{v_j\}_{j=1}^\infty$ be a bounded sequence in V, and let $\{t_j\}_{j=1}^\infty$ be a sequence of real or complex numbers, as appropriate, that converges to 0. Thus $\{t_j\, v_j\}_{j=1}^\infty$ converges to 0 in V, as in the previous section. This implies that $t_j\, T(v_j) = T(t_j\, v_j) \to 0$ in W as $j \to \infty$, because T is sequentially continuous at 0. Using this, one can check that T is a bounded linear mapping from V into W, as follows. If $E \subseteq V$ has the property that $T(E)$ is not bounded in W, then there is a sequence $\{v_j\}_{j=1}^\infty$ of elements of E such that $\{(1/j)\, T(v_j)\}_{j=1}^\infty$ does not converge to 0 in W, as in the previous section. This implies that E is not bounded in V, by the previous remark.

Suppose for the moment that there is a local base for the topology of V at 0 with only finitely or countably many elements. This implies that there is a sequence $U_1, U_2, U_3, \ldots$ of open subsets of V such that $0 \in U_j$ for every j, and for every open set $U \subseteq V$ with $0 \in U$ we have that $U_j \subseteq U$ for some j. We may also ask that $U_{j+1} \subseteq U_j$ for each j, since otherwise we can replace U_j with $\bigcap_{l=1}^j U_l$ for each j.

Let T be a linear mapping from V into W that is not continuous at 0. This means that there is an open set $U_W \subseteq W$ such that $0 \in U_W$ and

$$T(U_V) \not\subseteq U_W \tag{1.147}$$

for every open set $U_V \subseteq V$ with $0 \in U_V$. In particular, if j is a positive integer, then we can apply this to $(1/j)\, U_j$. Thus, for each positive integer j, we can choose $v_j \in U_j$ such that $(1/j)\, T(v_j) = T((1/j)\, v_j) \not\in U_W$. It follows that $T(v_j) \not\in j\, U_W$ for every j, so that $\{T(v_j)\}_{j=1}^\infty$ is not bounded in W.

Note that $\{v_j\}_{j=1}^\infty$ converges to 0 in V, because $v_j \in U_j$ for each j. This implies that $\{v_j\}_{j=1}^\infty$ is a bounded sequence in V, as in the preceding section. If T is a bounded linear mapping from V into W, then it follows that $\{T(v_j)\}_{j=1}^\infty$ is a bounded sequence in W, which is a contradiction. This shows the well-known fact that bounded linear mappings from V into W are continuous when there is a local base for the topology of V at 0 with only finitely or countable many elements.

Suppose for the moment again that the topologies on V and W are determined by norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. A linear mapping T from V into

W is said to be *bounded* with respect to these norms if there is a nonnegative real number C such that

$$\|T(v)\|_W \le C\,\|v\|_V \tag{1.148}$$

for every $v \in V$. It is easy to see that this is equivalent to the boundedness of T in the earlier sense in this context, because subsets of V and W are bounded in the sense of the previous section if and only if they are bounded with respect to $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. In particular, balls in V with respect to $\|\cdot\|_V$ are bounded in the sense of the previous section. One can also check directly that this notion of boundedness is equivalent to continuity in this situation.

Let V be a topological vector space over the real or complex numbers again, and let V' be the dual space of continuous linear functionals on V, as in Section 1.10. If E is a nonempty bounded subset of V, then put

$$N_E(\lambda) = \sup_{v \in E} |\lambda(v)| \tag{1.149}$$

for every $\lambda \in V'$. More precisely, if $\lambda \in V'$, then λ is bounded as a linear mapping from V into $\mathbf{R}$ or $\mathbf{C}$, as appropriate, as mentioned earlier. This implies that λ is bounded on E, so that (1.149) is finite. It is easy to see that (1.149) defines a seminorm on V'. The collection of these seminorms on V' is nondegenerate, because subsets of V with only one element are bounded. The topology determined on V' by this collection of seminorms is at least as strong as the topology described in Section 1.10, because compact subsets of V are bounded.

Suppose now that the topology on V is determined by a norm $\|\cdot\|_V$. A linear functional λ on V is said to be *bounded* with respect to $\|\cdot\|_V$ if there is a nonnegative real number C such that

$$|\lambda(v)| \le C\|v\|_V \tag{1.150}$$

for every $v \in V$. This is the same as (1.148), with $W = \mathbf{R}$ or $\mathbf{C}$, as appropriate, and where $\|\cdot\|_W$ is the corresponding absolute value function. Thus a bounded linear functional in this sense is the same as a continuous linear functional on V in this context, which also corresponds to (1.79) in Section 1.10. Put

$$\|\lambda\|_{V'} = \sup\{|\lambda(v)| : v \in V,\ \|v\|_V \le 1\} \tag{1.151}$$

for each $\lambda \in V'$, which is the same as (1.149), with E equal to the closed unit ball in V with respect to $\|\cdot\|_V$. This is also the smallest $C \ge 0$ such that (1.150) holds. Note that $\|\cdot\|_{V'}$ defines a norm on V', which is the *dual norm* associated to $\|\cdot\|_V$ on V. The topology determined on V' by the dual norm in this situation is the same as the topology determined by the collection of all seminorms (1.149) associated to nonempty bounded subsets of V, as in the preceding paragraph.

Let I be a nonempty set, and suppose that either V_j is a topological vector space over the real numbers for each $j \in I$, or that V_j is a topological vector space over the complex numbers for each $j \in I$. Thus the Cartesian product

$V = \prod_{j \in I} V_j$ is a topological vector space over the real or complex numbers, as appropriate, with respect to the associated product topology, as in Section 1.8. Let p_l be the standard coordinate projection from V onto V_l for each $l \in I$. Of course, p_l is a continuous linear mapping from V onto V_l for each $l \in I$. If E is a bounded subset of V, then it follows that $p_l(E)$ is bounded in V_l for each $l \in I$. Conversely, if $E \subseteq V$ has the property that $p_l(E)$ is bounded in V_l for each $l \in I$, then one can check that E is bounded in V. In particular, if E_j is a bounded subset of V_j for each $j \in I$, then $\prod_{j \in I} E_j$ is bounded in V.

1.19 Totally bounded sets

Let A be a commutative topological group. A subset E of A is said to be *totally bounded* in A as a commutative topological group if for every open set $U \subseteq A$ with $0 \in U$ there are finitely many elements $a_1, \ldots, a_n$ of A such that

$$E \subseteq \bigcup_{j=1}^{n} (a_j + U). \tag{1.152}$$

Suppose for the moment that $d(x, y)$ is a translation-invariant metric on A that determines the same topology on A. In this case, it is easy to see that E is totally bounded as a subset of A as a commutative topological group if and only if E is totally bounded in A as a metric space with respect to $d(x, y)$.

Let A be any commutative topological group again. If $E \subseteq A$ has only finitely many elements, then E is obviously totally bounded in A. Note that the converse holds when A is equipped with the discrete topology. It is easy to see that the union of finitely many totally bounded subsets of any commutative topological group A is totally bounded in A too. If $E \subseteq A$ is totally bounded in A, then every subset of E is totally bounded in A as well.

If $E \subseteq A$ is compact in A, then E is totally bounded in A. More precisely, if $U \subseteq A$ is an open set that contains 0, then E is covered by the translates of U by elements of E. If E is compact, then it follows that E can be covered by finitely many translates of U, as desired.

If $E \subseteq A$ is totally bounded in A, then one can check that the closure $\overline{E}$ of E in A is totally bounded in A, using the regularity of A as a topological space, as in Section 1.1. If $E_1, E_2 \subseteq A$ are totally bounded in A, then one can verify that $E_1 + E_2$ is totally bounded in A, using continuity of addition on A.

Let V be a topological vector space over the real or complex numbers, so that V is a commutative topological group with respect to addition in particular. If $E \subseteq V$ is totally bounded in V as a commutative topological group, then E is bounded in V as a topological vector space, as in Section 1.17. To see this, let $U_0 \subseteq V$ be an open set that contains 0, and let $U \subseteq V$ be a balanced open set that contains 0 and satisfies

$$U + U \subseteq U_0. \tag{1.153}$$

The existence of U uses the continuity of addition on V, and the fact that balanced open subsets of V that contain 0 form a local base for the topology of

V at 0, as in Section 1.6. If E is totally bounded in V, then there are finitely many elements $a_1, \ldots, a_n$ of V such that (1.152) holds. If $t \in \mathbf{R}$ or $\mathbf{C}$ satisfies $|t| \geq 1$, then $U \subseteq t\,U$, because U is balanced in V. It follows that

$$E \subseteq \bigcup_{j=1}^{n} (a_j + t\,U) \tag{1.154}$$

when $|t| \geq 1$. We also have that $a_j \in t\,U$ when $|t|$ is sufficiently large, as in Section 1.17. Combining this with (1.154), we get that

$$E \subseteq t\,U + t\,U = t\,(U + U) \subseteq t\,U_0 \tag{1.155}$$

when $|t|$ is sufficiently large, as desired.

Let A be a commutative topological group again, and let $U \subseteq A$ be an open set that contains 0. Let us say that a set $E_1 \subseteq U$ is *U-small* in A if

$$E_1 - E_1 \subseteq U, \tag{1.156}$$

so that $x - y \in U$ for every $x, y \in E_1$. This is also the same as saying that

$$E_1 \subseteq y + U \tag{1.157}$$

for every $y \in E_1$. In the other direction, if (1.157) holds for any $y \in A$, then we have that

$$E_1 - E_1 \subseteq (y + U) - (y + U) = U - U. \tag{1.158}$$

Note that subsets of U-small subsets of A are U-small as well.

If $E \subseteq A$ can be covered by finitely many U-small sets, then E can be covered by finitely many translates of U, by (1.157). If E has this property for every open set $U \subseteq A$ with $0 \in U$, then it follows that E is totally bounded in A. Conversely, if $E \subseteq A$ is totally bounded, and $U_0 \subseteq A$ is an open set that contains 0, then E can be covered by finitely many U_0-small sets in A. To see this, let $U \subseteq A$ be an open set such that $0 \in U$ and

$$U - U \subseteq U_0, \tag{1.159}$$

which exists by the continuity of the group operations on A. If E is totally bounded in A, then E can be covered by finitely many translates of U in A, each of which is U_0-small in A.

Suppose that $E \subseteq A$ is totally bounded, and that $U \subseteq A$ is an open set that contains 0. Thus E can be covered by finitely many U-small subsets of A, as in the preceding paragraph. More precisely, E can be expressed as the union of finitely many U-small sets, by taking the intersection of the previous U-small subsets of A with E. It follows from this that E can be covered by finitely many translates of U by elements of E, using (1.157). Thus we may take $a_1, \ldots, a_n$ to be elements of E in (1.152).

Let A, B be commutative topological groups, and let h be a continuous homomorphism from A into B. If $E \subseteq A$ is totally bounded, then it is easy to see that $h(E)$ is totally bounded in B. Suppose now that A is a subgroup of B, and that A is equipped with the topology that is induced by the one on B. In this case, $E \subseteq A$ is totally bounded in A if and only if E is totally bounded in B. The "only if" part follows from the previous statement, and the "if" part can be derived from the reformulations of total boundedness in the preceding paragraphs.

Let $U_1, \ldots, U_n$ be finitely many open subsets of A, with $0 \in U_j$ for each $j = 1, \ldots, n$, and put $U = \bigcap_{j=1}^n U_j$. Note that $E_0 \subseteq A$ is U-small in A if and only if E_0 is U_j-small in A for each $j = 1, \ldots, n$. If $E_j \subseteq A$ is U_j-small for each $j = 1, \ldots, n$, then $\bigcap_{j=1}^n E_j$ is U-small in A. If $E \subseteq A$ can be covered by finitely many U_j-small subsets of A for each $j = 1, \ldots, n$, then one can check that E can be covered by finitely many U-small subsets of A. More precisely, E can be covered by the intersections of the various U_j-small sets used to cover E for each $j = 1, \ldots, n$.

Let I be a nonempty set, and let A_j be a commutative topological group for each $j \in I$. Thus the Cartesian product $A = \prod_{j \in I} A_j$ is a commutative topological group with respect to the corresponding product topology, as in Section 1.8. Let p_l be the standard coordinate projection from A onto A_l for each $l \in I$, which is a continuous group homomorphism. If $E \subseteq A$ is totally bounded, then it follows that $p_l(E)$ is totally bounded in A_l for each $l \in I$. Conversely, if $E \subseteq A$ has the property that $p_l(E)$ is totally bounded in A_l for each $l \in I$, then one can verify that E is totally bounded in A. This uses the remarks in the previous paragraph, and the definition of the product topology on A. In particular, if $E_j \subseteq A_j$ is totally bounded in A_j for each $j \in I$, then $\prod_{j \in I} E_j$ is totally bounded in A.

Chapter 2

Borel measures

2.1 Summable functions

Let X be a nonempty set, and let f be a nonnegative real-valued function on X. The sum

$$\sum_{x \in X} f(x) \tag{2.1}$$

is defined as a nonnegative extended real number to be the supremum of the sums

$$\sum_{x \in A} f(x) \tag{2.2}$$

over all nonempty finite subsets A of X. It is sometimes convenient to allow f to be $+\infty$ at some points in X, in which case the sum (2.1) is automatically equal to $+\infty$. Observe that

$$\sum_{x \in X} t\, f(x) = t \sum_{x \in X} f(x) \tag{2.3}$$

for every positive real number t, which also works for $t = 0$ with the convention that $0 \cdot (+\infty) = 0$. If g is another nonnegative extended real-valued function on X, then one can verify that

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x). \tag{2.4}$$

If f is a real or complex-valued function on X, then we put

$$\|f\|_1 = \sum_{x \in X} |f(x)|, \tag{2.5}$$

and we say that f is *summable* on X when this is finite. As in (2.3), we have that

$$\|t\, f\|_1 = |t|\, \|f\|_1 \tag{2.6}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, where the right side is interpreted as being 0 when $t = 0$, even if f is not summable on X. Similarly, if g is another real or complex-valued function on X, then

$$\|f + g\|_1 \le \|f\|_1 + \|g\|_1, \tag{2.7}$$

using (2.4) and the triangle inequality for the absolute value function on $\mathbf{R}$ or $\mathbf{C}$. The spaces of real and complex-valued summable functions on X are denoted $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, respectively. These are vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, with respect to pointwise addition and scalar multiplication of functions, and (2.5) defines a norm on each of these spaces.

The *support* of a real or complex-valued function f on X is the set $\operatorname{supp} f$ of $x \in X$ such that $f(x) \ne 0$. Let $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ be the spaces of real and complex-valued functions on X with finite support, respectively. One can check that these are dense linear subspaces of $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, respectively, with respect to the ℓ^1 norm (2.5).

A real or complex-valued function f on X is said to *vanish at infinity* on X if for each $\epsilon > 0$ we have that

$$|f(x)| < \epsilon \tag{2.8}$$

for all but finitely many $x \in X$. Let $c_0(X, \mathbf{R})$ and $c_0(X, \mathbf{C})$ be the spaces of real and complex-valued functions on X that vanish at infinity, respectively. These are vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, with respect to pointwise addition and scalar multiplication of functions. If f is summable on X, then it is easy to see that f vanishes at infinity on X. If f vanishes at infinity on X, then the support of f has only finitely or countably many elements, as one can see by applying the definition with $\epsilon = 1/j$ for each positive integer j.

If f is a real or complex-valued summable function on X, then (2.1) can be defined as a real or complex number, as appropriate, by expressing f as a linear combination of summable nonnegative real-valued functions on X. One can check that the value of the sum does not depend on the particular expression of f of this type, using (2.3) and (2.4). Of course, if f has finite support in X, then the sum can be defined directly. Otherwise, if the support of f is countably infinite, then the sum can be treated as an absolutely convergent infinite series. One can also consider the sum as a suitable limit of finite subsums.

Another basic property of the sum is that

$$\Big|\sum_{x \in X} f(x)\Big| \le \sum_{x \in X} |f(x)| = \|f\|_1. \tag{2.9}$$

If f is real-valued, then (2.9) can be obtained by looking at the positive and negative parts of f. If one applies this argument directly to the real and imaginary parts of a complex-valued function on X, then one would get an additional constant factor on the right side of (2.9). If f has finite support in X, then (2.9) follows from the triangle inequality for the absolute value function on $\mathbf{R}$ and $\mathbf{C}$. One can get (2.9) for complex-valued summable functions on X without an additional constant factor by approximating the sum (2.1) by finite sums.

The mapping from a real or complex-valued summable function f on X to the sum (2.1) is linear in f, and so defines a linear functional on each of $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$. These linear functionals are continuous with respect to the topologies on these spaces determined by their ℓ^1 norms, because of (2.9). It follows that these linear functionals are uniquely determined by their restrictions to $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$, respectively, since these spaces are dense in the corresponding ℓ^1 spaces.

Let I be a nonempty set, and let E_j be a nonempty subset of X for each $j \in I$. If f is a nonnegative extended real-valued function on X, then

$$\sum_{x \in E_j} f(x) \tag{2.10}$$

can be defined as a nonnegative extended real number for each $j \in I$, and hence

$$\sum_{j \in I} \Big(\sum_{x \in E_j} f(x) \Big) \tag{2.11}$$

can be defined as a nonnegative extended real number too. If the E_j's are pairwise-disjoint in X, and

$$E = \bigcup_{j \in I} E_j, \tag{2.12}$$

then one can check that (2.11) is equal to

$$\sum_{x \in E} f(x). \tag{2.13}$$

In particular, if f is summable on E, then all of these sums are finite.

Now let f be a real or complex-valued summable function on X, or simply on E. Note that the restriction of f to any nonempty subset of E is summable as well. Thus (2.10) is defined as a real or complex number for each $j \in I$, and satisfies

$$\Big| \sum_{x \in E_j} f(x) \Big| \le \sum_{x \in E_j} |f(x)|, \tag{2.14}$$

as in (2.9). This implies that

$$\sum_{j \in I} \Big| \sum_{x \in E_j} f(x) \Big| \le \sum_{j \in I} \Big(\sum_{x \in E_j} |f(x)| \Big) = \sum_{x \in E} |f(x)|, \tag{2.15}$$

using the equality between (2.11) and (2.13) for $|f(x)|$ in the second step. The right side of (2.15) is finite, by hypothesis, so that (2.10) defines a summable function of j on I. This means that (2.11) is also defined as a real or complex number under these conditions. One can check that (2.11) is equal to (2.13) in this situation, by expressing f as a linear combination of nonnegative real-valued summable functions on X, to reduce to the previous case.

2.2 r-Summable functions

Let X be a nonempty set again, and let f be a real or complex-valued function on X. Put

$$\|f\|_r = \Big(\sum_{x \in X} |f(x)|^r\Big)^{1/r} \tag{2.16}$$

for every positive real number r, which is interpreted as being $+\infty$ when the sum is infinite. This is the same as (2.5) in the previous section when $r = 1$, and we say that f is *r-summable* when (2.16) is finite. Also put

$$\|f\|_\infty = \sup_{x \in X} |f(x)|, \tag{2.17}$$

which is finite exactly when f is bounded on X. If $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, then

$$\|t\, f\|_r = |t|\, \|f\|_r \tag{2.18}$$

for every $r > 0$, where the right side is interpreted as being equal to 0 when $t = 0$, even if $\|f\|_r$ is infinite. More precisely, this reduces to (2.3) when r is finite, and it can be verified directly when $r = \infty$. If g is another real or complex-valued function on X and $1 \le r \le \infty$, then

$$\|f + g\|_r \le \|f\|_r + \|g\|_r \tag{2.19}$$

by Minkowski's inequality for sums.

It is easy to see that

$$\|f\|_\infty \le \|f\|_r \tag{2.20}$$

for every $r > 0$. If $0 < r_1 < r_2 < \infty$, then we have that

$$\sum_{x \in X} |f(x)|^{r_2} \le \|f\|_\infty^{r_2 - r_1} \sum_{x \in X} |f(x)|^{r_1} \le \|f\|_{r_1}^{r_2 - r_1}\, \|f\|_{r_1}^{r_1} = \|f\|_{r_1}^{r_2}, \tag{2.21}$$

which implies that

$$\|f\|_{r_2} \le \|f\|_{r_1}. \tag{2.22}$$

In particular, if $a, b \ge 0$ and $0 < r < 1$, then

$$(a + b)^r \le a^r + b^r. \tag{2.23}$$

This follows from (2.22) by taking $r_1 = r$, $r_2 = 1$, and f to be a function with values a, b on a set X with two elements. Using (2.23), we get that

$$\|f + g\|_r^r = \sum_{x \in X} (f(x) + g(x))^r \le \sum_{x \in X} f(x)^r + \sum_{x \in X} g(x)^r = \|f\|_r^r + \|g\|_r^r \tag{2.24}$$

for all real or complex-valued functions f and g on X when $0 < r \le 1$.

Let $\ell^r(X, \mathbf{R})$ and $\ell^r(X, \mathbf{C})$ be the spaces of real and complex-valued functions f on X such that $\|f\|_r$ is finite, for each $r > 0$. These are vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, with respect to pointwise addition and scalar multiplication of functions, because of (2.18), (2.19), and (2.24). If $r \geq 1$, then $\|f\|_r$ defines a norm on each of these spaces, by (2.18) and (2.19). If $0 < r < 1$, then

$$\|f - g\|_r^r \tag{2.25}$$

defines a translation-invariant metric on each of $\ell^r(X, \mathbf{R})$ and $\ell^r(X, \mathbf{C})$, by (2.24). In this case, one can check that $\ell^r(X, \mathbf{R})$ and $\ell^r(X, \mathbf{C})$ are topological vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, with respect to the topologies determined by these metrics.

If $0 < r_1 < r_2 \leq \infty$, then $\ell^{r_1}(X, \mathbf{R})$ and $\ell^{r_1}(X, \mathbf{C})$ are contained in $\ell^{r_2}(X, \mathbf{R})$ and $\ell^{r_2}(X, \mathbf{C})$, respectively, by (2.20) and (2.22). These inclusions are continuous mappings with respect to the corresponding topologies on these spaces.

If a real or complex-valued function f on X vanishes at infinity, then it is easy to see that f is bounded on X. One can verify that $c_0(X, \mathbf{R})$ and $c_0(X, \mathbf{C})$ are closed linear subspaces of $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively, with respect to the topologies determined by the corresponding ℓ^∞ norms. More precisely, $c_0(X, \mathbf{R})$ and $c_0(X, \mathbf{R})$ are the same as the closures of $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ in $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively, with respect to the corresponding ℓ^∞ norms.

If $r < \infty$ and f is r-summable on X, then f vanishes at infinity on X. This is basically the same as the $r = 1$ case, mentioned in the previous section. We also have that $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ are dense in $\ell^r(X, \mathbf{R})$ and $\ell^r(X, \mathbf{C})$, respectively, when $r < \infty$, as in the $r = 1$ case.

It is well known that $\ell^r(X, \mathbf{R})$ and $\ell^r(X, \mathbf{C})$ are complete as metric spaces for every $r > 0$, by standard arguments. This uses the metric associated to the ℓ^r norm when $r \geq 1$, and (2.25) when $0 < r < 1$.

2.3 Some regularity conditions

By definition, the σ-algebra of in a topological space X is the smallest σ-algebra that contains all open subsets of X. Equivalently, this is the smallest σ-algebra that contains all closed subsets of X. A *Borel measure* on X is a countably-additive measure defined on the Borel sets in X. We shall restrict our attention to nonnegative Borel measures in this section, but we shall also consider real and complex-valued Borel measures later. Nonnegative Borel measures are allowed to take values in the nonnegative extended real numbers, as usual.

Remember that a subset E of X is said to be an F_σ *set* if E can be expressed as the union of countably many closed subsets of X. Similarly, E is said to be a G_δ *set* if E can be expressed as the intersection of countably many open subsets of X. Of course, F_σ sets and G_δ sets are Borel sets.

Let μ be a nonnegative Borel measure on a topological space X, and let $E \subseteq X$ be a Borel set. A basic *outer regularity* condition asks that

$$\mu(E) = \inf\{\mu(U) : U \subseteq X \text{ is an open set, and } E \subseteq U\}. \tag{2.26}$$

This holds trivially when $\mu(E) = \infty$. Otherwise, if $\mu(E) < \infty$, then (2.26) is equivalent to asking that

$$\text{for each } \epsilon > 0 \text{ there is an open set } U \subseteq X \\ \text{such that } E \subseteq U \text{ and } \mu(U \setminus E) < \epsilon. \tag{2.27}$$

Of course, (2.27) holds trivially when E is an open set in X. If E is a G_δ set, and $\mu(X) < \infty$, then it is easy to see that E satisfies (2.27). More precisely, it suffices to ask that there be an open set $V \subseteq X$ such that $E \subseteq V$ and $\mu(V) < \infty$, instead of $\mu(X) < \infty$. If $E_1, E_2, E_3, \ldots$ is a sequence of Borel subsets of X that satisfy (2.27), then one can check that $\bigcup_{j=1}^\infty E_j$ satisfies (2.27) too.

Similarly, a basic *inner regularity* condition asks that

$$\mu(E) = \sup\{\mu(A) : A \subseteq X \text{ is a closed set, and } A \subseteq E\}. \tag{2.28}$$

If $\mu(E) = \infty$, then (2.28) implies that every Borel set in X that contains E also satisfies (2.28). If $\mu(E) < \infty$, then (2.28) is equivalent to asking that

$$\text{for each } \epsilon > 0 \text{ there is a closed set } A \subseteq X \\ \text{such that } A \subseteq E \text{ and } \mu(E \setminus A) < \epsilon. \tag{2.29}$$

As before, (2.29) holds trivially when E is a closed set in X, and it is easy to see that (2.28) holds when E is an F_σ set. The union of finitely many Borel sets satisfying (2.29) also satisfies (2.29). If $E_1, E_2, E_3, \ldots$ is a sequence of Borel sets in X that satisfy (2.29), and if $\mu\Big(\bigcup_{j=1}^\infty E_j\Big) < \infty$, then $\bigcup_{j=1}^\infty E_j$ satisfies (2.29) as well. This uses the previous remark to reduce to the case where $E_j \subseteq E_{j+1}$ for each j. Otherwise, if $\mu\Big(\bigcup_{j=1}^\infty E_j\Big) = \infty$, then an analogous argument implies that $\bigcup_{j=1}^\infty E_j$ satisfies (2.28).

If $B \subseteq X$ is a Borel set, then it is easy to see that (2.27) holds with $E = B$ if and only if (2.29) holds with $E = X \setminus B$. In this situation, the closed set A in (2.29) corresponds exactly to the complement of the open set U in (2.28). Thus the condition $B \subseteq U$ in (2.27) corresponds exactly to $A \subseteq X \setminus B$ in (2.29). Similarly, $U \setminus B$ in (2.27) corresponds exactly to $(X \setminus B) \setminus A$ in (2.29). Hence the conditions that these two sets have μ-measure less than ϵ are the same.

Let us suppose for the rest of the section that

$$\text{every open subset of } X \text{ is an } F_\sigma \text{ set.} \tag{2.30}$$

This is equivalent to asking that

$$\text{every closed subset of } X \text{ is a } G_\delta \text{ set.} \tag{2.31}$$

It is well known that these conditions hold when the topology on X is determined by a metric.

Let μ be a nonnegative Borel measure on X again. If $E \subseteq X$ is an open set, then (2.30) implies (2.28), as before. It follows that (2.29) holds when E

is an open set and $\mu(E) < \infty$. If instead $E \subseteq X$ is a closed set, and if we also have that $\mu(X) < \infty$, then (2.31) implies (2.27). This works as well when E is contained in any open set $V \subseteq X$ with $\mu(V) < \infty$, as mentioned earlier.

Suppose for the moment that $\mu(X) < \infty$, and consider the collection $\mathcal{E}$ of Borel sets $E \subseteq X$ that satisfy (2.27) and (2.29). If E is either open or closed in X, then $E \in \mathcal{E}$, by some of the earlier remarks, including those in the previous paragraph. We also have seen that $E \in \mathcal{E}$ implies that $X \setminus E \in \mathcal{E}$. The earlier remarks about countable unions imply that $\mathcal{E}$ is closed under countable unions, and hence is a σ-algebra. It follows that $\mathcal{E}$ contains all Borel sets $E \subseteq X$ under these conditions.

If μ is any nonnegative Borel measure on X and $B \subseteq X$ is a Borel set, then

$$\mu_B(E) = \mu(B \cap E) \tag{2.32}$$

defines a nonnegative Borel measure on X too. Of course, if $\mu(B) < \infty$, then $\mu_B(X) < \infty$, so that the remarks in the previous paragraph can be applied to μ_B. More precisely, this means that for every Borel set $E \subseteq X$, we have that

$$\begin{aligned} &\text{for each } \epsilon > 0 \text{ there is an open set } U \subseteq X \\ &\text{such that } E \subseteq U \text{ and } \mu_B(U \setminus E) < \epsilon, \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} &\text{for each } \epsilon > 0 \text{ there is a closed set } A \subseteq X \\ &\text{such that } A \subseteq E \text{ and } \mu_B(E \setminus A) < \epsilon. \end{aligned} \tag{2.34}$$

Note that (2.34) is equivalent to (2.29) when $E \subseteq B$, and in particular when $E = B$. It follows that if μ is any nonnegative Borel measure on X, and $E \subseteq X$ is a Borel set with $\mu(E) < \infty$, then E satisfies (2.29).

Let μ be any nonnegative Borel measure on X again, and let $B \subseteq X$ be an open set. If $\mu(B) < \infty$, then (2.32) satisfies (2.33), as in the preceding paragraph. If $E \subseteq X$ is a Borel set with $E \subseteq B$, and $U \subseteq X$ is an open set such that $E \subseteq U$, then $U \cap B$ is also an open set in X that contains E. Thus (2.33) implies (2.27) when $E \subseteq B$. Now let $B_1, B_2, B_3, \ldots$ be a sequence of open subsets of X such that $\mu(B_j) < \infty$ for each j. If $E \subseteq X$ is a Borel set, then $E_j = E \cap B_j$ satisfies (2.27) for each j, by the previous remark. If we also have that $E \subseteq \bigcup_{j=1}^\infty B_j$, then it follows that $E = \bigcup_{j=1}^\infty E_j$ satisfies (2.27).

2.4 Locally compact Hausdorff spaces

A topological space X is said to be *locally compact* if for each point $x \in X$ there are an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U$ and $U \subseteq K$. If X is Hausdorff, then K is closed in X, so that the closure $\overline{U}$ of U in X is contained in K as well. This implies that $\overline{U}$ is compact, which may be used in the definition of local compactness in this case.

The *support* $\operatorname{supp} f$ of a real or complex-valued function f on a topological space X is defined to be the closure of the set of $x \in X$ such that $f(x) \neq 0$.

Let $C_{com}(X, \mathbf{R})$ and $C_{com}(X, \mathbf{C})$ be the spaces of continuous real and complex-valued functions on X with compact support, respectively. These are subalgebras of the spaces $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ of all continuous real and complex-valued functions on X, respectively, with respect to pointwise addition and scalar multiplication of functions. If X is equipped with the discrete topology, then this definition of the support of a real or complex-valued function on X reduces to the one used in Section 2.1, and $C_{com}(X, \mathbf{R})$, $C_{com}(X, \mathbf{C})$ are the same as the spaces $c_{00}(X, \mathbf{R})$, $c_{00}(X, \mathbf{C})$ defined there. If X is compact, then every real or complex-valued function on X has compact support, so that $C_{com}(X, \mathbf{R})$ and $C_{com}(X, \mathbf{C})$ are the same as $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$, respectively.

Let us suppose from now on in this section that X is a locally compact Hausdorff topological space. If $K \subseteq X$ is compact, $U \subseteq X$ is an open set, and $K \subseteq U$, then a well-known version of Urysohn's lemma implies that there is a real-valued continuous function f on X with compact support such that $f(x) = 1$ for every $x \in K$, $\operatorname{supp} f \subseteq U$, and $0 \le f(x) \le 1$ for every $x \in X$. In particular, this implies that $C_{com}(X, \mathbf{R})$ separates points in X. Note that X must be Hausdorff in order for continuous real or complex-valued functions on X to separate points in X, and that X has to be locally compact for such functions with compact support to separate points.

If f is any continuous real-valued function on X and $r \in \mathbf{R}$, then

$$\max(f(x), r) \tag{2.35}$$

and

$$\min(f(x), r) \tag{2.36}$$

are continuous on X too. The support of (2.35) is contained in the support of f when $r \le 0$, and the support of (2.36) is contained in the support of f when $r \ge 0$. If f has compact support in X, then it follows that (2.35) has compact support in X when $r \ge 0$, and that (2.36) has compact support in X when $r \ge 0$.

A linear functional λ on $C_{com}(X, \mathbf{R})$ as a vector space over $\mathbf{R}$ is said to be *nonnegative* if

$$\lambda(f) \ge 0 \tag{2.37}$$

for every $f \in C_{com}(X, \mathbf{R})$ such that $f(x) \ge 0$ for every $x \in X$. Similarly, a linear functional λ on $C_{com}(X, \mathbf{C})$ as a vector space over $\mathbf{C}$ is said to be *nonnegative* if for every nonnegative real-valued continuous function f on X with compact support, we have that

$$\lambda(f) \in \mathbf{R} \tag{2.38}$$

and that (2.37) holds. This implies that (2.38) holds for every $f \in C_{com}(X, \mathbf{R})$, because any continuous real-valued function on X with compact support can be expressed as a difference of continuous nonnegative real-valued functions on X with compact support. It follows that the restriction of λ to $C_{com}(X, \mathbf{R})$ is a nonnegative linear functional on $C_{com}(X, \mathbf{R})$. Any linear functional on $C_{com}(X, \mathbf{R})$ as a real vector space has a unique extension to a linear functional on $C_{com}(X, \mathbf{C})$ as a complex vector space, and this extension is automatically nonnegative when λ is nonnegative on $C_{com}(X, \mathbf{R})$.

Let μ be a nonnegative Borel measure on X such that

$$\mu(K) < \infty \tag{2.39}$$

for every compact set $K \subseteq X$. This implies that continuous real and complex-valued functions on X with compact support are integrable with respect to μ, because continuous real and complex-valued functions are bounded on compact sets. It follows that

$$\lambda(f) = \int_X f \, d\mu \tag{2.40}$$

defines a nonnegative linear functional on each of $C_{com}(X, \mathbf{R})$ and $C_{com}(X, \mathbf{C})$.

Now let λ be a nonnegative linear functional on $C_{com}(X, \mathbf{R})$ or $C_{com}(X, \mathbf{C})$. A version of the Riesz representation theorem implies that there is a unique nonnegative Borel measure μ on X that satisfies (2.39), (2.40), and some additional regularity properties. More precisely, μ should be outer regular, in the sense that (2.26) holds for every Borel set $E \subseteq X$. There are also some *inner regularity* conditions, of the form

$$\mu(E) = \sup\{\mu(K) : K \subseteq X \text{ is compact, and } K \subseteq E\}. \tag{2.41}$$

Namely, this should hold when E is an open set in X, and when E is a Borel set in X with $\mu(E) < \infty$. Note that (2.41) implies (2.28) in the previous section, because compact subsets of X are closed sets, since X is Hausdorff. If X is compact, then closed subsets of X are compact too, so that (2.28) implies (2.41).

2.5 Haar measure

Let A be a commutative topological group, in which the group operations are expressed additively, and suppose that A is locally compact as a topological space. In this situation, it suffices to ask that there be an open set in A that contains 0 and is contained in a compact set, because of translation-invariance. One may as well ask that the closure of this open set be compact, because A is Hausdorff, as in Section 1.1. If $E \subseteq A$ is a Borel set, then $E + a$ is a Borel set for every $a \in A$, because of continuity of translations. Similarly, $-E$ is a Borel set in A, because $x \mapsto -x$ is continuous on A.

Under these conditions, it is well known that there is a Borel measure H on A, known as *Haar measure*, that is invariant under translations and has some additional properties. *Invariance under translations* means that

$$H(E + a) = H(E) \tag{2.42}$$

for every Borel set $E \subseteq A$. This measure is supposed to be finite on compact subsets of A, and positive on nonempty open subsets of A. This measure should also be outer regular, in the sense that (2.26) in Section 2.3 holds with $\mu = H$ for all Borel sets $E \subseteq A$. Similarly, the inner regularity condition (2.41) in the previous section should hold with $\mu = H$, when $E \subseteq A$ is an open set, and when E is a Borel set with $H(E) < \infty$, as before. Of course, one can multiply H by a positive real number, and get another Borel measure on A with the same properties. It is well known that this is the only way to get another Borel measure on A with the same properties.

If A is any commutative group equipped with the discrete topology, then counting measure on A satisfies the requirements of Haar measure. If $A = \mathbf{R}$ as a commutative group with respect to addition, and with the standard topology, then one-dimensional Lebesgue measure satisfies the requirements of Haar measure. If $A = \mathbf{T}$, as a commutative group with respect to multiplication, and with the topology induced by the standard topology on $\mathbf{C}$, then the usual arc-length measure on $\mathbf{T}$ satisfies the requirements of Haar measure. This corresponds to one-dimensional Lebesgue measure on the interval $[0, 2\pi)$ in $\mathbf{R}$, using an arc-length parameterization of $\mathbf{T}$.

Let A be any locally compact commutative topological group again. If H is a Haar measure on A, then

$$H(-E) = H(E) \tag{2.43}$$

for every Borel set $E \subseteq A$. To see this, observe first that $H(-E)$ is a Borel measure on A that satisfies the same requirements as Haar measure. This implies that $H(-E)$ is equal to a positive constant multiple of $H(E)$, by the uniqueness of Haar measure mentioned earlier. In order to show that this constant multiple is equal to 1, it suffices to check that (2.43) holds for some Borel set $E \subseteq A$ such that $H(E)$ is positive and finite. Let U be an open set in A that contains 0 and has compact closure, and put

$$E = U \cap (-U). \tag{2.44}$$

Thus $E = -E$, by construction, so that (2.43) holds automatically. We also have that $H(E)$ is posiitive and finite, as desired, because E is nonempty, open, and contained in a compact set.

If f is a real or complex-valued Borel-measurable function on A, then

$$f_a(x) = f(x - a) \tag{2.45}$$

is Borel measurable on A for every $a \in A$. If f is a nonnegative real-valued Borel-measurable function on A, then f_a is nonnegative for every $a \in A$, and one can check that

$$\int_A f_a \, dH = \int_A f \, dH \tag{2.46}$$

for every $a \in A$, using invariance of H under translations. Similarly, if f is a real or complex-valued Borel-measurable function on A that is integrable with respect to H, then f_a is integrable with respect to H for every $a \in A$, and (2.46) holds for every $a \in A$. If f is a continuous real or complex-valued function on

A with compact support, then f_a is a continuous function on A with compact support for every $a \in A$. In this case, f is integrable with respect to H, because continuous functions are bounded on compact sets, and compact subsets of A have finite measure with respect to H.

A nonnegative linear functional λ on $C_{com}(A, \mathbf{R})$ or $C_{com}(A, \mathbf{C})$ is said to be *invariant under translations* if

$$\lambda(f_a) = \lambda(f) \tag{2.47}$$

for every $f \in C_{com}(A, \mathbf{R})$ or $C_{com}(A, \mathbf{C})$, as appropriate, and $a \in A$. As in the previous section, every nonnegative linear functional λ on $C_{com}(A, \mathbf{R})$ or $C_{com}(A, \mathbf{C})$ can be represented by a unique nonnegative Borel measure μ on A as in (2.40), where μ also satisfies (2.39) and some additional regularity properties. If λ is invariant under translations, then it follows that μ is invariant under translations on A as well, as in (2.42), because μ is uniquely determined by λ. Let us say that a nonnegative linear functional λ on $C_{com}(A, \mathbf{R})$ or $C_{com}(A, \mathbf{C})$ is *strictly positive* if

$$\lambda(f) > 0 \tag{2.48}$$

for every nonnegative real-valued function f on X with compact support such that $f(x) > 0$ for some $x \in A$. This implies that the corresponding nonnegative Borel measure μ on A is strictly positive in the sense that

$$\mu(U) > 0 \tag{2.49}$$

for every nonempty open set $U \subseteq A$.

A nonnegative linear functional λ on $C_{com}(A, \mathbf{R})$ or $C_{com}(A, \mathbf{C})$ that is strictly positive and invariant under translations is called a *Haar integral* on A. If H is a Haar measure on A, then

$$\lambda(f) = \int_A f \, dH \tag{2.50}$$

defines a Haar integral on A. Conversely, if λ is a Haar integral on A, then one can get a Haar measure H on A as in (2.50) using the Riesz representation theorem, as in the preceding paragraph. Similarly, uniqueness of Haar measure on A up to positive constant multiples corresponds to the uniqueness of the Haar integral up to positive constant multiples. If $A = \mathbf{R}$ or $\mathbf{T}$, for instance, then one can use ordinary Riemann integrals as Haar integrals.

2.6 σ-Compactness

A subset E of a topological space X is said to be *σ-compact* if E can be expressed as the union of countably many compact sets. If $E \subseteq X$ is σ-compact and $A \subseteq X$ is a closed set, then $E \cap A$ is σ-compact too, because $K \cap A$ is compact when K is compact. Similarly, if $E \subseteq X$ is σ-compact and $A \subseteq X$ is an F_σ set, then $E \cap A$ is σ-compact. In particular, if X is σ-compact, then F_σ sets in X are

σ-compact. If X is Hausdorff, then compact subsets of X are closed sets, and hence σ-compact subsets of X are F_σ sets.

Remember that $E \subseteq X$ is said to have the *Lindelöf propery* if every open covering of E in X can be reduced to a subcovering with only finitely or countably many elements. If E is σ-compact, then E has the Lindelöf property. If X is locally compact and X has the Lindelöf property, then X is σ-compact. If there is a base for the topoogy of X with only finitely or countably many elements, then X has the Lindelöf property, by Lindelöf's theorem. If X is a metric space with the Lindelöf property, then X can be covered by finitely or countably many open balls of any radius $r > 0$, which implies that X is separable.

Let us suppose from now on in this section that X is a Hausdorff topological space. Thus σ-compact subsets of X are F_σ sets, as before, and in particular they are Borel sets. Let μ be a nonnegative Borel measure on X. If $E \subseteq X$ is σ-compact, then E satisfies the inner regularity condition (2.41) in Section 2.4 with respect to compact subsets of X. Suppose for the moment that X is σ-compact, so that F_σ sets in X are σ-compact too. This implies that F_σ sets in X satisfy (2.41), as before. In this case, the inner regularity condition (2.28) in Section 2.3 with respect to closed subsets of X implies (2.41), because (2.41) holds for closed subsets of X.

If X is locally compact, then every compact subset of X is contained in an open set that is contained in another compact set. If X if locally compact and σ-compact, then it follows that X can be expressed as the union of countably many open sets, each of which is contained in a compact set. If μ is a nonnegative Borel measure on X that is finite on compact sets, then the previous statement implies that X can be expressed as the union of countably many open sets, each of which has finite measure with respect to μ.

Let μ be a nonnegative Borel measure on X again, and suppose that there is a σ-compact set $X_0 \subseteq X$ such that

$$\mu(X \setminus X_0) = 0. \tag{2.51}$$

If $E \subseteq X$ is a Borel set such that $E \cap X_0$ is σ-compact, then $E \cap X_0$ satisfies (2.41) in Section 2.4, as before, which implies that E satisfies (2.41) in this case. This implies that F_σ sets in X satisfy (2.41) in this situation. It follows that (2.28) in Section 2.3 implies (2.41), because (2.41) holds for closed sets, as before. If for each $\epsilon > 0$ there is a compact set $K \subseteq X$ such that

$$\mu(X \setminus K) < \epsilon, \tag{2.52}$$

then one can apply this condition to $\epsilon = 1/j$ for each positive integer j, to get that there is a σ-compact set $X_0 \subseteq X$ that satisfies (2.51).

2.7 Some nice subgroups

Let A be a commutative topological group. Note that the closure of a subgroup of A is a subgroup of A too. Let A_0 be a subgroup of A, and suppose that U is an open set in A such that $0 \in U$ and $U \subseteq A_0$. This implies that

$$A_0 = A_0 + U, \tag{2.53}$$

since each side of the equation is contained in the other, and hence that A_0 is an open subset of A. Using (2.53), we also get that A_0 is a closed set in A, because the closure of A_0 is contained in $A_0 + U$, as in (1.7) in Section 1.1. Alternatively, if A_0 is an open subgroup of A, then all of the cosets of A_0 in A are open sets as well, since they are translates of A_0. This implies that A_0 is a closed set in A, because the complement of A_0 in A can be expressed as a union of cosets of A_0, and hence is an open set.

Let E be a subset of A such that $0 \in E$ and E is symmetric about 0, in the sense that $-E = E$. Of course, if E is not already symmetric about 0, then one can consider $E \cap (-E)$ or $E \cup (-E)$ instead. Define $E_j \subseteq A$ recursively for each positive integer j by putting $E_1 = E$ and

$$E_{j+1} = E_j + E \tag{2.54}$$

for each j. Equivalently, E_j consists of the elements of A that can be expressed as sums of j elements of E. It is easy to see that

$$\bigcup_{j=1}^{\infty} E_j \tag{2.55}$$

is a subgroup of A, which is the subgroup of A generated by E. If E has only finitely or countably many elements, then E_j has only finitely or countably many elements for each j, which implies that (2.55) has only finitely or countably many elements. If E is compact, then E_j is compact for each j, by continuity of addition on A, and (2.55) is σ-compact. If 0 is an element of the interior of E, then (2.55) is an open subgroup of A, as in the preceding paragraph.

If A is locally compact, then there is a compact set $E \subseteq A$ that contains 0 in its interior, and one can also ask that E be symmetric about 0, as before. Under these conditions, (2.55) is a σ-compact open subgroup of A, as in the previous paragraph.

Let A be any commutative topological group again, and let A_0 be a subgroup of A. Any subset of A/A_0 with only finitely or countably many elements is contained in a subgroup of A/A_0 with only finitely or countably many elements, as before. This implies that any collection of finitely or countably many translates of A_0 in A is contained in a subgroup A_1 of A such that $A_0 \subseteq A_1$ and A_1/A_0 has only finitely or countably many elements. If A_0 is σ-compact in A, then it follows that A_1 is σ-compact as well.

Suppose from now on in this section that A_0 is an open subgroup of A. If $K \subseteq A$ is compact, then K is contained in the union of finitely many translates of A_0 in A. Similarly, σ-compact subsets of A are contained in the union of finitely or countably many translates of A_0 in A. More precisely, this works for subsets of A with the Lindelöf property.

Suppose that A is locally compact again, and let H be a Haar measure on A. Also let U be an open set in A such that $H(U) < \infty$. Note that $U \cap (a + A_0)$ is an open set in A for every $a \in A$, because A_0 is an open set in A. If $U \cap (a + A_0) \neq \emptyset$, then it follows that

$$H(U \cap (a + A_0)) > 0. \tag{2.56}$$

If n is a positive integer, then there are at most finitely many distinct cosets of A_0 in A whose intersection with U has Haar measure greater than or equal to $1/n$, because distinct cosets of A_0 are pairwise disjoint in A, and $H(U) < \infty$. This implies that (2.56) holds for at most finitely or countably many distinct cosets of A_0 in A. This shows that U is contained in the union of finitely or countably many translates of A_0 in A under these conditions.

If $E \subseteq A$ is a Borel set and $H(E) < \infty$, then there is an open set $U \subseteq A$ such that $E \subseteq U$ and $H(U) < \infty$, by the outer regularity of H on A. This implies that E is contained in the union of finitely or countably many translates of A_0 in A, as in the previous paragraph. The same conclusion holds when E is σ-finite with respect to H, in the sense that E can be expressed as the union of finitely or countably many Borel sets with finite measure with respect to H. If f is a complex-valued Borel measurable function on A that is integrable with respect to H, then the set of $x \in A$ such that $f(x) \neq 0$ is a Borel measurable set which is σ-finite with respect to H.

2.8 Real and complex measures

Let X be a set, and let $\mathcal{A}$ be a σ-algebra of measurable subsets of X. A *real or complex measure* on $(X, \mathcal{A})$ is a real or complex-valued function μ on $\mathcal{A}$, as appropriate, that is countably additive. This means that for each sequence $A_1, A_2, A_3, \ldots$ of pairwise-disjoint measurable subsets of X, we have that

$$\sum_{j=1}^{\infty} \mu(A_j) = \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big), \tag{2.57}$$

where the convergence of the series on the left side of the equation is part of the condition. It follows that this series should converge absolutely, since the same condition could be applied to any rearrangement of the A_j's. Note that this condition implies in particular that $\mu(\emptyset) = 0$, and that real measures are also known as *signed measures.*

Let μ be a real or complex measure on $(X, \mathcal{A})$, and let A be a measurable subset of X. The corresponding *total variation measure* $|\mu|$ is defined by putting

$$\begin{aligned} |\mu|(A) \quad = \quad & \sup\Big\{\sum_{j=1}^{\infty} |\mu(E_j)| : E_1, E_2, E_3, \ldots \text{ is a sequence of pairwise-} \\ & \qquad \text{disjoint measurable sets such that } \bigcup_{j=1}^{\infty} E_j = A\Big\}. \end{aligned} \tag{2.58}$$

Each of the sums in the supremum is finite, as in the preceding paragraph, and it is well known that (2.58) is finite. One can check that $|\mu|$ is countably additive, so that $|\mu|$ is a finite nonnegative measure on $(X, \mathcal{A})$. This measure can be characterized as the smallest nonnegative measure on $(X, \mathcal{A})$ such that

$$|\mu(A)| \le |\mu|(A) \tag{2.59}$$

for every measurable set $A \subseteq X$.

