Some lectures in basic analysis

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Preface

These informal notes are largely concerned with metric spaces, sequences and series, and continuous mappings. In particular, compactness, completeness, connectedness, and related notions will play central roles.

There are many textbooks in analysis, in which the reader can find much more information. These notes are intended to complement such textbooks, some of which can be found in the bibliography.

The abstract notion of a metric space is one of the main topics discussed in these notes, and is defined in Section 1.1. Basic examples include the real line, the complex plane, and n-dimensional Euclidean spaces, with their standard Euclidean metrics.

Some important properties of the real numbers that will be needed here are discussed in Sections 1.2 and 1.3. Some readers may be familiar with properties like these already, and otherwise more information can be found in many texts, including ones in the bibliography.

Many readers probably have at least some familiarity with sequences and series of real numbers. Here we shall consider convergence of sequences in arbitrary metric spaces, as well as some topics related to infinite series of real and complex numbers. As usual, we shall focus on some of the more theoretical aspects, rather than some of the typical examples that many readers may have seen previously.

One of the motivations for the study of continuity and convergence is the theory behind calculus. Some readers may have some familiarity with this already, and otherwise some of the results discussed here can be used to make precise well-known arguments related to differentiation and integration. Some related matters will be discussed in Chapter 5.

Some additional topics related to those discussed here will be mentioned in Section 4.15. Some related aspects of history may be found in [8, 13, 14, 15, 16, 29, 30, 32, 48, 49, 50, 52, 53, 54, 55, 56, 57, 59, 86, 108], for instance. Some related songs may be found in [90, 91, 92]. Some remarks concerning the clarity of explanations in mathematics may be found in [58].

A question came up about the axiom of choice, and we shall not deal with this much here. The axiom of choice is implicitly used in some proofs, particularly with a sequence of choices. Sometimes the axiom of choice is implicitly used, and a slightly different argument would not require it. A very nice reference for some topics like these is [71]. Some more information may be found in [63, 96], for instance.

Countability and uncountability of sets are discussed at the beginning of Chapter 2, without getting into cardinalities of sets too much. This is discussed further in [71] too. The reader may also be interested in Gödel's paper [42] on the continuum hypothesis. This is mentioned in [71] as well.

I would like to dedicate these notes to the memory of my undergraduate advisor, Dr. Richard M. Summerville.

Contents

Chapter 1

Metric spaces

1.1 The definition, and some examples

Definition 1.1.1 *A* metric space *is a set M together with a nonnegative real-valued function $d(x, y)$ defined for $x, y \in M$ that satisfies the following three conditions. First, for each $x, y \in M$, we have that*

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{1.1.2}$$

Second, $d(x, y)$ should be symmetric in x and y, so that

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in M. \tag{1.1.3}$$

The third condition is that

$$d(x, z) \leq d(x, y) + d(y, z) \quad \text{for every } x, y, z \in M, \tag{1.1.4}$$

which is called the triangle inequality. *Under these conditions, $d(\cdot, \cdot)$ is said to define a* metric *on M.*

As a basic example, the real line $\mathbf{R}$ is a metric space with respect to the standard *Euclidean metric*. If x is a real number, then the *absolute value* $|x|$ of x is defined as usual by

$$\begin{aligned} |x| &= x && \text{when } x \geq 0 \\ &= -x && \text{when } x \leq 0. \end{aligned} \tag{1.1.5}$$

It is well known and not difficult to see that

$$|x + y| \leq |x| + |y| \tag{1.1.6}$$

for every $x, y \in \mathbf{R}$. More precisely, we have equality in (1.1.6) when x and y have the same sign, and otherwise there is some cancellation in the left side. The standard Euclidean metric on $\mathbf{R}$ is defined by

$$d(x, y) = |x - y|. \tag{1.1.7}$$

One can check directly that this satisfies the requirements of a metric on $\mathbf{R}$. In particular, (1.1.4) follows from (1.1.6) in this situation.

Let n be a positive integer, and let $\mathbf{R}^n$ be the usual space of n-tuples of real numbers. Thus an element of $\mathbf{R}^n$ can be given as

$$x = (x_1, \ldots, x_n), \tag{1.1.8}$$

where $x_1, \ldots, x_n \in \mathbf{R}$. The standard *Euclidean metric* on $\mathbf{R}^n$ is defined by

$$d(x, y) = \Big(\sum_{j=1}^{n} (x_j - y_j)^2 \Big)^{1/2}, \tag{1.1.9}$$

using the nonnegative square root on the right side. It is easy to see that this satisfies the first two requirements (1.1.2), (1.1.3) of a metric. It is well known that (1.1.9) satisfies the triangle inequality (1.1.4), but this is more complicated when $n \geq 2$, and we shall not give a proof here. One can check that

$$d'(x, y) = \sum_{j=1}^{n} |x_j - y_j| \tag{1.1.10}$$

and

$$d''(x, y) = \max_{1 \leq j \leq n} |x_j - y_j| \tag{1.1.11}$$

also define metrics on $\mathbf{R}^n$. In particular, the triangle inequality for (1.1.10) and (1.1.11) can be obtained from (1.1.6). If $n = 1$, then (1.1.9), (1.1.10), and (1.1.11) are all the same as (1.1.7).

Let M be any set. If $x, y \in M$, then put

$$\begin{aligned} d(x, y) &= 0 \quad \text{when } x = y \\ &= 1 \quad \text{when } x \neq y. \end{aligned} \tag{1.1.12}$$

One can check that this defines a metric on M. This is called the *discrete metric* on M.

Now let $(M, d(x, y))$ be an arbitrary metric space, and let E be a subset of M. It is easy to see that the restriction of $d(x, y)$ to $x, y \in E$ defines a metric on E. Thus E may be considered as a metric space as well, using this metric.

1.2 Supremum and infimum

Definition 1.2.1 *Let A be a subset of the real line. A real number b is said to be an* upper bound *for A if for every $a \in A$ we have that $a \leq b$. A real number c is said to be the* least upper bound *or* supremum *of A in $\mathbf{R}$ if it satisfies the follwing two conditions. First,*

$$c \text{ is an upper bound for } A. \tag{1.2.2}$$

Second,

$$\text{if } b \in \mathbf{R} \text{ is an upper bound for } A, \text{ then } c \leq b. \tag{1.2.3}$$

It is easy to see that such a c is unique when it exists. More precisely, if $c', c'' \in \mathbf{R}$ both satisfy these two conditions, then one can check that $c' \leq c''$ and $c'' \leq c'$. This implies that $c' = c''$, as desired.

An important feature of the real numbers is the *least upper bound property*. This says that if A is a nonempty subset of $\mathbf{R}$ with an upper bound in $\mathbf{R}$, then A has a least upper bound in $\mathbf{R}$. As before, we shall also refer to this as the supremum of A in $\mathbf{R}$, which may be denoted $\sup A$.

The least upper bound property is an important difference between the real line and the set $\mathbf{Q}$ of rational numbers. If A is a nonempty subset of $\mathbf{Q}$ with an upper bound in $\mathbf{Q}$, then the supremum of A in $\mathbf{R}$ may not be an element of $\mathbf{Q}$.

There are well-known ways in which $\mathbf{R}$ can be obtained from $\mathbf{Q}$, but we shall not pursue this here.

Definition 1.2.4 *Let A be a subset of $\mathbf{R}$ again. A real number y is said to be a* lower bound *for A if for every $x \in A$ we have that $y \leq x$. A real number z is said to be the* greatest lower bound *or* infimum *of A in $\mathbf{R}$ if it satisfies the following two conditions. First,*

$$z \text{ is a lower bound for } A. \tag{1.2.5}$$

Second,

$$\text{if } y \in \mathbf{R} \text{ is a lower bound for } A, \text{ then } y \leq z. \tag{1.2.6}$$

One can check that such a z is unique when it exists, as before. In this case, z may be denoted $\inf A$.

If A is not the empty set $\emptyset$ and A has a lower bound in $\mathbf{R}$, then A has a greatest lower bound in $\mathbf{R}$. To see this, put

$$B = \{y \in \mathbf{R} : y \text{ is a lower bound for } A\}. \tag{1.2.7}$$

Thus $B \neq \emptyset$, by hypothesis, and every element of A is an upper bound for B. The least upper bound property for $\mathbf{R}$ implies that B has a least upper bound in $\mathbf{R}$. One can verify that $\sup B$ also satisfies the requirements of the greatest lower bound for A, as desired.

Alternatively, let $-A$ be the set of real numbers of the form $-a$, with $a \in A$. Of course, $-A \neq \emptyset$, because $A \neq \emptyset$. If $y \in \mathbf{R}$ is a lower bound for A, then $-y$ is an upper bound for $-A$. Hence $-A$ has a supremum in $\mathbf{R}$. One can check that $-\sup(-A)$ satisfies the requirements of the infimum of A.

1.3 Additional properties of R

The *archimedean property* of the real numbers is the following:

$$\begin{array}{l}\text{if } x \text{ and } y \text{ are positive real numbers, then} \\ \text{there is a positive integer } n \text{ such that } y < n\,x.\end{array} \tag{1.3.1}$$

Here is another important property of the real line:

(1.3.2) if $a, b \in \mathbf{R}$ and $a < b$, then there is an $r \in \mathbf{Q}$ such that $a < r < b$.

It is not too difficult to see that each of these two properties implies that other. There is a well-known argument by which the archimedean property can be obtained using the least upper bound property of $\mathbf{R}$. Alternatively, (1.3.2) can be obtained from standard constructions of $\mathbf{R}$ from $\mathbf{Q}$.

Let x be a positive real number, and let n be a positive integer. It is well known that there is a unique positive real number y such that

$$y^n = x. \tag{1.3.3}$$

The uniqueness of y can be verified directly, but the existence of y is more complicated. Let A be the set of nonnegative real numbers t such that

$$t^n < x. \tag{1.3.4}$$

Note that $0 \in A$, so that $A \neq \emptyset$. If $x \leq 1$, then it is easy to see that 1 is an upper bound for A. Otherwise, if $x > 1$, then one can check that x is an upper bound for A. In both cases, A has an upper bound in $\mathbf{R}$, which implies that A has a supremum in $\mathbf{R}$. It is well known that $y = \sup A$ satisfies (1.3.3), but we shall not prove this here.

If m is another positive integer, then one can define $x^{m/n}$ to be the positive nth root of $x^m > 0$. Using this, one can define x^r when $r \in \mathbf{Q}$ and $r > 0$. More precisely, one can verify that this does not depend on the particular representation for r as a ratio of positive integers. Note that $x^r \geq 1$ when $x \geq 1$, and $x^r \leq 1$ when $x \leq 1$.

If t is a positive real number, then one can define x^t as a positive real number too. More precisely, if $x \geq 1$, then one can take x^t to be the supremum of the set of positive real numbers of the form x^r, where $r \in \mathbf{Q}$ and $0 < r \leq t$. Otherwise, if $0 < x \leq 1$, then one can reduce to the previous case, by considering $1/x$.

1.4 Open balls and open sets

Let $(M, d(x, y))$ be a metric space.

Definition 1.4.1 *If x is an element of M and r is a positive real number, then the* open ball *in M centered at x with radius r is defined to be the set of $y \in M$ such that $d(x, y) < r$, i.e.,*

$$B(x, r) = \{y \in M : d(x, y) < r\}. \tag{1.4.2}$$

A subset U of M is said to be an open set *in M with respect to $d(\cdot, \cdot)$ if for every $x \in U$ there is a positive real number r such that*

$$B(x, r) \subseteq U. \tag{1.4.3}$$

Note that M is automatically an open subset of itself. The empty set is an open set in M as well.

Proposition 1.4.4 *If $z \in M$ and t is a positive real number, then $B(z,t)$ is an open set in M.*

To see this, let an element x of $B(z,t)$ be given. Thus $d(x,z) < t$, so that $r = t - d(x,z) > 0$. It suffices to show that

$$B(x,r) \subseteq B(z,t). \tag{1.4.5}$$

To do this, let y be an arbitrary point in $B(x,r)$, so that $d(x,y) < r$. Using the triangle inequality, we get that

$$d(y,z) \le d(y,x) + d(x,z) < r + d(x,z) = t. \tag{1.4.6}$$

This means that $d(y,z) < t$, and hence $y \in B(z,t)$. This shows that (1.4.5) holds, as desired.

Proposition 1.4.7 *If $z \in M$ and t is a nonnegative real number, then*

$$V(z,t) = \{w \in M : d(z,w) > t\} \tag{1.4.8}$$

is an open set in M.

To prove this, let $x \in V(z,t)$ be given, so that $r = d(x,z) - t > 0$. Under these conditions, one can use the triangle inequality to show that

$$B(x,r) \subseteq V(z,t). \tag{1.4.9}$$

The details are left as an exercise.

If a and b are real numbers with $a < b$, then

$$(a,b) = \{x \in \mathbf{R} : a < x < b\} \tag{1.4.10}$$

is the *open interval* in $\mathbf{R}$ from a to b. It is easy to see that this is an open set in $\mathbf{R}$, with respect to the standard Euclidean metric. This is the same as the open ball in $\mathbf{R}$ centered at the midpoint $(a+b)/2$ of the interval with radius $(b-a)/2$.

1.5 Limit points and closed sets

Let $(M, d(x,y))$ be a metric space, and let E be a subset of M.

Definition 1.5.1 *A point $p \in M$ is said to be a* limit point *of E in M if for every positive real number r there is a point $q \in E$ such that $d(p,q) < r$ and $p \neq q$.*

Note that a limit point of E in M may or may not be an element of E.

Proposition 1.5.2 *If $p \in M$ is a limit point of E, then for each $r > 0$ there are infinitely many elements of E in $B(p,r)$.*

Let $r > 0$ be given, and suppose for the sake of a contradiction that there are only finitely many elements of E in $B(p,r)$. Let $q_1, \ldots, q_n$ be a list of the elements of E in $B(p,r)$ that are not equal to p. Put

$$t = \min_{1 \le j \le n} d(p, q_j), \tag{1.5.3}$$

which is a positive real number. By construction, there are no elements of E in $B(p,t)$, except perhaps p, contradicting the hypothesis that p be a limit point of E in M.

Definition 1.5.4 *A subset E of M is said to be a* closed set *with respect to $d(\cdot,\cdot)$ if for every $p \in M$ such that p is a limit point of E, we have that $p \in E$.*

If $E \subseteq M$ has only finitely many elements, then the previous proposition implies that E has no limit points in M, and hence that E is a closed set in M. Observe that M is automatically a closed set in itself.

If $a, b \in \mathbf{R}$ and $a \le b$, then the *closed interval* in $\mathbf{R}$ from a to b is defined by

$$[a,b] = \{x \in \mathbf{R} : a \le x \le b\}. \tag{1.5.5}$$

It is easy to see that this is a closed set in $\mathbf{R}$, with respect to the standard Euclidean metric.

If A and B are sets, then let $A \setminus B$ be the set of points in A and not in B, i.e.,

$$A \setminus B = \{x \in A : x \notin B\}. \tag{1.5.6}$$

If M is a set and E is a subset of M, then $M \setminus E$ is called the *complement* of E in M. In this case, we have that

$$M \setminus (M \setminus E) = E. \tag{1.5.7}$$

Let $(M, d(x,y))$ be a metric space again.

Proposition 1.5.8 *A subset E of M is a closed set if and only if $M \setminus E$ is an open set in M.*

Suppose first that E is a closed set, and let us check that $M \setminus E$ is an open set. Let $p \in M \setminus E$ be given. Note that p is not a limit point of E, because $p \notin E$ and E is a closed set. This implies that there is an $r > 0$ such that $B(p,r)$ does not contain any elements of E, except perhaps p. It follows that $B(p,r)$ does not contain any elements of E, because $p \notin E$. Equivalently, this means that

$$B(p,r) \subseteq M \setminus E, \tag{1.5.9}$$

as desired.

Conversely, suppose that $M \setminus E$ is an open set in M, and let us show that E is a closed set. If $p \in M \setminus E$, then there is an $r > 0$ such that (1.5.9) holds, because $M \setminus E$ is an open set. This is the same as saying that

$$B(p, r) \cap E = \emptyset. \tag{1.5.10}$$

In particular, this means that p is not a limit point of E in M. It follows that every limit point of E in M is contained in E, as desired.

Using the proposition, we get that a subset U of M is an open set if and only if $M \setminus U$ is a closed set. More precisely, this follows from the proposition with $E = M \setminus U$.

1.6 Unions, intersections, and closures

Let $(M, d(x, y))$ be a metric space.

Proposition 1.6.1 *If $U_1, \ldots, U_n$ are finitely many open subsets of M, then their intersection $\bigcap_{j=1}^n U_j$ is an open set in M as well.*

Let x be an arbitrary element of $\bigcap_{j=1}^n U_j$, so that $x \in U_j$ for each $j = 1, \ldots, n$. Thus, for each $j = 1, \ldots, n$, there is a positive real number r_j such that

$$B(x, r_j) \subseteq U_j, \tag{1.6.2}$$

because U_j is an open set in M. Put

$$r = \min(r_1, \ldots, r_n), \tag{1.6.3}$$

and note that $r > 0$. Clearly

$$B(x, r) \subseteq B(x, r_j) \subseteq U_j \tag{1.6.4}$$

for each $j = 1, \ldots, n$, so that

$$B(x, r) \subseteq \bigcap_{j=1}^n U_j, \tag{1.6.5}$$

as desired.

Corollary 1.6.6 *If $E_1, \ldots, E_n$ are finitely many closed subsets of M, then their union $\bigcup_{j=1}^n E_j$ is a closed set too.*

It suffices to show that the complement $M \setminus \left(\bigcup_{j=1}^n E_j\right)$ of the union is an open set in M, by Proposition 1.5.8. It is well known and not difficult to check that

$$M \setminus \left(\bigcup_{j=1}^n E_j\right) = \bigcap_{j=1}^n \left(M \setminus E_j\right). \tag{1.6.7}$$

We also have that $M \setminus E_j$ is an open set in M for each $j = 1, \ldots, n$, because E_j is a closed set by hypothesis. Proposition 1.6.1 implies that the right side of (1.6.7) is an open set in M, as desired.

Proposition 1.6.8 *Let A be a nonempty set, and suppose that U_α is an open subset of M for each $\alpha \in A$. Under these conditions, $\bigcup_{\alpha \in A} U_\alpha$ is an open set in M.*

Let $x \in \bigcup_{\alpha \in A} U_\alpha$ be given, so that there is an index $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. Because U_{α_0} is an open set in M, by hypothesis, there is a positive real number r such that

$$B(x, r) \subseteq U_{\alpha_0}. \tag{1.6.9}$$

This implies that

$$B(x, r) \subseteq \bigcup_{\alpha \in A} U_\alpha, \tag{1.6.10}$$

as desired.

Corollary 1.6.11 *If A is a nonempty set, and $E_\alpha \subseteq M$ is a closed set for each $\alpha \in A$, then $\bigcap_{\alpha \in A} E_\alpha$ is a closed set in M.*

One can verify that

$$M \setminus \Big(\bigcap_{\alpha \in A} E_\alpha \Big) = \bigcup_{\alpha \in A} (M \setminus E_\alpha), \tag{1.6.12}$$

as before. Note that $M \setminus E_\alpha$ is an open set for every $\alpha \in A$, because E_α is a closed set, by hypothesis. Thus the right side of (1.6.12) is an open set too, by Proposition 1.6.8. This implies that $\bigcap_{\alpha \in A} E_\alpha$ is a closed set, by Proposition 1.5.8.

Alternatively, suppose that $p \in M$ is a limit point of $\bigcap_{\alpha \in A} E_\alpha$. If β is any element of A, then one can check that p is a limit point of E_β, because $\bigcap_{\alpha \in A} E_\alpha \subseteq E_\beta$. This implies that $p \in E_\beta$, because E_β is a closed set in M. It follows that $p \in \bigcap_{\alpha \in A} E_\alpha$, because the previous statement holds for every $\beta \in A$. This shows that $\bigcap_{\alpha \in A} E_\alpha$ contains all of its limit points in M, as desired.

Definition 1.6.13 *Let E be a subset of M. The* closure *of E in M is the set $\overline{E}$ consisting of all $p \in M$ such that $p \in E$, p is a limit point of E, or both.*

If $E \subseteq M$ is a closed set, then $\overline{E} = E$, because E contains its limit points in M. Conversely, if $\overline{E} = E$, then E contains all of its limit points in M, and hence E is a closed set.

Proposition 1.6.14 *If E is any subset of M, then $\overline{E}$ is a closed set in M.*

It is enough to show that $M \setminus \overline{E}$ is an open set in M, as in Proposition 1.5.8. Let $p \in M \setminus \overline{E}$ be given, so that $p \notin E$ and p is not a limit point of E. The latter implies that there is a positive real number r such that $B(p, r)$ does not contain any elements of E, other than p. This means that $B(p, r)$ does not contain any elements of E, because $p \notin E$. Equivalently, $B(p, r) \subseteq M \setminus E$. Using this, one can verify that

$$B(p, r) \subseteq M \setminus \overline{E}, \tag{1.6.15}$$

as desired. Alternatively, one can show that every limit point of $\overline{E}$ in M is a limit point of E, and hence an element of $\overline{E}$.

1.7 Compactness

Let $(M, d(x, y))$ be a metric space, and let K be a subset of M.

Definition 1.7.1 *A family $\{U_\alpha\}_{\alpha \in A}$ of open subsets of M is said to be an* open covering *of K in M if*

$$K \subseteq \bigcup_{\alpha \in A} U_\alpha. \tag{1.7.2}$$

If every open covering of K in M can be reduced to a finite subcovering of K, then K is said to be compact *in M. More precisely, K is compact in M if for every open covering $\{U_\alpha\}_{\alpha \in A}$ of K in M, there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that*

$$K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}. \tag{1.7.3}$$

It is easy to see that finite subsets of M are compact.

Let us consider a couple of examples in the real line, equipped with the standard Euclidean metric. Consider

$$K = \{0\} \cup \{1/j : j \in \mathbf{Z}_+\}, \tag{1.7.4}$$

where $\mathbf{Z}_+$ denotes the set of positive integers. We would like to show that this set is compact in $\mathbf{R}$. To do this, let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of K in $\mathbf{R}$. In particular, there is an $\alpha_0 \in A$ such that $0 \in U_{\alpha_0}$, because $0 \in K$. This implies that there is a positive real number r such that

$$(-r, r) \subseteq U_{\alpha_0}, \tag{1.7.5}$$

because U_{α_0} is an open set in $\mathbf{R}$. Let n be a positive integer such that $1/r < n$, as in the archimedean property for $\mathbf{R}$. Equivalently, $1/n < r$. If $n = 1$, then it follows that $K \subseteq U_{\alpha_0}$. Otherwise, if $n > 1$, then for each $j = 1, \ldots, n-1$, let α_j be an element of A such that $1/j \in U_{\alpha_j}$. Under these conditions, we get that

$$K \subseteq \bigcup_{j=0}^{n-1} U_{\alpha_j}, \tag{1.7.6}$$

as desired.

Next, let us show that the open unit interval $(0, 1)$ is not compact in $\mathbf{R}$. Thus we would like to find an open covering of $(0, 1)$ in $\mathbf{R}$ for which there is no finite subcovering. One way to do this is to use the family of open intervals of the form $(1/j, 1)$, where j is a positive integer greater than or equal to two. Each of these open intervals is an open set in $\mathbf{R}$ with respect to the standard Euclidean metric, and it is easy to see that

$$\bigcup_{j=2}^{\infty} (1/j, 1) = (0, 1). \tag{1.7.7}$$

However, one can check that $(0,1)$ is not contained in the union of finitely many of these intervals. More precisely,

$$(1/j, 1) \subseteq (1/(j+1), 1) \tag{1.7.8}$$

for each $j \geq 2$. This implies that the union of finitely many intervals of the form $(1/j, 1)$ is equal to a single such interval.

Let $(M, d(x,y))$ be any metric space again.

Definition 1.7.9 *A subset E of M is said to have the* limit point property *if for every subset L of E such that L has infinitely many elements, there is a point $p \in E$ that is a limit point of L in M.*

If $E \subseteq M$ has only finitely many elements, then E automatically has the limit point property.

Proposition 1.7.10 *If $K \subseteq M$ is compact, then K has the limit point property.*

Let L be an infinite subset of K, and suppose for the sake of a contradiction that L has no limit point in K. Let $p \in K$ be given, so that p is not a limit point of L. This means that there is a positive real number $r(p)$ such that $B(p, r(p))$ does not contain any elements of L, except perhaps p itself. The family of open balls $B(p, r(p))$ of this type, with $p \in K$, is an open covering of K, because $p \in B(p, r(p))$ for each $p \in K$. If K is compact, then there are finitely many elements $p_1, \ldots, p_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} B(p_j, r(p_j)). \tag{1.7.11}$$

In particular, it follows that

$$L \subseteq \bigcup_{j=1}^{n} B(p_j, r(p_j)), \tag{1.7.12}$$

because $L \subseteq K$, by hypothesis. This implies that L has at most n elements, because p_j is the only element of $B(p_j, r(p_j))$ that can be in L. This contradicts the hypothesis that L have infinitely many elements, as desired.

1.8 Bounded sets

Let $(M, d(x,y))$ be a metric space.

Definition 1.8.1 *A subset E of M is said to be* bounded *in M with respect to $d(\cdot,\cdot)$ if there is a point $p \in M$ and a positive real number r such that*

$$E \subseteq B(p, r). \tag{1.8.2}$$

The empty set is considered to be a bounded set in M, even when $M = \emptyset$.

Proposition 1.8.3 *If $K \subseteq M$ is compact and $p \in M$, then there is a positive real number r such that*

$$K \subseteq B(p, r). \tag{1.8.4}$$

Let $p \in M$ be given, and observe that

$$\bigcup_{j=1}^{\infty} B(p, j) = M. \tag{1.8.5}$$

It follows that the family of open balls $B(p, j)$, $j \in \mathbf{Z}_+$, is an open covering of K in M. If K is compact, then there are finitely many positive integers $j_1, \ldots, j_n$ such that

$$K \subseteq \bigcup_{l=1}^{n} B(p, j_l). \tag{1.8.6}$$

This implies that (1.8.4) holds with

$$r = \max(j_1, \ldots, j_n), \tag{1.8.7}$$

as desired.

Proposition 1.8.8 *If $E \subseteq M$ has the limit point property and $p \in M$, then there is a positive real number r such that $E \subseteq B(p, r)$.*

Suppose for the sake of a contradiction that for each positive integer j,

$$E \not\subseteq B(p, j). \tag{1.8.9}$$

Thus, for each positive integer j, there is an element of E whose distance to p is at least j. Let us now choose, for each positive integer j, such a point $x_j \in E$ with

$$d(p, x_j) \geq j. \tag{1.8.10}$$

Let

$$L = \{x_j : j \in \mathbf{Z}_+\} \tag{1.8.11}$$

be the set of points in E that have been chosen in this way.

Suppose for the sake of a contradiction that L has only finitely many elements. This implies that there is an $x \in E$ such that $x = x_j$ for infinitely many j. In this case, we would have that

$$d(p, x) \geq j \tag{1.8.12}$$

for infinitely many positive integers j, by (1.8.10). However, the archimedean property for $\mathbf{R}$ implies that there is a positive integer n such that $d(p, x) \leq n$, so that (1.8.12) holds for only finitely many $j \in \mathbf{Z}_+$.

Thus L has infinitely many elements. If E has the limit point property, then there is a point $q \in E$ such that q is a limit point of L in M. In particular, this implies that $B(q, 1)$ contains infinitely many elements of L, as in Proposition 1.5.2. It follows that

$$d(q, x_j) < 1 \tag{1.8.13}$$

for infinitely many $j \in \mathbf{Z}_+$, by the definition (1.8.11) of L. Using this and the triangle inequality, we get that

$$d(p, x_j) \le d(p, q) + d(q, x_j) < d(p, q) + 1 \tag{1.8.14}$$

for infinitely many $j \in \mathbf{Z}_+$. Combining this with (1.8.10), we obtain that

$$j < d(p, q) + 1 \tag{1.8.15}$$

for infinitely many $j \in \mathbf{Z}_+$. This is a contradiction, because (1.8.15) holds for only finitely many positive integers j, as before.

1.9 Compactness and closed sets

Let $(M, d(x, y))$ be a metric space.

Proposition 1.9.1 *If $K \subseteq M$ is compact, then K is a closed set in M.*

Let $p \in M \setminus K$ be given, and let us show that p is not a limit point of K. If j is a positive integer, then $V(p, 1/j)$ consists of the elements of M whose distance to p is larger than $1/j$, as in (1.4.8). Remember that this is an open set in M, as in Proposition 1.4.7. It is easy to see that

$$\bigcup_{j=1}^{\infty} V(p, 1/j) = M \setminus \{p\}. \tag{1.9.2}$$

This implies that the family of sets $V(p, 1/j)$, $j \in \mathbf{Z}_+$, is an open covering of K in M, because $p \in M \setminus K$, by hypothesis.

If K is compact, then there are finitely many positive integers $j_1, \ldots, j_n$ such that

$$K \subseteq \bigcup_{l=1}^{n} V(p, 1/j_l). \tag{1.9.3}$$

This means that

$$K \subseteq V(p, r), \tag{1.9.4}$$

where r is the positive real number defined by

$$1/r = \max(j_1, \ldots, j_n). \tag{1.9.5}$$

In particular, there are no elements of K in $B(p, r)$, so that p is not a limit point of K, as desired.

Proposition 1.9.6 *Suppose that $K \subseteq M$ is compact, and that $E \subseteq M$ is a closed set. If $E \subseteq K$, then E is compact in M.*

To see this, let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of E in M. Note that $M \setminus E$ is an open set, because E is a closed set, as in Proposition 1.5.8. We also have that

$$K \subseteq M = E \cup (M \setminus E) \subseteq \Big(\bigcup_{\alpha \in A} U_\alpha \Big) \cup (M \setminus E). \tag{1.9.7}$$

This implies that the collection of U_α, $\alpha \in A$, together with $M \setminus E$, forms an open covering of K in M.

If K is compact, then there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \Big(\bigcup_{j=1}^{n} U_{\alpha_j} \Big) \cup (M \setminus E). \tag{1.9.8}$$

In particular, this means that

$$E \subseteq \Big(\bigcup_{j=1}^{n} U_{\alpha_j} \Big) \cup (M \setminus E), \tag{1.9.9}$$

because $E \subseteq K$, by hypothesis. Of course, this implies that

$$E \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}, \tag{1.9.10}$$

as desired.

1.10 Relatively open sets

Let $(M, d(x, y))$ be a metric space, and let Y be a subset of M. Remember that the restriction of $d(x, y)$ to $x, y \in Y$ defines a metric on Y, so that Y may be considered as a metric space as well.

Definition 1.10.1 *A subset E of Y is said to be* relatively open *in Y if E is an open set in Y with respect to the restriction of $d(x, y)$ to $x, y \in Y$.*

Suppose for instance that $M = \mathbf{R}^2$, equipped with the standard Euclidean metric, and Y is a line in $\mathbf{R}^2$. One can have open line segments in Y that are relatively open in Y, and not open as subsets of M.

Let $(M, d(x, y))$ be any metric space again, and let Y be a subset of M. If $x \in Y$ and r is a positive real number, then let $B_Y(x, r)$ be the corresponding open ball in Y, with respect to the restriction of $d(\cdot, \cdot)$ to Y. Thus

$$B_Y(x, r) = \{y \in Y : d(x, y) < r\} = B(x, r) \cap Y, \tag{1.10.2}$$

where we continue to use $B(x, r)$ for the open ball in M centered at x with radius r. By definition, a subset E of Y is relatively open in Y if and only if for every $x \in E$ there is an $r > 0$ such that

$$B_Y(x, r) \subseteq E. \tag{1.10.3}$$

Proposition 1.10.4 *A subset E of Y is relatively open in Y if and only if there is an open subset U of M such that*

$$E = U \cap Y. \tag{1.10.5}$$

To prove the "if" part, suppose that $U \subseteq M$ is an open set, and let us check that $E = U \cap Y$ is relatively open in Y. Let $x \in E$ be given, so that $x \in U$ in particular. Because U is an open set in M, there is an $r > 0$ such that $B(x, r) \subseteq U$. This implies that

$$B_Y(x, r) = B(x, r) \cap Y \subseteq U \cap Y = E, \tag{1.10.6}$$

as desired.

Conversely, suppose that E is a relatively open subset of Y. Thus, for each $x \in E$, there is a positive real number $r(x)$ such that

$$B_Y(x, r(x)) \subseteq E. \tag{1.10.7}$$

Put

$$U = \bigcup_{x \in E} B(x, r(x)). \tag{1.10.8}$$

This may be interpreted as being the empty set when $E = \emptyset$. Observe that U is an open set in M, because open balls are open sets, and a union of open sets is an open set too, as in Propositions 1.4.4 and 1.6.8.

We would like to verify that (1.10.5) holds in this situation. It is easy to see that $E \subseteq U$, because $x \in B(x, r(x))$ for every $x \in E$. This implies that

$$E \subseteq U \cap Y, \tag{1.10.9}$$

because $E \subseteq Y$, by hypothesis. Using the definition (1.10.8) of U, we get that

$$U \cap Y = \Big(\bigcup_{x \in E} B(x, r(x))\Big) \cap Y = \bigcup_{x \in E} (B(x, r(x)) \cap Y). \tag{1.10.10}$$

It follows that

$$U \cap Y = \bigcup_{x \in E} B_Y(x, r(x)) \subseteq E, \tag{1.10.11}$$

as desired, using (1.10.2) in the first step, and (1.10.7) in the second step.

