Some topics in analysis related to Banach algebras, 2

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Abstract

These notes deal with some spaces of power series over fields with absolute value functions.

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Part I
Preliminaries

1 A few basic inequalities

If f is a nonnegative real-valued function on a nonempty finite set X, then put

$$\|f\|_r = \Big(\sum_{x \in X} f(x)^r \Big)^{1/r} \tag{1.1}$$

for every positive real number r, and

$$\|f\|_\infty = \max_{x \in X} f(x). \tag{1.2}$$

Clearly

$$\|f\|_\infty \le \|f\|_r \le (\#X)^{1/r} \, \|f\|_\infty \tag{1.3}$$

for every $r > 0$, where $\#X$ is the number of elements in X. Thus

$$\lim_{r \to \infty} \|f\|_r = \|f\|_\infty, \tag{1.4}$$

by the well-known fact that $a^{1/r} \to 1$ as $r \to \infty$ for every positive real number a. If $0 < r_1 \le r_2 < \infty$, then

$$\|f\|_{r_2}^{r_2} = \sum_{x \in X} f(x)^{r_2} \le \|f\|_\infty^{r_2 - r_1} \sum_{x \in X} f(x)^{r_1} = \|f\|_\infty^{r_2 - r_1} \, \|f\|_{r_1}^{r_1} \le \|f\|_{r_1}^{r_2}, \tag{1.5}$$

using (1.3) in the last step. It follows that

$$\|f\|_{r_2} \le \|f\|_\infty^{1 - (r_1/r_2)} \, \|f\|_{r_1}^{r_1/r_2} \le \|f\|_{r_1}. \tag{1.6}$$

If a, b are nonnegative real numbers and r is a positive real number, then

$$\max(a, b) \le (a^r + b^r)^{1/r}, \tag{1.7}$$

as in the first inequality in (1.3). Similarly, if $0 < r_1 \le r_2 < \infty$, then

$$(a^{r_2} + b^{r_2})^{1/r_2} \le (a^{r_1} + b^{r_1})^{1/r_1}, \tag{1.8}$$

by (1.6), where X has exactly two elements. If $0 < r \le 1$, then

$$(a + b)^r \le a^r + b^r. \tag{1.9}$$

This follows from (1.8), with $r_1 = r$ and $r_2 = 1$, and by taking rth powers of both sides of the inequality.

Let X be a nonempty finite set again, and let f and g be nonnegative real-valued functions on X. If $1 \le r \le \infty$, then *Minkowski's inequality* for finite sums says that

$$\|f + g\|_r \le \|f\|_r + \|g\|_r. \tag{1.10}$$

If $0 < r \le 1$, then one can check that

$$\|f + g\|_r^r \le \|f\|_r^r + \|g\|_r^r, \tag{1.11}$$

using (1.9). Of course, equality holds trivially in (1.10) and (1.11) when $r = 1$. If $r = \infty$, then it is easy to verify (1.10) directly.

2 q-Semimetrics

A nonnegative real-valued function $d(x,y)$ defined for x, y in a set X is said to be a *q-semimetric* on X for some positive real number q if it satisfies the following three conditions: first,

$$d(x,x) = 0 \quad \text{for every } x \in X; \tag{2.1}$$

second,

$$d(x,y) = d(y,x) \quad \text{for every } x, y \in X; \tag{2.2}$$

and third,

$$d(x,z)^q \le d(x,y)^q + d(y,z)^q \quad \text{for every } x, y, z \in X. \tag{2.3}$$

If, in addition,

$$d(x,y) > 0 \quad \text{when } x \ne y, \tag{2.4}$$

then $d(\cdot,\cdot)$ is a *q-metric* on X. In the case where $q = 1$, $d(\cdot,\cdot)$ is said to be a *metric* or *semimetric*, as appropriate.

A nonnegative real-valued function $d(x,y)$ defined for $x, y \in X$ is said to be a *semi-ultrametric* on X if it satisfies (2.1), (2.2), and

$$d(x,z) \le \max(d(x,y), d(y,z)) \quad \text{for every } x, y, z \in X. \tag{2.5}$$

If (2.4) holds too, then $d(\cdot,\cdot)$ is an *ultrametric* on X. An ultrametric or semi-ultrametric on X is also considered to be a q-metric or q-semimetric on X with $q = \infty$, respectively. As usual, the *discrete metric* is defined on X by (2.1) and putting $d(x,y) = 1$ when $x \ne y$,. It is easy to see that this is an ultrametric on X.

Note that (2.3) is the same as saying that

$$d(x,z) \le (d(x,y)^q + d(y,z)^q)^{1/q} \quad \text{for every } x, y, z \in X. \tag{2.6}$$

The right side of this inequality is automatically greater than or equal to the right side of the inequality in (2.5), as in (1.7). Similarly, the right side of the inequality in (2.6) decreases monotonically in q, as in (1.8). If $0 < q_1 \le q_2 \le \infty$ and $d(x,y)$ is a q_2-metric or q_2-semimetric on X, then it follows that $d(x,y)$ is a q_1-metric or q_1-semimetric on X as well, respectively.

Let $0 < q \le \infty$ be given, and let $d(x,y)$ be a q-metric or q-semimetric on X. If $0 < a < \infty$, then it is easy to see that

$$d(x,y)^a \tag{2.7}$$

is a (q/a)-metric or (q/a)-semimetric on X, as appropriate. As usual, q/a is interpreted as being ∞ when $q = \infty$.

Let $d(x,y)$ be a q-semimetric on X for some $q > 0$ again. If $x \in X$ and $0 < r < \infty$, then

$$B(x,r) = B_d(x,r) = \{y \in X : d(x,y) < r\} \tag{2.8}$$

is the *open ball* in X centered at x with radius r with respect to d. Similarly, if $x \in X$ and $0 \le r < \infty$, then

$$\overline{B}(x,r) = \overline{B}_d(x,r) = \{y \in X : d(x,y) \le r\} \tag{2.9}$$

is the *closed ball* in X centered at x with radius r with respect to d. If a is a positive real number, then we can also consider open and closed balls in X with respect to (2.7). Observe that

$$B_{d^a}(x, r^a) = B_d(x,r) \tag{2.10}$$

for every $x \in X$ and $r > 0$, and that

$$\overline{B}_{d^a}(x, r^a) = \overline{B}_d(x,r) \tag{2.11}$$

for every $x \in X$ and $r \ge 0$.

As usual, $U \subseteq X$ is said to be an *open set* with respect to d if for every $x \in U$ there is an $r > 0$ such that

$$B(x,r) \subseteq U. \tag{2.12}$$

This defines a topology on X, by standard arguments. If a is a positive real number, then the topologies determined on X by $d(x,y)$ and $d(x,y)^a$ are the same, because of (2.10).

One can check that open balls in X with respect to $d(\cdot,\cdot)$ are open sets with respect to the topology determined by $d(\cdot,\cdot)$. More precisely, one can adapt the well-known argument for $q = 1$ to any $q > 0$, or reduce to that case when $0 < q \le 1$ using the fact that $d(x,y)^q$ is an ordinary semimetric on X. If $d(\cdot,\cdot)$ is a q-metric on X, then X is Hausdorff with respect to the topology determined by $d(\cdot,\cdot)$. Similarly, one can verify that closed balls in X with respect to $d(\cdot,\cdot)$ are closed sets.

If $q = \infty$, so that $d(\cdot,\cdot)$ is a semi-ultrametric on X, then one can check that closed balls in X of positive radius with respect to $d(\cdot,\cdot)$ are open sets. One can also verify that open balls in X are closed sets in this case.

Suppose now that $d(\cdot,\cdot)$ is a q-metric on X for some $q > 0$. The notion of a Cauchy sequence in X with respect to $d(\cdot,\cdot)$ can be defined in the same way as for ordinary metrics. Similarly, X is said to be *complete* with respect to $d(\cdot,\cdot)$ if every Cauchy sequence in X with respect to $d(\cdot,\cdot)$ converges to an element of X with respect to the topology determined by $d(\cdot,\cdot)$. If a is a positive real number, then it is easy to see that a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is a Cauchy sequence with respect to $d(\cdot,\cdot)$ if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $d(\cdot,\cdot)^a$. It follows that X is complete with respect to $d(\cdot,\cdot)$ if and only if X is complete with respect to $d(\cdot,\cdot)^a$.

If X is not already complete with respect to $d(\cdot,\cdot)$, then one can pass to a completion using standard arguments. This is well known for ordinary metric spaces, and one can reduce to that case by considering $d(x,y)^q$ as a metric on X when $0 < q \le 1$.

If X, Y are sets, d_X, d_Y are q_X, q_Y-semimetrics on X, Y, respectively, for some $q_X, q_Y > 0$, and f is a mapping from X into Y, then uniform continuity of f with respect to d_X and d_Y can be defined in the usual way. If a, b are positive real numbers, then it is easy to see that f is uniformly continuous with respect to d_X and d_Y if and only if f is uniformly continuous with respect to d_X^a and d_Y^b. This can be used to reduce to the case of ordinary semimetrics, as before.

3 q-Absolute value functions

A nonnegative real-valued function $|x|$ on a field k is said to be a *q-absolute value function* on k for some positive real number q if it satisfies the following three conditions: first, for every $x \in k$,

$$|x| = 0 \quad \text{if and only if} \quad x = 0; \tag{3.1}$$

second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x, y \in k; \tag{3.2}$$

and third,

$$|x + y|^q \le |x|^q + |y|^q \quad \text{for every } x, y \in k. \tag{3.3}$$

If these conditions hold with $q = 1$, then $|\cdot|$ is said to be an *absolute value function* on k. The standard absolute value functions on the fields $\mathbf{R}$ of real numbers and $\mathbf{C}$ of complex numbers satisfy these conditions with $q = 1$.

A nonnegative real-valued function $|\cdot|$ on a field k is said to be an *ultrametric absolute value function* on k if it satisfies (3.1), (3.2), and

$$|x + y| \le \max(|x|, |y|) \quad \text{for every } x, y \in k. \tag{3.4}$$

An ultrametric absolute value function on k is also considered as a q-absolute value function with $q = \infty$. The *trivial absolute value function* is defined on a field k by (3.1) and putting $|x| = 1$ when $x \neq 0$. One can check that this defines an ultrametric absolute value function on k.

As before, (3.3) is the same as saying that

$$|x + y| \le (|x|^q + |y|^q)^{1/q} \quad \text{for every } x, y \in k. \tag{3.5}$$

The right side of this inequality is automatically greater than or equal to the right side of (3.4), as in (1.7). The right side of the inequality in (3.5) is also monotonically decreasing in q, as in (1.8). If $0 < q_1 \le q_2 \le \infty$ and $|\cdot|$ is a q_1-absolute value function on a field k, then it follows that $|x|$ is a q_2-absolute value function on k too.

Suppose that $|x|$ is a q-absolute value function on a field k for some $q > 0$. If a is a positive real number, then it is easy to see that

$$|x|^a \tag{3.6}$$

is a (q/a)-absolute value function on k. In particular, if $|x|$ is an ultrametric absolute value function on k, then (3.6) is an ultrametric absolute value function on k as well.

Let $|x|$ be a q-absolute value function on a field k for some $q > 0$ again. Using (3.1) and (3.2), one can check that $|1| = 1$, where the first 1 is the multiplicative identity element in k, and the second 1 is the multiplicative identity element in $\mathbf{R}$. If $x \in k$ satisfies $x^n = 1$ for some positive integer n, then it follows that $|x| = 1$ too. In particular, $|-1| = 1$.