If μ is a real measure on $(X, \mathcal{A})$, then

$$\mu_+ = (|\mu| + \mu)/2, \quad \mu_- = (|\mu| - \mu)/2 \tag{2.60}$$

are finite nonnegative measures on $(X, \mathcal{A})$, by (2.59). Observe that

$$\mu = \mu_+ - \mu_- \quad \text{and} \quad |\mu| = \mu_+ + \mu_-. \tag{2.61}$$

This expression for μ is known as the *Jordan decomposition* of μ. If μ is a complex measure on $(X, \mathcal{A})$, then the real and imaginary parts of μ are real measures on $(X, \mathcal{A})$. In particular, μ can be expressed as a linear combination of finite nonnegative measures on $(X, \mathcal{A})$, using the Jordan decompositions of the real and imaginary parts of μ.

Let μ be a real or complex measure on $(X, \mathcal{A})$. If f is a bounded real or complex-valued function on X that is measurable with respect to $\mathcal{A}$, then

$$\int_X f \, d\mu \tag{2.62}$$

can be defined as a real or complex number, as appropriate. More precisely, one can reduce to integrals with respect to nonnegative measures by expressing μ as a linear combination of finite nonnegative measures on $(X, \mathcal{A})$, as in the previous paragraph. We also have that

$$\left| \int_X f \, d\mu \right| \le \int_X |f| \, d|\mu| \tag{2.63}$$

for all such functions f. If μ is a real measure on X, then this can be derived from (2.61) and the analogous fact for integrals with respect to nonnegative measures. If μ is a complex measure on X, then one can get an inequality like this with an extra constant on the right side, by applying the previous argument to the real and imaginary parts of μ. If f is a measurable simple function on X, then the integral (2.62) can be defined in terms of μ directly, and (2.63) can be derived from (2.59). The same inequality for arbitrary bounded measurable functions on X can be obtained by approximation by simple functions. The integral (2.62) can also be defined for measurable functions f on X that are integrable with respect to $|\mu|$, with the same estimate (2.63). This uses the fact that μ can be expressed as a linear combination of nonnegative measures on $(X, \mathcal{A})$ that are bounded by $|\mu|$.

Let ν be a nonnegative measure on $(X, \mathcal{A})$. If g is a nonnegative real-valued function on X that is measurable with respect to $\mathcal{A}$, then

$$\nu_g(A) = \int_A g \, d\nu \tag{2.64}$$

defines a nonnegative measure on $(X, \mathcal{A})$, by the monotone convergence theorem. If f is another nonnegative real-valued measurable function on X, then

$$\int_X f \, d\nu_g = \int_X f \, g \, d\nu. \tag{2.65}$$

This follows directly from the definition of ν_g when f is a simple function, and otherwise one can approximate f by simple functions, as usual. In particular, this implies that f is integrable with respect to ν_g if and only if $f \, g$ is integrable with respect to ν. If f is any real or complex-valued measurable function on X, then it follows that f is integrable with respect to ν_g if and only if $f \, g$ is integrable with respect to ν. In this case, (2.65) can be derived from the analogous statement for nonnegative real-valued measurable functions on X, by expressing f as a linear combination of nonnegative functions that satisfy the same integrability conditions.

Now let g be a real or complex-valued function on X that is measurable with respect to $\mathcal{A}$ and integrable with respect to ν. One can verify that (2.64) defines a real or complex measure on $(X, \mathcal{A})$ under these conditions, using the dominated convergence theorem. Of course, $|g|$ is a nonnegative real-valued function on X that is measurable with respect to $\mathcal{A}$ and integrable with respect to ν, so that $\nu_{|g|}$ can be defined as a finite nonnegative measure on $(X, \mathcal{A})$ in the same way. It is well known that $\nu_{|g|}$ is the same as the total variation measure $|\nu_g|$ associated to ν_g. Note that ν_g can be expressed as a linear combination of finite nonnegative measures on $(X, \mathcal{A})$ by expressing g as a linear combination of nonnegative real-valued measurable functions that are integrable with respect to ν. If f is a bounded real or complex-valued measurable function on X, then the integral of f with respect to ν_g can be given as in (2.65). This also works when f is integrable with respect to $\nu_{|g|}$, which is the same as saying that $f \, g$ is integrable with respect to ν.

Let μ be any real or complex measure on $(X, \mathcal{A})$ again. It is well known that there is a real or complex-valued measurable function h on X such that $|h(x)| = 1$ for every $x \in X$ and

$$\mu(A) = |\mu|_h(A) = \int_A h \, d|\mu| \tag{2.66}$$

for every measurable set $A \subseteq X$. This uses the Radon–Nikodym theorem, and it is basically the same as the Hahn decomposition in the real case.

2.9 Vanishing at infinity

Let X be a locally compact Hausdorff topological space. A real or complex-valued function f on X is said to *vanish at infinity* on X if for each $\epsilon > 0$ there is a compact set $K \subseteq X$ such that

$$|f(x)| < \epsilon \tag{2.67}$$

for every $x \in X \setminus K$. If X is equipped with the discrete topology, then this reduces to the condition mentioned in Section 2.1. Let $C_0(X, \mathbf{R})$ and $C_0(X, \mathbf{C})$ be the spaces of real and complex-valued continuous functions on X, respectively, that vanish at infinity. These are subalgebras of the algebras $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$ of real and complex-valued bounded continuous functions on X, respectively. One can also check that $C_0(X, \mathbf{R})$ and $C_0(X, \mathbf{C})$ are closed sets in $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$, respectively, with respect to the topologies determined by the corresponding supremum norms. If X is equipped with the discrete topology, then $C_0(X, \mathbf{R})$ and $C_0(X, \mathbf{C})$ are the same as $c_0(X, \mathbf{R})$ and $c_0(X, \mathbf{C})$, respectively, and $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$ are the same as $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively.

If a real or complex-valued function f on X has compact support, then f vanishes at infinity on X. In particular, if X is compact, then every real or complex-valued function on X automatically vanishes at infinity, so that $C_0(X, \mathbf{R})$ and $C_0(X, \mathbf{C})$ are the same as $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$, respectively. Otherwise, $C_0(X, \mathbf{R})$ and $C_0(X, \mathbf{C})$ are the same as the closures of $C_{com}(X, \mathbf{R})$ and $C_{com}(X, \mathbf{C})$ in $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$, respectively, with respect to the topologies determined by the corresponding supremum norms. To see this, one can approximate a continuous real or complex-valued function f on X that vanishes at infinity uniformly by continuous functions on X with compact support using the version of Urysohn's lemma mentioned in Section 2.4. One can also approximate f by functions with compact support by composing f with suitable continuous functions on $\mathbf{R}$ or $\mathbf{C}$, as appropriate.

Let μ be a real or complex Borel measure on X, and let $|\mu|$ be the corresponding total variation measure on X, as in the previous section. If f is a real or complex-valued bounded Borel measurable function on X, as appropriate, then the integral of f with respect to μ on X can be defined in standard ways, as before. Using (2.63), we get that

$$\left| \int_X f \, d\mu \right| \le |\mu|(X) \sup_{x \in X} |f(x)|. \tag{2.68}$$

Remember that $|\mu|(X) < \infty$, as mentioned in the previous section.

Now let λ be a continuous linear functional on $C_0(X, \mathbf{R})$ or $C_0(X, \mathbf{C})$, with respect to the topology determined by the supremum norm. Another version of the Riesz representation theorem states that there is a unique real or complex Borel measure μ on X, as appropriate, such that

$$\lambda(f) = \int_X f \, d\mu \tag{2.69}$$

for every $f \in C_0(X, \mathbf{R})$ or $C_0(X, \mathbf{C})$, as appropriate, and where μ satisfies some additional regularity properties. More precisely, the corresponding total variation measure $|\mu|$ should be outer regular on X, in the sense that

$$|\mu|(E) = \inf\{|\mu|(U) : U \subseteq X \text{ is an open set, and } E \subseteq U\} \tag{2.70}$$

for every Borel set $E \subseteq X$. Similarly, $|\mu|$ should be inner regular on X, in the sense that

$$|\mu|(E) = \sup\{|\mu|(K) : K \subseteq X \text{ is compact, and } K \subseteq E\} \tag{2.71}$$

for every Borel set $E \subseteq X$. In this situation, $|\mu|(X)$ is equal to the dual norm of λ with respect to the supremum norm on $C_0(X, \mathbf{R})$ or $C_0(X, \mathbf{C})$, as appropriate, as in (1.151) in Section 1.18.

Let us suppose from now on in this section that X is equipped with the discrete topology. Let g be a real or complex-valued summable function on X, and put

$$\mu_g(E) = \sum_{x \in E} g(x) \tag{2.72}$$

for every subset E of X. This is interpreted as being equal to 0 when $E = \emptyset$, and otherwise the sum can be defined as in Section 2.1, since the restriction of g to E is summable on E. This defines a real or complex Borel measure on X, as appropriate, for which the corresponding total variation measure is given by

$$|\mu_g|(E) = \sum_{x \in E} |g(x)| \tag{2.73}$$

for every $E \subseteq X$. The outer regularity condition (2.70) is trivial in this case, because every subset of X is an open set, and the inner regularity condition (2.71) follows from the definition of the sum in (2.73) as the supremum of the corresponding finite subsums.

If f is a bounded real or complex-valued function on X, then $f\,g$ is summable on X, and we put

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x). \tag{2.74}$$

Note that f is automatically Borel measurable on X in this situation, and that (2.74) is the same as the integral of f on X with respect to (2.72). We also have that

$$|\lambda_g(f)| \le \sum_{x \in X} |f(x)|\, |g(x)| \le \|f\|_\infty \|g\|_1, \tag{2.75}$$

where $\|g\|_1$ is as in (2.5) in Section 2.1, and $\|f\|_\infty$ is as in (2.17) in Section 2.2. Thus λ_g defines a continuous linear functional on $\ell^\infty(X, \mathbf{R})$ or $\ell^\infty(X, \mathbf{C})$, as appropriate, with dual norm less than or equal to $\|g\|_1$. It is easy to see that the dual norm is equal to $\|g\|_1$, by considering a bounded function f with $\|f\|_\infty \le 1$ such that

$$f(x)\, g(x) = |g(x)| \tag{2.76}$$

for every $x \in X$.

Let us now consider the restriction of λ_g to $c_0(X, \mathbf{R})$ or $c_0(X, \mathbf{C})$, as appropriate. The restriction of λ_g defines a continuous linear functional on that space, with respect to the supremum norm again. The corresponding dual norm of λ_g is still less than or equal to $\|g\|_1$, because of (2.75). To check that the dual norm is equal to $\|g\|_1$ in this situation as well, one can consider real or complex-valued functions f on X with finite support such that $\|f\|_\infty \le 1$ and (2.76) holds when $f(x) \neq 0$. This implies that

$$\lambda_g(f) = \sum_{\substack{x \in X \\ f(x) \neq 0}} |g(x)|, \tag{2.77}$$

which can be used to approximate $\|g\|_1$.

Let λ be any continuous linear functional on $c_0(X, \mathbf{R})$ or $c_0(X, \mathbf{C})$, with respect to the supremum norm. It is well known that there is a real or complex-valued function g on X, as appropriate, such that g is summable on X, and

$$\lambda(f) = \lambda_g(f) \tag{2.78}$$

for every real or complex-valued function f on X, as appropriate, that vanishes at infinity on X. It is easy to first find a real or complex-valued function g on X, as appropriate, such that (2.78) holds when f has finite support in X, with $\lambda_g(f)$ defined as in (2.74). Using such functions f as in the previous paragraph, one can verify that g is summable on X, with $\|g\|_1$ less than or equal to the dual norm of λ. To show that (2.78) holds for every real or complex-valued function f on X that vanishes at infinity, one can approximate f by functions with finite support on X, and use the continuity of λ and λ_g.

2.10 Mappings and measurability

Let X and Y be sets, and let f be a mapping from X into Y. Suppose for the moment that $\mathcal{A}_X$ and $\mathcal{A}_Y$ are σ-algebras of subsets of X and Y, respectively. As usual, f is said to be *measurable* with respect to $\mathcal{A}_X$ and $\mathcal{A}_Y$ if for every $E \in \mathcal{A}_Y$, we have that $f^{-1}(E) \in \mathcal{A}_X$. Let Z be another set, and let $\mathcal{A}_Z$ be a σ-algebra of measurable subsets of Z. If $f : X \to Y$ is measurable with respect to $\mathcal{A}_X$ and $\mathcal{A}_Y$, and $g : Y \to Z$ is measurable with respect to $\mathcal{A}_Y$ and $\mathcal{A}_Z$, then their composition $g \circ f$ is measurable as a mapping from X into Z with respect to $\mathcal{A}_X$ and $\mathcal{A}_Z$, by standard arguments.

If $\mathcal{A}_Y$ is any σ-algebra of subsets of Y, then it is easy to see that

$$\{f^{-1}(E) : E \in \mathcal{A}_Y\} \tag{2.79}$$

is a σ-algebra of subsets of X. By construction, f is measurable with respect to (2.79) on X and $\mathcal{A}_Y$ on Y, and any other σ-algebra of subsets of X with this property contains (2.79).

Similarly, if $\mathcal{A}_X$ is any σ-algebra of subsets of X, then

$$\{E \subseteq Y : f^{-1}(E) \in \mathcal{A}_X\} \tag{2.80}$$

is a σ-algebra of subsets of Y. Clearly f is measurable with respect to $\mathcal{A}_X$ on X and (2.80) on Y, and any other σ-algebra of subsets of Y with this property is contained in (2.80).

Let $\mathcal{E}_Y$ be any collection of subsets of Y, and let $\mathcal{A}_Y$ be the smallest σ-algebra of subsets of Y that contains $\mathcal{E}_Y$. Also let $\mathcal{A}_X$ be any σ-algebra of subsets of X, and suppose that

$$f^{-1}(E) \in \mathcal{A}_X \tag{2.81}$$

for every $E \in \mathcal{E}_Y$. This is the same as saying that $\mathcal{E}_Y$ is contained in (2.80). It follows that $\mathcal{A}_Y$ is contained in (2.80), because (2.80) is a σ-algebra of subsets of Y, and using the definition of $\mathcal{A}_Y$. This shows that f is measurable with respect to $\mathcal{A}_X$ on X and $\mathcal{A}_Y$ on Y under these conditions.

Put

$$\mathcal{E}_X = \{f^{-1}(E) : E \in \mathcal{E}_Y\}, \tag{2.82}$$

and let $\mathcal{A}_X$ be the smallest σ-algebra of subsets of X that contains $\mathcal{E}_X$. Let us check that $\mathcal{A}_X$ is the same as (2.79) in this situation. The remarks in the previous paragraph imply that f is measurable with respect to $\mathcal{A}_X$ on X and $\mathcal{A}_Y$ on Y, which means that (2.79) is contained in $\mathcal{A}_X$. Of course, (2.82) is contained in (2.79), because $\mathcal{E}_Y \subseteq \mathcal{A}_Y$, by construction. This implies that $\mathcal{A}_X$ is contained in (2.79), because (2.79) is a σ-algebra.

Suppose for the moment that X and Y are topological spaces, and let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be the corresponding σ-algebras of Borel sets in X and Y, respectively. If f is continuous, then it is well known that f is Borel measurable, which is to say that f is measurable with respect to $\mathcal{A}_X$ and $\mathcal{A}_Y$. This follows from the criterion (2.81), with $\mathcal{E}_Y$ taken to be the topology on Y.

Let τ_Y be a topology on Y, and observe that

$$\{f^{-1}(V) : V \in \tau_Y\} \tag{2.83}$$

defines a topology on X. By construction, f is continuous with respect to (2.83) on X and τ_Y on Y, and (2.83) is the weakest topology on X with this property. If $\mathcal{B}_Y$ is a base for τ_Y, then

$$\{f^{-1}(V) : V \in \mathcal{B}_Y\} \tag{2.84}$$

is a base for (2.83). Similarly, if $\mathcal{B}_Y$ is a sub-base for τ_Y, then (2.84) is a sub-base for (2.83).

Let us take $\mathcal{E}_Y = \tau_Y$, so that $\mathcal{E}_X$ in (2.82) is the same as (2.83). Thus the smallest σ-algebra $\mathcal{A}_Y$ of subsets of Y that contains $\mathcal{E}_Y$ is the same as the σ-algebra of Borel sets in Y with respect to τ_Y, and the smallest σ-algebra $\mathcal{A}_X$ of subsets of X that contains $\mathcal{E}_X$ is the same as the σ-algebra of Borel sets in X with respect to the topology (2.83). Using the earlier remarks, we get that $\mathcal{A}_X$ is the same as (2.79), so that Borel sets in X are the same as inverse images of Borel sets in Y under f.

Suppose now that X is a subset of Y, and that $f : X \to Y$ is the corresponding inclusion mapping, so that $f(x) = x$ for every $x \in X$. If E is any subset of Y, then

$$f^{-1}(E) = E \cap X \tag{2.85}$$

in this case. Let τ_Y be a topology on Y, and observe that (2.83) is the same as the topology induced on X by τ_Y. The remarks in the previous paragraph imply that a subset of X is a Borel set with respect to the induced topology if and only if it can be expressed as $E \cap X$ for some Borel set E in Y. In particular, if E is a Borel set in Y, and $E \subseteq X$, then E is a Borel set in X with respect to the induced topology, because $E = E \cap X$.

Suppose that X is a Borel set in Y. If E is another Borel set in Y, then $E \cap X$ is a Borel set in Y too. This implies that every Borel set in X with respect to the induced topology is a Borel set in Y in this case, by the remarks in the preceding paragraph. It follows that the Borel sets in X with respect to the induced topology are the same as the Borel sets in Y that are also contained in X, since the other half of this statement always holds, as mentioned in the previous paragraph.

2.11 Pushing measures forward

Let X, Y be sets, and let $\mathcal{A}_X$, $\mathcal{A}_Y$ be σ-algebras of measurable subsets of X and Y, respectively. Also let $h : X \to Y$ be a mapping which is measurable with respect to $\mathcal{A}_X$ and $\mathcal{A}_Y$, in the sense that for every $E \in \mathcal{A}_Y$, we have that $h^{-1}(E) \in \mathcal{A}_X$, as in the previous section. If μ is a nonnegative measure on $(X, \mathcal{A}_X)$, then it is easy to see that

$$\nu(E) = \mu(h^{-1}(E)) \tag{2.86}$$

defines a nonnegative measure on $(Y, \mathcal{A}_Y)$. If f is a nonnegative measurable function on Y, then $f \circ h$ is measurable on X, and

$$\int_X f \circ h \, d\mu = \int_Y f \, d\nu. \tag{2.87}$$

More precisely, this follows from (2.86) when f is the indicator function on Y associated to a measurable subset of Y. If f is a nonnegative measurable simple function on Y, then (2.87) can be obtained from the previous statement by linearity. To get (2.87) for an arbitrary nonnegative measurable function f on Y, one can approximate f by simple functions in the usual way. If f is a real or complex-valued measurable function on Y, then $f \circ h$ is measurable on X too. Of course, $|f|$ is nonnegative and measurable on Y, and

$$\int_X |f \circ h| \, d\mu = \int_Y |f| \, d\nu, \tag{2.88}$$

by (2.87) applied to $|f|$. If these integrals are finite, so that f is integrable on Y with respect to ν and $f \circ h$ is integrable on X with respect to μ, then one can check that (2.87) holds.

Similarly, if μ is a real or complex measure on $(X, \mathcal{A}_X)$, then (2.86) defines a real or complex measure on $(Y, \mathcal{A}_Y)$, as appropriate. In this case, the total variation measure $|\mu|$ associated to μ defines a finite nonnegative measure on $(X, \mathcal{A}_X)$, so that

$$\sigma(E) = |\mu|(h^{-1}(E)) \tag{2.89}$$

defines a finite nonnegative measure on $(Y, \mathcal{A}_Y)$. If E is a measurable subset of Y, then

$$|\nu(E)| = |\mu(h^{-1}(E))| \leq |\mu|(h^{-1}(E)) = \sigma(E). \tag{2.90}$$

This implies that

$$|\nu|(E) \leq \sigma(E) \tag{2.91}$$

for every measurable set $E \subseteq Y$, as in Section 2.8. It follows that

$$\int_Y f\, d|\nu| \leq \int_Y f\, d\sigma = \int_X f\, d|\mu| \tag{2.92}$$

for every nonnegative measurable function f on Y, using the analogue of (2.87) for $|\mu|$ and σ in the second step. If f is a bounded real or complex-valued measurable function on Y, then $f \circ h$ is bounded and measurable on X, and (2.87) holds. The integrals in (2.87) can be defined as in Section 2.8, and one can get (2.87) by reducing to the case of nonnegative measures. This also works when f is a real or complex-valued measurable function on Y such that $f \circ h$ is integrable on X with respect to $|\mu|$, which is the same as saying that f is integrable on Y with respect to σ, as before.

Suppose now that X and Y are topological spaces, equipped with the corresponding σ-algebras of Borel sets. Let h be a continuous mapping from X into Y, which implies that h is Borel measurable, as in the previous section. Let us also ask that X and Y be Hausdorff, so that compact subsets of X and Y are closed sets. Let μ be a nonnegative Borel measure on X, and let ν be the corresponding Borel measure on Y, as in (2.86). Let $E \subseteq Y$ be a Borel set, and let K be a compact subset of X contained in $h^{-1}(E)$. Thus $h(K)$ is a compact subset of Y that is contained in E. Observe that

$$\nu(h(K)) = \mu(h^{-1}(h(K))) \geq \mu(K), \tag{2.93}$$

since $K \subseteq h^{-1}(h(K))$. If $\mu(K)$ approximates $\mu(h^{-1}(E))$, then it follows that $\nu(h(K))$ approximates $\nu(E)$. This is a basic way to get inner regularity conditions for ν from analogous conditions for μ. Similarly,

$$h^{-1}(E \setminus h(K)) = h^{-1}(E) \setminus h^{-1}(h(K)) \subseteq h^{-1}(E) \setminus K, \tag{2.94}$$

which implies that

$$\nu(E \setminus h(K)) = \mu(h^{-1}(E \setminus h(K))) \leq \mu(h^{-1}(E) \setminus K). \tag{2.95}$$

Suppose that h is also *proper*, in the sense that $h^{-1}(E)$ is a compact subset of X for every compact set $E \subseteq Y$. If μ is finite on compact subsets of X, then it follows that ν is finite on compact subsets of Y. Let us suppose now that X and Y are locally compact. If f is a continuous real or complex-valued function on Y with compact support, then $f \circ h$ is a continuous function with compact support on X. If λ is a nonnegative linear functional on $C_{com}(X, \mathbf{R})$ or $C_{com}(X, \mathbf{C})$, then

$$f \mapsto \lambda(f \circ h) \tag{2.96}$$

is a nonnegative linear functional on $C_{com}(Y, \mathbf{R})$ or $C_{com}(Y, \mathbf{C})$, as appropriate. Similarly, if f is a continuous real or complex-valued function on X that vanishes at infinity, then $f \circ h$ vanishes at infinity on Y. If λ is a continuous linear functional on $C_0(X, \mathbf{R})$ or $C_0(X, \mathbf{C})$ with respect to the supremum norm, then (2.96) defines a continuous linear functional on $C_0(Y, \mathbf{R})$ or $C_0(Y, \mathbf{C})$, as appropriate, with respect to the corresponding supremum norm. This uses the fact that the supremum norm of $f \circ h$ on X is less than or equal to the supremum norm of f on Y. This gives another way to look at pushing measures forward in this situation, in connection with the representation theorems mentioned in Sections 2.4 and 2.9.

2.12 Monotone classes

Let X be a set. A collection $\mathcal{M}$ of subsets of X is said to be a *monotone class* if it has the following two properties. First, if $A_1, A_2, A_3, \ldots$ is a sequence of elements of $\mathcal{M}$ such that $A_j \subseteq A_{j+1}$ for every $j \ge 1$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$. Second, if $B_1, B_2, B_3, \ldots$ is a sequence of elements of $\mathcal{M}$ such that $B_{j+1} \subseteq B_j$ for every $j \ge 1$, then $\bigcap_{j=1}^{\infty} B_j \in \mathcal{M}$. Clearly σ-algebras are monotone classes.

Let $\mathcal{E}$ be an algebra of subsets of X. It is well known that the smallest σ-algebra of subsets of X that contains $\mathcal{E}$ is the same as the smallest monotone class of subsets of X that contains $\mathcal{E}$. Of course, the former automatically contains the latter, because σ-algebras are monotone classes.

Let X and Y be sets, and suppose that $\mathcal{A}_X$ and $\mathcal{A}_Y$ are σ-algebras of subsets of X and Y, respectively. If $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, then $A \times B$ is said to be a *measurable rectangle* in $X \times Y$. Let $\mathcal{E}_{X \times Y}$ be the collection of subsets of $X \times Y$ that can be expressed as the union of finitely many pairwise-disjoint measurable rectangles, which are sometimes described as *elementary sets* in this situation. It is well known and not too difficult to check that $\mathcal{E}_{X \times Y}$ is an algebra of subsets of $X \times Y$. In the usual product measure construction, the collection $\mathcal{A}_{X \times Y}$ of measurable subsets of $X \times Y$ is defined to be the smallest σ-algebra that contains all measurable rectangles. This is clearly the same as the smallest σ-algebra of subsets of $X \times X$ that contains $\mathcal{E}$. Equivalently, this is the smallest monotone class of subsets of $X \times Y$ that contains $\mathcal{E}$, by the result mentioned in the preceding paragraph.

Let X be a set again, and let $\mathcal{A}$ be a σ-algebra of measurable subsets of X. Also let μ and ν be real or complex measures defined on $(X, \mathcal{A})$, and let $\mathcal{M}_0$ be the collection of measurable sets $A \in \mathcal{A}$ such that

$$\mu(A) = \nu(A). \tag{2.97}$$

It is easy to see that $\mathcal{M}_0$ is a monotone class of subsets of X. Suppose that $\mathcal{E}_0$ is an algebra of subsets of X contained in $\mathcal{A}$ such that (2.97) holds for every $A \in \mathcal{E}_0$, so that $\mathcal{E} \subseteq \mathcal{M}_0$. Let $\mathcal{A}_0$ be the smallest σ-algebra of subsets of X that contains $\mathcal{E}_0$, and observe that $\mathcal{A}_0 \subseteq \mathcal{A}$. Equivalently, $\mathcal{A}_0$ is the smallest monotone class of subsets of X that contains $\mathcal{E}$, as before. It follows that $\mathcal{A}_0 \subseteq \mathcal{M}_0$, which means that (2.97) holds for every $A \in \mathcal{A}_0$. Of course, we could reduce to the case where $\nu = 0$ here, by replacing μ with $\mu - \nu$. However, if μ and ν are finite nonnegative measures on $(X, \mathcal{A})$, then one can use the argument just given, without using signed measures.

Now let μ be a real or complex measure on $(X, \mathcal{A})$, and let ν be a finite nonnegative measure on $(X, \mathcal{A})$. As before, it is easy to see that the collection $\mathcal{M}_1$ of measurable sets $A \in \mathcal{A}$ such that

$$|\mu(A)| \le \nu(A) \tag{2.98}$$

is a monotone class of subsets of X. Let $\mathcal{E}_1$ be an algebra of subsets of X such that $\mathcal{E}_1 \subseteq \mathcal{A}$ and (2.98) holds for every $A \in \mathcal{E}_1$. Consider the smallest σ-algebra $\mathcal{A}_1$ of subsets of X that contains $\mathcal{E}_1$, so that $\mathcal{A}_1 \subseteq \mathcal{A}$ automatically. The result about monotone classes mentioned earlier implies that $\mathcal{A}_1$ is the smallest monotone class of subsets that contains $\mathcal{E}_1$. Thus $\mathcal{A}_1 \subseteq \mathcal{M}_1$, because $\mathcal{E}_1 \subseteq \mathcal{M}_1$, by construction. It follows that (2.98) holds for every $A \in \mathcal{A}_1$.

2.13 Product spaces

Let X and Y be sets, and let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be σ-algebras of measurable subsets of X and Y, respectively. This leads to a σ-algebra $\mathcal{A}_{X\times Y}$ of measurable subsets of $X \times Y$ in a standard way, as in the previous section. Let μ_X and μ_Y be nonnegative measures on $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$, respectively, and suppose that μ_X and μ_Y are σ-finite on X and Y. Under these conditions, the usual product measure construction leads to a nonnegative measure $\mu_x \times \mu_Y$ on $(X \times Y, \mathcal{A}_{X\times Y})$ such that

$$(\mu_X \times \mu_Y)(A \times B) = \mu_X(A)\, \mu_Y(B) \tag{2.99}$$

for every $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. As usual, the right side of (2.99) is interpreted as being equal to 0 when $\mu_X(A) = 0$ or $\mu_Y(B) = 0$, even if the other is equal to $+\infty$. Of course, $\mu_X \times \mu_Y$ is finite on $X \times Y$ when μ_X and μ_Y are finite. Otherwise, it is easy to see that $\mu_X \times \mu_Y$ is σ-finite on $X \times Y$, because of the σ-finiteness of μ_X and μ_Y.

The product measure $\mu_X \times \mu_Y$ is uniquely determined as a nonnegative measure on $(X \times Y, \mathcal{A}_{X\times Y})$ by (2.99). More precisely, if $\mathcal{E}_{X\times Y}$ is the corresponding algebra of elementary subsets of $X \times Y$, as in the previous section, then $\mu_X \times \mu_Y$ is determined on $\mathcal{E}_{X\times Y}$ by (2.99) and finite additivity. Suppose for the moment hat μ_X and μ_Y are finite on X and Y, so that any nonnegative measure on $X \times Y$ that satisfies (2.99) is finite on $X \times Y$. In this case, uniqueness of $\mu_X \times \mu_Y$ on $\mathcal{A}_{X\times Y}$ can be obtained from the remarks in the previous section. Otherwise, if μ_X and μ_Y are σ-finite on X and Y, respectively, then one can reduce to the finite case, by considering products of measurable sets with finite measure.

Now let X and Y be topological spaces, so that $X \times Y$ is a topological space too, with respect to the product topology. Let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be the corresponding

σ-algebras of Borel sets in X and Y, respectively. This leads to a σ-algebra $\mathcal{A}_{X\times Y}$ of measurable sets in $X \times Y$, as before, which is the smallest σ-algebra that contains all measurable rectangles in $X \times Y$. Let us compare this with the σ-algebra of Borel sets in $X \times Y$, corresponding to the product topology on $X \times Y$.

Let p_X and p_Y be the obvious coordinate projections from $X \times Y$ into X and Y, respectively, which are continuous mappings with respect to the product topology on $X \times Y$. As in Section 2.10, p_X and p_Y are Borel measurable, because they are continuous. If $A \subseteq X$ and $B \subseteq Y$ are Borel sets, then it follows that $p_X^{-1}(A) = A \times Y$ and $p_Y^{-1}(B) = X \times B$ are Borel sets in $X \times Y$. This implies that

$$A \times B = (A \times Y) \cap (X \times B) \tag{2.100}$$

is a Borel set in $X \times Y$, so that $\mathcal{A}_{X\times Y}$ is contained in the σ-algebra of Borel sets in $X \times Y$. If every open set in $X \times Y$ is an element of $\mathcal{A}_{X\times Y}$, then every Borel set in $X \times Y$ is an element of $\mathcal{A}_{X\times Y}$ too.

Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for the topologies on X and Y, respectively. This implies that

$$\mathcal{B}_{X\times Y} = \{U \times V : U \in \mathcal{B}_X,\, V \in \mathcal{B}_Y\} \tag{2.101}$$

is a base for the corresponding product topology on $X \times Y$, by standard arguments. Thus every open set in $X \times Y$ can be expressed as a union of elements of (2.101), and

$$\mathcal{B}_{X\times Y} \subseteq \mathcal{A}_{X\times Y}, \tag{2.102}$$

by construction. If $\mathcal{B}_X$ and $\mathcal{B}_Y$ has only finitely or countably many elements, then $\mathcal{B}_{X\times Y}$ has only finitely or countably many elements. This implies that $\mathcal{A}_{X\times Y}$ contains all open subsets of $X \times Y$ under these conditions.

If X, Y are locally compact Hausdorff topological spaces, then $X \times Y$ is a locally compact Hausdorff topological space as well, with respect to the product topology. If μ_X and μ_Y have suitable regularity properties, then a product measure can be constructed as a nonnegative Borel measure on $X \times Y$ with suitable regularity properties. One can look at this in terms of nonnegative linear functionals on spaces of continuous functions with compact support, as in Section 2.4. More precisely, if λ_X and λ_Y are nonnegative linear functionals on $C_{com}(X, \mathbf{R})$ and $C_{com}(Y, \mathbf{R})$, respectively, then one can define a product nonnegative linear functional on $C_{com}(X \times Y, \mathbf{R})$, basically by applying λ_X and λ_Y to a continuous function $f(x, y)$ on $X \times Y$ with compact support in each variable separately. This leads to a nonnegative Borel measure on $X \times Y$ with suitable regularity properties, as before.

If A, B are locally compact commutative topological groups, then $A \times B$ is a locally compact commutative topological group with respect to the product topology, and where the group operations are defined coordinatewise. Haar measure on $A \times B$ can be obtained from Haar measures on A and B as in the preceding paragraph.

2.14 Infinite products

It is well known that a product probability measure can be defined on any product of probability spaces. Let $X_1, X_2, X_3, \ldots$ be a sequence of topological spaces, and let $X = \prod_{j=1}^\infty X_j$ be their Cartesian product, equipped with the product topology. Also let $\mathcal{B}_j$ be a base for the topology of X_j for each j. If n is a positive integer, then let $\mathcal{B}(n)$ be the collection of subsets of X of the form $U = \prod_{j=1}^\infty U_j$, where $U_j \in \mathcal{B}_j$ when $j \le n$, and $U_j = X_j$ when $j > n$. The elements of $\mathcal{B}(n)$ are open sets in X for every n, and

$$\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}(n) \tag{2.103}$$

is a base for the product topology on X. If $\mathcal{B}_j$ has only finitely or countably many elements for each j, then $\mathcal{B}(n)$ has only finitely or countably many elements, and hence $\mathcal{B}$ has only finitely or countably many elements. If we use Borel sets in X_j as measurable sets for each j, then the elements of $\mathcal{B}$ are measurable in X with respect to the usual product construction. If $\mathcal{B}$ has only finitely or countably many elements, then it follows that every open set in X with respect to the product topology is measurable with respect to the usual product construction. This implies that Borel sets in X with respect to the product topology are measurable with respect to the usual product construction under these conditions.

Let I be a nonempty set, and let X_j be a compact Hausdorff topological space for each $j \in I$. Thus $X = \prod_{j \in I} X_j$ is a compact Hausdorff space with respect to the product topology, by Tychonoff's theorem. In this situation, the product of regular Borel probability measures on the X_j's can be defined as a regular Borel probability measure on X. One can look at this in terms of nonnegative linear functionals on the corresponding spaces of continuous functions, as in the previous section. Of course, the condition that the measure of a compact Hausdorff space be equal to 1 means that the associated linear functional take the value 1 on the constant function equal to 1 on the space.

If A_j is a compact commutative topological group for each $j \in I$, then $A = \prod_{j \in I} A_j$ is a compact commutative topological group with respect to the product topology, and where the group operations are defined coordinatewise. If Haar measure on A_j is normalized so that the measure of A_j is 1 for each $j \in I$, then Haar measure on A can be obtained as in the previous paragraph.

Let $X_1, X_2, X_3, \ldots$ be a sequence of locally compact Hausdorff topological spacesthat are not compact, and let Y_j be a one-point compactification of X_j for each j. A Borel probability measure on X_j with suitable regularity properties corresponds to a regular Borel probability measure on Y_j such that the set consisting of the point at infinity has measure 0. Given such probability measures for each j, one can get a regular Borel probability measure on $Y = \prod_{j=1}^\infty Y_j$ as before. One can consider $X = \prod_{j=1}^\infty X_j$ as a G_δ set in Y, whose complement has measure 0. Thus we get a Borel probability measure on X under these conditions.

Let $X_1, X_2, X_3, \ldots$ be a sequence of topological spaces, and let $X = \prod_{j=1}^\infty X_j$ be their product again, equipped with the product topology. Also let μ be a finite nonnegative Borel measure on X. Let $p_j : X \to X_j$ be the standard coordinate projection for each j, and let μ_j be the Borel measure defined on X_j by

$$\mu_j(E_j) = \mu(p_j^{-1}(E_j)) \tag{2.104}$$

for every Borel set $E_j \subseteq X_j$ and $j \geq 1$. This is the same as the Borel measure on X_j obtained by pushing μ forward using p_j, as in Section 2.11. If μ is given by a product of probability measures on the X_j's, then the μ_j's will be those probability measures. Let $E_j \subseteq X_j$ be a Borel set for each j, and observe that $p_j^{-1}(E_j)$ is a Borel set in X for each j, because p_j is continuous. This implies that

$$E = \prod_{j=1}^\infty E_j = \bigcap_{j=1}^\infty p_j^{-1}(E_j) \tag{2.105}$$

is a Borel set in X. If E_j is a compact subset of X_j for each j, then E is compact in X, by Tychonoff's theorem. Of course,

$$X \setminus E = X \setminus \Big(\bigcap_{j=1}^\infty p_j^{-1}(E_j)\Big) = \bigcup_{j=1}^\infty (X \setminus p_j^{-1}(E_j)) = \bigcup_{j=1}^\infty p_j^{-1}(X_j \setminus E_j). \tag{2.106}$$

It follows that

$$\mu(X \setminus E) \leq \sum_{j=1}^\infty \mu(p_j^{-1}(X_j \setminus E_j)) = \sum_{j=1}^\infty \mu_j(X_j \setminus E_j). \tag{2.107}$$

2.15 Completeness

Let X be a metric space, and let μ be a finite nonnegative Borel measure on X. If X is separable, then for each $r > 0$, X can be covered by only finitely or countably many closed balls of radius r. This implies that for each $r > 0$ and $\epsilon > 0$ there is a subset $E(r, \epsilon)$ of X such that $E(r, \epsilon)$ is the union of finitely many closed balls of radius r, and

$$\mu(X \setminus E(r, \epsilon)) < \epsilon. \tag{2.108}$$

Using this, one can check that for each $\epsilon > 0$ there is a closed set $E(\epsilon) \subseteq X$ such that $E(\epsilon)$ is totally bounded in X, and

$$\mu(X \setminus E(\epsilon)) < \epsilon. \tag{2.109}$$

If X is also complete as a metric space, then E is compact in X, which is a well-known way of getting this type of inner regularity.

Let (X_j, d_j) be a metric space for each positive integer j, and let $X = \prod_{j=1}^\infty X_j$ be the Cartesian product of the X_j's. Put

$$\widetilde{d}_l(x, y) = d_l(x_l, y_l) \tag{2.110}$$

for each $l \geq 1$ and $x, y \in X$, where x_l, y_l denote the lth coordinates of x, y in X_l. This defines a semimetric on X for each $l \geq 1$, as in Section 1.8, and the collection of these semimetrics determines a topology on X as in Section 1.3. This is the same as the product topology on X associated to the topologies on the X_j's determined by the d_j's, as before. Put

$$\widetilde{d_l'} = \min(d_l(x, y), 1/l) \tag{2.111}$$

for every $l \geq 1$ and $x, y \in X$, and

$$d(x, y) = \max_{l \geq 1} \widetilde{d_l'}(x, y) \tag{2.112}$$

for every $x, y \in X$. As in Section 1.4, (2.111) defines a semimetric on X that determines the same topology on X as (2.110) for each l, and (2.112) defines a metric on X which determines the same topology on X as the collection of semimetrics (2.110), which is the product topology on X in this case. One can check that a sequence of elements of X is a Cauchy sequence with respect to (2.112) if and only if the corresponding sequences of lth coordinates are Cauchy sequences in X_l for each l. If X_l is complete as a metric space with respect to d_l for each l, then it follows that X is complete as a metric space with respect to (2.112).

Let A be a commutative topological group. A sequence $\{x_j\}_{j=1}^{\infty}$ of elements of A is said to be a *Cauchy sequence* in A if

$$x_j - x_l \to 0 \tag{2.113}$$

in A as $j, l \to \infty$. More precisely, this means that for each open set $U \subseteq A$ with $0 \in U$ there is a positive integer L such that

$$x_j - x_l \in U \tag{2.114}$$

for every $j, l \geq L$. Convergent sequences in A are Cauchy sequences, as usual. If every Cauchy sequence in A converges to an element of A, then A is said to be *sequentially complete*. If the topology on A is determined by a translation-invariant metric $d(\cdot, \cdot)$, then a Cauchy sequence in A as a commutative topological group is the same as a Cauchy sequence in A with respect to $d(\cdot, \cdot)$. In this case, A is sequentially complete as a commutative topological group if and only if A is complete as a metric space with respect to $d(\cdot, \cdot)$.

Let I be a nonempty set, and let A_j be a commutative topological group for each $j \in I$. As in Section 1.8, the Cartesian product $A = \prod_{j \in I} A_j$ is a commutative topological group with respect to the corresponding product topology, where the group operations are defined coordinatewise. It is easy to see that a sequence of elements of A is a Cauchy sequence in A as a commutative topological group if and only if for each $j \in I$, the corresponding sequence of jth coordinates of the terms of the given sequence is a Cauchy sequence in A_j as a commutative topological group. Of course, a sequence of elements of A converges to some element of A if and only if for each $j \in I$, the corresponding sequence of jth coordinates of the terms of the given sequence converges in A_j to the jth coordinate of the given element of A. It follows that A is sequentially complete as a commutative topological group if and only if A_j is sequentially complete as a commutative topological group for each $j \in I$.

2.16 Measures and bounded sets

Let X be a topological space, and let $d(x, y)$ be a semimetric on X. Remember that d also determines a topology on X, as in Section 1.2. Let us say that d is *compatible with the given topology on* X if open sets in X with respect to d are open with respect to the given topology on X as well. Of course, it suffices to check that open balls in X with respect to d are open sets with respect to the given topology on X. More precisely, it is enough to verify that for each $x \in X$ and $r > 0$, we have that x is an element of the interior of $B_d(x, r)$ with respect to the given topology on X. Here $B_d(x, r)$ is the open ball in X centered at x with radius $r > 0$ with respect to d, as in (1.23) in Section 1.2. This condition is basically the same as saying that for every $x \in X$, $d(x, y)$ is continous as a real-valued function of y on X at x with respect to the given topology on X.

Let A be a commutative topological group, and let $d(x, y)$ be a translation-invariant semimetric on A. In order to check that d is compatible with the given topology on A, it suffices to verify that $d(0, y)$ is continuous as a real-valued function of y on A at 0 with respect to the given topology on A.

Let X be any topological space again, and let $d(x, y)$ be a semimetric on X that is compatible with the given topology on X. Remember that $\overline{B}_d(x, r)$ denotes the closed ball in X centered at $x \in X$ with radius $r \geq 0$ with respect to d, as in (1.24) in Section 1.2. This is a closed set in X with respect to the topology determined by d, as mentioned previously, and hence $\overline{B}_d(x, r)$ is a closed set in X with respect to the given topology on X, because d is supposed to be compatible with the given topology on X. Of course, for each $x \in X$, we have that

$$\bigcup_{n=1}^{\infty} B_d(x, n) = X. \tag{2.115}$$

If μ is a finite nonnegative Borel measure on X, then for every $x \in X$ and $\epsilon > 0$, there is a positive integer n such that

$$\mu(X \setminus \overline{B}_d(x, n)) < \epsilon. \tag{2.116}$$

Now let $d_1, d_2, d_3, \ldots$ be a sequence of semimetrics on X, each of which is compatible with the given topology on X. If μ is a finite nonnegative Borel measure on X, j is a positive integer, $x_j \in X$, and $\epsilon_j > 0$, then there is a positive integer n_j such that

$$\mu(X \setminus \overline{B}_{d_j}(x_j, n_j)) < \epsilon_j, \tag{2.117}$$

as in (2.116). If we do this for every positive integer j, then we get that

$$\mu\Big(X \setminus \Big(\bigcap_{j=1}^{\infty} \overline{B}_{d_j}(x_j, n_j)\Big)\Big) = \mu\Big(\bigcup_{j=1}^{\infty}(X \setminus \overline{B}_{d_j}(x_j, n_j))\Big) \le \sum_{j=1}^{\infty} \mu(X \setminus \overline{B}_{d_j}(x_j, n_j)) \le \sum_{j=1}^{\infty} \epsilon_j. \tag{2.118}$$

Let V be a topological vector space over the real or complex numbers. Let us say that a seminorm N on V is *compatible with the given topology on* V if the semimetric d_N on V associated to N as in (1.45) in Section 1.5 is compatible with the given topology on V, as before. This happens when N is continuous at 0 with respect to the given topology on V, because d_N is invariant under translations on V, by construction. In this situation, one might typically apply the remarks in the previous two paragraphs to balls centered at 0 in V with respect to the semimetrics associated to compatible seminorms on V.

Let $U \subseteq V$ be any open set that contains 0, and remember that

$$\bigcup_{n=1}^{\infty} n\, U = V, \tag{2.119}$$

as in (1.146) in Section 1.17. If μ is a finite nonnegative Borel measure on V, then for every $\epsilon > 0$ there is a positive integer n such that

$$\mu(V \setminus n\, U) < \epsilon. \tag{2.120}$$

Similarly, if $U_1, U_2, U_3, \ldots$ is a sequence of open subsets of V that contain 0, and if $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ is a sequence of positive real numbers, then there is a sequence $n_1, n_2, n_3, \ldots$ of positive integers such that

$$\mu(V \setminus n_j\, U_j) < \epsilon_j \tag{2.121}$$

for every positive integer j. This implies that

$$\mu\Big(V \setminus \bigcap_{j=1}^{\infty} (n_j\, U_j)\Big) = \mu\Big(\bigcup_{j=1}^{\infty} (V \setminus (n_j\, U_j))\Big) \le \sum_{j=1}^{\infty} \mu(V \setminus (n_j\, U_j)) \le \sum_{j=1}^{\infty} \epsilon_j. \tag{2.122}$$

Of course, if U is a bounded set in V, then $t\, U$ is bounded in V for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. Suppose now that $U_1, U_2, U_3, \ldots$ is a sequence of open sets in V that contain 0 and form a local base for the topology of V at 0. If $\{t_j\}_{j=1}^{\infty}$ is a sequence of real or complex numbers, as appropriate, then it is easy to see that $\bigcap_{j=1}^{\infty} t_j\, U_j$ is bounded in V.

2.17 Products of complex measures

Let X and Y be sets, and let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be σ-algebras of measurable subsets of X and Y, respectively. Also let $\mathcal{A}_{X\times Y}$ be the corresponding σ-algebra of measurable subsets of $X \times Y$, as in Section 2.12. Suppose that μ_X and μ_Y are complex measures on $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$, respectively. Under these conditions, one can get a complex measure $\mu_X \times \mu_Y$ on $(X \times Y, \mathcal{A}_{X\times Y})$ such that

$$(\mu_X \times \mu_Y)(A \times B) = \mu_X(A)\, \mu_Y(B) \tag{2.123}$$

for every $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. To see this, remember that μ_X and μ_Y can be expressed as linear combinations of finite nonnegative measures on $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$, respectively, as in Section 2.8. The products of these finite nonnegative measures on X and Y can be defined as finite nonnegative measures on $X \times Y$ in the usual way, as mentioned in Section 2.13. Thus $\mu_X \times \mu_Y$ can be obtained by combining these products of finite nonnegative measures in a simple way.

If μ_X and μ_Y are real measures on X and Y, respectively, then they can be expressed as differences of finite nonnegative measures on these spaces, as in Section 2.8 again. In this case, one can obtain $\mu_X \times \mu_Y$ from the products of these finite nonnegative measures a bit more easily. Note that $\mu_X \times \mu_Y$ is a real measure on $X \times Y$ in this situation.

As in Section 2.13, $\mu_X \times \mu_Y$ is uniquely determined as a complex measure on $(X \times Y, \mathcal{A}_{X\times Y})$ by (2.123). More precisely, let $\mathcal{E}_{X\times Y}$ be the corresponding algebra of elementary subsets of $X \times Y$, as in Section 2.12. As before, $\mu_X \times \mu_Y$ is clearly uniquely determined on $\mathcal{E}_{X\times Y}$ by (2.123) and finite additivity. This implies that $\mu_X \times \mu_Y$ is uniquely determined as a complex measure on $\mathcal{A}_{X\times Y}$, as in Section 2.12. This uses the fact that $\mathcal{A}_{X\times Y}$ is the same as the smallest monotone class of subsets of $X \times Y$ that contains $\mathcal{E}_{X\times Y}$.

Let $|\mu_X|$ and $|\mu_Y|$ be the total variation measures on $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ that correspond to μ_X and μ_Y as in Section 2.8, respectively. Thus $|\mu_X|$ and $|\mu_Y|$ are finite nonnegative measures on $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$, respectively, so that the corresponding product measure $|\mu_X| \times |\mu_Y|$ is defined as a finite nonnegative measure on $(X \times Y, \mathcal{A}_{X\times Y})$. Observe that

$$\begin{aligned} |(\mu_X \times \mu_Y)(A \times B)| &= |\mu_X(A)|\, |\mu_Y(B)| \\ &\le |\mu_X|(A)\, |\mu_Y|(B) = (|\mu_X| \times |\mu_Y|)(A \times B) \end{aligned} \tag{2.124}$$

for every $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. This uses (2.123) in the first step, and its analogue for $|\mu_X| \times |\mu_Y|$ in the third step. The second step uses (2.59) in Section 2.8, applied to both μ_X and μ_Y. Using (2.124), one can check that

$$|(\mu_X \times \mu_Y)(E)| \le (|\mu_X| \times |\mu_Y|)(E) \tag{2.125}$$

when E is an element of the algebra $\mathcal{E}_{X\times Y}$ of elementary subsets of $X \times Y$. This implies that (2.125) holds for every $E \in \mathcal{A}_{X\times Y}$, as in Section 2.12, because $\mathcal{A}_{X\times Y}$ is the smallest monotone class of subsets of $X \times Y$ that contains $\mathcal{E}_{X\times Y}$. It follows that

$$|\mu_X \times \mu_Y|(E) \le (|\mu_X| \times |\mu_Y|)(E) \tag{2.126}$$

for every $E \in \mathcal{A}_{X \times Y}$, where $|\mu_X \times \mu_Y|$ is the total variation measure associated to $\mu_X \times \mu_Y$ as in Section 2.8.