Proposition 1.10.12 *Let K be a subset of Y. Under these conditions, K is compact as a subset of M if and only K is compact as a subset of Y, with respect to the restriction of $d(\cdot, \cdot)$ to Y.*

Suppose first that K is compact as a subset of Y, and let us check that K is compact as a subset of M. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of K in M. If $\alpha \in A$, then U_α is an open subset of M, and hence $U_\alpha \cap Y$ is relatively open in Y, by the previous proposition. We also have that

$$K \subseteq \Big(\bigcup_{\alpha \in A} U_\alpha\Big) \cap Y = \bigcup_{\alpha \in A} (U_\alpha \cap Y), \tag{1.10.13}$$

because K is contained in each of $\bigcup_{\alpha \in A} U_\alpha$ and Y, by hypothesis. This shows that $\{U_\alpha \cap Y\}_{\alpha \in A}$ is an open covering of K in Y. If K is compact as a subset of Y, then there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{n} (U_{\alpha_j} \cap Y). \tag{1.10.14}$$

This implies that $K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$, as desired.

Conversely, suppose that K is compact as a subset of M, and let us show that K is compact as a subset of Y. To do this, let $\{E_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of K in Y. This means that E_α is a relatively open subset of Y for each $\alpha \in A$, and that

$$K \subseteq \bigcup_{\alpha \in A} E_\alpha. \tag{1.10.15}$$

If $\alpha \in A$, then there is an open subset U_α of M such that

$$E_\alpha = U_\alpha \cap Y, \tag{1.10.16}$$

by Proposition 1.10.4. This leads to an open covering $\{U_\alpha\}_{\alpha \in A}$ of K in M. If K is compact in M, then there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$. It follows that

$$K \subseteq \Big(\bigcup_{j=1}^{n} U_{\alpha_j}\Big) \cap Y = \bigcup_{j=1}^{n} (U_{\alpha_j} \cap Y) = \bigcup_{j=1}^{n} E_{\alpha_j}, \tag{1.10.17}$$

as desired, because $K \subseteq Y$.

1.11 Sequences of closed intervals

Proposition 1.11.1 *Let $I_1, I_2, I_3, \ldots$ be an infinite sequence of closed intervals in the real line. If*

$$I_{j+1} \subseteq I_j \tag{1.11.2}$$

for every positive integer j, then

$$\bigcap_{j=1}^{\infty} I_j \neq \emptyset. \tag{1.11.3}$$

If $j \in \mathbf{Z}_+$, then there are real numbers a_j, b_j such that

$$a_j \leq b_j \tag{1.11.4}$$

and $I_j = [a_j, b_j]$, by the definition of a closed interval in Section 1.5. We also have that

$$a_j \leq a_{j+1} \quad \text{and} \quad b_{j+1} \leq b_j \tag{1.11.5}$$

for every $j \geq 1$, because of (1.11.2). Put

$$A = \{a_j : j \in \mathbf{Z}_+\}. \tag{1.11.6}$$

Note that $A \neq \emptyset$, because $a_1 \in A$.

Let us check that

$$a_j \leq b_l \tag{1.11.7}$$

for all positive integers j and l. If $j \leq l$, then

$$a_j \leq a_l \leq b_l. \tag{1.11.8}$$

Similarly, if $j \geq l$, then

$$a_j \leq b_j \leq b_l. \tag{1.11.9}$$

Thus, for each positive integer l, b_l is an upper bound of A. This implies that the supremum of A exists in $\mathbf{R}$, and satisfies

$$\sup A \leq b_l \tag{1.11.10}$$

for every $l \geq 1$. Of course, $a_j \leq \sup A$ for every $j \geq 1$, by construction. It follows that

$$\sup A \in I_j \tag{1.11.11}$$

for every $j \geq 1$, as desired.

Put

$$B = \{b_l : l \in \mathbf{Z}_+\}, \tag{1.11.12}$$

which is another nonempty set of real numbers. Using (1.11.7), we get that a_j is a lower bound for B for each $j \geq 1$. This means that the infimum of B exists in $\mathbf{R}$, and satisfies

$$a_j \leq \inf B \tag{1.11.13}$$

for every $j \geq 1$. We also have that $\inf B \leq b_l$ for every $l \geq 1$, so that

$$\inf B \in I_j \tag{1.11.14}$$

for every $j \geq 1$. Observe that

$$\sup A \leq \inf B, \tag{1.11.15}$$

by (1.11.10) or (1.11.13). One can check that

$$\bigcap_{j=1}^{\infty} I_j = [\sup A, \inf B]. \tag{1.11.16}$$

However, this is not needed for Proposition 1.11.1.

1.12 Compactness of closed intervals

Theorem 1.12.1 *Let a and b be real numbers, with $a \le b$. Under these conditions, the closed interval $I = [a, b]$ is a compact subset of the real line, with respect to the standard Euclidean metric.*

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of I in $\mathbf{R}$, and suppose for the sake of a contradiction that I cannot be covered by finitely many U_α's. Put

$$L = [a, (a + b)/2] \quad \text{and} \quad R = [(a + b)/2, b], \tag{1.12.2}$$

which are the closed intervals corresponding to the left and right halves of I. In particular,

$$L \cup R = I, \tag{1.12.3}$$

and

$$\text{length}(L) = \text{length}(R) = \text{length}(I)/2, \tag{1.12.4}$$

where $\text{length}(I) = b - a$ is the usual length of an interval. It follows from (1.12.3) that at least one of L and R cannot be covered by finitely many U_α's, because of the analogous property for I. Let us now choose I_1 to be either L or R, in such a way that I_1 cannot be covered by finitely many U_α's.

Continuing in this way, we can find an infinite sequence $I_1, I_2, I_3, \ldots$ of closed intervals in the real line with the following properties. First,

$$I_1 \subseteq I \quad \text{and} \quad I_{j+1} \subseteq I_j \text{ for every } j \ge 1. \tag{1.12.5}$$

Second,

$$\begin{aligned} &\text{length}(I_1) = \text{length}(I)/2 \quad \text{and} \\ &\text{length}(I_{j+1}) = \text{length}(I_j)/2 \text{ for every } j \ge 1. \end{aligned} \tag{1.12.6}$$

Third,

(1.12.7) for each $j \in \mathbf{Z}_+$, I_j cannot be covered by finitely many U_α's.

More precisely, suppose that $I_j = [a_j, b_j]$ has been chosen in this way for some positive integer j. Put

$$L_j = [a_j, (a_j + b_j)/2] \quad \text{and} \quad R_j = [(a_j + b_j)/2, b_j], \tag{1.12.8}$$

so that

$$I_j = L_j \cup R_j \tag{1.12.9}$$

and

$$\text{length}(L_j) = \text{length}(R_j) = \text{length}(I_j)/2. \tag{1.12.10}$$

Using (1.12.7) and (1.12.9), we get that at least one of L_j and R_j cannot be covered by finitely many U_α's. We take I_{j+1} to be either L_j or R_j, in such a way that I_{j+1} cannot be covered by finitely many U_α's.

Note that $\bigcap_{j=1}^\infty I_j \neq \emptyset$, by (1.12.5) and Proposition 1.11.1. Let x be an element of $\bigcap_{j=1}^\infty I_j$, which is also an element of I, because $I_1 \subseteq I$. This implies that there is an $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$, because I is covered by the U_α's. It follows that

$$(x - r, x + r) \subseteq U_{\alpha_0} \tag{1.12.11}$$

for some positive real number r, because U_{α_0} is an open set in $\mathbf{R}$. If $j \in \mathbf{Z}_+$ is sufficiently large, then one can check that

$$\operatorname{length}(I_j) = 2^{-j} \operatorname{length}(I) < r. \tag{1.12.12}$$

In this case, we have that

$$I_j \subseteq (x - r, x + r) \subseteq U_{\alpha_0}, \tag{1.12.13}$$

because $x \in I_j$. This contradicts (1.12.7), as desired.

1.13 Cells in $\mathbf{R}^n$

Let n be a positive integer. If $A_1, \ldots, A_n$ are n sets, then their *Cartesian product* may be denoted

$$A_1 \times \cdots \times A_n \tag{1.13.1}$$

or

$$\prod_{j=1}^n A_j. \tag{1.13.2}$$

This is the set of n-tuples $a = (a_1, \ldots, a_n)$ such that $a_j \in A_j$ for each $j = 1, \ldots, n$.

Now let $a_1, b_1, \ldots, a_n, b_n$ be real numbers with $a_j \leq b_j$ for each $j = 1, \ldots, n$, and put $I_j = [a_j, b_j]$ for every $j = 1, \ldots, n$. The Cartesian product

$$C = \prod_{j=1}^n I_j \tag{1.13.3}$$

of $I_1, \ldots, I_n$ is called a *cell* in $\mathbf{R}^n$. The *diameter* of C is defined by

$$\operatorname{diam} C = \Big(\sum_{j=1}^n (b_j - a_j)^2\Big)^{1/2}. \tag{1.13.4}$$

This is the same as the maximum of the distances between elements of C, with respect to the standard Euclidean metric on $\mathbf{R}^n$.

Proposition 1.13.5 *If $C_1, C_2, C_3, \ldots$ is an infinite sequence of cells in $\mathbf{R}^n$ such that*

$$C_{l+1} \subseteq C_l \tag{1.13.6}$$

for every positive integer l, then

$$\bigcap_{l=1}^\infty C_l \neq \emptyset. \tag{1.13.7}$$

This is the same as Proposition 1.11.1 when $n = 1$, and we can reduce to that case otherwise. If $l \in \mathbf{Z}_+$, then there are n closed intervals $I_{1,l}, \ldots, I_{n,l}$ in $\mathbf{R}$ such that

$$C_l = \prod_{j=1}^{n} I_{j,l}, \tag{1.13.8}$$

as in (1.13.3). Using (1.13.6), we get that

$$I_{j,l+1} \subseteq I_{j,l} \tag{1.13.9}$$

for every positive integer l and $j = 1, \ldots, n$. This implies that

$$\bigcap_{l=1}^{\infty} I_{j,l} \neq \emptyset \tag{1.13.10}$$

for every $j = 1, \ldots, n$, by Proposition 1.11.1. If $x = (x_1, \ldots, x_n)$ has the property that

$$x_j \in \bigcap_{l=1}^{\infty} I_{j,l} \tag{1.13.11}$$

for each $j = 1, \ldots, n$, then

$$x \in \bigcap_{l=1}^{\infty} C_l, \tag{1.13.12}$$

as desired.

Proposition 1.13.13 *If C is any cell in $\mathbf{R}^n$, then C can be expressed as the union of 2^n cells, each of which has one-half the diameter of C.*

As before, C can be expressed as in (1.13.3), where $I_j = [a_j, b_j]$ is a closed interval in $\mathbf{R}$ for each $j = 1, \ldots, n$. Let $L_j = [a_j, (a_j + b_j)/2]$ and $R_j = [(a_j + b_j)/2, b_j]$ be the left and right halves of I_j for each $j = 1, \ldots, n$, so that $I_j = L_j \cup R_j$ for every $j = 1, \ldots, n$. Consider the cells in $\mathbf{R}^n$ that can be obtained by taking the Cartesian product of n closed intervals, where for each $j = 1, \ldots, n$, the jth interval is either L_j or R_j. It is easy to see that there are 2^n of these cells. One can check that the union of these 2^n cells is equal to C. One can also verify that the diameter of each of these 2^n cells is equal to $\operatorname{diam} C/2$. This uses the fact that the lengths of L_j and R_j are equal to $\operatorname{length}(I_j)/2$ for each $j = 1, \ldots, n$.

1.14 Compactness of cells

Theorem 1.14.1 *If C is a cell in $\mathbf{R}^n$ for some positive integer n, then C is compact, with respect to the standard Euclidean metric on $\mathbf{R}^n$.*

Of course, this is the same as Theorem 1.12.1 when $n = 1$, and we can use essentially the same argument here. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of C in $\mathbf{R}^n$, and suppose for the sake of a contradiction that C cannot be covered by finitely many U_α's. Under these conditions, we can find an infinite sequence of cells $C_1, C_2, C_3, \ldots$ with the following properties. First,

(1.14.2) $$C_l \subseteq C_{l-1} \text{ for every } l \geq 1,$$

where we put $C_0 = C$, for convenience. Second,

(1.14.3) $$\operatorname{diam} C_l = \operatorname{diam} C_{l-1}/2 \text{ for every } l \geq 1.$$

Third,

(1.14.4) for each $l \in \mathbf{Z}_+$, C_l cannot be covered by finitely many U_α's.

To see this, suppose that C_l has already been chosen in this way for some nonnegative integer l. Using Proposition 1.13.13, we can express C_l as the union of 2^n cells, each of which has one-half the diameter of C_l. At least one of these 2^n cells cannot be covered by finitely many U_α's, because otherwise C_l could be covered by finitely many U_α's. We take C_{l+1} to be one of these 2^n cells, in such a way that C_{l+1} cannot be covered by finitely many U_α's.

It follows that $\bigcap_{l=1}^\infty C_l \neq \emptyset$, by Proposition 1.13.5. Let x be an element of $\bigcap_{l=1}^\infty C_l$, which is an element of C in particular. Thus there is an $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. Because U_{α_0} is an open set in $\mathbf{R}^n$, there is a positive real number r such that

(1.14.5) $$B(x, r) \subseteq U_{\alpha_0}.$$

Here $B(x, r)$ is the open ball in $\mathbf{R}^n$ centered at x with radius r with respect to the standard Euclidean metric. Let l be a positive integer that is large enough so that

(1.14.6) $$\operatorname{diam} C_l = 2^{-l} \operatorname{diam} C < r.$$

It is easy to see that

(1.14.7) $$C_l \subseteq B(x, r) \subseteq U_{\alpha_0},$$

because $x \in C_l$. This contradicts (1.14.4), as desired.

Corollary 1.14.8 *If $E \subseteq \mathbf{R}^n$ is closed and bounded, with respect to the standard Euclidean metric, then E is compact.*

Indeed, one can find a cell $C \subseteq \mathbf{R}^n$ such that $E \subseteq C$, because E is bounded in $\mathbf{R}^n$. To get that E is compact, one can use the compactness of C and the hypothesis that E be a closed set, as in Proposition 1.9.6.

1.15 Totally bounded sets

Let $(M, d(x, y))$ be a metric space.

Definition 1.15.1 *A subset E of M is said to be* totally bounded *with respect to $d(\cdot, \cdot)$ if for every positive real number r there are finitely many elements $x_1, \ldots, x_n$ of M such that*

$$E \subseteq \bigcup_{j=1}^{n} B(x_j, r). \tag{1.15.2}$$

If $E = \emptyset$, then this condition is interpreted as holding with $n = 0$, even if $M = \emptyset$.

Proposition 1.15.3 *If $K \subseteq M$ is compact, then K is totally bounded.*

Let $r > 0$ be given. The collection of open balls $B(x, r)$ centered at elements x of K and with radius r forms an open covering of K in M. If K is compact, then there are finitely many elements $x_1, \ldots, x_n$ of K such that $K \subseteq \bigcup_{j=1}^{n} B(x_j, r)$, as desired.

Proposition 1.15.4 *If $E \subseteq M$ has the limit point property, then E is totally bounded in M.*

Let $r > 0$ be given again. We would like to show that E is contained in the union of finitely many open balls in M of radius r. Of course, this is trivial when $E = \emptyset$, and so we may suppose that there is an element x_1 of E. If

$$E \subseteq B(x_1, r), \tag{1.15.5}$$

then we can stop. Otherwise, we let x_2 be an element of $E \setminus B(x_1, r)$.

Suppose that $x_1, \ldots, x_n \in E$ have been chosen in this way for some positive integer n. If $E \subseteq \bigcup_{j=1}^{n} B(x_j, r)$, then we can stop. Otherwise, we let x_{n+1} be an element of

$$E \setminus \Big(\bigcup_{j=1}^{n} B(x_j, r) \Big). \tag{1.15.6}$$

Suppose for the sake of a contradiction that this process does not stop after finitely many steps. In this case, we get an infinite sequence $x_1, x_2, x_3, \ldots$ of elements of E such that

$$d(x_j, x_l) \geq r \tag{1.15.7}$$

for all positive integers j, l with $j < l$. Let

$$L = \{x_j : j \in \mathbf{Z}_+\} \tag{1.15.8}$$

be the set of points that have been chosen in this way. Note that L has infinitely many elements, because $x_j \neq x_l$ when $j < l$. Of course, $L \subseteq E$, by construction.

If E has the limit point property, then there is a point $p \in E$ such that p is a limit point of L in M. This implies that $B(p, r/2)$ contains infinitely many elements of L, as in Proposition 1.5.2. In particular, there are positive integers j, l such that $j < l$ and $x_j, x_l \in B(p, r/2)$. It follows that

$$d(x_j, x_l) \leq d(x_j, p) + d(p, x_l) < r/2 + r/2 = r, \tag{1.15.9}$$

using the triangle inequality in the first step. This contradicts (1.15.7), as desired.

Chapter 2

Metric spaces, continued

2.1 Countable sets

To say that a nonempty set A has finitely many elements means that there is a finite sequence $x_1, \ldots, x_n$ of elements of A in which every element of A occurs exactly once. In this case, the number of elements of A is the positive integer n. The empty set corresponds to taking $n = 0$.

Definition 2.1.1 *A set A is said to be* countably infinite *if there is a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of A in which every element of A occurs exactly once. More precisely, this means that $x_j \in A$ for every $j \in \mathbf{Z}_+$, and for every $a \in A$, there is a unique positive integer j such that $x_j = a$.*

Of course, the set $\mathbf{Z}_+$ of positive integers is countably infinite, since we can take $x_j = j$ for every $j \geq 1$. The set $\mathbf{Z}$ of all integers is countably infinite as well. To see this, one can take

$$\begin{aligned} x_j &= j/2 && \text{when } j \text{ is even} \\ &= -(j-1)/2 && \text{when } j \text{ is odd.} \end{aligned} \tag{2.1.2}$$

Proposition 2.1.3 *Let A and B be sets, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of B in which every element of A occurs at least once. More precisely, this means that $x_j \in B$ for every $j \geq 1$, and for every $a \in A$ there is a positive integer j such that $x_j = a$. Under these conditions, A has only finitely or countably many elements.*

Of course, if $A = \emptyset$, then A has only finitely many elements. Otherwise, let j_1 be the smallest positive integer such that $x_{j_1} \in A$. If this is the only element of A, then we can stop. Otherwise, let j_2 be the smallest positive integer such that $x_{j_2} \in A$ and $x_{j_2} \neq x_{j_1}$. Note that $j_1 < j_2$, by construction.

Suppose that positive integers $j_1 < \cdots < j_n$ have been chosen in this way for some $n \in \mathbf{Z}_+$. If $x_{j_1}, \ldots, x_{j_n}$ are all of the elements of A, then we can stop. Otherwise, let j_{n+1} be the smallest positive integer such that $x_{j_{n+1}} \in A$ and

$x_{j_{n+1}} \neq x_{j_l}$ for each $l = 1, \ldots, n$. By construction, $j_n < j_{n+1}$, since otherwise j_{n+1} would have been chosen earlier in the process.

If this process stops after finitely many steps, then A has only finitely many elements. Otherwise, we get an infinite sequence $\{x_{j_l}\}_{l=1}^{\infty}$ of elements of A in which every element of A occurs exactly once, so that A is countably infinite.

Corollary 2.1.4 *If A is a subset of a countably infinite set B, then A has only finitely or countably many elements.*

Lemma 2.1.5 *If $A_1, A_2, A_3, \ldots$ is an infinite sequence of finite sets, then their union $\bigcup_{j=1}^{\infty} A_j$ has only finitely or countably many elements.*

In this situation, it is not difficult to find a sequence in which every element of $\bigcup_{j=1}^{\infty} A_j$ occurs at least once, by listing the elements of A_j for each $j \in \mathbf{Z}_+$, one after the other. To get that $\bigcup_{j=1}^{\infty} A_j$ has only finitely of countably many elements, one can use Proposition 2.1.3.

Let us use the lemma to show that the set $\mathbf{Z}_+^2 = \mathbf{Z}_+ \times \mathbf{Z}_+$ of all ordered pairs of positive integers is countably infinite. Put

$$A_n = \{(j, l) \in \mathbf{Z}_+^2 : j + l = n + 1\} \tag{2.1.6}$$

for each positive integer n. It is easy to see that A_n has exactly n elements for every n, and that

$$\bigcup_{n=1}^{\infty} A_n = \mathbf{Z}_+^2. \tag{2.1.7}$$

It follows that $\mathbf{Z}_+^2$ is countably infinite, by Lemma 2.1.5, and because $\mathbf{Z}_+^2$ obviously has infinitely many elements. One can also use the fact that the A_n's are pairwise disjoint, to enumerate the elements of $\mathbf{Z}_+^2$ more directly.

Proposition 2.1.8 *Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of sets, each of which has only finitely or countably many elements. Under these conditions, $\bigcup_{j=1}^{\infty} A_j$ has only finitely or countably many elements as well.*

By hypothesis, for each positive integer j, there is an infinite sequence $\{x_{j,l}\}_{l=1}^{\infty}$ in which every element of A_j occurs at least once. Let $\{(j_n, l_n)\}_{n=1}^{\infty}$ be a sequence of elements of $\mathbf{Z}_+^2$ in which every element of $\mathbf{Z}_+^2$ occurs once. It follows that every element of $\bigcup_{j=1}^{\infty} A_j$ occurs at least once in the sequence $\{x_{j_n, l_n}\}_{n=1}^{\infty}$. This implies that $\bigcup_{j=1}^{\infty} A_j$ has only finitely or countably many elements, as in Proposition 2.1.3.

Let us now use the proposition to show that the set $\mathbf{Q}$ of rational numbers is countably infinite. If j is a positive integer, then let A_j be the set of integer multiples of $1/j$. It is easy to see that A_j is countably infinite for each $j \geq 1$, because the set $\mathbf{Z}$ of all integers is countably infinite. It follows that $\mathbf{Q} = \bigcup_{j=1}^{\infty} A_j$ is countably infinite, as desired.

Proposition 2.1.9 *Let n be a positive integer, and let $A_1, \ldots, A_n$ be n countably infinite sets. Under these conditions, $\prod_{j=1}^{n} A_j$ is countably infinite too.*

Of course, this is trivial when $n = 1$, and the $n = 2$ case corresponds to the countability of $\mathbf{Z}_+^2$. If $n \geq 2$, then there is a simple one-to-one correspondence between $\prod_{j=1}^n A_j$ and $\left(\prod_{j=1}^{n-1} A_j\right) \times A_n$. More precisely, if $a = (a_1, \ldots, a_n)$ is an element of $\prod_{j=1}^n A_j$, then

$$((a_1, \ldots, a_{n-1}), a_n) \tag{2.1.10}$$

defines an element of $\left(\prod_{j=1}^{n-1} A_j\right) \times A_n$, and every element of $\left(\prod_{j=1}^{n-1} A_j\right) \times A_n$ corresponds to a unique element of $\prod_{j=1}^n A_j$ in this way. Thus the countability of $\prod_{j=1}^n A_j$ can be obtained from the countability of $\prod_{j=1}^{n-1} A_j$ and the $n = 2$ case. This permits one to prove the proposition by induction.

2.2 Uncountable sets

Definition 2.2.1 *A set is said to be* uncountable *if it is neither finite nor countably infinite.*

Let $\mathcal{B}$ be the set of all infinite sequences $x = \{x_j\}_{j=1}^\infty$ such that for each $j \in \mathbf{Z}_+$, $x_j = 0$ or 1. It is well known that $\mathcal{B}$ is uncountable. To see this, let $x(1), x(2), x(3), \ldots$ be any infinite sequence of elements of $\mathcal{B}$. Thus, for each $l \in \mathbf{Z}_+$, $x(l) = \{x_j(l)\}_{j=1}^\infty$ is an infinite sequence with terms in $\{0, 1\}$. Put

$$y_j = 1 - x_j(j) \in \{0, 1\} \tag{2.2.2}$$

for each $j \in \mathbf{Z}_+$, so that $y = \{y_j\}_{j=1}^\infty$ defines an element of $\mathcal{B}$. If l is any positive integer, then $y \neq x(l)$, because $y_l \neq x_l(l)$, by construction. This shows that the elements of $\mathcal{B}$ cannot all be listed by a sequence, so that $\mathcal{B}$ is uncountable. This is Cantor's *diagonalization argument*.

There is a standard way to associate to each $x \in \mathcal{B}$ an element of the closed unit interval $[0, 1]$. It is well known that every element of $[0, 1]$ corresponds to an element of $\mathcal{B}$ in this way, but some elements of $[0, 1]$ may correspond to more than one element of $\mathcal{B}$. More precisely, it is well known and not too difficult to show that at most two elements of $\mathcal{B}$ can correspond to the same element of $[0, 1]$. In fact, this only happens for countably many elements of $\mathcal{B}$, and hence countably many elements of $[0, 1]$. One can use this to show that $[0, 1]$ is uncountable, because $\mathcal{B}$ is uncountable.

Definition 2.2.3 *Let $(M, d(x, y))$ be a metric space. A subset E of M is said to be* perfect *in M if E is a closed set, and if every element of E is a limit point of E.*

Closed intervals in the real line of positive length are perfect, for instance, with respect to the standard Euclidean metric on $\mathbf{R}$. If E is a nonempty perfect subset of $\mathbf{R}^n$ for some positive integer n, with respect to the standard Euclidean metric, then it can be shown that E is uncountable. This can be extended to a broad class of metric spaces using the Baire category theorem. Another type of example of a perfect set in $\mathbf{R}$ will be discussed in Section 2.9.

2.3 Separable metric spaces

Let $(M, d(x, y))$ be a metric space.

Definition 2.3.1 *A subset E of M is said to be* dense *in M if every element of M is either an element of E, a limit point of E, or both.*

Equivalently, $E \subseteq M$ is dense in M if the closure $\overline{E}$ of E in M is equal to M.

Definition 2.3.2 *If there is a dense set $E \subseteq M$ such that E has only finitely or countably many elements, then M is said to be* separable *as a metric space.*

It is easy to see that the set $\mathbf{Q}$ of rational numbers is dense in the real line, with respect to the standard Euclidean metric. Thus $\mathbf{R}$ is separable, with respect to the standard metric, because $\mathbf{Q}$ is countably infinite, as in Section 2.1.

Similarly, if n is a positive integer, then let $\mathbf{Q}^n$ be the set of $x \in \mathbf{R}^n$ such that $x_j \in \mathbf{Q}$ for each $j = 1, \ldots, n$. One can check that $\mathbf{Q}^n$ is dense in $\mathbf{R}^n$, with respect to the standard Euclidean metric. It follows from Proposition 2.1.9 that $\mathbf{Q}^n$ is countably infinite, so that $\mathbf{R}^n$ is separable with respect to the standard metric.

Proposition 2.3.3 *Separability of M is equivalent to the following condition: for every $r > 0$ there is a subset $E(r)$ of M such that $E(r)$ has only finitely or countably many elements and*

$$\bigcup_{x \in E(r)} B(x, r) = M. \tag{2.3.4}$$

If M is separable, then there is a dense set $E \subseteq M$ with only finitely or countably many elements. In this case, one can check that (2.3.4) holds with $E(r) = E$ for every $r > 0$.

Conversely, suppose that M satisfies the condition described in the statement of the proposition. This implies that for each positive integer j there is a subset $E(1/j)$ of M with only finitely or countably many elements such that (2.3.4) holds with $r = 1/j$. Put

$$E = \bigcup_{j=1}^{\infty} E(1/j), \tag{2.3.5}$$

which has only finitely or countably many elements, by Proposition 2.1.8. One can check that E is dense in M, using (2.3.4) for $r = 1/j$, $j \in \mathbf{Z}_+$. This means that M is separable, as desired.

Corollary 2.3.6 *If M is totally bounded, then M is separable.*

Indeed, if M is totally bounded, then for each $r > 0$ there is a finite subset $E(r)$ of M that satisfies (2.3.4).

2.4 Bases

The topics in this section are a bit abstract, and one may wish to skip this at first. We shall say more about this in the next section.

Let $(M, d(x, y))$ be a metric space.

Definition 2.4.1 *A collection $\mathcal{B}$ of open subsets of M is said to be a* base for the topology of M *if for every $x \in M$ and $r > 0$ there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq B(x, r)$.*

If $\mathcal{B}$ is a base for the topology of M and $W \subseteq M$ is an open set, then

$$W = \bigcup\{V : V \in \mathcal{B},\ V \subseteq W\}. \tag{2.4.2}$$

More precisely, the right side of (2.4.2) is automatically contained in W, by construction. In order to verify the opposite inclusion, let an arbitrary element x of W be given. Thus there is an $r > 0$ such that $B(x, r) \subseteq W$, because W is an open set in M. This implies that there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq B(x, r) \subseteq W$, as in Definition 2.4.1. It follows that x is an element of the right side of (2.4.2). This means that W is contained in the right side of (2.4.2), as desired.

Conversely, let $\mathcal{B}$ be a collection of open subsets of M, and suppose that every open subset of M can be expressed as the union of some elements of $\mathcal{B}$. Let $x \in M$ and $r > 0$ be given, and remember that $B(x, r)$ is an open set in M. By hypothesis, $B(x, r)$ can be expressed as a union of elements of $\mathcal{B}$. In particular, there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq B(x, r)$, because $x \in B(x, r)$. This shows that $\mathcal{B}$ is a base for the topology of M.

Of course, the collection of all open subsets of M is a base for the topology of M. The collection of all open balls in M is a base for the topology of M as well, because of the way that open sets in M are defined. The collection of open balls in M with radius of the form $1/j$ for some positive integer j is a base for the topology of M too, because of the archimedean property of the real numbers.

Proposition 2.4.3 *Let E be a subset of M, and let $\mathcal{B}_E$ be the collection of open balls in M of the form $B(x, 1/j)$, with $x \in E$ and $j \in \mathbf{Z}_+$. If E is dense in M, then $\mathcal{B}_E$ is a base for the topology of M.*

It is easy to see that E has to be dense in M in order for $\mathcal{B}_E$ to be a base for the topology of M.

Suppose that E is dense in M, and let us check that $\mathcal{B}_E$ is a base for the topology of M. Let $x \in M$ and $r > 0$ be given, as in Definition 2.4.1. The archimedean property for the real numbers implies that there is a positive integer j such that $2/j \le r$. Because E is dense in M, there is a $y \in E$ such that $d(x, y) < 1/j$. Thus $B(y, 1/j) \in \mathcal{B}_E$ and $x \in B(y, 1/j)$. One can also verify that

$$B(y, 1/j) \subseteq B(x, r), \tag{2.4.4}$$

using the triangle inequality. More precisely, if $z \in B(y, 1/j)$, then

$$d(x, z) \le d(x, y) + d(y, z) < 1/j + 1/j = 2/j \le r, \tag{2.4.5}$$

so that $z \in B(x, r)$, as desired.

Corollary 2.4.6 *If M is separable, then there is a base for the topology of M with only finitely or countably many elements.*

Suppose that M is separable, and let E be a dense subset of M with only finitely or countably many elements. It suffices to check that the collection $\mathcal{B}_E$ defined in Proposition 2.4.3 has only finitely or countably many elements. If j is a positive integer, then let $\mathcal{B}_{E,j}$ be the collection of open balls of the form $B(x, 1/j)$, with $x \in E$. Observe that

$$\mathcal{B}_E = \bigcup_{j=1}^{\infty} \mathcal{B}_{E,j}, \tag{2.4.7}$$

by the definition of $\mathcal{B}_E$. It is easy to see that $\mathcal{B}_{E,j}$ has only finitely or countably many elements for each $j \in \mathbf{Z}_+$, because E has only finitely or countably many elements, by hypothesis. More precisely, one can list the elements of E with a finite or infinite sequence, and use that to list the elements of $\mathcal{B}_{E,j}$ with a finite or infinite sequence. This implies that (2.4.7) has only finitely or countably many elements, by Proposition 2.1.8.

Proposition 2.4.8 *If there is a base $\mathcal{B}$ for the topology of M such that $\mathcal{B}$ has only finitely or countably many elements, then M is separable.*

Let $\mathcal{B}$ be a base for the topology of M. If $V \in \mathcal{B}$ and $V \ne \emptyset$, then let us pick a point x_V in V. Let

$$E = \{x_V : V \in \mathcal{B},\ V \ne \emptyset\} \tag{2.4.9}$$

be the set of points in M that have been chosen in this way. One can check that E is dense in M, because $\mathcal{B}$ is a base for the topology of M. If $\mathcal{B}$ has only finitely or countably many elements, then E has only finitely or countably many elements. More precisely, if one can list the elements of $\mathcal{B}$ with a finite or infinite sequence, then one can list the elements of E with a finite or infinite sequence too. In this case, we get that M is separable, as desired.

2.5 Lindelöf's theorem

The topics in this section are also a bit abstract, and one may wish to skip this at first. More precisely, the conclusion of the next theorem is quite interesting, and one may wish to skip the proof at first.

Let $(M, d(x, y))$ be a metric space.

Theorem 2.5.1 (Lindelöf's theorem) *Suppose that there is a base $\mathcal{B}$ for the topology of M with only finitely or countably many elements. If $\{U_\alpha\}_{\alpha \in A}$ is any family of open subsets of M, then there is a subset A_1 of A such that A_1 has only finitely or countably many elements and*

$$\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha. \tag{2.5.2}$$

Note that the hypothesis of Lindelöf's theorem is the same as saying that M is separable as a metric space, as in the previous section. Lindelöf's theorem will be used in the proof of Theorem 2.7.1.