Observe that

$$d(x, y) = |x - y| \tag{3.7}$$

defines a q-metric on k. If $|\cdot|$ is the trivial absolute value function on k, then (3.7) is the discrete metric on k. The standard Euclidean metrics on $\mathbf{R}$ and $\mathbf{C}$ correspond to their standard absolute value functions as in (3.7).

The *p-adic absolute value* $|x|_p$ of a rational number x is defined for each prime number p as follows. If $x = 0$, then $|x|_p = 0$. Otherwise, if $x \neq 0$, then x can be expressed as $p^j\,(a/b)$ for some integers a, b, and j, where $a, b \neq 0$ and neither a nor b is a multiple of p, in which case

$$|x|_p = p^{-j}. \tag{3.8}$$

One can verify that this is an ultrametric absolute value function on the field $\mathbf{Q}$ of rational numbers. The corresponding ultrametric

$$d_p(x, y) = |x - y|_p \tag{3.9}$$

is the *p-adic metric* on $\mathbf{Q}$.

Let $|x|$ be a q-absolute value function on any field k for some $q > 0$ again. If k is not complete with respect to the q-metric (3.7), then one can pass to a completion in a standard way. The field operations can be extended to the completion, so that the completion is a field as well. Similarly, $|x|$ can be extended to a q-absolute value function on the completion, which corresponds to the distance to 0 in the completion. The field $\mathbf{Q}_p$ of *p-adic numbers* is obtained by completing $\mathbf{Q}$ with respect to the p-adic absolute value function $|x|_p$ for each prime number p.

Let k be a field, and suppose that $|\cdot|_1$, $|\cdot|_2$ are q_1, q_2-absolute value functions on k for some $q_1, q_2 > 0$. We say that $|\cdot|_1$, $|\cdot|_2$ are *equivalent* if there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{3.10}$$

for every $x \in k$. In this case,

$$|x - y|_2 = |x - y|_1^a \tag{3.11}$$

for every $x, y \in k$, so that the topologies determined on k by the q_1, q_2-metrics associated to $|\cdot|_1$, $|\cdot|_2$ are the same. Conversely, it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k when the topologies determined on k by the associated q_1, q_2-metrics are the same. If $|\cdot|$ is a q-absolute value function on $\mathbf{Q}$ for some

$q > 0$, then a famous theorem of Ostrowski implies that $|\cdot|$ is either trivial on $\mathbf{Q}$, or $|\cdot|$ is equivalent to the standard (Euclidean) absolute value function on $\mathbf{Q}$, or $|\cdot|$ is equivalent to the p-adic absolute value function on $\mathbf{Q}$ for some prime number p.

4 Some additional properties

Let k be a field, and let $\mathbf{Z}_+$ be the set of positive integers, as usual. If $x \in k$ and $n \in \mathbf{Z}_+$, then we let $n \cdot x$ be the sum of n x's in k. Also let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. If there are $n \in \mathbf{Z}_+$ such that $|n \cdot 1|$ is arbitrarily large, where 1 is the multiplicative identity element in k, then $|\cdot|$ is said to be *archimedean* on k. Otherwise, $|\cdot|$ is said to be *non-archimedean* on k when $|n \cdot 1|$ is bounded for $n \in \mathbf{Z}_+$. If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that $|n \cdot 1| \le 1$ for every $n \in \mathbf{Z}_+$, so that $|\cdot|$ is non-archimedean on k. Conversely, if $|\cdot|$ is a non-archimedean q-absolute value function on k for some $q > 0$, then it is well known that $|\cdot|$ is an ultrametric absolute value function on k. If $|\cdot|$ is a q-absolute value function on k for some $q > 0$, and if $|n \cdot 1| > 1$ for some $n \in \mathbf{Z}_+$, then one can check more directly that $|\cdot|$ is archimedean on k, because $|n^j \cdot 1| = |(n \cdot 1)^j| = |n \cdot 1|^j \to \infty$ as $j \to \infty$.

Let $|\cdot|$ be a q-absolute value function on a field k for some $q > 0$ again. If k has positive characteristic, then $|\cdot|$ is automatically non-archimedean on k, because there are only finitely many elements of k of the form $n \cdot 1$ with $n \in \mathbf{Z}_+$. Otherwise, if k has characteristic 0, then there is a natural embedding of $\mathbf{Q}$ into k, and $|\cdot|$ induces a q-absolute value function on $\mathbf{Q}$. In this case, $|\cdot|$ is archimedean on k if and only if the induced q-absolute value function on $\mathbf{Q}$ is archimedean. If $|\cdot|$ is archimedean on k, and k is complete with respect to the associated q-metric, then another famous theorem of Ostrowski implies that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, with $|\cdot|$ corresponding to a q-absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard (Euclidean) absolute value function.

Let $|\cdot|$ be a q-absolute value function on any field k for some $q > 0$ again, so that

$$\{|x| : x \in k,\ x \neq 0\} \tag{4.1}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. We say that $|\cdot|$ is *discrete* on k if the real number 1 is not a limit point of (4.1) with respect to the standard topology on $\mathbf{R}$. Otherwise, one can verify that (4.1) is dense in $\mathbf{R}_+$ with respect to the topology induced by the standard topology on $\mathbf{R}$. Put

$$\rho_1 = \sup\{|x| : x \in k,\ |x| < 1\}, \tag{4.2}$$

which is a nonnegative real number less than or equal to 1. One can check that $\rho_1 = 0$ if and only if $|\cdot|$ is trivial on k. One can also verify that $\rho_1 < 1$ if and only if $|\cdot|$ is discrete on k. If $|\cdot|$ is nontrivial and discrete on k, then $0 < \rho_1 < 1$, and it is not difficult to see that the supremum in (4.2) is attained. In this case, it is not too hard to show that (4.1) consists exactly of integer powers of ρ_1.

Suppose that $|\cdot|$ is archimedean on k, so that k has characteristic 0, as before. In this situation, the induced q-absolute value function on $\mathbf{Q}$ is archimedean as well. Hence the induced q-absolute value function on $\mathbf{Q}$ is equivalent to the standard Euclidean absolute value function on $\mathbf{Q}$, by the theorem of Ostrowski mentioned in the previous section. In particular, the induced q-absolute value function on $\mathbf{Q}$ is not discrete, which implies that $|\cdot|$ is not discrete on k. If $|\cdot|$ is a discrete q-absolute value function on a field k, then it follows that $|\cdot|$ is non-archimedean on k.

5 q-Seminorms

Let k be a field, let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *q-seminorm* on V with respect to $|\cdot|$ on k for some positive real number q if it satisfies the following two conditions: first,

$$N(t\,v) = |t|\,N(v) \quad \text{for every } v \in V \text{ and } t \in k; \tag{5.1}$$

and second,

$$N(v+w)^q \le N(v)^q + N(w)^q \quad \text{for every } v, w \in V. \tag{5.2}$$

If, in addition,

$$N(v) > 0 \quad \text{when } v \ne 0, \tag{5.3}$$

then N is said to be a *q-norm* on V. Of course, (5.1) implies that $N(0) = 0$. If $q = 1$, then N is said to be a *norm* or *seminorm* on V, as appropriate.

A nonnegative real-valued function N on V is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k if it satisfies (5.1) and

$$N(v+w) \le \max(N(v), N(w)) \quad \text{for every } v, w \in V. \tag{5.4}$$

If (5.3) holds too, then N is an *ultranorm* on V. As usual, an ultranorm or semi-ultranorm may be considered as a q-norm or q-seminorm, respectively, with $q = \infty$.

As before, (5.2) is the same as saying that

$$N(v+w) \le (N(v)^q + N(w)^q)^{1/q} \quad \text{for every } v, w \in V. \tag{5.5}$$

The right side of this inequality is automatically greater than or equal to the right side of the inequality in (5.4), as in (1.7). The right side of the inequality in (5.5) also decreases monotonically in q, as in (1.8). If $0 < q_1 \le q_2 \le \infty$ and N is a q_2-norm or q_2-seminorm on V, then it follows that N is a q_1-norm or q_1-seminorm on V as well, respectively.

If N is a q-norm or q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, then

$$d(v,w) = d_N(v,w) = N(v-w) \tag{5.6}$$

is a q-metric or q-semimetric on V, as appropriate. Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k, and put $N(0) = 0$ and $N(v) = 1$ when $v \in V$ and $v \neq 0$. This is the *trivial ultranorm* on V, for which the corresponding ultrametric is the discrete metric on V.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$ again, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. Note that $|\cdot|$ has to be a q-absolute value function on k when $N(v) > 0$ for some $v \in V$. Let a be a positive real number, and remember that $|\cdot|^a$ is a (q_k/a)-absolute value function on k, as in Section 3. Similarly,

$$N(v)^a \tag{5.7}$$

is a (q/a)-seminorm on V with respect to $|\cdot|^a$ on k, and a (q/a)-norm on V when N is a q-norm on V.

Suppose now that N is a q-norm on V. If V is not already complete with respect to the associated q-metric, then one can pass to a completion of V, as usual. The vector space operations on V can be extended to the completion, so that the completion is also a vector space over k. One can extend N to a q-norm on the completion as well, which corresponds to the distance to 0 in the completion. If V is complete, and k is not complete with respect to the q_k-metric associated to $|\cdot|$, then scalar multiplication on V can be extended to the completion of k, so that V becomes a vector space over the completion of k, and N becomes a q-norm on V as a vector space over the completion of k.

If V is complete, then V is a *q-Banach space* with respect to N. If $q = 1$, then one may simply say that V is a *Banach space*. One may wish to include completeness of k in the definition of a q-Banach space.

Of course, k may be considered as a one-dimensional vector space over itself, and $|\cdot|$ may be considered as a q_k-norm on k with respect to itself.

6 Bounded linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let V, W be vector spaces over k, and let N_V, N_W be q_V, q_W-seminorms on V, W, respectively, with respect to $|\cdot|$ on k, and for some $q_V, q_W > 0$. A linear mapping T from V into W is *bounded* with respect to N_V, N_W if

$$N_W(T(v)) \le C\, N_V(v) \tag{6.1}$$

for some $C \ge 0$ and every $v \in V$. This implies that T is uniformly continuous with respect to the q_V, q_W-semimetrics on V, W associated to N_V, N_W, respectively, by a standard argument. If $|\cdot|$ is nontrivial on k, and if a linear mapping T from V into W is continuous at 0 with respect to the topologies determined on V, W by the q_V, q_W-semimetrics associated to N_V, N_W, respectively, then one can check that T is bounded as a linear mapping from V into W. More precisely, it suffices to ask that $N_W(T(v))$ be bounded on a ball in V centered at 0 with positive radius with respect to N_V, in place of continuity of T at 0.

Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W with respect to N_V, N_W. If $T \in \mathcal{BL}(V, W)$, then put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \geq 0 : (6.1) \text{ holds}\}, \tag{6.2}$$

and observe that the infimum is automatically attained. One can check that $\mathcal{BL}(V, W)$ is a vector space over k with respect to pointwise addition and scalar multiplication, and that (6.2) is a q_W-seminorm on $\mathcal{BL}(V, W)$ with respect to $|\cdot|$ on k. If N_W is a q_W-norm on W, then (6.2) is a q_W-norm on $\mathcal{BL}(V, W)$. If W is also complete with respect to the q_W-metric associated to N_W, then one can verify that $\mathcal{BL}(V, W)$ is complete with respect to the q_W-metric associated to (6.2).