Of course,

$$(|\mu_X| \times |\mu_Y|)(X \times Y) = |\mu_X|(X)\,|\mu_Y|(Y), \tag{2.127}$$

as in (2.123). Let $A_1, A_2, A_3, \ldots$ be a sequence of pairwise-disjoint elements of $\mathcal{A}_X$ such that $\bigcup_{j=1}^{\infty} A_j = X$, and let $B_1, B_2, B_3, \ldots$ be a sequence of pairwise-disjoint elements of $\mathcal{A}_Y$ such that $\bigcup_{l=1}^{\infty} B_l = Y$. Observe that

$$\begin{aligned}\Big(\sum_{j=1}^{\infty} |\mu_X(A_j)|\Big)\Big(\sum_{l=1}^{\infty} |\mu_Y(B_l)|\Big) &= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} |\mu_X(A_j)|\,|\mu_Y(B_l)| \\ &= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} |(\mu_X \times \mu_Y)(A_j \times B_l)|, \qquad (2.128)\end{aligned}$$

using (2.123) in the second step. We also have that

$$\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} |(\mu_X \times \mu_Y)(A_j \times B_l)| \le |\mu_X \times \mu_Y|(X \times Y), \tag{2.129}$$

by the definition of the total variation measure $|\mu_X \times \mu_Y|$. This uses the fact that the measurable rectangles $A_j \times B_l$ with $j, l \ge 1$ form a countable partition of $X \times Y$. Thus

$$\Big(\sum_{j=1}^{\infty} |\mu_X(A_j)|\Big)\Big(\sum_{l=1}^{\infty} |\mu_Y(B_l)|\Big) \le |\mu_X \times \mu_Y|(X \times Y), \tag{2.130}$$

by combining (2.128) and (2.129). This implies that

$$|\mu_X|(X)\,|\mu_Y|(Y) \le |\mu_X \times \mu_Y|(X \times Y), \tag{2.131}$$

by taking the supremum over all such sequences of A_j's and B_l's, and using the definition of total variation measures. Equivalently, this means that

$$(|\mu_X| \times |\mu_Y|)(X \times Y) \le |\mu_X \times \mu_Y|(X \times Y), \tag{2.132}$$

by (2.127). Using this and (2.126), one can verify that

$$|\mu_X \times \mu_Y|(E) = (|\mu_X| \times |\mu_Y|)(E) \tag{2.133}$$

for every $E \in \mathcal{A}_{X \times Y}$. Otherwise, if there is a strict inequality in (2.126) for any $E \in \mathcal{A}_{X \times Y}$, then one would get a strict inequality for $E = X \times Y$, which is not possible, by (2.132).

Alternatively, μ_X and μ_Y can be expressed as in (2.66) in Section 2.8. Using this, one can obtain $\mu_X \times \mu_Y$ from $|\mu_X| \times |mu_Y|$ in a similar way. This also gives another way to look at (2.133). In the real case, one can use Hahn decompositions for μ_X and μ_Y to get a Hahn decomposition for $\mu_X \times \mu_Y$ in a simple way. This gives another way to look at the total variation measures again.

Now let X, Y be locally compact Hausdorff topological spaces, so that $X \times Y$ is a locally compact Hausdorff topological space with respect to the product topology. As in Section 2.9, real or complex Borel measures on X and Y with suitable regularity properties correspond to continuous linear functionals on the spaces of continuous real or complex-valued functions on X and Y that vanish at infinity, as appropriate. In this situation, a product Borel measure on $X \times Y$ can be obtained from a corresponding product continuous linear functional on the space of continuous real or complex-valued functions on $X \times Y$ that vanish at infinity, as appropriate.

2.18 Hilbert spaces

Let V be a vector space over the real or complex numbers. An *inner product* on V is a real or complex-valued function $\langle v, w \rangle$, as appropriate, defined for $v, w \in V$, with the following three properties. First, for each $w \in V$, $\langle v, w \rangle$ should be a linear functional on V as a function of v. Second, we should have

$$\langle w, v \rangle = \langle v, w \rangle \tag{2.134}$$

for every $v, w \in V$ in the real case, and

$$\langle w, v \rangle = \overline{\langle v, w \rangle} \tag{2.135}$$

for every $v, w \in V$ in the complex case. Here $\overline{a}$ denotes the complex-conjugate of $a \in \mathbf{C}$, as usual. In the real case, (2.134) implies that for each $v \in V$, $\langle v, w \rangle$ defines a linear functional on V as a function of w. Similarly, in the complex case, (2.135) implies that for each $v \in V$, $\langle v, w \rangle$ is conjugate-linear as a function of w on V. Note that $\langle v, v \rangle$ is a real number for every $v \in V$ in the complex case, because of (2.135). The third condition is that

$$\langle v, v \rangle > 0 \tag{2.136}$$

for every $v \in V$ with $v \neq 0$. Of course, $\langle v, w \rangle = 0$ when $v = 0$ or $w = 0$, by the linear properties of the inner product.

Let $\langle v, w \rangle$ be an inner product on V, and put

$$\|v\| = \langle v, v \rangle^{1/2} \tag{2.137}$$

for every $v \in V$, using the nonnegative square root on the right side. It is easy to see that

$$\|t\, v\| = |t|\, \|v\| \tag{2.138}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, using the linearity properties of the inner product mentioned in the preceding paragraph. It is well known that

$$|\langle v, w \rangle| \leq \|v\|\, \|w\| \tag{2.139}$$

for every $v, w \in V$, which is the *Cauchy–Schwarz inequality.* This implies that

$$\|v + w\| \le \|v\| + \|w\| \tag{2.140}$$

for every $v, w \in V$, by a standard computation, so that (2.137) defines a norm on V as a vector space over $\mathbf{R}$ or $\mathbf{C}$, as appropriate. If V is complete with respect to the metric associated to this norm, then V is said to be a *Hilbert space.*

Let X be a nonempty set, and let f, g be a real or complex-valued functions that are 2-summable on X in the sense of Section 2.2, which is the same as saying that f, g are square-summable on X. If a, b are nonnegative real numbers, then it is well known that

$$a\, b \le (1/2)\, (a^2 + b^2), \tag{2.141}$$

because $(a - b)^2 \ge 0$. This implies that

$$|f(x)|\, |g(x)| \le (1/2)\, (|f(x)|^2 + |g(x)|^2) \tag{2.142}$$

for every $x \in X$, and it follows that $|f(x)|\, |g(x)|$ is summable on X. Put

$$\langle f, g \rangle = \langle f, g \rangle_{\ell^2(X, \mathbf{R})} = \sum_{x \in X} f(x)\, g(x) \tag{2.143}$$

in the real case, and

$$\langle f, g \rangle = \langle f, g \rangle_{\ell^2(X, \mathbf{C})} = \sum_{x \in X} f(x)\, \overline{g(x)} \tag{2.144}$$

in the complex case, where the sums are defined as in Section 2.1. It is easy to see that (2.143) and (2.143) define inner products on $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$, respectively. The corresponding norms on these spaces are the ℓ^2 norms discussed in Section 2.2. It is well known that these spaces are complete with respect to the metrics associated to the ℓ^2 norms, as before, so that $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ are Hilbert spaces.

Let $(X, \mathcal{A}, \mu)$ be a measure space, so that X is a set, $\mathcal{A}$ is a σ-algebra of measurable subsets of X, and μ is a nonnegative measure on $(X, \mathcal{A})$. Let us use $L^2(X, \mathbf{R})$ and $L^2(X, \mathbf{C})$ for the corresponding spaces of real and complex-valued square-integrable functions on X, respectively. If f, g are real or complex-valued square-integrable functions on X, then it is easy to see that $|f(x)|\, |g(x)|$ is integrable on X, using (2.141). As before, we put

$$\langle f, g \rangle = \langle f, g \rangle_{L^2(X, \mathbf{R})} = \int_X f(x)\, g(x)\, d\mu(x) \tag{2.145}$$

in the real case, and

$$\langle f, g \rangle = \langle f, g \rangle_{L^2(X, \mathbf{C})} = \int_X f(x)\, \overline{g(x)}\, d\mu(x) \tag{2.146}$$

in the complex case. These define inner products on $L^2(X, \mathbf{R})$ and $L^2(X, \mathbf{C})$, respectively, for which the corresponding norms are the usual L^2 norms,

$$\|f\|_2 = \Big(\int_X |f(x)|^2 \, d\mu(x) \Big)^{1/2}. \tag{2.147}$$

It is well known that $L^2(X, \mathbf{R})$ and $L^2(X, \mathbf{C})$ are complete with respect to the metrics associated to the L^2 norms, so that these spaces are Hilbert spaces. Of course, ℓ^2 spaces as in the previous paragraph may be considered as L^2 spaces with respect to counting measure.

2.19 Orthogonal vectors

Let V be a vector space over the real or complex numbers with an inner product $\langle v, w \rangle$, and let $\|v\|$ be the corresponding norm on V. As usual, two vectors $v, w \in V$ are said to be *orthogonal* when $\langle v, w \rangle = 0$. Similarly, a collection of vectors in V is said to be *orthogonal* if any two distinct vectors in the collection are orthogonal. If $v_1, \ldots, v_n$ are finitely many orthogonal vectors in V, then

$$\Big\| \sum_{j=1}^n v_j \Big\|^2 = \sum_{j=1}^n \|v_j\|^2, \tag{2.148}$$

by a standard computation. An orthogonal collection of vectors in V is said to be *orthonormal* if every vector in the collection has norm 1.

Suppose that $v_1, \ldots, v_n$ are finitely many orthonormal vectors in V. If $t_1, \ldots, t_n$ are real or complex numbers, as appropriate, then $t_1 \, v_1, \ldots, t_n \, v_n$ are orthogonal in V, and hence

$$\Big\| \sum_{j=1}^n t_j \, v_j \Big\|^2 = \sum_{j=1}^n |t_j|^2, \tag{2.149}$$

as in (2.148). Let $v \in V$ be given, and put

$$w = \sum_{j=1}^n \langle v, v_j \rangle \, v_j. \tag{2.150}$$

It is easy to see that $v - w$ is orthogonal to v_j for each $j = 1, \ldots, n$, by construction. This implies that $v - w$ is orthogonal to every element of the linear span of $v_1, \ldots, v_n$ in V. In particular, $v - w$ is orthogonal to w, so that

$$\|v\|^2 = \|v - w\|^2 + \|w\|^2 = \|v - w\|^2 + \sum_{j=1}^n |\langle v, v_j \rangle|^2 \geq \sum_{j=1}^n |\langle v, v_j \rangle|^2, \tag{2.151}$$

using (2.149) in the second step. If u is any element of the linear span of $v_1, \ldots, v_n$ in V, then $w - u$ is an element of the linear span of $v_1, \ldots, v_n$ too. Hence $v - w$ is orthogonal to $w - u$, so that

$$\|v - u\|^2 = \|v - w\|^2 + \|u - w\|^2 \geq \|v - w\|^2, \tag{2.152}$$

as in (2.148).

Let I be a nonempty set, and let $\{v_j\}_{j \in I}$ be an orthonormal family of vectors in V, indexed by I. If $v \in V$ and E is a nonempty finite subset of I, then

$$\sum_{j \in E} |\langle v, v_j \rangle|^2 \le \|v\|^2, \tag{2.153}$$

by (2.151). This implies that

$$\sum_{j \in I} |\langle v, v_j \rangle|^2 \le \|v\|^2, \tag{2.154}$$

where the sum on the left is defined as in Section 2.1. In particular, $\langle v, v_j \rangle$ is square-summable as a real or complex-valued function of j on I, as appropriate.

Let $v \in V$ be given, let E be a nonempty finite subset of I, and let t_j be a real or complex number, as appropriate, for each $j \in E$. Put

$$u = \sum_{j \in E} t_j \, v_j \tag{2.155}$$

and

$$w = \sum_{j \in E} \langle v, v_j \rangle \, v_j. \tag{2.156}$$

Using (2.152), we get that

$$\Big\| v - \sum_{j \in E} \langle v, v_j \rangle v_j \Big\| \le \Big\| v - \sum_{j \in E} t_j \, v_j \Big\|. \tag{2.157}$$

Similarly, (2.151) implies that

$$\|v\|^2 = \Big\| v - \sum_{j \in E} \langle v, v_j \rangle v_j \Big\|^2 + \sum_{j \in E} |\langle v, v_j \rangle|^2. \tag{2.158}$$

Suppose for the moment that $v \in V$ is an element of the closure of the linear span of the v_j's, $j \in I$, in V, with respect to the metric associated to the norm. This means that v can be approximated by finite sums of the form (2.155) with respect to $\| \cdot \|$. It follows that v can be approximated by finite sums of the form (2.156), because of (2.157). Under these conditions, we get that

$$\|v\|^2 = \sum_{j \in I} |\langle v, v_j \rangle|^2, \tag{2.159}$$

using (2.154) and (2.158). If the linear span of the v_j's, $j \in I$, is dense in V with respect to the metric associated to the norm, then it follows that (2.159) holds for every $v \in V$.

If f is a real or complex-valued function on I, as appropriate, with finite support, then

$$T(f) = \sum_{j \in I} f(j) \, v_j \tag{2.160}$$

defines an element of V. Let us use $c_{00}(I)$ to denote $c_{00}(I, \mathbf{R})$ or $c_{00}(I, \mathbf{C})$, depending on whether V is real or complex. Thus (2.160) defines a linear mapping from $c_{00}(I)$ into V. Observe that

$$\|T(f)\| = \|f\|_2 \tag{2.161}$$

for every $f \in c_{00}(I)$, as in (2.148), where $\|f\|_2$ is the ℓ^2 norm of f on I, as in Section 2.2. Similarly, if $g \in c_{00}(I)$ too, then one can verify that

$$\langle T(f), T(g)\rangle = \langle f, g\rangle_{\ell^2(I)}, \tag{2.162}$$

where $\langle f, g\rangle_{\ell^2(I)}$ is as in (2.143) or (2.144) in the previous section, as appropriate.

Let us also use $\ell^2(I)$ to denote $\ell^2(I, \mathbf{R})$ or $\ell^2(I, \mathbf{C})$, depending on whether V is real or complex. Remember that $c_{00}(I)$ is dense in $\ell^2(I)$, as in Section 2.2. If V is a Hilbert space, then there is a unique extension of T to a bounded linear mapping from $\ell^2(I)$ into V, by standard arguments. It is easy to see that this extension also satisfies (2.161) and (2.162) for every $f, g \in \ell^2(I)$. This extension may be used to define the sum on the right side of (2.160) as an element of V when $f \in \ell^2(I)$.

Alternatively, one can first define the sum on the right side of (2.160) as an element of V when $f \in \ell^2(I)$, and use this to extend T to $\ell^2(I)$. Of course, if I has only finitely many elements, then these sums are already defined. If $I = \mathbf{Z}_+$, then these sums may be treated as infinite series. One can show that the corresponding sequences of partial sums are Cauchy sequences in V under these conditions, which converge in V when V is complete. If f is square summable on any infinite set I, then f vanishes at infinity on I, as in Section 2.2. This implies that the support of f in I has only finitely or countably many elements, as in Section 2.1. This permits one to define the sum on the right side of (2.160) using finite sums or infinite series.

Of course, T maps $c_{00}(I)$ onto the linear span of the v_j's, $j \in I$, in V. If V is a Hilbert space, then the extension of T to $\ell^2(I)$ maps $\ell^2(I)$ onto the closure of the linear span of the v_j's, $j \in I$, in V. In particular, if $v \in V$ is an element of the closure of the linear span of the v_j's, $j \in I$, in V, then

$$v = \sum_{j \in I} \langle v, v_j\rangle\, v_j, \tag{2.163}$$

where the sum on the right can be defined as in the previous paragraphs. In this case, one can make sense of the sum on the right as a limit of finite subsums that converges to v more directly, as before. If V is a Hilbert space, and if the linear span of the v_j's, $j \in I$, is dense in V, then $\{v_j\}_{j \in I}$ is said to be an *orthonormal basis* in V.

Chapter 3

The Fourier transform

3.1 Complex Borel measures

Let A be a commutative topological group. The *Fourier transform* of a complex Borel measure μ on A is the complex-valued function $\widehat{\mu}$ defined on the dual group $\widehat{A}$ by

$$\widehat{\mu}(\phi) = \int_A \overline{\phi(x)}\, d\mu(x) \tag{3.1}$$

for every $\phi \in \widehat{A}$, where $\overline{\phi(x)}$ is the complex-conjugate of $\phi(x)$. Observe that

$$|\widehat{\mu}(\phi)| \le \int_A |\phi|\, d|\mu| = |\mu|(A) \tag{3.2}$$

for every $\phi \in \widehat{A}$, where $|\mu|$ is the total variation measure associated to μ on A. Similarly, if $\phi, \psi \in \widehat{A}$, then

$$|\widehat{\mu}(\phi) - \widehat{\mu}(\psi)| = \left| \int_A (\phi - \psi)\, d\mu \right| \le \int_A |\phi - \psi|\, d|\mu|. \tag{3.3}$$

Note that the right side of (3.3) is invariant under translations on $\widehat{A}$, which corresponds to multiplying ϕ and ψ by some other element of $\widehat{A}$.

If there is a nonempty compact set $K \subseteq A$ such that $|\mu|(A \setminus K) = 0$, then (3.3) implies that

$$|\widehat{\mu}(\phi) - \widehat{\mu}(\psi)| \le \int_K |\phi - \psi|\, d|\mu| \le |\mu|(A) \sup_{x \in K} |\phi(x) - \psi(x)| \tag{3.4}$$

for every $\phi, \psi \in \widehat{A}$. Suppose instead that for each $\epsilon > 0$ there is a nonempty compact set $K(\epsilon) \subseteq A$ such that

$$|\mu|(A \setminus K(\epsilon)) < \epsilon. \tag{3.5}$$

In this case, we can use (3.3) to get that

$$(3.6)\qquad \begin{aligned} |\widehat{\mu}(\phi) - \widehat{\mu}(\psi)| &\le \int_{K(\epsilon)} |\phi - \psi|\, d|\mu| + \int_{A\setminus K(\epsilon)} |\phi - \psi|\, d|\mu| \\ &< |\mu|(A) \sup_{x \in K(\epsilon)} |\phi(x) - \psi(x)| + 2\,\epsilon \end{aligned}$$

for every $\phi, \psi \in \widehat{A}$. As before, the right sides of (3.4) and (3.6) are invariant under translations on $\widehat{A}$. Using this, one can check that $\widehat{\mu}$ is uniformly continuous on $\widehat{A}$ as a commutative topological group with respect to the topology defined in Section 1.12 under these conditions, as in Section 1.14.

Let a be an element of A, and put

$$(3.7)\qquad \mu_a(E) = \mu(E - a)$$

for every Borel set $E \subseteq A$. It is easy to see that this defines a complex Borel measure on A. The total variation measure $|\mu_a|$ associated to μ can be obtained by translating $|\mu|$ in the same way. If g is a bounded complex-valued Borel measurable function on A, then

$$(3.8)\qquad \int_A g(x)\, d\mu_a(x) = \int_A g(x + a)\, d\mu(x).$$

To see this, suppose first that $g(x)$ is the indicator function $\mathbf{1}_E(x)$ on X associated to a Borel set $E \subseteq A$, which is equal to 1 when $x \in E$ and to 0 when $x \in A \setminus E$. In this case, $g(x+a)$ is the same as the indicator function associated to $E - a$, so that (3.8) follows from (3.7). This implies that (3.8) holds when g is a Borel measurable simple function on A, and the same result for bounded Borel measurable functions on A can be obtained by approximation by simple functions. If $\phi \in \widehat{A}$, then we get that

$$(3.9)\qquad \begin{aligned} \widehat{\mu_a}(\phi) = \int_A \overline{\phi(x)}\, d\mu_a(x) &= \int_A \overline{\phi(x + a)}\, d\mu(x) \\ &= \overline{\phi(a)} \int_A \overline{\phi(x)}\, d\mu(x) = \overline{\phi(a)}\, \widehat{\mu}(\phi). \end{aligned}$$

This uses (3.8) in the second step, and the fact that ϕ is a group homomorphism from A into $\mathbf{T}$ in the third step.

Similarly, let ν be the complex Borel measure on A defined by

$$(3.10)\qquad \nu(E) = \overline{\mu(-E)}$$

for every Borel set $E \subseteq A$. The total variation measure $|\nu|$ associated to ν is given by

$$(3.11)\qquad |\nu|(E) = |\mu|(-E)$$

for every Borel set $E \subseteq A$. If g is a bounded complex-valued Borel measurable function on A, then

$$(3.12)\qquad \int_A \overline{g(x)}\, d\nu(x) = \overline{\int_A g(-x)\, d\mu(x)}.$$

More precisely, this reduces to (3.10) when g is the indicator function of a Borel set $E \subseteq A$, in which case $g(-x)$ is the indicator function associated to $-E$. If g is a Borel measurable simple function on A, then (3.12) reduces to the case of indicator functions by linearity. It follows that (3.12) holds when g is a bounded complex-valued Borel measurable function on A, by approximation by simple functions. This implies that

$$\widehat{\nu}(\phi) = \int \overline{\phi(x)}\, d\nu(x) = \overline{\int_A \phi(-x)\, d\mu(x)} = \overline{\int_A \overline{\phi(x)}\, d\mu(x)} = \overline{\widehat{\mu}(\phi)} \tag{3.13}$$

for every $\phi \in \widehat{A}$. This uses the fact that ϕ is a group homomorphism from A into $\mathbf{T}$ in the third step, to get that $\phi(-x) = \phi(x)^{-1} = \overline{\phi(x)}$ for every $x \in A$.

Let δ_a be the Dirac measure on A associated to any $a \in A$, so that $\delta_a(E)$ is equal to 1 when $a \in E$ and to 0 when $a \notin E$. This may be considered as a Borel measure on A for each $a \in A$, and we have that

$$\widehat{\delta_a}(\phi) = \overline{\phi(a)} \tag{3.14}$$

for every $a \in A$ and $\phi \in \widehat{A}$.

3.2 Integrable functions

Let A be a locally compact commutative topological group, and let H be a Haar measure on A, as in Section 2.5. If f is a complex-valued Borel measurable function on A that is integrable with respect to H, then the *Fourier transform* of f is the function $\widehat{f}$ defined on $\widehat{A}$ by

$$\widehat{f}(\phi) = \int_A f(x)\, \overline{\phi(x)}\, dH(x) \tag{3.15}$$

for every $\phi \in \widehat{A}$. Remember that

$$\sigma_f(E) = \int_E f(x)\, dH(x) \tag{3.16}$$

defines a complex Borel measure on A under these conditions, for which the corresponding total variation measure is given by

$$|\sigma_f|(E) = \sigma_{|f|}(E) = \int_E |f(x)|\, dH(x), \tag{3.17}$$

as in Section 2.8. Thus (3.15) is the same as (3.1) applied to σ_f, and (3.2) corresponds to

$$|\widehat{f}(\phi)| \le \int_A |f(x)|\, dH(x) \tag{3.18}$$

for every $\phi \in \widehat{A}$ in this situation.

One can check that (3.17) is regular as a finite nonnegative Borel measure on A, using the regularity conditions for H mentioned in Section 2.5. More precisely, one can begin with the case where f is an integrable simple function on A, and deal with arbitrary integrable functions by approximation by simple functions. In particular, (3.17) satisfies (3.5). It follows that $\widehat{f}$ is uniformly continuous on $\widehat{A}$ as a commutative topological group with respect to the topology defined in Section 1.12, as in the previous section.

Let $a \in A$ be given, and put

$$f_a(x) = f(x - a), \tag{3.19}$$

as in Section 2.5. Remember that the integrability of f on A with respect to H implies that f_a is integrable on A with respect to H too, so that the Fourier transform of f_a can be defined as in (3.15). If $\phi \in \widehat{A}$, then we have that

$$\begin{aligned} \widehat{f_a}(\phi) = \int_A f(x-a)\,\overline{\phi(x)}\,dH(x) &= \int_A f(x)\,\overline{\phi(x+a)}\,dH(x) \\ &= \overline{\phi(a)} \int_A f(x)\,\overline{\phi(x)}\,dH(x) = \overline{\phi(a)}\,\widehat{f}(\phi). \end{aligned} \tag{3.20}$$

This uses translation-invariance of Haar measure in the second step, and the fact that ϕ is a group homomorphism from A into $\mathbf{T}$ in the third step. This can also be derived from (3.9), applied to (3.16). More precisely, if μ_{f_a} is the Borel measure on A that corresponds to f_a as in (3.16), then one can check that μ_{f_a} is the same as the measure obtained by translating σ_f by a, as in (3.7). This uses the invariance of Haar measure under translations.

Observe that

$$g(x) = \overline{f(-x)} \tag{3.21}$$

is also integrable on A with respect to H, because of (2.43) in Section 2.5. Thus the Fourier transform of g can be defined as in (3.15), and we have that

$$\widehat{g}(\phi) = \int_A \overline{f(-x)}\,\overline{\phi(x)}\,dH(x) = \overline{\int_A f(-x)\,\phi(x)\,dH(x)} \tag{3.22}$$

for every $\phi \in \widehat{A}$. It follows that

$$\widehat{g}(\phi) = \overline{\int_A f(x)\,\phi(-x)\,dH(x)} \tag{3.23}$$

for every $\phi \in \widehat{A}$, because H is invariant under $x \mapsto -x$, as in (2.43) in Section 2.5 again. Hence

$$\widehat{g}(\phi) = \overline{\int_A f(x)\,\overline{\phi(x)}\,dH(x)} = \overline{\widehat{f}(\phi)} \tag{3.24}$$

for every $\phi \in \widehat{A}$, because $\phi(-x) = \overline{\phi(x)}$ for every $x \in A$, as before. This can also be obtained from (3.13), applied to (3.16). In this case, the measure that corresponds to (3.16) as in (3.10) is the same as the measure that corresponds to g as in (3.16). This uses the invariance of H under $x \mapsto -x$, as in (2.43) in Section 2.5.

3.3 Compact groups

Let A be a compact commutative topological group, and let H be Haar measure on A, normalized so that $H(A) = 1$. If $\phi \in \widehat{A}$, then

$$\int_A \phi(x)\, dH(x) = \int_A \phi(x + a)\, dH(x) = \phi(a) \int_A \phi(x)\, dH(x) \tag{3.25}$$

for every $a \in A$, using translation-invariance of Haar measure in the first step. If $\phi(a) \neq 1$ for some $a \in A$, then it follows that

$$\int_A \phi\, dH = 0. \tag{3.26}$$

Let $L^2(A)$ be the usual space of complex-valued Borel measurable functions on A that are square-integrable with respect to H. If $f, g \in L^2(A)$, then we put

$$\langle f, g \rangle = \int_A f(x)\, \overline{g(x)}\, dH(x), \tag{3.27}$$

which is the usual integral inner product on $L^2(A)$. Let $\phi, \psi \in \widehat{A}$ be given, so that $\phi(x)\, \overline{\psi(x)}$ also defines an element of $\widehat{A}$, which is equal to 1 for every $x \in A$ exactly when $\phi = \psi$. If $\phi \neq \psi$ on A, then it follows that

$$\langle \phi, \psi \rangle = 0, \tag{3.28}$$

by (3.26) applied to $\phi(x)\, \overline{\psi(x)}$. The normalization $H(A) = 1$ implies that

$$\langle \phi, \phi \rangle = 1, \tag{3.29}$$

so that the elements of $\widehat{A}$ are orthonormal in $L^2(A)$ with respect to (3.27).

Let $f \in L^2(A)$ be given, and remember that f is integrable on A with respect to H, because $H(A)$ is finite. Thus the Fourier transform $\widehat{f}$ can be defined as in (3.15) in the previous section, and it can be reexpressed in terms of (3.27) as

$$\widehat{f}(\phi) = \langle f, \phi \rangle \tag{3.30}$$

for each $\phi \in \widehat{A}$. Using (3.30) and the orthonormality of the elements of $\widehat{A}$ in $L^2(A)$, we get that

$$\sum_{\phi \in \widehat{A}} |\widehat{f}(\phi)|^2 \leq \int_A |f|^2\, dH, \tag{3.31}$$

as in (2.154) in Section 2.19. In particular, $\widehat{f}$ is square-summable on $\widehat{A}$ when $f \in L^2(A)$.

It follows that $\widehat{f}$ vanishes at infinity on $\widehat{A}$ for every $f \in L^2(A)$, as in Section 2.2. In fact, $\widehat{f}$ vanishes at infinity on $\widehat{A}$ for every integrable function f on A with respect to H. This can be derived from the previous statement by approximating f by elements of $L^2(A)$ with respect to the L^1 norm, and using (3.18) in the previous section.

Let $\mathcal{E}$ be the linear span of $\widehat{A}$ in the space $C(A) = C(A, \mathbf{C})$ of all complex-valued continuous functions on A. It is easy to see that $\mathcal{E}$ is a subalgebra of $C(A)$ with respect to pointwise multiplication of functions, because $\widehat{A}$ is closed under multiplication of functions. Similarly, $\mathcal{E}$ is preserved by complex-conjugation, because $\overline{\phi} = 1/\phi \in \widehat{A}$ for every $\phi \in \widehat{A}$. Of course, the constant function on A equal to 1 at every point is an element of $\widehat{A}$, and hence of $\mathcal{E}$. It is well known that $\widehat{A}$ separates points in A when A is compact, so that $\mathcal{E}$ separates points in A too. Thus the theorem of Lebesgue, Stone, and Weierstrass implies that $\mathcal{E}$ is dense in $C(A)$ with respect to the supremum metric. It is also well known that $C(A)$ is dense in $L^2(A)$, because of the regularity properties of Haar measure on A. It follows that $\mathcal{E}$ is dense in $L^2(A)$, by combining the previous two statements.

This means that $\widehat{A}$ is an orthonormal basis for $L^2(A)$, as in Section 2.19. If $f \in L^2(A)$, then we get that

$$\sum_{\phi \in \widehat{A}} |\widehat{f}(\phi)|^2 = \int_A |f|^2 \, dH, \tag{3.32}$$

as in (2.159). We also have that

$$f = \sum_{\phi \in \widehat{A}} \widehat{f}(\phi) \, \phi, \tag{3.33}$$

as in (2.163), where the sum on the right converges in $L^2(A)$ in a suitable sense.

Let μ be a regular complex Borel measure on A, and suppose that

$$\widehat{\mu}(\phi) = 0 \tag{3.34}$$

for every $\phi \in \widehat{A}$. This implies that

$$\int_A g \, d\mu = 0 \tag{3.35}$$

for every $g \in \widehat{A}$, and hence for every $g \in \mathcal{E}$, by linearity. It follows that (3.35) holds for every $g \in C(A)$, because $\mathcal{E}$ is dense in $C(A)$ with respect to the supremum metric, as before. This means that $\mu = 0$ on A, because of regularity. Equivalently, one can look at this in terms of continuous linear functionals on $C(A)$, as in Section 2.9.

Now let f be a complex-valued Borel measurable function on A that is integrable with respect to H, and suppose that

$$\widehat{f}(\phi) = 0 \tag{3.36}$$

for every $\phi \in \widehat{A}$. If σ_f is the complex Borel measure on A corresponding to f as in (3.16) in the previous section, then (3.36) is the same as saying that (3.34) holds with $\mu = \sigma_f$. Remember that σ_f is regular, as mentioned in the previous section. It follows that $\sigma_f = 0$ on A, as in the preceding paragraph. This implies that $f = 0$ almost everywhere on A with respect to H. Alternatively, one can use (3.36) to get that

$$\int_A f\, g\, dH = 0 \tag{3.37}$$

for every $g \in \widehat{A}$, as in (3.35). This implies that (3.37) holds for every $g \in \mathcal{E}$, as before, and hence for every $g \in C(A)$. One can use this and standard approximation arguments to get that $f = 0$ almost everywhere with respect to H. Of course, these approximation arguments use the regularity properties of H on A.

Let B be a subgroup of $\widehat{A}$, and let $\mathcal{E}_B$ be the linear span of B in $C(A)$. This is a subalgebra of $C(A)$ that is invariant under complex-conjugation, as before. The constant function on A equal to 1 at every point is an element of B, since it is the identity element in $\widehat{A}$ as a group with respect to pointwise multiplication of functions on A. This implies that $\mathcal{E}_B$ contains the constant functions on A. If B separates points in A, then $\mathcal{E}_B$ separates points in A too. Under these conditions, $\mathcal{E}_B$ is dense in $C(A)$ with respect to the supremum metric, by the theorem of Lebesgue, Stone, and Weierstrass again. If $\phi \in \widehat{A} \setminus B$, then ϕ is orthogonal to every element of B, as in (3.28). This implies that

$$\langle \phi, g \rangle = 0 \tag{3.38}$$

for every $g \in \mathcal{E}_B$, by linearity. It follows that (3.38) holds for every continuous complex-valued function g on A, because $\mathcal{E}_B$ is dense in $C(A)$, as before. However, this does not work when $g = \phi$. This shows that

$$B = \widehat{A} \tag{3.39}$$

when B is a subgroup of $\widehat{A}$ that separates points in A, and A is compact.

3.4 Discrete groups

Let A be a commutative group, equipped with the discrete topology. Thus counting measure on A satisfies the requirements of Haar measure, so that integrability of complex-valued functions on A is the same as summability, as in Section 2.1. If f is a summable complex-valued function on A, then the expression (3.15) in Section 3.2 for the Fourier transform of f reduces to

$$\widehat{f}(\phi) = \sum_{x \in A} f(x)\, \overline{\phi(x)} \tag{3.40}$$

for each $\phi \in \widehat{A}$. The sum on the right is the sum of a summable function on A, because ϕ is bounded on A, and

$$|\widehat{f}(\phi)| \le \sum_{x \in A} |f(x)| \tag{3.41}$$

for each $\phi \in \widehat{A}$, as in (3.18) in Section 3.2. More precisely, (3.41) follows from (3.40) using (2.9) in Section 2.1 in this situation.

The summability of f on A implies that for each $\epsilon > 0$ there is a finite set $E(\epsilon) \subseteq A$ such that

$$\sum_{x \in A} |f(x)| < \sum_{x \in E(\epsilon)} |f(x)| + \epsilon, \tag{3.42}$$

because the sum on the left is defined to be the supremum of the corresponding finite subsums, as in Section 2.1. Equivalently, this means that

$$\sum_{x \in A \setminus E(\epsilon)} |f(x)| < \epsilon. \tag{3.43}$$

If $E \subseteq A$ is a finite set that contains $E(\epsilon)$, then we get that

$$\begin{aligned} \Big|\widehat{f}(\phi) - \sum_{x \in E} f(x)\,\overline{\phi(x)}\Big| &= \Big|\sum_{x \in A \setminus E} f(x)\,\overline{\phi(x)}\Big| \\ &\le \sum_{x \in A \setminus E} |f(x)| \le \sum_{x \in A \setminus E(\epsilon)} |f(x)| < \epsilon \end{aligned} \tag{3.44}$$

for every $\phi \in \widehat{A}$. This shows that the sum in (3.40) can be approximated by the finite subsums

$$\sum_{x \in E} f(x)\,\overline{\phi(x)} \tag{3.45}$$

uniformly over $\phi \in \widehat{A}$.

Let $x \in A$ be given, and put

$$\Psi_x(\phi) = \phi(x) \tag{3.46}$$

for each $\phi \in \widehat{A}$, which may be considered as a complex-valued function of ϕ on $\widehat{A}$. It is easy to see that Ψ_x is continuous with respect to the dual topology on $\widehat{A}$ discussed in Section 1.12, because $\{x\}$ is a compact subset of A. More precisely, Ψ_x is also a group homomorphism from $\widehat{A}$ into $\mathbf{T}$, and hence an element of the dual $\widehat{\widehat{A}}$ of $\widehat{A}$. In particular, Ψ_x is uniformly continuous on $\widehat{A}$ as a commutative topological group.

Observe that

$$\Psi_{-x}(\phi) = \phi(-x) = 1/\phi(x) = \overline{\phi(x)} \tag{3.47}$$

for every $x \in A$ and $\phi \in \widehat{A}$. If f is a complex-valued summable function on A, then (3.40) may be reexpressed as

$$\widehat{f}(\phi) = \sum_{x \in A} f(x)\,\Psi_{-x}(\phi) \tag{3.48}$$

for every $\phi \in \widehat{A}$, and the finite subsums (3.45) are the same as

$$\sum_{x \in E} f(x)\,\Psi_{-x}(\phi). \tag{3.49}$$

Thus (3.44) implies that $\widehat{f}$ can be approximated by finite sums of the form (3.49) uniformly on $\widehat{A}$. It follows that $\widehat{f}$ is uniformly continuous on $\widehat{A}$ as a commutative topological group, as in Section 1.14, because (3.46) is uniformly continuous on $\widehat{A}$ for each $x \in A$, as in the preceding paragraph. The uniform continuity of $\widehat{f}$ on $\widehat{A}$ can also be obtained as in Section 3.1, where (3.5) corresponds to (3.43) in this situation.

Similarly, we have that

$$\Psi_{x+y}(\phi) = \phi(x+y) = \phi(x)\,\phi(y) = \Psi_x(\phi)\,\Psi_y(\phi) \tag{3.50}$$

for every $x, y \in A$ and $\phi \in \widehat{A}$, so that

$$x \mapsto \Psi_x \tag{3.51}$$

defines a group homomorphism from A into $\widehat{\widehat{A}}$. In particular, it follows that

$$\{\Psi_x : x \in A\} \tag{3.52}$$

is a subgroup of $\widehat{\widehat{A}}$. Remember that $\widehat{A}$ is compact with respect to the dual topology when A is equipped with the discrete topology, as in Section 1.12. It is easy to see that (3.52) automatically separates points in $\widehat{A}$. Under these conditions, the remarks at the end of the previous section imply that (3.52) is equal to $\widehat{\widehat{A}}$. We also have that $\widehat{A}$ separates points in A when A is equipped with the discrete topology, as in Section 1.15. This implies that (3.51) is injective in this situation.

Let $H_{\widehat{A}}$ be Haar measure on $\widehat{A}$, normalized so that $H_{\widehat{A}}(\widehat{A}) = 1$. The elements of $\widehat{\widehat{A}}$ are orthonormal in $L^2(\widehat{A})$ with respect to the usual integral inner product corresponding to $H_{\widehat{A}}$, as in the previous section. If $x, y \in A$ and $x \neq y$, then $\Psi_x \neq \Psi_y$ as elements of $\widehat{\widehat{A}}$, by the injectivity of (3.51). Hence Ψ_x is orthogonal to Ψ_y in $L^2(\widehat{A})$ when $x \neq y$. If f is a complex-valued summable function on A, then one can use this to get that

$$f(y) = \int_{\widehat{A}} \widehat{f}(\phi)\,\overline{\Psi_{-y}(\phi)}\,dH_{\widehat{A}}(\phi) \tag{3.53}$$

for every $y \in A$. This also uses the fact that $\widehat{f}(\phi)$ can be approximated uniformly by the finite sums (3.49), as in (3.44), in order to interchange the order of integration and summation. Equivalently, this means that

$$f(y) = \int_{\widehat{A}} \widehat{f}(\phi)\,\Psi_y(\phi)\,dH_{\widehat{A}}(\phi) \tag{3.54}$$

for every $y \in A$, by (3.47).

Remember that summable functions on A are square-summable, as in Section 2.2. In this situation, we have that

$$\int_{\widehat{A}} |\widehat{f}(\phi)|^2\,dH_{\widehat{A}}(\phi) = \sum_{x \in A} |f(x)|^2, \tag{3.55}$$

because of the orthonormality of the Ψ_x's in $L^2(\widehat{A})$. This permits us to extend the Fourier transform to an isometric linear mapping from $\ell^2(A, \mathbf{C})$ into $L^2(\widehat{A})$, by standard arguments. More precisely, this mapping is surjective, because the Ψ_x's with $x \in A$ form an orthonormal basis for $L^2(\widehat{A})$. This follows from the fact that (3.52) separates points in $\widehat{A}$, as in the previous section.

3.5 Convolution of measures

Let A be a commutative topological group, and let μ, ν be complex Borel measures on A. Suppose that a product measure $\mu \times \nu$ can be defined as a complex Borel measure on $A \times A$ in a reasonable way, as in Sections 2.13 and 2.17. Let α be the mapping from $A \times A$ into A that corresponds to addition, so that

$$\alpha(x, y) = x + y \tag{3.56}$$

for every $x, y \in A$. Thus α is continuous with respect to the corresponding product topology on $A \times A$, by definition of a topological group, which implies that α is Borel measurable, as in Section 2.10. The *convolution* $\mu * \nu$ of μ and ν is defined as a complex Borel measure on A by putting

$$(\mu * \nu)(E) = (\mu \times \nu)(\alpha^{-1}(E)) \tag{3.57}$$

for every Borel set $E \subseteq A$. This is the same as the complex Borel on A obtained by pushing $\mu \times \nu$ forward from $A \times A$ to A using α, as in Section 2.11. If μ, ν are nonnegative real-valued Borel measures on A, then $\mu \times \nu$ is real-valued and nonnegative on $A \times A$, and hence (3.57) is real-valued and nonnegative on A. Note that $\alpha^{-1}(A) = A \times A$, so that

$$(\mu * \nu)(A) = (\mu \times \nu)(\alpha^{-1}(A)) = (\mu \times \nu)(A \times A) = \mu(A)\,\nu(A). \tag{3.58}$$

Let $|\mu|$ and $|\nu|$ be the total variation measures that correspond to μ and ν as in Section 2.8. Suppose that a product measure $|\mu| \times |\nu|$ can be defined as a nonnegative Borel measure on $A \times A$ in a reasonable way, as in Section 2.13. Remember that μ, ν can be expressed in terms of $|\mu|$, $|\nu|$ as in (2.66) in Section 2.8. Using this, $\mu \times \nu$ can be obtained from $|\mu| \times |\nu|$ by an analogous expression, as mentioned in Section 2.17. If $E \subseteq A$ is a Borel set, then

$$|(\mu * \nu)(E)| = |(\mu \times \nu)(\alpha^{-1}(E))| \le (|\mu| \times |\nu|)(\alpha^{-1}(E)) = (|\mu| * |\nu|)(E), \tag{3.59}$$

where $|\mu| * |\nu|$ is the convolution of $|\mu|$ and $|\nu|$ on A. This uses (2.125) in Section 2.17 in the second step. If $\mu \times \nu$ is expressed in terms of $|\mu| \times |\nu|$ as in (2.66) in Section 2.8, then this step can be derived from that. It follows that

$$|\mu * \nu|(E) \le (|\mu| * |\nu|)(E) \tag{3.60}$$

for every Borel set $E \subseteq A$, where $|\mu * \nu|$ is the total variation measure corresponding to $\mu * \nu$ on A. In particular,

$$|\mu * \nu|(A) \le (|\mu| * |\nu|)(A) = |\mu|(A)\,|\nu|(A), \tag{3.61}$$

using (3.58) in the second step.

It is easy to see that convolution of complex Borel measures is commutative, in the sense that

$$\mu * \nu = \nu * \mu \tag{3.62}$$

as complex Borel measures on A. This uses commutativity of addition on A, and "commutativity" in a natural sense of appropriate product measure constructions, up to suitable isomorphisms. Similarly, convolution of complex Borel measures is associative, because of associativity of addition on A, and natural "associativity" properties of appropriate product measure constructions, up to suitable isomorphisms.

If f is a bounded complex-valued Borel measurable function on A, then

$$\int_A f\, d(\mu * \nu) = \int_{A \times A} f \circ \alpha\, d(\mu \times \nu), \tag{3.63}$$

as in Section 2.11. If $\phi \in \widehat{A}$, then we get that

$$\begin{aligned}(\widehat{\mu * \nu})(\phi) = \int_A \overline{\phi}\, d(\mu * \nu) &= \int_{A \times A} \overline{\phi} \circ \alpha\, d(\mu \times \nu) \\ &= \int_A \int_A \overline{\phi(x+y)}\, d\mu(x)\, d\nu(y) \\ &= \int_A \int_A \overline{\phi(x)}\,\overline{\phi(y)}\, d\mu(x)\, d\nu(y) \\ &= \widehat{\mu}(\phi)\, \widehat{\nu}(\phi),\end{aligned} \tag{3.64}$$

using the appropriate version of Fubini's theorem.

Let K_1, K_2 be compact subsets of A, so that $K_1 \times K_2$ is a compact subset of $A \times A$, by Tychonoff's theorem. Observe that

$$(A \times A) \setminus (K_1 \times K_2) \subseteq ((A \setminus K_1) \times A) \cup (A \times (A \setminus K_2)), \tag{3.65}$$

which implies that

$$\begin{aligned}&(|\mu| \times |\nu|)((A \times A) \setminus (K_1 \times K_2)) \\ &\quad \le \quad (|\mu| \times |\nu|)((A \setminus K_1) \times A) + (|\mu| \times |\nu|)(A \times (A \setminus K_2)) \\ &\quad = \quad |\mu|(A \setminus K_1)\, |\nu|(A) + |\mu|(A)\, |\nu|(A \setminus K_2).\end{aligned} \tag{3.66}$$

Put

$$K_3 = \alpha(K_1 \times K_2) = K_1 + K_2, \tag{3.67}$$

which is a compact subset of A, because α is continuous. By construction,

$$K_1 \times K_2 \subseteq \alpha^{-1}(K_3), \tag{3.68}$$

and hence

$$\alpha^{-1}(A \setminus K_3) = \alpha^{-1}(A) \setminus \alpha^{-1}(K_3) \subseteq (A \times A) \setminus (K_1 \times K_2). \tag{3.69}$$

It follows that

$$
\begin{aligned}
|\mu * \nu|(A \setminus K_3) \le (|\mu| * |\nu|)(A \setminus K_3) &= (|\mu| \times |\nu|)(\alpha^{-1}(A \setminus K_3)) \\
&\le (|\mu| \times |\nu|)((A \times A) \setminus (K_1 \times K_2)).
\end{aligned} \tag{3.70}
$$

Let $E \subseteq A$ be a Borel set, and suppose that $K \subseteq A \times A$ is a compact set with $K \subseteq \alpha^{-1}(E)$. Thus $K_0 = \alpha(K)$ is a compact subset of A, because α is continuous, and $K \subseteq \alpha^{-1}(K_0)$. This implies that

$$\alpha^{-1}(E \setminus K_0) = \alpha^{-1}(E) \setminus \alpha^{-1}(K_0) \subseteq \alpha^{-1}(E) \setminus K, \tag{3.71}$$

so that

$$
\begin{aligned}
|\mu * \nu|(E \setminus K_0) \le (|\mu| * |\nu|)(E \setminus K_0) &= (|\mu| \times |\nu|)(\alpha^{-1}(E \setminus K_0)) \\
&\le (|\mu| \times |\nu|)(\alpha^{-1}(E) \setminus K).
\end{aligned} \tag{3.72}
$$

3.6 Measures and continuous functions

Let A be a commutative topological group. If ν is a nonnegative Borel measure on A, and f is a nonnegative real-valued Borel measurable function on A, then the *convolution* $f * \nu$ of f and ν can be defined as a nonnegative extended real-valued function on A by

$$(f * \nu)(x) = \int_A f(x - y)\, d\nu(y) \tag{3.73}$$

for every $x \in A$. In particular, if ν is a complex Borel measure on A, and f is a complex-valued Borel measurable function on A, then the convolution $|f| * |\nu|$ of $|f|$ and the total variation measure $|\nu|$ corresponding to ν can be defined as a nonnegative extended real-valued function on A by

$$(|f| * |\nu|)(x) = \int_A |f(x - y)|\, d|\nu|(y) \tag{3.74}$$

for every $x \in A$. In this case, if (3.74) is finite, then (3.73) can be defined as a complex number, and

$$|(f * \nu)(x)| \le (|f| * |\nu|)(x). \tag{3.75}$$

If f is bounded on A, then

$$(|f| * |\nu|)(x) \le \Big(\sup_{w \in A} |f(w)|\Big)\, |\nu|(A) \tag{3.76}$$

for every $x \in A$, so that (3.73) is defined and bounded on A.

Let ν be a complex Borel measure on A again, and let f be a complex-valued Borel measurable function on A such that (3.74) is finite for every $x \in A$. Thus (3.73) is defined as a complex number for every $x \in X$, and

$$(f * \nu)(x + a) - (f * \nu)(x) = \int_A (f(x + a - y) - f(x - y))\, d\nu(y) \tag{3.77}$$

for every $a, x \in A$. This implies that

$$\begin{aligned} |(f * \nu)(x + a) - (f * \nu)(x)| &\le \int_A |f(x + a - y) - f(x - y)| \, d|\nu|(y) \\ &\le \Big(\sup_{w \in A} |f(w + a) - f(w)|\Big) \, |\nu|(A) \qquad (3.78) \end{aligned}$$

for every $a, x \in A$. If f is uniformly continuous on A, then it follows that $f * \nu$ is uniformly continuous on A as well. If $\phi \in \widehat{A}$, then

$$(3.79) \quad (\phi * \nu)(x) = \int_A \phi(x - y) \, d\nu(y) = \int_A \phi(x) \, \overline{\phi(y)} \, d\nu(y) = \phi(x) \, \widehat{\nu}(\phi)$$

for every $x \in A$.

Suppose for the moment that there is a compact set $K \subseteq A$ such that $|\nu|(A \setminus K) = 0$. If f is a complex-valued Borel measurable function on A, then we get that

$$(3.80) \quad (|f| * |\nu|)(x) = \int_K |f(x - y)| \, d|\nu|(y) \le \Big(\sup_{y \in K} |f(x - y)|\Big) \, |\nu|(A)$$

for every $x \in A$, where the supremum on the right side is interpreted as being equal to 0 when $K = \emptyset$. If f is bounded on compact subsets of A, then the right side of (3.80) is finite for every $x \in A$. In fact, the right side of (3.80) is bounded on compact subsets of A in this case. This implies that (3.73) is defined for every $x \in A$, and that (3.73) is bounded on compact subsets of A.