To prove Lindelöf's theorem, let $\alpha \in A$ be given, and put

$$\mathcal{B}_\alpha = \{V \in \mathcal{B} : V \subseteq U_\alpha\}. \tag{2.5.3}$$

Note that

$$U_\alpha = \bigcup_{V \in \mathcal{B}_\alpha} V, \tag{2.5.4}$$

as in (2.4.2), because $\mathcal{B}$ is a base for the topology of M, and U_α is an open set in M. Put

$$\widetilde{\mathcal{B}} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha. \tag{2.5.5}$$

It is easy to see that

$$\bigcup_{V \in \widetilde{\mathcal{B}}} V = \bigcup_{\alpha \in A} \bigcup_{V \in \mathcal{B}_\alpha} V = \bigcup_{\alpha \in A} U_\alpha, \tag{2.5.6}$$

using the definition (2.5.5) of $\widetilde{\mathcal{B}}$ in the first step, and (2.5.4) in the second step.

If $V \in \widetilde{\mathcal{B}}$, then let us pick $\alpha(V) \in A$ such that $V \in \mathcal{B}_{\alpha(V)}$, which means that

$$V \subseteq U_{\alpha(V)}. \tag{2.5.7}$$

Let

$$A_1 = \{\alpha(V) : V \in \widetilde{\mathcal{B}}\} \tag{2.5.8}$$

be the collection of elements of A that have been chosen in this way. Observe that $\widetilde{\mathcal{B}}$ has only finitely or countably many elements, because $\mathcal{B}$ has only finitely of countably many elements, by hypothesis, and $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$, by construction. This implies that A_1 has only finitely or countably many elements as well. More precisely, the elements of $\widetilde{\mathcal{B}}$ can be listed by a finite or infinite sequence, which can be used to list the elements of A_1 by a finite or infinite sequence.

We also have that

$$\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{V \in \widetilde{\mathcal{B}}} U_{\alpha(V)} \supseteq \bigcup_{V \in \widetilde{\mathcal{B}}} V = \bigcup_{\alpha \in A} U_\alpha, \tag{2.5.9}$$

using the definition (2.5.8) of A_1 in the first step, (2.5.7) in the second step, and (2.5.6) in the third step. Of course,

$$\bigcup_{\alpha \in A_1} U_\alpha \subseteq \bigcup_{\alpha \in A} U_\alpha \tag{2.5.10}$$

automatically, because $A_1 \subseteq A$. Combining (2.5.9) and (2.5.10), we get (2.5.2), as desired.

2.6 Countable open coverings

Let $(M, d(x, y))$ be a metric space.

Proposition 2.6.1 *Suppose that $E \subseteq M$ has the limit point property. If $U_1, U_2, U_3, \ldots$ is a sequence of open subsets of M such that*

$$E \subseteq \bigcup_{j=1}^{\infty} U_j, \tag{2.6.2}$$

then there is a positive integer n such that

$$E \subseteq \bigcup_{j=1}^{n} U_j. \tag{2.6.3}$$

Suppose for the sake of a contradiction that

$$E \not\subseteq \bigcup_{j=1}^{n} U_j \tag{2.6.4}$$

for every $n \in \mathbf{Z}_+$. Let us choose, for each $n \in \mathbf{Z}_+$, a point

$$x_n \in E \setminus \Big(\bigcup_{j=1}^{n} U_j\Big). \tag{2.6.5}$$

Let

$$L = \{x_n : n \in \mathbf{Z}_+\} \tag{2.6.6}$$

be the set of points in E that have been chosen in this way.

Let us check that L has infinitely many elements. Otherwise, if L has only finitely many elements, then there is an $x \in E$ such that $x = x_n$ for infinitely many $n \in \mathbf{Z}_+$. Because $x \in E$, there is a positive integer j_0 such that $x \in U_{j_0}$, by (2.6.2). It follows that $x_n \neq x$ when $n \geq j_0$, by (2.6.5). This means that $x_n = x$ for only finitely many n, which is a contradiction, as desired.

If E has the limit point property, then there is a point $p \in E$ that is a limit point of L in M. Using (2.6.2), we get that there is a positive integer j_1 such that $p \in U_{j_1}$. Because U_{j_1} is an open set in M, by hypothesis, there is a positive real number r such that

$$B(p, r) \subseteq U_{j_1}. \tag{2.6.7}$$

We also have that $B(p, r)$ contains infinitely many elements of L, because p is a limit point of L, as in Proposition 1.5.2. This implies that $x_n \in B(p, r)$ for infinitely many $n \in \mathbf{Z}_+$.

It follows that

$$x_n \in U_{j_1} \tag{2.6.8}$$

for infinitely many $n \in \mathbf{Z}_+$, by (2.6.7). However, (2.6.8) can only hold when $n < j_1$, by (2.6.5). This is a contradiction, as desired.

2.7 Getting compactness

The proof of the next theorem uses Lindelöf's theorem, as well as other results discussed previously. One should not consider the next theorem as being completely established here unless one has gone through the proof of Lindelöf's theorem, as well as that of Corollary 2.4.6.

Let $(M, d(x, y))$ be a metric space.

Theorem 2.7.1 *If $E \subseteq M$ has the limit point property, then E is compact.*

Remember that the converse was given in Proposition 1.7.10.

Suppose first that M has the limit point property, and let us show that M is compact. Remember that M is totally bounded in this situation, as in Proposition 1.15.4. This implies that M is separable, as in Corollary 2.3.6. It follows that there is a base $\mathcal{B}$ for the topology of M with only finitely or countably many elements, as in Corollary 2.4.6.

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of M, as a subset of itself. Thus $\bigcup_{\alpha \in A} U_\alpha = M$. Lindelöf's theorem implies that there is a subset A_1 of A such that A_1 has only finitely or countably many elements, and

$$\bigcup_{\alpha \in A_1} U_\alpha = M. \tag{2.7.2}$$

If A_1 has only finitely many elements, then we can stop. Otherwise, suppose that A_1 is countably infinite. In this case, we can use Proposition 2.6.1 to find a finite subset A_2 of A_1 such that

$$M \subseteq \bigcup_{\alpha \in A_2} U_\alpha. \tag{2.7.3}$$

In both cases, we get a finite subcovering of M from $\{U_\alpha\}_{\alpha \in A}$, as desired.

Suppose now that E is a subset of M with the limit point property. Remember that E may be considered as a metric space as well, with respect to the restriction of $d(x, y)$ to $x, y \in E$. It is easy to see that E has the limit point property as a subset of itself, because of the corresponding property of E in M. Using this, the previous argument implies that E is compact as a subset of itself. It follows that E is compact as a subset of M, as in Proposition 1.10.12.

2.8 Connectedness

Let $(M, d(x, y))$ be a metric space.

Definition 2.8.1 *A pair A, B of subsets of M are said to be* separated *in M if*

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset. \tag{2.8.2}$$

If we take $A = (0, 1)$ and $B = (1, 2)$ in the real line, then their closures with respect to the standard Euclidean metric on $\mathbf{R}$ are given by $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$. It follows that A and B are separated in $\mathbf{R}$, although their closures are not disjoint in $\mathbf{R}$.

Definition 2.8.3 *A subset E of M is said to be* connected *in M if E cannot be expressed as the union of two nonempty separated subsets of M.*

Theorem 2.8.4 *A subset E of the real line is connected with respect to the standard metric on $\mathbf{R}$ if and only if it has the following property:*

$$\text{if } x, y \in E,\ w \in \mathbf{R}, \text{ and } x < w < y, \text{ then } w \in E. \tag{2.8.5}$$

Suppose first that (2.8.5) does not hold, and let us show that E is not connected in $\mathbf{R}$. By hypothesis, there are $x, y \in E$ and $w \in \mathbf{R}$ such that $x < w < y$ and $w \notin E$. Put

$$A = \{u \in E : u < w\} \quad \text{and} \quad B = \{u \in E : w < u\}. \tag{2.8.6}$$

Thus $x \in A$ and $y \in B$, so that $A, B \neq \emptyset$, and $E = A \cup B$, because $w \notin E$. It is easy to see that

$$\overline{A} \subseteq \{u \in \mathbf{R} : u \leq w\} \quad \text{and} \quad \overline{B} \subseteq \{u \in \mathbf{R} : w \leq u\}. \tag{2.8.7}$$

This implies that (2.8.2) holds, so that A and B are separated in $\mathbf{R}$. It follows that E is not connected in $\mathbf{R}$, as desired.

Before proceeding to the other half of the theorem, let us mention the following useful fact.

Proposition 2.8.8 *If A is a nonempty subset of $\mathbf{R}$ with an upper bound in $\mathbf{R}$, then $\sup A \in \overline{A}$. Similarly, if B is a nonempty subset of $\mathbf{R}$ with a lower bound in $\mathbf{R}$, then $\inf B \in \overline{B}$.*

This can be verified directly from the definitions, and the details are left as an exercise.

Let us now prove the "if" part of Theorem 2.8.4. Thus we suppose that (2.8.5) holds, and we would like to show that E is connected in $\mathbf{R}$. Suppose for the sake of a contradiction that E is not connected, so that there are nonempty separated sets $A, B \subseteq \mathbf{R}$ such that $E = A \cup B$. Let x be an element of A, and let y be an element of B. We may as well suppose that $x < y$, since otherwise we could interchange the roles of A and B.

Note that $A \cap [x, y]$ is a nonempty subset of $\mathbf{R}$ with an upper bound in $\mathbf{R}$, and put

$$z = \sup(A \cap [x, y]). \tag{2.8.9}$$

Clearly $x \le z \le y$, because $x \in A \cap [x, y]$, and y is an upper bound for $A \cap [x, y]$. Using Proposition 2.8.8, we get that z is in the closure of $A \cap [x, y]$ in $\mathbf{R}$. It is easy to see that this implies that $z \in \overline{A}$. It follows that $z \notin B$, because $\overline{A} \cap B = \emptyset$, by hypothesis.

Using (2.8.5), we get that $z \in E$. This implies that $z \in A$, because $z \notin B$. Of course, $z < y$, because $z \le y$ and $z \ne y$, since $y \in B$ and $z \notin B$. We can use (2.8.5) again to get that $(z, y) \subseteq E$, because $x \le z$. It follows that

$$(z, y) \subseteq B, \tag{2.8.10}$$

because $E = A \cup B$, and (z, y) is disjoint from A, by the definition (2.8.9) of z. This means that $z \in \overline{B}$. This contradicts the hypothesis that $A \cap \overline{B} = \emptyset$, as desired.

2.9 The Cantor set

Put $E_0 = [0, 1]$ and $E_1 = [0, 1/3] \cup [2/3, 1]$. Equivalently, E_1 is obtained from E_0 be removing the open middle third $(1/3, 2/3)$. Similarly, we can remove the open middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ from $[0, 1/3]$ and $[2/3, 1]$, respectively, to get

$$E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]. \tag{2.9.1}$$

Continuing in this way, we can define a subset E_j of $\mathbf{R}$ for every nonnegative integer j with the following properties. First, for each $j \ge 0$, E_j is the union of 2^j pairwise-disjoint closed intervals of length 3^{-j}. Second, E_{j+1} is obtained from E_j by removing the open middle thirds of each of the 2^j closed intervals in E_j just mentioned. In particular,

$$E_{j+1} \subseteq E_j \tag{2.9.2}$$

for every $j \ge 0$.

The middle-thirds *Cantor set* is defined to be

$$E = \bigcap_{j=0}^{\infty} E_j. \tag{2.9.3}$$

Note that E_j is a closed set in $\mathbf{R}$ with respect to the standard Euclidean metric for every $j \ge 0$, because it is the union of finitely many closed sets. It follows that E is a closed set in $\mathbf{R}$ too.

If j is a nonnegative integer, then let A_j be the set of 2^{j+1} elements of $[0, 1]$ that occur as endpoints of the 2^j closed intervals that make up E_j. Thus $A_0 = \{0, 1\}$, $A_1 = \{0, 1/3, 2/3, 1\}$, $A_2 = \{0, 1/9, 2/9, 1/3, 2/3, 7/9, 8/9, 1\}$, etc. By construction,

$$A_j \subseteq E_j \tag{2.9.4}$$

for every $j \ge 0$. We also have that

$$A_j \subseteq A_{j+1} \tag{2.9.5}$$

for every $j \ge 0$. More precisely, A_{j+1} consists of the elements of A_j, together with the endpoints of the open middle thirds of the closed intervals that make up E_j.

Let us check that

$$A_j \subseteq E_l \tag{2.9.6}$$

for all nonnegative integers j, l. If $j \le l$, then

$$A_j \subseteq A_l \subseteq E_l, \tag{2.9.7}$$

using (2.9.5) in the first step, and (2.9.4) in the second step. Similarly, if $j \ge l$, then

$$A_j \subseteq E_j \subseteq E_l, \tag{2.9.8}$$

using (2.9.4) in the first step, and (2.9.2) in the second step. It follows that

$$A_j \subseteq E \tag{2.9.9}$$

for every $j \ge 0$.

Observe that

$$A = \bigcup_{j=0}^{\infty} A_j \tag{2.9.10}$$

is countably infinite. Clearly

$$A \subseteq E, \tag{2.9.11}$$

by (2.9.9). Let us check that every element of E is a limit point of A, with respect to the standard metric on $\mathbf{R}$.

If $x \in E$ and j is a nonnegative integer, then $x \in E_j$, and hence x is contained in one of the 2^j closed intervals I that make up E_j. The endpoints of I are elements of $A_j \subseteq A$, at least one of which is different from x. The distance from x to the endpoints of I is less than or equal to the length of I, which is 3^{-j}. Using this, it is easy to see that x is a limit point of A, as desired.

In particular, every element of E is a limit point of E, because of (2.9.11). This shows that E is perfect with respect to the standard metric on $\mathbf{R}$, because E is a closed set.

Chapter 3

Sequences and series

3.1 Complex numbers

Every complex number z can be expressed in a unique way as $z = x + y\,i$, where $x, y \in \mathbf{R}$ and $i^2 = -1$. In this case, x and y are known as the *real* and *imaginary parts* of z, respectively. The *complex conjugate* of z is the complex number

$$\overline{z} = x - y\,i. \tag{3.1.1}$$

The *absolute value* or *modulus* of z is the nonnegative real number

$$|z| = (x^2 + y^2)^{1/2}. \tag{3.1.2}$$

Note that the complex conjugate of $\overline{z}$ is z, and that $|\overline{z}| = |z|$.

Addition and multiplication of real numbers can be extended to the set $\mathbf{C}$ of complex numbers in a standard way. If $z, w \in \mathbf{C}$, then one can check that

$$\overline{z + w} = \overline{z} + \overline{w} \tag{3.1.3}$$

and

$$\overline{z\,w} = \overline{z}\,\overline{w}. \tag{3.1.4}$$

We also have that

$$z\,\overline{z} = |z|^2 \tag{3.1.5}$$

for every $z \in \mathbf{C}$. It follows that

$$|z\,w|^2 = z\,w\,\overline{z\,w} = z\,\overline{z}\,w\,\overline{w} = |z|^2\,|w|^2 \tag{3.1.6}$$

for every $z, w \in \mathbf{C}$, so that

$$|z\,w| = |z|\,|w|. \tag{3.1.7}$$

If $z \in \mathbf{C}$ and $z \neq 0$, then $|z| > 0$, and

$$z\,(\overline{z}/|z|^2) = 1, \tag{3.1.8}$$

which means that $\overline{z}/|z|^2$ is the multiplicative inverse of z in $\mathbf{C}$.

It is well known that

$$|z + w| \le |z| + |w| \tag{3.1.9}$$

for every $z, w \in \mathbf{C}$, which can be verified directly. If we identify $(x, y) \in \mathbf{R}^2$ with $z = x + y\,i \in \mathbf{C}$, then the absolute value of a complex number corresponds exactly to the standard Euclidean norm on $\mathbf{R}^2$. Thus (3.1.9) is the same as the triangle inequality for the standard Euclidean norm on $\mathbf{R}^2$. It is easy to see that

$$d(z, w) = |z - w| \tag{3.1.10}$$

defines a metric on $\mathbf{C}$, which is the standard metric on $\mathbf{C}$. This corresponds exactly to the standard Euclidean metric on $\mathbf{R}^2$.

3.2 Convergent sequences

Let $(M, d(x, y))$ be a metric space.

Definition 3.2.1 *A sequence $\{x_j\}_{j=1}^\infty$ of elements of M is said to* converge *to an element x of M if for every positive real number ϵ there is a positive integer L such that*

$$d(x_j, x) < \epsilon \tag{3.2.2}$$

for every $j \ge L$.

Proposition 3.2.3 *If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of M that converges to $x \in M$ and to $x' \in M$, then $x = x'$.*

Suppose for the sake of a contradiction that $x \ne x'$, so that $d(x, x') > 0$. Because $\{x_j\}_{j=1}^\infty$ converges to x in M, there is a positive integer L such that

$$d(x_j, x) < d(x, x')/2 \tag{3.2.4}$$

for every $j \ge L$. Similarly, there is an $L' \in \mathbf{Z}_+$ such that

$$d(x_j, x') < d(x, x')/2 \tag{3.2.5}$$

for every $j \ge L'$, because $\{x_j\}_{j=1}^\infty$ converges to x' in M. If $j \ge \max(L, L')$, then it follows that

$$d(x, x') \le d(x, x_j) + d(x_j, x') < d(x, x')/2 + d(x, x')/2 = d(x, x'), \tag{3.2.6}$$

using the triangle inequality in the first step. In particular, this holds when $j = \max(L, L')$, which is a contradiction, as desired.

If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of M that converges to an element x of M, then x is called the *limit* of $\{x_j\}_{j=1}^\infty$, and we put

$$\lim_{j \to \infty} x_j = x. \tag{3.2.7}$$

We may also say that x_j tends to x as $j \to \infty$ in this case, or $x_j \to x$ as $j \to \infty$.

Definition 3.2.8 *A sequence* $\{x_j\}_{j=1}^\infty$ *of elements of* M *is said to be* bounded *in* M *if the set*

$$\{x_j : j \in \mathbf{Z}_+\} \tag{3.2.9}$$

of its terms is bounded in M.

Proposition 3.2.10 *If* $\{x_j\}_{j=1}^\infty$ *is a sequence of elements of* M *that converges to an element* x *of* M, *then* $\{x_j\}_{j=1}^\infty$ *is bounded in* M.

Because $\{x_j\}_{j=1}^\infty$ converges to x in M, there is a positive integer L such that

$$d(x_j, x) < 1 \tag{3.2.11}$$

for every $j \geq L$. If $L = 1$, then it follows that the set of the terms in this sequence is contained in $B(x, 1)$, as desired. Otherwise, put

$$r = \max\{d(x_j, x) : j = 1, \ldots, L - 1\}, \tag{3.2.12}$$

which is a nonnegative real number. It is easy to see that

$$d(x_j, x) < r + 1 \tag{3.2.13}$$

for every $j \geq 1$, so that the set of terms in the sequence is contained in $B(x, r+1)$, as desired.

Proposition 3.2.14 *(a) Let* $\{x_j\}_{j=1}^\infty$ *be a sequence of real numbers that increases monotonically, in the sense that*

$$x_j \leq x_{j+1} \tag{3.2.15}$$

for every $j \geq 1$. *If the set of* x_j*'s,* $j \in \mathbf{Z}_+$, *has an upper bound in* $\mathbf{R}$, *then* $\{x_j\}_{j=1}^\infty$ *converges to a real number* x, *with respect to the standard Euclidean metric on* $\mathbf{R}$.

(b) Let $\{y_j\}_{j=1}^\infty$ *be a monotonically decreasing sequence of real numbers, so that*

$$y_{j+1} \leq y_j \tag{3.2.16}$$

for every $j \geq 1$. *If the set of* y_j*'s has a lower bound in* $\mathbf{R}$, *then* $\{y_j\}_{j=1}^\infty$ *converges to a real number* y, *with respect to the standard metric on* $\mathbf{R}$.

To prove (a), put $A = \{x_j : j \in \mathbf{Z}_+\}$, and $x = \sup A$. Let $\epsilon > 0$ be given, and note that $x - \epsilon$ is not an upper bound for A, by definition of the supremum. Thus there is a positive integer L such that

$$x - \epsilon < x_L. \tag{3.2.17}$$

Of course, $x_j \leq x$ for every $j \geq 1$, because x is an upper bound for A. If $j \geq L$, then

$$x_j \geq x_L > x - \epsilon, \tag{3.2.18}$$

because the x_j's increase monotonically. It follows that

$$|x_j - x| = x - x_j < \epsilon \tag{3.2.19}$$

for every $j \geq L$, as desired. Part (b) can be shown in essentially the same way, or by reducing to the previous case.

3.3 Sums and products

In this section, we consider some basic properties of convergent sequences of complex numbers. Of course, this uses the standard metric on $\mathbf{C}$.

Proposition 3.3.1 *Let $\{z_j\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ be sequences of complex numbers that converge to complex numbers z and w, respectively. Under these conditions, $\{z_j + w_j\}_{j=1}^\infty$ converges to $z + w$ in $\mathbf{C}$.*

Thus

$$\lim_{j \to \infty} (z_j + w_j) = \lim_{j \to \infty} z_j + \lim_{j \to \infty} w_j, \tag{3.3.2}$$

where more precisely the limits on the right exist by hypothesis, and the existence of the limit on the left is part of the conclusion. To see this, let $\epsilon > 0$ be given. Because $\{z_j\}_{j=1}^\infty$ converges to z in $\mathbf{C}$, there is a positive integer L_1 such that

$$|z_j - z| < \epsilon/2 \tag{3.3.3}$$

for every $j \ge L_1$. Similarly, because $\{w_j\}_{j=1}^\infty$ converges to w in $\mathbf{C}$, there is an $L_2 \in \mathbf{Z}_+$ such that

$$|w_j - w| < \epsilon/2 \tag{3.3.4}$$

for every $j \ge L_2$. If $j \ge \max(L_1, L_2)$, then it follows that

$$\begin{aligned} |(z_j + w_j) - (z + w)| &= |(z_j - z) + (w_j - w)| \\ &\le |z_j - z| + |w_j - w| < \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned} \tag{3.3.5}$$

as desired.

Lemma 3.3.6 *If $\{a_j\}_{j=1}^\infty$ is a sequence of complex numbers that converges to 0, and $\{b_j\}_{j=1}^\infty$ is a bounded sequence of complex numbers, then $\{a_j\, b_j\}_{j=1}^\infty$ converges to 0 in $\mathbf{C}$.*

The boundedness of $\{b_j\}_{j=1}^\infty$ in $\mathbf{C}$ implies that there is a positive real number B such that

$$|b_j| \le B \tag{3.3.7}$$

for every $j \ge 1$. Let $\epsilon > 0$ be given. Because $\{a_j\}_{j=1}^\infty$ converges to 0 in $\mathbf{C}$, there is a positive integer L such that

$$|a_j| < \epsilon/B \tag{3.3.8}$$

for every $j \ge L$. This implies that

$$|a_j\, b_j| = |a_j|\, |b_j| \le B\, |a_j| < B\, (\epsilon/B) = \epsilon \tag{3.3.9}$$

for every $j \ge L$, as desired.

Lemma 3.3.10 *Let $\{z_j\}_{j=1}^\infty$ be a sequence of complex numbers that converges to a complex number z. If $c \in \mathbf{C}$, then $\{c\, z_j\}_{j=1}^\infty$ converges to $c\, z$ in $\mathbf{C}$.*

Equivalently, this says that

$$\lim_{j\to\infty} (c\, z_j) = c \Big(\lim_{j\to\infty} z_j\Big), \tag{3.3.11}$$

where the limit on the right exists by hypothesis, and the existence of the limit on the left is part of the conclusion.

Of course, if $c = 0$, then $c\, z_j = 0$ for every $j \ge 1$, $c\, z = 0$, and the lemma is trivial. Thus we may suppose that $c \ne 0$. Let $\epsilon > 0$ be given. Because $\{z_j\}_{j=1}^\infty$ converges to z in $\mathbf{C}$, there is a positive integer L such that

$$|z_j - z| < \epsilon/|c| \tag{3.3.12}$$

for every $j \ge L$. This implies that

$$|c\, z_j - c\, z| = |c|\, |z_j - z| < |c|\, (\epsilon/|c|) = \epsilon \tag{3.3.13}$$

for every $j \ge L$, as desired.

Proposition 3.3.14 *If $\{z_j\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ are sequences of complex numbers that converge to $z, w \in \mathbf{C}$, respectively, then $\{z_j\, w_j\}_{j=1}^\infty$ converges to $z\, w$ in $\mathbf{C}$.*

As before, this can be summarized by saying that

$$\lim_{j\to\infty} (z_j\, w_j) = \Big(\lim_{j\to\infty} z_j\Big) \Big(\lim_{j\to\infty} w_j\Big), \tag{3.3.15}$$

where the limits on the right exist by hypothesis, and the existence of the limit on the left is part of the conclusion.

Observe that

$$z_j\, w_j = z_j\, (w_j - w) + z_j\, w \tag{3.3.16}$$

for every $j \ge 1$. The hypothesis that $\{w_j\}_{j=1}^\infty$ converges to w in $\mathbf{C}$ is equivalent to saying that $\{w_j - w\}_{j=1}^\infty$ converges to 0. We also have that $\{z_j\}_{j=1}^\infty$ is a bounded sequence in $\mathbf{C}$, because it converges, as in Proposition 3.2.10. It follows that $\{z_j\, (w_j - w)\}_{j=1}^\infty$ converges to 0 in $\mathbf{C}$, as in Lemma 3.3.6. Using Lemma 3.3.10, we get that $\{z_j\, w\}_{j=1}^\infty$ converges to $z\, w$ in $\mathbf{C}$. Thus the right side of (3.3.16) is the sum of two convergent sequences in $\mathbf{C}$. This implies that $\{z_j\, w_j\}_{j=1}^\infty$ converges to $z\, w$, by Proposition 3.3.1, as desired.

Proposition 3.3.17 *If $\{z_j\}_{j=1}^\infty$ is a sequence of nonzero complex numbers that converges to a nonzero complex number z, then $\{1/z_j\}_{j=1}^\infty$ converges to $1/z$ in $\mathbf{C}$.*

This means that

$$\lim_{j\to\infty} (1/z_j) = 1/\Big(\lim_{j\to\infty} z_j\Big), \tag{3.3.18}$$

where the existence of the limit on the right is part of the hypothesis, and the existence of the limit on the left is part of the conclusion.

Let us begin by observing that

$$1/z_j - 1/z = (z - z_j)/(z_j\, z) \tag{3.3.19}$$

for every $j \ge 1$. Thus

$$|1/z_j - 1/z| = |z - z_j|/(|z_j|\,|z|) \tag{3.3.20}$$

for every $j \ge 1$. We should first deal with the factor of $|z_j|$ in the denominator.

Because $\{z_j\}_{j=1}^\infty$ converges to z and $z \ne 0$, there is a positive integer L_0 such that

$$|z_j - z| < |z|/2 \tag{3.3.21}$$

for every $j \ge L_0$. Using the triangle inequality, we get that

$$|z| \le |z_j| + |z - z_j| < |z_j| + |z|/2 \tag{3.3.22}$$

when $j \ge L_0$. This implies that

$$|z|/2 < |z_j| \tag{3.3.23}$$

for every $j \ge L_0$. Combining this with (3.3.20), we get that

$$|1/z_j - 1/z| \le (2/|z|^2)\,|z - z_j| \tag{3.3.24}$$

for every $j \ge L_0$.

Let $\epsilon > 0$ be given. Because $\{z_j\}_{j=1}^\infty$ converges to z, there is a positive integer L such that

$$|z_j - z| < (|z|^2/2)\,\epsilon \tag{3.3.25}$$

for every $j \ge L$. If $j \ge \max(L_0, L)$, then it follows that

$$|1/z_j - 1/z| < (2/|z|^2)\,(|z|^2/2)\,\epsilon = \epsilon, \tag{3.3.26}$$

as desired.

3.4 Subsequences and sequential compactness

Definition 3.4.1 *Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of some set, and let $\{j_l\}_{l=1}^\infty$ be a strictly increasing sequence of positive integers, so that $j_l < j_{l+1}$ for every $l \ge 1$. Under these conditions, $\{x_{j_l}\}_{l=1}^\infty$ is called a* subsequence *of $\{x_j\}_{j=1}^\infty$.*

Note that a sequence may be considered as a subsequence of itself.

Let $(M, d(x,y))$ be a metric space. If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of M that converges to an element x of M, and if $\{x_{j_l}\}_{l=1}^\infty$ is a subsequence of $\{x_j\}_{j=1}^\infty$, then it is easy to see that $\{x_{j_l}\}_{l=1}^\infty$ converges to x in M as well. This uses the fact that $j_l \ge l$ for every $l \ge 1$.

Definition 3.4.2 *A subset K of M is said to be* sequentially compact *if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of K there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ and an element x of K such that $\{x_{j_l}\}_{l=1}^\infty$ converges to x in M.*

Proposition 3.4.3 *A subset K of M is sequentially compact if and only if K has the limit point property in M.*

Suppose first that K has the limit point property, and let us show that K is sequentially compact. Let $\{x_j\}_{j=1}^\infty$ be any sequence of elements in K, and let $L = \{x_j : j \in \mathbf{Z}_+\}$ be the set of terms in this sequence. If L has only finitely many elements, then there is an $x \in K$ such that $x_j = x$ for infinitely many $j \in \mathbf{Z}_+$. Equivalently, this means that there is subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that $x_{j_l} = x$ for every $l \geq 1$. Of course, $\{x_{j_l}\}_{l=1}^\infty$ converges to x in this case.

Suppose now that L has infinitely many elements. If K has the limit point property, then there is a point $x \in K$ that is a limit point of L in M. This implies that for every $r > 0$, there are infinitely many elements of L in $B(x, r)$, as in Proposition 1.5.2. It follows that for each $r > 0$,

$$d(x, x_j) < r \tag{3.4.4}$$

for infinitely many $j \in \mathbf{Z}_+$.

Using this, we can find a subsequence $\{x_{j_n}\}_{n=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that

$$d(x, x_{j_n}) < 1/n \tag{3.4.5}$$

for every $n \geq 1$. More precisely, we can first choose $j_1 \in \mathbf{Z}_+$ so that (3.4.5) holds with $n = 1$. If $j_n \in \mathbf{Z}_+$ has been chosen for some positive integer n, then we can choose $j_{n+1} \in \mathbf{Z}_+$ so that $j_{n+1} > j_n$ and (3.4.5) holds with n replaced with $n + 1$. It is easy to see that $\{x_{j_n}\}_{n=1}^\infty$ converges to x in M, because of (3.4.5).

Conversely, suppose that $K \subseteq M$ is sequentially compact, and let us show that K has the limit point property. Let L be an infinite subset of K, and let $\{x_j\}_{j=1}^\infty$ be an infinite sequence of distinct elements of L. If K is sequentially compact, then there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to an element x of K in M. Let $r > 0$ be given, so that $d(x_{j_l}, x) < r$ for all but finitely many $l \in \mathbf{Z}_+$. This implies that there are infinitely many elements of L in $B(x, r)$, so that x is a limit point of L in M, as desired.

3.5 Cauchy sequences and completeness

Let $(M, d(x, y))$ be a metric space.

Definition 3.5.1 *A sequence $\{x_j\}_{j=1}^\infty$ of elements of M is said to be a* Cauchy sequence *if for every $\epsilon > 0$ there is a positive integer L such that*

$$d(x_j, x_l) < \epsilon \tag{3.5.2}$$

for every $j, l \geq L$.

Proposition 3.5.3 *If a sequence $\{x_j\}_{j=1}^\infty$ of elements of M converges to an element x of M, then $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in M.*

Let $\epsilon > 0$ be given. Because $\{x_j\}_{j=1}^\infty$ converges to x in M, there is a positive integer L such that

$$d(x_j, x) < \epsilon/2 \tag{3.5.4}$$

for every $j \geq L$. If $j, l \geq L$, then it follows that

$$d(x_j, x_l) \leq d(x_j, x) + d(x, x_l) < \epsilon/2 + \epsilon/2 = \epsilon, \tag{3.5.5}$$

as desired.

Proposition 3.5.6 *If $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in M, then $\{x_j\}_{j=1}^\infty$ is bounded in M.*

By hypothesis, there is a positive integer L such that

$$d(x_j, x_l) < 1 \tag{3.5.7}$$

for every $j, l \geq L$. If $L = 1$, then we get that $x_j \in B(x_1, 1)$ for every $j \geq 1$, as desired. Otherwise, put

$$r = \max\{d(x_j, x_L) : j = 1, \ldots, L-1\}, \tag{3.5.8}$$

which is a nonnegative real number. In this case, we have that $x_j \in B(x_L, r+1)$ for every $j \geq 1$, as desired.

Proposition 3.5.9 *Let $\{x_j\}_{j=1}^\infty$ be a Cauchy sequence of elements of M. If there is a subsequence $\{x_{j_n}\}_{n=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to an element x of M, then $\{x_j\}_{j=1}^\infty$ converges to x in M.*

Let $\epsilon > 0$ be given. Because $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in M, there is a positive integer L such that

$$d(x_j, x_l) < \epsilon/2 \tag{3.5.10}$$

for every $j, l \geq L$. Similarly, there is a positive integer N such that

$$d(x_{j_n}, x) < \epsilon/2 \tag{3.5.11}$$

for every $n \geq N$, because $\{x_{j_n}\}_{n=1}^\infty$ converges to x in M. If $l \geq L$, $n \geq N$, and $j_n \geq L$, then we get that

$$d(x_l, x) \leq d(x_l, x_{j_n}) + d(x_{j_n}, x) < \epsilon/2 + \epsilon/2 = \epsilon. \tag{3.5.12}$$

As before, $j_n \geq n$ for every positive integer n, so that $j_n \geq L$ when $n \geq L$. In particular, if $n = \max(L, N)$, then $j_n \geq L$ and $n \geq N$. Using this, we obtain that

$$d(x_l, x) < \epsilon \tag{3.5.13}$$

for every $l \geq L$, as desired.