Let Z be another vector space over k, with a q_Z-seminorm N_Z with respect to $|\cdot|$ on k for some $q_Z > 0$. If T_1, T_2 are bounded linear mappings from V into W and from W into Z with respect to N_V, N_W, and N_Z, as appropriate, then their composition $T_2 \circ T_1$ is a bounded linear mapping from V into Z, with

$$\|T_2 \circ T_1\|_{op,VZ} \leq \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}. \tag{6.3}$$

In particular, the space $\mathcal{BL}(V) = \mathcal{BL}(V, V)$ of bounded linear mappings from V into itself is closed under compositions. Of course, the identity mapping $I = I_V$ on V is bounded, with $\|I\|_{op} = 1$ when $N_V(v) > 0$ for some $v \in V$.

Suppose that N_V, N_W are q_V, q_W-norms on V, W, respectively, and that V_0 is a dense linear subspace of V with respect to the q_V-metric associated to N_V. Let T_0 be a bounded linear mapping from V_0 into W, with respect to the restriction of N_V to V_0. As before, T_0 is uniformly continuous with respect to the q_V, q_W-metrics associated to N_V, N_W. If W is complete with respect to the q_W-metric associated to N_W, then T_0 has a unique extension to a uniformly continuous mapping from V into W. This follows from a well-known result about metric spaces when $q_V = q_W = 1$, and otherwise one can reduce to the case of metric spaces or use analogous arguments for q-metric spaces. One can check that this extension is a bounded linear mapping from V into W in this situation, with the same operator q_W-norm as on V_0. More precisely, completeness of W is only needed to get the existence of the extension, and ordinary continuity is sufficient for uniqueness.

7 Submultiplicative q-seminorms

Let k be a field, and let $\mathcal{A}$ be an (associative) *algebra* over k. This means that $\mathcal{A}$ is a vector space over k equipped with a binary operation of multiplication, which is bilinear as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, and which satisfies the associative law. Also let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q-seminorm on $\mathcal{A}$ as a vector space over k with respect to $|\cdot|$ on k for some $q > 0$. To say that N is *submultiplicative* on $\mathcal{A}$ means that

$$N(x\,y) \leq N(x)\,N(y) \tag{7.1}$$

for every $x, y \in \mathcal{A}$. Similarly, N is said to be *multiplicative* on $\mathcal{A}$ if

$$N(x\,y) = N(x)\,N(y) \tag{7.2}$$

for every $x, y \in \mathcal{A}$.

Suppose that $\|\cdot\|$ is a submultiplicative q-norm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. If $\mathcal{A}$ is complete with respect to the q-metric associated to $\|\cdot\|$, then $(\mathcal{A}, \|\cdot\|)$ is said to be a *q-Banach algebra* over k. As usual, one may simply say that $(\mathcal{A}, \|\cdot\|)$ is a *Banach algebra* when $q = 1$. Sometimes one also asks that $\mathcal{A}$ have a multiplicative identity element e such that $\|e\| = 1$, and we shall normally follow that convention here. If $\mathcal{A}$ is not complete, then one can pass to a completion, as in Section 5.

Let X be a nonempty topological space, and let $C(X, k)$ be the space of continuous k-valued functions on X, using the topology determined on k by the q_k-metric associated to $|\cdot|$. Of course, $C(X, k)$ contains the constant k-valued functions on X, including the function $\mathbf{1}_X$ whose value at every point in X is the multiplicative identity element 1 in k. Note that $C(X, k)$ is a commutative algebra over k with respect to pointwise addition and multiplication of functions, and that $\mathbf{1}_X$ is the multiplicative identity element in $C(X, k)$. If E is a nonempty compact subset of X and $f \in C(X, k)$, then put

$$\|f\|_{sup,E} = \sup_{x \in E} |f(x)|, \tag{7.3}$$

where in fact the supremum is attained, by standard results. One can check that this defines a submultiplicative q_k-seminorm on $C(X, k)$, which is the *supremum q_k-seminorm* associated to E.

Similarly, let $C_b(X, k)$ be the space of continuous k-valued functions f on X that are bounded, in the sense that $|f(x)|$ has a finite upper bound on X. This is a subalgebra of $C(X, k)$ that contains the constant functions. If $f \in C_b(X, k)$, then we put

$$\|f\|_{sup} = \sup_{x \in X} |f(x)|, \tag{7.4}$$

where now the supremum is finite by hypothesis. This defines a submultiplicative q_k-norm on $C_b(X, k)$, which is the *supremum q_k-norm.* If X is compact, then $C(X, k) = C_b(X, k)$, and (7.4) is the same as (7.3) with $E = X$. Note that $\|\mathbf{1}_X\|_{sup} = 1$. If k is complete with respect to the q_k-metric associated to $|\cdot|$, then $C_b(X, k)$ is complete with respect to the q_k-metric associated to (7.4), by standard arguments. Thus $C_b(X, k)$ is a q_k-Banach algebra with respect to (7.4).

Let V be a vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. The space $\mathcal{BL}(V)$ of bounded linear mappings from V into itself with respect to N_V is an algebra over k, with composition of linear mappings as multiplication, and the corresponding operator q_V-seminorm $\|\cdot\|_{op}$ is submultiplicative on $\mathcal{BL}(V)$. Suppose that N_V is a q_V-norm on V, and that $V \neq \{0\}$, so that $\|\cdot\|_{op}$ is a q_V-norm on $\mathcal{BL}(V)$, and $\|I\|_{op} = 1$. If V is complete with respect to the q_V-metric associated to N_V, then $\mathcal{BL}(V)$ is complete with respect to the q_V-metric associated to $\|\cdot\|_{op}$, as before. Under these conditions, $(\mathcal{BL}(V), \|\cdot\|_{op})$ is a q_V-Banach algebra over k.

8 Nonnegative functions

Let X be a nonempty set, and let f be a nonnegative real-valued function on X. The *support* of f is the set of $x \in X$ such that $f(x) > 0$. Let us say that f *vanishes at infinity* on X if for every $\epsilon > 0$ we have that

$$f(x) < \epsilon \tag{8.1}$$

for all but finitely many $x \in X$. In particular, this holds when the support of f has only finitely many elements. If f vanishes at infinity on X, then it is easy to see that the support of f has only finitely or countably many elements, by applying the previous definition to $\epsilon = 1/j$, $j \in \mathbf{Z}_+$.

The sum

$$\sum_{x \in X} f(x) \tag{8.2}$$

is defined as a nonnegative extended real number to be the supremum of the sums

$$\sum_{x \in A} f(x) \tag{8.3}$$

over all nonempty finite subsets A of X. If the supremum is finite, then f is said to be *summable* on X. Of course, if f has finite support in X, then (8.2) reduces to a finite sum. If f is summable on X, then one can check that f vanishes at infinity on X. Let us also permit f to take values in the set of nonnegative extended real numbers, with (8.2) equal to $+\infty$ when $f(x) = +\infty$ for any $x \in X$.

If t is a positive real number, then

$$\sum_{x \in X} t\, f(x) = t \sum_{x \in X} f(x), \tag{8.4}$$

where $t \cdot (+\infty)$ is interpreted as being $+\infty$, as usual. If g is another nonnegative extended real-valued function on X, then

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x), \tag{8.5}$$

where the sum of any nonnegative extended real number and $+\infty$ is interpreted as being $+\infty$. These statements can be verified directly from the definitions.

Let f be a nonnegative real-valued function on X again, and let r be a positive real number. If $f(x)^r$ is summable on X, then f is *r-summable* on X, and

$$\|f\|_r = \Big(\sum_{x \in X} f(x)^r\Big)^{1/r} \tag{8.6}$$

is defined as a nonnegative real number. Otherwise, (1.1) may be interpreted as being $+\infty$. Similarly,

$$\|f\|_\infty = \sup_{x \in X} f(x) \tag{8.7}$$

is defined as a nonnegative extended real number, which is finite when f is bounded on X. Note that f is bounded on X when it vanishes at infinity on X, in which case the supremum on the right side of (8.7) is automatically attained.

If t is a positive real number, then

$$\|t\, f\|_r = t\, \|f\|_r \tag{8.8}$$

for every $r > 0$, with $t \cdot (+\infty) = +\infty$, as before. If $0 < r_1 \le r_2 \le \infty$, then

$$\|f\|_{r_2} \le \|f\|_\infty^{1-(r_1/r_2)}\, \|f\|_{r_1}^{r_1/r_2} \le \|f\|_{r_1}, \tag{8.9}$$

as in (1.3) and (1.6). If f is r-summable on X for any positive real number r, then $f(x)^r$ vanishes at infinity on X, and hence f vanishes at infinity on X. In this case, one can use (8.9) to check that

$$\lim_{r \to \infty} \|f\|_r = \|f\|_\infty, \tag{8.10}$$

as in (1.4).

If g is another nonnegative real-valued function on X and $1 \le r \le \infty$, then *Minkowski's inequality* for arbitrary sums states that

$$\|f + g\|_r \le \|f\|_r + \|g\|_r. \tag{8.11}$$

If $0 < r \le 1$, then

$$\|f + g\|_r^r \le \|f\|_r^r + \|g\|_r^r, \tag{8.12}$$

as in (1.11). In both cases, it follows that $\|f + g\|_r < \infty$ when $\|f\|_r, \|g\|_r < \infty$, which can also be verified more directly.

Let $\{f_j\}_{j=1}^\infty$ be a sequence of nonnegative real-valued functions on X, and suppose that $\{f_j\}_{j=1}^\infty$ converges pointwise to a nonnegative real-valued function f on X. Under these conditions,

$$\sum_{x \in X} f(x) \le \sup_{j \ge 1} \Big(\sum_{x \in X} f_j(x) \Big), \tag{8.13}$$

which is a simplified version of Fatou's lemma for sums. To see this, let A be a nonempty finite subset of X, and observe that

$$\sum_{x \in A} f(x) = \lim_{j \to \infty} \Big(\sum_{x \in A} f_j(x) \Big) \le \sup_{j \ge 1} \Big(\sum_{x \in A} f_j(x) \Big) \le \sup_{j \ge 1} \Big(\sum_{x \in X} f_j(x) \Big). \tag{8.14}$$

This implies (8.13), by taking the supremum over A. If $\{f_j\}_{j=1}^\infty$ increases monotonically in j, so that $f_j(x) \le f_{j+1}(x)$ for every $j \ge 1$ and $x \in X$, then

$$\sum_{x \in X} f_j(x) \to \sum_{x \in X} f(x) \tag{8.15}$$

as $j \to \infty$. This is the analogue of the monotone convergence theorem for sums, which can be obtained from (8.13). Similarly, one can check that

$$\sup_{x \in X} f_j(x) \to \sup_{x \in X} f(x) \tag{8.16}$$

as $j \to \infty$ in this case.