In particular, if f is continuous on A, then f is bounded on compact subsets of A. Observe that

$$\begin{aligned} |(f * \nu)(x + a) - (f * \nu)(x)| &\le \int_K |f(x + a - y) - f(x - y)| \, d|\nu|(y) \\ &\le \Big(\sup_{y \in K} |f(x + a - y) - f(x - y)|\Big) \, |\nu|(A) \qquad (3.81) \end{aligned}$$

for every $a, x \in A$. Remember that continuous functions on A are uniformly continuous along compact subsets of A, as in Section 1.14. If f is continuous on A and $x \in A$, then it follows that the right side of (3.81) tends to 0 as $a \to 0$ in A. This implies that $f * \nu$ is continuous at x, by (3.81).

Suppose now that for each $\epsilon > 0$ there is a compact set $K(\epsilon) \subseteq A$ such that

$$(3.82) \qquad |\nu|(A \setminus K(\epsilon)) < \epsilon.$$

If f is a bounded complex-valued Borel measurable function on A, then

$$\begin{aligned} (3.83) \quad & |(f * \nu)(x + a) - (f * \nu)(x)| \\ &\le \int_{K(\epsilon)} |f(x + a - y) - f(x - y)| \, d|\nu|(y) \\ &\quad + \int_{A \setminus K(\epsilon)} |f(x + a - y) - f(x - y)| \, d|\nu|(y) \\ &\le \Big(\sup_{y \in K(\epsilon)} |f(x + a - y) - f(x - y)|\Big) \, |\nu|(A) + 2 \Big(\sup_{w \in A} |f(w)|\Big) \, \epsilon \end{aligned}$$

for every $a, x \in A$ and $\epsilon > 0$. Suppose that f is also continuous on A, so that f is uniformly continuous along compact subsets of A, as before. This implies that for each $x \in A$ and $\epsilon > 0$, the first term on the right side of (3.83) tends to 0 as $a \to 0$ in A. It follows that $f * \nu$ is continuous at every $x \in A$ when f is bounded and continuous on A, using (3.83).

Alternatively, if f is bounded and continuous on A, and ν is any complex Borel measure on A, then one can check that $f * \nu$ is sequentially continuous on A, using the dominated convergence theorem. This implies that $f * \nu$ is continuous on A when there is a local base for the topology of A at 0 with only finitely or countably many elements.

Remember that the support of a complex-valued function f on A is defined to be the closure in A of the set where $f \neq 0$. If $x \in A$ and

$$|\nu|(x - \operatorname{supp} f) = 0, \tag{3.84}$$

then one can check that $(|f| * |\nu|)(x) = 0$, so that $(f * \nu)(x)$ is defined and equal to 0. If $K \subseteq A$ is compact, then $K + \operatorname{supp} f$ is a closed set in A, as in Section 1.1, because $\operatorname{supp} f$ is a closed set in A, by definition. If $|\nu|(A \setminus K) = 0$, then (3.84) holds whenever

$$x - \operatorname{supp} f \subseteq A \setminus K, \tag{3.85}$$

which is the same as saying that $x \in A \setminus (K + \operatorname{supp} f)$. Under these conditions, we get that

$$\operatorname{supp}(f * \nu) \subseteq K + \operatorname{supp} f, \tag{3.86}$$

at least if $f * \nu$ is defined everywhere on A, so that its support is defined in the usual way.

Suppose for the moment that A is locally compact. If $K \subseteq A$ is compact, $|\nu|(A \setminus K) = 0$, and f has compact support in A, then (3.86) implies that $f * \nu$ has compact support in A as well. Suppose instead that for each $\epsilon > 0$ there is a compact set $K(\epsilon) \subseteq A$ that satisfies (3.82). If f is a bounded complex-valued Borel measurable function on A, then

$$\begin{aligned} |(f * \nu)(x)| &\le \int_{K(\epsilon)} |f(x-y)|\, d|\nu|(y) + \int_{A \setminus K(\epsilon)} |f(x-y)|\, d|\nu|(y) \\ &\le \Big(\sup_{y \in K(\epsilon)} |f(x-y)| \Big)\, |\nu|(A) + \Big(\sup_{w \in A} |f(w)| \Big)\, \epsilon \end{aligned} \tag{3.87}$$

for every $x \in A$ and $\epsilon > 0$. If f also vanishes at infinity on A, then one can use (3.87) to show that $f * \nu$ vanishes at infinity on A too.

Let f be a bounded continuous complex-valued function on A, and let μ be a complex Borel measure on A. Thus $f * \mu$ is defined and bounded on A, as before. Suppose that $f * \mu$ is continuous on A, as in various situations mentioned earlier. Let ν be another complex Borel measure on A, so that

$$\begin{aligned} ((f * \mu) * \nu)(x) &= \int_A (f * \mu)(x - z)\, d\nu(z) \\ &= \int_A \Big(\int_A f(x - w - z)\, d\mu(w) \Big)\, d\nu(z) \end{aligned} \tag{3.88}$$

for every $x \in A$. Suppose also that $\mu \times \nu$ can be defined in a reasonable way on $A \times A$, so that $\mu * \nu$ can be defined on A as in the previous section. It follows that $f * (\mu * \nu)$ can be defined on A in the usual way, with

$$(3.89) \qquad (f * (\mu * \nu))(x) = \int_A f(x - y)\, d(\mu * \nu)(y) = \int_{A \times A} f(x - w - z)\, d(\mu \times \nu)(w, z)$$

for each $x \in A$, where the second step is as in (3.63) in the previous section. Using the appropriate version of Fubini's theorem, we get that the right sides of (3.88) and (3.89) are the same.

Suppose now that there are compact sets $K_1, K_2 \subseteq A$ such that $|\mu|(A \setminus K_1) = |\nu|(A \setminus K_2) = 0$. This implies that

$$(3.90) \qquad (|\mu| \times |\nu|)((A \times A) \setminus (K_1 \times K_2)) = 0,$$

by (3.66) in the previous section. In this case, $K_3 = K_1 + K_2$ is compact, and $(|\mu| * |\nu|)(A \setminus K_3) = 0$, by (3.70). Under these conditions, the remarks in the preceding paragraph can be applied to continuous functions f on A, without asking that f be bounded on A. In particular, the relevant convolutions will be continuous on A, as before.

3.7 Convolution of integrable functions

Let A be a locally compact commutative topological group, and let H be a Haar measure on A. Also let f be a nonnegative real-valued Borel measurable function on A, and let ν be a finite nonnegative Borel measure on A. Thus $f * \nu$ can be defined as a nonnegative extended real-valued function on A, as in (3.73) in the previous section. Under suitable conditions, Fubini's theorem implies that

$$(3.91) \qquad \begin{aligned} \int_A (f * \nu)(x)\, dH(x) &= \int_A \int_A f(x - y)\, d\nu(y)\, dH(x) \\ &= \int_A \int_A f(x - y)\, dH(x)\, d\nu(y) \\ &= \int_A \int_A f(x)\, dH(x)\, d\nu(y) \\ &= \nu(A) \int_A f(x)\, dH(x), \end{aligned}$$

using also the translation-invariance of Haar measure in the third step. In particular, this is finite when f is integrable on A with respect to H.

Now let f be a complex-valued Borel measurable function on A that is integrable with respect to H, and let ν be a complex Borel measure on A.

Applying (3.91) to $|f|$ and $|\nu|$, we get that

$$\int_A (|f| * |\nu|)(x)\, dH(x) = |\nu|(A) \int_A |f(x)|\, dH(x), \tag{3.92}$$

so that $(|f| * |\nu|)(x) < \infty$ for almost every $x \in A$ with respect to H. This permits us to define $(f * \nu)(x)$ for almost every $x \in A$ with respect to H, as in the previous section. We also get that

$$\int_A |(f * \nu)(x)|\, dH(x) \le |\nu|(A) \int_A |f(x)|\, dH(x), \tag{3.93}$$

using (3.75) and (3.92), so that $f * \nu$ is integrable on A with respect to H.

Remember that

$$\sigma_f(E) = \int_E f(x)\, dH(x) \tag{3.94}$$

defines a complex Borel measure on A, as in Section 2.8, for which the corresponding total variation measure is given by $|\sigma_f| = \sigma_{|f|}$. As mentioned in Section 3.2, $\sigma_{|f|}$ is a regular Borel measure on A, because of the regularity properties of Haar measure on A. Under suitable conditions, $\sigma_f \times \nu$ can be defined as a complex Borel measure on $A \times A$, as in Section 2.13. This means that $\sigma_f * \nu$ can be defined as a complex Borel measure on A, as in Section 3.5. If h is a bounded complex-valued Borel measurable function on A, then

$$\begin{aligned} \int_A h\, d(\sigma_f * \nu) &= \int_A \int_A h(x+y)\, d\sigma_f(x)\, d\nu(y) \\ &= \int_A \int_A h(x+y)\, f(x)\, dH(x)\, d\nu(y) \\ &= \int_A \int_A h(x)\, f(x-y)\, dH(x)\, d\nu(y), \end{aligned} \tag{3.95}$$

using (3.63) in Section 3.5 in the first step, and translation-invariance of Haar measure in the second step. It follows that

$$\begin{aligned} \int_A h\, d(\sigma_f * \nu) &= \int_A \int_A h(x)\, f(x-y)\, d\nu(y)\, dH(x) \\ &= \int_A h(x)\, (f * \nu)(x)\, dH(x), \end{aligned} \tag{3.96}$$

by interchanging the order of integration in the first step, and using the definition of $f * \nu$ in the second step. Thus

$$\sigma_f * \nu = \sigma_{f*\nu} \tag{3.97}$$

as complex Borel measures on A, where $\sigma_{f*\nu}$ is defined as in (3.94).

If f, g are nonnegative Borel measurable functions on A, then their *convolution* is defined as a nonnegative extended real-valued function on A by

$$(f * g)(x) = \int_A f(x-y)\, g(y)\, dH(y) \tag{3.98}$$

for each $x \in A$. If f, g are complex-valued Borel measurable functions on A, and if

$$\int_A |f(x-y)|\,|g(y)|\,dH(y) < \infty, \tag{3.99}$$

then (3.98) is defined as a complex number. Suppose that g is integrable with respect to H, so that σ_g can be defined as a complex Borel measure on A as in (3.94), with total variation measure $|\sigma_g| = \sigma_{|g|}$, as before. In this case, (3.99) corresponds exactly to the finiteness (3.74) in the previous section with $\nu = \sigma_g$, and (3.98) corresponds exactly to (3.73).

3.8 Continuous homomorphisms

Let A and B be commutative topological groups, and let h be a continuous homomorphism from A into B. If μ is a complex Borel measure on A, then

$$\nu(E) = \mu(h^{-1}(E)) \tag{3.100}$$

defines a complex Borel measure on B, as in Section 2.11. Let $\phi \in \widehat{B}$ be given, so that $\widehat{h}(\phi) = \phi \circ h$ is in $\widehat{A}$, as in (1.113) in Section 1.12. Using the definition (3.1) of the Fourier transform of a measure in Section 3.1, we get that

$$\widehat{\nu}(\phi) = \int_B \overline{\phi}\,d\nu = \int_A \overline{\phi} \circ h\,d\mu = \widehat{\mu}(\widehat{h}(\phi)). \tag{3.101}$$

This also uses (2.87) in Section 2.11 in the second step.

As a basic class of examples, suppose that A is a subgroup of B, equipped with the induced topology. In this case, we can take $h : A \to B$ to be the obvious inclusion mapping, so that $h(x) = x$ for every $x \in X$. If E is a Borel set in B, then

$$h^{-1}(E) = A \cap E \tag{3.102}$$

is a Borel set in A with respect to the induced topology, because h is continuous with respect to the induced topology on A. If μ is a complex Borel measure on A, then (3.100) is the same as defining ν as a complex Borel measure on B by putting

$$\nu(E) = \mu(A \cap E) \tag{3.103}$$

for every Borel set E in B. Of course, $\widehat{h}(\phi)$ is the same as the restriction of $\phi \in \widehat{B}$ to A in this situation.

Now let A be any commutative topological group, let I be a nonempty set, and let B be $\mathbf{T}^I$, the Cartesian product of copies of $\mathbf{T}$ indexed by I. As in Section 1.8, $\mathbf{T}^I$ is a commutative topological group with respect to coordinatewise multiplication and the product topology. This is the same as the group $c(I, \mathbf{T})$ of $\mathbf{T}$-valued functions on I with respect to pointwise multiplication of functions, as in Section 1.9. A continuous homomorphism h from A into B corresponds exactly to a family of continuous homomorphisms h_j from A into $\mathbf{T}$ for each

$j \in I$. Equivalently, $h_j \in \widehat{A}$ for every $j \in I$. If $j_1, \ldots, j_n$ are finitely many elements of I, and $l_1, \ldots, l_n$ are the same number of integers, then

$$\phi(z) = z_{j_1}^{l_1} \cdots z_{j_n}^{l_n} \tag{3.104}$$

defines an element of $\widehat{B}$, where $z_j \in \mathbf{T}$ denotes the jth component of $z \in B = \mathbf{T}^I$ for each $j \in I$. Conversely, every element of $\widehat{B}$ is of this form, by the discussions of the dual of $\mathbf{T}$ and duals of Cartesian products in Section 1.11. If $\phi \in \widehat{B}$ is as in (3.104), then

$$\widehat{h}(\phi) = \phi \circ h = h_{j_1}^{l_1} \cdots h_{j_n}^{l_n}. \tag{3.105}$$

Let V be a topological vector space over the real numbers, so that V is a commutative topological group with respect to addition in particular. Also let h be a continuous linear mapping from V into $\mathbf{R}^n$ for some positive integer n, where $\mathbf{R}^n$ is equipped with its standard topology. Thus h is a continuous group homomorphism from V into $\mathbf{R}^n$ as commutative topological groups with respect to addition. Note that the jth coordinate h_j of h is a continuous linear functional on V for each $j = 1, \ldots, n$. Conversely, if $h_1, \ldots, h_n$ are continuous linear functionals on V, then they can be combined to get a continuous linear mapping from V into $\mathbf{R}^n$.

3.9 Some additional properties

Let A be a locally compact commutative topological group, and let H be a Haar measure on A. Suppose for the moment that $K \subseteq A$ is compact, $U \subseteq A$ is an open set, $0 \in U$, and $\overline{U}$ is compact. Under these conditions, $K + U$ is an open set in A and $K + \overline{U}$ is a compact subset of A, as in Section 1.1. In particular, $H(K + \overline{U}) < \infty$.

If f is a continuous real or complex-valued function on A, then f is uniformly continuous along compact subsets of A, as in Section 1.14. In particular, if f also has compact support in A, then one can use this to show that f is uniformly continuous on A. Similarly, if f is a continuous real or complex-valued continuous function on A that vanishes at infinity, then f is uniformly continuous on A. This can be derived from the fact that f is uniformly continuous along compact subsets of A, or by approximating f by continuous functions on A with compact support uniformly on A.

Now let f be a real or complex-valued Borel measurable function on A that is integrable with respect to Haar measure. Under these conditions, it is well known that

$$\lim_{a \to 0} \int_A |f(x) - f(x - a)| \, dH(x) = 0, \tag{3.106}$$

where more precisely the limit is taken as $a \in A$ tends to 0 with respect to the given topology on A. To see this, suppose first that f is a continuous function on A with compact support. In this case, (3.106) can be obtained from uniform continuity. This also uses the fact that we can restrict our attention to integrals over fixed compact subsets of A, by the earlier remarks. If f is an integrable function on A, then it is well known that f can be approximated by continuous functions on A with compact support with respect to the L^1 norm associated to H, because of the regularity properties of H on A. This permits us to get (3.106) from the analogous statement for continuous functions with compact support. Note that translation-invariance of Haar measure is used here, to ensure that translations of f are approximated by translations of approximations to f.

Let f be a complex-valued integrable function on A with respect to H, and put $f_a(x) = f(x - a)$ for every $a, x \in A$, as before. Remember that

$$\widehat{f_a}(\phi) = \overline{\phi(a)}\, \widehat{f}(\phi) \tag{3.107}$$

for every $a \in A$ and $\phi \in \widehat{A}$, as in (3.20) in Section 3.2. Thus

$$|1 - \phi(a)|\, |\widehat{f}(\phi)| = |1 - \overline{\phi(a)}|\, |\widehat{f}(\phi)| = |\widehat{f}(\phi) - \widehat{f_a}(\phi)| \tag{3.108}$$

for every $a \in A$ and $\phi \in \widehat{A}$. It follows that

$$|1 - \phi(a)|\, |\widehat{f}(\phi)| \le \int_A |f - f_a|\, dH \tag{3.109}$$

for every $a \in A$ and $\phi \in \widehat{A}$, using (3.18) in Section 3.2, and the linearity of the Fourier transform. It is well known that $\widehat{A}$ is locally compact with respect to the dual topology defined in Section 1.12 when A is locally compact. In this case, it is also well known that $\widehat{f}$ vanishes at infinity on $\widehat{A}$ when f is integrable on A with respect to H. We shall return to these matters in Section 4.4.

3.10 Some related continuity conditions

Let V be a topological vector space over the real numbers, so that V is also a commutative topological group with respect to addition. If λ is a continuous linear functional on V, then

$$\phi_\lambda(v) = \exp(i\, \lambda(v)) \tag{3.110}$$

defines a continuous group homomorphism from V as a commutative topological group with respect to addition into $\mathbf{T}$, as in Section 1.11. We have also seen that every continuous group homomorphism from V into $\mathbf{T}$ is of this form. If μ is a complex Borel measure on V, then the Fourier transform of μ can be expressed as

$$\widehat{\mu}(\phi_\lambda) = \int_V \exp(-i\, \lambda(v))\, d\mu(v) \tag{3.111}$$

for every $\lambda \in V'$. One may simply consider $\widehat{\mu}$ as a complex-valued function on V', which is sometimes expressed as $\widehat{\mu}(\lambda)$ for each $\lambda \in V'$.

It is well known that

$$|\exp(i\, t_1) - \exp(i\, t_2)| \le |t_1 - t_2| \tag{3.112}$$

for every $t_1, t_2 \in \mathbf{R}$. This can be derived from the fact that the derivative of $\exp(i\,t)$ is equal to $i\,\exp(i\,t)$, which has modulus equal to 1 for every $t \in \mathbf{R}$. If $\lambda_1, \lambda_2 \in V'$, then it follows that

$$|\phi_{\lambda_1}(v) - \phi_{\lambda_2}(v)| = |\exp(i\,\lambda_1(v)) - \exp(i\,\lambda_2(v))| \le |\lambda_1(v) - \lambda_2(v)| \tag{3.113}$$

for every $v \in V$.

Suppose for the moment that $E \subseteq V$ is a nonempty Borel set which is also bounded, in the sense of Section 1.17, and that

$$|\mu|(V \setminus E) = 0. \tag{3.114}$$

One can simply take E to be a closed set in V, since the closure of a bounded subset of V is bounded as well, as in Section 1.17. If $\lambda_1, \lambda_2 \in V'$, then

$$|\widehat{\mu}(\phi_{\lambda_1}) - \widehat{\mu}(\phi_{\lambda_2})| \le |\mu|(V) \sup_{v \in E} |\phi_{\lambda_1}(v) - \phi_{\lambda_2}(v)|, \tag{3.115}$$

as in (3.4) in Section 3.1. Combining this with (3.113), we get that

$$|\widehat{\mu}(\phi_{\lambda_1}) - \widehat{\mu}(\phi_{\lambda_2})| \le |\mu|(V) \sup_{v \in E} |\lambda_1(v) - \lambda_2(v)| \tag{3.116}$$

for every $\lambda_1, \lambda_2 \in V'$. It follows that $\widehat{\mu}(\phi_\lambda)$ is uniformly continuous as a function of $\lambda \in V'$ with respect to the topology defined on V' by the collection of supremum seminorms associated to nonempty bounded subsets of V, as in Section 1.18.

Similarly, let $\epsilon > 0$ be given, and suppose that $E(\epsilon) \subseteq V$ is a nonempty bounded Borel set such that

$$|\mu|(V \setminus E(\epsilon)) < \epsilon. \tag{3.117}$$

As before, one can simply take $E(\epsilon)$ to be a closed set in V, by replacing it with its closure in V. If $\lambda_1, \lambda_2 \in V'$, then

$$|\widehat{\mu}(\phi_{\lambda_1}) - \widehat{\mu}(\phi_{\lambda_2})| \le |\mu|(V) \sup_{v \in E(\epsilon)} |\phi_{\lambda_1}(v) - \phi_{\lambda_2}(v)| + 2\,\epsilon, \tag{3.118}$$

as in (3.6) in Section 3.1. This implies that

$$|\widehat{\mu}(\phi_{\lambda_1}) - \widehat{\mu}(\phi_{\lambda_2})| \le |\mu|(V) \sup_{v \in E(\epsilon)} |\lambda_1(v) - \lambda_2(v)| + 2\,\epsilon \tag{3.119}$$

for every $\lambda_1, \lambda_2 \in V'$, using (3.113), as before. If we can do this for every $\epsilon > 0$, then we get that $\widehat{\mu}(\phi_\lambda)$ is uniformly continuous as a function of $\lambda \in V'$ with respect to the topology defined on V' by the collection of supremum seminorms associated to nonempty bounded subsets of V again.

Of course, one might prefer to have compact sets E, $E(\epsilon)$ in V, as in Section 3.1.

3.11 Some additional continuity conditions

Let A be a commutative topological group, and let μ be a complex Borel measure on A. Suppose that $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of elements of $\widehat{A}$ that converges pointwise to another element ϕ of $\widehat{A}$ everywhere on A. Under these conditions, we have that

$$\lim_{j\to\infty} \widehat{\mu}(\phi_j) = \widehat{\mu}(\phi), \tag{3.120}$$

by the dominated convergence theorem. More precisely, the dominated convergence theorem implies that

$$\lim_{j\to\infty} \int_A |\phi_j - \phi|\, d|\mu| = 0, \tag{3.121}$$

and (3.120) follows from this, as in (3.3) in Section 3.1. In particular, (3.120) implies that $\widehat{\mu}$ is sequentially continuous with respect to the topology defined on $\widehat{A}$ in Section 1.12.

Let V be a topological vector space over the real or complex numbers, and let N be a seminorm on V that is compatible with the given topology on V, as in Section 2.16. Let us say that a linear functional λ on V is *bounded with respect to* N if there is a nonnegative real number C such that

$$|\lambda(v)| \le C\, N(v) \tag{3.122}$$

for every $v \in V$. This implies that λ is continuous at 0 on V, and hence everywhere on V, because N is supposed to be compatible with the given topology on V. Let V'_N be the collection of linear functionals on V that are bounded with respect to N, which one can check is a linear subspace of the dual space V'. Put

$$\|\lambda\|_{V'_N} = \sup\{|\lambda(v)| : v \in V,\, N(v) \le 1\} \tag{3.123}$$

for every $\lambda \in V'_N$, which is the same as the smallest $C \ge 0$ such that (3.122) holds. It is easy to see that this defines a norm on V'_N. If N is a norm on V, and the given topology on V is the same as the one determined by N, then $V'_N = V'$, and (3.123) is the same as the dual norm (1.151) in Section 1.18.

Let N be any compatible seminorm on V again, and let E be a nonempty bounded subset of V, as in Section 1.17. It is easy to see that N is bounded on E under these conditions. If $\lambda \in V'_N$, then

$$\sup_{v\in E} |\lambda(v)| \le \sup_{v\in E}(\|\lambda\|_{V'_N}\, N(v)) = \|\lambda\|_{V'_N} \sup_{v\in E} N(v). \tag{3.124}$$

This implies that the topology on V'_N determined by $\|\lambda\|_{V'_N}$ is at least as strong as the topology induced on V'_N by the topology determined on V' by the collection of supremum seminorms associated to nonempty bounded subsets of V, as in Section 1.18.

Suppose now that V is a topological vector space over the real numbers, and let N be a compatible seminorm on V again. If $v \in V$ and $r \ge 0$, then let $\overline{B}_N(v,r)$ be the closed ball in V centered at v with radius r with respect to the semimetric d_N associated to N as in (1.45) in Section 1.5. Remember that closed balls were defined in (1.24) in Section 1.2, so that $\overline{B}_N(0,r)$ consists of $w \in V$ such that $N(w) \le r$. Let μ be a complex Borel measure on V, and let $\lambda_1, \lambda_2 \in V'_N$ be given. If ϕ_λ is as in (3.110) in the previous section, then we have that

$$|\widehat{\mu}(\phi_{\lambda_1}) - \widehat{\mu}(\phi_{\lambda_2})| \le \int_V |\phi_{\lambda_1} - \phi_{\lambda_2}|\, d|\mu|, \tag{3.125}$$

as in (3.3) in Section 3.1. If r is any nonnegative real number, then we can split the integral over V into integrals over $\overline{B}_N(0,r)$ and $V \setminus \overline{B}_N(0,r)$, to get that

$$\begin{aligned} |\widehat{\mu}(\phi_{\lambda_1}) - \widehat{\mu}(\phi_{\lambda_2})| \quad \le \quad & |\mu|(\overline{B}_N(0,r)) \sup\{|\phi_{\lambda_1}(v) - \phi_{\lambda_2}(v)| : v \in V,\, N(v) \le r\} \\ & + 2\,|\mu|(V \setminus \overline{B}_N(0,r)). \end{aligned} \tag{3.126}$$

We also have that

$$|\phi_{\lambda_1}(v) - \phi_{\lambda_2(v)}| \le |\lambda_1(v) - \lambda_2(v)| \le \|\lambda_1 - \lambda_2\|_{V'_N}\, N(v) \tag{3.127}$$

for every $v \in V$, using (3.113) in the previous section in the first step, and the definition of $\|\lambda\|_{V'_N}$ in the second step. Combining this with (3.126), we get that

$$|\widehat{\mu}(\phi_{\lambda_1}) - \widehat{\mu}(\phi_{\lambda_2})| \le |\mu|(\overline{B}_N(0,r))\, r\, \|\lambda_1 - \lambda_2\|_{V'_N} + 2\,|\mu|(V \setminus \overline{B}(0,r)) \tag{3.128}$$

for every $r \ge 0$. Of course,

$$\lim_{r \to \infty} |\mu|(V \setminus \overline{B}(0,r)) = 0, \tag{3.129}$$

by standard arguments. Using (3.128) and (3.129), we get that $\widehat{\mu}(\phi_\lambda)$ is uniformly continuous as a function of $\lambda \in V'_N$ with respect to $\|\lambda\|_{V'_N}$.

Let $\lambda_1, \ldots, \lambda_n$ be finitely many continuous linear functionals on V, and put

$$N(v) = \max_{1 \le j \le n} |\lambda_j(v)| \tag{3.130}$$

for each $v \in V$. It is easy to see that this defines a seminorm on V, and that N is compatible with the given topology on V, because the λ_j's are continuous. By construction, λ_j is bounded on V with respect to N for each j, and linear combinations of the λ_j's are bounded on V with respect to N as well. Conversely, if λ is any linear functional on V that is bounded with respect to N, then λ can be expressed as a linear combination of the λ_j's. This uses the fact that the kernel of λ contains the intersection of the kernels of the λ_j's, as in Section 1.13. If $t \in \mathbf{R}^n$, then put

$$\lambda^t(v) = \sum_{j=1}^n t_j\, \lambda_j(v) \tag{3.131}$$

for each $v \in V$, so that λ^t is a linear functional on V that is bounded with respect to N. Let μ be a complex Borel measure on V, and let ϕ_λ be as in (3.110) again. The remarks in the previous paragraph imply that $\widehat{\mu}(\phi_{\lambda^t})$ is uniformly continuous as a function of t with respect to the standard metric on $\mathbf{R}^n$.

Chapter 4

Equicontinuity and related topics

4.1 Equicontinuity

Let X be a topological space. A collection $\mathcal{E}$ of complex-valued functions on X is said to be *equicontinuous* at a point $x \in X$ if for every $\epsilon > 0$ there is an open set $U(x, \epsilon) \subseteq X$ such that $x \in U(x, \epsilon)$ and for every $f \in \mathcal{E}$ and $y \in U(x, \epsilon)$ we have that

$$|f(x) - f(y)| < \epsilon. \tag{4.1}$$

Of course, this implies that every $f \in \mathcal{E}$ is continuous at x. If $\mathcal{E}$ has only finitely many elements, and if every element of $\mathcal{E}$ is continuous at x, then it is easy to see that $\mathcal{E}$ is equicontinuous at x.

Let V be a topological vector space over the real or complex numbers, and let $\mathcal{E}$ be a collection of linear functionals on V. If $\mathcal{E}$ is equicontinuous at 0, then $\mathcal{E}$ is equicontinuous at every point in V, by linearity. Suppose that there is an open set $U_1 \subseteq V$ such that $0 \in U_1$ and

$$|\lambda(v)| < 1 \tag{4.2}$$

for every $\lambda \in \mathcal{E}$ and $v \in U_1$. If ϵ is any positive real number, then it follows that

$$|\lambda(v)| < \epsilon \tag{4.3}$$

for every $\lambda \in \mathcal{E}$ and $v \in \epsilon\, U_1$, so that $\mathcal{E}$ is equicontinuous at 0 on V. Conversely, the existence of U_1 is obviously necessary for $\mathcal{E}$ to be equicontinuous at 0 on V.

Now let A be a commutative topological group, and let $\mathcal{E}$ be a collection of group homomorphisms from A into $\mathbf{T}$. If $\mathcal{E}$ is equicontinuous at 0, then one can check that $\mathcal{E}$ is equicontinuous at every point in A, as before. Suppose that there is an open set $U_0 \subseteq A$ such that $0 \in U_0$ and

$$\operatorname{Re} \phi(y) \geq 0 \tag{4.4}$$

for every $\phi \in \mathcal{E}$ and $y \in U_0$. Using continuity of addition on A, one can get a sequence $U_1, U_2, U_3, \ldots$ of open subsets of A such that $0 \in U_j$ and

$$U_j + U_j \subseteq U_{j-1} \tag{4.5}$$

for each $j \geq 1$. In particular, this implies that $U_j \subseteq U_{j-1}$ for every $j \geq 1$. Let $n \cdot a$ be the sum of n a's in A for each $a \in A$ and positive integer n, as before. If $y \in U_j$ for some positive integer j, then

$$2 \cdot y \in U_{j-1}, \tag{4.6}$$

by (4.5). If l is a nonnegative integer less than or equal to j, then it follows that

$$2^l \cdot y \in U_{j-l} \subseteq U_0. \tag{4.7}$$

Combining this with (4.4), we get that

$$\operatorname{Re} \phi(y)^{2^l} = \operatorname{Re} \phi(2^l \cdot y) \geq 0 \tag{4.8}$$

for every $y \in U_j$ and $l = 0, \ldots, j$. This condition implies that $\phi(y)$ is in a small neighborhood of 1 in $\mathbf{T}$, where this neighborhood of 1 in $\mathbf{T}$ can be made arbitrarily small by taking j large enough. Thus (4.4) implies that $\mathcal{E}$ is equicontinuous at 0 on A. Conversely, If $\mathcal{E}$ is a collection of group homomorphisms from A into $\mathbf{T}$ that is equicontinuous at 0, then it is easy to see that there is an open set $U_0 \subseteq A$ such that $0 \in U_0$ and (4.4) holds for every $\phi \in \mathcal{A}$ and $y \in U_0$.

Let V be a topological vector space over the real numbers, so that V is a commutative topological group with respect to addition in particular. If λ is a linear functional on V, then

$$\phi_\lambda(v) = \exp(i\,\lambda(v)) \tag{4.9}$$

defines a group homomorphism from V as a group with respect to addition into $\mathbf{T}$. If λ is continuous on V, then ϕ_λ is continuous on V as well, and we have seen in Section 1.11 that every continuous group homomorphism from V into $\mathbf{T}$ is of this form. Let $\mathcal{E}$ be a collection of linear functionals on V, and let

$$\mathcal{E}_1 = \{\phi_\lambda : \lambda \in \mathcal{E}\} \tag{4.10}$$

be the corresponding collection of group homomorphisms from V into $\mathbf{T}$. If $\mathcal{E}$ is equicontinuous at 0 on V, then it is easy to see that $\mathcal{E}_1$ is equicontinuous at 0 on V, using the continuity of the exponential function. Conversely, if $\mathcal{E}_1$ is equicontinuous at 0 on V, then $\mathcal{E}$ is equicontinuous at 0 on V. More precisely, the equicontinuity of $\mathcal{E}_1$ on V means that there are neighborhoods U of 0 in V such that each $\phi_\lambda \in \mathcal{E}_1$ is close to 1 on U. This implies that each $\lambda \in \mathcal{E}$ is close to 0 modulo $2\,\pi$ on U. We can also take U to be balanced in V, as in Section 1.6. This permits us to conclude that each $\lambda \in \mathcal{E}$ is small on U, as desired, and as in Section 1.11.

4.2 Equicontinuity and compactness

Let X be a nonempty topological space, and let $\mathcal{E}$ be a collection of complex-valued functions on X. Suppose that $\mathcal{E}$ is equicontinuous at every point in X, so that for each $x \in X$ and $\epsilon > 0$ there is an open set $U(x, \epsilon) \subseteq X$ such that $x \in U(x, \epsilon)$ and (4.1) holds for every $f \in \mathcal{E}$ and $y \in U(x, \epsilon)$. In particular, this implies that every $f \in \mathcal{E}$ is continuous on X, so that $\mathcal{E}$ may be considered as a subset of the space $C(X, \mathbf{C})$ of all continuous complex-valued functions on X. We may also consider $\mathcal{E}$ as a subset of the space $c(X, \mathbf{C})$ of all complex-valued functions on X. As usual, we take $C(X, \mathbf{C})$ and $c(X, \mathbf{C})$ to be equipped with the topologies defined in Sections 1.7 and 1.9, respectively.

Suppose that g is a complex-valued function on X which is in the closure of $\mathcal{E}$ in $c(X, \mathbf{C})$. This basically means that g can be approximated by elements of $\mathcal{E}$ on finite subsets of X. Using the equicontinuity condition described in the previous paragraph, we get that

$$|g(x) - g(y)| \leq \epsilon \tag{4.11}$$

when $y \in U(x, \epsilon)$ for some $x \in X$ and $\epsilon > 0$. This implies that g is continuous on X, so that the closure of $\mathcal{E}$ in $c(X, \mathbf{C})$ is contained in $C(X, \mathbf{C})$ under these conditions. More precisely, this shows that the closure of $\mathcal{E}$ in $c(X, \mathbf{C})$ is equicontinuous at every point in X as well.

Let g be an element of the closure of $\mathcal{E}$ in $c(X, \mathbf{C})$ again, and let K be a nonempty compact subset of X. Using an argument of the usual Arzelà–Ascoli type, one can verify that g can be approximated by elements of $\mathcal{E}$ uniformly on K. More precisely, this can be obtained from approximations of g by elements of $\mathcal{E}$ on suitable finite subsets of K, using the compactness of K and the equicontinuity conditions. This means that g is in the closure of $\mathcal{E}$ in $C(X, \mathbf{C})$, so that the closure of $\mathcal{E}$ in $c(X, \mathbf{C})$ is contained in the closure of $\mathcal{E}$ in $C(X, \mathbf{C})$. The opposite inclusion holds automatically, because this topology on $C(X, \mathbf{C})$ is at least as strong as the topology induced on $C(X, \mathbf{C})$ by the one on $c(X, \mathbf{C})$. Thus the closures of $\mathcal{E}$ in $c(X, \mathbf{C})$ and in $C(X, \mathbf{C})$ are the same under these conditions. If $\mathcal{E}$ is a closed set in $C(X, \mathbf{C})$, then it follows that $\mathcal{E}$ is a closed set in $c(X, \mathbf{C})$. The converse is trivial, as before, because of the way that the topologies are defined.

Similarly, one can check that the topology induced on $\mathcal{E}$ by the topology on $C(X, \mathbf{C})$ is the same as the topology induced on $\mathcal{E}$ by the topology on $c(X, \mathbf{C})$. The topology induced on $\mathcal{E}$ by the topology on $C(X, \mathbf{C})$ is automatically at least as strong as the topology induced on $\mathcal{E}$ by the topology on $c(X, \mathbf{C})$, and so it suffices to show the opposite inclusion. Let $f_0 \in \mathcal{E}$ and a nonempty compact set $K \subseteq X$ be given. The same type of argument as in the previous paragraph implies that $f \in \mathcal{E}$ is uniformly close to f_0 on K when f is close enough to f_0 on a suitable finite subset of K. This means that relative neighborhoods of f_0 in $\mathcal{E}$ with respect to the topology on $C(X, \mathbf{C})$ contain relative neighborhoods of f_0 in $\mathcal{E}$ with respect to the topology on $c(X, \mathbf{C})$, as desired.

Suppose now that $\mathcal{E}$ is also bounded pointwise on X, so that

$$\mathcal{E}_x = \{f(x) : f \in \mathcal{E}\} \tag{4.12}$$

is a bounded set in $\mathbf{C}$ for every $x \in X$. Equivalently, this means that there is a nonnegative real-valued function A on X such that

$$|f(x)| \le a(x) \tag{4.13}$$

for every $f \in \mathcal{E}$ and $x \in X$. Observe that

$$\{f \in c(X, \mathbf{C}) : |f(x)| \le a(x)\} \tag{4.14}$$

is a compact subset of $c(X, \mathbf{C})$, by Tychonoff's theorem, since (4.14) may be considered as a product of closed disks in $\mathbf{C}$. If $\mathcal{E}$ is a closed set in $c(X, \mathbf{C})$, then it follows that $\mathcal{E}$ is compact in $c(X, \mathbf{C})$, because $\mathcal{E}$ is contained in (4.14) by construction.

Of course, if $\mathcal{E}$ is compact in $c(X, \mathbf{C})$, then $\mathcal{E}$ is compact as a subset of itself, with respect to the topology induced on $\mathcal{E}$ by the one on $c(X, \mathbf{C})$. This is the same as the topology induced on $\mathcal{E}$ by the topology on $C(X, \mathbf{C})$ in this situation, as before. It follows that $\mathcal{E}$ is compact as a subset of $C(X, \mathbf{C})$ when $\mathcal{E}$ is compact as a subset of $c(X, \mathbf{C})$ under these conditions. The converse is trivial, because the topology induced on $\mathcal{E}$ by the one on $C(X, \mathbf{C})$ is at least as strong as the topology induced on $\mathcal{E}$ by the one on $c(X, \mathbf{C})$.

If $\mathcal{E}$ is a closed set in $C(X, \mathbf{C})$, then we have seen that $\mathcal{E}$ is a closed set in $c(X, \mathbf{C})$. Thus $\mathcal{E}$ is compact in $C(X, \mathbf{C})$ when $\mathcal{E}$ is closed in $C(X, \mathbf{C})$ and $\mathcal{E}$ is bounded pointwise on X, by the remarks in the previous paragraphs.

4.3 Some related arguments

Let X be a nonempty topological space, and let E be a nonempty subset of X. Remember that $c(X, \mathbf{C})$ denotes the space of all complex-valued functions on X, and similarly $c(E, \mathbf{C})$ denotes the space of complex-valued functions on E. There is a natural mapping from $C(X, \mathbf{C})$ into $c(E, \mathbf{C})$, which sends a complex-valued function f on X to its restriction on E. This mapping is linear, and more precisely an algebra homomorphism with respect to pointwise multiplication of functions. This mapping is also continuous with respect to the topologies defined on $c(X, \mathbf{C})$ and $c(E, \mathbf{C})$ in Section 1.9.

Let $\mathcal{E}$ be a collection of complex-valued functions on X, and suppose that $\mathcal{E}$ is equicontinuous at every point in X. Suppose also that E is dense in X, which implies that the restriction mapping from $c(X, \mathbf{C})$ into $c(E, \mathbf{C})$ mentioned in the previous paragraph is injective on $\mathcal{E}$. Consider the topology on $\mathcal{E}$ that corresponds to the topology induced on the image of $\mathcal{E}$ in $c(E, \mathbf{C})$ by the usual topology on $c(E, \mathbf{C})$. Under these conditions, one can check that this is the same as the topology induced on $\mathcal{E}$ by the usual topology on $c(X, \mathbf{C})$. Of course, the topology induced on $\mathcal{E}$ by the usual topology on $c(X, \mathbf{C})$ is automatically at least as strong as the topology on $\mathcal{E}$ that corresponds to $c(E, \mathbf{C})$. To get the opposite inclusion, let $x \in X$ and $f, g \in \mathcal{E}$ be given, and let y be an element of E. If y is sufficiently close to x in X, and if $f(y)$ is sufficiently close to $g(y)$ in $\mathbf{C}$, then one can get $f(x)$ to be as close to $g(x)$ in $\mathbf{C}$ as one wants, because $\mathcal{E}$ is equicontinuous at x.

Let $\{f_j\}_{j=1}^\infty$ be a sequence of elements of $\mathcal{E}$ that converges pointwise on E. This implies that $\{f_j(y)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{C}$ for every $y \in E$. If $x \in X$, then one can verify that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{C}$ too, using the equicontinuity of $\mathcal{E}$ at x, and by approximating x by $y \in E$. Thus $\{f_j\}_{j=1}^\infty$ converges pointwise to a complex-valued function f on X, because $\mathbf{C}$ is complete. As in Section 4.2, f is continuous on X in this situation, and $\{f_j\}_{j=1}^\infty$ converges to f with respect to the topology defined on $C(X, \mathbf{C})$ in Section 1.7.

Remember that $\mathcal{E}_x$ is the subset of $\mathbf{C}$ defined in (4.12) in Section 4.2 for each $x \in E$. Suppose that $\mathcal{E}$ is bounded pointwise on E, so that $\mathcal{E}_y$ is a bounded subset of $\mathbf{C}$ for each $y \in E$. If $x \in X$, then it is easy to see that $\mathcal{E}_x$ is bounded in $\mathbf{C}$ too, using the equicontinuity of $\mathcal{E}$ at x, and by approximating x by $y \in E$. Thus $\mathcal{E}$ is bounded pointwise on X under these conditions.

Let $\{f_j\}_{j=1}^\infty$ be a sequence of complex-valued functions on X. If $\{f_j(x)\}_{j=1}^\infty$ is a bounded sequence of complex numbers for some $x \in X$, then there is a subsequence $\{f_{j_l}(x)\}_{l=1}^\infty$ of $\{f_j(x)\}_{j=1}^\infty$ that converges in $\mathbf{C}$, because closed and bounded subsets of $\mathbf{C}$ are sequentially compact. If $\{f_j(y)\}_{j=1}^\infty$ is a bounded sequence of complex numbers for every $y \in E$, and if E has only finitely or countably many elements, then it is well known that there is a subsequence $\{f_{j_l}\}_{l=1}^\infty$ of $\{f_j\}_{j=1}^\infty$ that converges pointwise on E. This is easy to do when E has only finitely many elements, using the previous statement for individual points in E repeatedly. If E is countably infinite, then the desired subsequence can be obtained from a diagonalization argument.

Suppose that $\mathcal{E}$ is bounded pointwise on E, and let $\{f_j\}_{j=1}^\infty$ be a sequence of elements of $\mathcal{E}$. If E has only finitely or countably many elements, then there is a subsequence $\{f_{j_l}\}_{l=1}^\infty$ of $\{f_j\}_{j=1}^\infty$ that converges pointwise on E, as in the preceding paragraph. It follows that $\{f_{j_l}\}_{l=1}^\infty$ converges to a continuous function f on X with respect to the usual topology on $C(X, \mathbf{C})$, as before. If $\mathcal{E}$ is also a closed set in $C(X, \mathbf{C})$, then $f \in \mathcal{E}$. This implies that $\mathcal{E}$ is sequentially compact when $\mathcal{E}$ is pointwise bounded on E and a closed set in $C(X, \mathbf{C})$ and $\mathcal{E}$, and E has only finitely or countably many elements.

If E has only finitely or countably many elements, then the usual topology on $c(E, \mathbf{C})$ discussed in Section 1.9 is determined by a collection of finitely or countably many seminorms on $c(E, \mathbf{C})$. This implies that this topology on $c(E, \mathbf{C})$ is determined by a translation-invariant metric, as in Section 1.4.

Let A be a commutative topological group, and let $\mathcal{E}$ be an equicontinuous collection of group homomorphisms from A into $\mathbf{T}$. Also let E_0 be a subset of A such that the subgroup E of A generated by E_0 is dense in A. In this case, one can get the same topology on $\mathcal{E}$ using E_0 instead of E. If E_0 has only finitely or countably many elements, then E has the same property, as in Section 2.7.

Similarly, let V be a topological vector space over the real or complex numbers, and let $\mathcal{E}$ be an equicontinuous collection of linear functionals on V. Also let E_1 be a subset of V whose linear span E is dense in V. As before, one can get the same topology on $\mathcal{E}$ using E_1 instead of E. As a variant of this, let E_2 be the linear span of E_1 with rational coefficients in the real case. In the complex case, let E_2 be the linear span of E_1 whose coefficients are complex numbers with rational real and imaginary parts. In both cases, it is easy to see that E_2 is dense in E, and hence in V. If E_1 has only finitely or countably many elements, then one can check that E_2 has only finitely or countably many elements as well.

4.4 Subsets of $\widehat{A}$

Let A be a commutative topological group, and let $\widehat{A}$ be the corresponding dual group, with the topology described in Section 1.12. This is the same as the topology induced on $\widehat{A}$ by the topologies defined on $C(A, \mathbf{C})$ and $C(A, \mathbf{T})$ in Section 1.7. It is easy to see that $\widehat{A}$ is a closed set in $C(A, \mathbf{C})$ and $C(A, \mathbf{T})$, with respect to these topologies. More precisely, the collection of all group homomorphisms from A into $\mathbf{T}$ is a closed set in $c(A, \mathbf{C})$ and $c(A, \mathbf{T})$ with respect to the topologies defined in Section 1.9. This implies the previous statement, because the topologies on $C(A, \mathbf{C})$ and $C(A, \mathbf{T})$ defined in Section 1.7 are at least as strong as the topologies induced on them by the topologies defined on $c(A, \mathbf{C})$ and $c(A, \mathbf{T})$ in Section 1.9.

Let $U_0 \subseteq A$ be an open set with $0 \in U_0$, and let $\mathcal{E}_0$ be the collection of all group homomorphisms ϕ from A into $\mathbf{T}$ such that

$$\operatorname{Re} \phi(y) \geq 0 \tag{4.15}$$

for every $y \in U_0$. Thus $\mathcal{E}_0$ is equicontinuous at 0 in A, and hence at every point in A, as in Section 4.1. In particular, the elements of $\mathcal{E}_0$ are continuous on A, so that $\mathcal{E}_0 \subseteq \widehat{A}$. It is easy to see that $\mathcal{E}_0$ is a closed set in $c(A, \mathbf{C})$ and $c(A, \mathbf{T})$ with respect to the topologies defined in Section 1.9. Remember that $c(A, \mathbf{T})$ is compact with respect to this topology, by Tychonoff's theorem, as in Section 1.9. It follows that $\mathcal{E}_0$ is a compact subset of $c(A, \mathbf{T})$ with respect to this topology, because closed subsets of compact sets are compact. In fact, $\mathcal{E}_0$ is compact in $C(A, \mathbf{C})$, as in the Section 4.2. This implies that $\mathcal{E}_0$ is compact as a subset of $\widehat{A}$ with respect to the topology induced by the usual one on $C(A, \mathbf{C})$.

Let $\mathbf{1}_A$ be the indicator function associated to A on A, which is the constant function on A equal to 1 at every point in A. Of course, this is the identity element in the dual group $\widehat{A}$. Also let K be a nonempty compact subset of A, and consider

$$\Big\{\phi \in \widehat{A} : \sup_{x \in K} |\phi(x) - 1| < 1\Big\}. \tag{4.16}$$

This is the same as the open unit ball in $\widehat{A}$ centered at $\mathbf{1}_A$ with respect to the supremum semimetric associated to K as in (1.106) in Section 1.12. In particular, (4.16) is an open set in $\widehat{A}$ with respect to the topology defined in Section 1.12.

Suppose now that $U_0 \subseteq A$ is an open set, $K \subseteq A$ is compact, $0 \in U_0$, and $U_0 \subseteq K$. Note that there exist subsets U_0 and K of A with these properties exactly when A is locally compact. Under these conditions, (4.16) is contained in the collection $\mathcal{E}_0$ of all group homomorphisms ϕ from A into $\mathbf{T}$ that satisfy (4.15) for every $y \in U_0$. This uses the simple fact that $\operatorname{Re} z > 0$ when z is a complex number such that $|z - 1| < 1$. It follows that $\widehat{A}$ is locally compact with respect to the topology described in Section 1.12 when A is locally compact, because $\mathcal{E}_0$ is compact in $\widehat{A}$, as before.

Suppose that A is locally compact, and let H be a Haar measure on A. Also let f be a complex-valued Borel measurable function on A that is integrable with respect to H, and let $\epsilon > 0$ be given. Under these conditions, there is an open set $U_0 \subseteq A$ such that $0 \in U_0$ and

$$\int_A |f(x) - f(x - a)| \, dH(x) < \epsilon \tag{4.17}$$

for every $a \in A$, as in (3.106) in Section 3.9. Let $\mathcal{E}_0$ be the collection of group homomorphisms ϕ from A into $\mathbf{T}$ that satisfy (4.15) for every $y \in U_0$, as before. If $\phi \in \widehat{A} \setminus \mathcal{E}_0$, then there is an $a \in U_0$ such that $\operatorname{Re} \phi(a) < 0$, which implies that

$$|\phi(a) - 1| > 1. \tag{4.18}$$

Combining (4.17) and (4.18) with (3.109) in Section 3.9, we get that

$$|\widehat{f}(\phi)| < \epsilon \tag{4.19}$$

for every $\phi \in \widehat{A} \setminus \mathcal{E}_0$. This implies that $\widehat{f}$ vanishes at infinity on $\widehat{A}$, because $\mathcal{E}_0$ is compact in $\widehat{A}$, as before.