Corollary 3.5.14 *If $K \subseteq M$ is sequentially compact, $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in M, and $x_j \in K$ for every $j \geq 1$, then $\{x_j\}_{j=1}^\infty$ converges to an element of K in M.*

Definition 3.5.15 *If every Cauchy sequence of elements of M converges to an element of M, then M is said to be* complete *as a metric space.*

Corollary 3.5.16 *If n is a positive integer, then $\mathbf{R}^n$ is complete with respect to the standard Euclidean metric.*

Let $\{x_j\}_{j=1}^\infty$ be a Cauchy sequence of elements of $\mathbf{R}^n$. It is easy to see that there is a cell C in $\mathbf{R}^n$ such that $x_j \in C$ for every $j \geq 1$, because $\{x_j\}_{j=1}^\infty$ is bounded in $\mathbf{R}^n$, as in Proposition 3.5.6. Remember that C is compact in $\mathbf{R}^n$, so that C has the limit point property and is sequentially compact, by Proposition 1.7.10, Theorem 1.14.1, and Proposition 3.4.3. Thus $\{x_j\}_{j=1}^\infty$ converges to an element of C, as in Corollary 3.5.14.

3.6 Some particular sequences

In this section, we consider the convergence of some particular sequences of real or complex numbers, with respect to the standard metrics on $\mathbf{R}$ and $\mathbf{C}$.

Let p be a positive real number, so that a^p can be defined as a positive real number for every positive real number a, as in Section 1.3. It is well known that

$$\lim_{j \to \infty} 1/j^p = 0. \tag{3.6.1}$$

To see this, let $\epsilon > 0$ be given. Observe that

$$1/j^p < \epsilon \tag{3.6.2}$$

if and only if

$$j > (1/\epsilon)^{1/p}. \tag{3.6.3}$$

This holds for all but finitely many positive integers j, by the archimedean property of the real numbers. This is all a bit simpler when $p = 1/k$ for some positive integer k. If p is any positive real number, then there is a positive integer k such that $1/k \leq p$, by the archimedean property, which can be used to reduce to that case.

If a is a nonnegative real number with $a < 1$, then

$$\lim_{j \to \infty} a^j = 0. \tag{3.6.4}$$

This is trivial when $a = 0$, and so we may as well suppose that $a > 0$. Put $b = 1/a - 1 > 0$. It is easy to see that

$$(1 + b)^j \geq 1 + b\, j \tag{3.6.5}$$

for every positive integer j, using induction, for instance. This implies that

$$a^j = (1+b)^{-j} \le 1/(1+b\,j) \tag{3.6.6}$$

for every $j \ge 1$, and (3.6.4) follows easily from this.

Let α be a positive real number, and let us show that

$$\lim_{j \to \infty} j^\alpha\, a^j = 0. \tag{3.6.7}$$

This follows from (3.6.6) when $a > 0$ and $\alpha < 1$. Otherwise, the archimedean property for the real numbers implies that there is a positive integer k such that $\alpha < k$. Note that $a^{1/k} < 1$, so that

$$\lim_{j \to \infty} j^{\alpha/k}\, a^{j/k} = 0, \tag{3.6.8}$$

as before. This implies that

$$j^\alpha\, a^j = (j^{\alpha/k}\, a^{j/k})^k \to 0 \quad \text{as } j \to \infty, \tag{3.6.9}$$

as desired.

If p is a positive real number, then

$$\lim_{j \to \infty} p^{1/j} = 1. \tag{3.6.10}$$

Suppose first that $p \ge 1$, so that $p^{1/j} \ge 1$ for every $j \in \mathbf{Z}_+$. Let $\epsilon > 0$ be given, and observe that

$$p^{1/j} < 1 + \epsilon \tag{3.6.11}$$

if and only if

$$p < (1+\epsilon)^j. \tag{3.6.12}$$

This holds for all but finitely many $j \in \mathbf{Z}_+$, by (3.6.5), with $b = \epsilon$. Thus (3.6.10) holds when $p \ge 1$. If $0 < p \le 1$, then we can apply the previous argument to $1/p$, to get that

$$\lim_{j \to \infty} 1/p^{1/j} = 1. \tag{3.6.13}$$

This implies (3.6.10), by Proposition 3.3.17.

In fact,

$$\lim_{j \to \infty} j^{1/j} = 1. \tag{3.6.14}$$

Let $\epsilon > 0$ be given, and note that $j^{1/j} \ge 1$ for every $j \in \mathbf{Z}_+$. As before,

$$j^{1/j} < 1 + \epsilon \tag{3.6.15}$$

if and only if

$$j < (1+\epsilon)^j, \tag{3.6.16}$$

which is the same as saying that

$$j\,(1+\epsilon)^{-j} < 1. \tag{3.6.17}$$

The left side of (3.6.17) tends to 0 as $j \to \infty$, as in (3.6.7). This implies that (3.6.17) holds for all but finitely many $j \in \mathbf{Z}_+$, as desired.

If z is a complex number such that $|z| < 1$, then

$$\lim_{j \to \infty} z^j = 0. \tag{3.6.18}$$

More precisely,

$$|z^j| = |z|^j \to 0 \quad \text{as } j \to \infty, \tag{3.6.19}$$

by (3.6.4).

3.7 Infinite series

Definition 3.7.1 *Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of complex numbers. The infinite series $\sum_{j=1}^{\infty} a_j$ is said to* converge *if the corresponding sequence of partial sums*

$$s_n = \sum_{j=1}^{n} a_j \tag{3.7.2}$$

converges as a sequence of complex numbers, with respect to the standard metric on $\mathbf{C}$. *In this case, we put*

$$\sum_{j=1}^{\infty} a_j = \lim_{n \to \infty} s_n. \tag{3.7.3}$$

Sometimes we may wish to consider infinite series $\sum_{j=0}^{\infty} a_j$ starting at $j = 0$, or some other integer.

Proposition 3.7.4 *Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be convergent series of complex numbers.*

(a) Under these conditions, $\sum_{j=1}^{\infty} (a_j + b_j)$ converges too, with

$$\sum_{j=1}^{\infty} (a_j + b_j) = \sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j. \tag{3.7.5}$$

(b) If c is a complex number, then $\sum_{j=1}^{\infty} c\, a_j$ converges, with

$$\sum_{j=1}^{\infty} c\, a_j = c \sum_{j=1}^{\infty} a_j. \tag{3.7.6}$$

If n is any positive integer, then

$$\sum_{j=1}^{n} (a_j + b_j) = \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j \tag{3.7.7}$$

and

$$\sum_{j=1}^{n} c\, a_j = c \sum_{j=1}^{n} a_j. \tag{3.7.8}$$

Thus Proposition 3.7.4 follows from the analogous statements for sequences of complex numbers.

Proposition 3.7.9 *An infinite series $\sum_{j=1}^{\infty} a_j$ of complex numbers converges if and only if the following condition holds: for every $\epsilon > 0$ there is a positive integer L such that*

$$\left| \sum_{j=l}^{n} a_j \right| < \epsilon \tag{3.7.10}$$

for all positive integers l, n with $n \ge l \ge L$.

One can check that this condition is equivalent to saying that the corresponding sequence of partial sums (3.7.2) is a Cauchy sequence with respect to the standard metric on $\mathbf{C}$. Thus the proposition follows from the completeness of $\mathbf{C}$ with respect to the standard metric, which is the same as the completeness of $\mathbf{R}^2$ with respect to the standard Euclidean metric.

Corollary 3.7.11 *If $\sum_{j=1}^{\infty} a_j$ is a convergent series of complex numbers, then $\{a_j\}_{j=1}^{\infty}$ converges to 0 as a sequence of complex numbers.*

This follows by taking $l = n$ in (3.7.10).

Definition 3.7.12 *An infinite series $\sum_{j=1}^{\infty} a_j$ of complex numbers is said to* converge absolutely *if $\sum_{j=1}^{\infty} |a_j|$ converges as an infinite series of nonnegative real numbers.*

Proposition 3.7.13 *If $\sum_{j=1}^{\infty} a_j$ is an absolutely convergent series of complex numbers, then $\sum_{j=1}^{\infty} a_j$ converges.*

Observe that

$$\left| \sum_{j=l}^{n} a_j \right| \le \sum_{j=l}^{n} |a_j| \tag{3.7.14}$$

for all positive integers l, n with $l \le n$, by the triangle inequality. One can use this to obtain Proposition 3.7.13 from Proposition 3.7.9.

Proposition 3.7.15 *Let $\sum_{j=1}^{\infty} a_j$ be an infinite series of nonnegative real numbers. Under these conditions, $\sum_{j=1}^{\infty} a_j$ converges if and only if the corresponding sequence of partial sums (3.7.2) is bounded.*

If $\sum_{j=1}^{\infty} a_j$ is a convergent series of complex numbers, then the corresponding sequence of partial sums is bounded, as in Proposition 3.2.10. If a_j is a nonnegative real number for each $j \ge 1$, then it is easy to see that the partial sums are monotonically increasing. In this case, the boundedness of the partial sums implies that they converge as a sequence of real numbers, as in Proposition 3.2.14.

Proposition 3.7.16 (Comparison test) *Let $\sum_{j=1}^\infty a_j$ be an infinite series of complex numbers, and let $\sum_{j=1}^\infty b_j$ be an infinite series of nonnegative real numbers. If $|a_j| \le b_j$ for every $j \ge 1$ and $\sum_{j=1}^\infty b_j$ converges, then $\sum_{j=1}^\infty a_j$ converges absolutely.*

Observe that

$$\sum_{j=1}^n |a_j| \le \sum_{j=1}^n b_j \tag{3.7.17}$$

for every positive integer n. If $\sum_{j=1}^\infty b_j$ converges, then its sequence of partial sums is bounded. This implies that the partial sums of $\sum_{j=1}^\infty |a_j|$ are bounded as well, so that this series converges too.

Let z be a complex number, and let us consider the corresponding *geometric series* $\sum_{j=0}^\infty z^j$. Here z^j is interpreted as being equal to 1 when $j = 0$. If n is a nonnegative integer, then

$$(1-z)\sum_{j=0}^n z^j = \sum_{j=0}^n z^j - \sum_{j=0}^n z^{j+1} = \sum_{j=0}^n z^j - \sum_{j=1}^{n+1} z^j = 1 - z^{n+1}. \tag{3.7.18}$$

This implies that

$$\sum_{j=0}^n z^j = (1 - z^{n+1})/(1-z). \tag{3.7.19}$$

when $z \ne 1$. If $|z| < 1$, then we get that $\sum_{j=0}^\infty z^j$ converges, with

$$\sum_{j=0}^\infty z^j = 1/(1-z), \tag{3.7.20}$$

by (3.6.18). Note that $\sum_{j=0}^\infty z^j$ converges absolutely in this case, by the same argument for $|z|$. If $|z| \ge 1$, then

$$|z^j| = |z|^j \ge 1 \tag{3.7.21}$$

for every $j \ge 0$. This means that $\{z^j\}_{j=0}^\infty$ does not converge to 0, so that $\sum_{j=0}^\infty z^j$ does not converge.

3.8 Power series

Let $a_0, a_1, a_2, a_3, \ldots$ be an infinite sequence of complex numbers, and consider the corresponding *power series* $\sum_{j=0}^\infty a_j\, z^j$, for $z \in \mathbf{C}$.

Proposition 3.8.1 *If $\sum_{j=0}^\infty a_j\, z^j$ converges absolutely for some $z \in \mathbf{C}$, then $\sum_{j=0}^\infty a_j\, w^j$ converges absolutely for every $w \in \mathbf{C}$ with $|w| \le |z|$.*

Indeed, if $|w| \le |z|$, then

$$|a_j\, w_j| = |a_j|\,|w|^j \le |a_j|\,|z|^j = |a_j\, z^j| \tag{3.8.2}$$

for every nonnegative integer j. Thus Proposition 3.8.1 follows from the comparison test.

Proposition 3.8.3 *If $\sum_{j=0}^{\infty} a_j\, z^j$ converges for some nonzero complex number z, then $\sum_{j=0}^{\infty} a_j\, w^j$ converges absolutely for every $w \in \mathbf{C}$ with $|w| < |z|$.*

If $\sum_{j=0}^{\infty} a_j\, z^j$ converges, then $\{a_j\, z^j\}_{j=0}^{\infty}$ converges to 0, by Corollary 3.7.11. This implies that $\{a_j\, z^j\}_{j=0}^{\infty}$ is a bounded sequence in $\mathbf{C}$, as in Proposition 3.2.10. It follows that there is a nonnegative real number C such that

$$|a_j\, z^j| \le C \tag{3.8.4}$$

for every $j \ge 0$. Let $w \in \mathbf{C}$ be given, and observe that

$$|a_j\, w^j| = |a_j\, z^j|\,(|w|^j/|z|^j) \le C\,(|w|/|z|)^j \tag{3.8.5}$$

for every $j \ge 0$, by (3.8.4). If $|w| < |z|$, then $\sum_{j=0}^{\infty}(|w|/|z|)^j$ is a convergent geometric series, as in the previous section. Of course, this implies that $\sum_{j=0}^{\infty} C\,(|w|/|z|)^j$ converges too, as in Proposition 3.7.4. Proposition 3.8.3 now follows from the comparison test.

Let E be the set of complex numbers z such that $\sum_{j=0}^{\infty} a_j\, z^j$ does not converge, and put

$$E_1 = \{|z| : z \in E\}. \tag{3.8.6}$$

This is a set of nonnegative real numbers, which is not empty when $E \ne \emptyset$.

Definition 3.8.7 *The* radius of convergence ρ *of $\sum_{j=0}^{\infty} a_j\, z^j$ is defined by*

$$\rho = \inf E_1 \tag{3.8.8}$$

when $E \ne \emptyset$, and by $\rho = +\infty$ when $E = \emptyset$.

Proposition 3.8.9 *(a) If $w \in \mathbf{C}$ satisfies $|w| > \rho$, then $\sum_{j=0}^{\infty} a_j\, w^j$ does not converge.*

(b) If $w \in \mathbf{C}$ satisfies $|w| < \rho$, then $\sum_{j=0}^{\infty} a_j\, w^j$ converges absolutely.

(c) The radius of convergence of $\sum_{j=0}^{\infty} a_j\, z^j$ is uniquely determined by the properties in (a) and (b).

To prove (a), let w be a complex number such that $\sum_{j=0}^{\infty} a_j\, w^j$ converges. If $z \in \mathbf{C}$ satisfies $|z| < |w|$, then $\sum_{j=0}^{\infty} a_j\, z^j$ converges absolutely, by Proposition 3.8.3. This implies that $|w|$ is a lower bound for E_1, so that $|w| \le \inf E_1 = \rho$ when $E \ne \emptyset$. If $E = \emptyset$, then $|w| < \rho = +\infty$ holds trivially.

If $\sum_{j=0}^{\infty} a_j\, w^j$ does not converge absolutely, then $\sum_{j=0}^{\infty} a_j\, z^j$ does not converge for any $z \in \mathbf{C}$ with $|w| < |z|$, by Proposition 3.8.3 again. This implies that $E \ne \emptyset$, and that $\rho = \inf E_1 \le |w|$.

If $0 \le \rho \le +\infty$ satisfies the conditions in (a) and (b), then $E = \emptyset$ when $\rho = +\infty$, and $E \ne \emptyset$, $\inf E_1 = \rho$ when $\rho < \infty$. Alternatively, if $0 \le \rho', \rho'' \le +\infty$ both satisfy both (a) and (b) in place of ρ, then one can check directly that $\rho' = \rho''$.

3.9 Cauchy's condensation test

Theorem 3.9.1 (Cauchy's condensation test) *Let $\{a_j\}_{j=1}^\infty$ be a monotonically decreasing sequence of nonnegative real numbers. Under these conditions, $\sum_{j=1}^\infty a_j$ converges if and only if $\sum_{k=0}^\infty 2^k\, a_{2^k}$ converges.*

Suppose first that $\sum_{k=0}^\infty 2^k\, a_{2^k}$ converges, so that the corresponding sequence of partial sums $\sum_{k=0}^n 2^k\, a_{2^k}$ are bounded. We would like to show that the partial sums for $\sum_{j=1}^\infty a_j$ are bounded, which would imply that the series converges.

If k is any nonnegative integer, then

$$\sum_{j=2^k}^{2^{k+1}-1} a_j \le \sum_{j=2^k}^{2^{k+1}-1} a_{2^k} = 2^k\, a_{2^k}. \tag{3.9.2}$$

If l is a positive integer, n is a nonnegative integer, and $l < 2^{n+1}$, then we get that

$$\sum_{j=1}^{l} a_j \le \sum_{k=0}^{n} \sum_{j=2^k}^{2^{k+1}-1} a_j \le \sum_{k=0}^{n} 2^k\, a_{2^k}. \tag{3.9.3}$$

Thus the boundedness of the partial sums of $\sum_{k=0}^\infty 2^k\, a_{2^k}$ implies the boundedness of the partial sums for $\sum_{j=1}^\infty a_j$, as desired.

Similarly, if k is a positive integer, then

$$2^{k-1}\, a_{2^k} \le \sum_{j=2^{k-1}+1}^{2^k} a_j. \tag{3.9.4}$$

If n is a positive integer, then we obtain that

$$\sum_{k=0}^{n} 2^k\, a_{2^k} = a_1 + \sum_{k=1}^{n} 2^k\, a_{2^k} \le a_1 + 2 \sum_{k=1}^{n} \sum_{j=2^{k-1}+1}^{2^k} a_j \le 2 \sum_{j=1}^{2^n} a_j. \tag{3.9.5}$$

It follows that the partial sums of $\sum_{k=0}^\infty 2^k\, a_{2^k}$ are bounded when the partial sums of $\sum_{j=1}^\infty a_j$ are bounded, as desired.

Let p be a positive real number, and note that $a_j = 1/j^p$ defines a monotonically descreasing sequence of positive real numbers. In this situation,

$$2^k\, a_{2^k} = 2^{(1-p)\,k} = (2^{1-p})^k. \tag{3.9.6}$$

Thus the Cauchy condensation test implies that $\sum_{j=1}^\infty 1/j^p$ converges exactly when

$$\sum_{k=0}^{\infty} (2^{1-p})^k \tag{3.9.7}$$

converges. The latter is a geometric series that converges exactly when

$$2^{1-p} < 1, \tag{3.9.8}$$

which holds if and only if $p > 1$.

3.10 Another criterion for convergence

Theorem 3.10.1 *Let $\{a_j\}_{j=0}^{\infty}$ be a sequence of complex numbers, and suppose that the sequence of sums*

$$A_n = \sum_{j=0}^{n} a_j \tag{3.10.2}$$

is bounded. Also let $\{b_j\}_{j=0}^{\infty}$ be a monotonically decreasing sequence of nonnegative real numbers that converges to 0*. Under these conditions, $\sum_{j=0}^{\infty} a_j\, b_j$ converges.*

Put $A_{-1} = 0$, so that $a_j = A_j - A_{j-1}$ for every nonnegative integer j. Let n be a nonnegative integer, and observe that

$$\sum_{j=0}^{n} a_j\, b_j = \sum_{j=0}^{n} (A_j - A_{j-1})\, b_j = \sum_{j=0}^{n} A_j\, b_j - \sum_{j=0}^{n} A_{j-1}\, b_j. \tag{3.10.3}$$

The last sum on the right can be reexpressed as

$$\sum_{j=0}^{n} A_{j-1}\, b_j = \sum_{j=-1}^{n-1} A_j\, b_{j+1} = \sum_{j=0}^{n} A_j\, b_{j+1} - A_n\, b_{n+1}. \tag{3.10.4}$$

Thus

$$\begin{aligned} \sum_{j=0}^{n} a_j\, b_j &= \sum_{j=0}^{n} A_j\, b_j - \sum_{j=0}^{n} A_j\, b_{j+1} + A_n\, b_{n+1} \\ &= \sum_{j=0}^{n} A_j\, (b_j - b_{j+1}) + A_n\, b_{n+1}. \end{aligned} \tag{3.10.5}$$

Lemma 3.3.6 implies that $\{A_n\, b_{n+1}\}_{n=0}^{\infty}$ converges to 0, because $\{A_n\}_{n=0}^{\infty}$ is a bounded sequence, and $\{b_{n+1}\}_{n=0}^{\infty}$ converges to 0.

It suffices to show that

$$\sum_{j=0}^{\infty} A_j\, (b_j - b_{j+1}) \tag{3.10.6}$$

converges, by the remarks in the preceding paragraph. If n is a nonnegative integer, then

$$\sum_{j=0}^{n} (b_j - b_{j+1}) = \sum_{j=0}^{n} b_j - \sum_{j=0}^{n} b_{j+1} = \sum_{j=0}^{n} b_j - \sum_{j=1}^{n+1} b_j = b_0 - b_{n+1}. \tag{3.10.7}$$

This implies that $\sum_{j=0}^{\infty} (b_j - b_{j+1})$ converges, because $\{b_{n+1}\}_{n=0}^{\infty}$ converges as a sequence of real numbers. Because $\{A_j\}_{j=0}^{\infty}$ is bounded, there is a nonnegative real number C such that

$$|A_j| \le C \tag{3.10.8}$$

for every $j \geq 0$. It follows that

$$|A_j\,(b_j - b_{j+1})| \leq C\,(b_j - b_{j+1}) \tag{3.10.9}$$

for every $j \geq 0$, because $b_j - b_{j+1} \geq 0$, by hypothesis. Of course,

$$\sum_{j=0}^{\infty} C\,(b_j - b_{j+1}) \tag{3.10.10}$$

converges, because $\sum_{j=0}^{\infty}(b_j - b_{j+1})$ converges. Thus the comparison test implies that (3.10.6) converges absolutely, as desired.

If $a_j = (-1)^j$ for every nonnegative integer j, then A_n is equal to 1 when j is even, and to 0 when j is odd. In particular, $\{A_n\}_{n=0}^{\infty}$ is bounded. In this case, Theorem 3.10.1 corresponds to Leibniz' alternating series test.

Let z be a complex number, and suppose that

$$a_j = z^j \tag{3.10.11}$$

for every nonnegative integer j. If $|z| < 1$, then the comparison test implies that $\sum_{j=0}^{\infty} b_j\, z^j$ converges absolutely when $\{b_j\}_{j=0}^{\infty}$ is bounded. Suppose that $z \neq 1$, so that

$$A_n = \sum_{j=0}^{n} z^j = (1 - z^{n+1})/(1 - z) \tag{3.10.12}$$

for every nonnegative integer n, as in (3.7.19). If $|z| = 1$, then

$$|A_n| = |1 - z^{n+1}|/|1 - z| \leq (1 + |z|^{n+1})/|1 - z| = 2/|1 - z| \tag{3.10.13}$$

for every $n \geq 0$, so that $\{A_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers.

3.11 Extended real numbers

The set of *extended real numbers* is defined to be the set of real numbers together with two additional elements, denoted $+\infty$ and $-\infty$. The standard ordering on the real line can be extended to the set of extended real numbers, by putting

$$-\infty < x < +\infty \tag{3.11.1}$$

for every $x \in \mathbf{R}$. If A is any set of extended real numbers, then the notions of upper and lower bounds for A in the set of extended real numbers can be defined in the usual way. In particular, $+\infty$ is automatically an upper bound for A, and $-\infty$ is automatically a lower bound for A.

The notions of supremum or least upper bound and infimum or greatest lower bound for A in the set of extended real numbers can be defined in the same way as before too. The supremum and infimum of A in the set of extended real numbers always exists, as follows. If $+\infty \in A$, or if $A \cap \mathbf{R}$ has no upper bound in $\mathbf{R}$, then $\sup A = +\infty$. If $+\infty \notin A$, and $A \cap \mathbf{R}$ is nonempty and has an upper bound in $\mathbf{R}$, then $\sup A$ is the same as the supremum of $A \cap \mathbf{R}$ in $\mathbf{R}$. Otherwise, if $A \subseteq \{-\infty\}$, then $\sup A = -\infty$. Similarly, if $-\infty \in A$ or $A \cap \mathbf{R}$ has no lower bound in $\mathbf{R}$, then $\inf A = -\infty$. If $-\infty \not\in A$ and $A \cap \mathbf{R}$ is nonempty and has a lower bound in $\mathbf{R}$, then $\inf A$ is the same as the infimum of $A \cap \mathbf{R}$ in $\mathbf{R}$. If $A \subseteq \{+\infty\}$, then $\inf A = +\infty$. Note that

$$\inf A \le \sup A \tag{3.11.2}$$

when $A \neq \emptyset$.

Sums and products of extended real numbers are defined in some situations, as follows. If $x \in \mathbf{R}$, then we put

$$x + (+\infty) = (+\infty) + x = +\infty, \quad x + (-\infty) = (-\infty) + x = -\infty. \tag{3.11.3}$$

We also put

$$(+\infty) + (+\infty) = +\infty, \quad (-\infty) + (-\infty) = -\infty. \tag{3.11.4}$$

The sum of $+\infty$ and $-\infty$ is not defined. If x is a nonzero extended real number, then we put

$$x\,(+\infty) = (+\infty)\,x = +\infty, \quad x\,(-\infty) = (-\infty)\,x = -\infty \tag{3.11.5}$$

when $x > 0$, and

$$x\,(+\infty) = (+\infty)\,x = -\infty, \quad x\,(-\infty) = (-\infty)\,x = +\infty \tag{3.11.6}$$

when $x < 0$. The product of 0 with $\pm\infty$ is not defined. We put $x/(+\infty) = x/(-\infty) = 0$ when $x \in \mathbf{R}$, and these quotients are not defined when $x = \pm\infty$. Although $1/0$ is not defined, it may be appropriate to interpret it as being $+\infty$, when dealing with nonnegative extended real numbers.

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of real numbers.

Definition 3.11.7 *We say that x_j* tends to $+\infty$ as $j \to \infty$*, or $x_j \to +\infty$ as $j \to \infty$, if for every nonnegative real number R there is a positive integer L such that*

$$x_j > R \quad \text{for every } j \ge L. \tag{3.11.8}$$

Similarly, we say that x_j tends to $-\infty$ as $j \to \infty$*, or $x_j \to -\infty$ as $j \to \infty$, if for every nonnegative real number R there is a positive integer L such that*

$$x_j < -R \quad \text{for every } j \ge L. \tag{3.11.9}$$

Thus, if x is any extended real number, then the condition that $x_j \to x$ as $j \to \infty$ is defined, using convergence with respect to the standard Euclidean metric on $\mathbf{R}$ when $x \in \mathbf{R}$, and the previous definition when $x = \pm\infty$. It is easy to see that this can hold for at most one x.

Proposition 3.11.10 *Let $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$ be sequences of real numbers, let x, y be extended real numbers, and suppose that $x_j \to x$ and $y_j \to y$ as $j \to \infty$.*

(a) If $x + y$ is defined as an extended real number, then $x_j + y_j \to x + y$ as $j \to \infty$.

(b) If $x\,y$ is defined as an extended real number, then $x_j\,y_j \to x\,y$ and $j \to \infty$.

Of course, this follows from previous results when $x, y \in \mathbf{R}$. It is not difficult to prove (a) and (b) directly in the other cases. Suppose that $x = +\infty$, for instance. If the set of y_j's has a lower bound in $\mathbf{R}$, then it is easy to see that $x_j + y_j \to \infty$ as $j \to \infty$. In particular, this holds when $y_j \to y$ as $j \to \infty$ and $y \neq -\infty$. Similarly, suppose that there is a positive real number a and a positive integer L_0 such that

$$y_j \geq a \tag{3.11.11}$$

for every $j \geq L_0$. Using this, one can check that $x_j\,y_j \to +\infty$ as $j \to \infty$. If $y_j \to y$ as $j \to \infty$, then this condition holds when $y > 0$.

If $\{x_j\}_{j=1}^\infty$ is a sequence of nonzero real numbers such that $|x_j| \to \infty$ as $j \to \infty$, then

$$1/x_j \to 0 \quad \text{as } j \to \infty. \tag{3.11.12}$$

If $\{x_j\}_{j=1}^\infty$ is a sequence of positive real numbers that converges to 0, then

$$1/x_j \to +\infty \quad \text{as } j \to \infty. \tag{3.11.13}$$

Proposition 3.11.14 *If $\{x_j\}_{j=1}^\infty$ is any sequence of real numbers, then there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ and an extended real number x such that $x_{j_l} \to x$ as $l \to \infty$.*

Suppose first that there are real numbers a and b such that

$$a \leq x_j \leq b \quad \text{for every } j \geq 1. \tag{3.11.15}$$

Remember that $[a, b]$ is a compact subset of the real line with respect to the standard Euclidean metric, which implies that $[a, b]$ has the limit point property, and hence is sequentially compact, by previous results. This means that there is a subsequence of $\{x_j\}_{j=1}^\infty$ that converges to an element of $[a, b]$, as desired.

Next, suppose that the set of x_j's has no upper bound in $\mathbf{R}$. This implies that for each positive integer n,

$$x_j > n \quad \text{for infinitely many } j \geq 1, \tag{3.11.16}$$

since otherwise the set of x_j's would have a finite upper bound. We would like to find a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that

$$x_{j_l} > l \quad \text{for every } l \geq 1, \tag{3.11.17}$$

which implies that $x_{j_l} \to \infty$ as $j \to \infty$. We can start by choosing j_1 to be any positive integer such that $x_{j_1} > 1$. If j_l has been chosen for some positive integer l, then can choose j_{l+1} to be a positive integer such that $j_{l+1} > j_l$ and $x_{j_{l+1}} > l + 1$, using (3.11.16).

Similarly, if the set of x_j's does not have a lower bound in $\mathbf{R}$, then one can find a subsequence that tends to $-\infty$. This can be shown in essentially the same way, or by reducing to the previous case.

3.12 Limits superior and inferior

Let $\{x_j\}_{j=1}^\infty$ be a sequence of real numbers, and let E be the set of extended real numbers x for which there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that

$$x_{j_l} \to x \text{ as } l \to \infty. \tag{3.12.1}$$

Note that $E \neq \emptyset$, by Proposition 3.11.14.

Definition 3.12.2 *The* upper limit *or* limit superior *of* $\{x_j\}_{j=1}^\infty$ *is defined by*

$$\limsup_{j\to\infty} x_j = \sup E. \tag{3.12.3}$$

Similarly, the lower limit *or* limit inferior *of* $\{x_j\}_{j=1}^\infty$ *is defined by*

$$\liminf_{j\to\infty} x_j = \inf E. \tag{3.12.4}$$

Thus

$$\liminf_{j\to\infty} x_j \le \limsup_{j\to\infty} x_j. \tag{3.12.5}$$

If there is an extended real number x such that $x_j \to x$ as $j \to \infty$, then

$$E = \{x\}, \tag{3.12.6}$$

and

$$\limsup_{j\to\infty} x_j = \liminf_{j\to\infty} x_j = x. \tag{3.12.7}$$

Proposition 3.12.8 *Put* $y = \limsup_{j\to\infty} x_j$.

(a) If $z \in \mathbf{R}$ *satisfies* $y < z$*, then* $x_j < z$ *for all but finitely many* $j \ge 1$.

(b) If $w \in \mathbf{R}$ *satisfies* $w < y$*, then* $x_j > w$ *for infinitely many* $j \ge 1$.

(c) There is only one extended real number y *that satisfies the conditions in (a) and (b).*

Suppose for the sake of a contradiction that $z \in \mathbf{R}$, $y < z$, and $x_j \ge z$ for infinitely many $j \ge 1$. Equivalently, this means that there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that

$$x_{j_l} \ge z \quad \text{for every } l \ge 1. \tag{3.12.9}$$

Using Proposition 3.11.14, we can get a subsequence $\{x_{j_{l_n}}\}_{n=1}^\infty$ of $\{x_{j_l}\}_{l=1}^\infty$ and an extended real number x such that $x_{j_{l_n}} \to x$ as $n \to \infty$. It is easy to see that $\{x_{j_{l_n}}\}_{n=1}^\infty$ may also be considered as a subsequence of $\{x_j\}_{j=1}^\infty$, so that $x \in E$. One can also check that $x \ge z$, because $x_{j_{l_n}} \ge z$ for every $n \ge 1$, by (3.12.9). Thus $x > y$, because $y < z$, by hypothesis. This contradicts the fact that $y = \sup E$, as desired.

If $w < y$, then w is not an upper bound for E, by definition of the supremum. This means that there is an $x \in E$ such that $w < x$. In this case, there is a subsequence $\{x_{j_l}\}_{l=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ such that $x_{j_l} \to x$ as $l \to \infty$. It is easy to see that $x_{j_l} > w$ for all but finitely many $l \geq 1$, because $w < x$. In particular, $x_j > w$ for infinitely many $j \geq 1$, as desired.

Suppose that for the sake of a contradiction that y' and y'' are distinct extended real numbers that satisfy the conditions in (a) and (b) in place of y. We may as well suppose that $y' < y''$, since otherwise we can interchange the roles of y' and y''. This implies that there is a real number r such that $y' < r < y''$. Using (a) for y' and $z = r$, we get that $x_j < r$ for all but finitely many $j \geq 1$. However, part (b) for y'' and $w = r$ implies that $x_j > r$ for infinitely many $j \geq 1$, which is a contradiction.