9 Infinite series

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q-norm N with respect to $|\cdot|$ on k for some $q > 0$. An infinite series $\sum_{j=1}^{\infty} v_j$ with $v_j \in V$ for each $j \geq 1$ is said to *converge* in V with respect to N if the corresponding sequence of partial sums $\sum_{j=1}^{n} v_j$ converges in V with respect to the q-metric associated to N, in which case the value of the infinite sum is defined to be the limit of the sequence of partial sums. If $\sum_{j=1}^{\infty} v_j$ converges in V and $t \in k$, then $\sum_{j=1}^{\infty} t\, v_j$ converges in V too, with

$$\sum_{j=1}^{\infty} t\, v_j = t \sum_{j=1}^{\infty} v_j. \tag{9.1}$$

Similarly, if $\sum_{j=1}^{\infty} w_j$ is another convergent series in V, then $\sum_{j=1}^{\infty}(v_j + w_j)$ converges as well, with

$$\sum_{j=1}^{\infty}(v_j + w_j) = \sum_{j=1}^{\infty} v_j + \sum_{j=1}^{\infty} w_j. \tag{9.2}$$

If $\sum_{j=1}^{\infty} a_j$ is an infinite series of nonnegative real numbers, then the sequence $\sum_{j=1}^{n} a_j$ increases monotonically. In this case, $\sum_{j=1}^{\infty} a_j$ converges in $\mathbf{R}$ with respect to the standard absolute value function if and only if the partial sums $\sum_{j=1}^{n} a_j$ have a finite upper bound. Otherwise, if the partial sums are unbounded, then one may interpret $\sum_{j=1}^{\infty} a_j$ as being $+\infty$. This is equivalent to considering the sum $\sum_{j \in \mathbf{Z}_+} a_j$ as a nonnegative extended real number as in the previous section.

Let $\sum_{j=1}^{\infty} v_j$ be an infinite series with terms in V again. The sequence $\sum_{j=1}^{n} v_j$ is a Cauchy sequence in V with respect to the q-metric associated to N if and only if for every $\epsilon > 0$ there is an $L \in \mathbf{Z}_+$ such that

$$N\Big(\sum_{j=l}^{n} v_j\Big) < \epsilon. \tag{9.3}$$

for every $n \geq l \geq L$. Of course, if V is complete with respect to the q-metric associated to N, then it follows that $\sum_{j=1}^{\infty} v_j$ converges in V with respect to N. Note that the Cauchy condition just mentioned implies that

$$\lim_{j \to \infty} N(v_j) = 0, \tag{9.4}$$

by taking $l = n$.

If $q < \infty$, then

$$N\Big(\sum_{j=l}^{n} v_j\Big)^q \leq \sum_{j=l}^{n} N(v_j)^q \tag{9.5}$$

when $n \geq l \geq 1$, by the q-norm version (5.2) of the triangle inequality. In this situation, $\sum_{j=1}^{\infty} v_j$ is said to converge *q-absolutely* with respect to N when

$\sum_{j=1}^{\infty} N(v_j)^q$ converges as an infinite series of nonnegative real numbers. This implies that the sequence $\sum_{j=1}^{n} v_j$ of partial sums is a Cauchy sequence in V with respect to the q-metric associated to N, using (9.3) and (9.5). If V is complete with respect to the q-metric associated to N, so that $\sum_{j=1}^{\infty} v_j$ converges in V with respect to N, then we can also use (9.5) to get that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big)^q \le \sum_{j=1}^{\infty} N(v_j)^q \tag{9.6}$$

under these conditions.

If $q = \infty$, then

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \max_{l \le j \le n} N(v_j) \tag{9.7}$$

when $n \ge l \ge 1$, by the ultranorm version (5.4) of the triangle inequality. Thus (9.4) imples that the sequence of partial sums $\sum_{j=1}^{n} v_j$ is a Cauchy sequence in V with respect to the ultrametric associated to N in this case, as in (9.3). If V is complete with respect to the ultrametric associated to N, then $\sum_{j=1}^{\infty} v_j$ converges in V with respect to N, and we have that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \max_{j \ge 1} N(v_j), \tag{9.8}$$

using (9.7) again. Note that the maximum on the right side of (9.8) is attained when (9.4) holds.

10 Weighted ℓ^r spaces

Let X be a nonempty set, and let $w(x)$ be a nonnegative real-valued function on X. Also let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let V be a vector space over k, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. If r is a positive real number, then we let

$$\ell^r_w(X, V) = \ell^r_{w,N}(X, V) \tag{10.1}$$

be the space of V-valued functions f on X such that $N(f(x))\, w(x)$ is r-summable as a nonnegative real-valued function on X, which is to say that $N(f(x))^r\, w(x)^r$ is summable on X. If f has this property, then we put

$$\|f\|_{r,w} = \|f\|_{\ell^r_w(X,V)} = \Big(\sum_{x \in X} N(f(x))^r\, w(x)^r\Big)^{1/r}, \tag{10.2}$$

where the sum is as defined in Section 8.

Similarly, let

$$\ell^\infty_w(X, V) = \ell^\infty_{w,N}(X, V) \tag{10.3}$$

be the space of V-valued functions f on X such that $N(f(x))\,w(x)$ is bounded on X. In this case, we put

$$\|f\|_{\infty,w} = \|f\|_{\ell^\infty_w(X,V)} = \sup_{x \in X} (N(f(x))\,w(x)). \tag{10.4}$$

If f is a V-valued function on X that is not in $\ell^r_w(X,V)$ for some $r > 0$, then we may interpret $\|f\|_{r,w}$ as being $+\infty$, as usual. Equivalently,

$$\|f\|_{r,w} = \|N(f)\,w\|_r \tag{10.5}$$

for every V-valued function f on X and $r > 0$, where $N(f)$ is the nonnegative real-valued function $N(f(x))$ on X, and $\|\cdot\|_r$ is as in Section 8. If $w(x) = 1$ for every $x \in X$, then we may drop w from the notation in (10.1), (10.2), (10.3), and (10.4).

If $f \in \ell^r_w(X,V)$ for some $r > 0$ and $t \in k$, then $t\,f \in \ell^r_w(X,V)$ too, with

$$\|t\,f\|_{r,w} = |t|\,\|f\|_{r,w}. \tag{10.6}$$

One can check directly that $\ell^r_w(X,V)$ is also closed under finite sums for each $r > 0$, so that $\ell^r_w(X,V)$ is a vector space with respect to pointwise addition and scalar multiplication. More precisely, if f, g are V-valued functions on X and $q = \infty$, then it is easy to see that

$$\|f+g\|_{\infty,w} \le \max(\|f\|_{\infty,w}, \|g\|_{\infty,w}). \tag{10.7}$$

Similarly, if $0 < r \le q$ and $r < \infty$, then

$$\|f+g\|^r_{r,w} \le \|f\|^r_{r,w} + \|g\|^r_{r,w}. \tag{10.8}$$

Remember that N may be considered as an r-seminorm on V when $0 < r \le q$, as in Section 5. This permits (10.8) to be obtained directly from the definitions. If $q \le r$ and $q < \infty$, then

$$\|f+g\|^q_{r,w} \le \|f\|^q_{r,w} + \|g\|^q_{r,w}. \tag{10.9}$$

This can be derived from Minkowski's inequality for sums, with exponent $r/q \ge 1$. It follows that $\|\cdot\|_{r,w}$ defines an r-seminorm on $\ell^r_w(X,V)$ when $0 < r \le q$, and a q-seminorm when $q \le r$. If N is a q-norm on V, and $w(x) > 0$ for every $x \in X$, then $\|\cdot\|_{r,w}$ is an r-norm on $\ell^r_w(X,V)$ when $0 < r \le q$, and a q-norm when $q \le r$. If V is also complete with respect to the q-metric associated to N, then one can verify that $\ell^r_w(X,V)$ is complete with respect to the q or r-metric associated to $\|\cdot\|_{r,w}$, using standard arguments.

If f is a V-valued function on X and $0 < r_1 \le r_2 \le \infty$, then

$$\|f\|_{r_2,w} \le \|f\|_{r_1,w}, \tag{10.10}$$

by (8.9) and (10.5). Hence

$$\ell^{r_1}_w(X,V) \subseteq \ell^{r_2}_w(X,V) \tag{10.11}$$

when $r_1 \le r_2$. If $f \in \ell^r_w(X, V)$ for some positive real number r, then

$$\lim_{r \to \infty} \|f\|_{r,w} = \|f\|_{\infty,w}, \tag{10.12}$$

by (8.10) and (10.5). Similarly, if w_1, w_2 are nonnegative real-valued functions on X such that $w_1(x) \le w_2(x)$ for every $x \in X$, then

$$\|f\|_{r,w_1} \le \|f\|_{r,w_2} \tag{10.13}$$

for every $f \in c(X, V)$ and $r > 0$. This implies that

$$\ell^r_{w_2}(X, V) \subseteq \ell^r_{w_1}(X, V) \tag{10.14}$$

for every $r > 0$ in this case.

11 Vanishing at infinity

Let X be a nonempty set, let k be a field, and let V be a vector space over k. The space

$$c(X, V) \tag{11.1}$$

of all V-valued functions on X is a vector space over k with respect to pointwise addition and scalar multiplication. As before, the *support* of a $f \in c(X, V)$ is the set of $x \in X$ such that $f(x) \neq 0$. Let

$$c_{00}(X, V) \tag{11.2}$$

be the space of $f \in c(X, V)$ such that the support of f has only finitely many elements. This is a linear subspace of $c(X, V)$, which is the same as $c(X, V)$ when X has only finitely many elements.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. If $f \in c(X, V)$, then the support of $N(f(x))$, as a real-valued function on X, is contained in the support of $f \in c(X, V)$. Let

$$c_{00,N}(X, V) \tag{11.3}$$

be the space of $f \in c(X, V)$ such that $N(f(x))$ has finite support in X. It is easy to see that $c_{00,N}(X, V)$ is a linear subspace of $c(X, V)$ that contains $c_{00}(X, V)$. If N is a q-norm on V, then the support of $f \in c(X, V)$ and $N(f(x))$ are the same, and hence $c_{00,N}(X, V)$ is the same as $c_{00}(X, V)$.

Let $w(x)$ be a nonnegative real-valued function on X. A V-valued function f on X is said to *vanish at infinity* on X with respect to N and w if for every $\epsilon > 0$ we have that

$$N(f(x))\, w(x) < \epsilon \tag{11.4}$$

for all but finitely many $x \in X$. Equivalently, this means that $N(f(x))\, w(x)$ vanishes at infinity as a nonnegative real-valued function on X, as in Section 8. Let

$$c_{0,w}(X, V) = c_{0,w,N}(X, V) \tag{11.5}$$

be the space of $f \in c(X, V)$ that vanish with respect to N and w. As before, we may drop w from this terminology and notation when $w(x) = 1$ for every $x \in X$.

It is easy to see that $c_{0,w}(X, V)$ is a linear subspace of $c(X, V)$. Observe that

$$c_{0,w}(X, V) \subseteq \ell^\infty_w(X, V), \tag{11.6}$$

because $N(f(x))\, w(x)$ is bounded on X when it vanishes at infinity on X. In this case, the supremum on the right side of (10.4) is automatically attained. One can also check that $c_{0,w}(X, V)$ is a closed set in $\ell^\infty_w(X, V)$, with respect to the q-semimetric associated to (10.4).

Clearly

$$c_{00}(X, V) \subseteq c_{00,N}(X, V) \subseteq c_{0,w}(X, V). \tag{11.7}$$

One can verify that $c_{0,w}(X, V)$ is the closure of $c_{00}(X, V)$ in $\ell^\infty_w(X, V)$, with respect to the q-semimetric associated to (10.4).

If $0 < r < \infty$, then

$$\ell^r_w(X, V) \subseteq c_{0,w}(X, V). \tag{11.8}$$

More precisely, if $f \in \ell^r_w(X, V)$, then $N(f(x))\, w(x)$ is r-summable on X, and hence vanishes at infinity on X, as in Section 8.