4.5 Boundedness and total boundedness

Let X be a nonempty set, and let $c(X, \mathbf{C})$ be as in Section 1.9. Thus $c(X, \mathbf{C})$ is a complex topological vector space with respect to the topology defined there. Note that a subset $\mathcal{E}$ of $c(X, \mathbf{C})$ is a bounded subset of $c(X, \mathbf{C})$ as a topological vector space, as in Section 1.17, if and only if $\mathcal{E}$ is bounded pointwise on X. This is easy to see directly from the definitions, and it can also be derived from the characterization of bounded subsets of topological vector spaces V for which the topology on V is defined by a collection of seminorms on V mentioned in Section 1.17. This can be obtained from the remarks about bounded subsets of Cartesian products of topological vector spaces in Section 1.18 as well.

If $\mathcal{E}$ is bounded pointwise on X, then $\mathcal{E}$ is totally bounded as a subset of $c(X, \mathbf{C})$ as a commutative topological group with respect to addition, as in Section 1.19. As before, this can be verified directly from the definitions, or using the remarks about totally bounded subsets of Cartesian products of commutative topological groups in Section 1.19. The main point is that bounded subsets of $\mathbf{C}$ are totally bounded, and similarly for $\mathbf{C}^n$ for each positive integer

n. Conversely, if $\mathcal{E}$ is totally bounded in $c(X, \mathbf{C})$ as a commutative topological group with respect to addition, then $\mathcal{E}$ is bounded pointwise on X. This can be seen directly, or using the fact that totally bounded subsets of a topological vector space are bounded, as in Section 1.19.

In particular, $c(X, \mathbf{T})$ is totally bounded as a subset of $c(X, \mathbf{C})$, considered as a commutative group with respect to addition. Remember that $c(X, \mathbf{T})$ is also a commutative topological group with respect to pointwise multiplication of functions and the usual topology, as in Section 1.9. One can check that $c(X, \mathbf{T})$ is totally bounded as a subset of itself, considered as a commutative topological group. This can be verified directly from the definitions, or using the remarks about Cartesian products in Section 1.19. Of course, this uses the fact that $\mathbf{T}$ is totally bounded.

Suppose now that X is equipped with a topology, and let $C(X, \mathbf{C})$ be as in Section 1.7, so that $C(X, \mathbf{C})$ is a complex topological vector space with respect to the usual topology. Observe that $\mathcal{E} \subseteq C(X, \mathbf{C})$ is bounded in the sense of Section 1.17 if and only if the elements of $\mathcal{E}$ are uniformly bounded on every nonempty compact subset of X. This follows from the characterization of bounded subsets of a topological space V when the topology on V is determined by a collection of seminorms mentioned in Section 1.17, since the usual topology on $C(X, \mathbf{C})$ is determined by the collection of supremum seminorms associated to nonempty compact subsets of X.

Suppose for the moment that $\mathcal{E}$ is equicontinuous at every point in X, and that $\mathcal{E}$ is bounded pointwise on X. In this case, it is easy to see that for each $x \in X$ there is an open set $U \subseteq X$ such that $x \in U$ and the elements of $\mathcal{E}$ are uniformly bounded on U. This implies that the elements of $\mathcal{E}$ are uniformly bounded on compact subsets of X. Under these conditions, one can also check that $\mathcal{E}$ is totally bounded in $C(X, \mathbf{C})$ as a commutative topological group with respect to addition, using standard arguments of the Arzelà–Ascoli type. The main point is to use equicontinuity to reduce uniform approximations of elements of $\mathcal{E}$ on a nonempty compact set $K \subseteq X$ to approximations on suitable finite subsets of K, as in Section 4.2.

Similarly, suppose that for each compact set $K \subseteq X$ and $x \in K$, the restrictions of the elements of $\mathcal{E}$ to K are equicontinuous at x, with respect to the topology induced on K by the one on X. If $\mathcal{E}$ is bounded pointwise on X, then the elements of $\mathcal{E}$ are uniformly bounded on compact subsets of X. In this case, $\mathcal{E}$ is totally bounded in $C(X, \mathbf{C})$ as a commutative topological group with respect to addition as well. These statements can be verified in basically the same way as in the preceding paragraph. One can also reduce to the previous situation, by restricting the elements of $\mathcal{E}$ to a compact set K.

Conversely, let $\mathcal{E}$ be a totally bounded subset of $C(X, \mathbf{C})$ as a commutative topological group with respect to addition. If $K \subseteq X$ is compact, then the restrictions of the elements of $\mathcal{E}$ to K are equicontinuous at every point in K, with respect to the topology induced on K by the one on X. Indeed, total boundedness of $\mathcal{E}$ in $C(X, \mathbf{C})$ implies that the restrictions of elements of $\mathcal{E}$ to K can be approximated uniformly on K by finitely many continuous functions on K. Using this, the equicontinuity of the restrictions of the elements of $\mathcal{E}$ to a point $x \in K$ can be obtained from the continuity of these approximations.

Remember that $C(X, \mathbf{T})$ is a commutative topological group with respect to pointwise multiplication of functions and the usual topology, as in Section 1.7. If $\mathcal{E}$ is any subset of $C(X, \mathbf{T})$, then one can check that $\mathcal{E}$ is totally bounded as a subset of $C(X, \mathbf{T})$ as a commutative topological group with respect to multiplication if and only if $\mathcal{E}$ is totally bounded as a subset of $C(X, \mathbf{C})$ as a commutative topological group with respect to addition. Of course, the usual topology on $C(X, \mathbf{T})$ is the same as the topology induced by the usual topology on $C(X, \mathbf{C})$. The identity element in $C(X, \mathbf{T})$ is the constant function $\mathbf{1}_X$ on X equal to 1 at every point, so that relatively open subsets of $C(X, \mathbf{T})$ that contain $\mathbf{1}_X$ as an element are used in the definition of totally bounded subsets of $C(X, \mathbf{T})$. One also considers translations of these relatively open sets with respect to multiplication in $C(X, \mathbf{T})$, instead of translations in $C(X, \mathbf{C})$ with respect to addition of open sets that contain 0.

4.6 Equicontinuity along convergent sequences

Let X be a topological space, and let $\mathcal{E}$ be a collection of complex-valued functions on X. Also let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of X that converges to a point $x \in X$. Let us say that $\mathcal{E}$ is *equicontinuous along* $\{x_j\}_{j=1}^\infty$, x if for every $\epsilon > 0$ there is a positive integer L such that

$$|f(x_j) - f(x)| < \epsilon \tag{4.20}$$

for every $f \in \mathcal{E}$ and $j \geq L$. In particular, this implies that

$$\lim_{j \to \infty} f(x_j) = f(x) \tag{4.21}$$

for every $f \in \mathcal{E}$. If $\mathcal{E}$ has only finitely many elements, each of which satisfies (4.21), then $\mathcal{E}$ is equicontinuous along $\{x_j\}_{j=1}^\infty$, x.

Let us say that $\mathcal{E}$ is *equicontinuous along convergent sequences at* $x \in X$ if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of X that converges to x, $\mathcal{E}$ is equicontinuous along $\{x_j\}_{j=1}^\infty$, x. This implies that every element of $\mathcal{E}$ is sequentially continuous at x. If $\mathcal{E}$ has only finitely many elements, each of which is sequentially continuous at x, then $\mathcal{E}$ is equicontinuous along convergent sequences at x. If $\mathcal{E}$ is any collection of complex-valued functions on X that is equicontinuous at x, then it is easy to see that $\mathcal{E}$ is equicontinuous along convergent sequences at x.

Suppose that there is a local base for the topology of X at a point x with only finitely or countably many elements. This implies that there is a sequence $U_1, U_2, U_3, \ldots$ of open subsets of X such that $x \in U_j$ for every j, and for every open set $U \subseteq X$ with $x \in U$, we have that $U_j \subseteq U$ for some j. As usual, we may also ask that $U_{j+1} \subseteq U_j$ for each j, by replacing U_j with $\bigcap_{l=1}^j U_l$ for each j. Let $\mathcal{E}$ be a collection of complex-valued functions on X, and suppose that $\mathcal{E}$ is not equicontinuous at x. This means that there is an $\epsilon > 0$ such that for each open set $U \subseteq X$ with $x \in U$, there is a point $y \in U$ and an $f \in \mathcal{E}$ with

$$|f(x) - f(y)| \geq \epsilon. \tag{4.22}$$

It follows that for each positive integer j, we can choose $x_j \in U_j$ and $f_j \in \mathcal{E}$ so that

$$|f_j(x_j) - f_j(x)| \geq \epsilon. \tag{4.23}$$

By construction, $\{x_j\}_{j=1}^{\infty}$ converges to x in X. However, $\mathcal{E}$ is not equicontinuous along $\{x_j\}_{j=1}^{\infty}$, x. This shows that $\mathcal{E}$ is equicontinuous at x when $\mathcal{E}$ is equicontinuous along convergent sequences at x and there is a local base for the topology of X at x with only finitely or countably many elements.

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of X that converges to a point $x \in X$. If K is the subset of X consisting of the x_j's for each positive integer j and x, then it is easy to see that K is compact in X. Let $\mathcal{E}$ be a collection of complex-valued functions on X again. If the restrictions of the elements of $\mathcal{E}$ to K are equicontinuous at x with respect to the topology induced on K by the one on X, then it is easy to see that $\mathcal{E}$ is equicontinuous along $\{x_j\}_{j=1}^{\infty}$, x. Suppose now that for every compact set $K \subseteq X$ and point $x \in K$ we have that the restrictions of the elements of $\mathcal{E}$ to K are equicontinuous at x, with respect to the topology induced on K by the one on X. This implies that $\mathcal{E}$ is equicontinuous along convergent sequences at every point in X, by the previous remark. If for every $x \in X$ there is a local base for the topology of X at x with only finitely or countably many elements, then it follows that $\mathcal{E}$ is equicontinuous at every point in X, as in the preceding paragraph. Of course, if X is locally compact, and if the restrictions of elements of $\mathcal{E}$ to every compact set $K \subseteq X$ are equicontinuous at every point in K, then $\mathcal{E}$ is equicontinuous at every point $x \in X$.

4.7 Strong σ-compactness

Let X be a topological space again. Let us say that X is *strongly σ-compact* if there is a family $\{K_j\}_{j \in I}$ of finitely or countably many compact subsets of X such that every compact subset of X is contained in the union of finitely many K_j's. In particular, this implies that $\bigcup_{j \in I} K_j = X$, because subsets of X with only one element are compact. It follows that X is σ-compact, and that X is compact when I has only finitely many elements. Of course, we may as well take I to be the set $\mathbf{Z}_+$ of positive integers. We may also ask that $K_j \subseteq K_{j+1}$ for every $j \in \mathbf{Z}_+$, since otherwise we can replace K_j with $\bigcup_{l=1}^{j} K_l$ for each $j \in \mathbf{Z}_+$. Under these conditions, we get that every compact subset of X is contained in K_j for some $j \in \mathbf{Z}_+$.

Suppose for the moment that X is locally compact. In this case, it is easy to see that every compact subset of X is contained in the interior of another compact subset of X. Suppose that X is σ-compact, so that there is a sequence $K_1, K_2, K_3, \ldots$ of compact subsets of X such that $X = \bigcup_{j=1}^{\infty} K_j$. We may also ask that $K_j \subseteq K_{j+1}$ for each j, as usual. More precisely, we can use the previous remark to modify the K_j's to get that K_j is contained in the interior of K_{j+1} for each j. This implies that the interiors of the K_j's form an increasing sequence of open subsets of X whose union is equal to X. If K is any compact subset of X, then it follows that K is contained in the interior of K_j for some j. This shows that X is strongly σ-compact when X is locally compact and σ-compact.

Consider the set $\mathbf{Q}$ of rational numbers, equipped with the topology induced by the standard topology on $\mathbf{R}$. Of course, $\mathbf{Q}$ is σ-compact, because $\mathbf{Q}$ is countable. If K is any compact subset of $\mathbf{Q}$, $a, b \in \mathbf{Q}$, and $a < b$, then there is an $x \in \mathbf{Q}$ such that $a < x < b$ and $x \not\in K$. Let $K_1, K_2, K_3, \ldots$ be any sequence of compact subsets of $\mathbf{Q}$. If j is any positive integer, then we can choose $x_j \in \mathbf{Q}$ such that $0 \le x_j < 1/j$ and $x_j \not\in \bigcup_{l=1}^{j} K_j$, by the previous remark. Let E be the subset of $\mathbf{Q}$ consisting of the x_j's, $j \in \mathbf{Z}_+$, together with 0. Note that E is a compact subset of $\mathbf{Q}$. It follows that $\mathbf{Q}$ is not strongly σ-compact.

Let A be a commutative topological group, and suppose that there is a local base for the topology of A at 0 with only finitely or countably many elements. This implies that there is a sequence $U_1, U_2, U_3, \ldots$ of open subsets of A such that $0 \in U_j$ for every j, and for every open set $U \subseteq A$ with $0 \in U$ we have that $U_j \subseteq U$ for some j. If $j \in \mathbf{Z}_+$, then let $\mathcal{E}_j$ be the collection of group homomorphisms ϕ from A into $\mathbf{T}$ such that

$$\operatorname{Re} \phi(x) \ge 0 \tag{4.24}$$

for every $x \in U_j$. As in Section 4.4, $\mathcal{E}_j$ is a compact subset of $\widehat{A}$ for each j, with respect to the topology defined on $\widehat{A}$ in Section 1.12.

Now let $\mathcal{E}$ be any compact subset of $\widehat{A}$ with respect to this topology. This implies that $\mathcal{E}$ is totally bounded in $\widehat{A}$ as a commutative topological group, as in Section 1.19. It follows that $\mathcal{E}$ is totally bounded in $C(X, \mathbf{T})$ as a commutative topological group, with respect to the topology defined in Section 1.7 and pointwise multiplication of functions. Equivalently, this means that $\mathcal{E}$ is totally bounded in $C(X, \mathbf{C})$ as a commutative topological group with respect to the topology defined in Section 1.7 and pointwise addition of functions, as in Section 4.5. If K is a compact subset of A and $x \in K$, then the restrictions of the elements of $\mathcal{E}$ to K are equicontinuous on K at x with respect to the topology induced on K by the one on A, as in Section 4.5 again. This implies that $\mathcal{E}$ is equicontinuous at every point $x \in K$, as in the previous section, because there is a local base for the topology of A at 0 with only finitely or countably many elements, and hence at every point $x \in A$. Of course, it suffices to consider $x = 0$ here, since the elements of $\mathcal{E}$ are group homomorphisms from A into $\mathbf{T}$. Using the equicontinuity of $\mathcal{E}$ at 0, we get that there is an open set $U \subseteq A$ such that $0 \in U$ and (4.24) holds for every $\phi \in \mathcal{E}$ and $x \in U$. Because the U_j's form a local base for the topology of A at 0, there is a positive integer j_0 such that $U_{j_0} \subseteq U$. It follows that

$$\mathcal{E} \subseteq \mathcal{E}_{j_0}, \tag{4.25}$$

so that $\widehat{A}$ is strongly σ-compact under these conditions.

4.8 Subsets of V'

Let V be a topological vector space over the real or complex numbers, and let V' be the corresponding dual space of continuous linear functionals on V, as usual. It is easy to see that the collection of all linear functionals on V is a closed set in $c(V, \mathbf{R})$ or $c(V, \mathbf{C})$, as appropriate, with respect to the topology discussed in Section 1.9. Of course, V' is the same as the intersection of the collection of all linear functionals on V with $C(V, \mathbf{R})$ or $C(V, \mathbf{C})$, as appropriate. In particular, V' is a closed set in $C(V, \mathbf{R})$ or $C(V, \mathbf{C})$, as appropriate, with respect to the topology defined in Section 1.7. Note that the topology induced on V' by the topology just mentioned on $c(V, \mathbf{R})$ or $c(V, \mathbf{C})$, as appropriate, is the same as the weak* topology on V', which was defined in Section 1.13.

Let U be an open set in V with $0 \in U$, and let $\mathcal{E}$ be the collection of linear functionals λ on V such that

$$|\lambda(v)| \le 1 \tag{4.26}$$

for every $v \in U$. This implies that

$$|\lambda(v)| \le |t| \tag{4.27}$$

for every $v \in t\,U$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate. It follows that $\mathcal{E}$ is equicontinuous at 0 on V, and hence at every point in V, as in Section 4.1. Observe that $\mathcal{E}$ is a closed set in $c(V, \mathbf{R})$ or $c(V, \mathbf{C})$, as appropriate, with respect to the topology defined in Section 1.9. This uses the fact that the collection of all linear functionals on V is a closed set with respect to this topology, as in the previous paragraph, and the way that $\mathcal{E}$ is defined. If v is any element of V, then $r\,v \in U$ when $r \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, is sufficiently small, because of the continuity of $r \mapsto r\,v$ at $r = 0$. This implies that $v \in t\,U$ when $|t|$ is sufficiently large, and hence that $\mathcal{E}$ is bounded pointwise on V, by (4.27). It follows that $\mathcal{E}$ is compact in $c(V, \mathbf{R})$ or $c(V, \mathbf{C})$, as appropriate, by Tychonoff's theorem, as in Section 4.2. Of course, each element of $\mathcal{E}$ is continuous on V, by equicontinuity, so that $\mathcal{E} \subseteq V'$. Thus $\mathcal{E}$ is compact with respect to the weak* topology on V', which is the famous theorem of Banach and Alaoglu.

Suppose for the moment that the topology on V is determined by a nondegenerate collection $\mathcal{N}$ of seminorms, as in Section 1.5. Let $N_1, \ldots, N_l$ be finitely many elements of $\mathcal{N}$, and let C be a nonnegative real number. It is easy to see that the collection of linear functionals λ on V such that

$$|\lambda(v)| \le C \max_{1 \le j \le l} N_j(v) \tag{4.28}$$

for every $v \in V$ is equicontinuous at 0 on V, and hence at every element of V. Conversely, if $\mathcal{E}$ is any collection of linear functionals on V that is equicontinuous at 0, then there are finitely many seminorms $N_1, \ldots, N_l \in \mathcal{N}$ and a $C \ge 0$ such that (4.28) holds for every $\lambda \in \mathcal{E}$ and $v \in V$. This is analogous to the characterization of continuous linear functionals on V in this situation in Section 1.10. Suppose now that the topology on V is determined by a single norm $\|v\|_V$, and let $\|\lambda\|_{V'}$ be the corresponding dual norm on V', as in (1.151) in

Section 1.18. In this case, a subset $\mathcal{E}$ of V' is equicontinuous at 0 if and only if the elements of $\mathcal{E}$ have uniformly bounded dual norms. The Banach–Alaoglu theorem implies that closed balls in V' with respect to $\|\lambda\|_{V'}$ are compact subsets of V' with respect to the weak* topology.

Let V be any topological vector space over $\mathbf{R}$ or $\mathbf{C}$ again, and let $\mathcal{E}$ be a collection of continuous linear functionals on V. Note that $\mathcal{E}$ is bounded pointwise on V if and only if $\mathcal{E}$ is bounded as a subset of V' with respect to the weak* topology on V', as in Section 1.17. If $\mathcal{E}$ is equicontinuous at 0 and $E \subseteq V$ is a bounded set, then one can check that the elements of $\mathcal{E}$ are uniformly bounded on E. In particular, this implies that $\mathcal{E}$ is bounded pointwise on V, because subsets of V with only one element are bounded, as in Section 1.17.

Now let $\mathcal{E}$ be a collection of continuous linear functionals on V that is bounded pointwise on a subset of second category in V, in the sense of Baire category. In this case, the theorem of Banach and Steinhaus implies that $\mathcal{E}$ is equicontinuous at 0 on V. Remember that complete metric spaces are of second category as subsets of themselves, by the Baire category theorem.

Suppose for the rest of the section that there is a local base for the topology of V at 0 with only finitely or countably many elements. This means that there is a sequence $U_1, U_2, U_3, \ldots$ of open subsets of V such that $0 \in U_j$ for every j, and that every open set $U \subseteq V$ with $0 \in U$ contains U_j for some j. If $j \in \mathbf{Z}_+$, then let $\mathcal{E}_j$ be the collection of linear functionals λ on V that satisfy (4.26) for every $v \in U_j$. As before, $\mathcal{E}_j$ is equicontinuous at 0 on V for each j, and $\mathcal{E}_j$ is a compact subset of V' with respect to the weak* topology. In particular, V' is σ-compact with respect to the weak* topology, because $\bigcup_{j=1}^{\infty} \mathcal{E}_j = V$.

As in Section 1.2, it is well known that there is a translation-invariant metric $d(\cdot,\cdot)$ on V that determines the same topology on V, because there is a local base for the topology of V at 0 with only finitely or countably many elements. Suppose now that V is also complete as a metric space with respect to $d(\cdot,\cdot)$. This is the same as asking that V be sequentially complete as a commutative topological group with respect to addition, as in Section 2.15. If $\mathcal{E} \subseteq V'$ is pointwise bounded on V, then $\mathcal{E}$ is equicontinuous on V, by the Banach–Steinhaus theorem. This implies that there is an open set $U \subseteq V$ such that $0 \in U$ and every $\lambda \in \mathcal{E}$ satisfies (4.26) for every $v \in U$. We also have that $U_{j_0} \subseteq U$ for some $j_0 \in \mathbf{Z}_+$, because the U_j's form a local base for the topology of V at 0, and hence $\mathcal{E} \subseteq \mathcal{E}_{j_0}$. If $\mathcal{E} \subseteq V'$ is compact with respect to the weak* topology, then $\mathcal{E}$ is a bounded set in V with respect to the weak* topology, as in Section 1.17. This means that $\mathcal{E}$ is bounded pointwise on V, as before. It follows that V' is strongly σ-compact with respect to the weak* topology under these conditions.

4.9 The dual of $\ell^1(X)$

Let X be a nonempty set, and remember that $\ell^1(X,\mathbf{R})$ and $\ell^1(X,\mathbf{C})$ denote the spaces of real and complex-valued summable functions on X, respectively, as in Section 2.1. Also let g be a bounded real or complex-valued function on X. If f is a real or complex-valued summable function on X, as appropriate, then $f\,g$ is summable on X as well, and we put

$$\lambda_g(f) = \sum_{x \in X} f(x)\,g(x). \tag{4.29}$$

Observe that

$$|\lambda_g(f)| \le \sum_{x \in X} |f(x)|\,|g(x)| \le \|f\|_1\,\|g\|_\infty, \tag{4.30}$$

where $\|f\|_1$ is as in (2.5) in Section 2.1, and $\|g\|_\infty$ is as in (2.17) in Section 2.2. This implies that λ_g defines a continuous linear functional on $\ell^1(X, \mathbf{R})$ or $\ell^1(X, \mathbf{C})$, as appropriate, with dual norm less than or equal to $\|g\|_\infty$. If $y \in X$ and $f_y(x)$ is the function on X equal to 1 when $x = y$ and to 0 when $x \ne y$, then $\|f_y\|_1 = 1$ and

$$\lambda_g(f_y) = g(y). \tag{4.31}$$

Using this, one can check that the dual norm of λ_g on $\ell^1(X, \mathbf{R})$ or $\ell^1(X, \mathbf{C})$, as appropriate, is equal to $\|g\|_\infty$.

It is well known that every continuous linear functional λ on $\ell^1(X, \mathbf{R})$ or $\ell^1(X, \mathbf{C})$ is of the form λ_g for some bounded real or complex-valued function g on X, as appropriate. More precisely, put

$$g(y) = \lambda(f_y) \tag{4.32}$$

for every $y \in X$, where f_y is as in the preceding paragraph. It is easy to see that $|g(y)|$ is less than or equal to the dual norm of λ for every $y \in X$, so that g is bounded on X. Thus λ_g may be defined as a bounded linear functional on $\ell^1(X, \mathbf{R})$ or $\ell^1(X, \mathbf{C})$, as in the preceding paragraph. By construction,

$$\lambda(f) = \lambda_g(f) \tag{4.33}$$

when f has finite support in X, because such functions f can be expressed as linear combinations of the f_y's. This implies that (2.78) holds for every real or complex-valued summable function f on X, as appropriate, by approximating f by functions with finite support in X with respect to the ℓ^1 norm. This also uses the fact that both λ and λ_g are continuous on $\ell^1(X, \mathbf{R})$ or $\ell^1(X, \mathbf{C})$, as appropriate.

Thus the duals of $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$ can be identified with the spaces $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ of bounded real and complex-valued functions on X, respectively. In particular, this permits us to define the corresponding weak* topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$. More precisely, if f is a real or complex-valued summable function on X, then

$$N_f(g) = |\lambda_g(f)| \tag{4.34}$$

defines a seminorm on $\ell^\infty(X, \mathbf{R})$ or $\ell^\infty(X, \mathbf{C})$, as appropriate. The weak* topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ as the duals of $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, respectively, are the same as the topologies determined on these spaces by the corresponding collections of seminorms of the form (4.34), as in Section 1.5.

If $y \in X$ and f_y is as before, then

$$N_{f_y}(g) = |\lambda_g(f_y)| = |g(y)| \tag{4.35}$$

for every bounded real or complex-valued function g on X, using (4.31) in the second step. Thus the weak* topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ as the duals of $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, respectively, are at least as strong as the topologies determined on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ by the corresponding collections of seminorms of the form (4.35) with $y \in X$. Of course, $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ are linear subspaces of the spaces $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ of all real and complex-valued functions on X, respectively. The topologies determined on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ by the corresponding collections of seminorms (4.35) with $y \in X$ are the same as the topologies induced on these spaces by the topologies defined on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, respectively, in Section 1.9.

Let r be a nonnegative real number, and consider the closed balls

$$\{g \in \ell^\infty(X, \mathbf{R}) : \|g\|_\infty \le r\} \tag{4.36}$$

and

$$\{g \in \ell^\infty(X, \mathbf{C}) : \|g\|_\infty \le r\} \tag{4.37}$$

centered at 0 with radius r in $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively, and with respect to the ℓ^∞ norm. Note that (4.36) and (4.37) correspond to closed balls centered at 0 with radius r in the duals of $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, respectively, and that these closed balls in the dual spaces are equicontinuous at 0. One can check that the topologies induced on (4.36) and (4.37) by the weak* topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ are the same as the topologies induced on (4.36) and (4.37) by the topologies determined on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ by the corresponding collections of seminorms of the form (4.35) with $y \in X$. More precisely, one can first verify that the topologies induced on (4.36) and (4.37) by the weak* topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively, are the same as the topologies induced on (4.36) and (4.37) by the topologies determined on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively, by the collections of seminorms of the form (4.34), where f has finite support in X. This uses the fact that arbitrary summable functions f on X can be approximated by functions with finite support in X with respect to the ℓ^1 norm. This also uses the fact that

$$|\lambda_g(f) - \lambda_g(f')| = |\lambda_g(f - f')| \le \|f - f'\|_1 \, \|g\|_\infty \tag{4.38}$$

for all summable real or complex-valued functions f, f' on X and bounded real or complex-valued functions g on X, as appropriate, to deal with the approximations of f just mentioned on (4.36) and (4.37). It is easy to see that the topologies determined on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ by seminorms of the form (4.34) where f has finite support in X are the same as the topologies determined on these spaces by seminorms of the form (4.35) with $y \in X$, so that the induced topologies on (4.36) and (4.37) are the same as well.

Equivalently, (4.36) is the same as

$$\{g \in c(X, \mathbf{R}) : |g(x)| \le r \text{ for every } x \in X\}, \tag{4.39}$$

and (4.37) is the same as

$$\{g \in c(X, \mathbf{C}) : |g(x)| \le r \text{ for every } x \in X\}. \tag{4.40}$$

These are compact subsets of $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, respectively, with respect to the topologies defined in Section 1.9, by Tychonoff's theorem. This implies that (4.36) and (4.37) are compact with respect to the weak* topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively, by the remarks in the preceding paragraph. This corresponds to the Banach–Alaoglu theorem in this situation.

Of course, if X has only finitely many elements, then one gets the same topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$ using the corresponding ℓ^∞ norms, the collection of seminorms of the form (4.34), or the collection of seminorms of the form (4.35). If X is countably infinite, then the topologies defined on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ as in Section 1.9 can be described by translation-invariant metrics, as in Section 1.4. This implies that the topologies induced on (4.36) and (4.37) by the weak* topologies on $\ell^\infty(X, \mathbf{R})$ and $\ell^\infty(X, \mathbf{C})$, respectively, can be described by the restrictions of the metrics just mentioned to these sets.

Chapter 5

Countability and compatibility conditions

5.1 Small sets

Let X be a set, and let $d(x, y)$ be a semimetric on X. The *diameter* of a nonempty subset E of X is defined as a nonnegative extended real number by

$$\operatorname{diam} E = \sup\{d(x, y) : x, y \in E\}, \tag{5.1}$$

which may be interpreted as being 0 when $E = \emptyset$. Let us say that E is ϵ*-small* for some $\epsilon > 0$ if

$$d(x, y) < \epsilon \tag{5.2}$$

for every $x, y \in E$. If E is ϵ-small, then $\operatorname{diam} E \leq \epsilon$. If $\operatorname{diam} E < \epsilon$, then E is ϵ-small. If E is ϵ-small and $x \in E$, then $E \subseteq B_d(x, \epsilon)$, where $B_d(x, \epsilon)$ is the open ball in X centered at x with radius ϵ, as in (1.23) in Section 1.2. Observe that $B_d(x, \epsilon)$ is $(2\,\epsilon)$-small for every $x \in X$, by the triangle inequality.

Suppose for the moment that A is a commutative group, and that $d(x, y)$ is a translation-invariant semimetric on A. In this case, $E \subseteq A$ is ϵ-small for some $\epsilon > 0$ with respect to $d(\cdot, \cdot)$ if and only if E is U-small with $U = B_d(0, \epsilon)$ in the sense of Section 1.19.

Let X be any set with a semimetric $d(x, y)$ again. As usual, a subset E of X is said to be *totally bounded* with respect to d if for each $\epsilon > 0$, E can be covered by finitely many open balls of radius ϵ with respect to d. Equivalently, this means that for each $\epsilon > 0$, E can be covered by finitely many ϵ-small sets with respect to d, by the earlier remarks. One can also take these ϵ-small sets to be subsets of E, by replacing them with their intersections with E, if necessary. Thus E is totally bounded with respect to d if and only if for each $\epsilon > 0$, E can be expressed as the union of finitely many ϵ-small sets.

Suppose that for each $\epsilon > 0$ there is a set $A(\epsilon) \subseteq X$ with only finitely or countably many elements such that

$$X = \bigcup_{x \in A(\epsilon)} B_d(x, \epsilon). \tag{5.3}$$

Using (5.3), one can check that

$$A = \bigcup_{j=1}^{\infty} A(1/j) \tag{5.4}$$

is a dense set in X with respect to the topology determined by d. This implies that X is separable with respect to the topology determined by d, because A has only finitely or countably many elements. Conversely, if X is separable with respect to d, then we can take $A(\epsilon)$ to be any dense subset of X with only finitely or countably many elements, for each $\epsilon > 0$. If for each $\epsilon > 0$, X can be covered by finitely or countably many ϵ-small sets with respect to d, then X satisfies the criterion for separability just mentioned.

Let $d_1, \ldots, d_l$ be finitely many semimetrics on X, and remember that

$$d(x, y) = \max_{1 \le j \le l} d_j(x, y) \tag{5.5}$$

defines a semimetric on X as well, as in Section 1.3. Observe that $E_0 \subseteq X$ is ϵ-small with respect to d for some $\epsilon > 0$ if and only if E_0 is ϵ-small with respect to d_j for each $j = 1, \ldots, l$. If $E_j \subseteq X$ is ϵ-small with respect to d_j for each $j = 1, \ldots, l$, then $\bigcap_{j=1}^{l} E_j$ is ϵ-small with respect to d.

If $E \subseteq X$ can be covered by finitely many ϵ-small sets with respect to d_j for some $\epsilon > 0$ and each $j = 1, \ldots, l$, then E can be covered by finitely many ϵ-small sets with respect to d. More precisely, one can cover E by intersections of the ϵ-small sets with respect each d_j that are given, by hypothesis. If E is totally bounded with respect to d_j for each $j = 1, \ldots, l$, then it follows that E is totally bounded with respect to d. The converse is trivial, because $d_j \le d$ for each $j = 1, \ldots, l$.

Similarly, if $E \subseteq X$ can be covered by finitely or countably many ϵ-small sets with respect to d_j for some $\epsilon > 0$ and each $j = 1, \ldots, n$, then E can be covered by finitely or countably many ϵ-small sets with respect to d. If X is separable with respect to d_j for each $j = 1, \ldots, n$, then it follows that X is separable with respect to d. The converse is trivial, as in the preceding paragraph.

Let $\mathcal{M}$ be a nonempty collection of semimetrics on X, which determines a topology on X, as in Section 1.3. One can check that a subset E of X is dense in X with respect to the topology determined by $\mathcal{M}$ if and only if for every collection $d_1, \ldots, d_l$ of finitely many elements of $\mathcal{M}$, E is dense in X with respect to the topology determined on X by $d_1, \ldots, d_l$. Equivalently, this means that for every collection $d_1, \ldots, d_l$ of finitely many elements of $\mathcal{M}$, E is dense in X with respect to the topology determined by (5.5).

Suppose that for each $d \in \mathcal{M}$, X is separable with respect to the topology determined by d. If $d_1, \ldots, d_l$ are finitely many elements of $\mathcal{M}$, then X is separable with respect to the topology determined by $d_1, \ldots, d_l$, as before. If $d_1, d_2, d_3, \ldots$ is an infinite sequence of elements of $\mathcal{M}$, then one can check that X is separable with respect to the topology determined by $d_1, d_2, d_3, \ldots$, by taking the union of dense subsets of X with respect to $d_1, \ldots, d_l$ with only finitely or countably many elements for each positive integer l. This shows that X is separable with respect to the topology determined by $\mathcal{M}$ when $\mathcal{M}$ has only finitely or countably many elements. One can also look at this in terms of the discussion in Section 1.4, as follows.

Let d be any semimetric on X again, and put

$$d_t(x, y) = \min(d(x, y), t) \tag{5.6}$$

for every $x, y \in X$ and $t > 0$, as in Section 1.4. Remember that d_t is also a semimetric on X for every $t > 0$, and that d_t determines the same topology on X as d. By construction, every set $E \subseteq X$ is ϵ-small with respect to d_t when $t < \epsilon$. If $\epsilon \leq t$, then E is ϵ-small with respect to d_t if and only if E is ϵ-small with respect to d. It follows that for any $t > 0$, $E \subseteq X$ is totally bounded with respect to d_t if and only if E is totally bounded with respect to d.

Let $d_1, d_2, d_3, \ldots$ be a sequence of semimetrics on X, and put

$$d'_j(x, y) = \min(d_j(x, y), 1/j) \tag{5.7}$$

for every $x, y \in X$ and $j \geq 1$, as in Section 1.4 again. Thus d'_j is a semimetric on X that determines the same topology on X as d_j for each j, as before. Remember that

$$d'(x, y) = \max_{j \geq 1} d'_j(x, y) \tag{5.8}$$

defines a semimetric on X, where the maximum on the right side of (5.8) is always attained. We have also seen that the topology determined on X by (5.8) is the same as the topology determined by the collection of d_j's, $j \geq 1$.

A subset E of X is ϵ-small with respect to (5.8) for some $\epsilon > 0$ if and only if E is ϵ-small with respect to d'_j for every $j \geq 1$, because the maximum on the right side of (5.8) is attained. As before, E is automatically ϵ-small with respect to d'_j when $1/j < \epsilon$. If $\epsilon \leq 1/j$, then E is ϵ-small with respect to d'_j if and only if E is ϵ-small with respect to d_j. Thus E is ϵ-small with respect to (5.8) if and only if E is ϵ-small with respect to d_j for every $j \leq 1/\epsilon$. This holds automatically when $\epsilon > 1$, and otherwise if $0 < \epsilon \leq 1$, then this holds if and only if E is ϵ-small with respect to (5.5), with $l = l(\epsilon)$ taken to be the largest positive integer less than or equal to $1/\epsilon$.

If $E \subseteq X$ is totally bounded with respect to d_j for each $j \geq 1$, then E is totally bounded with respect to (5.5) for every $l \geq 1$, as before. This implies that E is totally bounded with respect to (5.8), by the remarks in the preceding paragraph. Conversely, if E is totally bounded with respect to (5.8), then E is totally bounded with respect to d_j for every $j \geq 1$.

Similarly, if X is separable with respect to d_j for every $j \geq 1$, then we have seen that X is separable with respect to (5.5) for every $l \geq 1$. In this case, one can check that X is separable with respect to (5.8) as well. Conversely, if X is separable with respect to (5.8), then it is easy to see that X is separable with respect to d_j for each $j \geq 1$.

5.2 Countable bases

Let X be a topological space, and let $\mathcal{B}$ be a base for the topology of X. If $U \in \mathcal{B}$ and $U \neq \emptyset$, then let x_U be an element U, and let

$$E = \{x_U : U \in \mathcal{B},\, U \neq \emptyset\} \tag{5.9}$$

be the set of points in X that have been chosen in this way. It is easy to see that E is dense in X, because $\mathcal{B}$ is a base for the topology of X. If $\mathcal{B}$ has only finitely or countably many elements, then E has only finitely or countably many elements, so that X is separable.

Let $d(x, y)$ be a semimetric on X, and let E be a dense subset of X with respect to the topology determined by d. Under these conditions, it is well known and not difficult to check that

$$\mathcal{B}(d, E) = \{B_d(x, 1/n) : x \in E,\, n \in \mathbf{Z}_+\} \tag{5.10}$$

is a base for the topology determined on X by d. If E has only finitely or countably many elements, then it is easy to see that (5.10) has only finitely or countably many elements too.

It is well known that the set of finite subsets of $\mathbf{Z}_+$ is countably infinite. More precisely, every finite subset of $\mathbf{Z}_+$ is contained in $\{1, \ldots, n\}$ for some positive integer n, so that the set of finite subsets of $\mathbf{Z}_+$ can be expressed as a countable union of finite sets. This implies that the set of finite subsets of $\mathbf{Z}_+$ has only finitely or countably many elements. The set of finite subsets of $\mathbf{Z}_+$ is obviously infinite, and hence it is countably infinite. It follows that the set of finite subsets of any countably infinite set is countably infinite.

Now let $\mathcal{M}$ be a nonempty collection of semimetrics on X, and let X be equipped with the topology determined by $\mathcal{M}$ as in Section 1.3. Also let E be a dense subset of X with respect to this topology, and let $\mathcal{B}(E)$ be the collection of subsets of X of the form

$$\bigcap_{j=1}^{l} B_{d_j}(x, 1/n), \tag{5.11}$$

where $d_1, \ldots, d_l$ are finitely many elements of $\mathcal{M}$, $x \in E$, and $n \in \mathbf{Z}_+$. One can check that $\mathcal{B}(E)$ is a base for the topology determined on X by $\mathcal{M}$, using standard arguments. If $\mathcal{M}$ and E both have only finitely or countably many elements, then one can verify that $\mathcal{B}(E)$ has only finitely or countably many elements as well. This uses the fact that the set of finite subsets of $\mathcal{M}$ has only finitely or countably many elements, as in the previous paragraph.

Of course, if $\mathcal{M}$ consists of a single semimetric d, then $\mathcal{B}(E)$ is the same as (5.10). If $\mathcal{M}$ has only finitely or countably many elements, then one can reduce to the case where $\mathcal{M}$ consists of a single semimetric on X, as in Section 1.4. Note that for any nonempty collection $\mathcal{M}$ of semimetrics on X,

$$\bigcup_{d \in \mathcal{M}} \mathcal{B}(d, E) \tag{5.12}$$

is a sub-base for the topology determined on X by $\mathcal{M}$.

Let X be a topological space, and let $\mathcal{B}$ be a sub-base for the topology of X. This means that

$$\widetilde{\mathcal{B}} = \left\{ \bigcap_{j=1}^{n} U_j : U_1, \ldots, U_l \text{ are finitely many elements of } \mathcal{B} \right\} \tag{5.13}$$

is a base for the topology of X. If $\mathcal{B}$ has only finitely or countably many elements, then the set of finite subsets of $\mathcal{B}$ has only finitely or countably many elements, as before. This implies that $\widetilde{\mathcal{B}}$ has only finitely or countably many elements.

Let I be a nonempty set, and let τ_j be a topology on X for each $j \in I$. This leads to a topology τ on X generated by the τ_j's, $j \in I$, in the sense that

$$\bigcup_{j \in I} \tau_j \tag{5.14}$$

is a sub-base for τ. If $\mathcal{B}_j$ is a sub-base for τ_j for each $j \in I$, then

$$\bigcup_{j \in I} \mathcal{B}_j \tag{5.15}$$

is a sub-base for τ. If I has only finitely or countably many elements, and $\mathcal{B}_j$ has only finitely or countably many elements, then (5.15) has only finitely or countably many elements. In this case, there is a base for τ with only finitely or countably many elements, as in the previous paragraph.

5.3 Sequences of topologies

Let X be a set, and let $\tau_1, \tau_2, \tau_3, \ldots$ be an infinite sequence of topologies on X such that

$$\tau_n \subseteq \tau_{n+1} \tag{5.16}$$

for every $n \in \mathbf{Z}_+$. Of course, any collection of subsets of X is a sub-base for a topology on X. Let τ be the topology on X for which

$$\bigcup_{n=1}^{\infty} \tau_n \tag{5.17}$$

is a sub-base. Note that (5.17) is closed under finite intersections and unions, because of (5.16) and the definition of a topology. This implies that (5.17) is a base for τ in this situation. Similarly, if $\mathcal{B}_n$ is a base for τ_n for each $n \in \mathbf{Z}_+$, then one can check that

$$\bigcup_{n=1}^{\infty} \mathcal{B}_n \tag{5.18}$$

is a base for τ. If $E \subseteq X$ is dense in X with respect to τ_n for each $n \in \mathbf{Z}_+$, then one can verify that E is dense in X with respect to τ as well. If $E_n \subseteq X$ is dense in X with respect to τ_n for each $n \in \mathbf{Z}_+$, then it follows that $\bigcup_{n=1}^{\infty} E_n$ is dense in X with respect to τ. If X is separable with respect to τ_n for each $n \in \mathbf{Z}_+$, then it is easy to see that X is separable with respect to τ, using the previous statement.

Let $\mathrm{Int}_{\tau_n} E$ denote the interior of $E \subseteq X$ with respect to τ_n for each $n \in \mathbf{Z}_+$, and let $\mathrm{Int}_{\tau} E$ denote the interior of E with respect to τ. Observe that

$$\mathrm{Int}_{\tau_n} E \subseteq \mathrm{Int}_{\tau_{n+1}} E \tag{5.19}$$

for each n, because of (5.16), and that

$$\mathrm{Int}_{\tau} E = \bigcup_{n=1}^{\infty} \mathrm{Int}_{\tau_n} E, \tag{5.20}$$

by the definition of τ. Similarly, let $\overline{E}_{\tau_n}$ denote the closure of E in X with respect to τ_n for each $n \in \mathbf{Z}_+$, and let $\overline{E}_{\tau}$ be the closure of E in X with respect to τ. Using (5.16) and the definition of τ again, one can check that

$$\overline{E}_{\tau_n} \supseteq \overline{E}_{\tau_{n+1}} \tag{5.21}$$

for each $n \in \mathbf{Z}_+$, and

$$\overline{E}_{\tau} = \bigcap_{n=1}^{\infty} \overline{E}_{\tau_n}. \tag{5.22}$$

Of course, (5.19) and (5.20) correspond to (5.21) and (5.22) in standard ways, by taking complements in E.

Let $\mathcal{A}(\tau_n)$ be the σ-algebra of Borel sets in X with respect to τ_n for each $n \in \mathbf{Z}_+$, and let $\mathcal{A}(\tau)$ be the σ-algebra of Borel sets in X with respect to τ. Thus

$$\mathcal{A}(\tau_n) \subseteq \mathcal{A}(\tau_{n+1}) \tag{5.23}$$

for every $n \in \mathbf{Z}_+$, by (5.16), and

$$\mathcal{A}(\tau_n) \subseteq \mathcal{A}(\tau) \tag{5.24}$$

for each $n \in \mathbf{Z}_+$, because $\tau_n \subseteq \tau$, by construction. It is easy to see that

$$\bigcup_{n=1}^{\infty} \mathcal{A}(\tau_n) \tag{5.25}$$

is an algebra of subsets of X, because $\mathcal{A}(\tau_n)$ is a σ-algebra of subsets of X for each n, and hence an algebra of subsets of X for every n, and using (5.23). We also have that (5.25) is contained in $\mathcal{A}(\tau)$, because of (5.24).

Let $\mathcal{A}$ be any σ-algebra of subsets of X that contains (5.25), and let us check that

$$\tau \subseteq \mathcal{A}. \tag{5.26}$$

Every element of τ can be expressed as the union of a sequence of elements of the τ_n's, as in (5.20). Each of these elements of the τ_n's is contained in (5.25), by construction, and hence is contained in $\mathcal{A}$. This implies that every element of τ is contained in $\mathcal{A}$, because $\mathcal{A}$ is a σ-algebra. Thus (5.26) holds. It follows that

$$\mathcal{A}(\tau) \subseteq \mathcal{A}, \tag{5.27}$$

which means that $\mathcal{A}(\tau)$ is the smallest σ-algebra of subsets of X that contains (5.25). This is basically the same as saying that $\mathcal{A}(\tau)$ is the smallest σ-algebra of subsets of X that contains (5.17), which can be shown in the same way.

5.4 Comparing topologies

Let X be a set, let τ_1, τ_2 be topologies on X, and let $\mathcal{A}(\tau_1)$, $\mathcal{A}(\tau_2)$ be the corresponding σ-algebras of Borel sets in X, respectively. If

$$\tau_1 \subseteq \tau_2, \tag{5.28}$$

then

$$\mathcal{A}(\tau_1) \subseteq \mathcal{A}(\tau_2). \tag{5.29}$$

If

$$\tau_2 \subseteq \mathcal{A}(\tau_1), \tag{5.30}$$

then

$$\mathcal{A}(\tau_2) \subseteq \mathcal{A}(\tau_1). \tag{5.31}$$

Thus (5.28) and (5.30) imply that

$$\mathcal{A}(\tau_1) = \mathcal{A}(\tau_2). \tag{5.32}$$

We shall see some examples of this later.

Suppose for the moment that $\mathcal{B}_2$ is a base for τ_2, and that

$$\mathcal{B}_2 \subseteq \mathcal{A}(\tau_1). \tag{5.33}$$

If $\mathcal{B}_2$ has only finitely or countably many elements, then it follows that (5.30) holds. More precisely, this works when every element of τ_2 can be expressed as the union of finitely or countably many elements of $\mathcal{B}_2$.

Suppose now that $\mathcal{B}_2$ is a sub-base for τ_2, and let $\widetilde{\mathcal{B}}_2$ be the collection of finite intersections of elements of $\mathcal{B}_2$, as in (5.13) in Section 5.2. Thus $\widetilde{\mathcal{B}}_2$ is a base for τ_2, as before. Clearly (5.33) implies that

$$\widetilde{\mathcal{B}}_2 \subseteq \mathcal{A}(\tau_1). \tag{5.34}$$

If $\mathcal{B}_2$ has only finitely or countably many elements, then we have seen that $\widetilde{\mathcal{B}}_2$ has only finitely or countably many elements as well. In this case, (5.34) implies (5.30), as in the previous paragraph.

Consider the condition

$$\text{every element of } \tau_2 \text{ is an } F_\sigma \text{ set with respect to } \tau_1. \tag{5.35}$$

This condition implies (5.30), because F_σ sets with respect to τ_1 are elements of $\mathcal{A}(\tau_1)$. If we have both (5.28) and (5.35), then it follows that

$$\text{every element of } \tau_1 \text{ is an } F_\sigma \text{ set with respect to } \tau_1. \tag{5.36}$$

Similarly, (5.28) and (5.35) imply that

$$\text{every element of } \tau_2 \text{ is an } F_\sigma \text{ set with respect to } \tau_2. \tag{5.37}$$

This is because (5.28) implies that closed sets with respect to τ_1 are closed sets with respect to τ_2, so that F_σ sets with respect to τ_1 are F_σ sets with respect to τ_2.

Note that (5.36) and (5.37) correspond exactly to (2.30) in Section 2.3, for the topologies τ_1 and τ_2, respectively. Of course, if (5.32) holds, then a Borel measure on X with respect to τ_1 is the same as a Borel measure on X with respect to τ_2. If (5.28) holds as well, then the regularity conditions discussed in Section 2.3 for τ_1 would imply the analogous conditions for τ_2. Suppose that (5.28) holds, and that X is Hausdorff with respect to τ_1 and hence τ_2, so that compact sets with respect to τ_1 and τ_2 are closed sets, and Borel sets in particular. In this case, inner regularity conditions using compact sets for τ_2 would imply analogous conditions for τ_1, since compact sets with respect to τ_2 are compact with respect to τ_1 when (5.28) holds.

Let $\mathcal{B}_2$ be a base for τ_2 again, and suppose that

$$\text{every element of } \mathcal{B}_2 \text{ is an } F_\sigma \text{ set with respect to } \tau_1. \tag{5.38}$$

If $\mathcal{B}_2$ has only finitely or countably many elements, then this implies (5.35).

Now let $\mathcal{B}_2$ be a sub-base for τ_2, and let $\widetilde{\mathcal{B}}_2$ be the collection of finite intersections of elements of $\mathcal{B}_2$, as before. If (5.38) holds, then

$$\text{every element of } \widetilde{\mathcal{B}}_2 \text{ is an } F_\sigma \text{ set with respect to } \tau_1, \tag{5.39}$$

because finite intersections of F_σ sets are F_σ sets too. If $\mathcal{B}_2$ has only finitely or countably many elements, then $\widetilde{\mathcal{B}}_2$ has only finitely or countably many elements, and (5.39) implies (5.35), because $\widetilde{\mathcal{B}}_2$ is a base for τ_2.