Proposition 3.12.10 *Put $u = \liminf_{j\to\infty} x_j$.*

(a) If $t \in \mathbf{R}$ satisfies $t < u$, then $x_j > t$ for all but finitely many $j \geq 1$.

(b) If $v \in \mathbf{R}$ satisfies $u < v$, then $x_j < v$ for infinitely many $j \geq 1$.

(c) There is only one extended real number u that satisfies the conditions in (a) and (b).

This can be shown in essentially the same way as before. Alternatively, one can check that

$$-\liminf_{j\to\infty} x_j = \limsup_{j\to\infty}(-x_j), \tag{3.12.11}$$

and use this to reduce to the previous proposition.

Corollary 3.12.12 *Suppose that*

$$\limsup_{j\to\infty} x_j = \liminf_{j\to\infty} x_j, \tag{3.12.13}$$

and let x be their common value. Under these conditions, $x_j \to x$ as $j \to \infty$.

This can be verified using part (a) of each of Propositions 3.12.8 and 3.12.10.

The definitions of the upper and lower limits used here correspond to Definition 3.16 on p56 of [112]. The characterization of the upper limit in Proposition 3.12.8 is analogous to Theorem 3.17 on p56 of [112]. However, instead of part (b) of Proposition 3.12.8, part (a) of Theorem 3.17 in [112] states that

$$y = \limsup_{j\to\infty} x_j \in E. \tag{3.12.14}$$

It is easy to see that this implies part (b) of Proposition 3.12.8. Conversely, it is not too difficult to show that (3.12.14) can be obtained from parts (a) and (b) of Proposition 3.12.8.

Although part (c) of Proposition 3.12.8 is stated in the same way as the last part of Theorem 3.17 in [112], its meaning is a bit different, because it uses part (b) of Proposition 3.12.8 instead of (3.12.14).

Sometimes upper and lower limits are defined or characterized in other ways, as in Definition 1.13 on p14 of [111]. In that definition, it is mentioned that the upper limit satisfies (3.12.14) and is the largest extended real number with this property, which means that the definition in [111] is equivalent to the one used in [112]. This could also be obtained from Proposition 3.12.8.

If $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ are sequences of real numbers, then it is well known that

$$\limsup_{j \to \infty}(a_j + b_j) \le \limsup_{j \to \infty} a_j + \limsup_{j \to \infty} b_j, \tag{3.12.15}$$

as long as the right side is defined as an extended real number, which is to say that it is not a sum of $+\infty$ and $-\infty$. This corresponds to part (b) of Exercise 4 on p32 of [111], and to Exercise 5 on p78 of [112].

3.13 Applications for infinite series

Proposition 3.13.1 (The root test) *Let*

$$\sum_{j=1}^\infty a_j \tag{3.13.2}$$

be an infinite series of complex numbers, and put

$$\alpha = \limsup_{j \to \infty} |a_j|^{1/j}. \tag{3.13.3}$$

The series (3.13.2) converges absolutely when $\alpha < 1$, and does not converge when $\alpha > 1$.

Suppose first that $\alpha < 1$, and let β be a real number such that $\alpha < \beta < 1$. Using part (a) of Proposition 3.12.8, we get that

$$|a_j|^{1/j} < \beta \tag{3.13.4}$$

for all but finitely many $j \ge 1$. Equivalently, this means that

$$|a_j| < \beta^j \tag{3.13.5}$$

for all but finitely many $j \ge 1$. It is easy to see that $\sum_{j=1}^\infty |a_j|$ converges in this case, in comparison with the convergent geometric series $\sum_{j=1}^\infty \beta^j$.

Now suppose that $\alpha > 1$, and let γ be a real number such that $1 \le \gamma < \alpha$. Part (b) of Proposition 3.12.8 implies that

$$|a_j|^{1/j} > \gamma \tag{3.13.6}$$

for infinitely many $j \ge 1$. This is the same as saying that

$$|a_j| > \gamma^j \tag{3.13.7}$$

for infinitely many $j \ge 1$. If we take $\gamma = 1$, then we get that $\{a_j\}_{j=1}^\infty$ does not converge to 0 as a sequence of complex numbers, so that $\sum_{j=1}^\infty a_j$ does not converge. We can also take $\gamma > 1$, to obtain that $\{a_j\}_{j=1}^\infty$ is not even bounded.

If $a_j = 1$ for every $j \ge 1$, then $\alpha = 1$, and $\sum_{j=1}^\infty a_j$ does not converge. If $a_j = 1/j^2$ for every $j \ge 1$, then $\sum_{j=1}^\infty a_j$ converges, and $a_j^{1/j} = (j^{1/j})^{-2} \to 1$ as $j \to \infty$, so that $\alpha = 1$.

As an application of the root test, let

$$\sum_{j=0}^{\infty} a_j\, z^j \tag{3.13.8}$$

be a power series with complex coefficients, which obviously converges absolutely when $z = 0$. Also let α be as in (3.13.3) again, which can be defined without using the $j = 0$ term. If z is a nonzero complex number, then one can check that

$$\limsup_{j \to \infty} |a_j\, z^j|^{1/j} = |z|\, \alpha. \tag{3.13.9}$$

The root test implies that (3.13.8) converges absolutely when $|z|\alpha < 1$, and that (3.13.8) does not converge when $|z|\alpha > 1$. It follows that

$$\rho = 1/\alpha \tag{3.13.10}$$

is the radius of convergence of (3.13.8), which is interpreted as being $+\infty$ when $\alpha = 0$.

Proposition 3.13.11 (The ratio test) *An infinite series $\sum_{j=1}^\infty a_j$ of nonzero complex numbers converges absolutely when*

$$\limsup_{j \to \infty} (|a_{j+1}|/|a_j|) < 1, \tag{3.13.12}$$

and does not converge when

$$\liminf_{j \to \infty} (|a_{j+1}|/|a_j|) > 1. \tag{3.13.13}$$

If (3.13.12) holds, then let β be a real number such that

$$\limsup_{j \to \infty} (|a_{j+1}|/|a_j|) < \beta < 1. \tag{3.13.14}$$

Part (a) of Proposition 3.12.8 implies that there is a positive integer L_1 such that

$$|a_{j+1}|/|a_j| < \beta \tag{3.13.15}$$

for every $j \ge L_1$. This means that

$$|a_{j+1}| < \beta\, |a_j| \tag{3.13.16}$$

for every $j \ge L_1$, so that

$$|a_j| \le \beta^{j - L_1}\, |a_{L_1}| \tag{3.13.17}$$

for every $j \ge L_1$. Using this, the convergence of $\sum_{j=1}^\infty |a_j|$ can be obtained from the convergence of the geometric series $\sum_{j=1}^\infty \beta^j$.

If (3.13.13) holds, then let γ be a real number such that

$$\liminf_{j\to\infty}(|a_{j+1}|/|a_j|) > \gamma \geq 1. \tag{3.13.18}$$

Part (a) of Proposition 3.12.10 implies that there is a positive integer L_2 such that

$$|a_{j+1}|/|a_j| > \gamma \tag{3.13.19}$$

for every $j \geq L_2$. Thus

$$|a_{j+1}| > \gamma\,|a_j| \tag{3.13.20}$$

for every $j \geq L_2$, and hence

$$|a_j| \geq \gamma^{j-L_2}\,|a_{L_2}| \tag{3.13.21}$$

for every $j \geq L_2$. In particular, this implies that $\{a_j\}_{j=1}^\infty$ does not converge to 0 when $\gamma = 1$, so that $\sum_{j=1}^\infty a_j$ does not converge. If we take $\gamma > 1$, then we get that $|a_j| \to \infty$ as $j \to \infty$.

3.14 Rearrangements of infinite series

Let π be a one-to-one mapping from the set of positive integers onto itself. Thus $\pi(j) \in \mathbf{Z}_+$ for every $j \in \mathbf{Z}_+$, and every positive integer k can be expressed as $\pi(j)$ for exactly one $j \in \mathbf{Z}_+$, which may be denoted $\pi^{-1}(k)$. If $\sum_{j=1}^\infty a_j$ is an infinite series of complex numbers, then $\sum_{j=1}^\infty a_{\pi(j)}$ is called a *rearrangement* of $\sum_{j=1}^\infty a_j$. If $a_j = 0$ for all but finitely many $j \geq 1$, then it is easy to see that $a_{\pi(j)} = 0$ for all but finitely many $j \geq 1$ too, and that

$$\sum_{j=1}^\infty a_{\pi(j)} = \sum_{j=1}^\infty a_j. \tag{3.14.1}$$

Suppose for the moment that a_j is a nonnegative real number for each $j \geq 1$. If l, n are positive integers such that

$$\max\{\pi(j) : 1 \leq j \leq l\} \leq n, \tag{3.14.2}$$

then

$$\sum_{j=1}^l a_{\pi(j)} \leq \sum_{j=1}^n a_j. \tag{3.14.3}$$

Similarly, if

$$\max\{\pi^{-1}(j) : 1 \leq j \leq l\} \leq n, \tag{3.14.4}$$

then

$$\sum_{j=1}^l a_j \leq \sum_{j=1}^n a_{\pi(j)}. \tag{3.14.5}$$

If $\sum_{j=1}^\infty a_j$ converges, then one can use (3.14.3) and (3.14.5) to check that $\sum_{j=1}^\infty a_{\pi(j)}$ converges, and that (3.14.1) holds.

If $\sum_{j=1}^\infty a_j$ is an absolutely convergent series of complex numbers, then it is easy to see that $\sum_{j=1}^\infty a_{\pi(j)}$ is absolutely convergent as well, by applying the remarks in the preceding paragraph to $\sum_{j=1}^\infty |a_j|$. In order to show that (3.14.1) holds in this situation, suppose first that a_j is a real number for each $j \geq 1$. In this case, one can express $\sum_{j=1}^\infty a_j$ as a difference of convergent series of nonnegative real numbers, to which the previous remarks can be applied. If the a_j's are complex numbers, then one can reduce to the real case, by considering the real and imaginary parts.

3.15 Cauchy products of infinite series

Let $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ be infinite series of complex numbers. Put

$$c_n = \sum_{j=0}^n a_j\, b_{n-j} \tag{3.15.1}$$

for every nonnegative integer n. The corresponding infinite series $\sum_{n=0}^\infty c_n$ is called the *Cauchy product* of $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$. If $a_j = 0$ for all but finitely many $j \geq 0$, and $b_l = 0$ for all but finitely many $l \geq 0$, then one can verify that $c_n = 0$ for all but finitely many $n \geq 0$, and that

$$\sum_{n=0}^\infty c_n = \Big(\sum_{j=0}^\infty a_j\Big)\Big(\sum_{l=0}^\infty b_l\Big). \tag{3.15.2}$$

Suppose for the moment that a_j is a nonnegative real number for every $j \geq 0$, and that b_l is a nonnegative real number for every $l \geq 0$, so that c_n is a nonnegative real number for every $n \geq 0$. If N is a nonnegative integer, then it is easy to see that

$$\sum_{n=0}^N c_n \leq \Big(\sum_{j=0}^N a_j\Big)\Big(\sum_{l=0}^N b_l\Big). \tag{3.15.3}$$

Similarly, if N_1 and N_2 are nonnegative integers, then one can check that

$$\Big(\sum_{j=0}^{N_1} a_j\Big)\Big(\sum_{l=0}^{N_2} b_l\Big) \leq \sum_{n=0}^{N_1+N_2} c_n. \tag{3.15.4}$$

If $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ converge, then one can use (3.15.3) to get that $\sum_{n=0}^\infty c_n$ converges, with

$$\sum_{n=0}^\infty c_n \leq \Big(\sum_{j=0}^\infty a_j\Big)\Big(\sum_{l=0}^\infty b_l\Big). \tag{3.15.5}$$

In this case, one can also use (3.15.4) to get that

$$\Big(\sum_{j=0}^{\infty} a_j\Big)\Big(\sum_{l=0}^{\infty} b_l\Big) \le \sum_{n=0}^{\infty} c_n, \tag{3.15.6}$$

so that (3.15.2) holds.

If the a_j's and b_l's are complex numbers, then

$$|c_n| \le \sum_{j=0}^{n} |a_j|\,|b_{n-j}| \tag{3.15.7}$$

for every $n \ge 0$. The sum on the right is the same as the nth term of the Cauchy product of $\sum_{j=0}^{\infty} |a_j|$ and $\sum_{l=0}^{\infty} |b_l|$. If these two series converge, then their Cauchy product converges too, as in the preceding paragraph. Under these conditions, $\sum_{n=0}^{\infty} |c_n|$ converges as well, by the comparison test. More precisely, we have that

$$\sum_{n=0}^{\infty} |c_n| \le \Big(\sum_{j=0}^{\infty} |a_j|\Big)\Big(\sum_{l=0}^{\infty} |b_l|\Big). \tag{3.15.8}$$

Thus $\sum_{n=0}^{\infty} c_n$ converges absolutely when $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ converge absolutely. One can check that (3.15.2) holds in this situation, by reducing to the case of convergent series of nonnegative real numbers, as in the previous section.

Chapter 4

Continuous mappings

4.1 Continuity at a point

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces, and let f be a function defined on M with values in N. This is also known as a *mapping* from M into N, which may be expressed by $f : M \to N$.

Definition 4.1.1 *We say that f is* continuous *at a point $x \in M$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\rho(f(x), f(y)) < \epsilon \tag{4.1.2}$$

for every $y \in M$ with $d(x, y) < \delta$.

Proposition 4.1.3 *A mapping f from M into N is continuous at $x \in M$ if and only if for every sequence $\{x_j\}_{j=1}^{\infty}$ of elements of M that converges to x, $\{f(x_j)\}_{j=1}^{\infty}$ converges to $f(x)$ in N.*

Suppose first that f is continuous at x, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of M that converges to x. Also let $\epsilon > 0$ be given, and let $\delta > 0$ be as in Definition 4.1.1. Because $\{x_j\}_{j=1}^{\infty}$ converges to x in M, there is a positive integer L such that

$$d(x, x_j) < \delta \tag{4.1.4}$$

for every $j \geq L$. This implies that

$$\rho(f(x), f(x_j)) < \epsilon \tag{4.1.5}$$

for every $j \geq L$, by (4.1.2), as desired.

Conversely, we would like to show that f is continuous at x when it has this property in terms of sequences. Let $\epsilon > 0$ be given, and suppose for the sake of a contradiction that there is no $\delta > 0$ as in Definition 4.1.1. This means that for every $\delta > 0$ there is a point $x(\delta) \in M$ such that $d(x, x(\delta)) < \delta$ and

$$\rho(f(x), f(x(\delta))) \geq \epsilon. \tag{4.1.6}$$

If j is a positive integer, then we can apply this with $\delta = 1/j$, to get a point $x_j \in M$ such that

$$d(x, x_j) < 1/j \tag{4.1.7}$$

and

$$\rho(f(x), f(x_j)) \geq \epsilon. \tag{4.1.8}$$

This leads to a sequence $\{x_j\}_{j=1}^\infty$ of elements of M that converges to x in M, and for which $\{f(x_j)\}_{j=1}^\infty$ does not converge to $f(x)$ in N, as desired.

Proposition 4.1.9 *Let $(M, d(\cdot,\cdot))$ be a metric space, and let f, g be complex-valued functions on M. Suppose that f and g are continuous at a point $x \in M$, with respect to the standard metric on $\mathbf{C}$. Under these conditions, $f + g$ and $f\,g$ are continuous at x as well. If, for every $w \in M$, $f(w) \neq 0$, then $1/f$ is continuous at x too.*

This follows from Proposition 4.1.3, and previous results about convergent sequences of complex numbers.

Definition 4.1.10 *Let $(M, d(\cdot,\cdot))$ and $(N, \rho(\cdot,\cdot))$ be metric spaces. A mapping f from M into N is said to be* continuous *on M if f is continuous at every point $x \in M$.*

One can use Proposition 4.1.9 to show that

$$\text{polynomial functions on } \mathbf{R}^n \text{ are continuous,} \tag{4.1.11}$$

with respect to the standard Euclidean metric on $\mathbf{R}^n$. Similarly,

$$\text{rational functions are continuous,} \tag{4.1.12}$$

when the denominator is not zero.

If z and w are complex numbers, then one can check that

$$\big||z| - |w|\big| \leq |z - w|, \tag{4.1.13}$$

using the triangle inequality. One can use this to show that the absolute value defines a continuous real-valued function on the complex plane, with respect to the standard Euclidean metrics on $\mathbf{R}$ and $\mathbf{C}$.

4.2 Compositions and inverse images

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces.

Proposition 4.2.1 *A mapping f from M into N is continuous if and only if for every open set $V \subseteq N$, its* inverse image

$$f^{-1}(V) = \{x \in M : f(x) \in V\} \tag{4.2.2}$$

is an open set in M.

Suppose that f is continuous, and let an open set $V \subseteq N$ be given. Let $x \in f^{-1}(V)$ be given, so that $f(x) \in V$. Because V is an open set in N, there is an $\epsilon > 0$ such that

$$\{z \in N : \rho(f(x), z) < \epsilon\} \subseteq V. \tag{4.2.3}$$

Using the continuity of f at x, we get that there is a $\delta > 0$ such that if $y \in M$ satisfies $d(x, y) < \delta$, then $\rho(f(x), f(y)) < \epsilon$, and hence $f(y) \in V$. This shows that

$$\{y \in M : d(x, y) < \delta\} \subseteq f^{-1}(V), \tag{4.2.4}$$

as desired.

Conversely, suppose that f has the property described in the proposition, and let us show that f is continuous on M. Let $x \in M$ be given, and let us show that f is continuous at x. To do this, let $\epsilon > 0$ be given, and note that

$$V = \{z \in N : \rho(f(x), z) < \epsilon\} \tag{4.2.5}$$

is an open set in N, because it is an open ball. Thus $f^{-1}(V)$ is an open set in M, by hypothesis. Of course, $f(x) \in V$, by construction, so that $x \in f^{-1}(V)$. It follows that there is a $\delta > 0$ such that (4.2.4) holds, by the definition of an open set. If $y \in M$ satisfies $d(x, y) < \delta$, then we get that $y \in f^{-1}(V)$, so that $f(y) \in V$, as desired.

Corollary 4.2.6 *A mapping f from M into N is continuous if and only if for every closed set $E \subseteq N$, $f^{-1}(E)$ is a closed set in M.*

It is easy to see that

$$f^{-1}(N \setminus E) = M \setminus f^{-1}(E) \tag{4.2.7}$$

for every $E \subseteq N$. Using this, one can check that the condition in Corollary 4.2.6 if equivalent to the one in Proposition 4.2.1, because a subset of a metric space is a closed set if and only if its complement is an open set.

Proposition 4.2.8 *Let $(M_1, d_1(\cdot, \cdot))$, $(M_2, d_2(\cdot, \cdot))$, and $(M_3, d_3(\cdot, \cdot))$ be metric spaces, let f be a continuous mapping from M_1 into M_2, and let g be a continuous mapping from M_2 into M_3. Under these conditions, the* composition $g \circ f$ *of f and g is continuous as a mapping from M_1 into M_3.*

More precisely, $g \circ f$ is defined by putting $(g \circ f)(w) = g(f(w))$ for every $w \in M_1$.

To prove the proposition, let $x \in M_1$ be given, and let us check that $g \circ f$ is continuous at x. Let $\epsilon > 0$ be given, and let us use the continuity of g at $f(x)$ to get that there is an $\eta > 0$ such that

$$d_3(g(f(x)), g(u)) < \epsilon \tag{4.2.9}$$

for every $u \in M_2$ with $d_2(f(x), u) < \eta$. The continuity of f at x implies that there is a $\delta > 0$ such that

$$d_2(f(x), f(y)) < \eta \tag{4.2.10}$$

for every $y \in M_1$ with $d_1(x, y) < \delta$. In this case, we can take $u = f(y)$ in (4.2.9), to get that

$$d_3(g(f(x)), g(f(y))) < \epsilon, \tag{4.2.11}$$

as desired.

Alternatively, if $\{x_j\}_{j=1}^{\infty}$ is any sequence of elements of M_1 that converges to x, then

$$\{f(x_j)\}_{j=1}^{\infty} \text{ converges to } f(x) \tag{4.2.12}$$

in M_2, because f is continuous at x. This implies that

$$\{g(f(x_j))\}_{j=1}^{\infty} \text{ converges to } g(f(x)) \tag{4.2.13}$$

in M_3, because g is continuous at $f(x)$, as desired.

We can also use the characterization of continuity in Proposition 4.2.1. Let W be an open subset of M_3, so that

$$g^{-1}(W) \text{ is an open subset of } M_2, \tag{4.2.14}$$

by the continuity of g. This implies that

$$f^{-1}(g^{-1}(W)) \text{ is an open set in } M_1, \tag{4.2.15}$$

by the continuity of f. It is easy to see that

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)), \tag{4.2.16}$$

directly from the definitions. Thus

$$(g \circ f)^{-1}(W) \text{ is an open set in } M_1, \tag{4.2.17}$$

as desired.

4.3 Images of compact sets

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces.

Theorem 4.3.1 *If f is a continuous mapping from M into N and K is a compact subset of M, then*

$$f(K) = \{f(x) : x \in K\} \tag{4.3.2}$$

is a compact subset of N.

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $f(K)$ in N. Thus U_α is an open subset of N for every $\alpha \in A$, so that $f^{-1}(U_\alpha)$ is an open set in M for every $\alpha \in A$, by Proposition 4.2.1. One can check that

$$K \subseteq \bigcup_{\alpha \in A} f^{-1}(U_\alpha), \tag{4.3.3}$$

because $f(K) \subseteq \bigcup_{\alpha \in A} U_\alpha$, by hypothesis. This shows that $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open covering of K in M. If K is compact in M, then it follows that there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j}). \tag{4.3.4}$$

Using this, one can verify that

$$f(K) \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}, \tag{4.3.5}$$

as desired.

Alternatively, suppose that K is sequentially compact in M, and let us show that

$$f(K) \text{ is sequentially compact in } N. \tag{4.3.6}$$

Let $\{z_j\}_{j=1}^\infty$ be an arbitrary sequence of elements of $f(K)$. If j is any positive integer, then let us choose $x_j \in K$ such that

$$f(x_j) = z_j. \tag{4.3.7}$$

Because K is sequentially compact, there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to an element x of K in M. Thus

$$\{z_{j_l}\}_{l=1}^\infty = \{f(x_{j_l})\}_{l=1}^\infty \tag{4.3.8}$$

is a subsequence of $\{z_j\}_{j=1}^\infty$ that converges to $f(x) \in f(K)$ in N, because f is continuous at x, by hypothesis.

Corollary 4.3.9 (Extreme value theorem) *Suppose that f is a continuous real-valued function on M, with respect to the standard Euclidean metric on $\mathbf{R}$. If K is a nonempty compact subset of M, then there are points $p, q \in K$ such that*

$$f(p) \le f(x) \le f(q) \tag{4.3.10}$$

for every $x \in M$.

Theorem 4.3.1 implies that $f(K)$ is a compact subset of the real line, so that $f(K)$ is closed and bounded. Of course, $f(K) \neq \emptyset$, because $K \neq \emptyset$, by hypothesis, so that the supremum and infimum of $f(K)$ exist in $\mathbf{R}$. In this situation, the supremum and infimum of $f(K)$ are elements of $f(K)$, because $f(K)$ is a closed set in $\mathbf{R}$, as in Proposition 2.8.8. This is the same as saying that the maximum and minimum of f are attained on K, as desired.

4.4 Uniform continuity

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces.

Definition 4.4.1 *A mapping f from M into N is said to be* uniformly continuous *on M if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\rho(f(x), f(y)) < \epsilon \tag{4.4.2}$$

for every $x, y \in M$ with $d(x, y) < \delta$.

Note that uniform continuity implies ordinary continuity. In order to get an example where the converse does not hold, one can take $M = N = \mathbf{R}$ with the standard Euclidean metric, and put $f(x) = x^2$ for every $x \in \mathbf{R}$. One can check that f is not uniformly continuous on $\mathbf{R}$, using the fact that

$$f(x) - f(y) = x^2 - y^2 = (x + y)\,(x - y) \tag{4.4.3}$$

for every $x, y \in \mathbf{R}$.

Alternatively, let us take M to be the open unit interval $(0, 1)$ in the real line, equipped with the restriction of the standard Euclidean metric on $\mathbf{R}$ to $(0, 1)$. If we put $f(x) = 1/x$ for every $x \in (0, 1)$, then one can verify that f is not uniformly continuous on $(0, 1)$.

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be arbitrary metric spaces again.

Theorem 4.4.4 *If f is a continuous mapping from M into N, and if M is compact, then f is uniformly continuous on M.*

Let $\epsilon > 0$ be given. If $x \in M$, then there is a positive real number $\delta(x)$ such that

$$\rho(f(x), f(y)) < \epsilon/2 \tag{4.4.5}$$

for every $y \in M$ with $d(x, y) < \delta(x)$. Let $B(x)$ be the open ball in M centered at x with radius $\delta(x)/2$. The collection of these open balls $B(x)$, $x \in M$, is an open covering of M, because $x \in B(x)$ for every $x \in M$, and open balls in M are open sets. If M is compact, then there are finitely many elements $x_1, \ldots, x_l$ of M such that

$$M \subseteq \bigcup_{j=1}^{l} B(x_j). \tag{4.4.6}$$

Put

$$\delta = \min\{\delta(x_j)/2 : 1 \le j \le l\}, \tag{4.4.7}$$

which is a positive real number. Let w and y be elements of M such that $d(w, y) < \delta$. Using (4.4.6), we get that there is a $j \in \{1, \ldots, l\}$ such that $w \in B(x_j)$, so that $d(x_j, w) < \delta(x_j)/2$. It follows that

$$d(x_j, y) \le d(x_j, w) + d(w, y) < \delta(x_j)/2 + \delta \le \delta(x_j). \tag{4.4.8}$$

This implies that

$$\rho(f(w), f(y)) \le \rho(f(w), f(x_j)) + \rho(f(x_j), f(y)) < \epsilon/2 + \epsilon/2 = \epsilon, \tag{4.4.9}$$

as desired.

Alternatively, let $\epsilon > 0$ be given again, and suppose for the sake of a contradiction that there is no $\delta > 0$ that satisfies the condition in the definition of uniform continuity. This means that for every $\delta > 0$ there are points $x(\delta), y(\delta) \in M$ such that $d(x(\delta), y(\delta)) < \delta$ and

$$\rho(f(x(\delta)), f(y(\delta))) \ge \epsilon. \tag{4.4.10}$$

If j is a positive integer, then we can take $\delta = 1/j$, to get points $x_j, y_j \in M$ such that $d(x_j, y_j) < 1/j$ and

$$\rho(f(x_j), f(y_j)) \ge \epsilon. \tag{4.4.11}$$

Because M is compact, and thus sequentially compact, there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to $x \in M$. Using the same sequence of indices $\{j_l\}_{l=1}^\infty$, we get a corresponding subsequence $\{y_{j_l}\}_{l=1}^\infty$ of $\{y_j\}_{j=1}^\infty$. It is easy to see that $\{y_{j_l}\}_{l=1}^\infty$ converges to x in M in this situation as well. This implies that $\{f(x_{j_l})\}_{l=1}^\infty$ and $\{f(y_{j_l})\}_{l=1}^\infty$ both converge to $f(x)$ in N, because f is continuous at x, by hypothesis. Of course,

$$\rho(f(x_{j_l}), f(y_{j_l})) \le \rho(f(x_{j_l}), f(x)) + \rho(f(x), f(y_{j_l})) \tag{4.4.12}$$

for every $l \ge 1$, by the triangle inequality. The right side of (4.4.12) is as small as we want when l is sufficiently large, because $\{f(x_{j_l})\}_{l=1}^\infty$ and $\{f(y_{j_l})\}_{l=1}^\infty$ converge to $f(x)$ in N. This contradicts (4.4.11), as desired.

4.5 Images of connected sets

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces.

Theorem 4.5.1 *If f is a continuous mapping from M into N, and if E is a connected subset of M, then $f(E)$ is connected as a subset of N.*

Suppose for the sake of a contradiction that $f(E)$ is not connected in N. This means that there are nonempty separated subsets A, B of N such that $f(E) = A \cup B$. Put

$$A_1 = f^{-1}(A) \cap E, \quad B_1 = f^{-1}(B) \cap E. \tag{4.5.2}$$

It is easy to see that $A_1 \cup B_1 = E$, by construction. We also have that $A_1, B_1 \ne \emptyset$, because A and B are nonempty subsets of $f(E)$.

We would like to check that A_1 and B_1 are separated in M. Observe that $A_1 \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{A})$, where $\overline{A}$ is the closure of A in N. Remember that $\overline{A}$ is a closed set in N, so that $f^{-1}(\overline{A})$ is a closed set in M, as in Corollary 4.2.6.

Using this, one can check that the closure $\overline{A_1}$ of A_1 in M is contained in $f^{-1}(\overline{A})$. This implies that

$$\overline{A_1} \cap B_1 \subseteq f^{-1}(\overline{A}) \cap f^{-1}(B). \tag{4.5.3}$$

However, $f^{-1}(\overline{A}) \cap f^{-1}(B) = f^{-1}(\overline{A} \cap B) = f^{-1}(\emptyset) = \emptyset$, because A and B are separated in N. It follows that $\overline{A_1} \cap B_1 = \emptyset$. One can show that $A_1 \cap \overline{B_1} = \emptyset$ in the same way, so that A_1 and B_1 are separated in M. This means that E is not connected in M, as desired.

Corollary 4.5.4 (Intermediate value theorem) *Let a and b be real numbers with $a < b$, and let f be a continuous real-valued function on the closed interval $[a, b]$, with respect to the standard Euclidean metric on $\mathbf{R}$ and its restriction to $[a, b]$. If $t \in \mathbf{R}$ satisfies $f(a) < t < f(b)$ or $f(b) < t < f(a)$, then there is an $x \in (a, b)$ such that $f(x) = t$.*

Let us first extend f to a real-valued function on the real line, by putting $f(y) = f(a)$ when $y \in \mathbf{R}$ satisfies $y < a$, and $f(z) = f(b)$ when $z \in \mathbf{R}$ satisfies $z > b$. It is easy to see that this extension is continuous on $\mathbf{R}$. Remember that $[a, b]$ is connected as a subset of the real line, as in Theorem 2.8.4. It follows that $f([a, b])$ is connected in $\mathbf{R}$ as well, by the previous theorem. If t is as in the statement of the corollary, then we get that $t \in f([a, b])$, using Theorem 2.8.4 again.

4.6 Uniform convergence

Let M be a set, and let $(N, \rho(u, v))$ be a metric space. Also let $\{f_j\}_{j=1}^\infty$ be a sequence of mappings from M into N, and let f be another mapping from M into N.

Definition 4.6.1 *We say that $\{f_j\}_{j=1}^\infty$ converges to f* pointwise *on M if for every $x \in M$, $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ in N. We say that $\{f_j\}_{j=1}^\infty$ converges to f* uniformly *on M if for every $\epsilon > 0$ there is a positive integer L such that*

$$\rho(f_j(x), f(x)) < \epsilon \tag{4.6.2}$$

for every $x \in M$ and $j \geq L$.

It is easy to see that uniform convergence implies pointwise convergence. To get an example where the converse does not hold, let us take M to be the closed unit interval $[0, 1]$, $N = \mathbf{R}$ with the standard Euclidean metric, and $f_j(x) = x^j$ for every $x \in [0, 1]$ and $j \geq 1$. We have seen that $\{f_j\}_{j=1}^\infty$ converges pointwise to the real-valued function f defined on $[0, 1]$ by putting $f(x) = 0$ when $0 \leq x < 1$, and $f(1) = 1$. However, one can check that $\{f_j\}_{j=1}^\infty$ does not converge to f uniformly on $[0, 1]$.

Theorem 4.6.3 *Let $(M, d(\cdot, \cdot))$ and $(N, \rho(\cdot, \cdot))$ be metric spaces, and let*

$$\{f_j\}_{j=1}^\infty \tag{4.6.4}$$

be a sequence of mappings from M into N that converges uniformly to a mapping f from M into N. If $x \in M$ and f_j is continuous at x for every $j \geq 1$, then f is continuous at x too.

Let $\epsilon > 0$ be given. Because $\{f_j\}_{j=1}^\infty$ converges to f uniformly on M, there is a positive integer L such that

$$\rho(f_j(w), f(w)) < \epsilon/3 \tag{4.6.5}$$

for every $w \in M$ and $j \geq L$. By hypothesis, f_L is continuous at x, and so there is a $\delta_L > 0$ such that

$$\rho(f_L(x), f_L(y)) < \epsilon/3 \tag{4.6.6}$$

for every $y \in M$ with $d(x, y) < \delta_L$. It follows that

$$\begin{aligned} \rho(f(x), f(y)) &\leq \rho(f(x), f_L(x)) + \rho(f_L(x), f_L(y)) + \rho(f_L(y), f(y)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned} \tag{4.6.7}$$

for every $y \in M$ with $d(x, y) < \delta_L$, as desired.

The following criterion for uniform convergence is due to Weierstrass.

Proposition 4.6.8 *Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of complex-valued functions on a set M, and let $A_1, A_2, A_3, \ldots$ be an infinite sequence of nonnegative real numbers. Suppose that*

$$|a_j(x)| \leq A_j \tag{4.6.9}$$

for every $x \in M$ and $j \geq 1$, and that $\sum_{j=1}^\infty A_j$ converges. Note that $\sum_{j=1}^\infty a_j(x)$ converges absolutely for every $x \in M$, by the comparison test. Under these conditions, the sequence of partial sums $\sum_{j=1}^n a_j(x)$ converges to $\sum_{j=1}^\infty a_j(x)$ uniformly on M as $n \to \infty$, with respect to the standard metric on $\mathbf{C}$.