We also have that

$$c_{00}(X, V) \subseteq c_{00,N}(X, V) \subseteq \ell^r_w(X, V) \tag{11.9}$$

for every $r > 0$. If $r < \infty$, then one can check that $c_{00}(X, V)$ is dense in $\ell^r_w(X, V)$, with respect to the q or r-semimetric associated to (10.2), as appropriate.

Let w_1, w_2 be nonnegative real-valued functions on X such that $w_1(x) \le w_2(x)$ for every $x \in X$. If $f \in c(X, V)$ vanishes at infinity on X with respect to N and w_2, then f vanishes at infinity with respect to N and w_1 as well. Hence

$$c_{0,w_2}(X, V) \subseteq c_{0,w_1}(X, V) \tag{11.10}$$

in this situation.

12 Multiplication operators

Let k be a field, and let $\mathcal{A}$ be an algebra over k. If $a \in \mathcal{A}$, then

$$M_a(x) = a\, x \tag{12.1}$$

defines a linear mapping from $\mathcal{A}$ into itself, which is the (left) *multiplication operator* associated to a. One can check that

$$M_a \circ M_b = M_{a\, b} \tag{12.2}$$

for every $a, b \in \mathcal{A}$, using associativity of multiplication on $\mathcal{A}$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. Observe that M_a is a bounded linear mapping from $\mathcal{A}$ into itself with respect to N on $\mathcal{A}$ for each $a \in \mathcal{A}$, with

$$\|M_a\|_{op} \le N(a), \tag{12.3}$$

where $\|\cdot\|_{op}$ is the operator q-seminorm corresponding to N. If $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, then

$$\|M_a\|_{op} = N(a) \tag{12.4}$$

for every $a \in \mathcal{A}$.

Now let X be a nonempty set, and let V be a vector space over k. If $a \in c(X,k)$ and $f \in c(X,V)$, then put

$$(M_a(f))(x) = a(x)\, f(x) \tag{12.5}$$

for each $x \in X$, so that $M_a(f) = a\, f \in c(X,V)$. This defines a linear mapping from $c(X,V)$ into itself, which is the *multiplication operator* associated to a. Of course, M_a maps $c_{00}(X,V)$ into itself as well. Note that (12.2) holds for every $a, b \in c(X,k)$ in this situation too. If $V = k$, then this definition of M_a corresponds to the previous one with $\mathcal{A} = c(X,k)$, as a commutative algebra with respect to pointwise multiplication of functions. If $a(x) \ne 0$ for every $x \in X$, then M_a is a one-to-one linear mapping from $c(X,V)$ onto itself, with inverse $M_{1/a}$.

Let$|\cdot|$ be a q_k-absolute value function on k again, and let w_1, w_2 be nonnegative real-valued functions on X. Suppose that $a \in c(X,k)$ satisfies

$$|a(x)|\, w_1(x) \le C\, w_2(x) \tag{12.6}$$

for some $C \ge 0$ and every $x \in X$. Let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. If $f \in c(X,V)$, then

$$N_V((M_a(f))(x))\, w_1(x) = |a(x)|\, N_V(f(x))\, w_1(x) \le C\, N_V(f(x))\, w_2(x) \tag{12.7}$$

for every $x \in X$. This implies that

$$\|M_a(f)\|_{r,w_1} \le C\, \|f\|_{r,w_2}. \tag{12.8}$$

for every $r > 0$. Thus M_a defines a bounded linear mapping from $\ell^r_{w_2}(X,V)$ into $\ell^r_{w_1}(X,V)$ for every $r > 0$, with operator q_V or r-seminorm less than or equal to C. Similarly, (12.7) implies that M_a maps $c_{0,w_2}(X,V)$ into $c_{0,w_1}(X,V)$.

Suppose now that $a \in c(X,k)$ satisfies

$$w_2(x) \le C'\, |a(x)|\, w_1(x) \tag{12.9}$$

for some $C' \ge 0$ and every $x \in X$. This implies that

$$\begin{aligned} N_V(f(x))\, w_2(x) &\le C'\, |a(x)|\, N_V(f(x))\, w_1(x) \\ &\le C'\, N_V((M_a(f))(x))\, w_1(x) \end{aligned} \tag{12.10}$$

for every $f \in c(X, V)$ and $x \in X$. It follows that

$$\|f\|_{r,w_2} \le C' \, \|M_a(f)\|_{r,w_1} \tag{12.11}$$

for every $f \in c(X, V)$ and $r > 0$. If $a(x) \ne 0$ for every $x \in X$, then (12.9) is the same as saying that

$$(1/|a(x)|) \, w_2(x) \le C' \, w_1(x) \tag{12.12}$$

for every $x \in X$. In this case, $M_{1/a}$ defines a bounded linear mapping from $\ell^r_{w_1}(X, V)$ into $\ell^r_{w_2}(X, V)$ for every $r > 0$, as in the previous paragraph.

13 Sums of vectors

Let X be a nonempty set, let k be a field, and let V be a vector space over k again. If $f \in c_{00}(X, V)$, then

$$\sum_{x \in X} f(x) \tag{13.1}$$

defines an element of V, by reducing to a finite sum. This defines a linear mapping

$$f \mapsto \sum_{x \in X} f(x) \tag{13.2}$$

from $c_{00}(X, V)$ into V.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. If $f \in c(X, V)$, then we let $\|f\|_q$ be as in (10.2) and (10.4), with $w(x) = 1$ for every $x \in X$. Observe that

$$N\Big(\sum_{x \in X} f(x)\Big) \le \|f\|_q \tag{13.3}$$

for every $f \in c_{00}(X, V)$, by the q-seminorm version of the triangle inequality. Thus (13.2) is a bounded linear mapping from $c_{00}(X, V)$ into V, using $\|f\|_q$ on $c_{00}(X, V)$ and N on V. More precisely, the operator q-seminorm of (13.2) is equal to 1 when $N(v) > 0$ for some $v \in V$.

Suppose that N is a q-norm on V, and that V is complete with respect to the associated q-metric. If $q < \infty$, then there is a unique extension of (13.2) to a bounded linear mapping from $\ell^q(X, V)$ into V, as in Section 6. If $q = \infty$, then these is a unique extension of (13.2) to a bounded linear mapping from $c_0(X, V)$ into V, with respect to the supremum ultranorm $\|f\|_\infty$ associated to N on $c_0(X, V)$. These extensions can be used to define (13.1) as an element of V in these cases. Note that these extensions also satisfy (13.3), as before.

Alternatively, (13.1) can be obtained from convergent infinite series in these situations. More precisely, let $\{x_j\}_{j=1}^\infty$ be a sequence of distinct elements of X, and suppose that $f \in c(X, V)$ has support contained in the set of x_j's. Under these conditions, the sum (13.1) corresponds formally to the infinite series

$$\sum_{j=1}^\infty f(x_j). \tag{13.4}$$

Of course, if f has finite support in in X, then all but finitely many terms in this series are equal to 0, so that the sum can be defined as before. If $q < \infty$ and $f \in \ell^q(X, V)$, then (13.4) converges q-absolutely with respect to N, and hence converges in V with respect to N, as in Section 9. If $q = \infty$ and $f \in c_0(X, V)$, then $N(f(x_j)) \to 0$ as $j \to \infty$, and (13.4) converges in V with respect to N as in Section 9 again. Note that the support of every element of $c_0(X, V)$ has only finitely or countable many elements, as in Section 8. In particular, this holds when $f \in \ell^q(X, V)$ and $q < \infty$.

If f is a summable real or complex-valued function on X, then f can be expressed as a linear combination of nonnegative real-valued summable functions on X, whose sums can be defined as in Section 8. Of course, all of these approaches to sums over X are based on suitable approximations by finite sums.

14 Sums of sums

Let X be a nonempty set, and let $\{E_j\}_{j \in I}$ be a family of pairwise-disjoint nonempty subsets of X indexed by a nonempty set I. If f is a nonnegative extended real-valued function on X, then

$$\sum_{x \in E_j} f(x) \tag{14.1}$$

can be defined as a nonnegative extended real number for each $j \in I$, as in Section 8. Hence

$$\sum_{j \in I} \Big(\sum_{x \in E_j} f(x) \Big) \tag{14.2}$$

can be defined as a nonnegative extended real number as in Section 8 too. If we put

$$E = \bigcup_{j \in I} E_j, \tag{14.3}$$

then one can check that (14.2) is equal to

$$\sum_{x \in E} f(x), \tag{14.4}$$

which is defined as a nonnegative extended real number as in Section 8 as well.

Let k be a field, and let V be a vector space over k. If $f \in c_{00}(X, V)$, then (14.1) can be defined as an element of V for each $j \in I$. This defines a V-valued function on I with finite support, so that (14.2) can also be defined as an element of V. Of course, (14.4) can be defined as an element of V too, and one can check that (14.2) is equal to (14.4).

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, let N be a q-norm on V with respect to $|\cdot|$ on k for some $q > 0$, and suppose that V is complete with respect to the q-metric associated to N. If $q < \infty$ and $f \in \ell^q(X, V)$, then the restriction of $N(f(x))$ to any nonempty subset of X is

q-summable on that set. This permits us to define (14.1) as an element of V for each $j \in I$, as in the previous section, with

$$N\Big(\sum_{x \in E_j} f(x)\Big)^q \le \sum_{x \in E_j} N(f(x))^q \tag{14.5}$$

for every $j \in I$. Hence

$$\sum_{j \in I} N\Big(\sum_{x \in E_j} f(x)\Big)^q \le \sum_{j \in I} \Big(\sum_{x \in E_j} N(f(x))^q\Big) = \sum_{x \in E} N(f(x))^q, \tag{14.6}$$

which implies that (14.2) defines an element of $\ell^q(I, V)$. Thus (14.2) can be defined as an element of V, as in the previous section, and one can verify that (14.2) is equal to (14.4) in this case as well.

If $q = \infty$ and $f \in c_0(X, V)$, then the restriction of f to any nonempty subset of X vanishes at infinity on that set. As before, this permits us to define (14.1) as an element of V for every $j \in I$, with

$$N\Big(\sum_{x \in E_j} f(x)\Big) \le \max_{x \in E_j} N(f(x)) \tag{14.7}$$

for every $j \in I$. One can use (14.7) to check that (14.1) vanishes at infinity as a V-valued function on I, so that (14.2) can be defined as an element of V, as in the previous section. As usual, one can check that (14.2) is equal to (14.4) under these conditions.

If X is the Cartesian product $Y \times Z$ of nonempty sets Y, Z, then X can be partitioned into the families of subsets of the form $\{y\} \times Z$, $y \in Y$, and $Y \times \{z\}$, $z \in Z$. The previous remarks can be used to show that sums over X are the same as iterated sums over Y and Z under suitable conditions.