Let $\overline{E}_{\tau_1}$, $\overline{E}_{\tau_2}$ be the closures of $E \subseteq X$ with respect to τ_1, τ_2, respectively. If (5.28) holds, then

$$\overline{E}_{\tau_2} \subseteq \overline{E}_{\tau_1} \tag{5.40}$$

for every $E \subseteq X$. Consider the condition

$$\begin{gathered}\text{for each } x \in X \text{ and } W \in \tau_2 \text{ there is a } U \in \tau_2 \\ \text{such that } x \in U \text{ and } \overline{U}_{\tau_1} \subseteq W.\end{gathered} \tag{5.41}$$

This would be the same as regularity of X with respect to τ_2 in the strict sense if we used the closure of U with respect to τ_2 instead of τ_1. If (5.28) holds, then this condition implies that X is regular in the strict sense with respect to τ_2, because of (5.40).

If there is a base for τ_2 with only finitely or countably many elements, and if (5.41) holds, then (5.35) holds. To see this, let $W \in \tau_2$ be given. Using (5.41), we get that W can be expressed as a union of open sets U with respect to τ_2 whose closures with respect to τ_1 are contained in W. If there is a base for τ_2 with only finitely or countably many elements, then Lindelöf's theorem implies that W can be expressed as the union of finitely or countably many such U's. It follows that W is an F_σ set with respect to τ_1, as desired.

5.5 Upper and lower semicontinuity

There is a standard topology on the set of extended real numbers, which is defined as follows. Let U be a set of extended real numbers, and let x be an element of U. If $x \in \mathbf{R}$, then we say that x is an element of the interior of U if there are $a, b \in \mathbf{R}$ such that $a < x < b$ and $(a, b) \subseteq U$. If $x = +\infty$, then we say that x is an element of the interior of U if there is an $a \in \mathbf{R}$ such that $(a, +\infty] \subseteq U$. If $x = -\infty$, then we say that x is an element of the interior of U if there is a $b \in \mathbf{R}$ such that $[-\infty, b) \subseteq U$. If every element of U is an element of the interior of U, then U is an open set. It is easy to see that this defines a topology on the set of extended real numbers. Note that $\mathbf{R}$ is an open set with respect to this topology, and that the corresponding induced topology on $\mathbf{R}$ is the standard topology.

Let X be a topological space, and let f be a function on X with values in the set of extended real numbers. We say that f is *upper semincontinuous* on X if for every $b \in \mathbf{R}$ we have that

$$\{x \in X : f(x) < b\} \tag{5.42}$$

is an open set in X. Similarly, f is said to be *lower semincontinuous* on X if for every $a \in \mathbf{R}$,

$$\{x \in X : f(x) > a\} \tag{5.43}$$

is an open set in X. Observe that f is both upper and lower semicontinuous on X if and only if f is continuous on X, using the standard topology on the set of extended real numbers. If f is either upper or lower semicontinuous on X, then f is Borel measurable on X.

If x_0 is any element of X, then pointwise versions of upper and lower semicontinuity of f at x_0 can be defined as follows. We say that f is upper semicontinuous at x_0 if for every $b \in \mathbf{R}$ with $f(x_0) < b$ there is an open set $U \subseteq X$ such that $x_0 \in U$ and $f(x) < b$ for every $x \in U$. This is vacuous when $f(x_0) = \infty$, and if $f(x_0) \in \mathbf{R}$, then we can take b to be of the form $f(x_0) + \epsilon$, with $\epsilon > 0$. Similarly, f is lower semicontinuous at x_0 if for every $a \in \mathbf{R}$ with $f(x_0) > a$ there is an open set $U \subseteq X$ such that $x_0 \in U$ and $f(x) > a$ for every $x \in U$.

As before, this is vacuous when $f(x_0) = -\infty$, and if $f(x_0) \in \mathbf{R}$, then we can take a to be of the form $f(x_0) - \epsilon$, with $\epsilon > 0$. One can check that f is upper semicontinuous on X if and only if f is upper semicontinuous at every point in X, and that f is lower semicontinuous on X if and only if f is lower semicontinuous at every point in X. As before, f is both upper and lower semicontinuous at a point $x_0 \in X$ if and only if f is continuous at x_0.

It is well known and not too difficult to check that the supremum of any nonempty collection of lower semicontinuous extended real-valued functions on X is lower semicontinuous on X as well. Similarly, the infimum of any nonempty collection of upper semicontinuous extended real-valued functions on X is upper semicontinuous.

Let $d(x, y)$ be a semimetric on X. As in Section 2.16, $d(\cdot, \cdot)$ is compatible with the given topology on X if and only if for each $x \in X$, $d(x, y)$ is continuous as a real-valued function of y on X at x. In this situation, $d(x, y)$ is automatically lower semicontinuous as a real-valued function of y on X at x, because $d(\cdot, \cdot)$ is nonnegative, and $d(x, x) = 0$. Thus $d(\cdot, \cdot)$ is compatible with the topology on X if and only if for each $x \in X$, $d(x, y)$ is upper semicontinuous as a real-valued function of y on X at x. If A is a commutative topological group, and $d(\cdot, \cdot)$ is a translation-invariant semimetric on A, then $d(\cdot, \cdot)$ is compatible with the topology on A exactly when $d(0, y)$ is upper semicontinuous as a real-valued function of y on A at 0.

Let X be an arbitrary topological space again, and let f be an extended real-valued function on X. Observe that f is upper semicontinuous on X if and only if for every $b \in \mathbf{R}$,

$$\{x \in X : f(x) \geq b\} \tag{5.44}$$

is a closed set in X. Similarly, f is lower semicontinuous on X if and only if for every $a \in \mathbf{R}$,

$$\{x \in X : f(x) \leq a\} \tag{5.45}$$

is a closed set in X. These statements follow from the earlier definitions by taking complements, as usual.

5.6 Semicompatible semimetrics and seminorms

Let X be a set with a topology τ_1, and let $d(x, y)$ be a semimetric on X. Let us say that $d(\cdot, \cdot)$ is *semicompatible with* τ_1 if for every $x \in X$, $d(x, y)$ is lower semicontinuous as a real-valued function of y on X with respect to τ_1. This is the same as saying that closed balls in X with respect to $d(\cdot, \cdot)$ are closed sets with respect to τ_1, because of the characterization (5.45) of lower semicontinuity. If $d(\cdot, \cdot)$ is compatible with τ_1, then $d(\cdot, \cdot)$ is semicompatible with τ_1, because open sets in X with respect to the topology determined by $d(\cdot, \cdot)$ are open with respect to τ_1, and hence closed sets in X with respect to the topology determined by $d(\cdot, \cdot)$ are closed sets with respect to τ_1. Note that the remarks in Section 2.16 about finite Borel measures and compatible semimetrics work for semicompatible semimetrics as well.

Let A be a commutative group with a topology τ_1 that makes A a commutative topological group, and let $d(x, y)$ be a translation-invariant semimetric on A. In this case, $d(\cdot, \cdot)$ is semicompatible with τ_1 when $d(0, y)$ is lower semicontinuous as a function of y on A with respect to τ_1. This is the same as saying that closed balls in A centered at 0 with respect to $d(\cdot, \cdot)$ are closed sets with respect to τ_1.

Let V be a vector space over the real or complex numbers with a topology τ_1 that makes V a topological vector space, and let N be a seminorm on V. Let us say that N is *semicompatible with* τ_1 if N is lower semicontinuous with respect to τ_1. Equivalently, this means that the semimetric d_N on V associated to N as in (1.45) in Section 1.5 is semicompatible with τ_1. If N is compatible with τ_1, then N is semicompatible with τ_1, as before.

Let X be a nonempty set, and consider the spaces $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ of real and complex-valued functions on X with the topologies defined in Section 1.9. Remember that $\|f\|_r$ is defined for real and complex-valued functions f on X and $r > 0$ as in (2.16) and (2.17) in Section 2.2. It is easy to see that

$$f \mapsto \|f\|_r \tag{5.46}$$

is lower semicontinuous on each of $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ for every $r > 0$. More precisely, if $r = \infty$, then (5.46) is the supremum of a collection of continuous functions on each of these spaces, basically by construction. Similarly, if r is a positive real number and E is a nonempty finite subset of X, then

$$f \mapsto \Big(\sum_{x \in E} |f(x)|^r\Big)^{1/r} \tag{5.47}$$

is continuous on each of $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$. This implies that (5.46) is lower semicontinuous on each of these spaces, by taking the supremum over E. In this case, we also have that

$$f \mapsto \|f\|_r^r \tag{5.48}$$

is lower semicontinuous on each of $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, for essentially the same reasons.

Let $r > 0$ be given, and let τ_1 denote the topology induced on $\ell^r(X, \mathbf{R})$ or $\ell^r(X, \mathbf{C})$ by the topology defined on $c(X, \mathbf{R})$ or $c(X, \mathbf{C})$, respectively, in Section 1.9. If $r \geq 1$, then $\|f\|_r$ defines a norm on each of $\ell^r(X, \mathbf{R})$ and $\ell^r(X, \mathbf{C})$, and this norm is semicompatible with τ_1, because of the lower semicontinuity of (5.46). If $0 < r \leq 1$, then $\|f - g\|_r^r$ defines a translation-invariant metric on each of $\ell^r(X, \mathbf{R})$ and $\ell^r(X, \mathbf{C})$, as in (2.25) in Section 2.2. This metric is semicompatible with τ_1 on each of these spaces, because of the lower semicontinuity of (5.48).

Let V be a topological vector space over the real or complex numbers again, and let N be a seminorm on V that is compatible with the given topology on V. Also let $\mathcal{E}(N)$ be the collection of linear functionals λ on V such that

$$|\lambda(v)| \leq N(v) \tag{5.49}$$

for every $v \in V$. Because N is supposed to be compatible with the given topology on V, the elements of $\mathcal{E}(N)$ are continuous on V, and in fact they are equicontinuous at 0. It is well known that for every $v \in V$ there is a $\lambda \in \mathcal{E}(N)$ such that

$$\lambda(v) = N(v), \tag{5.50}$$

by the Hahn–Banach theorem. This implies that

$$N(v) = \sup\{|\lambda(v)| : \lambda \in \mathcal{E}(N)\} \tag{5.51}$$

for every $v \in V$, where the supremum is actually attained. Of course, every $\lambda \in V'$ is continuous with respect to the weak topology on V, by definition of the weak topology, as in Section 1.13. Thus $|\lambda(v)|$ defines a continuous function on V with respect to the weak topology as well. It follows from (5.51) that N is lower semicontinuous with respect to the weak topology on V, which means that N is semicompatible with the weak topology on V.

Similarly, for every $v \in V$,

$$\lambda \mapsto |\lambda(v)| \tag{5.52}$$

defines a continuous function on V' with respect to the weak* topology on V', which was defined in Section 1.13 too. Let E be a nonempty bounded subset of V, as in Section 1.17, and let $N_E(\lambda)$ be the corresponding supremum seminorm on V', as in (1.149) in Section 1.18. Observe that N_E is lower semicontinuous with respect to the weak* topology on V', because it is the supremum of a nonempty collection of continuous functions on V' with respect to the weak* topology, by construction. Thus N_E is semicompatible with the weak* topology on V'.

5.7 Comparing topologies again

Let X be a set, and let τ_1 be a topology on X. Also let $\mathcal{M}_2$ be a nonempty collection of semimetrics on X, and let τ_2 be the topology determined on X by $\mathcal{M}_2$, as in Section 1.3. Suppose that the elements of $\mathcal{M}_2$ are semicompatible with τ_1, as in the previous section, so that closed balls in X with respect to elements of $\mathcal{M}_2$ are closed sets with respect to τ_1. Using this, one can check that X satisfies the regularity condition (5.41) in Section 5.4 with respect to τ_1 and τ_2. This is very similar to showing that X is regular in the strict sense with respect to τ_2 in this case.

If $x \in X$, $r > 0$, and $d \in \mathcal{M}_2$, then the open ball $B_d(x, r)$ in X centered at x with radius r with respect to d is defined as in (1.23) in Section 1.2, and is an open set in X with respect to τ_2. Let E be a subset of X, and consider the collection $\mathcal{B}_2(E)$ of subsets of X of the form

$$\bigcap_{j=1}^{l} B_{d_j}(x, 1/n), \tag{5.53}$$

where $x \in E$, $d_1, \ldots, d_l$ are finitely many elements of $\mathcal{M}_2$, and $n \in \mathbf{Z}_+$. Thus the elements of $\mathcal{B}_2(E)$ are open subsets of X with respect to τ_2. Suppose that $\mathcal{M}_2$ has only finitely or countably many elements, E is dense in X with respect to τ_2, and E has only finitely or countably many elements. Under these conditions, $\mathcal{B}_2(E)$ is a base for τ_2, as mentioned in Section 5.2, and $\mathcal{B}_2(E)$ has only finitely or countably many elements as well. If the elements of $\mathcal{M}_2$ are also semicompatible with τ_1, then it is easy to see that the sets (5.53) are F_σ sets with respect to τ_1. It follows that (5.38) in Section 5.4 holds in this situation with $\mathcal{B}_2 = \mathcal{B}_2(E)$, so that (5.35) holds too.

Let V be a topological vector space over the real or complex numbers, and suppose that the topology on V is determined by a nonempty collection $\mathcal{N}$ of finitely or countably many seminorms on V. Each element of $\mathcal{N}$ is semicompatible with the corresponding weak topology on V, as in the previous section. Let τ_2 be the initial topology on V, and let τ_1 be the corresponding weak topology. Of course, τ_2 is the same as the topology determined on V by the collection $\mathcal{M}$ of semimetrics on V associated to elements of $\mathcal{N}$. Note that the elements of $\mathcal{M}$ are semicompatible with τ_1, because of the corresponding property of $\mathcal{N}$. If V is separable with respect to τ_2, then the remarks in the previous paragraph imply that (5.35) in Section 5.4 holds with $X = V$. If V is a locally convex topological vector space with a countable base for its topology at 0, then it is well known that there is a collection of finitely or countably many seminorms on V that determines the same topology on V.

Now let X be a countably-infinite set, let r be a positive real number, and let V be $\ell^r(X, \mathbf{R})$ or $\ell^r(X, \mathbf{C})$. If $r \geq 1$, then $\|f\|_r$ defines a norm on V, and we take τ_2 to be the corresponding topology on V. If $0 < r \leq 1$, then $\|f - g\|_r^r$ defines a metric on V, and we take τ_2 to be the corresponding topology on V. Let τ_1 denote the topology induced on V by the topology defined on $c(X, \mathbf{R})$ or $c(X, \mathbf{C})$, as appropriate, in Section 1.9. As in the previous section, $\|f\|_r$ is semicompatible with τ_1 when $r \geq 1$, and $\|f - g\|_r$ is semicompatible with τ_1 when $0 < r \leq 1$. It is well known and not too difficult to show that V is separable with respect to τ_2 under these conditions. This uses the fact that functions with finite support in X are dense in these spaces. Thus (5.35) in Section 5.4 holds in this situation, as before. There are analogous statements for $V = c_0(X, \mathbf{R})$ or $c_0(X, \mathbf{C})$, with τ_2 taken to be the topology corresponding to $\|f\|_\infty$.

5.8 Some refinements

Let V be a topological vector space over the real or complex numbers, and let N be a seminorm on V that is compatible with the given topology on V. Also let $\mathcal{E}(N)$ be the collection of linear functionals on V that satisfy (5.49) for every $v \in V$, as in Section 5.6. If $\Lambda(N)$ is a nonempty subset of $\mathcal{E}(N)$, then

$$\sup\{|\lambda(v)| : \lambda \in \Lambda(N)\} \leq N(v) \tag{5.54}$$

for every $v \in V$, by definition of $\mathcal{E}(N)$. We shall be interested in situations where

$$\sup\{|\lambda(v)| : \lambda \in \Lambda(N)\} = N(v) \tag{5.55}$$

for every $v \in V$. Remember that this holds when $\Lambda(N) = \mathcal{E}(N)$, as in (5.51) in Section 5.6.

Note that the linear functional on V that is identically equal to 0 is automatically an element of $\mathcal{E}(N)$. Thus we can always take $\Lambda(N)$ to be nonempty, by including this linear functional. If $\Lambda(N) \subseteq \Lambda_1(N) \subseteq \mathcal{E}(N)$ and (5.55) holds for every $v \in V$, then it is easy to see that (5.55) also holds with $\Lambda(N)$ replaced by $\Lambda_1(N)$.

Let E be a subset of V, let $\overline{E}$ be the closure of E in V with respect to the given topology on V, and let $\overline{E}_N$ be the closure of E in V with respect to the topology determined on V by the semimetric associated to N. The hypothesis that N be compatible with the given topology on V means that the given topology on V is at least as strong as the topology determined by N, so that

$$\overline{E} \subseteq \overline{E}_N. \tag{5.56}$$

If (5.55) holds for every $v \in E$, then one can check that (5.55) also holds for every $v \in \overline{E}_N$. This implies that (5.55) holds for every $v \in \overline{E}$, by (5.56). In particular, if E is dense in V, then it follows that (5.55) holds for every $v \in V$.

As in Section 5.6, the Hahn–Banach theorem implies that for each $v \in V$ there is a $\lambda \in \mathcal{E}(N)$ such that $\lambda(v) = N(v)$. If $E \subseteq V$ has only finitely or countably many elements, then it follows that there is a subset $\Lambda(N)$ of $\mathcal{E}(N)$ such that $\Lambda(N)$ has only finitely or countably many elements and (5.55) holds for every $v \in E$. If V is separable, then we can also choose E to be dense in V. This implies that (5.55) holds for every $v \in V$, as in the previous paragraph. Thus, if V is separable, then there is a nonempty subset $\Lambda(N)$ of $\mathcal{E}(N)$ such that $\Lambda(N)$ has only finitely or countably many elements and (5.55) holds for every $v \in V$.

Remember that the elements of $\mathcal{E}(N)$ are continuous with respect to the given topology on V, so that $\mathcal{E}(N) \subseteq V'$. Let $\Lambda(N)$ be a nonempty subset of $\mathcal{E}(N)$ that satisfies (5.55) for every $v \in V$. Also let τ_1 be any topology on V such that every element of $\Lambda(N)$ is continuous on V with respect to τ_1. Using (5.55), we get that N is lower semicontinuous on V with respect to τ_1, as in Section 5.5. If V is a topological vector space with respect to τ_1, then it follows that N is semicompatible with τ_1 as a seminorm on V.

Let $\mathcal{N}$ be a nonempty collection of finitely or countably many seminorms on V, each of which is compatible with the given topology on V. If V is separable, then for each $N \in \mathcal{N}$, there is a nonempty subset $\Lambda(N)$ of $\mathcal{E}(N)$ such that $\Lambda(N)$ has only finitely or countably many elements and (5.55) holds for every $v \in V$, as before. Put

$$\Lambda = \bigcup_{N \in \mathcal{N}} \Lambda(N), \tag{5.57}$$

which is a subset of V' with only finitely or countably many elements. By construction,

$$\Lambda(N) \subseteq \Lambda \cap \mathcal{E}(N) \tag{5.58}$$

for every $N \in \mathcal{N}$. If $\mathcal{N}$ is nondegenerate on V, then it is easy to see that Λ is nondegenerate on V as well.

Let τ_2 be the given topology on V, and suppose now that this is the same as the topology determined on V by $\mathcal{N}$ as in Section 1.5. Thus each $N \in \mathcal{N}$ is automatically compatible with τ_2. Let us continue to suppose that $\mathcal{N}$ has only finitely or countably many elements, and that V is separable with respect to τ_2. Let Λ be as in the preceding paragraph, and let τ_1 be the weak topology on V associated to Λ as in Section 1.13. Note that $\mathcal{N}$ is nondegenerate on V under these conditions, so that Λ is nondegenerate on V too. Every $N \in \mathcal{N}$ is semicompatible with τ_1 on V, because of (5.55), as before. It follows that (5.35) in Section 5.4 holds in this situation with $X = V$, as in the previous section.

5.9 Inner products

Let V be a vector space over the real or complex numbers, let $\langle v, w\rangle$ be an inner product on V, and let $\|v\|$ be the corresponding norm on V, as in Section 2.18. If $w \in V$, then

$$\lambda_w(v) = \langle v, w\rangle \tag{5.59}$$

defines a bounded linear functional on V, by the Cauchy–Schwarz inequality (2.139). More precisely, the Cauchy–Schwarz inequality implies that the dual norm of λ_w on V with respect to $\|\cdot\|$ is less than or equal to $\|w\|$. It is easy to see that the dual norm of λ_w on V is actually equal to $\|w\|$, using the fact that

$$\lambda_w(w) = \langle w, w\rangle = \|w\|^2. \tag{5.60}$$

If V is a Hilbert space, then it is well known that every bounded linear functional on V is of the form (5.59) for some $w \in V$.

If $v \in V$, then

$$\|v\| = \sup\{|\langle v, w\rangle| : w \in V,\ \|w\| \le 1\}. \tag{5.61}$$

Indeed, the right side of (5.61) is less than or equal to $\|v\|$, by the Cauchy–Schwarz inequality. To get the opposite inequality, one can take $w = v/\|v\|$ when $v \neq 0$, and of course (5.61) is trivial when $v = 0$. Equivalently,

$$\|v\| = \sup\{|\lambda_w(v)| : w \in V,\ \|w\| \le 1\} \tag{5.62}$$

for every $v \in V$, using the notation in (5.59). This implies (5.51) in Section 5.6 in this context.

Let τ_2 be the topology determined on V by the metric associated to the norm. Also let τ_1 be the corresponding weak topology on V, as in Section 1.13, at least for the moment. As in Section 5.6, (5.62) implies that $\|\cdot\|$ is lower semicontinuous with respect to τ_1, so that $\|\cdot\|$ is semicompatible with τ_1 as a norm on V. This situation is simpler than before, because we get (5.62) directly from the inner product. If V is separable with respect to τ_2, then it follows that (5.35) in Section 5.4 holds with $X = V$, as in Section 5.7.

Let E be a subset of V whose closure in V with respect to the metric associated to the norm is the closed unit ball in V. Under these conditions,

$$\|v\| = \sup\{|\langle v, w\rangle| : w \in E\} \tag{5.63}$$

for every $v \in V$. More precisely, the right side of (5.63) is automatically less than or equal to the right side of (5.61), because E is contained in the closed unit ball in V. To get the opposite inequality, one can approximate $w \in V$ with $\|w\| \leq 1$ by elements of E with respect to the metric associated to the norm, by hypothesis. As before, (5.63) is the same as saying that

$$\|v\| = \sup\{|\lambda_w(v)| : w \in E\} \tag{5.64}$$

for every $v \in V$.

Let $\Lambda(E)$ be the collection of linear functionals on V of the form λ_w with $w \in E$. Note that $\Lambda(E)$ is nondegenerate on V under these conditions. Let τ_2 be the topology on V determined by the norm again, and now let τ_1 be the weak topology on V associated to $\Lambda(E)$ as in Section 1.13. The usual weak topology on V, using all continuous linear functionals on V, is at least as strong as the weak topology corresponding to $\Lambda(E)$, because $\Lambda(E) \subseteq V'$. Using (5.64), we get that $\|\cdot\|$ is lower semicontinuous on V with respect to τ_1, as in Section 5.5. If V is separable with respect to τ_2, then it follows that (5.35) in Section 5.4 holds with $X = V$, as in Section 5.7 again. Separability of V also means that we can take E to have only finitely or countably many elements.

Let I be a nonempty set, and let $\{v_j\}_{j\in I}$ be an orthonormal family of vectors in V indexed by I. Suppose that the linear span of the v_j's, $j \in I$, is dense in V with respect to the metric associated to the norm. This implies that

$$\|v\| = \Big(\sum_{j\in I} |\langle v, v_j\rangle|^2\Big)^{1/2} \tag{5.65}$$

for every $v \in V$, as in (2.159) in Section 2.19. Let Λ_I be the collection of linear functionals on V of the form λ_{v_j}, $j \in I$, which is nondegenerate on V under these conditions. Let τ_2 be the topology determined on V by the norm again, and let τ_1 be the weak topology on V associated to Λ_I as in Section 1.13. As before, the usual weak topology on V, using all continuous linear functionals on V, is at least as strong as the weak topology associated to Λ_I, because $\Lambda_I \subseteq V'$. It follows from (5.65) that $\|\cdot\|$ is lower semicontinuous on V with respect to τ_1, as in Section 5.6.

If V is separable with respect to the metric associated to the norm, then (5.35) in Section 5.4 holds with $X = V$, as in Section 5.7. In this case, it is well known that I can have only finitely or countably many elements, by standard arguments. This uses the fact that

$$\|v_j - v_l\|^2 = \|v_j\|^2 + \|v_l\|^2 = 2 \tag{5.66}$$

for every $j, l \in I$ with $j \neq l$, because of orthonormality.

Of course, if V has finite dimension, then there is an orthonormal basis for V with only finitely many elements, by the Gram–Schmidt process. Similarly, suppose that there is a sequence of vectors in V whose linear span in V is dense in V with respect to the metric associated to the norm. The existence of such a sequence is equivalent to the separability of V with respect to the metric associated to the norm. In this case, the Gram–Schmidt process leads to a finite or infinite sequence of orthonormal vectors in V whose linear span is dense in V with respect to the metric associated to the norm.

If V is a Hilbert space, then there is a well-known argument for getting the existence of an orthonormal basis in V, using the axiom of choice. More precisely, one can use Zorn's lemma or Hausdorff's maximality principle to get a maximal collection of orthonormal vectors in V. Using maximality, one can show that the linear span of this orthonormal collection is dense in V.

5.10 Supremum seminorms

Let X be a nonempty set, and put

$$\lambda_x(f) = f(x) \tag{5.67}$$

for every $x \in X$ and real or complex-valued function f on X. This defines a continuous linear functional on each of the spaces $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ of real and complex-valued functions on X, with respect to the topologies defined on these spaces in Section 1.9. These topologies on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ are the same as the weak topologies on these spaces associated to the collections of linear functionals of the form λ_x with $x \in X$, as mentioned in Section 1.13. If E is a nonempty subset of X, then put

$$N_E(f) = \sup_{x \in E} |f(x)| = \sup_{x \in E} |\lambda_x(f)| \tag{5.68}$$

for every real or complex-valued function f on X, where the supremum is defined as a nonnegative extended real number. Note that N_E is lower semicontinuous on each of $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, with respect to the topologies defined in Section 1.9.

Now let X be a nonempty topological space. Of course, the restriction of λ_x to each of the spaces $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ of continuous real and complex-valued functions on X defines a continuous linear functional with respect to the topology defined in Section 1.7 for every $x \in X$. If E is a nonempty subset of X, then N_E is lower semicontinuous with respect to the topologies induced on $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ by the topologies defined on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ in Section 1.9, respectively, as before. In particular, N_E is lower semicontinuous with respect to the usual weak topologies on $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$, determined by the collections of all continuous linear functionals on these spaces. If E is contained in a compact subset of X, and f is a continuous real or complex-valued function on X, then f is bounded on E, so that (5.68) is finite. If E is compact, then N_E is the same as the supremum seminorm associated to E on $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$, as in (1.58) in Section 1.7. Observe that (5.68) implies (5.51) in Section 5.6 in this situation. If E is any nonempty subset of X, then it is easy to see that

$$N_{\overline{E}}(f) = N_E(f) \tag{5.69}$$

for every continuous real or complex-valued function f on X, where $\overline{E}$ is the closure of E in X.

Remember that $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$ denote the spaces of bounded continuous real and complex-valued functions on X, respectively. If E is any nonempty subset of X, then N_E defines a seminorm on each of these spaces, which is the same as the supremum norm $\|f\|_{sup}$ when $E = X$. As before, these seminorms are lower semicontinuous with respect to the topologies induced on $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$ by the topologies defined on $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$, respectively, in Section 1.9. This implies that these seminorms are lower semicontinuous with respect to the usual weak topologies on $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$, because λ_x is continuous with respect to the supremum norm on these spaces for every $x \in X$. In this situation as well, (5.68) implies (5.51) in Section 5.6.

Let E be a dense subset of X, so that

$$N_X(f) = N_E(f) \tag{5.70}$$

for every continuous real or complex-valued function f on X, as in (5.69). Note that

$$\{\lambda_x : x \in E\} \tag{5.71}$$

is a nondegenerate collection of linear functionals on each of $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$. Using (5.70), we get that N_X is lower semicontinuous with respect to the weak topologies on $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ associated to (5.71) as in Section 1.13. In particular, the supremum norm is lower semicontinuous with respect to the weak topologies on $C_b(X, \mathbf{R})$ and $C_b(X, \mathbf{C})$ associated to (5.71), considered now as a collection of linear functionals on each of these spaces. Of course, if X is separable, then we can take E to have only finitely or countably many elements.

5.11 Some separability conditions

Let X be a nonempty compact metric space. Remember that continuous real and complex-valued functions on X are bounded and uniformly continuous. In this case, one can verify that the spaces $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ of continuous real and complex-valued functions on X are separable with respect to the supremum metric. More precisely, it suffices to cover $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ by countably many sets of arbitrarily small diameter with respect to the supremum metric, as in Section 5.1. Let $\delta > 0$ be given, so that X can be covered by finitely many open balls of radius δ, by compactness. If a given real or complex-valued function f on X does not vary too much on these balls, then f will be determined to within a uniformly small amount on X by its values on the finite set of centers of these balls of radius δ. Because $\mathbf{R}$ and $\mathbf{C}$ are separable with respect to their standard metrics, one can use this to cover sets of functions like these by countably many sets of small diameter with respect to the supremum metric. Every continuous real or complex-valued function on X satisfies conditions like this for some $\delta > 0$, by uniform continuity. If one takes $\delta = 1/l$ for each positive integer l, then one can cover $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ by countably many sets of small diameter with respect to the supremum metric, as desired.

Now let X be a nonempty compact Hausdorff topological space. If there is a base $\mathcal{B}$ for the topology of X with only finitely or countably many elements, then $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ are separable with respect to the supremum metric. In this situation, Urysohn's metrization theorem implies that there is a metric on X that determines the same topology, so that one can reduce to the remarks in the previous paragraph. One can also give a more direct argument that is analogous to the earlier one, as follows. As before, we would like to cover $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ by countably many sets of arbirarily small diameter with respect to the supremum metric. Let $U_1, \ldots, U_n$ be finitely many nonempty elements of $\mathcal{B}$ that covers X, and let x_j be an element of U_j for each $j = 1, \ldots, n$. If f is a real or complex-valued function that does not vary too much on U_j for each $j = 1, \ldots, n$, then f will be approximately determined uniformly on X by its values on $x_1, \ldots, x_n$, and one can deal with these functions as before. If f is any continuous real or complex-valued function on X, then there are finite coverings of X by elements of $\mathcal{B}$ on which f does not vary too much, by continuity, compactness, and the definition of a base for the topology of X. If $\mathcal{B}$ has only finitely or countably many elements, then there are only finitely or countably many coverings of X by finitely many elements of $\mathcal{B}$. One can use this to cover $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$ by finitely or countably many sets with small diameter with respect to the supremum norm, as desired.

Of course, if X is a compact metric space, then it is well known that X is separable, and hence has a base for its topology with only finitely or countably many elements. Thus the conditions in the previous two paragraphs are equivalent. Suppose that X satisfies either of these equivalent conditions, and let E be a dense subset of X, so that (5.71) is a nondegenerate collection of linear functionals on each of $C(X, \mathbf{R})$ and $C(X, \mathbf{C})$. Let us take $V = C(X, \mathbf{R})$ or $C(X, \mathbf{C})$, and τ_2 to be the topology determined on V by the supremum norm. Also let τ_1 be the weak topology on V associated to (5.71) as in Section 1.13. Remember that the supremum norm on V is semicompatible with τ_1, as in the preceding section. Using the separability of V with respect to τ_2, as in the previous paragraphs, we get that (5.35) in Section 5.4 holds, as in Section 5.7. Note that we can take E to have only finitely or countably many elements in this situation, because X is separable, so that (5.71) has only finitely or countably many elements.

Let X be a nonempty compact Hausdorff topological space again, so that continuous real and complex-valued functions on X are bounded. If $C(X, \mathbf{R})$ is separable with respect to the supremum metric, then it is well known that there is a base for the topology of X with only finitely or countably many elements.

To see this, let $\mathcal{E}$ be a countable dense subset of $C(X, \mathbf{R})$, and put

$$U_f = \{x \in X : f(x) < 1/2\} \tag{5.72}$$

for every $f \in \mathcal{E}$. Thus U_f is an open set in X for every $f \in \mathcal{E}$, so that

$$\mathcal{B}(\mathcal{E}) = \{U_f : f \in \mathcal{E}\} \tag{5.73}$$

is a collection of open subsets of X with only finitely or countably many elements. We would like to verify that (5.73) is a base for the topology of X. Let $p \in X$ and an open set $U \subseteq X$ be given, with $p \in U$. By Urysohn's lemma, there is a continuous real-valued function $f_{p,U}$ on X such that $f_{p,U}(p) = 0$ and $f_{p,U}(x) = 1$ when $x \in X \setminus U$. Because $\mathcal{E}$ is dense in $C(X, \mathbf{R})$, there is an $f \in \mathcal{E}$ such that

$$|f(x) - f_{p,U}(x)| < 1/2 \tag{5.74}$$

for every $x \in X$. This implies that $f(p) < 1/2$ and $f(x) > 1/2$ for every $x \in X \setminus U$, because of the corresponding properties of $f_{p,U}$. It follows that $p \in U_f$ and $U \subseteq U_f$, so that (5.73) is a base for the topology of X, as desired.

5.12 Countable products

Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of topological spaces, and let $Y = \prod_{j=1}^{\infty} Y_j$ be their Cartesian product. Also let P_n be the natural mapping from Y into $\prod_{j=1}^{n} Y_j$ for each positive integer n, which simply keeps the first n coordinates of every element of Y. Of course, Y can be identified with

$$\Big(\prod_{j=1}^{n} Y_j\Big) \times \Big(\prod_{j=n+1}^{\infty} Y_j\Big) \tag{5.75}$$

in an obvious way for each $n \in \mathbf{Z}_+$, and P_n corresponds to the projection of (5.75) onto the first factor. If $E_n \subseteq \prod_{j=1}^{n} Y_j$, then $P_n^{-1}(E_n)$ corresponds to

$$E_n \times \Big(\prod_{j=n+1}^{\infty} Y_j\Big) \tag{5.76}$$

in (5.75). Put

$$\tau_{Y,n} = \Big\{P_n^{-1}(U_n) : U_n \subseteq \prod_{j=1}^{n} Y_j \text{ is an open set}\Big\} \tag{5.77}$$

for each $n \in \mathbf{Z}_+$, where more precisely U_n should be an open set in $\prod_{j=1}^{n} Y_j$ with respect to the product topology associated to the given topologies on $Y_1, \ldots, Y_n$. This defines a topology on Y for each $n \in \mathbf{Z}_+$, which is the weakest topology on Y with respect to which P_n is continuous, using the product topology on $\prod_{j=1}^{n} Y_j$. Equivalently, $\tau_{Y,n}$ consists of subsets of Y that correspond to subsets of (5.75) of the form (5.76), where E_n is an open set in $\prod_{j=1}^n Y_j$ with respect to the product topology.

Let $P_{n+1,n}$ be the natural projection from $\prod_{j=1}^{n+1} Y_j$ into $\prod_{j=1}^n Y_j$ for each $n \in \mathbf{Z}_+$, which keeps the first n coordinates of each element of the domain. This mapping is clearly continuous with respect to the corresponding product topologies. As before, we can identify $\prod_{j=1}^{n+1} Y_j$ with

$$\Big(\prod_{j=1}^n Y_j\Big) \times Y_{n+1} \tag{5.78}$$

for each $n \in \mathbf{Z}_+$, so that $P_{n+1,n}$ corresponds to the projection onto the first factor in (5.78). If $E_n \subseteq \prod_{j=1}^n Y_j$, then $P_{n+1,n}^{-1}(E_n)$ corresponds to $E_n \times Y_{n+1}$ in (5.78). If $U_n \subseteq \prod_{j=1}^n Y_j$ is an open set with respect to the product topology, then $P_{n+1,n}^{-1}(U_n)$ is an open set in $\prod_{j=1}^{n+1} Y_j$ with respect to its product topology, because $P_{n+1,n}$ is continuous.

As usual, we can identify Y with

$$\Big(\prod_{j=1}^n Y_j\Big) \times Y_{n+1} \times \Big(\prod_{j=n+2}^\infty Y_j\Big) \tag{5.79}$$

for each $n \in \mathbf{Z}_+$. Using this identification, P_n corresponds to the projection of (5.79) onto the first factor, and P_{n+1} corresponds to the projection of (5.79) onto the product of the first two factors. By construction,

$$P_n = P_{n+1,n} \circ P_{n+1} \tag{5.80}$$

for each $n \in \mathbf{Z}_+$ as mappings on Y, which can also be seen in terms of (5.79). Observe that

$$P_n^{-1}(E_n) = (P_{n+1,n} \circ P_{n+1})^{-1}(E_n) = P_{n+1}^{-1}(P_{n+1,n}^{-1}(E_n)) \tag{5.81}$$

for every $E_n \subseteq \prod_{j=1}^n Y_j$, by (5.80). This corresponds to

$$E_n \times Y_{n+1} \times \Big(\prod_{j=n+2}^\infty Y_j\Big) \tag{5.82}$$

as a subset of (5.79).

Using (5.81), we get that

$$\tau_{Y,n} \subseteq \tau_{Y,n+1} \tag{5.83}$$

for every $n \in \mathbf{Z}_+$. One can also look at this in terms of the identification of Y with (5.79), and the other identifications mentioned earlier. Let τ_Y be the product topology on Y, associated to the given topologies on the Y_j's. Note that

$$\tau_{Y,n} \subseteq \tau_Y \tag{5.84}$$

for every $n \in \mathbf{Z}_+$, which is basically the same as saying that P_n is continuous with respect to τ_Y on Y. In fact, one can check that

$$\bigcup_{n=1}^{\infty} \tau_{Y,n} \tag{5.85}$$

is a base for τ_Y, using the definition of the product topology.

Let $\mathcal{A}(\tau_{Y,n})$ be the σ-algebra of Borel sets in Y with respect to $\tau_{Y,n}$ for each $n \in \mathbf{Z}_+$. Equivalently,

$$\mathcal{A}(\tau_{Y,n}) = \Big\{P_n^{-1}(E_n) : E_n \subseteq \prod_{j=1}^{n} Y_j \text{ is a Borel set}\Big\} \tag{5.86}$$

for each $n \in \mathbf{Z}_+$, where more precisely E_n should be a Borel set in $\prod_{j=1}^n Y_j$ with respect to the associated product topology. The equivalence of these two descriptions of $\mathcal{A}(\tau_{Y,n})$ follows from remarks in Section 2.10. Using (5.83) and the first description of $\mathcal{A}(\tau_{Y,n})$, we get that

$$\mathcal{A}(\tau_{Y,n}) \subseteq \mathcal{A}(\tau_{Y,n+1}) \tag{5.87}$$

for every $n \in \mathbf{Z}_+$. This can also be obtained from the second description (5.86) of $\mathcal{A}(\tau_{Y,n})$, using the mapping $P_{n+1,n}$ mentioned earlier.

Let $\mathcal{A}(\tau_Y)$ be the σ-algebra of Borel sets in Y with respect to τ_Y. Note that

$$\mathcal{A}(\tau_{Y,n}) \subseteq \mathcal{A}(\tau_Y) \tag{5.88}$$

for every $n \in \mathbf{Z}_+$, because of (5.84) and the first description of $\mathcal{A}(\tau_{Y,n})$ in the preceding paragraph. As before, this can also be obtained from (5.87) and the fact that P_n is continuous, and hence Borel measurable. It is easy to see that

$$\bigcup_{n=1}^{\infty} \mathcal{A}(\tau_{Y,n}) \tag{5.89}$$

is an algebra of subsets of Y, using (5.88) and the fact that $\mathcal{A}(\tau_{Y,n})$ is a σ-algebra of subsets of Y for each n. As in Section 5.3, $\mathcal{A}(\tau_Y)$ is the same as the smallest σ-algebra of subsets of Y that contains (5.85), or that contains (5.89).

5.13 Collections of mappings

Let I be a nonempty set, and let Y_j be a set for each $j \in I$. Also let X be a set, and let f_j be a mapping from X into Y_j for each $j \in I$. Suppose that τ_{Y_j} is a topology on Y_j for each $j \in I$, so that

$$\{f_j^{-1}(V_j) : V_j \in \tau_{Y_j}\} \tag{5.90}$$

is the weakest topology on X with respect to which f_j is continuous, as in Section 2.10. Consider the topology τ_X on X generated by the topologies (5.90) with $j \in I$, so that

$$\bigcup_{j \in I} \{f_j^{-1}(V_j) : V_j \in \tau_{Y_j}\} \tag{5.91}$$

is a sub-base for τ_X. By construction, f_j is continuous with respect to τ_X for each $j \in I$, and τ_X is the weakest topology on X with this property.

Put $Y = \prod_{j \in I} Y_j$, and let τ_Y be the product topology on Y corresponding to τ_{Y_j} on Y_j for each j. Also let f be the mapping from X into Y whose jth coordinate is equal to f_j for each j. It is easy to see that f is continuous with respect to τ_X and τ_Y, and that τ_X is the weakest topology on X with this property. Equivalently,

$$\tau_X = \{f^{-1}(V) : V \in \tau_Y\}. \tag{5.92}$$

as in (2.83). It follows that the σ-algebra of Borel sets in X with respect to τ_X is the same as

$$\{f^{-1}(E) : E \subseteq Y \text{ is a Borel set}\}, \tag{5.93}$$

as in Section 2.10, and using the Borel sets in Y with respect to τ_Y in (5.93).

Let us now take $I = \mathbf{Z}_+$, as in the previous section. If $n \in \mathbf{Z}_+$, then we let F_n be the mapping from X into $\prod_{j=1}^n Y_j$ whose jth coordinate is equal to f_j for each $j = 1, \ldots, n$. Equivalently,

$$F_n = P_n \circ f, \tag{5.94}$$

where P_n is the natural mapping from Y into $\prod_{j=1}^n Y_j$, as before. Note that F_n is continuous with respect to τ_X on X and the product topology on $\prod_{j=1}^n Y_j$ corresponding to τ_{Y_j} on Y_j. This is basically the same as saying that f_j is continuous with respect to τ_X for $j = 1, \ldots, n$, which holds by construction.

Put

$$\tau_{X,n} = \left\{F_n^{-1}(U_n) : U_n \subseteq \prod_{j=1}^n Y_j \text{ is an open set}\right\}, \tag{5.95}$$

for each $n \in \mathbf{Z}_+$, where more precisely U_n should be an open set in $\prod_{j=1}^n Y_j$ with respect to the product topology associated to the given topologies on $Y_1, \ldots, Y_n$. This defines a topology on X for each $n \in \mathbf{Z}_+$, which is the weakest topology on X with respect to which F_n is continuous, using the product topology on $\prod_{j=1}^n Y_j$. Equivalently,

$$\tau_{X,n} = \{f^{-1}(V_n) : V_n \in \tau_{Y,n}\} \tag{5.96}$$

for each $n \in \mathbf{Z}_+$, where $\tau_{Y,n}$ is as in (5.77), because of (5.94). Observe that

$$\tau_{X,n} \subseteq \tau_X \tag{5.97}$$

for each $n \in \mathbf{Z}_+$, because F_n is continuous with respect to τ_X, as in the preceding paragraph. This could also be obtained from (5.96), using (5.84) and (5.92).

Let $P_{n+1,n}$ be the natural projection from $\prod_{j=1}^{n+1} Y_j$ into $\prod_{j=1}^{n} Y_j$ for each $n \in \mathbf{Z}_+$ again, so that

$$P_{n+1,n} \circ F_{n+1} = F_n \tag{5.98}$$

for every n. One can check that

$$\tau_{X,n} \subseteq \tau_{X,n+1} \tag{5.99}$$

for each $n \in \mathbf{Z}_+$, using (5.81), (5.95), and (5.98). This can also be obtained from (5.83) and (5.96).

Of course,

$$\bigcup_{n=1}^{\infty} \tau_{X,n} \tag{5.100}$$

is contained in τ_X, by (5.97), and one can verify that (5.100) is a base for τ_X. More precisely, (5.91) is contained in (5.100), which implies that (5.100) is a sub-base for τ_X. To get that (5.100) is a base for τ_X, one can use the fact that (5.100) is closed under finite intersections. Alternatively, (5.100) is the same as

$$\left\{f^{-1}(V) : V \in \bigcup_{n=1}^{\infty} \tau_{Y,n}\right\}, \tag{5.101}$$

because of (5.96). This is a base for τ_X, because of (5.92) and the fact that (5.85) is a base for τ_Y.

Let $\mathcal{A}(\tau_{X,n})$ be the σ-algebra of Borel sets in X with respect to $\tau_{X,n}$ for each $n \in \mathbf{Z}_+$. Equivalently,

$$\mathcal{A}(\tau_{X,n}) = \left\{F_n^{-1}(E_n) : E_n \subseteq \prod_{j=1}^{n} Y_j \text{ is a Borel set}\right\}, \tag{5.102}$$

as in Section 2.10, and using the product topology on $\prod_{j=1}^{n} Y_j$. Similarly,

$$\mathcal{A}(\tau_{X,n}) = \{f^{-1}(A_n) : A_n \in \mathcal{A}(\tau_{Y,n})\}, \tag{5.103}$$

where $\mathcal{A}(\tau_{Y,n})$ is the σ-algebra of Borel sets in Y with respect to $\tau_{Y,n}$, as in the previous section. This uses (5.96) and the remarks in Section 2.10 again. Note that

$$\mathcal{A}(\tau_{X,n}) \subseteq \mathcal{A}(\tau_{X,n+1}) \tag{5.104}$$

for each $n \in \mathbf{Z}_+$, by (5.99).

Let $\mathcal{A}(\tau_X)$ be the σ-algebra of Borel sets in X with respect to τ_X. This is the same as

$$\mathcal{A}(\tau_X) = \{f^{-1}(A) : A \in \mathcal{A}(\tau_Y)\}, \tag{5.105}$$

where $\mathcal{A}(\tau_Y)$ is the σ-algebra of Borel sets in Y with respect to τ_Y, as in the previous section. This uses (5.92) and the remarks in Section 2.10, as usual. We also have that

$$\mathcal{A}(\tau_{X,n}) \subseteq \mathcal{A}(\tau_X) \tag{5.106}$$

for each $n \in \mathbf{Z}_+$, because of (5.97). This corresponds to (5.88) as well, using (5.103) and (5.105).

Observe that

$$\bigcup_{n=1}^{\infty} \mathcal{A}(\tau_{X,n}) \tag{5.107}$$

is an algebra of subsets of X, because of (5.104) and the fact that $\mathcal{A}(\tau_{X,n})$ is a σ-algebra of subsets of X for each n. This is the same as

$$\Big\{f^{-1}(A) : A \in \bigcup_{n=1}^{\infty} \mathcal{A}(\tau_{Y,n})\Big\}, \tag{5.108}$$

because of (5.103). Of course, (5.107) is contained in $\mathcal{A}(\tau_X)$, because of (5.106). As in Section 5.3, $\mathcal{A}(\tau)$ is the same as the smallest σ-algebra of subsets of X that contains (5.100), or equivalently that contains (5.107). This corresponds to the analogous statement for $\mathcal{A}(\tau_Y)$ mentioned in the previous section, using the remarks in Section 2.10 again.

5.14 Some more comparisons of topologies

Let X be a topological space, and let $X_1, X_2, X_3, \ldots$ be a sequence of Borel sets in X, so that $\bigcup_{j=1}^{\infty} X_j$ is a Borel set in X as well. If $E \subseteq X$ is a Borel set, then $E \cap X_j$ is a Borel set for each j. In the other direction, if $E \subseteq X$ has the property that $E \cap X_j$ is a Borel set for each j, then

$$E \cap \Big(\bigcup_{j=1}^{\infty} X_j\Big) = \bigcup_{j=1}^{\infty} (E \cap X_j) \tag{5.109}$$

is a Borel set in X too. In particular, if $E \subseteq \bigcup_{j=1}^{\infty} X_j$, then it follows that E is a Borel set in X. Remember that a subset of X_j is a Borel set in X if and only if it is Borel set in X_j with respect to the induced topology, as in Section 2.10.

Let X_0 be a subset of X. If $E \subseteq X$ is an F_σ set in X, then $E \cap X_0$ is an F_σ set in X_0, with respect to the induced topology. If X_0 is a closed set in X, then every closed subset of X_0 with respect to the induced topology is a closed set in X. In this case, every subset of X_0 that is an F_σ set with respect to the induced topology is an F_σ set in X as well. Similarly, if X_0 is an F_σ set in X, then every subset of X_0 that is an F_σ set with respect to the induced topology is an F_σ set in X too.

Let $X_1, X_2, X_3, \ldots$ be a sequence of F_σ sets in X. If $E \subseteq X$ is an F_σ set, then $E \cap X_j$ is an F_σ in X for each j. If E is any subset of X with the property that $E \cap X_j$ is an F_σ set in X for each j, then (5.109) is an F_σ set in X. It follows that E is an F_σ set in X when $E \subseteq \bigcup_{j=1}^{\infty} X_j$, as before.

Suppose for the moment that

(5.110) every open subset of X is an F_σ set.

If X_0 is any subset of X, then it is easy to see that X_0 satisfies the analogous condition with respect to the induced topology. More precisely, if $E \subseteq X_0$ is relatively open, then there is an open set $U \subseteq X$ such that $E = U \cap X_0$. By hypothesis, U is an F_σ set in X, so that E can be expressed as the union of countably many relatively closed sets in X_0. If X_0 is an F_σ set in X, then it follows that E is an F_σ set in X, as mentioned earlier.