If $x \in M$ and n is a positive integer, then one can verify that

$$\left|\sum_{j=1}^\infty a_j(x) - \sum_{j=1}^n a_j(x)\right| = \left|\sum_{j=n+1}^\infty a_j(x)\right| \leq \sum_{j=n+1}^\infty |a_j(x)| \leq \sum_{j=n+1}^\infty A_j. \tag{4.6.10}$$

Of course,

$$\sum_{j=n+1}^\infty A_j = \sum_{j=1}^\infty A_j - \sum_{j=1}^n A_j \tag{4.6.11}$$

tends to 0 as $n \to \infty$, because $\sum_{j=1}^\infty A_j$ converges. The proposition follows easily from these two statements.

4.7 Continuity of power series

Let $\sum_{j=0}^{\infty} c_j\, z^j$ be a power series with coefficients in $\mathbf{C}$.

Proposition 4.7.1 *Suppose that $\sum_{j=0}^{\infty} |c_j|\, r^j$ converges for some positive real number r. Consider the complex-valued function defined on the closed disk*

$$\{z \in \mathbf{C} : |z| \le r\} \tag{4.7.2}$$

by

$$f(z) = \sum_{j=0}^{\infty} c_j\, z^j, \tag{4.7.3}$$

where the series converges absolutely by the comparison test. Under these conditions, f is continuous on (4.7.2), with respect to the standard metric on $\mathbf{C}$ and its restriction to (4.7.2).

If j is a nonnegative integer, then

$$|c_j\, z^j| = |c_j|\, |z|^j \le |c_j|\, r^j \tag{4.7.4}$$

on (4.7.2). Thus the sequence of partial sums

$$\sum_{j=0}^{n} c_j z^j \tag{4.7.5}$$

converges to $f(z)$ uniformly on (4.7.2), by Weierstrass' criterion. We also have that (4.7.5) is continuous on the complex plane for every $n \ge 0$, and in particular the restrictions of these functions to (4.7.2) are continuous. The continuity of f on (4.7.2) now follows from Theorem 4.6.3.

Proposition 4.7.6 *Let R be a positive extended real number, and suppose that $\sum_{j=0}^{\infty} |c_j|\, r^j$ converges for every positive real number r with $r < R$. Consider the complex-valued function defined on*

$$\{z \in \mathbf{C} : |z| < R\} \tag{4.7.7}$$

by (4.7.3), where the series converges absolutely by the comparison test. Under these conditions, f is continuous on (4.7.7), with respect to the standard metric on $\mathbf{C}$, and its restriction to (4.7.7).

Let a point $z_0 \in \mathbf{C}$ with $|z_0| < R$ be given, and let us check that f is continuous at z_0, as a complex-valued function defined on (4.7.7). The restriction of f to (4.7.2) is continuous when $0 < r < R$, as in the previous proposition. If $|z_0| < r < R$, then one can check that the continuity of f at z_0 as a function on (4.7.2) implies the continuity of f at z_0 as a function on (4.7.7).

4.8 The supremum metric

Let M be a set, and let $(N, \rho(u, v))$ be a metric space.

Definition 4.8.1 *A mapping f from M into N is said to be* bounded *if $f(M)$ is a bounded set in N.*

Now let $(M, d(x, y))$ be a metric space as well. The space of all continuous mappings from M into N may be denoted $C(M, N)$. The space of all mappings from M into N that are both bounded and continuous may be denoted $C_b(M, N)$. If M is compact, then every continuous mapping from M into N is bounded, by Theorem 4.3.1.

Suppose from now on in this section that $M \neq \emptyset$. If f and g are bounded continuous mappings from M into N, then one can check that $\rho(f(x), g(x))$ is bounded as a real-valued function on M. Put

$$\theta(f, g) = \sup\{\rho(f(x), g(x)) : x \in M\}, \tag{4.8.2}$$

which is a nonnegative real number. If $f(x) = g(x)$ for every $x \in M$, then $\rho(f(x), g(x)) = 0$ for every $x \in M$, and $\theta(f, g) = 0$. Conversely, if $\theta(f, g) = 0$, then $\rho(f(x), g(x)) = 0$ for every $x \in M$, so that $f(x) = g(x)$ for every $x \in M$.

It is easy to see that

$$\theta(f, g) = \theta(g, f), \tag{4.8.3}$$

because $\rho(\cdot, \cdot)$ is symmetric. Let h be another bounded continuous mapping from M into N, and observe that

$$\rho(f(x), h(x)) \le \rho(f(x), g(x)) + \rho(g(x), h(x)) \le \theta(f, g) + \theta(g, h) \tag{4.8.4}$$

for every $x \in M$. This implies that

$$\theta(f, h) \le \theta(f, g) + \theta(g, h). \tag{4.8.5}$$

Thus $\theta(\cdot, \cdot)$ defines a metric on $C_b(M, N)$, which is known as the *supremum metric.*

Proposition 4.8.6 *A sequence $\{f_j\}_{j=1}^{\infty}$ of bounded continuous mappings from M into N converges to $f \in C_b(M, N)$ with respect to the supremum metric if and only if $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on M.*

Suppose first that $\{f_j\}_{j=1}^{\infty}$ converges to f with respect to the supremum metric, and let $\epsilon > 0$ be given. By definition of convergence of a sequence in a metric space, there is a positive integer L such that

$$\theta(f_j, f) < \epsilon \tag{4.8.7}$$

for every $j \ge L$. If $x \in M$, then it follows that

$$\rho(f_j(x), f(x)) \le \theta(f_j, f) < \epsilon \tag{4.8.8}$$

for every $j \geq L$. This means that $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on M, as desired.

Conversely, suppose that $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on M. Thus, for each $\epsilon > 0$, there is a positive integer $L(\epsilon)$ such that

$$\rho(f_j(x), f(x)) < \epsilon \tag{4.8.9}$$

for every $x \in M$ and $j \geq L(\epsilon)$. It follows that

$$\theta(f_j, f) \leq \epsilon \tag{4.8.10}$$

for every $j \geq L(\epsilon)$, by the definition of the supremum metric. This means that

$$\theta(f_j, f) \leq \epsilon/2 < \epsilon \tag{4.8.11}$$

for every $j \geq L(\epsilon/2)$, so that $\{f_j\}_{j=1}^{\infty}$ converges to f with respect to the supremum metric.

4.9 Completeness of $C_b(M,N)$

Let $(M, d(x,y))$ and $(N, \rho(u,v))$ be nonempty metric spaces.

Theorem 4.9.1 *If N is complete as a metric space with respect to $\rho(\cdot,\cdot)$, then $C_b(M,N)$ is complete with respect to the supremum metric.*

Let $\{f_j\}_{j=1}^{\infty}$ be a Cauchy sequence of elements of $C_b(M,N)$, with respect to the supremum metric. This means that for every $\epsilon > 0$ there is a positive integer $L(\epsilon)$ such that

$$\theta(f_j, f_l) < \epsilon \tag{4.9.2}$$

for every $j, l \geq L(\epsilon)$. If $x \in M$, then we get that

$$\rho(f_j(x), f_l(x)) \leq \theta(f_j, f_l) < \epsilon \tag{4.9.3}$$

for every $j, l \geq L(\epsilon)$. This shows that $\{f_j(x)\}_{j=1}^{\infty}$ is a Cauchy sequence in N. It follows that $\{f_j(x)\}_{j=1}^{\infty}$ converges in N, because N is complete by hypothesis.

Put

$$f(x) = \lim_{j \to \infty} f_j(x) \tag{4.9.4}$$

for every $x \in M$, which defines a mapping from M into N. One can check that

$$\rho(f(x), f_l(x)) \leq \epsilon \tag{4.9.5}$$

for every $x \in M$ and $l \geq L(\epsilon)$, using (4.9.3) and (4.9.4). More precisely, this uses the fact that

$$\rho(f(x), f_l(x)) \leq \rho(f(x), f_j(x)) + \rho(f_j(x), f_l(x)) \tag{4.9.6}$$

for all $j \geq 1$, by the triangle inequality. This implies that

$$\{f_l\}_{l=1}^{\infty} \text{ converges to } f \text{ uniformly on } M. \tag{4.9.7}$$

It follows that

$$f \text{ is continuous on } M, \tag{4.9.8}$$

by Theorem 4.6.3, and because f_l is continuous on M for every $l \geq 1$, by hypothesis.

One can verify that

$$f \text{ is bounded on } M, \tag{4.9.9}$$

using (4.9.5) with $\epsilon = 1$ and $l = L(1)$, and the boundedness of $f_{L(1)}$ on M, by hypothesis. Thus $f \in C_b(M, N)$, and $\{f_l\}_{l=1}^{\infty}$ converges to f with respect to the supremum metric, because of uniform convergence, as in the previous section.

4.10 The limit of a function

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces, let E be a subset of M, and suppose that $p \in M$ is a limit point of E. Also let f be a function defined on E with values in N, and let q be an element of N.

Definition 4.10.1 *We say that the* limit of $f(x)$, as $x \in E$ approaches p, is equal to q, *if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\rho(f(x), q) < \epsilon \tag{4.10.2}$$

for every $x \in E$ with $d(p, x) < \delta$ and $x \neq p$.

In this case, we put

$$\lim_{\substack{x \in E \\ x \to p}} f(x) = q, \tag{4.10.3}$$

or simply

$$\lim_{x \to p} f(x) = q, \tag{4.10.4}$$

when $E = M$, or it is clear which set E being used.

Proposition 4.10.5 *Under the conditions just mentioned, (4.10.3) holds if and only if for every sequence $\{x_j\}_{j=1}^{\infty}$ of elements of E that converges to p in M and satisfies $x_j \neq p$ for each $j \geq 1$, we have that $\{f(x_j)\}_{j=1}^{\infty}$ converges to q in N.*

This is analogous to Proposition 4.1.3, and we omit the details. One can use this to get the uniqueness of the limit, when it exists, from the corresponding statement for convergent sequences.

Proposition 4.10.6 *Let f and g be complex-valued functions on E, and suppose that the limits of $f(x)$ and $g(x)$, as $x \in E$ approaches p, exist, with respect to the standard metric on $\mathbf{C}$. Under these conditions,*

$$\lim_{\substack{x \in E \\ x \to p}} (f(x) + g(x)) = \lim_{\substack{x \in E \\ x \to p}} f(x) + \lim_{\substack{x \in E \\ x \to p}} g(x) \tag{4.10.7}$$

and

$$\lim_{\substack{x \in E \\ x \to p}} (f(x)\, g(x)) = \Big(\lim_{\substack{x \in E \\ x \to p}} f(x)\Big) \Big(\lim_{\substack{x \in E \\ x \to p}} g(x)\Big). \tag{4.10.8}$$

We also have that

$$\lim_{\substack{x \in E \\ x \to p}} 1/f(x) = \Big(\lim_{\substack{x \in E \\ x \to p}} f(x)\Big)^{-1} \tag{4.10.9}$$

when $f(x) \neq 0$ for every $x \in E$, and the limit on the right is not zero.

This follows from the previous proposition, and the corresponding statements for convergent sequences of complex numbers.

Let f be a mapping from M into a metric space N again. If $p \in M$ is a limit point of M, then it is easy to see that f is continuous at p if and only if

$$\lim_{x \to p} f(x) = f(p). \tag{4.10.10}$$

If p is not a limit point of M, then one can check that f is automatically continuous at p.

4.11 One-sided limits

Let a and b be real numbers with $a < b$, and let $(N, \rho(u, v))$ be a metric space. Also let f be a function defined on the open interval (a, b) in the real line with values in N. If $p \in \mathbf{R}$ satisfies $a \le p < b$, then put

$$E_+(p) = \{x \in \mathbf{R} : p < x < b\}. \tag{4.11.1}$$

The *limit of $f(x)$ as $x \in (a, b)$ approaches p from the right* is defined by

$$f(p+) = \lim_{x \to p+} f(x) = \lim_{\substack{x \in E_+(p) \\ x \to p}} f(x), \tag{4.11.2}$$

when the limit on the right side exists, with respect to the standard Euclidean metric on $\mathbf{R}$. Similarly, if $p \in \mathbf{R}$ satisfies $a < p \le b$, then put

$$E_-(p) = \{x \in \mathbf{R} : a < x < p\}. \tag{4.11.3}$$

The *limit of $f(x)$ as $x \in (a, b)$ approaches p from the left* is defined by

$$f(p-) = \lim_{x \to p-} f(x) = \lim_{\substack{x \in E_-(p) \\ x \to p}} f(x), \tag{4.11.4}$$

when the limit on the right side exists. If $p \in (a, b)$, then it is easy to see that

$$\lim_{x \to p} f(x) \tag{4.11.5}$$

exists if and only if the one-sided limits (4.11.2) and (4.11.4) exist and are equal, in which case (4.11.5) is equal to the common value of the one-sided limits.

Definition 4.11.6 *A real-valued function f on (a, b) is said to be* monotonically increasing *on (a, b) if for every $x, y \in (a, b)$ with $x < y$, we have that*

$$f(x) \le f(y). \tag{4.11.7}$$

Proposition 4.11.8 *Let f be a monotonically increasing real-valued function on (a, b). If $p \in (a, b)$, then the one-sided limits of f at p exist, with respect to the standard Euclidean metric on* $\mathbf{R}$*, and satisfy*

$$f(p-) \le f(p) \le f(p+). \tag{4.11.9}$$

If q is another element of (a, b), and $p < q$, then

$$f(p+) \le f(q-). \tag{4.11.10}$$

Let $p \in (a, b)$ be given, and put

$$A_+(p) = \{f(x) : p < x < b\} \tag{4.11.11}$$

and

$$A_-(p) = \{f(x) : a < x < p\}. \tag{4.11.12}$$

Note that $f(p)$ is a lower bound for $A_+(p)$, and an upper bound for $A_-(p)$, because f is monotonically increasing on (a, b). In this situation, the one-sided limits of f at p are given by

$$f(p+) = \inf A_+(p) \tag{4.11.13}$$

and

$$f(p-) = \sup A_-(p). \tag{4.11.14}$$

To prove (4.11.13), let $\epsilon > 0$ be given. Observe that there is a point $x(\epsilon)$ in (p, b) such that

$$f(x(\epsilon)) < \inf A_+(p) + \epsilon, \tag{4.11.15}$$

by definition of the infimum. If $p < x \le x(\epsilon)$, then

$$\inf A_+(p) \le f(x) \le f(x(\epsilon)) < \inf A_+(p) + \epsilon, \tag{4.11.16}$$

using the monotonicity of f in the second step. This implies (4.11.13), and the argument for (4.11.14) is analogous.

It is easy to obtain (4.11.9) from (4.11.13) and (4.11.14). If $p < q < b$, then let x be an element of (p, q). Using (4.11.13) and the analogue of (4.11.14) for q, we get that

$$f(p+) \le f(x) \le f(q-). \tag{4.11.17}$$

More precisely, this also uses the fact that $f(x)$ is an element of $A_+(p)$ and $A_-(q)$. This implies (4.11.10).

Theorem 4.11.18 *If f is a monotonically increasing real-valued function on (a, b), then f is continuous at all but finitely or countably many points in (a, b), with respect to the standard Euclidean metric on $\mathbf{R}$ and its restriction to (a, b).*

Because the one-sided limits of f exist at every point in (a, b), f is continuous at $p \in (a, b)$ if and only if

$$f(p+) = f(p-) = f(p). \tag{4.11.19}$$

It follows that f is not continuous at p exactly when

$$f(p-) < f(p+), \tag{4.11.20}$$

by (4.11.9). In this case, we can choose a rational number $r(p) \in (f(p-), f(p+))$. If f is not continuous at $q \in (a, b)$ and $p < q$, then

$$r(p) < f(p+) \le f(q-) < r(q), \tag{4.11.21}$$

by (4.11.10). Thus we get a one-to-one correspondence between the set of discontinuities of f in (a, b) and a set of rational numbers, which implies the theorem.

4.12 Prescribing jump discontinuities

Consider the real-valued function defined on the real line by

$$\begin{aligned} I(x) &= 0 \quad \text{when } x \le 0 \\ &= 1 \quad \text{when } x > 0. \end{aligned} \tag{4.12.1}$$

This may be called the *unit step function* on $\mathbf{R}$, as in Definition 6.14 on p129 of [112]. Note that $I(x)$ is monotonically increasing on $\mathbf{R}$, and that it is discontinuous at 0.

If $u \in \mathbf{R}$, then

$$\begin{aligned} I(x-u) &= 0 \quad \text{when } x \le u \\ &= 1 \quad \text{when } x > u \end{aligned} \tag{4.12.2}$$

is monotonically increasing on $\mathbf{R}$, and discontinuous at u. One can use linear combinations of these functions with positive coefficients to get monotonically increasing functions on $\mathbf{R}$ with discontinuities at any finite set of points.

Let $\{t_j\}_{j=1}^\infty$ be an infinite sequence of real numbers, and let $\{c_j\}_{j=1}^\infty$ be an infinite sequence of complex numbers such that

$$\sum_{j=1}^\infty |c_j| \tag{4.12.3}$$

converges. Put

$$f(x) = \sum_{j=1}^\infty c_j \, I(x - t_j) \tag{4.12.4}$$

for each $x \in \mathbf{R}$, where the series on the right converges absolutely, by the comparison test. If c_j is a nonnegative real number for each j, then it is easy to see that f is a monotonically increasing real-valued function on $\mathbf{R}$.

The sequence of partial sums

$$\sum_{j=1}^{n} c_j \, I(x - t_j) \tag{4.12.5}$$

converges to f uniformly on $\mathbf{R}$, by Weierstrass' criterion, as in Section 4.6. If $x \in \mathbf{R}$ and $x \neq t_j$ for each j, then it follows that

$$f \text{ is continuous at } x, \tag{4.12.6}$$

as before. If the t_j's are distinct elements of $\mathbf{R}$, and $c_j \neq 0$ for each j, then one can check that f is discontinuous at t_l for each l. More precisely, the one-sided limits of f at t_l exist for each l, and satisfy

$$f(t_l+) - f(t_l-) = c_l \tag{4.12.7}$$

in this case.

This type of construction is described another way in Remark 4.31 on p97 of [112]. This description is mentioned in Theorem 6.16 on p130 of [112], and some of its properties are indicated in Exercise 8 on p166 of [112].

4.13 Path connectedness

Let $(M, d(x, y))$ be a metric space.

Proposition 4.13.1 *Let a and b be real numbers with $a < b$, and let p be a continuous mapping from the closed interval $[a, b]$ into M, with respect to the restriction of the standard Euclidean metric on $\mathbf{R}$ to $[a, b]$. Under these conditions, $p([a, b])$ is a connected subset of M.*

As in an earlier proof, it is helpful to extend p to a mapping from the real line into M, by putting $p(t) = p(a)$ when $t < a$, and $p(t) = p(b)$ when $t > b$. It is easy to see that this extension is continuous as a mapping from $\mathbf{R}$ into M. It follows that $p([a, b])$ is a connected subset of M, because $[a, b]$ is connected in $\mathbf{R}$.

Definition 4.13.2 *A subset E of M is said to be* path connected *if for every pair of points $x, y \in E$ there is a continuous mapping p from the closed unit interval $[0, 1]$ into M such that $p(0) = x$, $p(1) = y$, and $p(t) \in E$ for every $t \in [0, 1]$. This uses the restriction of the standard Euclidean metric on $\mathbf{R}$ to $[0, 1]$.*

Proposition 4.13.3 *If $E \subseteq M$ is path connected, then E is connected in M.*

Suppose for the sake of a contradiction that E is not connected in M, so that there are nonempty separated subsets A, B of M such that $A \cup B = E$. Let x be an element of A, and let y be an element of B. If E is path connected in M, then there is a continuous mapping p from $[0, 1]$ into M such that $p(0) = x$, $p(1) = y$, and $p([0, 1]) \subseteq E$.

Put

$$A_1 = A \cap p([0,1]), \quad B_1 = B \cap p([0,1]). \tag{4.13.4}$$

Observe that

$$A_1 \cup B_1 = p([0,1]), \tag{4.13.5}$$

because $p([0, 1]) \subseteq E = A \cup B$. By construction, $x \in A_1$ and $y \in B_1$, so that $A_1, B_1 \neq \emptyset$. One can check that A_1 and B_1 are separated in M, because A and B are separated in M, and $A_1 \subseteq A$, $B_1 \subseteq B$. This implies that $p([0, 1])$ is not connected in M, which is a contradiction, as desired.

Definition 4.13.6 *Let n be a positive integer. A subset E of $\mathbf{R}^n$ is said to be* convex *if for every $x, y \in E$ and $t \in \mathbf{R}$ with $0 \le t \le 1$, we have that*

$$(1 - t)\, x + t\, y \in E. \tag{4.13.7}$$

It is easy to see that convex subsets of $\mathbf{R}^n$ are path connected, with respect to the standard Euclidean metric on $\mathbf{R}^n$. Note that connected subsets of the real line are convex.

Proposition 4.13.8 *Let M and N be metric spaces, and let f be a continuous mapping from M into N. If E is a path-connected subset of M, then $f(E)$ is path connected in N.*

This follows from the definition of path connectedness, and the fact that compositions of continuous mappings are continuous.

It is well known and not too difficult to show that connected open subsets of $\mathbf{R}^n$ are connected. However, there are examples of connected subsets of $\mathbf{R}^2$ that are not path connected.

4.14 Integral metrics

In this section, the reader is supposed to be familiar with the Riemann integral of a continuous function on a closed interval in the real line. Let a, b be real numbers, with $a < b$, and let

$$C([a,b]) = C([a,b], \mathbf{R}) \tag{4.14.1}$$

be the space of continuous real-valued functions on $[a, b]$. More precisely, this uses the standard Euclidean metric on $\mathbf{R}$, and its restriction to $[a, b]$.

If $f, g \in C([a, b])$, then put

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx. \tag{4.14.2}$$

One can check that this defines a metric on $C([a,b])$. In particular, $d_1(f,g) > 0$ unless $f = g$ on $[a,b]$.

Remember that continuous real-valued functions on $[a,b]$ are automatically bounded, because $[a,b]$ is compact. The supremum metric on $C([a,b])$ is defined by

$$\theta(f,g) = \sup\{|f(x) - g(x)| : a \le x \le b\}, \tag{4.14.3}$$

as in Section 4.8. It is easy to see that

$$d_1(f,g) \le (b-a)\,\theta(f,g) \tag{4.14.4}$$

for every $f, g \in C([a,b])$.

In particular, if $\{f_j\}_{j=1}^\infty$ is a sequence of continuous real-valued functions on $[a,b]$ that converges uniformly to a continuous real-valued function f on $[a,b]$, then $\{f_j\}_{j=1}^\infty$ converges to f with respect to $d_1(\cdot,\cdot)$.

However, $C([a,b])$ is not complete as a metric space with respect to (4.14.2). To get a complete metric space, one can use the *Lebesgue integral*.

4.15 Some additional topics

Let $(M, d(x,y))$ and $(N, \rho(u,v))$ be metric spaces. A mapping ϕ from M into N is said to be an *isometry* if

$$\rho(\phi(x), \phi(y)) = d(x,y) \tag{4.15.1}$$

for every $x, y \in M$. A *completion* of M may be defined as an isometry from M onto a dense subset of a complete metric space. It is well known that every metric space has a completion, which is unique up to isometric equivalence.

Suppose that M is complete. If $U_1, U_2, U_3, \ldots$ is an infinite sequence of dense open subsets of M, then the *Baire category theorem* states that $\bigcap_{j=1}^\infty U_j$ is dense in M too.

A mapping ϕ from M into itself is said to be a *contraction* if there is a nonnegative real number $c < 1$ such that

$$d(\phi(x), \phi(y)) \le c\, d(x,y) \tag{4.15.2}$$

for every $x, y \in M$. In this case, if M is also nonempty and complete, then the *contraction mapping theorem* states that there is a unique $w \in M$ such that $\phi(w) = w$.

Let n be a positive integer, and let

$$B_n = \left\{ x \in \mathbf{R}^n : \sum_{j=1}^n x_j^2 \le 1 \right\} \tag{4.15.3}$$

be the closed unit ball in $\mathbf{R}^n$. Suppose that f is a continuous mapping from B_n into itself, with respect to the restriction of the standard Euclidean metric on $\mathbf{R}^n$ to B_n. *Brouwer's fixed-point theorem* says that there is an $x \in B_n$ such that $f(x) = x$. This can be obtained from the intermediate value theorem when $n = 1$.

Let a and b be real numbers, with $a < b$. Suppose that f is a continuous real-valued function on $[a, b]$, with respect to the standard Euclidean metric on $\mathbf{R}$ and its restriction to $[a, b]$. Under these conditions, a famous theorem of Weierstrass says that f can be uniformly approximated by polynomials on $[a, b]$.

Chapter 5

Some derivatives and integrals

5.1 Functions on subsets of R

Let E be a nonempty subset of the real line, and let f be a real-valued function on E. In this and the next sections, we always use the standard Euclidean metric on $\mathbf{R}$, and its restriction to E. If

$$x \in E \text{ is a limit point of } E, \tag{5.1.1}$$

then the *derivative* of f at x as a function on E may be defined as usual by

$$f'(x) = \lim_{\substack{w \in E \\ w \to x}} \frac{f(w) - f(x)}{w - x}, \tag{5.1.2}$$

when this limit exists. In this case, f is said to be *differentiable* at x, as a function on E.

Of course, one is often particularly interested in functions defined on something like an interval in the real line. Some of the usual arguments work for other types of subsets of $\mathbf{R}$, and some arguments involve more properties of E. See [76, 77] for some topics related to functions defined on rational numbers.

If f is differentiable at x as a function on E, then

$$f \text{ is continuous at } x \text{ as a function on } E. \tag{5.1.3}$$

Indeed, under these conditions, we have that

$$\begin{aligned} \lim_{\substack{w \in E \\ w \to x}} (f(w) - f(x)) &= \lim_{\substack{w \in E \\ w \to x}} \Big(\Big(\frac{f(w) - f(x)}{w - x}\Big)(w - x)\Big) \\ &= \Big(\lim_{\substack{w \in E \\ w \to x}} \Big(\frac{f(w) - f(x)}{w - x}\Big)\Big)\Big(\lim_{\substack{w \in E \\ w \to x}} (w - x)\Big) \\ &= f'(x) \cdot 0 = 0, \end{aligned} \tag{5.1.4}$$

where the second step is as in Section 4.10.

Let g be another real-valued function on E that is differentiable at x. It is easy to see that

$$f + g \text{ is differentiable at } x, \tag{5.1.5}$$

with

$$(f + g)'(x) = f'(x) + g'(x). \tag{5.1.6}$$

This uses the analogous statement for limits of sums in Section 4.10.

One can also check that

$$f\,g \text{ is differentiable at } x, \tag{5.1.7}$$

with

$$(f\,g)'(x) = f'(x)\,g(x) + f(x)\,g'(x), \tag{5.1.8}$$

which is the usual *product rule* for the derivative. This can be obtained in a standard way from the usual results about limits of products in Section 4.10.

Suppose that

$$f(w) \neq 0 \tag{5.1.9}$$

for each $w \in E$, so that

$$1/f \tag{5.1.10}$$

defines a real-valued function on E too. In this case, one can show that

$$1/f \text{ is differentiable at } x, \tag{5.1.11}$$

with

$$(1/f)'(x) = -f'(x)/f(x)^2. \tag{5.1.12}$$

This uses the fact that $1/f$ is continuous at x as a function on E, because of (5.1.3). Note that (5.1.12) could be obtained from the product rule, if one knows (5.1.11).

More precisely, one can consider $1/f$ as a real-valued function on the set

$$\{w \in E : f(w) \neq 0\}, \tag{5.1.13}$$

even if $f(w) = 0$ for some $w \in E$. If $f(x) \neq 0$, then (5.1.13) contains every $w \in E$ that is sufficiently close to x, because f is continuous at x. In particular, this means that x is a limit point of (5.1.13), because x is a limit point of E in $\mathbf{R}$, by hypothesis.

Of course, the *quotient rule* for the derivative can be obtained from (5.1.12) and the product rule.

5.2 The chain rule

Let E be a nonempty subset of the real line again, and let f be a real-valued function on E. Suppose that $x \in E$ is a limit point of E, and that f is differentiable at x. Let E_1 be another subset of the real line, and suppose that

(5.2.1) $$f(E) \subseteq E_1,$$

and that

(5.2.2) $$f(x) \text{ is a limit point of } E_1.$$

Also let g be a real-valued function on E_1, and put

(5.2.3) $$h = g \circ f,$$

which is a real-valued function on E. If

(5.2.4) $$g \text{ is differentiable at } f(x),$$

then the *chain rule* states that

(5.2.5) $$h \text{ is differentiable at } x,$$

with

(5.2.6) $$h'(x) = g'(f(x))\, f'(x).$$

This is very easy to see when g is a linear function on $\mathbf{R}$. Using this, one can reduce to the case where

(5.2.7) $$g'(f(x)) = 0,$$

which is a bit simpler.

Suppose for the moment that f is a one-to-one function on E, and that g is the inverse of f on

(5.2.8) $$E_1 = f(E),$$

so that

(5.2.9) $$h(w) = g(f(w)) = w$$

for every $w \in E$. It is easy to see that (5.2.2) holds in this case, because f is continuous at x, as in the previous section. If (5.2.4) holds, then

(5.2.10) $$g'(f(x))\, f'(x) = h'(x) = 1.$$

This means that $f'(x) \neq 0$, and that

(5.2.11) $$g'(f(x)) = 1/f'(x).$$

Suppose for the moment again that

(5.2.12) $$E = (a, b)$$

for some real numbers a, b with $a < b$, although one could also permit $a = -\infty$ or $b = +\infty$. Suppose also that f is differentiable at every $w \in (a, b)$, with

$$f'(w) > 0. \tag{5.2.13}$$

One can use the mean value theorem to get that f is strictly increasing on (a, b). One can also check that $f((a, b))$ is an open interval in $\mathbf{R}$, or an open half-line, or $\mathbf{R}$. If g is the inverse of f on $f((a, b))$, then part of Exercise 2 on p114 of [112] states that

$$g \text{ is differentiable at every point in } f((a, b)). \tag{5.2.14}$$

Let E be any nonempty subset of $\mathbf{R}$ again, and suppose that $x \in E$ is a limit point of E, as before. Suppose that f is one-to-one on E, and let g be the inverse of f on (5.2.8). Suppose also that f is differentiable at x, as usual, with $f'(x) \neq 0$. If

$$g \text{ is continuous at } f(x), \tag{5.2.15}$$

as a real-valued function on (5.2.8), then one can show that (5.2.4) holds. This corresponds to Theorem 7.5 E on p174 of [47], at least when E is an interval in $\mathbf{R}$, which may be unbounded, as well as open, closed, or half-open and half-closed, as on p2 of [47].

More precisely, (5.2.15) implies that if $w \in E$ and $f(w)$ is close to $f(x)$, then

$$w = g(f(w)) \text{ is close to } x = g(f(x)). \tag{5.2.16}$$

Otherwise, let r_0 be a positive real number, and let g_0 be the restriction of g to

$$f(E \cap (x - r_0, x + r_0)). \tag{5.2.17}$$

If r_0 is small enough, then one can check that g_0 is continuous at $f(x)$ as a function on (5.2.17), using the hypothesis that $f'(x) \neq 0$.

5.3 One-sided derivatives

Let E be a nonempty subset of $\mathbf{R}$ again, and let f be a real-valued function on E. If $x \in E$, then put

$$E^+(x) = \{w \in E : w \geq x\} \tag{5.3.1}$$

and

$$E^-(x) = \{w \in E : w \leq x\}. \tag{5.3.2}$$

If

$$x \text{ is a limit point of } E^+(x), \tag{5.3.3}$$

then the *derivative of f at x from the right* as a function on E is defined by

$$f'_+(x) = \lim_{\substack{w \in E^+(x) \\ w \to x}} \frac{f(w) - f(x)}{w - x}, \tag{5.3.4}$$

when this limit exists.

Similarly, if

$$x \text{ is a limit point of } E^-(x), \tag{5.3.5}$$

then the *derivative of f at x from the left* as a function on E is defined by

$$f'_-(x) = \lim_{\substack{w \in E^-(x) \\ w \to x}} \frac{f(w) - f(x)}{w - x}, \tag{5.3.6}$$

when this limit exists. These are the same as the derivatives at x of the restrictions of f to $E^+(x)$ and $E^-(x)$, respectively, when they exist.

If x is a limit point of $E^+(x)$ or $E^-(x)$, then it is easy to see that x is a limit point of E. One can check that the converse holds as well.

Suppose for the moment that

$$x \text{ is a limit point of } E^+(x) \text{ and } E^-(x), \tag{5.3.7}$$

so that x is a limit point of E in particular. If f is differentiable at x as a function on E, then it is easy to see that the derivatives of f at x from the left and right exist, with

$$f'_+(x) = f'_-(x) = f'(x). \tag{5.3.8}$$

Conversely, if the derivatives of f at x from the left and right exist and are equal to each other, then f is differentiable at x as a function on E, with the derivative of f at x as in (5.3.8). This is very similar to a remark in Section 4.11.