15 Cauchy products

Let k be a field, let $\mathcal{A}$ be an algebra over k, and let $\sum_{j=0}^\infty a_j$, $\sum_{l=0}^\infty b_l$ be infinite series with terms in $\mathcal{A}$, considered formally for the moment. Thus

$$c_n = \sum_{j=0}^n a_j \, b_{n-j} \tag{15.1}$$

is defined as an element of $\mathcal{A}$ for every nonnegative integer n, and the corresponding series $\sum_{n=0}^\infty c_n$ is the *Cauchy product* of $\sum_{j=0}^\infty a_j$, $\sum_{l=0}^\infty b_l$. By construction,

$$\sum_{n=0}^\infty c_n = \Big(\sum_{j=0}^\infty a_j\Big) \Big(\sum_{l=0}^\infty b_l\Big) \tag{15.2}$$

formally, and this can be made precise in some situations. Indeed, let f be the $\mathcal{A}$-valued function defined on

$$X = (\mathbf{Z}_+ \cup \{0\}) \times (\mathbf{Z}_+ \cup \{0\}) \tag{15.3}$$

by

$$f(j,l) = a_j\, b_l \tag{15.4}$$

for each $j, l \ge 0$. Put

$$E_n = \{(j,l) \in X : j + l = n\} \tag{15.5}$$

for each nonnegative integer n, so that the E_n's are pairwise-disjoint nonempty finite subsets of X such that

$$\bigcup_{n=0}^{\infty} E_n = X. \tag{15.6}$$

Observe that

$$c_n = \sum_{(j,l) \in E_n} f(j,l) \tag{15.7}$$

for every $n \ge 0$, so that

$$\sum_{n=0}^{\infty} c_n = \sum_{(j,l) \in X} f(j,l) \tag{15.8}$$

formally. Similarly, the sum over X can be identified formally with iterated sums over j and l, to get that

$$\sum_{(j,l) \in X} f(j,l) = \Big(\sum_{j=0}^{\infty} a_j\Big)\Big(\sum_{l=0}^{\infty} b_l\Big). \tag{15.9}$$

If $a_j = 0$ for all but finitely many $j \ge 0$, and $b_l = 0$ for all but finitely many $l \ge 0$, then f has finite support in X, and $c_n = 0$ for all but finitely many $n \ge 0$. In this case, the various infinite sums mentioned in the preceding paragraph reduce to finite sums, and (15.8) and (15.9) hold, as in the previous section.

If $\mathcal{A} = k = \mathbf{R}$ and $a_j, b_l \ge 0$ for every $j, l \ge 0$, then the infinite sums mentioned earlier can be defined as nonnegative extended real numbers, as in Section 8. We also have (15.8) in this situation, as in the previous section, and that the left side of (15.9) can be expressed in terms of iterated sums. The iterated sums are equal to the right side of (15.9) when the sums on the right side of (15.9) are both positive or both finite. If either of the sums on the right side of (15.9) are equal to 0, then the left side of (15.9) is equal to 0 too.

Now let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let $\mathcal{A}$ be an algebra over k with a submultiplicative q-norm N with respect to $|\cdot|$ on k for some $q > 0$, and suppose that $\mathcal{A}$ is complete with respect to the q-norm associated to N. Suppose for the moment that $q < \infty$, and that

$$\sum_{j=0}^{\infty} N(a_j)^q,\ \sum_{l=0}^{\infty} N(b_l)^q < \infty. \tag{15.10}$$

This implies that

$$\sum_{(j,l)\in X} N(f(j,l))^q \le \sum_{(j,l)\in X} N(a_j)^q\, N(b_l)^q = \Big(\sum_{j=0}^{\infty} N(a_j)^q\Big)\Big(\sum_{l=0}^{\infty} N(b_l)^q\Big), \tag{15.11}$$

using the remarks in the previous paragraph in the second step. We also have that

$$N(c_n)^q \le \sum_{j=0}^{n} N(a_j\, b_{n-j})^q = \sum_{(j,l)\in E_n} N(f(j,l))^q \tag{15.12}$$

for each $n \ge 0$, by the q-norm version of the triangle inequality, so that

$$\sum_{n=0}^{\infty} N(c_n)^q \le \sum_{n=0}^{\infty}\Big(\sum_{(j,l)\in E_n} N(f(j,l))^q\Big) = \sum_{(j,l)\in X} N(f(j,l))^q. \tag{15.13}$$

It follows that the infinite sums mentioned at the beginning of the section are defined as elements of $\mathcal{A}$, as in Sections 9 and 13, and that they satisfy (15.8) and (15.9), as in the previous section.

Similarly, if $q = \infty$ and

$$\lim_{j\to\infty} N(a_j) = \lim_{l\to\infty} N(b_l) = 0, \tag{15.14}$$

then it is easy to see that $f(j,l)$ vanishes at infinity on X with respect to N. In this case,

$$N(c_n) \le \max_{0\le j\le n} N(a_j\, b_{n-j}) = \max_{(j,l)\in E_n} N(f(j,l)) \tag{15.15}$$

for each $n \ge 0$, by the ultranorm version of the triangle inequality, which can be used to verify that

$$\lim_{n\to\infty} N(c_n) = 0. \tag{15.16}$$

Hence the infinite sums mentioned earlier in the section are defined as elements of $\mathcal{A}$, as in Sections 9 and 13, and they satisfy (15.7) and (15.9), as before.

Part II
Power series

16 Formal power series

Let k be a field, and let T be an indeterminate. As in [4, 12], we shall normally use upper-case letters like T for indeterminates, and lower-case letters like t for elements of k. As usual, a *formal power series* in T with coefficients in k may be expressed as

$$f(T) = \sum_{j=0}^{\infty} f_j \, T^j, \tag{16.1}$$

where $f_j \in k$ for each nonnegative integer j, and the space of these formal power series is denoted $k[[T]]$. More precisely, f_j is a k-valued function of j on the set of nonnegative integers, so that $k[[T]]$ can be defined as $c(\mathbf{Z}_+ \cup \{0\}, k)$. This is a vector space over k with respect to pointwise addition and scalar multiplication of functions on $\mathbf{Z}_+ \cup \{0\}$, which corresponds to termwise addition and scalar multiplication of formal power series expressed as in (16.1). A *formal polynomial* in T with coefficients in k may be considered as a formal power series $f(T) \in k[[T]]$ with $f_j = 0$ for all but finitely many $j \geq 0$. The space of these formal polynomials is denoted $k[T]$, which corresponds to the linear subspace $c_{00}(\mathbf{Z}_+ \cup \{0\}, k)$ of $c(\mathbf{Z}_+ \cup \{0\}, k)$.

If $f(T)$ and $g(T) = \sum_{j=0}^{\infty} g_j \, T^j$ are elements of $k[[T]]$, then their product is defined to be the formal power series

$$f(T) \, g(T) = h(T) = \sum_{n=0}^{\infty} h_n \, T^n, \tag{16.2}$$

where

$$h_n = \sum_{j=0}^{n} f_j \, g_{n-j} \tag{16.3}$$

for each $n \geq 0$. This corresponds to the Cauchy product of $f(T)$ and $g(T)$, as in Section 15, and one can verify that $k[[T]]$ is a commutative algebra over k with respect to this definition of multiplication. Note that $k[T]$ is a subalgebra of $k[[T]]$, and we can identify k with the subalgebra of $k[T]$ of formal polynomials for which all but the first coefficient is equal to 0. The multiplicative identity element 1 in k corresponds to the multiplicative identity element in $k[[T]]$ in this way.

If $a(T) \in k[[T]]$ and $l \in \mathbf{Z}_+$, then the lth power $a(T)^l$ of $a(T)$ can be defined as an element of $k[[T]]$ in the the preceding paragraph. Let us interpret $a(T)^l$ as being equal to 1 when $l = 0$, as usual. If j, n are nonnegative integers with $j \leq n$, then the coefficient of T^j in

$$\sum_{l=0}^{n} a(T)^l \, T^l \tag{16.4}$$

does not depend on n. Hence

$$\sum_{l=0}^{\infty} a(T)^l \, T^l \tag{16.5}$$

can be defined as an element of $k[[T]]$. Note that

$$(1 - a(T) \, T) \sum_{l=0}^{n} a(T)^l \, T^l = 1 - a(T)^{n+1} \, T^{n+1} \tag{16.6}$$

for every $n \geq 0$, by a standard argument. One can use this to check that

$$(1 - a(T)\,T) \sum_{l=0}^{\infty} a(T)^l\, T^l = 1. \tag{16.7}$$

This shows that (16.5) is the multiplicative inverse of $1 - a(T)\,T$ in $k[[T]]$.

If $f(T) \in k[[T]]$ satisfies $f_0 \neq 0$, then $f(T)$ can be expressed as f_0 times $(1 - a(T)\,T)$ for some $a(T) \in k[[T]]$. This implies that $f(T)$ has a multiplicative inverse in $k[[T]]$, by the remarks in the previous paragraph. Conversely, if $f(T) \in k[[T]]$ has a multiplicative inverse in $k[[T]]$, then $f_0 \neq 0$. This follows from the fact that $f(T) \mapsto f_0$ is an algebra homomorphism from $k[[T]]$ onto k.

17 Additional formal series

Let k be a field, and let T be an indeterminate again. As before, a *formal Laurent series* in T with coefficients in k may be expressed as

$$f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j, \tag{17.1}$$

where $f_j \in k$ for each $j \in \mathbf{Z}$. The space of these formal Laurent series may be defined precisely as $c(\mathbf{Z}, k)$. This is a vector space over k with respect to pointwise addition and scalar multiplication of functions on $\mathbf{Z}$, which corresponds to termwise addition and scalar multiplication of formal Laurent series expressed as in (17.1). A formal power series in T may be identified with a formal Laurent series $f(T)$ with $f_j = 0$ when $j < 0$, which corresponds to identifying $c(\mathbf{Z}_+ \cup \{0\}, k)$ with the subspace of $c(\mathbf{Z}, k)$ consisting of functions that vanish on negative integers.

Let $k((t))$ be the space of formal Laurent series $f(T)$ with coefficients in k such that $f_j = 0$ for all but finitely many negative integers j. As in [4], we may use

$$f(T) = \sum_{j>>-\infty} f_j\, T^j \tag{17.2}$$

to indicate that $f_j = 0$ for all but finitely many $j < 0$. As before, $k((T))$ can be defined precisely as the linear subspace of $c(\mathbf{Z}, k)$ consisting of functions that are equal to 0 at all but finitely many negative integers, and $k[[T]]$ can be identified with a linear subspace of $k((T))$.

If $f(T)$ and $g(T) = \sum_{j>>-\infty} g_j\, T^j$ are elements of $k((T))$, then put

$$f(T)\, g(T) = h(T) = \sum_{n=-\infty}^{\infty} h_n\, T^n, \tag{17.3}$$

where

$$h_n = \sum_{j=-\infty}^{\infty} f_j\, g_{n-j} \tag{17.4}$$

for each $n \in \mathbf{Z}$. All but finitely many terms in the sum on the right side of (17.4) are equal to 0, because $f_j = g_j = 0$ for all but finitely many $j < 0$, so that h_n is defined as an element of k for every $n \in \mathbf{Z}$. Similarly, $h_n = 0$ for all but finitely many $n < 0$, so that $h(T) \in k((T))$. One can check that $k((T))$ is a commutative algebra over k with respect to this definition of multiplication. This definition of multiplication is compatible with the previous one for $k[[T]]$, so that $k[[T]]$ may be considered as a subalgebra of $k((T))$.

In fact, $k((T))$ is a field with respect to this definition of multiplication. Indeed, every nonzero element of $k((T))$ can be expressed as $T^l f(T)$, where $f(T) \in k[[T]]$, $f_0 \neq 0$, and $l \in \mathbf{Z}$. Under these conditions, $f(T)$ has a multiplicative inverse in $k[[T]]$, as in the previous section. Thus $T^{-l} f(T)^{-1}$ is the multiplicative inverse of $T^l f(T)$ in $k((T))$, as desired.