Let X be any topological space again, and let $X_1, X_2, X_3, \ldots$ be a sequence of closed or F_σ subsets of X such that

$$X = \bigcup_{j=1}^{\infty} X_j. \tag{5.111}$$

Consider the following condition, for each $j \in \mathbf{Z}_+$:

(5.112) every relatively open subset of X_j is an F_σ set in X.

If (5.110) holds, then (5.112) holds for every $j \in \mathbf{Z}_+$, as in the previous paragraph. Conversely, if (5.112) holds for every $j \in \mathbf{Z}_+$, then (5.110) holds. Indeed, if $U \subseteq X$ is an open set, then $U \cap X_j$ is a relatively open set in X_j for each j. This implies that $U \cap X_j$ is an F_σ set in X for each j, by hypothesis. It follows that U is an F_σ set in X, as desired, because

$$U = \bigcup_{j=1}^{\infty} (U \cap X_j), \tag{5.113}$$

by (5.111). Note that (5.112) is equivalent to asking that

(5.114) every relatively open subset of X_j is an F_σ set with respect to the induced topology on X_j,

because X_j is supposed to be an F_σ set in X.

Let X be a set, and let us suppose for the rest of the section that τ_1, τ_2 are topologies on X such that

$$\tau_1 \subseteq \tau_2, \tag{5.115}$$

so that that Borel sets in X with respect to τ_1 are Borel sets with respect to τ_2. Let X_0 be a subset of X that is a Borel set with respect to τ_1, and hence with respect to τ_2. Under these conditions, a subset E of X_0 is a Borel set with respect to the topology induced on X_0 by τ_1 if and only if E is a Borel set in X with respect to τ_1, and similarly for τ_2, as in Section 2.10. If

(5.116) the topologies induced on X_0 by τ_1 and τ_2 are the same,

then we get the same σ-algebras of Borel subsets of X_0 with respect to these induced topologies. In this case, it follows that $E \subseteq X_0$ is a Borel set in X with respect to τ_1 if and only if E is a Borel set in X with respect to τ_2.

Suppose now that $X_0 \subseteq X$ is an F_σ set with respect to τ_1, and hence with respect to τ_2. Remember that a subset E of X_0 is an F_σ set with respect to the topology induced on X_0 by τ_1 if and only if E is an F_σ set in X with respect to τ_1, and similarly for τ_2. If (5.116) holds, then the collections of F_σ sets in X_0 with respect to these induced topologies are the same. This implies that $E \subseteq X_0$ is an F_σ set in X with respect to τ_1 if and only if E is an F_σ set in X with respect to τ_2 in this situation.

Suppose for the moment that X is Hausdorff with respect to τ_1, and hence with respect to τ_2. If $X_0 \subseteq X$ is compact with respect to τ_2, then X_0 is compact with respect to τ_1 as well. This implies that X_0 is a closed set in X with respect to τ_1 and τ_2, and in particular that X_0 is a Borel set in X with respect to τ_1 and τ_2. It is well known that (5.116) holds in this case. More precisely, if $E \subseteq X_0$ is a closed set with respect to τ_2, then E is compact with respect to τ_2, because X_0 is compact with respect to τ_2. This implies that E is compact with respect to τ_1, because of (5.115). It follows that E is a closed set with respect to τ_1, because τ_1 is supposed to be Hausdorff. This shows that the topology induced on X_0 by τ_1 is at least as strong as the topology induced on X_0 by τ_2 in this situation, so that (5.116) holds, using (5.115) again.

Let $X_1, X_2, X_3, \ldots$ be a sequence of subsets of X, and suppose that for each $j \in \mathbf{Z}_+$,

(5.117) the topologies induced on X_j by τ_1 and τ_2 are the same.

Suppose also that for each $j \in \mathbf{Z}_+$,

(5.118) X_j is a Borel set in X with respect to τ_1,

and hence with respect to τ_2. In particular, this means that $\bigcup_{j=1}^\infty X_j$ is a Borel set in X with respect to τ_1, and hence with respect to τ_2. Let E be a subset of $\bigcup_{j=1}^\infty X_j$ that is a Borel set in X with respect to τ_2, so that $E \cap X_j$ is a Borel set in X with respect to τ_2 for each $j \in \mathbf{Z}_+$. Using (5.117), we get that $E \cap X_j$ is a Borel set in X with respect to τ_1 for every $j \in \mathbf{Z}_+$, as before. It follows that

$$E = \bigcup_{j=1}^\infty (E \cap X_j) \tag{5.119}$$

is also a Borel set in X with respect to τ_1 in this situation. Suppose now that for each $j \in \mathbf{Z}_+$,

(5.120) X_j is an F_σ set in X with respect to τ_1,

and hence with respect to τ_2. If E is a subset of $\bigcup_{j=1}^\infty X_j$ that is an F_σ set with respect to τ_2, then $E \cap X_j$ is an F_σ set in X with respect to τ_2 for every $j \in \mathbf{Z}_+$. This implies that $E \cap X_j$ is an F_σ set in X with respect to τ_1 for each $j \in \mathbf{Z}_+$, using (5.117) again. Thus E is an F_σ set in X with respect to τ_1 too, because of (5.119).

Let us continue to ask that $X_1, X_2, X_3, \ldots$ satisfy (5.117) and (5.120) for each $j \in \mathbf{Z}_+$, and suppose now that they also satisfy (5.111). This implies that

(5.121) F_σ sets in X with respect to τ_1 and τ_2 are the same, because of (5.115) and the remarks in the preceding paragraph. In addition, we ask that

(5.122) every element of τ_1 is an F_σ set with respect to τ_1,

which is to say that (5.110) holds with respect to τ_1. This is equivalent to asking that for each $j \in \mathbf{Z}_+$,

(5.123) every relatively open set in X_j with respect to τ_1 is an F_σ set in X with respect to τ_1,

which is the same as (5.112) with respect to τ_1. Thus the equivalence of (5.122) and (5.123) is the same as the equivalence of (5.110) and (5.112) discussed earlier, using τ_1 as the topology on X. Of course, (5.123) is the same as

(5.124) every relatively open set in X_j with respect to τ_2 is an F_σ set in X with respect to τ_1

in this situation, because of (5.117). Using this, we get that

(5.125) every element of τ_2 is an F_σ set with respect to τ_1,

in essentially the same way that (5.110) was obtained from (5.112) earlier.

Chapter 6

Direct sums

6.1 The strong product topology

Let I be nonempty set, and let X_j be a topological space for each $j \in I$. A subset W of the Cartesian product $X = \prod_{j \in I} X_j$ is said to be an open set with respect to the *strong product topology* if for every $x \in W$ there are open sets $U_j \subseteq X_j$ for each $j \in I$ such that $x \in \prod_{j \in I} U_j$ and $\prod_{j \in I} U_j \subseteq W$. It is easy to see that this defines a topology on X. Equivalently, if $U_j \subseteq X_j$ is an open set for every $j \in I$, then $U = \prod_{j \in I} U_j$ is an open set in X with respect to the strong product topology, and the collection of these open sets forms a base for the strong product topology. Of course, the strong product topology on X is always at least as strong as the ordinary product topology on X. If I has only finitely many elements, then the ordinary and strong product topologies on X are the same. If I is any nonempty set and X_j is equipped with the discrete topology for every $j \in I$, then the strong product topology on X is the same as the discrete topology.

Let X_j be any topological space for each $j \in I$ again. If X_j satisfies the first separation condition for each $j \in I$, then it is easy to see that X satisfies the first separation condition with respect to the product topology. This implies that X satisfies the first separation condition with respect to the strong product topology, since the strong product topology is at least as strong as the product topology. Similarly, if X_j is Hausdorff for each $j \in I$, then X is Hausdorff with respect to the product topology. This implies that X is Hausdorff with respect to the strong product topology.

If $E_j \subseteq X_j$ is a closed set in X for every $j \in I$, then $E = \prod_{j \in I} E_j$ is a closed set in X with respect to the product topology. This implies that E is a closed set with respect to the strong product topology on X as well. Let A_j be any subset of X_j for each $j \in I$, and let $\overline{A_j}$ be the closure of A_j in X_j. One can check that every element of $\prod_{j \in I} \overline{A_j}$ is an element of the closure of $A = \prod_{j \in I} A_j$ in X with respect to the strong product topology, and hence with respect to the product topology on X. It follows that $\prod_{j \in I} \overline{A_j}$ is the same as the closure of A

in X with respect to both the product and strong product topologies.

Suppose for the moment that X_j is regular as a topological space in the strict sense for every $j \in I$. Let $W \subseteq X$ be an open set with respect to the product or strong product topology, and let x be an element of W. Thus there are open sets $V_j \subseteq X_j$ for each $j \in I$ such that $x \in \prod_{j \in I} V_j$, $\prod_{j \in I} V_j \subseteq W$, and, in the case of the ordinary product topology, $V_j = X_j$ for all but finitely many $j \in I$. As in Section 1.8, let us use x_j to denote the jth coordinate of x in X_j for each $j \in I$. Because X_j is regular in the strict sense for each $j \in I$, there is an open set $U_j \subseteq X_j$ for each $j \in I$ such that $x_j \in U_j$ and $\overline{U_j} \subseteq V_j$, where $\overline{U_j}$ is the closure of U_j in X_j again. If $V_j = X_j$, then we can simply take $U_j = X_j$ too. Thus $x \in \prod_{j \in I} U_j$, and $\prod_{j \in I} U_j$ is an open set in X with respect to the product or strong product topology, as appropriate. In both cases, the closure of $\prod_{j \in I} U_j$ in X is equal to $\prod_{j \in I} \overline{U_j}$, which is contained in $\prod_{j \in I} V_j$, and hence W, by construction. This shows that X is also regular in the strict sense with respect to the product and strong product topologies under these conditions.

6.2 Sums of commutative groups

Let I be a nonempty set again, and let A_j be a commutative group for each $j \in I$, with the group operations expressed additively. As in Section 1.8, the Cartesian product $\prod_{j \in I} A_j$ is a commutative group as well, where the group operations are defined coordinatewise. This is known as the *direct product* of the A_j's, $j \in I$. Put

$$\sum_{j \in I} A_j = \Big\{ a \in \prod_{j \in I} A_j : a_j = 0 \text{ for all but finitely many } j \in I \Big\}, \tag{6.1}$$

where a_j denotes the jth coordinate of $a \in \prod_{j \in I} A_j$ in A_j for each $j \in I$, as usual. This is a subgroup of $\prod_{j \in I} A_j$, which is known as the *direct sum* of the A_j's, $j \in I$. Of course, the direct sum and product of the A_j's are the same when I has only finitely many elements. More precisely, this holds when $A_j = \{0\}$ for all but finitely many $j \in I$.

Suppose now that A_j is a commutative topological group for each $j \in I$. One can check that $\prod_{j \in I} A_j$ is a commutative topological group with respect to the corresponding strong product topology on $\prod_{j \in I} A_j$. It follows that $\sum_{j \in I} A_j$ is a commutative topological group with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$. If $A_j = \{0\}$ for all but finitely many $j \in I$, then the strong product topology on $\prod_{j \in I} A_j$ is the same as the ordinary product topology.

Let I_1 be a subset of I, and put

$$W(I_1) = \Big\{ a \in \prod_{j \in I} A_j : a_j \neq 0 \text{ for every } j \in I_1 \Big\}. \tag{6.2}$$

Equivalently,

$$W(I_1) = \prod_{j \in I} U_j(I_1), \tag{6.3}$$

where $U_j(I_1) = A_j \setminus \{0\}$ when $j \in I_1$ and $U_j(I_1) = A_j$ when $j \in I \setminus I_1$. This implies that $W(I_1)$ is an open set in $\prod_{j \in I} A_j$ with respect to the strong product topology for every $I_1 \subseteq I$, since A_j and $A_j \setminus \{0\}$ are open subsets of A_j for every $j \in I$. The complement of $\sum_{j \in I} A_j$ in $\prod_{j \in I} A_j$ is the same as the union of $W(I_1)$ over all infinite subsets I_1 of I. Thus the complement of $\sum_{j \in I} A_j$ in $\prod_{j \in I} A_j$ is an open set with respect to the strong product topology, which means that $\sum_{j \in I} A_j$ is a closed set in $\prod_{j \in I} A_j$ with respect to the strong product topology. By contrast, $\sum_{j \in I} A_j$ is dense in $\prod_{j \in I} A_j$ with respect to the ordinary product topology. Note that $W(I_1)$ is an open set in $\prod_{j \in I} A_j$ with respect to the ordinary product topology when I_1 has only finitely many elements.

Let I_1 be a subset of I again, and put

$$A(I_1) = \Big\{a \in \prod_{j \in I} A_j : a_j = 0 \text{ for every } j \in I \setminus I_1\Big\}. \tag{6.4}$$

This is a subgroup of $\prod_{j \in I} A_j$, and a closed set with respect to the product topology. If $I_1 = \emptyset$, then $A(I_1) = \{0\}$, and otherwise there is a natural one-to-one correspondence between $A(I_1)$ and $\prod_{j \in I_1} A_j$, which is a group isomorphism. If $I_1 \neq \emptyset$, then this isomorphism is also a homeomorphism with respect to the topology induced on $A(I_1)$ by the product topology on $\prod_{j \in I} A_j$ and the product topology on $\prod_{j \in I_1} A_j$. Similarly, if $I_1 \neq \emptyset$, then this isomorphism is a homeomorphism with respect to the topology induced on $A(I_1)$ by the strong product topology on $\prod_{j \in I} A_j$ and the strong product topology on $\prod_{j \in I_1} A_j$. If I_1 has only finitely many elements, then $A(I_1)$ is contained in $\sum_{j \in I} A_j$, and the topologies induced on $A(I_1)$ by the product and strong product topologies on $\prod_{j \in I} A_j$ are the same. By construction, $\sum_{j \in I} A_j$ is the same as the union of $A(I_1)$ over all finite subsets I_1 of I.

6.3 Sums of vector spaces

Let I be a nonempty set, and suppose that either V_j is a vector space over the real numbers for every $j \in I$, or a vector space over the complex numbers for every $j \in I$. Under these conditions, $\prod_{j \in I} V_j$ is a vector space over the real or complex numbers, as appropriate, where the vector space operations are defined coordinatewise, as in Section 1.8. This may be described as the *direct product* of the V_j's, $j \in I$. Put

$$\sum_{j \in I} V_j = \Big\{v \in \prod_{j \in I} V_j : v_j = 0 \text{ for all but finitely many } j \in I\Big\}, \tag{6.5}$$

as in the previous section. This is a linear subspace of $\prod_{j \in I} V_j$, which may be described as the *direct sum* of the V_j's, $j \in I$. These definitions correspond to those in the preceding section by considering vector spaces as commutative groups with respect to addition. As before, the direct sum and product are the same when I has only finitely many elements, or when $V_j = \{0\}$ for all but finitely many $j \in I$.

Suppose that V_j is a topological vector space for each $j \in I$. In particular, this means that V_j is a commutative topological group with respect to addition for each $j \in I$. Thus $\prod_{j \in I} V_j$ is a commutative topological group with respect to the corresponding strong product topology, as in the preceding section.

If U_j is a balanced open subset of V_j that contains 0 for each $j \in I$, then

$$U = \prod_{j \in I} U_j \tag{6.6}$$

is a balanced open set in $\prod_{j \in I} V_j$ with respect to the strong product topology. Remember that balanced open subsets of V_j that contain 0 form a local base for the topology of V_j at 0 for every $j \in I$, as in Section 1.6. This implies that subsets of V as in (6.6) form a local base for the topology of $\prod_{j \in I} V_j$ at 0. In particular, this shows that balanced open sets in V that contain 0 form a local base for the topology of V at 0.

As usual, scalar multiplication on $\prod_{j \in I} V_j$ corresponds to a mapping from

$$\mathbf{R} \times \Big(\prod_{j \in I} V_j\Big) \quad \text{or} \quad \mathbf{C} \times \Big(\prod_{j \in I} V_j\Big) \tag{6.7}$$

into $\prod_{j \in I} V_j$, as appropriate. Using the standard topology on $\mathbf{R}$ or $\mathbf{C}$, and the strong product topology on $\prod_{j \in I} V_j$, we get an associated product topology on the appropriate product in (6.7). The remarks in the previous paragraph imply that scalar multiplication on $\prod_{j \in I} V_j$ is continuous as a mapping from the appropriate product in (6.7) into $\prod_{j \in I} V_j$ at $(0, 0)$, using the product topology on the appropriate product in (6.7) just mentioned. In particular, for each $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, $v \mapsto t\,v$ defines a continuous mapping from $\prod_{j \in I} V_j$ into itself, with respect to the strong product topology on $\prod_{j \in I} V_j$.

Let $v \in \prod_{j \in I} V_j$ be given, and consider

$$t \mapsto t\,v \tag{6.8}$$

as a mapping from $\mathbf{R}$ or $\mathbf{C}$, as appropriate, into $\prod_{j \in I} V_j$. If $v_j \neq 0$ for infinitely many $j \in I$, then (6.8) is not continuous at $t = 0$ with respect to the standard topology on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, and the strong product topology on $\prod_{j \in I} V_j$. This implies that $\prod_{j \in I} V_j$ is not a topological vector space with respect to the strong product topology when $V_j \neq \{0\}$ for infinitely many $j \in I$.

One can check that $\sum_{j \in I} V_j$ is a topological vector space over $\mathbf{R}$ or $\mathbf{C}$, as appropriate, with respect to the topology induced on $\sum_{j \in I} V_j$ by the strong product topology on $\prod_{j \in I} V_j$. We have already seen that $\sum_{j \in I} V_j$ is a commutative topological group with respect to addition with respect to this topology. One can get a local base for this topology on $\sum_{j \in I} V_j$ at 0 consisting of balanced open sets, using the intersections of the sets (6.6) mentioned earlier with $\sum_{j \in I} V_j$. If $v \in \sum_{j \in I} V_j$, then (6.8) is continuous as a mapping from $\mathbf{R}$ or $\mathbf{C}$, as appropriate, into $\sum_{j \in I} V_j$ with respect to this topology. This can be used to show that scalar multiplication on $\sum_{j \in I} V_j$ is continuous with respect to this topology, as desired.

Let X be a nonempty set, and remember that the spaces $c(X, \mathbf{R})$ and $c(X, \mathbf{C})$ of real and complex-valued functions on X, respectively, correspond to Cartesian products of copies of $\mathbf{R}$ and $\mathbf{C}$ indexed by X, as in Section 1.9. Similarly, the spaces $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ of real and complex-valued functions on X, respectively, with finite support correspond to direct sums of copies of $\mathbf{R}$ and $\mathbf{C}$ indexed by X. Of course, $\mathbf{R}$ and $\mathbf{C}$ may be considered as one-dimensional topological vector spaces over themselves, with respect to their standard topologies. Thus $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ may be considered as topological vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, using the topologies induced by the corresponding strong product topologies on the Cartesian products, as in the previous paragraph.

6.4 Associated Borel sets

Let I be a nonempty set, and let A_j be a commutative topological group for each $j \in I$. Because the strong product topology on $\prod_{j \in I} A_j$ is at least as strong as the ordinary product topology, the σ-algebra of Borel sets in $\prod_{j \in I} A_j$ with respect to the strong product topology contains the σ-algebra of Borel sets with respect to the ordinary product topology. Of course, if I has only finitely many elements, then the strong product topology on $\prod_{j \in I} A_j$ is the same as the ordinary product topology, so the corresponding σ-algebras of Borel sets are the same as well. This also happens when $A_j = \{0\}$ for all but finitely many $j \in I$.

Remember that $\sum_{j \in I} A_j$ is a closed set in $\prod_{j \in I} A_j$ with respect to the strong product topology, as in Section 6.2. In particular, this implies that $\sum_{j \in I} A_j$ is a Borel set in $\prod_{j \in I} A_j$ with respect to the strong product topology. It follows that the Borel sets in $\sum_{j \in I} A_j$ with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$ are the same as Borel sets in $\prod_{j \in I} A_j$ with respect to the strong product topology that are also contained in $\sum_{j \in I} A_j$, as in Section 2.10.

A subset of $\sum_{j \in I} A_j$ is a Borel set with respect to the topology induced by the product topology on $\prod_{j \in I} A_j$ if and only if it can be expressed as the intersection of $\sum_{j \in I} A_j$ with a Borel set in $\prod_{j \in I} A_j$ with respect to the product topology, as in Section 2.10. These sets are automatically Borel sets with respect to the strong product topology on $\prod_{j \in I} A_j$, by the previous remarks.

Let I_1 be a subset of I, and let $A(I_1)$ be the subgroup of $\prod_{j \in I} A_j$ defined in (6.4). Remember that $A(I_1)$ is a closed set in $\prod_{j \in I} A_j$ with respect to the product topology, and hence with respect to the strong product topology. In particular, $A(I_1)$ is a Borel set in $\prod_{j \in I} A_j$ with respect to the product and strong product topologies. As in Section 2.10, a subset of $A(I_1)$ is a Borel set with respect to the topology induced on $A(I_1)$ by the product topology on $\prod_{j \in I} A_j$ if and only if it is a Borel set in $\prod_{j \in I} A_j$ with respect to the product topology. Similarly, a subset of $A(I_1)$ is a Borel set with respect to the topology induced on $A(I_1)$ by the strong product topology on $\prod_{j \in I} A_j$ if and only if it is a Borel set in $\prod_{j \in I} A_j$ with respect to the strong product topology.

If I_1 has only finitely many elements, then the topologies induced on $A(I_1)$ by the product and strong product topologies on $\prod_{j \in I} A_j$ are the same, as before. Hence the Borel sets in $A(I_1)$ with respect to these induced topologies are the same as well. This implies that a subset of $A(I_1)$ is a Borel set with respect to the product topology on $\prod_{j \in I} A_j$ if and only if it is a Borel set with respect to the strong product topology on $\prod_{j \in I} A_j$, by the remarks in the preceding paragraph.

Let us now take $I = \mathbf{Z}_+$ for the rest of the section. Of course, if I is any countably-infinite set, then one can reduce to this case. Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of commutative topological groups, where the group operations are expressed additively. In this case, the Cartesian product $\prod_{j=1}^{\infty} A_j$ may be considered as the set of infinite sequences $a = \{a_j\}_{j=1}^{\infty}$ with $a_j \in A_j$ for each j. Similarly, the direct sum may be considered as

$$\sum_{j=1}^{\infty} A_j = \Big\{a \in \prod_{j=1}^{\infty} A_j : a_j = 0 \text{ for all but finitely many } j \geq 1\Big\}. \tag{6.9}$$

Let n be a positive integer, and put

$$A(n) = \Big\{a \in \prod_{j=1}^{\infty} A_j : a_j = 0 \text{ for each } j > n\Big\}. \tag{6.10}$$

This corresponds to $A(I_1)$ in (6.4), with $I_1 = \{1, \ldots, n\}$. Observe that $A(n)$ is a subgroup of $\sum_{j=1}^{\infty} A_j$ for each n,

$$A(n) \subseteq A(n+1) \tag{6.11}$$

for each n, and

$$\sum_{j=1}^{\infty} A_j = \bigcup_{n=1}^{\infty} A(n). \tag{6.12}$$

As before, $A(n)$ is a closed set in $\prod_{j=1}^{\infty} A_j$ with respect to the product topology for each n, and the topologies induced on $A(n)$ by the product and strong product topologies on $\prod_{j=1}^{\infty} A_j$ are the same for each n. There is an obvious group isomorphism between $A(n)$ and $\prod_{j=1}^{n} A_j$ for each n, which is a homeomorphism with respect to the topology induced on $A(n)$ by either of the product or strong product topologies on $\prod_{j=1}^{\infty} A_j$ and the product topology on $\prod_{j=1}^{n} A_j$.

It follows that $\sum_{j=1}^{\infty} A_j$ is an F_σ set in $\prod_{j=1}^{\infty} A_j$ with respect to the product topology, because of (6.12) and the fact that $A(n)$ is a closed set in $\prod_{j=1}^{\infty} A_j$ with respect to the product topology for each n. In particular, $\sum_{j=1}^{\infty} A_j$ is a Borel set in $\prod_{j=1}^{\infty} A_j$ with respect to the product topology. This implies that the Borel sets in $\sum_{j=1}^{\infty} A_j$ with respect to the topology induced by the product topology on $\prod_{j=1}^{\infty} A_j$ are the same as the Borel sets in $\prod_{j=1}^{\infty} A_j$ with respect to the product topology that are contained in $\sum_{j=1}^{\infty} A_j$, as in Section 2.10.

If n is any positive integer, then a subset of $A(n)$ is a Borel set in $\prod_{j \in I} A_j$ with respect to the strong product topology if and only if it is a Borel set in $\prod_{j \in I} A_j$ with respect to the product topology. This follows from the analogous statement for $A(I_1)$ when I_1 is finite mentioned earlier.

If $E \subseteq \sum_{j=1}^{\infty} A_j$ is a Borel set with respect to the strong product topology on $\prod_{j=1}^{\infty} A_j$, then $E \cap A(n)$ is a Borel set with respect to the strong product topology on $\prod_{j=1}^{\infty} A_j$ for each n, because $A(n)$ is a Borel set in $\prod_{j=1}^{\infty} A_j$ with respect to the strong product topology for each n. This implies that $E \cap A(n)$ is a Borel set with respect to the product topology on $\prod_{j=1}^{\infty} A_j$ for each n, as in the preceding paragraph. Thus

$$E = \bigcup_{n=1}^{\infty} (E \cap A(n)) \tag{6.13}$$

is a Borel set in $\prod_{j=1}^{\infty} A_j$ with respect to the product topology. This shows that Borel sets in $\sum_{j=1}^{\infty} A_j$ with respect to the product and strong product topologies on $\prod_{j=1}^{\infty} A_j$ are the same.

6.5 Homomorphisms on sums

Let I be a nonempty set, and let A_j be a commutative group for each $j \in I$, with the group operations expressed additively. Also let B be another commutative group, with the group operations expressed additively. Suppose that ϕ_j is a group homomorphism from A_j into B for every $j \in I$. Let $a \in \sum_{j \in I} A_j$ be given, so that $a_j \in A_j$ is equal to 0 for all but finitely many $j \in I$. This implies that $\phi_j(a_j) = 0$ in B for all but finitely many $j \in I$, so that

$$\phi(a) = \sum_{j \in I} \phi_j(a_j) \tag{6.14}$$

defines an element of B. This defines a group homomorphism ϕ from $\sum_{j \in I} A_j$ into B. If $\phi_j \equiv 0$ on A_j for all but finitely many $j \in I$, then (6.14) defines a group homomorphism from $\prod_{j \in I} A_j$ into B.

Let $l \in I$ and $x_l \in A_l$ be given, and let $\widehat{x}$ be the element of $\sum_{j \in I} A_j$ such that $\widehat{x}_j = 0$ when $j \neq l$ and $\widehat{x}_l = x_l$. If ϕ is expressed on $\sum_{j \in I} A_j$ as in (6.14), then

$$\phi_l(x_l) = \phi(\widehat{x}). \tag{6.15}$$

Thus the ϕ_j's are uniquely determined by ϕ. If ϕ is any group homomorphism from $\sum_{j \in I} A_j$ into B, (6.15) defines a group homomorphism from A_l into B for each l. Using these homomorphisms, ϕ can be expressed as in (6.14) for every $a \in \sum_{j \in I} A_j$.

Suppose now that A_j is a commutative topological group for each $j \in I$, and that B is a commutative topological group as well. If ϕ is a continuous homomorphism from $\sum_{j \in I} A_j$ into B, with respect to the topology induced on $\sum_{j \in I} A_j$ by the strong product topology on $\prod_{j \in I} A_j$, then (6.15) defines a continuous homomorphism from A_l into B for each $l \in I$. In the other direction, if ϕ_j is a continuous homomorphism from A_j into B for each $j \in I$, and if $\phi_j \equiv 0$ on A_j for all but finitely many $j \in I$, then (6.14) defines a continuous homomorphism from $\prod_{j \in I} A_j$ into B, with respect to the product topology on $\prod_{j \in I} A_j$.

Let W be any open set in B that contains 0. Using continuity of addition on B at 0, one can get an open set $W_1 \subseteq B$ such that $0 \in W_1$ and

$$W_1 + W_1 \subseteq W. \tag{6.16}$$

Repeating the process, one can get open sets $W_j \subseteq B$ for $j \geq 2$ such that $0 \in W_j$ and

$$W_j + W_j \subseteq W_{j-1} \tag{6.17}$$

for each j. Using induction, one can verify that

$$W_1 + W_2 + \cdots + W_{n-1} + W_n + W_n \subseteq W \tag{6.18}$$

for every $n \geq 1$. This implies that

$$W_1 + W_2 + \cdots + W_{n-1} + W_n \subseteq W \tag{6.19}$$

for every $n \geq 1$, because $0 \in W_n$, by construction.

Let us now take $I = \mathbf{Z}_+$, although one could deal with any countable set I in a similar way. Let $A_1, A_2, A_3, \ldots$ be a sequence of commutative topological groups, as before, and let ϕ_j be a continuous homomorphism from A_j into B for each j. If $a \in \sum_{j=1}^\infty A_j$, then $a_j = 0$ in A_j for all but finitely many j, and

$$\phi(a) = \sum_{j=1}^\infty \phi_j(a_j) \tag{6.20}$$

defines an element of B, as in (6.14). This defines a group homomorphism from $\sum_{j=1}^\infty A_j$ into B, as before, and we would like to show that ϕ is continuous with respect to the topology induced on $\sum_{j=1}^\infty A_j$ by the strong product topology on $\prod_{j=1}^\infty A_j$. Of course, it suffices to check that ϕ is continuous at 0, because ϕ is a group homomorphism.

Let W be an open subset of B that contains 0, and let $W_1, W_2, W_3, \ldots$ be as before. If j is any positive integer, then there is an open set $U_j \subseteq A_j$ such that $0 \in U_j$ and

$$\phi_j(U_j) \subseteq W_j, \tag{6.21}$$

because ϕ_j is continuous. Observe that

$$U = \Big(\sum_{j=1}^\infty A_j\Big) \cap \Big(\prod_{j=1}^\infty U_j\Big) \tag{6.22}$$

is an open set in $\sum_{j=1}^\infty A_j$ with respect to the topology induced by the strong product topology on $\prod_{j=1}^\infty A_j$. Of course, (6.22) contains 0, because $0 \in U_j$ for each j, by construction. Let us verify that

$$\phi(U) \subseteq W. \tag{6.23}$$

Let $a \in U$ be given, so that $a_j \in U_j$ for each j, and $a_j = 0$ for all but finitely many j. Hence there is an $n \in \mathbf{Z}_+$ such that $a_j = 0$ when $j > n$. It follows that

$$\phi(a) = \phi_1(a_1) + \cdots + \phi_n(a_n) \in W_1 + \cdots + W_n \subseteq W, \tag{6.24}$$

using (6.20) in the first step, (6.21) in the second step, and (6.19) in the third step. Thus we get (6.23), which implies that ϕ is continuous at 0 with respect to the topology induced on $\sum_{j=1}^\infty A_j$ by the strong product topology on $\prod_{j=1}^\infty A_j$, as desired.

Let us now take $B = \mathbf{T}$, for which the group operations are normally expressed multiplicatively. Let $A_1, A_2, A_3, \ldots$ be a sequence of commutative topological groups again, and let ϕ_j be a continuous homomorphism from A_j into $\mathbf{T}$ for each j. Equivalently, ϕ_j is an element of the dual group $\widehat{A_j}$ associated to A_j for each j. If $a \in \sum_{j=1}^\infty A_j$, then $a_j = 0$ in A_j for all but finitely many j, so that $\phi_j(a_j) = 1$ in $\mathbf{T}$ for all but finitely many j. Thus

$$\phi(a) = \prod_{j=1}^\infty \phi_j(a_j) \tag{6.25}$$

defines an element of $\mathbf{T}$, which corresponds to (6.20) in this situation. This defines a group homomorphism from $\sum_{j=1}^\infty A_j$ into $\mathbf{T}$, and ϕ is also continuous with respect to the topology induced on $\sum_{j=1}^\infty A_j$ by the strong product topology on $\prod_{j=1}^\infty A_j$, by the previous remarks. This means that ϕ defines an element of the dual of $\sum_{j=1}^\infty A_j$, as a commutative topological group with respect to the topology induced by the strong product topology on $\prod_{j=1}^\infty A_j$. Conversely, if ϕ is any continuous homomorphism from $\sum_{j=1}^\infty A_j$ into $\mathbf{T}$, with respect to the topology induced on $\sum_{j=1}^\infty A_j$ by the strong product topology on $\prod_{j=1}^\infty A_j$, then ϕ can be expressed as in (6.25) for some $\phi_j \in \widehat{A_j}$, $j \in \mathbf{Z}_+$, as before. We have also seen that the ϕ_j's are uniquely determined by ϕ, as in (6.15). This shows that the dual of $\sum_{j=1}^\infty A_j$ as a commutative topological group with the topology induced by the strong product topology on $\prod_{j=1}^\infty A_j$ is isomorphic as a commutative group to the direct product $\prod_{j=1}^\infty \widehat{A_j}$ of the corresponding dual groups in a natural way.

6.6 Subsets of sums

Let I be a nonempty set, let A_j be a commutative group for each $j \in I$, and let E be a subset of the direct sum $\sum_{j \in I} A_j$. Put

$$I_E = \{j \in I : \text{there is an } a \in E \text{ such that } a_j \neq 0\}, \tag{6.26}$$

so that $a_j = 0$ for every $a \in E$ and $j \in I \setminus I_E$. This means that

$$E \subseteq A(I_E), \tag{6.27}$$

where $A(I_E)$ is as in (6.4) in Section 6.2. Suppose that A_j is a commutative topological group for each $j \in I$, so that $\sum_{j \in I} A_j$ is a commutative topological group with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$. If E is totally bounded in $\sum_{j \in I} A_j$ with respect to this topology, as in Section 1.19, then we would like to check that I_E has only finitely many elements.

Let $j \in I_E$ be given, so that there is an element $a(j)$ of E such that $a_j(j) \neq 0$ in A_j. Let U_j be an open subset of A_j such that $0 \in U_j$ and $a_j(j) \notin U_j$, such as $U_j = A_j \setminus \{a_j(j)\}$. Put $U_j = A_j$ when $j \in I \setminus I_E$, and

$$U = \Big(\sum_{j \in I} A_j\Big) \cap \Big(\prod_{j \in I} U_j\Big). \tag{6.28}$$

Thus U is an open set in $\sum_{j \in I} A_j$ with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$, because U_j is an open set in A_j for each $j \in I$. We also have that $0 \in U$, because $0 \in U_j$ for each $j \in I$.

Suppose that E is totally bounded with respect to the topology induced on $\sum_{j \in I} A_j$ by the strong product topology on $\prod_{j \in I} A_j$, as before. By definition, this means that there are finitely many elements $x(1), \ldots, x(n)$ of $\sum_{j \in I} A_j$ such that

$$E \subseteq \bigcup_{l=1}^{n} (x(l) + U). \tag{6.29}$$

Of course, for each $l = 1, \ldots, n$, $x_j(l) = 0$ in A_j for all but finitely many $j \in I$, by definition of the direct sum. This implies that

$$I_0 = \{j \in I : x_j(l) \neq 0 \text{ for some } l = 1, \ldots, n\}, \tag{6.30}$$

is a finite subset of I. If we can verify that

$$I_E \subseteq I_0, \tag{6.31}$$

then it follows that I_E has only finitely many elements, as desired.

Let $j \in I_E$ be given, and let $a(j)$ be the corresponding element of E mentioned earlier. Because of (6.29), there is an $l \in \{1, \ldots, n\}$ such that

$$a(j) \in x(l) + U. \tag{6.32}$$

This implies that

$$a_j(j) \in x_j(l) + U_j, \tag{6.33}$$

by passing to the jth coordinate in A_j. If $j \notin I_0$, then $x_j(l) = 0$, so that (6.33) implies that $a_j(j) \in U_j$. This contradicts the way that U_j was chosen when $j \in I_E$, so that $j \in I_0$, as desired.

If I_1 is any nonempty subset of I and $A(I_1)$ is as in (6.4) in Section 6.2, then there is a natural one-to-one correspondence between $A(I_1)$ and $\prod_{j \in I_1} A_j$, as before. This leads to a natural one-to-one correspondence between

$$A(I_1) \cap \Big(\sum_{j \in I} A_j\Big) \tag{6.34}$$

and $\sum_{j \in I_1} A_j$. More precisely, this correspondence is a group isomorphism, and a homeomorphism with respect to the topologies induced by the corresponding strong product topologies on $\prod_{j \in I} A_j$ and $\prod_{j \in I_1} A_j$. If E is a subset of (6.34), then we can identify E with a subset of $\sum_{j \in I_1} A_j$ in this way. It follows that E is totally bounded in (6.34) if and only if the corresponding subset of $\sum_{j \in I_1} A_j$ is totally bounded, using the topologies induced by the strong product topologies on $\prod_{j \in I} A_j$ and $\prod_{j \in I_1} A_j$ again.

Suppose that I_1 has only finitely many elements, so that $A(I_1)$ is contained in $\sum_{j \in I} A_j$, $\sum_{j \in I_1} A_j$ is the same as $\prod_{j \in I_1} A_j$, and the product and strong product topologies on $\prod_{j \in I_1} A_j$ are the same. This permits us to reduce to the characterization of totally bounded subsets of $\prod_{j \in I_1} A_j$ with respect to the product topology mentioned in Section 1.19. Combining this with the earlier discussion, we get that a subset E of $\sum_{j \in I} A_j$ is totally bounded with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$ if and only if I_E is finite and the projection of E into A_j under the standard coordinate projection is totally bounded in A_j for each $j \in I$. Of course, this condition on the projections holds trivially when $j \in I \setminus I_E$, in which case the projection of E into A_j is contained in $\{0\}$.

Remember that compact subsets of commutative topological groups are automatically totally bounded, as in Section 1.19. If $E \subseteq \sum_{j \in I} A_j$ is compact with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$, then it follows that I_E has only finitely many elements, as before. If $I_1 \subseteq I$ is nonempty and finite, then the topologies induced on $A(I_1)$ by the product and strong product topologies on $\prod_{j \in I} A_j$ are the same, as usual. Thus compact subsets of $A(I_1)$ with respect to the topology induced by the strong product topology on $\prod_{j \in I} A_j$ correspond to compact subsets of $\prod_{j \in I_1} A_j$ with respect to the product topology.

6.7 Bounded subsets

Let I be a nonempty set, and suppose that either V_j is a topological vector space over the real numbers for each $j \in I$, or that V_j is a topological vector spave over the complex numbers for each $j \in I$. Thus $\sum_{j \in I} V_j$ is a topological vector space over the real or complex numbers, as appropriate, with respect to the topology induced by the corresponding strong product topology on $\prod_{j \in I} V_j$, as in Section 6.3. Let E be a subset of $\sum_{j \in I} V_j$, and put

$$I_E = \{j \in I : \text{there is a } v \in E \text{ such that } v_j \neq 0\}, \tag{6.35}$$

as in the previous section. If E is bounded in $\sum_{j \in I} V_j$, as in Section 1.17, and with respect to the topology induced by the strong product topology on $\prod_{j \in I} V_j$, then we would like to show that I_E has only finitely many elements. This includes the case of totally bounded subsets of $\sum_{j \in I} V_j$ as a commutative topological group with respect to addition and the same topology, as in Section 1.19.

Suppose for the sake of a contradiction that I_E has infinitely many elements, and let $\{j_l\}_{l=1}^\infty$ be an infinite sequence of distinct elements of I_E. Let $l \in \mathbf{Z}_+$ be given, and let $v(l)$ be an element of E such that $v_{j_l}(l) \neq 0$ in V_{j_l}. Let $U_{j_l} \subseteq V_{j_l}$ be an open set such that $0 \in U_{j_l}$ and

$$v_{j_l}(l) \notin l\, U_{j_l}. \tag{6.36}$$

We can also take U_{j_l} to be balanced in V_{j_l}, so that

$$v_{j_l}(l) \notin t\, U_{j_l} \tag{6.37}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t| \leq l$. If $j \in I$ and $j \neq j_l$ for every $l \in \mathbf{Z}_+$, then we take $U_j = V_j$.

Put

$$U = \Big(\sum_{j \in I} V_j\Big) \cap \Big(\prod_{j \in I} U_j\Big), \tag{6.38}$$

where $U_j \subseteq V_j$ is as in the previous paragraph for every $j \in I$. Thus U is an open set in $\sum_{j \in I} V_j$ with respect to the topology induced by the strong product topology on $\prod_{j \in I} V_j$, because U_j is an open set in V_j for every $j \in I$. We also have that $0 \in U$, because $0 \in U_j$ for every $j \in I$, by construction. If we choose U_{j_l} to be balanced in V_{j_l} for every $l \in \mathbf{Z}_+$, then U_j is balanced in V_j for every $j \in I$, and hence U is balanced in $\sum_{j \in I} V_j$. Using (6.36), we get that

$$v(l) \notin l\, U \tag{6.39}$$

for each $l \in \mathbf{Z}_+$. If we take U_{j_l} to be balanced in V_{j_l}, then we have that

$$v(l) \notin t\, U \tag{6.40}$$

for every $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, such that $|t| \leq l$. This implies that E is not bounded in $\sum_{j \in I} V_j$ with respect to the topology induced by the strong product topology on $\prod_{j \in I} V_j$, as desired, because $v(l) \in E$ for every $l \in \mathbf{Z}_+$, by construction.

If $I_1 \subseteq I$, then

$$V(I_1) = \Big\{v \in \prod_{j \in I} V_j : v_j = 0 \text{ for every } j \in I \setminus I_1\Big\} \tag{6.41}$$

is a linear subspace of $\prod_{j \in I} V_j$, which corresponds to (6.4) in Section 6.2. If I_1 has only finitely many elements, then $V(I_1)$ is contained in $\sum_{j \in I} V_j$. As in (6.27), we have that

$$E \subseteq V(I_E) \tag{6.42}$$

automatically, for any subset E of $\prod_{j \in I} V_j$. Of course, if $E \subseteq \sum_{j \in I} V_j$ is bounded with respect to the topology induced by the strong product topology on $\prod_{j \in I} V_j$, then E is bounded with respect to the topology induced by the product topology on $\prod_{j \in I} V_j$. This implies that for each $j \in I$, the standard coordinate projection onto V_j maps E to a bounded set. Conversely, suppose that $E \subseteq \sum_{j \in I} V_j$ has the property that I_E has only finitely many elements, and that for each $j \in I$, the standard coordinate projection onto V_j maps E to a bounded set. Under these conditions, one can check that E is bounded in $\sum_{j \in I} V_j$ with respect to the topology induced by the strong product topology on $\prod_{j \in I} V_j$. More precisely, the condition that I_E be finite basically permits one to reduce to finite products, for which the product and strong product topologies are the same.

6.8 Semimetrics on products

Let I be a nonempty set, let X_j be a set for each $j \in I$, and let $X = \prod_{j \in I} X_j$ be the corresponding Cartesian product. Also let $d_j(x_j, y_j)$ be a semimetric on X_j for each $j \in I$, and let a_j be a positive real number for each $j \in I$. Put

$$d(x, y) = \sup_{j \in I} (a_j \, d_j(x_j, y_j)) \tag{6.43}$$

for every $x, y \in X$, where the supremum is defined as a nonnegative extended real number. As usual, x_j and y_j are the jth components of x and y in X_j, respectively, for each $j \in I$. It is easy to see that (6.43) satisfies the requirements of a semimetric on X, with suitable interpretations for nonnegative extended real numbers. Otherwise, one can consider

$$d'(x, y) = \min(d(x, y), t) \tag{6.44}$$

for any positive real number t, as in Section 1.4. This is finite by construction, and defines a semimetric on X.

Let $\overline{B}_{X_j}(x_j, r_j)$ be the closed ball in X_j centered at $x_j \in X_j$ with radius $r_j \geq 0$ with respect to d_j, as in (1.24) in Section 1.2. Similarly, let $\overline{B}_X(x, r)$ be the closed ball in X centered at $x \in X$ with radius $r \geq 0$ with respect to d in (6.43), which can be defined in the usual way, even if d may not be quite a semimetric in the normal sense. Observe that

$$\overline{B}_X(x, r) = \prod_{j \in I} \overline{B}_{X_j}(x_j, r/a_j) \tag{6.45}$$

for every $x \in X$ and $r \geq 0$. Suppose for the moment that X_j is equipped with a topology for each $j \in I$, and that d_j is semicompatible with this topology for every $j \in I$, as in Section 5.6. This means that closed balls in X_j with respect to d_j are closed sets for each $j \in I$. It follows that (6.45) is a closed set in X with respect to the corresponding product topology for every $x \in X$ and $r \geq 0$. Thus d is semicompatible with the product topology on X in the same sense under these conditions.

Let $B_{X_j}(x_j, r_j)$ be the open ball in X_j centered at $x_j \in X_j$ with radius $r_j > 0$ with respect to d_j, as in (1.23) in Section 1.2. As before, the open ball $B_X(x, r)$ in X centered at $x \in X$ with radius $r > 0$ with respect to d can be defined in the same way, even if d may not be quite a semimetric on X. One can check that

$$B_X(x,r) = \bigcup_{0<t<1} \Big(\prod_{j \in I} B_{X_j}(x_j, t\,r/a_j)\Big) \tag{6.46}$$

for every $x \in X$ and $r > 0$, where more precisely the union is taken over all $t \in \mathbf{R}$ such that $0 < t < 1$. Suppose for the moment again that X_j is equipped with a topology for each $j \in I$, and now that d_j is compatible with this topology for every $j \in I$, as in Section 2.16. This implies that open balls in X_j with respect to d_j are open sets for each $j \in I$. In this case, (6.46) is an open set in X with respect to the corresponding strong product topology for every $x \in X$ and $r > 0$. This basically means that d is compatible with the strong product topology on X in the same sense, even if d may not be quite a semimetric on X.

Let $\mathcal{M}_j$ be a nonempty collection of semimetrics on X_j for each $j \in I$. Thus $\mathcal{M}_j$ determines a topology on X_j for each $j \in I$, as in Section 1.3. If B_j is an open ball in X_j with respect to an element of $\mathcal{M}_j$ for each $j \in I$, then

$$\prod_{j \in I} B_j \tag{6.47}$$

is an open set in X with respect to the corresponding strong product topology. Let us also ask that open balls in X_j with respect to elements of $\mathcal{M}_j$ form a base for this topology for each $j \in I$, instead of only a sub-base. This can be arranged by including in $\mathcal{M}_j$ the maximum of any finite collection of its elements. Under these conditions, the collection of subsets of X of the form (6.47) as before is a base for the strong product topology on X. Similarly, if $d_j \in \mathcal{M}_j$, $x_j \in X_j$, and $a_j > 0$ are given for each $j \in I$, then (6.46) is an open set in X with respect to the strong product topology for every $r > 0$. The collection of subsets of X of this form is a base for the strong product topology on X too in this situation.

6.9 Semimetrics on sums

Let I be a nonempty set, and let A_j be a commutative group for each $j \in I$. If d_j is a semimetric on A_j and a_j is a positive real number for each $j \in I$, then we can defined d on $\prod_{j \in I} A_j$ as in (6.43). Note that d is invariant under translations on $\prod_{j \in I} A_j$ when d_j is invariant under translations on A_j for each $j \in I$. If x, y are elements of $\sum_{j \in I} A_j$, then $x_j = y_j = 0$ in A_j for all but finitely many $j \in I$, so that $d_j(x_j, y_j) = 0$ for all but finitely many $j \in I$, and (6.43) reduces to

$$d(x,y) = \max_{j \in I} (a_j\, d_j(x_j, y_j)). \tag{6.48}$$

In particular, (6.48) is finite, and defines a semimetric on $\sum_{j \in I} A_j$.

Let $B_{A_j}(x_j, r_j)$ be the open ball in A_j centered at $x_j \in A_j$ with radius $r_j > 0$ with respect to d_j for each $j \in I$, and let $B(x,r)$ be the open ball in $\sum_{j \in I} A_j$

centered at $x \in \sum_{j \in I} A_j$ with radius $r > 0$ with respect to (6.48). Observe that

$$B(x,r) = \Big(\sum_{j \in I} A_j\Big) \cap \Big(\prod_{j \in I} B_{A_j}(x_j, r/a_j)\Big) \tag{6.49}$$

for every $x \in \sum_{j \in I} A_j$ and $r > 0$. Equivalently, $B(x,r)$ is the same as the intersection of $\sum_{j \in I} A_j$ with the open ball in $\prod_{j \in I} A_j$ centered at x with radius r with respect to (6.43). Thus (6.49) is a simpler version of (6.46) in this situation. Suppose for the moment that A_j is equipped with a topology for each $j \in I$, and that d_j is compatible with this topology for every $j \in I$. This means that open balls in A_j with respect to d_j are open sets for each $j \in I$, so that (6.49) is an open set in $\sum_{j \in I} A_j$ with respect to the topology induced by the corresponding strong product topology on $\prod_{j \in I} A_j$. Hence (6.48) is compatible with the topology induced on $\sum_{j \in I} A_j$ by the strong product topology on $\prod_{j \in I} A_j$ under these conditions.