5.4 More on one-sided derivatives

Let us continue with the same notation and hypotheses as in the previous section. Suppose for the moment again that

$$f \text{ has a local maximum at } x, \tag{5.4.1}$$

as a function on E, so that

$$f(w) \le f(x) \tag{5.4.2}$$

for all $w \in E$ that are sufficiently close to x. If x is a limit point of $E^+(x)$, and if the derivative of f at x from the right as a function on E exists, then it is easy to see that

$$f'_+(x) \le 0. \tag{5.4.3}$$

Similarly, if x is a limit point of $E^-(x)$, and if the derivative of f at x from the left as a function on E exists, then

$$f'_-(x) \ge 0. \tag{5.4.4}$$

If (5.3.7) holds, and if f is differentiable at x as a function on E, then it follows that

$$f'(x) = 0, \tag{5.4.5}$$

because of (5.3.8).

Similarly, suppose for the moment that

$$f \text{ has a local minimum at } x, \tag{5.4.6}$$

as a function on E, so that

$$f(x) \le f(w) \tag{5.4.7}$$

for all $w \in E$ that are sufficiently close to x. If x is a limit point of $E^+(x)$, and if the derivative of f at x from the right as a function on E exists, then

$$f'_+(x) \ge 0. \tag{5.4.8}$$

If x is a limit point of $E^-(x)$, and if the deriviative of f at x from the left as a function on E exists, then

$$f'_-(x) \le 0. \tag{5.4.9}$$

If (5.3.7) holds, and if f is differentiable at x as a function on E, then (5.4.5) holds, because of (5.3.8), as before. Of course, these statements correspond to those in the preceding paragraph for $-f$.

These remarks are related to Rolle's theorem and the mean value theorem when E is a closed interval in $\mathbf{R}$. Remember that Rolle's theorem uses the extreme value theorem, as in Corollary 4.3.9. One might consider analogous types of arguments when E is the Cantor set, as in Section 2.9, for instance.

See [6, 24, 27, 75, 123] for some additional topics concerning the mean value theorem. Some related matters are discussed in [34, 36].

Let a and b be real numbers with $a < b$, and let f, g be continuous real-valued functions on $[a, b]$ that are differentiable at every point in (a, b). Under these conditions, the *generalized mean value theorem* states that there is an $x \in (a, b)$ such that

$$(f(b) - f(a))\, g'(x) = (g(b) - g(a))\, f'(x), \tag{5.4.10}$$

as in Theorem 5.9 on p107 of [112]. To see this, put

$$h(t) = (f(b) - f(a))\, g(t) - (g(b) - g(a))\, f(t) \tag{5.4.11}$$

for every $t \in [a, b]$. This is a continuous real-valued function on $[a, b]$ that is differentiable at every point in (a, b), with

$$h(a) = f(b)\, g(a) - f(a)\, g(b) = h(b). \tag{5.4.12}$$

One can use Rolle's theorem to get a point $x \in (a, b)$ such that $h'(x) = 0$, and (5.4.10) follows from this.

5.5 The Darboux property

Let a and b be real numbers with $a < b$, and let f be a real-valued function on $[a, b]$ that is differentiable at every point in $[a, b]$. Suppose that $\lambda \in \mathbf{R}$ satisfies

$$f'(a) < \lambda < f'(b) \tag{5.5.1}$$

or

$$f'(b) < \lambda < f'(a). \tag{5.5.2}$$

Under these conditions, there is an $x \in (a,b)$ such that

$$f'(x) = \lambda. \tag{5.5.3}$$

To see this, it is convenient to suppose that (5.5.1) holds, and otherwise one can use an analogous argument when (5.5.2) holds, or reduce to this case. Let g be the real-valued function defined on $[a,b]$ by

$$g(w) = f(w) - \lambda\, w, \tag{5.5.4}$$

and observe that g is differentiable on $[a,b]$, with

$$g' = f' - \lambda. \tag{5.5.5}$$

In particular,

$$g'(a) = f'(a) - \lambda < 0 \tag{5.5.6}$$

and

$$g'(b) = f'(b) - \lambda > 0, \tag{5.5.7}$$

because of (5.5.1).

The extreme value theorem implies that g attains its minimum at a point $x \in [a,b]$. One can check that $x \in (a,b)$ in this case, using (5.5.6) and (5.5.7), as in the previous section. It follows that

$$g'(x) = 0, \tag{5.5.8}$$

as in the previous section again. This implies (5.5.3), by (5.5.5). Of course, the same conclusion can be obtained from the intermediate value theorem when f' is continuous on $[a,b]$.

Let f be a real-valued function on the real line that is differentiable at every point in $\mathbf{R}$. Put

$$f_j(x) = j\,(f(x + (1/j)) - f(x)) \tag{5.5.9}$$

for every $x \in \mathbf{R}$ and positive integer j, and observe that

$$\lim_{j \to \infty} f_j(x) = f'(x) \tag{5.5.10}$$

for every $x \in \mathbf{R}$. Pointwise limits of continuous functions are said to be of *Baire class one*. Thus f' is of Baire class one. Some related references include [12, 19, 20, 21, 31].

5.6 Some examples

Let l be a positive integer, and consider the function f_l defined on the real line by

(5.6.1)
$$\begin{aligned} f_l(x) &= x^2 \sin(x^{-l}) \quad \text{when } x \neq 0 \\ &= 0 \quad\quad\quad\quad\quad \text{when } x = 0. \end{aligned}$$

We shall use standard properties of trigonometric functions here, although one could avoid this by using other nonconstant periodic differentiable functions on $\mathbf{R}$. If $x \neq 0$, then f_l is differentiable at x, with

(5.6.2)
$$\begin{aligned} f_l'(x) &= 2\,x\,\sin(x^{-l}) + x^2\,(-l\,x^{-l-1})\,\cos(x^{-l}) \\ &= 2\,x\,\sin(x^{-l}) - l\,x^{1-l}\,\cos(x^{-l}). \end{aligned}$$

One can check directly that f_l is differentiable at 0, with

(5.6.3)
$$f_l'(0) = 0.$$

Note that

(5.6.4)
$$\lim_{x \to 0} (2\,x\,\sin(x^{-l})) = 0,$$

because $\sin(x^{-l})$ is bounded for $x \neq 0$. If $l = 1$, then (5.6.2) reduces to

(5.6.5)
$$f_1'(x) = 2\,x\,\sin(1/x) - \cos(1/x)$$

for $x \neq 0$. It is easy to see that f_1' is not continuous at 0. If $l > 1$, then $f_l'(x)$ is not even bounded for $x \neq 0$ near 0.

Clearly (5.6.5) is bounded on bounded subsets of $\mathbf{R} \setminus \{0\}$. Of course, it is well known that

(5.6.6)
$$\lim_{t \to 0} (t^{-1}\,\sin(t)) = \sin'(0) = 1.$$

One can use this with $t = 1/x$ to get that

(5.6.7)
$$f_1'(x) \to 1 \text{ as } x \to \pm\infty.$$

In particular, this implies that (5.6.5) is bounded on $\mathbf{R} \setminus \{0\}$. Similarly, if $l > 1$, then one can use (5.6.6) with $t = x^{-l}$ to get that

(5.6.8)
$$f_l'(x) \to 0 \text{ as } x \to \pm\infty.$$

See [72] for more on some related examples, and their properties.

5.7 Lipschitz mappings

Let $(M, d(\cdot,\cdot))$ and $(N, \rho(\cdot,\cdot))$ be metric spaces. A mapping f from M into N is said to be *Lipschitz* if there is a nonnegative real number C such that

(5.7.1)
$$\rho(f(x), f(w)) \leq C\,d(x, w)$$

for all $x, w \in M$. In this case, we may say that f is *Lipschitz with constant* C, to indicate the role of C. It is easy to see that

(5.7.2) Lipschitz mappings are uniformly continuous.

Note that

(5.7.3) f is Lipschitz with constant $C = 0$ on M if and only if f is constant on M.

Let E be a subset of the real line, and let f be a real-valued function on E. Note that f is Lipschitz with constant $C \geq 0$ on E exactly when

$$|f(x) - f(w)| \leq C\,|x - w| \tag{5.7.4}$$

for all $x, w \in E$. This uses the standard Euclidean metric on $\mathbf{R}$, and its restriction to E.

Suppose that $x \in E$ is a limit point of E, and that f is differentiable at x. If f is Lipschitz with constant C on E, then one can check that

$$|f'(x)| \leq C. \tag{5.7.5}$$

Let a and b be real numbers with $a < b$, and let f be a continuous real-valued function on $[a, b]$. Suppose that f is differentiable at every point in (a, b), and that there is a nonnegative real number C such that (5.7.5) holds for every $x \in (a, b)$. Under these conditions, one can verify that

(5.7.6) f is Lipschitz with constant C on $[a, b]$,

using the mean value theorem.

Let E be a subset of the real line again, and let f be a real-valued function on E. Suppose that $x_0 \in E$ is a limit point of E, and that f is differentiable at x_0. If ϵ_0 is a positive real number, then one can check that there is a positive real number δ_0 such that

$$\left| \frac{f(w) - f(x_0)}{w - x_0} \right| < |f'(x_0)| + \epsilon_0 \tag{5.7.7}$$

for every $w \in E$ with $w \neq x_0$ and $|w - x_0| < \delta_0$. This implies that

$$|f(w) - f(x_0)| \leq (|f'(x_0)| + \epsilon_0)\,|w - x_0| \tag{5.7.8}$$

for every $w \in E$ with $|w - x_0| < \delta_0$, because (5.7.8) holds trivially when $w = x_0$. This may be considered as a type of pointwise Lipschitz condition for f at x_0. In particular, this implies that f is continuous at x_0. Remember that the continuity of f at x_0 was also mentioned in Section 5.1.

5.8 Suprema and infima of functions

Let E be a nonempty set, and let f be a real-valued function on E. Suppose that f is bounded on E, so that $f(E)$ is a nonempty bounded subset of the real line. Under these conditions, we may use the notation

$$\sup_{x \in E} f(x) = \sup f(E) = \sup\{f(x) : x \in E\} \tag{5.8.1}$$

and

$$\inf_{x \in E} f(x) = \inf f(E) = \inf\{f(x) : x \in E\}. \tag{5.8.2}$$

Of course,

$$\inf_{x \in E} f(x) \le \sup_{x \in E} f(x). \tag{5.8.3}$$

If $t \in \mathbf{R}$, then it is easy to see that

$$t\,f \text{ is a bounded real-valued function on } E \tag{5.8.4}$$

too. If $t > 0$, then one can check that

$$\sup_{x \in E}(t\,f(x)) = t\Big(\sup_{x \in E} f(x)\Big) \tag{5.8.5}$$

and

$$\inf_{x \in E}(t\,f(x)) = t\Big(\inf_{x \in E} f(x)\Big). \tag{5.8.6}$$

Similarly, if $t < 0$, then

$$\sup_{x \in E}(t\,f(x)) = t\Big(\inf_{x \in E} f(x)\Big) \tag{5.8.7}$$

and

$$\inf_{x \in E}(t\,f(x)) = t\Big(\sup_{x \in E} f(x)\Big). \tag{5.8.8}$$

Let g be another bounded real-valued function on E, and observe that

$$f + g \text{ is bounded on } E \tag{5.8.9}$$

as well. One can verify that

$$\sup_{x \in E}(f(x) + g(x)) \le \Big(\sup_{x \in E} f(x)\Big) + \Big(\sup_{x \in E} g(x)\Big) \tag{5.8.10}$$

and

$$\inf_{x \in E}(f(x) + g(x)) \ge \Big(\inf_{x \in E} f(x)\Big) + \Big(\inf_{x \in E} g(x)\Big). \tag{5.8.11}$$

5.9 More on suprema and infima of functions

Let us continue with the same notation and hypotheses as in the previous section. We also have that

$$\sup_{x \in E}(f(x) + g(x)) \ge \Big(\sup_{x \in E} f(x)\Big) + \Big(\inf_{x \in E} g(x)\Big) \tag{5.9.1}$$

and

$$\inf_{x \in E}(f(x) + g(x)) \le \Big(\inf_{x \in E} f(x)\Big) + \Big(\sup_{x \in E} g(x)\Big). \tag{5.9.2}$$

This can be verified directly, or using the previous inequalities and some of the earlier remarks with $t = -1$.

Suppose for the moment that

$$f, g \ge 0 \text{ on } E. \tag{5.9.3}$$

Under these conditions, one can check that

$$\sup_{x \in E}(f(x)\, g(x)) \le \Big(\sup_{x \in E} f(x)\Big) \Big(\sup_{x \in E} g(x)\Big) \tag{5.9.4}$$

and

$$\inf_{x \in E}(f(x)\, g(x)) \ge \Big(\inf_{x \in E} f(x)\Big) \Big(\inf_{x \in E} g(x)\Big). \tag{5.9.5}$$

Suppose for the moment again that

$$f(x) \le g(x) \tag{5.9.6}$$

for every $x \in E$. In this case, it is easy to see that

$$\sup_{x \in E} f(x) \le \sup_{x \in E} g(x) \tag{5.9.7}$$

and

$$\inf_{x \in E} f(x) \le \inf_{x \in E} g(x). \tag{5.9.8}$$

If E_0 is a nonempty subset of E, then

$$\sup_{x \in E_0} f(x) \le \sup_{x \in E} f(x) \tag{5.9.9}$$

and

$$\inf_{x \in E} f(x) \le \inf_{x \in E_0} f(x). \tag{5.9.10}$$

5.10 Darboux sums

Let a and b be real numbers, with $a \le b$, and let

$$\mathcal{P} = \{t_j\}_{j=0}^{l} \tag{5.10.1}$$

be a *partition* of $[a, b]$. More precisely, this is a finite sequence of real numbers with

$$a = t_0 \le t_1 \le \cdots \le t_{l-1} \le t_l = b. \tag{5.10.2}$$

It is convenient to include the case where $a = b$ here, but the various sums and integrals that we shall consider will be equal to 0 when this happens. Similarly, if $t_{j-1} = t_j$ for some $j \ge 1$, then the corresponding term in each of the sums that we shall consider will be equal to 0.

Let f be a bounded real-valued function on $[a, b]$, and put

$$M_j = \sup\{f(x) : t_{j-1} \le x \le t_j\} \tag{5.10.3}$$

and

$$m_j = \inf\{f(x) : t_{j-1} \le x \le t_j\} \tag{5.10.4}$$

for each $j = 1, \ldots, l$. Consider

$$U(\mathcal{P}, f) = \sum_{j=1}^{l} M_j\,(t_j - t_{j-1}) \tag{5.10.5}$$

and

$$L(\mathcal{P}, f) = \sum_{j=1}^{l} m_j\,(t_j - t_{j-1}), \tag{5.10.6}$$

as on p121 of [112], for instance. These are the *upper and lower Darboux sums* associated to f and $\mathcal{P}$, respectively.

If

$$w_j \in [t_{j-1}, t_j],\ 1 \le j \le l, \tag{5.10.7}$$

then

$$L(\mathcal{P}, f) \le \sum_{j=1}^{l} f(w_j)\,(t_j - t_{j-1}) \le U(\mathcal{P}, f). \tag{5.10.8}$$

The sum in the middle is the *Riemann sum* associated to f, $w_1, \ldots, w_l$, and the partition $\mathcal{P}$.

5.11 Darboux–Stieltjes sums

Let us continue with the same notation and hypotheses as in the previous section. Also let α be a monotonically increasing real-valued function on $[a, b]$, so that

$$\alpha(t_j) - \alpha(t_{j-1}) \ge 0 \tag{5.11.1}$$

for each $j = 1, \ldots, l$. Consider

$$U(\mathcal{P}, f, \alpha) = \sum_{j=1}^{l} M_j\,(\alpha(t_j) - \alpha(t_{j-1})) \tag{5.11.2}$$

and

$$L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{l} m_j \, (\alpha(t_j) - \alpha(t_{j-1})), \tag{5.11.3}$$

as on p122 of [112]. These are the *upper and lower Darboux sums* or *Darboux–Stieltjes sums* associated to f, $\mathcal{P}$, and α, respectively.

If $w_1, \ldots, w_l$ are as in (5.10.7), then

$$L(\mathcal{P}, f, \alpha) \le \sum_{j=1}^{l} f(w_j) \, (\alpha(t_j) - \alpha(t_{j-1})) \le U(\mathcal{P}, f, \alpha). \tag{5.11.4}$$

The sum in the middle is the *Riemann–Stieltjes sum* associated to f, the points $w_1, \ldots, w_l$, the partition $\mathcal{P}$, and α.

Of course, if $\alpha(x) = x$ on $[a, b]$, then (5.11.2) and (5.11.3) are the same as (5.10.5) and (5.10.6), respectively. Note that

$$\begin{aligned} \sum_{j=1}^{l} (\alpha(t_j) - \alpha(t_{j-1})) &= \sum_{j=1}^{l} \alpha(t_j) - \sum_{j=1}^{l} \alpha(t_{j-1}) \\ &= \sum_{j=1}^{l} \alpha(t_j) - \sum_{j=0}^{l-1} \alpha(t_j) = \alpha(b) - \alpha(a). \end{aligned} \tag{5.11.5}$$

5.12 More on Darboux sums

Let us continue with the same notation and hypotheses as in the previous two sections. Clearly

$$m_j \le M_j \tag{5.12.1}$$

for each j. This implies that

$$L(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, f, \alpha), \tag{5.12.2}$$

which could also be obtained from (5.11.4). In particular,

$$L(\mathcal{P}, f) \le U(\mathcal{P}, f), \tag{5.12.3}$$

which follows from (5.10.8) as well.

If we put

$$M = \sup\{f(x) : a \le x \le b\} \tag{5.12.4}$$

and

$$m = \inf\{f(x) : a \le x \le b\}, \tag{5.12.5}$$

then

$$M = \max_{1 \le j \le l} M_j \tag{5.12.6}$$

and

$$m = \min_{1 \le j \le l} m_j. \tag{5.12.7}$$

Using this, we get that

$$U(\mathcal{P}, f, \alpha) \le \sum_{j=1}^{l} M\,(\alpha(t_j) - \alpha(t_{j-1})) = M\,(\alpha(b) - \alpha(a)) \tag{5.12.8}$$

and

$$L(\mathcal{P}, f, \alpha) \ge \sum_{j=1}^{l} m\,(\alpha(t_j) - \alpha(t_{j-1})) = m\,(\alpha(b) - \alpha(a)). \tag{5.12.9}$$

In particular,

$$U(\mathcal{P}, f) \le M\,(b - a) \tag{5.12.10}$$

and

$$L(\mathcal{P}, f) \ge m\,(b - a), \tag{5.12.11}$$

as on p121 of [112]. Note that equality holds in these four inequalities when $\mathcal{P}$ is the partition of $[a, b]$ consisting only of a and b, so that $l = 1$.

Let $\mathcal{P}^* = \{t_k^*\}_{k=0}^{l^*}$ be another partition of $[a, b]$. We say that

$$\mathcal{P}^* \text{ is a } \textit{refinement} \text{ of } \mathcal{P} \tag{5.12.12}$$

if for each $j = 0, 1 \ldots, l$ there is a $k \in \{0, 1, \ldots, l^*\}$ such that

$$t_j = t_k^*. \tag{5.12.13}$$

This means that $[t_{j-1}, t_j]$ may be partitioned further by $\mathcal{P}^*$ for each $j = 1, \ldots, l$. In this case, one can show that

$$U(\mathcal{P}^*, f, \alpha) \le U(\mathcal{P}, f, \alpha) \tag{5.12.14}$$

and

$$L(\mathcal{P}, f, \alpha) \le L(\mathcal{P}^*, f, \alpha), \tag{5.12.15}$$

as in Theorem 6.4 on p123 of [112].

5.13 Upper and lower integrals

Let us continue with the same notation and hypotheses as in the previous three sections. The *upper and lower Riemann integrals* of f on $[a, b]$ are defined by

$$\overline{\int_a^b} f\,dx = \inf_{\mathcal{P}} U(\mathcal{P}, f) \tag{5.13.1}$$

and

$$\underline{\int_a^b} f\,dx = \sup_{\mathcal{P}} L(\mathcal{P}, f), \tag{5.13.2}$$

respectively, as on p121 of [112]. More precisely, this infimum and supremum are taken over all partitions $\mathcal{P}$ of $[a, b]$.

Remember that

$$m\,(b-a) \le L(\mathcal{P}, f) \le U(\mathcal{P}, f) \le M\,(b-a) \tag{5.13.3}$$

for all such partitions $\mathcal{P}$, as in (5.12.3), (5.12.10), and (5.12.11), where M and m are as in (5.12.4) and (5.12.5), respectively. This implies that the infimum and supremum in (5.13.1) and (5.13.2) exist in $\mathbf{R}$, with

$$m\,(b-a) \le \overline{\int_a^b} f\,dx \tag{5.13.4}$$

and

$$\underline{\int_a^b} f\,dx \le M\,(b-a). \tag{5.13.5}$$

We also have that

$$\overline{\int_a^b} f\,dx \le M\,(b-a) \tag{5.13.6}$$

and

$$\underline{\int_a^b} f\,dx \ge m\,(b-a). \tag{5.13.7}$$

Similarly, the *upper and lower Riemann–Stieltjes integrals* of f with respect to α on $[a, b]$ are defined by

$$\overline{\int_a^b} f\,d\alpha = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha) \tag{5.13.8}$$

and

$$\underline{\int_a^b} f\,d\alpha = \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha), \tag{5.13.9}$$

respectively, as on p122 of [112]. These are the same as in (5.13.1) and (5.13.2) when $\alpha(x) = x$ on $[a, b]$.

Note that

$$m\,(\alpha(b)-\alpha(a)) \le L(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, f, \alpha) \le M\,(\alpha(b)-\alpha(a)) \tag{5.13.10}$$

for all partitions $\mathcal{P}$ of $[a, b]$, as in (5.12.2), (5.12.8), and (5.12.9). This implies that the infimum and supremum in (5.13.8) and (5.13.9) exist in $\mathbf{R}$, with

$$m\,(\alpha(b)-\alpha(a)) \le \overline{\int_a^b} f\,d\alpha \tag{5.13.11}$$

and

$$\underline{\int_a^b} f\,d\alpha \le M\,(\alpha(b)-\alpha(a)). \tag{5.13.12}$$

In addition,

$$\overline{\int_a^b} f\,d\alpha \le M\,(\alpha(b) - \alpha(a)) \tag{5.13.13}$$

and

$$\underline{\int_a^b} f\,d\alpha \ge m\,(\alpha(b) - \alpha(a)). \tag{5.13.14}$$

5.14 Common refinements

Let us continue with the same notation and hypotheses as in the previous four sections. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any two partitions of $[a, b]$, as in the previous section. It is not too difficult to see that there is a partition $\mathcal{P}^*$ of $[a, b]$ such that

$$\mathcal{P}^* \text{ is a refinement of both } \mathcal{P}_1,\ \mathcal{P}_2. \tag{5.14.1}$$

The smallest such *common refinement* of $\mathcal{P}_1$ and $\mathcal{P}_2$ can be obtained using all of the points in $[a, b]$ that occur in $\mathcal{P}_1$ and $\mathcal{P}_2$, listed in order.

Under these conditions, we have that

$$L(\mathcal{P}_1, f, \alpha) \le L(\mathcal{P}^*, f, \alpha) \le U(\mathcal{P}^*, f, \alpha) \le U(\mathcal{P}_2, f, \alpha). \tag{5.14.2}$$

This uses (5.12.15) in the first step, (5.12.2) in the second step, and (5.12.14) in the third step. One can use this to get that

$$\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha, \tag{5.14.3}$$

as in Theorem 6.5 on p124 of [112]. In particular,

$$\underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx. \tag{5.14.4}$$

If $\mathcal{P} = \{t_j\}_{j=0}^l$ is a partition of $[a, b]$, then

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{l} (M_j - m_j)\,(\alpha(t_j) - \alpha(t_{j-1})), \tag{5.14.5}$$

by (5.11.2) and (5.11.3). Remember that M_j and m_j are as in (5.10.3) and (5.10.4), respectively, so that

$$M_j - m_j \ge 0 \tag{5.14.6}$$

for each j in particular.

Let $\mathcal{P}$, $\mathcal{P}^*$ be partitions of $[a, b]$, and suppose that $\mathcal{P}^*$ is a refinement of $\mathcal{P}$. Observe that

$$U(\mathcal{P}^*, f, \alpha) - L(\mathcal{P}^*, f, \alpha) \le U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha), \tag{5.14.7}$$

because of (5.12.14) and (5.12.15). This is related to part (a) of Theorem 6.7 on p125 of [112], and its proof.

5.15 Riemann–Stieltjes integrals

We continue with the same notation and hypotheses as in the previous sections. If

$$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx, \tag{5.15.1}$$

then f is said to be *Riemann integrable* on $[a, b]$, as on p121 of [112]. In this case, the *Riemann integral* of f on $[a, b]$ is denoted

$$\int_a^b f\,dx, \tag{5.15.2}$$

and defined to be the common value in (5.15.1). The Riemann integral may also be expressed as

$$\int_a^b f(x)\,dx. \tag{5.15.3}$$

Similarly, if

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha, \tag{5.15.4}$$

then f is said to be *Riemann–Stieltjes integrable* with respect to α on $[a, b]$, as on p122 of [112]. The corresponding *Riemann–Stieltjes integral*

$$\int_a^b f\,d\alpha \tag{5.15.5}$$

is defined to be the common value in (5.15.4), and this may also be expressed as

$$\int_a^b f(x)\,d\alpha(x). \tag{5.15.6}$$

This is the same as in the previous paragraph when $\alpha(x) = x$ on $[a, b]$, as usual.

One can show that f is Riemann–Stieltjes integrable with respect to α on $[a, b]$ if and only if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon, \tag{5.15.7}$$

as in Theorem 6.6 on p124 of [112]. More precisely, it is somewhat easier to see that f is Riemann–Stieltjes integrable with respect to α on $[a, b]$ if and only if for every $\epsilon > 0$ there are partitions $\mathcal{P}_1$, $\mathcal{P}_2$ of $[a, b]$ such that

$$U(\mathcal{P}_2, f, \alpha) - L(\mathcal{P}_1, f, \alpha) < \epsilon. \tag{5.15.8}$$

In order to get (5.15.7) from (5.15.8), one can take $\mathcal{P}$ to be a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$, and use (5.12.14), (5.12.15).

5.16 Continuity and integrability

Let us continue with the same notation and hypotheses as in the previous sections. If

$$f \text{ is continuous on } [a,b], \tag{5.16.1}$$

then one can show that

$$f \text{ is Riemann–Stieltjes integrable with respect to } \alpha \text{ on } [a,b], \tag{5.16.2}$$

as in Theorem 6.8 on p125 of [112]. This uses the fact that f is uniformly continuous on $[a,b]$, because $[a,b]$ is compact. If c is a real number, then c may be considered as a constant function on $[a,b]$, and it is easy to see that

$$\int_a^b c\,d\alpha = c\,(\alpha(b) - \alpha(a)). \tag{5.16.3}$$

If

$$f \text{ is continuous at all but finitely many points in } [a,b], \tag{5.16.4}$$

and if

$$\alpha \text{ is continuous at those points}, \tag{5.16.5}$$

then one can show that (5.16.2) holds, as in Theorem 6.10 on p126 of [112]. If

$$f \text{ is monotonic on } [a,b], \tag{5.16.6}$$

and

$$\alpha \text{ is continuous on } [a,b], \tag{5.16.7}$$

then (5.16.2) holds, as in Theorem 6.9 on p126 of [112].

If

$$\begin{aligned} f(x) &= 0 \quad \text{when } x \in [a,b] \text{ is irrational} \\ &= 1 \quad \text{when } x \in [a,b] \text{ is rational}, \end{aligned} \tag{5.16.8}$$

then can check that

$$\overline{\int_a^b} f\,d\alpha = \alpha(b) - \alpha(a) \tag{5.16.9}$$

and

$$\underline{\int_a^b} f\,d\alpha = 0. \tag{5.16.10}$$

This implies that f is not Riemann–Stieltjes integrable with respect to α on $[a,b]$ when α is not constant on $[a,b]$. In particular, f is not Riemann integrable on $[a,b]$ when $a < b$, as in Exercise 4 on p138 of [112].

5.17 More on Riemann–Stieltjes integrals

We continue with the same notation and hypotheses as in the previous sections. Let $I(x)$ be the unit step function on the real line, as in Section 4.12. Also let w be a real number with

(5.17.1) $$a \le w < b,$$

and put

(5.17.2) $$\alpha_w(x) = I(x - w)$$

for $x \in [a, b]$, which is a monotonically increasing real-valued function on $[a, b]$. If

(5.17.3) $$f \text{ is continuous at } w,$$

then (5.16.2) holds, with $\alpha = \alpha_w$, and

(5.17.4) $$\int_a^b f \, d\alpha_w = f(w).$$

This corresponds to Theorem 6.15 on p130 of [112], which is stated for $w \ne a$.

The argument for $w = a$ is very similar, but if $w = b$, then $\alpha_w \equiv 0$ on $[a, b]$, because of the way that $I(x)$ is defined. Some related results are mentioned in Exercise 3 on p138 of [112]. In particular, one can consider functions defined on $\mathbf{R}$ in the same way as $I(x)$ when $x \ne 0$, and equal to 1 or 1/2 at $x = 0$.

Suppose that (5.16.2) holds, and let $\mathcal{P} = \{t_j\}_{j=0}^l$ be a partition of $[a, b]$. Of course,

(5.17.5) $$L(\mathcal{P}, f, \alpha) \le \int_a^b f \, d\alpha \le U(\mathcal{P}, f, \alpha),$$

by definition of the integral. If $w_j \in [t_{j-1}, t_j]$, $1 \le j \le l$, then one can use (5.11.4) and (5.17.5) to get that

(5.17.6) $$\left| \sum_{j=1}^l f(w_j) \, (\alpha(t_j) - \alpha(t_{j-1})) - \int_a^b f \, d\alpha \right| \le U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha).$$

In particular, if (5.15.7) holds for some $\epsilon > 0$, then

(5.17.7) $$\left| \sum_{j=1}^l f(w_j) \, (\alpha(t_j) - \alpha(t_{j-1})) - \int_a^b f \, d\alpha \right| < \epsilon.$$

This corresponds to part (c) of Theorem 6.7 on p125 of [112].

5.18 Multiplying functions by constants

Let us continue with the same notation and hypotheses as in the previous sections again. Let r be a positive real number, so that

(5.18.1) $$r \, f \text{ is a bounded real-valued function on } [a, b].$$

If $\mathcal{P}$ is a partition of $[a, b]$, then one can check that

$$U(\mathcal{P}, r\, f, \alpha) = r\, U(\mathcal{P}, f, \alpha) \tag{5.18.2}$$

and

$$L(\mathcal{P}, r\, f, \alpha) = r\, L(\mathcal{P}, f, \alpha), \tag{5.18.3}$$

using some remarks in Section 5.8. This implies that

$$\overline{\int_a^b} r\, f\, d\alpha = r \overline{\int_a^b} f\, d\alpha \tag{5.18.4}$$

and

$$\underline{\int_a^b} r\, f\, d\alpha = r \underline{\int_a^b} f\, d\alpha, \tag{5.18.5}$$

using the same remarks from Section 5.8. If f is Riemann–Stieltjes integrable with respect to α on $[a, b]$, then it follows that

(5.18.6) $r\, f$ is Riemann–Stieltjes integrable with respect to α on $[a, b]$

too, with

$$\int_a^b r\, f\, d\alpha = r \int_a^b f\, d\alpha, \tag{5.18.7}$$

as in part (a) of Theorem 6.12 on p128 of [112].

Of course,

(5.18.8) $-f$ is a bounded real-valued function on $[a, b]$

as well. If $\mathcal{P}$ is a partition of $[a, b]$, then one can verify that

$$U(\mathcal{P}, -f, \alpha) = -L(\mathcal{P}, f, \alpha) \tag{5.18.9}$$

and

$$L(\mathcal{P}, -f, \alpha) = -U(\mathcal{P}, f, \alpha), \tag{5.18.10}$$

using some other remarks in Section 5.8. This means that

$$\overline{\int_a^b} - f\, d\alpha = -\underline{\int_a^b} f\, d\alpha \tag{5.18.11}$$

and

$$\underline{\int_a^b} - f\, d\alpha = -\overline{\int_a^b} f\, d\alpha, \tag{5.18.12}$$

using these same remarks from Section 5.8. If f is Riemann–Stieltjes integrable with respect to α on $[a, b]$, then we get that

(5.18.13) $-f$ is Riemann–Stieltjes integrable with respect to α on $[a, b]$, with

$$\int_a^b -f\,d\alpha = -\int_a^b f\,d\alpha. \tag{5.18.14}$$

Let r be a positive real number again, and note that

(5.18.15) $r\,\alpha$ is a monotonically increasing real-valued function on $[a,b]$.

If $\mathcal{P}$ is a partition of $[a,b]$, then it is easy to see that

$$U(\mathcal{P}, f, r\,\alpha) = r\,U(\mathcal{P}, f, \alpha) \tag{5.18.16}$$

and

$$L(\mathcal{P}, f, r\,\alpha) = r\,L(\mathcal{P}, f, \alpha). \tag{5.18.17}$$

This implies that

$$\overline{\int_a^b} f\,d(r\,\alpha) = r\,\overline{\int_a^b} f\,d\alpha \tag{5.18.18}$$

and

$$\underline{\int_a^b} f\,d(r\,\alpha) = r\,\underline{\int_a^b} f\,d\alpha, \tag{5.18.19}$$

using some remarks from Section 5.8, as before. If f is Riemann–Stieltjes integrable with respect to α on $[a,b]$, then we get that

(5.18.20) f is Riemann–Stieltjes integrable with respect to $r\,\alpha$ on $[a,b]$,

with

$$\int_a^b f\,d(r\,\alpha) = r\int_a^b f\,d\alpha. \tag{5.18.21}$$

This corresponds to part of part (e) of Theorem 6.12 on p128 of [112].