18 Absolute values on $k((T))$

Let k be a field, let T be an indeterminate, and let $0 < r \leq 1$ be given. If $f(T) \in k((T))$ and $f(T) \neq 0$, then we let $j_0(f)$ be the smallest integer such that $f_{j_0} \neq 0$, so that $f_j = 0$ when $j < j_0$. Put

(18.1) $$|f(T)|_r = r^{j_0(f)}$$

when $f(T) \neq 0$, and $|f(T)|_r = 0$ when $f(T) = 0$. Let us check that $|\cdot|_r$ defines an ultrametric absolute value function on $k((T))$. If $f(T), g(T) \in k((T))$ and $f(T), g(T), f(T) + g(T) \neq 0$, then it is easy to see that

(18.2) $$j_0(f(T) + g(T)) \geq \min(j_0(f(T)), j_0(g(T))).$$

This implies that

(18.3) $$|f(T) + g(T)|_r \leq \max(|f(T)|_r, |g(T)|_r)$$

for every $f(T), g(T) \in k((T))$. Similarly,

(18.4) $$j_0(f(T)\, g(T)) = j_0(f(T)) + j_0(g(T))$$

when $f(T), g(T) \neq 0$, so that

(18.5) $$|f(T)\, g(T)|_r = |f(T)|_r\, |g(T)|_r$$

for every $f(T), g(T) \in k((T))$.

Observe that

(18.6) $$|f(T)|_r^a = |f(T)|_{r^a}$$

for every $f(T) \in k((T))$, $0 < r \leq 1$, and $0 < a < \infty$. Hence the absolute value functions $|\cdot|_r$ are equivalent on $k((T))$ when $0 < r < 1$, as in Section 3. If $r = 1$, then $|\cdot|_r$ is the trivial absolute value function on $k((T))$.

Let $0 < r < 1$ and $l \in \mathbf{Z}$ be given, and let $\overline{B}_r(0, r^l)$ be the closed ball in $k((T))$ centered at 0 with radius r^l with respect to the ultrametric associated to $|\cdot|_r$. In this situation, we have that

$$\overline{B}_r(0, r^l) = \{f(T) \in k((T)) : f_j = 0 \text{ for every } j < l\}. \tag{18.7}$$

This can be identified with the Cartesian product of copies of k indexed by $j \in \mathbf{Z}$ with $j \geq l$. One can check that the topology determined on $\overline{B}_r(0, r^l)$ by the ultrametric associated to $|\cdot|_r$ corresponds exactly to the product topology on this Cartesian product, using the discrete topology on k in each factor. In particular, a sequence of elements of $\overline{B}_r(0, r^l)$ converges with respect to this topology if and only if for each $j \in \mathbf{Z}$ with $j \geq l$, the corresponding sequence of coefficients of T^j is eventually equal to the coefficient of T^j of the limit.

It is not difficult to verify that $k((T))$ is complete with respect to the ultrametric associated to $|\cdot|_r$ for every $0 < r \leq 1$. This is trivial when $r = 1$, because the associated ultrametric on $k((T))$ is the discrete metric. If $0 < r < 1$, then it is helpful to begin by observing that a Cauchy sequence in $k((T))$ with respect to the ultrametric associated to $|\cdot|_r$ is contained in $\overline{B}_r(0, r^l)$ for some $l \in \mathbf{Z}$, because Cauchy sequences are bounded. In this case, the Cauchy condition also implies that for each $j \in \mathbf{Z}$, the corresponding sequence of coefficients of T^j is eventually constant. The limiting values of these coefficients of T^j determine another element of $k((T))$, which is the limit of the Cauchy sequence, as in the preceding paragraph.

19 Weighted ℓ^q norms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let T be an indeterminate. If r is a nonnegative real number, q is a positive real number, and $f(T) \in k[[T]]$, then put

$$\|f(T)\|_{q,r} = \Big(\sum_{j=0}^{\infty} |f_j|^q \, r^{q\,j}\Big)^{1/q}, \tag{19.1}$$

which is defined as a nonnegative extended real number. Similarly, put

$$\|f(T)\|_{\infty,r} = \sup_{j \geq 0}(|f_j| \, r^j), \tag{19.2}$$

which is also defined as a nonnegative extended real number. Here r^j is interpreted as being equal to 1 when $j = 0$, even when $r = 0$. Thus

$$\|f(T)\|_{q,0} = |f_0| \tag{19.3}$$

for every $f(T) \in k[[T]]$ and $q > 0$.

Let r be a nonnegative real number again, and put

$$w_r(j) = r^j \tag{19.4}$$

for every nonnegative integer j. Remember that $k[[T]]$ can be defined precisely as $c(\mathbf{Z}_+ \cup \{0\}, k)$, as in Section 16. As in Section 10, $\|f\|_{q,w_r}$ can be defined as a nonnegative extended real number for every $f \in c(\mathbf{Z}_+ \cup \{0\}, k)$ and $q > 0$. Let $f(T)$ be given, and let f be the corresponding element of $c(\mathbf{Z}_+ \cup \{0\}, k)$, whose value at $j \geq 0$ is the coefficient f_j of T^j in $f(T)$. Observe that

$$\|f(T)\|_{q,r} = \|f\|_{q,w_r} \tag{19.5}$$

for every $q > 0$, where the left side of (19.5) is as in (19.1) and (19.2), and the right side of (19.5) is as in Section 10.

Here r corresponds to the radius on which a power series might be considered. Of course, r^j is increasing as a function of r for each $j \in \mathbf{Z}_+$. If $0 \leq r_1 \leq r_2 < \infty$, then

$$\|f(T)\|_{q,r_1} \leq \|f(T)\|_{q,r_2} \tag{19.6}$$

for every $f(T) \in k[[T]]$ and $q > 0$, as in (10.13). If $0 < q_1 \leq q_2 \leq \infty$, then

$$\|f(T)\|_{q_2,r} \leq \|f(T)\|_{q_1,r} \tag{19.7}$$

for every $f(T) \in k[[T]]$ and $r \geq 0$, as in (10.10). If $f(T) \in k[[T]]$ and $\|f(T)\|_{q,r}$ is finite for some positive real number q and $r \geq 0$, then

$$\lim_{q \to \infty} \|f(T)\|_{q,r} = \|f(T)\|_{\infty,r}, \tag{19.8}$$

as in (10.12).

Let $a \in k$ and $f(T) \in k[[T]]$ be given, so that $a\, f(T) \in k[[T]]$ too. If $a \neq 0$, then

$$\|a\, f(T)\|_{q,r} = |a|\, \|f(T)\|_{q,r} \tag{19.9}$$

for every $q > 0$ and $r \geq 0$. If $a = 0$, then $a\, f(T) = 0$, and the left side of (19.9) is equal to 0.

Now let $f(T), g(T) \in k[[T]]$ be given. If $q_k = \infty$, then

$$\|f(T) + g(T)\|_{\infty,r} \leq \max(\|f(T)\|_{\infty,r}, \|g(T)\|_{\infty,r}) \tag{19.10}$$

for every $r \geq 0$, as in (10.7). If $0 < q \leq q_k$ and $q < \infty$, then

$$\|f(T) + g(T)\|_{q,r}^q \leq \|f(T)\|_{q,r}^q + \|g(T)\|_{q,r}^q \tag{19.11}$$

for every $r \geq 0$, as in (10.8). If $q_k \leq q$ and $q_k < \infty$, then

$$\|f(T) + g(T)\|_{q,r}^{q_k} \leq \|f(T)\|_{q,r}^{q_k} + \|g(T)\|_{q,r}^{q_k} \tag{19.12}$$

for every $r \geq 0$, as in (10.9).

20 Weighted ℓ^q spaces

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let T be an indeterminate, and let r be a positive real number. If $0 < q \le \infty$, then put

$$PS_r^q(k) = \{f(T) \in k[[T]] : \|f(T)\|_{q,r} < \infty\}, \tag{20.1}$$

where $\|f(T)\|_{q,r}$ is as in (19.1) and (19.2). As before, $k[[T]]$ can be defined precisely as $c(\mathbf{Z} \cup \{0\}, k)$, and w_r can be defined as a positive real-valued function on $\mathbf{Z}_+ \cup \{0\}$ as in (19.4). Using (19.5), we have that $PS_r^q(k)$ corresponds exactly to the space $\ell^q_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ defined in Section 10. Thus $PS_r^q(k)$ is a linear subspace of $k[[T]]$ for every $q > 0$, and $\|f(T)\|_{q,r}$ defines a q-norm on $PS_r^q(k)$ when $q \le q_k$, and a q_k-norm on $PS_r^q(k)$ when $q_k \le q$. It is sometimes convenient to allow $r = 0$ here, in which case $PS_r^q(k)$ reduces to $k[[T]]$ and $\ell^q_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ reduces to $c(\mathbf{Z}_+ \cup \{0\}, k)$ for every $q > 0$. Note that $\|f(T)\|_{q,0}$ is a q_k-seminorm on $k[[T]]$ for every $q > 0$, by (19.3).

If $0 \le r_1 \le r_2 < \infty$, then

$$PS_{r_2}^q(k) \subseteq PS_{r_1}^q(k) \tag{20.2}$$

for every $q > 0$. This follows from (19.6), and also corresponds to (10.14), as in the preceding paragraph. Similarly, if $0 < q_1 \le q_2 \le \infty$, then

$$PS_r^{q_1}(k) \subseteq PS_r^{q_2}(k) \tag{20.3}$$

for every $r \ge 0$. This follows from (19.7), and corresponds to (10.11) as well.

If r is a nonnegative real number, then put

$$PS_{0,r}(k) = \big\{f(T) \in k[[T]] : \lim_{j\to\infty} |f_j|\, r^j = 0\big\}. \tag{20.4}$$

This is a linear subspace of $k[[T]]$ for every $r \ge 0$, which corresponds to the subspace $c_{0,w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ of $c(\mathbf{Z}_+ \cup \{0\}, k)$ defined in Section 11. As before, $PS_{0,r}(k)$ reduces to $k[[T]]$ and $c_{0,w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ reduces to $c(\mathbf{Z}_+ \cup \{0\}, k)$ when $r = 0$. Of course,

$$PS_{0,r}(k) \subseteq PS_r^\infty(k) \tag{20.5}$$

for every $r \ge 0$, because convergent sequences are bounded, and as in (11.6). We also have that $PS_{0,r}(k)$ is a closed set in $PS_r^\infty(k)$ with respect to the topology determined by the q_k-semimetric associated to $\|f(T)\|_{\infty,r}$ for every $r \ge 0$, as in Section 11.

If $0 \le r_1 \le r_2 < \infty$, then it is easy to see that

$$PS_{0,r_2}(k) \subseteq PS_{0,r_1}(k). \tag{20.6}$$

This may be considered as an instance of (11.10) as well. If $0 < q < \infty$, then

$$PS_r^q(k) \subseteq PS_{0,r}(k) \tag{20.7}$$

for every $r \ge 0$, because the terms of a convergent series converge to 0. This may also be considered as an instance of (11.8).