Let $\mathcal{M}_j$ be a nonempty collection of semimetrics on A_j for each $j \in I$, which leads to a topology on A_j in the usual way. If $d_j \in \mathcal{M}_j$, $x_j \in A_j$, and $a_j > 0$ are given for each $j \in I$, then (6.49) is an open set in $\sum_{j \in I} A_j$ with respect to the topology induced by the corresponding strong product topology on $\prod_{j \in I} A_j$ for each $r > 0$. As before, let us also ask that for each $j \in I$, the collection of open balls in A_j associated to elements of $\mathcal{M}_j$ form a base for this topology on A_j. In this case, the collection of subsets of $\sum_{j \in I} A_j$ corresponding to $d_j \in \mathcal{M}_j$ for each $j \in I$ is a base for the topology induced on $\sum_{j \in I} A_j$ by the strong product topology on $\prod_{j \in I} A_j$. This uses the analogous statements for the strong product topology on $\prod_{j \in I} A_j$ mentioned in the previous section. If the elements of $\mathcal{M}_j$ are invariant under translations on A_j, then it suffices to ask that the open balls in A_j centered at 0 with respect to elements of $\mathcal{M}_j$ form a local base for this topology on A_j at 0. If this holds for every $j \in I$, then one can simply say that the collection of subsets of $\sum_{j \in I} A_j$ for the form (6.49) with $x = 0$ and $d_j \in \mathcal{M}_j$ for each $j \in I$ is a local base for the topology induced on $\sum_{j \in I} A_j$ by the strong product topology on $\prod_{j \in I} A_j$ at 0.

Let $\mathcal{M}$ be the collection of semimetrics on $\sum_{j \in I} A_j$ of the form (6.48) with $d_j \in \mathcal{M}_j$ and $a_j > 0$ for every $j \in I$. It follows from the remarks in the preceding paragraph that the topology determined on $\sum_{j \in I} A_j$ by $\mathcal{M}$ is the same as the topology induced on $\sum_{j \in I} A_j$ by the strong product topology on $\prod_{j \in I} A_j$.

6.10 Seminorms on sums

Let I be a nonempty set, and suppose that either V_j is a vector space over the real numbers for each $j \in I$, or that V_j is a vector space over the complex numbers for each $j \in I$. Thus $\prod_{j \in I} V_j$ is a vector space over the real or complex numbers, as appropriate, with respect to coordinatewise addition and scalar multiplication, and $\sum_{j \in I} V_j$ is a linear subspace of $\prod_{j \in I} V_j$. Let N_j be a seminorm on V_j for each $j \in I$, and let a_j be a positive real number for each

$j \in I$. Put

$$N(v) = \sup_{j \in I} (a_j \, N_j(v_j)) \tag{6.50}$$

for every $v \in \prod_{j \in I} V_j$, where the supremum is defined as a nonnegative extended real number. As usual, v_j denotes the jth coordinate of v in V_j for each $j \in I$. It is easy to see that N satisfies many of the properties of a seminorm on $\prod_{j \in I} V_j$, as in Section 1.5. More precisely, $N(0) = 0$, and N satisfies the triangle inequality (1.43), with suitable interpretations for nonnegative extended real numbers. Similarly, N satisfies the homogeneity property (1.42) when $t \neq 0$, with suitable interpretations for nonnegative extended real numbers.

If $v \in \sum_{j \in I} A_j$, then $v_j = 0$ for all but finitely many $j \in I$, so that $N_j(v_j) = 0$ for all but finitely many $j \in I$. In this case, (6.50) is finite, and it reduces to

$$N(v) = \max_{j \in I} (a_j \, N_j(v_j)). \tag{6.51}$$

This defines a seminorm on $\sum_{j \in I} V_j$, with no additional qualifications.

As usual,

$$d_j(v_j - w_j) = N_j(v_j - w_j) \tag{6.52}$$

defines a translation-invariant semimetric on V_j for each $j \in I$. Similarly, put

$$d(v - w) = N(v - w) \tag{6.53}$$

for every $v, w \in \prod_{j \in I} V_j$, where N is as in (6.50). This corresponds to the d_j's as in (6.43) in Section 6.8. If $v, w \in \sum_{j \in I} V_j$, then $v - w \in \sum_{j \in I} V_j$, and $N(v - w)$ can be given as in (6.51). In this case, (6.53) corresponds to the d_j's as in (6.48).

Let $\mathcal{N}_j$ be a nonempty collection of seminorms on V_j for each $j \in I$, which leads to a topology on V_j in the usual way. As before, we ask that for each $j \in I$, the collection of open balls in V_j associated to elements of $\mathcal{N}_j$ form a base for this topology on V_j. This is the same as saying that the open balls in V_j associated to elements of $\mathcal{N}_j$ centered at 0 form a local base for this topology on V_j at 0, by invariance under translations. This can be arranged by including maxima of finite collections of elements of $\mathcal{N}_j$ in $\mathcal{N}_j$ for each $j \in I$. Let $\mathcal{N}$ be the collection of seminorms on $\sum_{j \in I} V_j$ of the form (6.51) with $N_j \in \mathcal{N}_j$ and $a_j > 0$ for every $j \in I$. One can check that the topology determined on $\sum_{j \in I} V_j$ by $\mathcal{N}$ is the same as the topology induced on $\sum_{j \in I} V_j$ by the strong product topology on $\prod_{j \in I} V_j$, using the topology determined on V_j by $\mathcal{N}_j$ for each $j \in I$. This follows from the analogous statement for collections of semimetrics mentioned in the previous section, using the correspondence between these seminorms and semimetrics, as in the preceding paragraph.

6.11 Sums of seminorms

Let I be a nonempty set, and suppose again that either V_j is a vector space over the real numbers for every $j \in I$, or that V_j is a complex vector space for every

$j \in I$. Also let N_j be a seminorm on V_j for each $j \in I$, and let a_j be a positive real number for each $j \in I$. Put

$$\widetilde{N}(v) = \sum_{j\in I} a_j \, N_j(v_j) \tag{6.54}$$

for every $v \in \prod_{j\in I} V_j$, where the sum is defined as a nonnegative extended real number as in Section 2.1. As in the previous section, this satisfies many of the same properties as a seminorm on $\prod_{j\in I} V_j$. Namely, $\widetilde{N}(0) = 0$, and $\widetilde{N}$ satisfies the triangle inequality, with suitable interpretations for nonnegative extended real numbers. The usual homogeneity condition also holds for nonzero scalars, with suitable interpretations for nonnegative extended real numbers. If $v \in \sum_{j\in I} V_j$, then the right side of (6.54) reduces to a finite sum, and in particular the sum is finite. The restriction of $\widetilde{N}$ to $\sum_{j\in I} V_j$ defines a seminorm on $\sum_{j\in I} V_j$ in the usual sense.

If N is as in (6.50), then

$$N(v) \le \widetilde{N}(v) \tag{6.55}$$

for every $v \in \prod_{j\in I} V_j$. Suppose for the moment that I has only finitely or countably many elements. Let b_j be a positive real number for each $j \in I$, and suppose that

$$\sum_{j\in I} 1/b_j < \infty. \tag{6.56}$$

This condition holds automatically when I is finite, and it corresponds to the convergence of an infinite series when I is countably infinite. Observe that

$$\widetilde{N}(v) = \sum_{j\in I} (1/b_j)\,(a_j\, b_j\, N_j(v_j)) \le \Big(\sum_{j\in I} 1/b_j\Big) \sup_{j\in I}\,(a_j\, b_j\, N_j(v_j)) \tag{6.57}$$

for every $v \in \prod_{j\in I} V_j$.

Let $\mathcal{N}_j$ be a nonempty collection of seminorms on V_j for each $j \in I$. As in the previous section, let $\mathcal{N}$ be the collection of seminorms on $\sum_{j\in I} V_j$ of the form (6.51), where $N_j \in \mathcal{N}_j$ and $a_j > 0$ for each $j \in I$. Similarly, let $\widetilde{\mathcal{N}}$ be the collection of seminorms on $\sum_{j\in I} V_j$ of the form (6.54), where $N_j \in \mathcal{N}_j$ and $a_j > 0$ for every $j \in I$. The topology determined on $\sum_{j\in I} V_j$ by $\widetilde{\mathcal{N}}$ is always at least as strong as the topology determined on $\sum_{j\in I} V_j$ by $\mathcal{N}$, because of (6.55). If I has only finitely or countably many elements, then these two topologies on $\sum_{j\in I} V_j$ are the same, because of (6.57).

Let V_j be equipped with the topology determined by $\mathcal{N}_j$ for each $j \in I$. Suppose that for each $j \in I$, the open balls in V_j centered at 0 with respect to elements of $\mathcal{N}_j$ form a local base for this topology on V_j at 0, as in the previous section. Let λ_j be a continuous linear functional on V_j for each $j \in I$. Under these conditions, for each $j \in I$ there is an $N_j \in \mathcal{N}_j$ and a positive real number a_j such that

$$|\lambda_j(v_j)| \le a_j\, N_j(v_j) \tag{6.58}$$

for every $v_j \in V_j$. This corresponds to (1.79) in Section 1.10 with $l = 1$, which can be obtained in the same way as before, using the additional hypothesis on $\mathcal{N}_j$. Put

$$\lambda(v) = \sum_{j \in I} \lambda_j(v_j) \tag{6.59}$$

for every $v \in \sum_{j \in I} V_j$, where the sum on the right reduces to a finite sum in $\mathbf{R}$ or $\mathbf{C}$, as appropriate. This defines a linear functional on $\sum_{j \in I} V_j$, as in Section 6.5. Using (6.58) and (6.59), we get that

$$|\lambda(v)| \le \sum_{j \in I} |\lambda_j(v_j)| \le \sum_{j \in I} a_j \, N_j(v_j) \tag{6.60}$$

for every $v \in \sum_{j \in I} V_j$. Of course, the right side is the same as (6.54), with these a_j's and N_j's. Thus the right side is an element of $\widetilde{\mathcal{N}}$. This implies that λ is continuous with respect to the topology determined on $\sum_{j \in I} V_j$ by $\widetilde{\mathcal{N}}$, as in Section 1.10 again. If I has only finitely or countably many elements, then we can choose positive real numbers b_j for each $j \in I$ so that (6.56) holds. In this case, we can combine (6.57) and (6.60) to get that

$$|\lambda(v)| \le \Big(\sum_{j \in I} 1/b_j\Big) \max_{j \in I} (a_j \, b_j \, N_j(v_j)) \tag{6.61}$$

for every $v \in \sum_{j \in I} V_j$. Note that we can replace the supremum on the right side of (6.57) with the maximum here because $v \in \sum_{j \in I} V_j$. As before, the second factor on the right side is an element of $\mathcal{N}$. This shows more directly that λ is continuous with respect to the topology determined on $\sum_{j \in I} V_j$ by $\mathcal{N}$ when I has only finitely or countably many elements.

6.12 Supremum semimetrics and seminorms

Let X be a set. If E is a nonempty subset of X, then put

$$N_E(f) = \sup_{x \in E} |f(x)| \tag{6.62}$$

for every complex-valued function f on X, where the supremum is defined as a nonnegative extended real number. Let B be a nonempty set, let E_β be a nonempty subset of X for each $\beta \in B$, and put

$$E = \bigcup_{\beta \in B} E_\beta. \tag{6.63}$$

Under these conditions, we have that

$$N_E(f) = \sup_{\beta \in B} N_{E_\beta}(f) \tag{6.64}$$

for every complex-valued function f on X. Thus

$$N_E(f - g) = \sup_{\beta \in B} N_{E_\beta}(f - g) \tag{6.65}$$

for all complex-valued functions f, g on X.

Let A be a commutative group, with the group operations expressed additively. Also let $a_1, \ldots, a_l$ be finitely many elements of A, and put

$$a = a_1 + \cdots + a_l. \tag{6.66}$$

Suppose that ϕ, ψ are homomorphisms from A into $\mathbf{T}$, so that

$$\phi(a) = \prod_{j=1}^{l} \phi(a_j), \quad \psi(a) = \prod_{j=1}^{l} \psi(a_j). \tag{6.67}$$

Observe that

$$\phi(a) - \psi(a) = \sum_{j=1}^{l} \phi(a_1) \cdots \phi(a_{j-1})\, (\phi(a_j) - \psi(a_j))\, \psi(a_{j+1}) \cdots \psi(a_l), \tag{6.68}$$

and hence

$$|\phi(a) - \psi(a)| \le \sum_{j=1}^{l} |\phi(a_j) - \psi(a_j)|. \tag{6.69}$$

Let $E_1, \ldots, E_l$ be finitely many nonempty subsets of A, and put

$$E = E_1 + \cdots + E_l. \tag{6.70}$$

Using (6.69), we get that

$$N_E(\phi - \psi) \le \sum_{j=1}^{l} N_{E_j}(\phi - \psi), \tag{6.71}$$

where N_{E_j}, N_E are as in (6.62). If $0 \in E_j$ for each $j = 1, \ldots, l$, then $E_j \subseteq E$ for each j, and

$$\max_{1 \le j \le l} N_{E_j}(f) \le N_E(f) \tag{6.72}$$

for every complex-valued function f on A.

Let V be a vector space over the real or complex numbers. Also let E be a nonempty subset of V, let N_E be as in (6.62), and let λ be a linear functional on V. If $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and $t \ne 0$, then

$$N_{t\,E}(\lambda) = \sup_{v \in E} |\lambda(t\, v)| = \sup_{v \in E} (|t|\, |\lambda(v)|) = |t|\, N_E(\lambda). \tag{6.73}$$

This works when $t = 0$ and $N_E(\lambda) < \infty$ too. Put

$$\widetilde{E} = \bigcup_{|t| \le 1} t\, E, \tag{6.74}$$

as in (1.56) in Section 1.6. Thus $\widetilde{E}$ is a balanced set in V that contains E. It is easy to see that

$$N_{\widetilde{E}}(\lambda) = N_E(\lambda), \tag{6.75}$$

as in (6.73).

Let $E_1, \ldots, E_l$ be finitely many nonempty subsets of V, and let E be as in (6.70). If $v \in E$, then there are $v_j \in E_j$ for each $j = 1, \ldots, l$ such that $v = \sum_{j=1}^l v_j$. If λ is a linear functional on V, then we have that

$$|\lambda(v)| = \left| \sum_{j=1}^l \lambda(v_j) \right| \le \sum_{j=1}^l |\lambda(v_j)|, \tag{6.76}$$

and thus

$$N_E(\lambda) \le \sum_{j=1}^l N_{E_j}(\lambda). \tag{6.77}$$

Suppose now that $E_1, \ldots, E_l$ are balanced subsets of V, and let $v_j \in E_j$ be given for each $j = 1, \ldots, l$. Also let λ be a linear functional on V, and choose $t_1, \ldots, t_l \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, so that $|t_j| = 1$ and

$$t_j\, \lambda(v_j) = |\lambda(v_j)| \tag{6.78}$$

for each $j = 1, \ldots, l$. If we put $w = \sum_{j=1}^l t_j\, v_j$, then we get that

$$\lambda(w) = \sum_{j=1}^l t_j\, \lambda(v_j) = \sum_{j=1}^l |\lambda(v_j)|. \tag{6.79}$$

Note that $t_j\, v_j \in E_j$ for each j, because E_j is balanced, so that $w \in E$. This implies that

$$\sum_{j=1}^l |\lambda(v_j)| \le N_E(\lambda). \tag{6.80}$$

Taking the supremum over $v_j \in E_j$, we get that

$$\sum_{j=1}^l N_{E_j}(\lambda) \le N_E(\lambda). \tag{6.81}$$

It follows that

$$N_E(\lambda) = \sum_{j=1}^l N_{E_j}(\lambda) \tag{6.82}$$

under these conditions, by combining (6.77) and (6.81).

6.13 Countable sums

Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of commutative topological groups, with the group operations expressed additively. Thus the direct sum $\sum_{j=1}^\infty A_j$ is also a commutative topological group, with respect to the topology induced by the strong product topology on $\prod_{j=1}^\infty A_j$. Let $A(n)$ be as in (6.10) in Section 6.4 for each positive integer n, so that $\sum_{j=1}^\infty A_j$ is equal to $\bigcup_{n=1}^\infty A(n)$, as in (6.12). Remember that $A(n)$ is a closed set in $\prod_{j=1}^\infty A_j$ with respect to the product topology for each n, and that there is a natural isomorphism between $A(n)$ and $\prod_{j=1}^n A_j$. The topologies induced on $A(n)$ by the product and strong product topologies on $\prod_{j=1}^\infty A_j$ are the same for each n, and correspond to the product topology on $\prod_{j=1}^n A_j$.

Suppose for the moment that for each $j \in \mathbf{Z}_+$, the topology on A_j is determined by a metric. This implies that for each $n \in \mathbf{Z}_+$, the product topology on $\prod_{j=1}^n A_j$ is determined by a metric. It follows that the topology induced on $A(n)$ by the product and strong product topologies on $\prod_{j=1}^\infty A_j$ is determined by a metric for each $n \in \mathbf{Z}_+$. Hence relatively open sets in $A(n)$ are F_σ sets with respect to this topology for each $n \in \mathbf{Z}_+$. More precisely, relatively open sets in $A(n)$ are F_σ sets with respect to the product topology on $\prod_{j=1}^\infty A_j$, because $A(n)$ is a closed set with respect to the product topology on $\prod_{j=1}^\infty A_j$. Let U be an open set in $\sum_{j=1}^\infty A_j$ with respect to the topology induced by the strong product topology on $\prod_{j=1}^\infty A_j$. Thus $U \cap A(n)$ is a relatively open set in $A(n)$ with respect to the topology induced by the strong product topology on $\prod_{j=1}^\infty A_j$ for each $n \in \mathbf{Z}_+$. This implies that $U \cap A(n)$ is an F_σ set with respect to the product topology on $\prod_{j=1}^\infty A_j$ for each $n \in \mathbf{Z}_+$, as before. It follows that U is an F_σ set with respect to the product topology on $\prod_{j=1}^\infty A_j$. In particular, this shows that $\sum_{j=1}^\infty A_j$ satisfies (2.30) in Section 2.3 with respect to the topology induced by the strong product topology on $\prod_{j=1}^\infty A_j$ under these conditions.

Similarly, if A_j is σ-compact for each j, then it is easy to see that $\prod_{j=1}^n A_j$ is σ-compact with respect to the corresponding product topology for each n. This implies that $A(n)$ is σ-compact with respect to the product topology on $\prod_{j=1}^\infty A_j$ for each n. It follows that $\sum_{j=1}^\infty A_j$ is σ-compact with respect to the product topology on $\prod_{j=1}^\infty A_j$ in this case.

Conditions like these can be helpful in dealing with suitable Borel measures on $\sum_{j=1}^\infty A_j$. One can also look at properties of Borel measures on $\sum_{j=1}^\infty A_j$ in terms of their behavior on $A(n)$ for each $n \in \mathbf{Z}_+$.

Chapter 7

Some additional topics

7.1 Topological dimension 0

Let X be a topological space, and let x be an element of X. We say that X has *topological dimension* 0 at x if for every open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$, $U \subseteq W$, and U is also a closed set in X. Equivalently, this means that

$$\{U \subseteq X : x \in U,\ U \text{ is both open and closed in } X\} \tag{7.1}$$

is a local base for the topology of X at x. In order to check that X has topological dimension 0 at x, it suffices to show that there is a local base for the topology of X at x consisting of subsets of X that are both open and closed. More precisely, it is enough to show that there is a sub-base for the local topology of X at x consisting of subsets of X that are both open and closed.

If X has topological dimension 0 at every $x \in X$, then X is said to have *topological dimension* 0 as a topological space. Sometimes X is required to be nonempty as well. Equivalently, X has topological dimension 0 when

$$\{U \subseteq X : U \text{ is both open and closed in } X\} \tag{7.2}$$

is a base for the topology of X. If there is a base for the topology of X consisting of subsets of X that are both open and closed, then (7.2) is a base for the topology of X. Similarly, if there is a sub-base for the topology of X consisting of subsets of X that are both open and closed, then (7.2) is a base for the topology of X.

Note that X has topological dimension 0 at a point $x \in X$ if and only if for every closed set $E \subseteq X$ with $x \notin E$ there is an open set $U \subseteq X$ such that $x \in U$, $E \cap U = \emptyset$, and U is a closed set in X. Here E corresponds to the complement of W in the previous formulation. Alternatively, one can say that there are disjoint open sets $U, V \subseteq X$ such that $x \in U$, $E \subseteq V$, and $U \cap V = X$. This is the same as asking that there be an open set $V \subseteq X$ such that $E \subseteq V$,

$x \notin V$, and V is closed in X. Thus X has topological dimension 0 when any of these reformulations holds for every $x \in X$.

We say that X is *totally separated* if for every $x, y \in X$ with $x \neq y$ there is an open set $U \subseteq X$ such that $x \in U$, $y \notin U$, and U is also a closed set in X. This is the same as saying that there are disjoint open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cup V = X$, which is more clearly symmetric in x and y. Note that totally separated spaces are Hausdorff in particular. If X has topological dimension 0, then X is obviously regular as a topological space in the strict sense. If X has topological dimension 0 and X satisfies the first separation condition, then it is easy to see that X is totally separated.

If X has topological dimension 0 at x, $Y \subseteq X$, and $x \in Y$, then Y has topological dimension 0 at x as well, with respect to the induced topology. This uses the fact that if $U \subseteq X$ is both open and closed, then $U \cap Y$ is both relatively open and relatively closed in Y. If X has topological dimension 0 and $Y \subseteq X$, then Y has topological dimension 0 with respect to the induced topology, at least if $Y \neq \emptyset$ when that is included in the definition. If X is totally separated and $Y \subseteq X$, then Y is totally separated with respect to the induced topology.

Remember that X is said to be *totally disconnected* if X does not contain any connected subsets with at least two elements. If X is totally separated, and X has at least two elements, then X is not connected. If X is totally separated, $Y \subseteq X$, and Y has at least two elements, then Y is not connected, because Y is totally separated with respect to the induced topology. It follows that totally separated spaces are totally disconnected. If X is locally compact, Hausdorff, and totally disconnected, then is it well known that X has topological dimension 0.

Suppose that X is totally separated, $x \in X$, $K \subseteq X$ is compact, and $x \notin K$. If $y \in K$, then $x \neq y$, and so there is an open set $V(y) \subseteq X$ such that $y \in V(y)$, $x \notin V(y)$, and $V(y)$ is a closed set in X. Using compactness, one can cover K by finitely many $V(y)$'s, to get an open set $V \subseteq X$ such that $K \subseteq V$, $x \notin V$, and V is a closed set in X. If X is compact, then one can use this to get that X has topological dimension 0, because closed subsets of X are compact. Similarly, if X is locally compact, then one can show that X has topological dimension 0.

Let A be a commutative topological group. If A has topological dimension 0 at 0, then A has topological dimension 0, because translations on A are homeomorphisms. Similarly, in order to check that A is totally separated, it suffices to verify that for each $y \in A$ with $y \neq 0$, there is an open set $U \subseteq A$ such that $0 \in U$, $y \notin U$, and U is a closed set in A.

7.2 Semi-ultrametrics

Let X be a set. A semimetric $d(x, y)$ on X is said to be a *semi-ultrametric* on X if

$$d(x, z) \leq \max(d(x, y), d(y, z)) \tag{7.3}$$

for every $x, y, z \in X$. Note that (7.3) implies the ordinary triangle inequality for semimetrics. Similarly, a metric $d(x, y)$ on X is said to be an *ultrametric* if it satisfies (7.3). It is easy to see that the discrete metric on X is an ultrametric on X.

Let $d(x, y)$ be a semi-ultrametric on X, and let r be a nonnegative real number. If $x, y \in X$ satisfy $d(x, y) \le r$, then it is easy to see that

$$\overline{B}_d(x, r) \subseteq \overline{B}_d(y, r), \tag{7.4}$$

where these closed balls are defined as in (1.24) in Section 1.2. This implies that

$$\overline{B}_d(x, r) = \overline{B}_d(y, r) \tag{7.5}$$

under these conditions, by interchanging the roles of x and y. In particular, it follows that closed balls in X with respect to d of positive radius are open sets with respect to the usual topology determined by d. This implies that X has topological dimension 0 with respect to this topology.

Similarly, if $x, y \in X$ satisfy $d(x, y) < r$, then

$$B_d(x, r) \subseteq B_d(y, r), \tag{7.6}$$

where these open balls are defined as in (1.23) in Section 1.2. Hence

$$B_d(x, r) = B_d(y, r), \tag{7.7}$$

by interchanging the roles of x and y again. One can use this to show that open balls in X with respect to d are closed sets with respect to the topology determined by d, by verifying that open balls contain all of their limit points. Alternatively, one can check that complements of open balls in X with respect to d are open sets. It follows that X has topological dimension 0 with respect to this topology, as before.

Now let $\mathcal{M}$ be a nonempty collection of semi-ultrametrics on X. Consider the topology determined on X by $\mathcal{M}$, as in Section 1.3. It is easy to see that X has topological dimension 0 with respect to this topology, using the remarks in the previous paragraphs.

If $\mathcal{M}$ has only finitely or countably many elements, then the arguments in Sections 1.3 and 1.4 lead to a single semi-ultrametric on X that determines the same topology on X. More precisely, if $d_1, \ldots, d_l$ are finitely many semi-ultrametrics on X, then one can check that their maximum is a semi-ultrametric on X as well. If d is any semi-ultrametric on X and t is a positive real number, then the minimum of d and t defines a semi-ultrametric on X too. Similarly, if $d_1, d_2, d_3, \ldots$ is an infinite sequence of semi-ultrametrics on X, then (1.38) in Section 1.4 defines a semi-ultrametric on X.

Let A be a commutative group, with the group operations expressed additively, and let $d(x, y)$ be a semi-ultrametric on A that is invariant under translations on A. Observe that

$$\begin{aligned} d(x_0 + y_0, x + y) &\le \max(d(x_0 + y_0, x + y_0), d(x + y_0, x + y)) \\ &= \max(d(x_0, x), d(y_0, y)) \end{aligned} \tag{7.8}$$

for every $x_0, x, y_0, y \in A$ under these conditions. Hence

$$B_d(x_0, r) + B_d(y_0, r) \subseteq B_d(x_0 + y_0, r) \tag{7.9}$$

for every $x_0, y_0 \in A$ and $r > 0$, and

$$\overline{B}_d(x_0, r) + \overline{B}_d(y_0, r) \subseteq \overline{B}_d(x_0 + y_0, r) \tag{7.10}$$

for every $r > 0$. Remember that $d(x, y)$ is invariant under $x \mapsto -x$ on A, as in (1.27) in Section 1.2. It follows that $B_d(0, r)$ is a subgroup of A for every $r > 0$, and that $\overline{B}_d(0, r)$ is a subgroup of A for every $r \geq 0$.

Let B be a subgroup of A, and let A/B be the corresponding quotient group. Also let q be the associated quotient homomorphism from A onto A/B, and let $d_{A/B}$ be the discrete metric on A/B. Observe that

$$d_{A/B}(q(x), q(y)) \tag{7.11}$$

defines a translation-invariant semi-ultrametric on A. The open ball in A with respect to (7.11) centered at 0 with radius $r > 0$ is equal to B when $r \leq 1$, and is equal to A when $r > 1$. Similarly, the closed ball in A with respect to (7.11) centered at 0 with radius $r \geq 0$ is equal to B when $r < 1$, and to A when $r > 1$.

7.3 Uniformly separated sets

Let A be a commutative topological group, in which the group operations are expressed additively. Let us say that $E_1, E_2 \subseteq A$ are *uniformly separated in* A if there is an open set $U \subseteq A$ such that $0 \in U$ and

$$(E_1 + U) \cap E_2 = \emptyset. \tag{7.12}$$

In this case, we may also say that E_1 and E_2 are U*-separated* in A. Note that (7.12) holds if and only if

$$E_1 \cap (E_2 - U) = \emptyset. \tag{7.13}$$

This implies that the property of being uniformly separated in A is symmetric in E_1 and E_2.

A subset B of A is said to be *symmetric about* 0 in A if

$$-B = B. \tag{7.14}$$

If $U \subseteq A$ is an open set that contains 0, then $U \cap (-U)$ is an open set that contains 0, is symmetric about 0, and is contained in U. Thus we may restrict our attention here to open sets $U \subseteq A$ that contain 0 and are symmetric about 0 in the preceding paragraph. In this case, (7.12) is equivalent to (7.13), so that (7.12) is symmetric in E_1 and E_2.

Let $d(x, y)$ be a translation-invariant semimetric on A, and let ϵ be a positive real number. Let us say that $E_1, E_2 \subseteq A$ are ϵ*-separated* with respect to d if

$$d(x, y) \geq \epsilon \tag{7.15}$$

for every $x \in E_1$ and $y \in E_2$. It is easy to see that this happens if and only if (7.12) holds with $U = B_d(0, \epsilon)$. Note that $B_d(0, \epsilon)$ is symmetric about 0 in A, by (1.27) in Section 1.2. If d is compatible with the given topology on A, as in Section 2.16, then it follows that E_1 and E_2 are uniformly separated in A.

Suppose for the moment that the topology on A is determined by a nondegenerate collection $\mathcal{M}$ of translation-invariant semimetrics on A. Every element of $\mathcal{M}$ is compatible with the topology on A, so that the remarks in the previous paragraph can be applied. Similarly, if $d_1, \ldots, d_l$ are finitely many elements of $\mathcal{M}$, then their maximum d is a translation-invariant semimetric on A that is compatible with the topology on A. If $E_1, E_2 \subseteq A$ are ϵ-separated with respect to d for some $\epsilon > 0$, then it follows that E_1 and E_2 are uniformly separated in A, as before.

Conversely, suppose that $E_1, E_2 \subseteq A$ are uniformly separated in A. Thus there is an open set $U \subseteq A$ such that $0 \in U$ and (7.12) holds. Because the topology on A is determined by $\mathcal{M}$, there are finitely many elements $d_1, \ldots, d_l$ of $\mathcal{M}$ and a positive real number ϵ such that

$$\bigcap_{j=1}^{l} B_{d_j}(0, \epsilon) \subseteq U, \tag{7.16}$$

as in (1.30) in Section 1.3. Let d be the maximum of $d_1, \ldots, d_l$ again, so that

$$B_d(0, \epsilon) \subseteq U, \tag{7.17}$$

by (7.16) and (1.33) in Section 1.3. This implies that (7.15) holds for every $x \in E_1$ and $y \in E_2$, as before, so that E_1 and E_2 are ϵ-separated with respect to d.

Suppose that $E_1, E_2 \subseteq A$ are uniformly separated in A, so that there is an open set $U \subseteq A$ that contains 0 and satisfies (7.12). Using the continuity of the group operations at 0 on A, we get that there is an open set $U_0 \subseteq A$ such that $0 \in U_0$ and

$$U_0 + U_0 - U_0 \subseteq U. \tag{7.18}$$

Combining this with (7.12), we obtain that

$$(E_1 + U_0 + U_0 - U_0) \cap E_2 = \emptyset. \tag{7.19}$$

Equivalently, this means that

$$(E_1 + U_0 + U_0) \subseteq (E_2 + U_0) = \emptyset. \tag{7.20}$$

Remember that the closures $\overline{E_1}$, $\overline{E_2}$ of E_1, E_2 in A are contained in $E_1 + U_0$, $E_2 + U_0$, respectively, as in (1.7) in Section 1.1. Thus (7.20) implies that

$$(\overline{E_1} + U_0) \cap \overline{E_2} = \emptyset. \tag{7.21}$$

In particular, this means that $\overline{E_1}$, $\overline{E_2}$ are uniformly separated in A.

Suppose that $K \subseteq A$ is compact, $E \subseteq A$ is a closed set, and $K \cap E = \emptyset$. Thus $W = A \setminus E$ is an open set that contains K. As in (1.13) in Section 1.1, there is an open set $U \subseteq A$ such that $0 \in U$ and $K + U \subseteq W$. It follows that

$$(K + U) \cap E = \emptyset, \tag{7.22}$$

so that K and E are uniformly separated in A.

Suppose that $E \subseteq A$ is uniformly separated from $A \setminus E$ in A. This means that there is an open set $U \subseteq A$ such that $0 \in U$ and

$$(E + U) \cap (A \setminus E) = \emptyset, \tag{7.23}$$

so that

$$E + U \subseteq E. \tag{7.24}$$

It follows that

$$E = E + U \tag{7.25}$$

in this situation, because $E \subseteq E + U$ when $0 \in U$. In particular, this implies that E is both open and closed in A. If $K \subseteq A$ is compact and open, then K is uniformly separated from its complement in A, as in the preceding paragraph.

Suppose that $E \subseteq A$ is uniformly separated from its complement in A again, and let U be as in the previous paragraph. As before, we may also take U to be symmetric about 0 in A. Put $U_1 = U$, and define U_j recursively for $j \in \mathbf{Z}_+$ by

$$U_{j+1} = U_j + U. \tag{7.26}$$

Using (7.24), we get that

$$E + U_j \subseteq E \tag{7.27}$$

for each $j \in \mathbf{Z}_+$. If we put

$$B = \bigcup_{j=1}^{\infty} U_j, \tag{7.28}$$

then we obtain that

$$E + B \subseteq E, \tag{7.29}$$

by (7.27). More precisely,

$$E = E + B, \tag{7.30}$$

because $0 \in B$. We also have that B is an open subgroup of A under these conditions, as in Section 2.7.

Of course, if B is any subgroup of A, then

$$B + B = B. \tag{7.31}$$

If B is an open subgroup of A, then it follows that B is uniformly separated from its complement in A.

7.4 Compact and open subgroups

Let A be a commutative topological group, where the group operations are expressed additively, and let B be an open subgroup of A. In this case, B is a closed set in A too, as in Section 2.7. Let A/B be the corresponding quotient group, and let q be the associated quotient homomorphism from A onto A/B, whose kernel is equal to B. Of course, A/B is a commutative topological group with respect to the discrete topology. It is easy to see that q is continuous with respect to the discrete topology on A/B, because B is an open set in A.

Remember that $\widehat{A}$ denotes the dual group associated to A as in Section 1.11, which consists of all continuous group homomorphisms from A into $\mathbf{T}$. Of course, every homomorphism from A/B into any commutative topological group is continuous, because A/B is equipped with the discrete topology. Thus the dual group $(\widehat{A/B})$ associated to A/B consists of all group homomorphisms from A/B into $\mathbf{T}$. The quotient homomorphism q leads to a dual homomorphism $\widehat{q}$ from $(\widehat{A/B})$ into $\widehat{A}$, as in Section 1.12. More precisely, if ψ is a group homomorphism from A/B into $\mathbf{T}$, then

$$\widehat{q}(\psi) = \psi \circ q, \tag{7.32}$$

as in (1.113).

If ψ is any group homomorphism from A/B into $\mathbf{T}$, then $\psi \circ q$ is a group homomorphism from A into $\mathbf{T}$ whose kernel contains B. Conversely, if ϕ is a group homomorphism from A into $\mathbf{T}$ whose kernel contains B, then ϕ can be expressed as $\psi \circ q$ for some group homomorphism ψ from A/B into $\mathbf{T}$. Note that the condition that the kernel of ϕ contain B implies that ϕ is continuous on A, because B is an open subgroup of A. It is easy to see that $\widehat{q}$ is injective, because q is surjective, by construction.

Remember that $\widehat{q}$ is continuous with respect to the usual dual topologies on $\widehat{A}$ and $(\widehat{A/B})$, as in Section 1.12. In this case, $(\widehat{A/B})$ is compact with respect to its dual topology, because A/B is equipped with the discrete topology. Thus $\widehat{q}$ maps $(\widehat{A/B})$ onto a compact subgroup of $\widehat{A}$, with respect to the dual topology on $\widehat{A}$.

Let C be a subgroup of A, and let ϕ be a group homomorphism from A into $\mathbf{T}$ such that

$$\operatorname{Re} \phi(x) > 0 \tag{7.33}$$

for every $x \in C$. Equivalently, this means that $\phi(C)$ is contained in

$$\{z \in \mathbf{T} : \operatorname{Re} z > 0\}. \tag{7.34}$$

However, $\{1\}$ is the only subgroup of $\mathbf{T}$ contained in (7.34), as mentioned in Section 1.11. It follows that $\phi(x) = 1$ for every $x \in C$ under these conditions, because $\phi(C)$ is a subgroup of $\mathbf{T}$. Note that (7.33) holds when

$$|\phi(x) - 1| < 1 \tag{7.35}$$

for every $x \in C$.

In particular, if

$$\sup_{x \in C} |\phi(x) - 1| < 1, \tag{7.36}$$

then (7.35) and hence (7.33) hold for every $x \in C$. If C is any nonempty compact subset of A, then

$$\Big\{\phi \in \widehat{A} : \sup_{x \in C} |\phi(x) - 1| < 1\Big\} \tag{7.37}$$

is an open set in $\widehat{A}$ with respect to the dual topology defined in Section 1.12. If C is a subgroup of A, then (7.37) is the same as

$$\{\phi \in \widehat{A} : \phi(x) = 1 \text{ for every } x \in C\}, \tag{7.38}$$

by the earlier remarks. Of course, (7.38) is automatically a subgroup of $\widehat{A}$.

Let C be a subgroup of A again, equipped with the topology induced by the topology on A, and let h be the inclusion mapping from C into A, so that $h(x) = x$ for every $x \in C$. Thus C may be considered as a commutative topological group as well, and h is a continuous group homomorphism from C into A. This leads to a dual homomorphism $\widehat{h}$ from $\widehat{A}$ into $\widehat{C}$, as before. If $\phi \in \widehat{A}$, then

$$\widehat{h}(\phi) = \phi \circ h \tag{7.39}$$

is the same as the restriction of ϕ to C. Note that the kernel of $\widehat{h}$ is equal to (7.38).

If C is a compact subgroup of A, then (7.38) is an open subgroup of $\widehat{A}$ with respect to the dual topology, as before. Alternatively, we have seen that the dual topology on $\widehat{C}$ is the same as the discrete topology when C is compact. We also have that $\widehat{h}$ is continuous with respect to the corresponding dual topologies on $\widehat{A}$ and $\widehat{C}$, as in Section 1.12. It follows that the trivial subgroup of $\widehat{C}$ is an open subgroup when C is compact, so that the kernel of $\widehat{h}$ is an open subgroup of $\widehat{A}$, by continuity.

7.5 Strongly 0-dimensional groups

Let A be a commutative topological group, with the group operations expressed additively. Let us say that A is *strongly* 0*-dimensional* if the collection of open subgroups of A forms a local base for the topology of A at 0. This implies that A has topological dimension 0 at 0, because open subgroups of A are closed sets in A. In this case, it follows that A has topological dimension 0 at every point, as in Section 7.1. Of course, if A is equipped with the discrete topology, then A is strongly 0-dimensional.

The rational numbers $\mathbf{Q}$ form a commutative topological group with respect to addition and the topology induced by the standard topology on $\mathbf{R}$. It is easy to see that $\mathbf{Q}$ has topological dimension 0 with respect to this topology.

However, one can check that $\mathbf{Q}$ is not strongly 0-dimensional. More precisely, if U is an open subgroup of $\mathbf{Q}$, then one can verify that $U = \mathbf{Q}$.

Let A be any commutative topological group again. In order to check that A is strongly 0-dimensional, it suffices to show that there is a local base for the topology of A at 0 consisting of open subgroups of A. In fact, it is enough to show that there is a local sub-base for the topology of A at 0 consisting of open subgroups of A. Of course, the intersection of any family of subgroups of A is a subgroup of A too. The intersection of finitely many open subgroups of A is an open subgroup of A.

Let A be a commutative group, and let $\mathcal{M}$ be a nondegenerate collection of translation-invariant semimetrics on A. Thus A is a commutative topological group with respect to the topology determined by $\mathcal{M}$, as in Section 1.3. If the elements of $\mathcal{M}$ are semi-ultrametrics on A, then A is strongly 0-dimensional with respect to this topology, because of the remarks in Section 7.2.

Conversely, let A be a commutative topological group that is strongly 0-dimensional, and let $\mathcal{B}_0$ be a local sub-base for the topology of A at 0 consisting of open subgroups of A. In particular, one can simply take $\mathcal{B}_0$ to be the collection of all open subgroup of A. Let $\mathcal{M}_0$ be the collection of semi-ultrametrics of the form (7.11) in Section 7.2 corresponding to elements of $\mathcal{B}_0$. Remember that these semi-ultrametrics on A are invariant under translations. It is easy to see that $\mathcal{M}_0$ determines the same topology on A under these conditions.

Let A be any commutative topological group, and suppose that $V \subseteq A$ is an open set that contains 0 and has the property that V is uniformly separated from its complement in A, as in Section 7.3. This implies that there is an open subgroup B of A such that

$$V + B \subseteq V, \tag{7.40}$$

as in (7.29). It follows that

$$B \subseteq V, \tag{7.41}$$

because $0 \in V$, by hypothesis.

Suppose that for every open set $W \subseteq A$ with $0 \in W$ there is an open set $V \subseteq A$ such that $0 \in V$ and V is uniformly separated from its complement in A. This implies that every open set $W \subseteq A$ with $0 \in W$ contains an open subgroup B of A, as in the preceding paragraph. This means that A is strongly 0-dimensional. Conversely, if A is strongly 0-dimensional, then A has the property just mentioned. This uses the fact that open subgroups of A are uniformly separated from their complements in A, as in Section 7.3.

Let A be any commutative topological group again, and let K be a compact open subset of A. This implies that K is uniformly separated from its complement in A, as in Section 7.3. Hence there is an open subgroup B of A such that

$$K + B \subseteq K, \tag{7.42}$$

as in (7.29). If $0 \in K$, then we get that

$$B \subseteq K. \tag{7.43}$$

If A is locally compact, and if A has topological dimension 0, then A is strongly 0-dimensional. To see this, let $W \subseteq A$ be an open set that contains 0. We may also suppose that W is contained in a compact subset of A, because A is locally compact. If A has topological dimension 0, then there is an open set $V \subseteq A$ such that $0 \in V$, $V \subseteq W$, and V is a closed set in A. It follows that V is compact, because V is a closed set that is contained in a compact set. This implies that V is uniformly separated from its complement in A, as before. Hence there is an open subgroup B in A that is contained in V, as in (7.41). In particular, $B \subseteq W$, as desired.

7.6 Equicontinuity and open subgroups

Let A be a commutative topological group, with the group operations expressed additively. Suppose that B is an open subgroup of A, and let $\mathcal{E}_B$ be the collection of group homomorphisms ϕ from A into $\mathbf{T}$ such that

$$\phi(x) = 1 \tag{7.44}$$

for every $x \in B$. It is easy to see that $\mathcal{E}_B$ is equicontinuous at 0 on A, as in Section 4.1. Of course, it follows that every subcollection of $\mathcal{E}_B$ is equicontinuous at 0 as well. Note that $\mathcal{E}_B$ is a subgroup of $\widehat{A}$ too.

Suppose for the moment that A is strongly 0-dimensional. Let $\mathcal{E}$ be a collection of group homomorphisms from A into $\mathbf{T}$ that is equicontinuous at 0. Thus there is an open set $U \subseteq A$ such that $0 \in U$ and

$$\operatorname{Re} \phi(x) > 0 \tag{7.45}$$

for every $\phi \in \mathcal{E}$ and $x \in U$. Because A is strongly 0-dimensional, there is an open subgroup B of A such that $B \subseteq U$. Thus (7.45) holds for every $\phi \in \mathcal{E}$ and $x \in B$. This implies that (7.44) holds for every $\phi \in \mathcal{E}$ and $x \in B$, because B is a subgroup of A, as in Section 7.4. This shows that

$$\mathcal{E} \subseteq \mathcal{E}_B \tag{7.46}$$

under these conditions.

Let A be any commutative topological group again, and let $\mathcal{E}$ be a collection of group homomorphisms from A into $\mathbf{T}$ that is equicontinuous at 0. As before, there is an open set $U \subseteq A$ that contains 0 and for which (7.45) holds for every $\phi \in \mathcal{E}$ and $x \in U$. Suppose that $\mathcal{E}$ also has the property that

$$\phi^n \in \mathcal{E} \tag{7.47}$$

for every $\phi \in \mathcal{E}$ and $n \in \mathbf{Z}_+$. In particular, this holds when $\mathcal{E}$ is a subgroup of $\widehat{A}$, with respect to pointwise multiplication of $\mathbf{T}$-valued functions on A, as usual. Combining this with (7.45), we get that

$$\operatorname{Re} \phi(x)^n > 0 \tag{7.48}$$

for every $\phi \in \mathcal{E}$, $n \in \mathbf{Z}_+$, and $x \in U$. This implies that (7.44) holds for every $\phi \in \mathcal{E}$ and $x \in U$. More precisely, it suffices to ask that (7.47) and hence (7.48) hold for positive integers n that are integer powers of 2.

Put

$$B = \bigcap_{\phi \in \mathcal{E}} \phi^{-1}(\{1\}) = \{x \in A : \phi(x) = 1 \text{ for every } \phi \in \mathcal{E}\}, \tag{7.49}$$

which is a subgroup of A. Under the conditions described in the previous paragraph, we have that

$$U \subseteq B. \tag{7.50}$$

This implies that B is an open subgroup of A, as in Section 2.7. Of course, (7.46) holds automatically in this situation, by the definition of B.

7.7 Collections of subgroups

Let A be any commutative group, with the group operations expressed additively, and let $\mathcal{B}_0$ be a nonempty collection of subgroups of A. Let us say that $U \subseteq A$ is an open set with respect to $\mathcal{B}_0$ if for every $x \in U$ there are finitely many elements $B_1, \ldots, B_l$ of $\mathcal{B}_0$ such that

$$x + \bigcap_{j=1}^{l} B_j \subseteq U. \tag{7.51}$$

It is easy to see that this defines a topology on A. Of course, if B is any subgroup of A, then

$$x + B = B \tag{7.52}$$

for every $x \in B$. This implies that the elements of $\mathcal{B}_0$ are open sets in A with respect to this topology. Similarly, if $a \in A$ and $B \in \mathcal{B}_0$, then $a + B$ is an open set in A with respect to this topology. By construction,

$$\{a + B : a \in A,\ B \in \mathcal{B}_0\} \tag{7.53}$$

is a sub-base for this topology on A. In particular, $\mathcal{B}_0$ is a local sub-base for this topology on A at 0.

Translations on A are obviously homeomorphisms with respect to this topology determined by $\mathcal{B}_0$. If $x \in A$ and B is any subgroup of A, then

$$-(x + B) = -x - B = -x + B. \tag{7.54}$$

This implies that $x \mapsto -x$ is continuous on A with respect to the topology determined on A by $\mathcal{B}_0$. Similarly, if $x, y \in A$ and B is a subgroup of A, then

$$(x + B) + (y + B) = x + y + B + B = x + y + B. \tag{7.55}$$

This implies that addition on A is continuous as a mapping from $A \times A$ into A, with respect to the topology determined on A by $\mathcal{B}_0$, and the associated product topology on $A \times A$.

Let us say that $\mathcal{B}_0$ is *nondegenerate* if

$$\bigcap_{B \in \mathcal{B}_0} B = \{0\}. \tag{7.56}$$

This implies that $\{0\}$ is a closed set in A with respect to the topology determined by $\mathcal{B}_0$. In this case, it follows that A is a commutative topological group with respect to this topology. Of course, A is strongly 0-dimensional with respect to this topology, because the elements of $\mathcal{B}_0$ are open subgroups of A. If A is any commutative topological group that is strongly 0-dimensional, and if $\mathcal{B}_0$ is a local base for the topology of A at 0 consisting of open subgroups of A, then the given topology on A is the same as the one determined by $\mathcal{B}_0$ as before.

Let A be any commutative group again. If B is a subgroup of A, then let d_B be the translation-invariant semi-ultrametric on A associated to B as in (7.11) in Section 7.2. Let $\mathcal{B}_0$ be a nonempty collection of subgroups of A, and put

$$\mathcal{M}_0 = \{d_B : B \in \mathcal{B}_0\}, \tag{7.57}$$

which is a nonempty collection of translation-invariant semi-ultrametrics on A. One can check that the topology determined on A by $\mathcal{M}_0$ as in Section 1.3 is the same as the topology determined on A by $\mathcal{B}_0$. If $\mathcal{B}_0$ is nonndegenerate, then $\mathcal{M}_0$ is nondegenerate as a collection of semimetrics on A.

7.8 Strongly totally separated groups

Let A be a commutative topological group, with the group operations expressed additively, as usual. Let us say that A is *strongly totally separated* if for every $x, y \in A$ with $x \neq y$ there is an open set $V \subseteq A$ such that $x \in V$, $y \notin V$, and V is uniformly separated from its complement in A. In particular, this implies that V is a closed set in A, as in Section 7.3. If A is strongly totally separated, then it follows that A is totally separated, as in Section 7.1. In order to check that A is strongly totally separated, one can use translations on A to reduce to the case where $x = 0$ in the previous definition.

Suppose that A is strongly totally separated, and let $y \in A$ with $y \neq 0$ be given. By hypothesis, there is an open set $V \subseteq A$ such that $0 \in V$, $y \notin V$, and V is uniformly separated from its complement in A. This implies that there is an open subgroup B of A such that $B \subseteq V$, as in Sections 7.3 and 7.5. It follows that $y \notin B$, because $B \subseteq V$ and $y \notin V$. This shows that

$$\bigcap \{B : B \text{ is an open subgroup of } A\} = \{0\} \tag{7.58}$$

when A is strongly totally separated. Conversely, if a commutative topological group A satisfies (7.58), then A is strongly totally separated. This uses the fact that an open subgroup B of A is uniformly separated from its complement in A, as in Section 7.3.

Remember that a commutative topological group A satisfies the first separation condition as a topological space. If A is strongly 0-dimensional, then it follows that A is strongly totally separated. The group of rational numbers $\mathbf{Q}$ is not strongly totally separated with respect to the topology induced by the standard topology on $\mathbf{R}$, because $\mathbf{Q}$ is the only open subgroup of itself.

Let A be a commutative topological group, and let τ be the given topology on A. Also let $\mathcal{B}_0$ be the collection of all open subgroups of A. Note that $A \in \mathcal{B}_0$, so that $\mathcal{B}_0 \neq \emptyset$. Let τ_0 be the topology on A associated to $\mathcal{B}_0$ as in the previous section. Thus

$$\tau_0 \subseteq \tau, \tag{7.59}$$

because the elements of $\mathcal{B}_0$ are open subgroups of A with respect to τ, by hypothesis. The condition that A be strongly totally separated with respect to τ corresponds exactly to the nondegeneracy of $\mathcal{B}_0$, as in (7.58).

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