5.19 Adding two functions

We continue with the same notation and hypotheses as in the previous sections. Let g be another bounded real-valued function on $[a,b]$, so that

(5.19.1) $f + g$ is bounded on $[a,b]$

as well. If $\mathcal{P}$ is a partition of $[a,b]$, then one can check that

$$U(\mathcal{P}, f+g, \alpha) \le U(\mathcal{P}, f, \alpha) + U(\mathcal{P}, g, \alpha) \tag{5.19.2}$$

and

$$L(\mathcal{P}, f+g, \alpha) \ge L(\mathcal{P}, f, \alpha) + L(\mathcal{P}, g, \alpha), \tag{5.19.3}$$

using the remarks about suprema and infima of sums of functions in Section 5.8.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be arbitrary partitions of $[a, b]$, and let $\mathcal{P}^*$ be a common refinement of them, as in Section 5.14. Observe that

$$\begin{aligned} \overline{\int_a^b} (f+g)\,d\alpha &\le U(\mathcal{P}^*, f+g, \alpha) \le U(\mathcal{P}^*, f, \alpha) + U(\mathcal{P}^*, g, \alpha) \\ &\le U(\mathcal{P}_1, f, \alpha) + U(\mathcal{P}_2, g, \alpha), \end{aligned} \tag{5.19.4}$$

where the first step is as in Section 5.13, the second step is as in (5.19.2), and the third step is as in Section 5.12. One can use this to get that

$$\overline{\int_a^b} (f+g)\,d\alpha \le \overline{\int_a^b} f\,d\alpha + \overline{\int_a^b} g\,d\alpha. \tag{5.19.5}$$

Similarly,

$$\begin{aligned} \underline{\int_a^b} (f+g)\,d\alpha &\ge L(\mathcal{P}^*, f+g, \alpha) \ge L(\mathcal{P}^*, f, \alpha) + L(\mathcal{P}^*, g, \alpha) \\ &\ge L(\mathcal{P}_1, f, \alpha) + L(\mathcal{P}_2, g, \alpha). \end{aligned} \tag{5.19.6}$$

This implies that

$$\underline{\int_a^b} (f+g)\,d\alpha \ge \underline{\int_a^b} f\,d\alpha + \underline{\int_a^b} g\,d\alpha. \tag{5.19.7}$$

If f and g are Riemann–Stieltjes integrable with respect to α on $[a, b]$, then it follows that

(5.19.8) $f + g$ is Riemann–Stieltjes integrable with respect to α on $[a, b]$,

with

$$\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha. \tag{5.19.9}$$

This corresponds to another part of part (a) of Theorem 6.12 on p128 of [112].

Let f and g be any two bounded real-valued functions on $[a, b]$ again. If $\mathcal{P}$ is a partition of $[a, b]$, then one can verify that

$$U(\mathcal{P}, f+g, \alpha) \ge U(\mathcal{P}, f, \alpha) + L(\mathcal{P}, g, \alpha) \tag{5.19.10}$$

and

$$L(\mathcal{P}, f+g, \alpha) \le L(\mathcal{P}, f, \alpha) + U(\mathcal{P}, g, \alpha), \tag{5.19.11}$$

using the remarks at the beginning of Section 5.9. These inequalities can also be obtained from those mentioned at the beginning of the section using some of the remarks in the previous section.

One can use these inequalities to get that

$$\overline{\int_a^b} (f+g)\,d\alpha \ge \overline{\int_a^b} f\,d\alpha + \underline{\int_a^b} g\,d\alpha \tag{5.19.12}$$

and

$$\underline{\int_a^b} (f+g)\, d\alpha \le \underline{\int_a^b} f\, d\alpha + \overline{\int_a^b} g\, d\alpha, \tag{5.19.13}$$

as before. This could also be obtained from the analogous inequalities mentioned earlier and some of the remarks in the previous section. If g is Riemann–Stieltjes integrable with respect to α on $[a, b]$, then we get that

$$\overline{\int_a^b} (f+g)\, d\alpha = \overline{\int_a^b} f\, d\alpha + \int_a^b g\, d\alpha \tag{5.19.14}$$

and

$$\underline{\int_a^b} (f+g)\, d\alpha = \underline{\int_a^b} f\, d\alpha + \int_a^b g\, d\alpha. \tag{5.19.15}$$

5.20 Another monotonically increasing function

We continue with the same notation and hypotheses as in the previous sections, starting in Section 5.10. Let β be another monotonically increasing real-valued function on $[a, b]$, so that

$$\alpha + \beta \text{ is monotonically increasing on } [a, b] \tag{5.20.1}$$

too. If $\mathcal{P}$ is a partition of $[a, b]$, then it is easy to see that

$$U(\mathcal{P}, f, \alpha + \beta) = U(\mathcal{P}, f, \alpha) + U(\mathcal{P}, f, \beta) \tag{5.20.2}$$

and

$$L(\mathcal{P}, f, \alpha + \beta) = L(\mathcal{P}, f, \alpha) + L(\mathcal{P}, f, \beta). \tag{5.20.3}$$

This implies that

$$U(\mathcal{P}, f, \alpha + \beta) \ge \overline{\int_a^b} f\, d\alpha + \overline{\int_a^b} f\, d\beta \tag{5.20.4}$$

and

$$L(\mathcal{P}, f, \alpha + \beta) \le \underline{\int_a^b} f\, d\alpha + \underline{\int_a^b} f\, d\beta. \tag{5.20.5}$$

It follows that

$$\overline{\int_a^b} f\, d(\alpha + \beta) \ge \overline{\int_a^b} f\, d\alpha + \overline{\int_a^b} f\, d\beta \tag{5.20.6}$$

and

$$\underline{\int_a^b} f\, d(\alpha + \beta) \le \underline{\int_a^b} f\, d\alpha + \underline{\int_a^b} f\, d\beta. \tag{5.20.7}$$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be arbitrary partitions of $[a, b]$, and let $\mathcal{P}^*$ be a common refinement of them. Note that

$$\begin{aligned} \overline{\int_a^b} f\,d(\alpha+\beta) &\leq U(\mathcal{P}^*, f, \alpha+\beta) = U(\mathcal{P}^*, f, \alpha) + U(\mathcal{P}^*, f, \beta) \\ &\leq U(\mathcal{P}_1, f, \alpha) + U(\mathcal{P}_2, f, \beta), \end{aligned} \tag{5.20.8}$$

which implies that

$$\overline{\int_a^b} f\,d(\alpha+\beta) \leq \overline{\int_a^b} f\,d\alpha + \overline{\int_a^b} f\,d\beta. \tag{5.20.9}$$

Combining this with (5.20.6) we obtain that

$$\overline{\int_a^b} f\,d(\alpha+\beta) = \overline{\int_a^b} f\,d\alpha + \overline{\int_a^b} f\,d\beta. \tag{5.20.10}$$

Similarly,

$$\begin{aligned} \underline{\int_a^b} f\,d(\alpha+\beta) &\geq L(\mathcal{P}^*, f, \alpha+\beta) = L(\mathcal{P}^*, f, \alpha) L(\mathcal{P}^*, f, \beta) \\ &\geq L(\mathcal{P}_1, f, \alpha) + L(\mathcal{P}_2, f, \beta), \end{aligned} \tag{5.20.11}$$

and thus

$$\underline{\int_a^b} f\,d(\alpha+\beta) \geq \underline{\int_a^b} f\,d\alpha + \underline{\int_a^b} f\,d\beta. \tag{5.20.12}$$

This means that

$$\underline{\int_a^b} f\,d(\alpha+\beta) = \underline{\int_a^b} f\,d\alpha + \underline{\int_a^b} f\,d\beta, \tag{5.20.13}$$

because of (5.20.7).

One can use (5.20.10) and (5.20.13) to get that

(5.20.14) f is Riemann–Stieltjes integrable with respect to $\alpha + \beta$ on $[a, b]$

if and only if

(5.20.15) f is Riemann–Stieltjes integrable with respect to both α and β on $[a, b]$.

In this case, we have that

$$\int_a^b f\,d(\alpha+\beta) = \int_a^b f\,d\alpha + \int_a^b f\,d\beta, \tag{5.20.16}$$

as in part (e) of Theorem 6.12 on p128 of [112].

5.21 Some inequalities with integrals

Let us continue with the same notation and hypotheses as in the previous sections, starting in Section 5.10. Let g be another bounded real-valued function on $[a, b]$ again, and suppose that

$$f(x) \le g(x) \tag{5.21.1}$$

for every $x \in [a, b]$. If $\mathcal{P}$ is a partition of $[a, b]$, then it is easy to see that

$$U(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, g, \alpha) \tag{5.21.2}$$

and

$$L(\mathcal{P}, f, \alpha) \le L(\mathcal{P}, g, \alpha). \tag{5.21.3}$$

This implies that

$$\overline{\int_a^b} f \, d\alpha \le U(\mathcal{P}, g, \alpha) \tag{5.21.4}$$

and

$$L(\mathcal{P}, f, \alpha) \le \underline{\int_a^b} g \, d\alpha, \tag{5.21.5}$$

by definition of the upper and lower integrals, as in Section 5.13. It follows that

$$\overline{\int_a^b} f \, d\alpha \le \overline{\int_a^b} g \, d\alpha \tag{5.21.6}$$

and

$$\underline{\int_a^b} f \, d\alpha \le \underline{\int_a^b} g \, d\alpha, \tag{5.21.7}$$

because $\mathcal{P}$ is arbitrary. If f and g are Riemann–Stieltjes integrable with respect to α on $[a, b]$, then we get that

$$\int_a^b f \, d\alpha \le \int_a^b g \, d\alpha. \tag{5.21.8}$$

This corresponds to part (b) of Theorem 6.12 on p128 of [112].

If f is Riemann–Stieltjes integrable with respect to α on $[a, b]$, then

$$\Big(\inf_{a \le x \le b} f(x)\Big)(\alpha(b) - \alpha(a)) \le \int_a^b f \, d\alpha \le \Big(\sup_{a \le x \le b} f(x)\Big)(\alpha(b) - \alpha(a)) \tag{5.21.9}$$

as in Section 5.13. The first inequality implies that

$$\begin{aligned} -\int_a^b f \, d\alpha &\le -\Big(\inf_{a \le x \le b} f(x)\Big)(\alpha(b) - \alpha(a)) \\ &= \Big(\sup_{a \le x \le b} (-f(x))\Big)(\alpha(b) - \alpha(a)), \end{aligned} \tag{5.21.10}$$

where the second step is as in Section 5.8. One can check that

$$\sup_{a \le x \le b} |f(x)| = \max\Big(\sup_{a \le x \le b} f(x), \sup_{a \le x \le b} (-f(x))\Big). \tag{5.21.11}$$

It follows that

$$\Big|\int_a^b f\,d\alpha\Big| \le \Big(\sup_{a \le x \le b} |f(x)|\Big)\,(\alpha(b) - \alpha(a)). \tag{5.21.12}$$

This corresponds to part (d) of Theorem 6.12 on p128 of [112].

5.22 Subintervals of $[a, b]$

We continue with the same notation and hypotheses as in the previous sections again, starting in Section 5.10. Let c be a real number with

$$a \le c \le b, \tag{5.22.1}$$

so that $[a, c]$ and $[c, b]$ are closed intervals in $\mathbf{R}$ with

$$[a, c] \cup [c, b] = [a, b]. \tag{5.22.2}$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of $[a, c]$ and $[c, b]$, respectively, then we can combine them to get a partition $\mathcal{P}_{1,2}$ of $[a, b]$. Of course, the restrictions of f to $[a, c]$ and $[c, b]$ define bounded real-valued functions on the two intervals. Similarly, the restrictions of α to $[a, c]$ and $[c, b]$ define monotonically increasing real-valued functions on these intervals. It is easy to see that

$$U(\mathcal{P}_{1,2}, f, \alpha) = U(\mathcal{P}_1, f, \alpha) + U(\mathcal{P}_2, f, \alpha) \tag{5.22.3}$$

and

$$L(\mathcal{P}_{1,2}, f, \alpha) = L(\mathcal{P}_1, f, \alpha) + L(\mathcal{P}_2, f, \alpha). \tag{5.22.4}$$

This implies that

$$\overline{\int_a^b} f\,d\alpha \le U(\mathcal{P}_1, f, \alpha) + U(\mathcal{P}_2, f, \alpha) \tag{5.22.5}$$

and

$$\underline{\int_a^b} f\,d\alpha \ge L(\mathcal{P}_1, f, \alpha) + L(\mathcal{P}_2, f, \alpha). \tag{5.22.6}$$

One can use this to get that

$$\overline{\int_a^b} f\,d\alpha \le \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha \tag{5.22.7}$$

and

$$\underline{\int_a^b} f\,d\alpha \ge \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha. \tag{5.22.8}$$

Let $\mathcal{P}$ be any partition of $[a, b]$, and let $\mathcal{P}^*$ be a partition of $[a, b]$ that is a refinement of $\mathcal{P}$ that contains c, which can be obtained by adding c to $\mathcal{P}$, if necessary. Using $\mathcal{P}^*$, we get partitions $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ of $[a, c]$ and $[c, b]$, respectively. Observe that

$$\begin{aligned} U(\mathcal{P}, f, \alpha) \ge U(\mathcal{P}^*, f, \alpha) &= U(\mathcal{P}_1^*, f, \alpha) + U(\mathcal{P}_2^*, f, \alpha) \\ &\ge \overline{\int_a^c} f\, d\alpha + \overline{\int_c^b} f\, d\alpha, \end{aligned} \tag{5.22.9}$$

where the first step is as in Section 5.12, the second step is as in (5.22.3), and the third step is as in Section 5.13. This implies that

$$\overline{\int_a^b} f\, d\alpha \ge \overline{\int_a^c} f\, d\alpha + \overline{\int_c^b} f\, d\alpha, \tag{5.22.10}$$

so that

$$\overline{\int_a^b} f\, d\alpha = \overline{\int_a^c} f\, d\alpha + \overline{\int_c^b} f\, d\alpha. \tag{5.22.11}$$

Similarly,

$$\begin{aligned} L(\mathcal{P}, f, \alpha) \le L(\mathcal{P}^*, f, \alpha) &= L(\mathcal{P}_1^*, f, \alpha) + L(\mathcal{P}_2^*, f, \alpha) \\ &\le \underline{\int_a^c} f\, d\alpha + \underline{\int_c^b} f\, d\alpha, \end{aligned} \tag{5.22.12}$$

which implies that

$$\underline{\int_a^b} f\, d\alpha \le \underline{\int_a^c} f\, d\alpha + \underline{\int_c^b} f\, d\alpha, \tag{5.22.13}$$

and thus

$$\underline{\int_a^b} f\, d\alpha = \underline{\int_a^c} f\, d\alpha + \underline{\int_c^b} f\, d\alpha. \tag{5.22.14}$$

These statements could also be obtained from the previous ones using the remarks in Section 5.18.

One can use (5.22.11) and (5.22.14) to get that f is Riemann–Stieltjes integrable with respect to α on $[a, b]$ if and only if

(5.22.15) the restrictions of f to $[a, c]$ and $[c, b]$ are Riemann–Stieltjes integrable with respect to α.

In this case, we have that

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha, \tag{5.22.16}$$

as in part (c) of Theorem 6.12 on p128 of [112].

If f is Riemann–Stieltjes integrable with respect to α on $[a, b]$, then one can repeat the argument in the preceding paragraph to get that

(5.22.17) the restriction of f to any closed subinterval of $[a, b]$ is Riemann–Stieltjes integrable with respect to α

as well.

5.23 Derivatives of integrals

We continue with the same notation and hypotheses as before, starting in Section 5.10. Suppose that f is Riemann–Stieltjes integrable with respect to α on $[a, b]$. If $x \in [a, b]$, then the restriction of f to $[a, x]$ is Riemann–Stieltjes integrable with respect to α, as in the previous section, and we put

$$F(x) = \int_a^x f\,d\alpha. \tag{5.23.1}$$

Similarly, if $y \in [x, b]$, then the restrictions of f to $[a, y]$ and $[x, y]$ are Riemann–Stieltjes integrable with respect to α, and

$$F(y) - F(x) = \int_x^y f\,d\alpha, \tag{5.23.2}$$

as in the previous section.

Observe that

$$\begin{aligned} |F(y) - F(x)| = \left|\int_x^y f\,d\alpha\right| &\le \Big(\sup_{x \le v \le y} |f(v)|\Big)\,(\alpha(y) - \alpha(x)) \\ &\le \Big(\sup_{a \le v \le b} |f(v)|\Big)\,(\alpha(y) - \alpha(x)), \end{aligned} \tag{5.23.3}$$

where the second step is as in Section 5.21. One can use this to get that

(5.23.4) F is continuous at any point in $[a, b]$ at which α is continuous.

This basically corresponds to the first part of Theorem 6.20 on p133 of [112].

Suppose from now on in this section that $a < b$, and that $\alpha(u) = u$ on $[a, b]$, so that f is Riemann integrable on $[a, b]$. Put

$$F(x) = \int_a^x f(t)\,dt \tag{5.23.5}$$

for each $x \in [a, b]$, as before. If $y \in [x, b]$, then

$$F(y) - F(x) = \int_x^y f(t)\,dt, \tag{5.23.6}$$

as before. If

$$f \text{ is continuous at a point } w \in [a, b], \tag{5.23.7}$$

then it is well known and not too difficult to show that

$$F \text{ is differentiable at } w, \tag{5.23.8}$$

with

$$F'(w) = f(w). \tag{5.23.9}$$

This is the second part of Theorem 6.20 on p133 of [112].

More precisely, suppose first that

$$a \le w < b, \tag{5.23.10}$$

and let $h > 0$ be given, with $h \le b - w$. Observe that

$$\frac{F(w+h) - F(w)}{h} = \frac{1}{h} \int_w^{w+h} f(t)\, dt, \tag{5.23.11}$$

because of (5.23.6). If f is continuous at w, then one can check that

$$\lim_{h \to 0+} \frac{F(w+h) - F(x)}{h} = f(w). \tag{5.23.12}$$

Of course, if f is constant on $[a, b]$, then (5.23.11) is the same as the constant value of f. One can use this to reduce to the case where $f(w) = 0$, which is a bit simpler.

Similarly, suppose now that

$$a < w \le b, \tag{5.23.13}$$

and let $h > 0$ be given, with $h \le w - a$. In this case,

$$\frac{F(w) - F(w-h)}{h} = \frac{1}{h} \int_{w-h}^{w} f(t)\, dt, \tag{5.23.14}$$

using (5.23.6) again. If f is continuous at w, then one can verify that

$$\lim_{h \to 0+} \frac{F(w) - F(w-h)}{h} = f(w), \tag{5.23.15}$$

as before.

5.24 Integrals of derivatives

Let a and b be real numbers with $a < b$, and let F_1 be a real-valued function on $[a, b]$. Suppose that

$$F_1 \text{ is differentiable at every point in } [a, b], \tag{5.24.1}$$

and put

$$f = F_1' \tag{5.24.2}$$

on $[a, b]$. If

$$f \text{ is Riemann integrable on } [a, b], \tag{5.24.3}$$

then Theorem 6.21 on p134 of [112] says that

$$\int_a^b f(x)\,dx = F_1(b) - F_1(a). \tag{5.24.4}$$

This is a version of the *fundamental theorem of calculus*. Of course, (5.24.3) includes the condition that f be bounded on $[a, b]$.

If f is continuous on $[a, b]$, and F is as in (5.23.5), then F is differentiable at every point in $[a, b]$, as in the previous section, and we have that

$$(F_1 - F)' = F_1' - F' = 0 \tag{5.24.5}$$

on $[a, b]$. This means that $F_1 - F$ is constant on $[a, b]$, by the mean value theorem. In this case, (5.24.4) follows from the definition of F.

Otherwise, let $\mathcal{P} = \{t_j\}_{j=0}^l$ be a partition of $[a, b]$, and note that

$$\sum_{j=1}^l (F_1(t_j) - F_1(t_{j-1})) = F_1(b) - F_1(a), \tag{5.24.6}$$

as in (5.11.5). The mean value theorem implies that for each $j = 1, \ldots, l$ there is an $r_j \in [t_{j-1}, t_j]$ such that

$$F_1(t_j) - F_1(t_{j-1}) = F_1'(r_j)\,(t_j - t_{j-1}). \tag{5.24.7}$$

This means that

$$\sum_{j=1}^l F_1'(r_j)\,(t_j - t_{j-1}) = F_1(b) - F_1(a), \tag{5.24.8}$$

by (5.24.6).

If f is bounded on $[a, b]$, then

$$L(\mathcal{P}, f) \le \sum_{j=1}^l F_1'(r_j)\,(t_j - t_{j-1}) \le U(\mathcal{P}, f), \tag{5.24.9}$$

by definition of the upper and lower Darboux sums associated to f and $\mathcal{P}$, as in Section 5.10. Thus

$$L(\mathcal{P}, f) \le F_1(b) - F_1(a) \le U(\mathcal{P}, f), \tag{5.24.10}$$

by (5.24.8). It follows that

$$\underline{\int_a^b} f(x)\,dx \le F_1(b) - F_1(a) \le \overline{\int_a^b} f(x)\,dx, \tag{5.24.11}$$

by definition of the upper and lower integrals, as in Section 5.13. If (5.24.3) holds, then we get that (5.24.4) holds.

5.25 More on Riemann–Stieltjes integrability

Let us continue with the same notation and hypotheses as before, starting in Section 5.10. Let a_1, b_1 be real numbers with $a_1 \le b_1$, and suppose that

$$a_1 \le f \le b_1 \tag{5.25.1}$$

on $[a, b]$. Also let ϕ be a continuous real-valued function on $[a_1, b_1]$, and note that ϕ is bounded on $[a_1, b_1]$, because $[a_1, b_1]$ is compact. Thus $\phi \circ f$ is a bounded real-valued function on $[a, b]$.

If f is Riemann–Stieltjes integrable with respect to α on $[a, b]$, then

(5.25.2) $\phi \circ f$ is Riemann–Stieltjes integrable with respect to α on $[a, b]$

too. This is Theorem 6.11 on p127 of [112]. This uses the fact that ϕ is uniformly continuous on $[a_1, b_1]$, because $[a_1, b_1]$ is compact. Basically, one can show that

$$U(\mathcal{P}, \phi \circ f, \alpha) - L(\mathcal{P}, \phi \circ f, \alpha) \tag{5.25.3}$$

is arbitrarily small when

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) \tag{5.25.4}$$

is sufficiently small.

In particular, if f is a bounded real-valued function on $[a, b]$ that is Riemann–Stieltjes integrable with respect to α, then

(5.25.5) f^2 is Riemann–Stieltjes integrable with respect to α on $[a, b]$.

Let g be another bounded real-valued function on $[a, b]$ that is Riemann–Stieltjes integrable with respect to α. Under these conditions, part (a) of Theorem 6.13 on p129 of [112] says that

(5.25.6) $f\,g$ is Riemann–Stieltjes integrable with respect to α on $[a, b]$.

This uses the fact that

$$f\,g = (1/2)\,((f + g)^2 - f^2 - g^2). \tag{5.25.7}$$

Similarly,

(5.25.8) $|f|$ is Riemann–Stieltjes integrable with respect to α on $[a, b]$,

as in part (b) of Theorem 6.13 on p129 of [112]. Note that

$$\int_a^b f\,d\alpha,\ -\int_a^b f\,d\alpha \le \int_a^b |f|\,d\alpha, \tag{5.25.9}$$

because $f, -f \le |f|$ on $[a, b]$. This means that

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha, \tag{5.25.10}$$

as in [112].

5.26 Differentiability of α

Let us continue with the same notation and hypotheses as in the previous sections. Suppose that $a < b$,

$$\alpha \text{ is differentiable at every point in } [a, b], \tag{5.26.1}$$

α' is bounded on $[a, b]$, and

$$\alpha' \text{ is Riemann integrable on } [a, b]. \tag{5.26.2}$$

Under these conditions, Theorem 6.17 on 131 of [112] states that a bounded real-valued function f on $[a, b]$ is Riemann–Stieltjes integrable with respect to α on $[a, b]$ if and only if

$$f\,\alpha' \text{ is Riemann integrable on } [a, b], \tag{5.26.3}$$

in which case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx. \tag{5.26.4}$$

Note that (5.26.3) holds when f is Riemann integrable on $[a, b]$, because of (5.26.2), as in the previous section.

5.27 Changing variables

Let a_0 and b_0 be real numbers with $a_0 \le b_0$, and let ϕ be a continuous real-valued function on $[a_0, b_0]$ that is also monotonically increasing on $[a_0, b_0]$. If y, z are real numbers with $a_0 \le y \le z \le b_0$, then

$$\phi([y, z]) = [\phi(y), \phi(z)]. \tag{5.27.1}$$

More precisely,

$$\phi([y, z]) \subseteq [\phi(y), \phi(z)], \tag{5.27.2}$$

because ϕ is monotonically increasing on $[a_0, b_0]$, and

$$[\phi(y), \phi(z)] \subseteq \phi([y, z]), \tag{5.27.3}$$

by the intermediate value theorem.

Put $a = \phi(a_0)$ and $b = \phi(b_0)$, so that $a \le b$. Let α be a monotonically increasing real-valued function on $[a, b]$, as usual, and put

$$\beta = \alpha \circ \phi. \tag{5.27.4}$$

It is easy to see that this is a monotonically increasing real-valued function on $[a_0, b_0]$. Also let f be a bounded real-valued function on $[a, b]$, as before, so that

$$g = f \circ \phi \tag{5.27.5}$$

is a bounded real-valued function on $[a_0, b_0]$.

Let $\mathcal{P}_0 = \{t_{0,j}\}_{j=0}^{l_0}$ be a partition of $[a_0, b_0]$. Note that

$$\mathcal{P}_0^{\phi} = \{\phi(t_{0,j})\}_{j=0}^{l_0} \tag{5.27.6}$$

is a partition of $[a, b]$. One can check that

$$U(\mathcal{P}_0, g, \beta) = U(\mathcal{P}_0^{\phi}, f, \alpha) \tag{5.27.7}$$

and

$$L(\mathcal{P}_0, g, \beta) = L(\mathcal{P}_0^{\phi}, f, \alpha), \tag{5.27.8}$$

as in (38) on p133 of [112]. This uses the fact that

$$\phi([t_{0,j-1}, t_{0,j}]) = [\phi(t_{0,j-1}), \phi(t_{0,j})] \tag{5.27.9}$$

for each $j = 1, \ldots, l_0$, as in (5.27.1).

Observe that every partition of $[a, b]$ is of the form (5.27.6) for some partition $\mathcal{P}_0$ of $[a_0, b_0]$. One can use this and (5.27.7), (5.27.8) to get that

$$\overline{\int_{a_0}^{b_0}} g\, d\beta = \overline{\int_{a}^{b}} f\, d\alpha \tag{5.27.10}$$

and

$$\underline{\int_{a_0}^{b_0}} g\, d\beta = \underline{\int_{a}^{b}} f\, d\alpha. \tag{5.27.11}$$

In particular, if f is Riemann–Stieltjes integrable with respect to α on $[a, b]$, then

(5.27.12) g is Riemann–Stieltjes integrable with respect to β on $[a_0, b_0]$,

with

$$\int_{a_0}^{b_0} g\, d\beta = \int_{a}^{b} f\, d\alpha, \tag{5.27.13}$$

as in Theorem 6.19 on p132 of [112]. Although this theorem is stated for the case where ϕ is strictly increasing on $[a_0, b_0]$, we do not really need this here. If ϕ is strictly increasing on $[a_0, b_0]$, then we get a one-to-one correspondence between partitions of $[a_0, b_0]$ and $[a, b]$, but this is not really needed.

Of course, if

$$\phi(y) = \phi(z) \tag{5.27.14}$$

for some $y, z \in [a_0, b_0]$ with $y < z$, then ϕ is constant on $[y, z]$, because ϕ is monotonically increasing on $[a_0, b_0]$. This implies that g and β are constant on $[y, z]$ as well.

More precisely, the previous argument shows that (5.27.12) holds if and only if f is Riemann–Stieltjes integrable with respect to α on $[a, b]$. If ϕ is strictly increasing on $[a_0, b_0]$, then one can also look at the "only if" part in terms of the inverse of ϕ.

5.28 Some sums of step functions

We continue with the same notation and hypotheses as before, starting in Section 5.10 again. Let $I(x)$ be the unit step function on the real line, as in Section 4.12, and let $\{w_j\}_{j=1}^{\infty}$ be a sequence of elements of (a, b). Also let $\{c_j\}_{j=1}^{\infty}$ be an infinite sequence of nonnegative real numbers such that

$$\sum_{j=1}^{\infty} c_j \text{ converges,} \tag{5.28.1}$$

and put

$$\alpha(x) = \sum_{j=1}^{\infty} c_j\, I(x - w_j) \tag{5.28.2}$$

for each $x \in [a, b]$, where the series on the right converges by the comparison test. This defines a monotonically increasing real-valued function on $[a, b]$. If f is continuous on $[a, b]$, then Theorem 6.16 on p130 of [112] states that

$$\int_a^b f\, d\alpha = \sum_{j=1}^{\infty} c_j\, f(w_j). \tag{5.28.3}$$

Note that the series on the right converges absolutely, by the comparison test, because f is bounded on $[a, b]$. More precisely, in [112], the w_j's are asked to be distinct, but this does not really seem to be needed, although there may not be any reason to consider repetitions. One could permit w_j to be equal to a for some j, but not b, because of the way that $I(x)$ is defined, as in Section 5.17. One could use analogous arguments to get that f is Riemann–Stieltjes integrable with respect to α on $[a, b]$ when

$$f \text{ is continuous at } w_j \text{ for each } j, \tag{5.28.4}$$

in addition to being bounded on $[a, b]$, and that (5.28.3) holds in this case.

Let n be a positive integer, and put

$$\alpha_n(x) = \sum_{j=1}^{n} c_j\, I(x - w_j) \tag{5.28.5}$$

for each $x \in [a, b]$. This is a monotonically increasing real-valued function on $[a, b]$. If f is continuous at each of $w_1, \dots, w_n$, then it is easy to see that f is Riemann–Stieltjes integrable with respect to α_n on $[a, b]$, with

$$\int_a^b f\, d\alpha_n = \sum_{j=1}^{n} c_j\, f(w_j), \tag{5.28.6}$$

using remarks in Section 5.17 and 5.20.

Similarly,

(5.28.7) $$\beta_n(x) = \sum_{j=n+1}^{\infty} c_j \, I(x - w_j)$$

defines a monotonically increasing real-valued function on $[a, b]$, with

(5.28.8) $$\alpha_n(x) + \beta_n(x) = \alpha(x)$$

for each $x \in [a, b]$. Observe that

(5.28.9) $$\overline{\int_a^b} f \, d\beta_n \le M \, (\beta_n(b) - \beta_n(a)) = M \sum_{j=n+1}^{\infty} c_j$$

and

(5.28.10) $$\underline{\int_a^b} f \, d\beta_n \ge m \, (\beta_n(b) - \beta_n(a)) = m \sum_{j=n+1}^{\infty} c_j,$$

where the first steps are as in Section 5.13, and the second steps follow from the definition (5.28.7) of β_n. Of course, the right sides of (5.28.9) and (5.28.10) tend to 0 as $n \to \infty$, because $\sum_{j=1}^{\infty} c_j$ converges, by hypothesis. To get the conclusions mentioned earlier, one can use (5.28.8) and the remarks in Section 5.20.

5.29 Limits of integrals

Let a and b be real numbers with $a < b$, and let $\{f_j\}_{j=1}^{\infty}$ be a sequence of bounded real-valued functions on $[a, b]$ that converges uniformly to a real-valued function f on $[a, b]$. Note that

(5.29.1) $$f \text{ is bounded on } [a, b]$$

as well, as mentioned in Section 4.9. Let α be a monotonically increasing real-valued function on $[a, b]$, as before.

Of course, $\{f_j - f\}_{j=1}^{\infty}$ converges to 0 uniformly on $[a, b]$. It is easy to see that

(5.29.2) $$\lim_{j \to \infty} \overline{\int_a^b} (f_j - f) \, d\alpha = 0$$

and

(5.29.3) $$\lim_{j \to \infty} \underline{\int_a^b} (f_j - f) \, d\alpha = 0,$$

using the remarks in Section 5.13.

It is not too difficult to show that

(5.29.4) $$\lim_{j \to \infty} \overline{\int_a^b} f_j \, d\alpha = \overline{\int_a^b} f \, d\alpha$$

and

$$\lim_{j \to \infty} \underline{\int_a^b} f_j \, d\alpha = \underline{\int_a^b} f \, d\alpha, \tag{5.29.5}$$

using the remarks in Section 5.19. This also uses (5.29.2), (5.29.3), the facts that

$$f_j = f + (f_j - f) \tag{5.29.6}$$

and

$$f = f_j + (f - f_j) \tag{5.29.7}$$

for each j.

If

(5.29.8) f_j is Riemann–Stieltjes integrable with respect to α on $[a, b]$

for each j, then it follows that

(5.29.9) f is Riemann–Stieltjes integrable with respect to α on $[a, b]$,

with

$$\lim_{j \to \infty} \int_a^b f_j \, d\alpha = \int_a^b f \, d\alpha, \tag{5.29.10}$$

as in Theorem 7.16 on p151 of [112].

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Index