21 Radius of convergence

Let k be a field with a q_k-absolute value function $|\cdot|$ again, and let T be an indeterminate. If $f(T) \in k[[T]]$, then the *radius of convergence* of $f(T)$ may be defined as a nonnegative extended real number by

$$\text{(21.1)} \qquad \begin{aligned} \operatorname{rad}(f(T)) &= \sup\{r \geq 0 : f(T) \in PS_r^\infty(k)\} \\ &= \sup\{r \geq 0 : \|f(T)\|_{\infty,r} < \infty\}. \end{aligned}$$

This is the supremum of a nonempty set of nonnegative real numbers r, because $r = 0$ is automatically in this set. If $0 \leq r_1 < \operatorname{rad}(f(T))$, then there is an $r \geq r_1$ such that $f(T) \in PS_r^\infty(k)$, by the definition of the supremum. This implies that

$$\text{(21.2)} \qquad f(T) \in PS_{r_1}^\infty(k)$$

when $r_1 < \operatorname{rad}(f(T))$, by (20.2). Of course, if $r_2 > \operatorname{rad}(f(T))$, then

$$\text{(21.3)} \qquad f(T) \notin PS_{r_2}^\infty(k),$$

by definition of $\operatorname{rad}(f(T))$. Clearly $\operatorname{rad}(f(T))$ is uniquely determined by these two conditions.

Suppose that $f(T) \in PS_r^\infty(k)$ for some $r > 0$, so that $\|f(T)\|_{\infty,r} < \infty$ and

$$\text{(21.4)} \qquad |f(j)| \leq \|f(T)\|_{\infty,r}\, r^{-j}$$

for every nonnegative integer j. This implies that

$$\text{(21.5)} \qquad |f(j)|\, r_1^j \leq \|f(T)\|_{\infty,r}\, (r_1/r)^j$$

for every nonnegative real number r_1 and nonnegative integer j. In particular, it follows that

$$\text{(21.6)} \qquad \lim_{j \to \infty} |f(j)|\, r_1^j = 0$$

when $0 \leq r_1 < r$, so that $f(T) \in PS_{0,r_1}(k)$. Similarly, if $0 \leq r_1 < r$ and q is a positive real number, then

$$\text{(21.7)} \qquad \sum_{j=0}^\infty |f(j)|^q\, r_1^{q\,j} \leq \|f\|_{\infty,r}^q \sum_{j=0}^\infty (r_1/r)^{q\,j} = \|f\|_{\infty,r}^q\, (1 - (r_1/r)^q)^{-1}.$$

This means that $f(T) \in PS_{r_1}^q(k)$, with

$$\text{(21.8)} \qquad \|f(T)\|_{q,r_1} \leq \|f\|_{\infty,r}\, (1 - (r_1/r)^q)^{-1/q}$$

under these conditions.

If r_1 is any nonnegative real number with $r_1 < \operatorname{rad}(f(T))$, then there is a real number r such that $r_1 < r < \operatorname{rad}(f(T))$. Thus $f(T) \in PS_r^\infty(k)$, so that the remarks in the preceding paragraph can be applied. It follows that $f(T) \in PS_{0,r_1}(k)$, and that $f(T) \in PS_{r_1}^q(k)$ for every $q > 0$, as before. This implies that the radius of convergence could be defined equivalently in terms of $PS_r^q(k)$ for any $q > 0$ instead of $PS_r^\infty(k)$, or in terms of $PS_{0,r}(k)$. This also uses the fact that $PS_r^q(k)$ and $PS_{0,r}(k)$ are contained in $PS_r^\infty(k)$ for every $r \geq 0$ and $q > 0$, as in (20.3) and (20.5).

If r is a nonnegative extended real number, then put

$$PS_r(k) = \{f(T) \in k[[T]] : \mathrm{rad}(f(T)) \geq r\}. \tag{21.9}$$

This is the same as $k[[T]]$ when $r = 0$, as usual. If $r > 0$, then

$$PS_r = \bigcap_{0 \leq r_1 < r} PS_{r_1}^q(k) \tag{21.10}$$

for every $q > 0$, and

$$PS_r(k) = \bigcap_{0 \leq r_1 < r} PS_{0,r_1}(k). \tag{21.11}$$

In particular, $PS_r(k)$ is a linear subspace of $k[[T]]$ for every $r \geq 0$. Of course, $PS_r(k)$ increases as r decreases.

22 Submultiplicativity conditions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let T be an indeterminate, and let r be a positive real number. Also let $f(T), g(T) \in PS_r^q(k)$ be given, where $0 < q \leq q_k$, and $PS_r^q(k)$ is as in (20.1). We would like to check that $h(T) = f(T)\, g(T) \in PS_r^q(T)$ too, with

$$\|h(T)\|_{q,r} \leq \|f(T)\|_{q,r}\, \|g(T)\|_{q,r}. \tag{22.1}$$

This means that $PS_r^q(k)$ is a subalgebra of $k[[T]]$ under these conditions, and that $\|f(T)\|_{q,r}$ is submultiplicative as a q-norm on $PS_r^q(k)$. Note that

$$\|1\|_{q,r} = 1 \tag{22.2}$$

for every $q > 0$ and $r \geq 0$, where the 1 on the left is the constant power series with constant term equal to the multiplicative identity element of k, and the 1 on the right is the usual real number. If $r = 0$, then equality holds in (22.1), because of (19.3). This also uses the fact that $f(T) \mapsto f_0$ is an algebra homomorphism from $k[[T]]$ into k, as in Section 16.

Suppose first that $q < \infty$, and remember that $|\cdot|$ may be considered as a q-absolute value function on k, as in Section 3, because $q \leq q_k$. Remember also that the nth coefficient h_n of $h(T)$ is given as in (16.3) for each nonnegative integer n. It follows that

$$|h_n|^q \leq \sum_{j=0}^{n} |f_j|^q\, |g_{n-j}|^q \tag{22.3}$$

for every $n \ge 0$. This implies that

$$|h_n|^q \, r^{q\,n} \le \sum_{j=0}^{n} (|f_j|^q \, r^{q\,j}) \, (|g_j| \, r^{q\,(n-j)}) \tag{22.4}$$

for every $n \ge 0$. The right side of (22.4) is the same as the nth term of the Cauchy product of the series $\sum_{j=0}^{\infty} |f_j|^q \, r^{q\,j}$ and $\sum_{l=0}^{\infty} |g_l|^q \, r^{q\,l}$, as in Section 15. These series are convergent as series of nonnegative real numbers, because $f(T), g(T) \in PS_r^q(k)$, by hypothesis. Hence

$$\begin{aligned} \sum_{n=0}^{\infty} |h_n|^q \, r^{q\,n} &\le \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} (|f_j|^q \, r^{q\,j}) \, (|g_{n-j}|^q \, r^{q\,(n-j)}) \Big) \\ &= \Big(\sum_{j=0}^{\infty} |f_j|^q \, r^{q\,j} \Big) \Big(\sum_{l=0}^{\infty} |g_l|^q \, r^{q\,l} \Big), \end{aligned} \tag{22.5}$$

where the second step is as in as in Section 15. This is the same as (22.1), by taking the qth roots of both sides.

Suppose now that $q = \infty$, so that $q_k = \infty$. Using (16.3) again, we get that

$$|h_n| \le \max_{0 \le j \le n} (|f_j| \, |g_{n-j}|) \tag{22.6}$$

for each $n \ge 0$, by the ultrametric version of the triangle inequality. Thus

$$|h_n| \, r^n \le \max_{0 \le j \le n} (((|f_j| \, r^j) \, (|g_{n-j}| \, r^{n-j})) \tag{22.7}$$

for every $n \ge 0$, which implies (22.1) when $q = \infty$. Similarly, suppose that $f(T), g(T) \in PS_{0,r}(k)$, so that

$$\lim_{j \to \infty} |f_j| \, r^j = \lim_{l \to \infty} |g_l| \, r^l = 0. \tag{22.8}$$

In this case, one can use (22.7) to verify that

$$\lim_{n \to \infty} |h_n| \, r^n = 0, \tag{22.9}$$

so that $h(T) \in PS_{0,r}(k)$ too.

If r is a nonnegative extended real number, then $PS_r(k)$ as in (21.9) is a subalgebra of $k[[T]]$ as well. This follows from the previous remarks and the characterization of $\mathrm{rad}(f(T))$ in terms of ℓ^{q_k} norms and spaces, as in the preceding section.

23 Multiplicativity conditions

Let k be a field with an ultrametric absolute value function $|\cdot|$, let T be an indeterminate, and let r be a positive real number. If $f(T), g(T) \in PS_r^\infty(k)$, then $h(T) = f(T)\, g(T) \in PS_r^\infty(k)$, as in the previous section. In fact, we have that

$$\|h(T)\|_{\infty,r} = \|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} \tag{23.1}$$

in this situation. It suffices to check that

$$\|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} \le \|h(T)\|_{\infty,r}, \tag{23.2}$$

because the opposite inequality follows from (22.1). We may as well suppose that $f(T), g(T) \ne 0$, since otherwise this is trivial.

Let us begin with the case where $f(T), g(T) \in PS_{0,r}(k)$, so that (22.8) holds. In particular, this implies that the supremum in the definition (19.2) of each of $\|f(T)\|_{\infty,r}$ and $\|g(T)\|_{\infty,r}$ is attained. The standard argument is to take $j_r(f)$, $j_r(g)$ to be the smallest nonnegative integers such that

$$|f_{j_r(f)}|\, r^{j_r(f)} = \|f(T)\|_{\infty,r}, \quad |g_{j_r(g)}|\, r^{j_r(g)} = \|g(T)\|_{\infty,r}. \tag{23.3}$$

Thus

$$|f_j|\, r^j < \|f(T)\|_{\infty,r} \tag{23.4}$$

for all nonnegative integers $j < j_r(f)$, and

$$|g_j|\, r^j < \|g(T)\|_{\infty,r} \tag{23.5}$$

for all nonnegative integers $j < j_r(g)$. We would like to verify that

$$\|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} \le |h_{j_r(f)+j_r(g)}|\, r^{j_r(f)+j_r(g)} \tag{23.6}$$

under these conditions.

Remember that

$$h_{j_r(f)+j_r(g)} = \sum_{j=0}^{j_r(f)+j_r(g)} f_j\, g_{j_r(f)+j_r(g)-j}, \tag{23.7}$$

as in (16.3). The sum on the right side is equal to

$$\sum_{j=0}^{j_r(f)-1} f_j\, g_{j_r(f)+j_r(g)-j} + f_{j_r(f)}\, g_{j_r(g)} + \sum_{j=j_r(f)+1}^{j_r(f)+j_r(g)} f_j\, g_{j_r(f)+j_r(g)-j}, \tag{23.8}$$

where the first sum is interpreted as being 0 when $j_r(f) = 0$, and the second sum is interpreted as being 0 when $j_r(g) = 0$. Taking $l = j_r(f) + j_r(g) - j$, the second sum can be reexressed as

$$\sum_{l=0}^{j_r(g)-1} f_{j_r(f)+j_r(g)-l}\, g_l, \tag{23.9}$$

which is also interpreted as being 0 when $j_r(g) = 0$. Thus

$$(23.10) \qquad f_{j_r(f)}\, g_{j_r(g)} = h_{j_r(f)+j_r(g)} - \sum_{j=0}^{j_r(f)-1} f_j\, g_{j_r(f)+j_r(g)-j} - \sum_{l=0}^{j_r(g)-1} f_{j_r(f)+j_r(g)-l}\, g_l,$$

with the same intepretations of the sums on the right when $j_r(f) = 0$ or $j_r(g) = 0$. The ultrametric version of the triangle inequality implies that

$$(23.11) \qquad |f_{j_r(f)}|\,|g_{j_r(g)}| \le \max\Big(|h_{j_r(f)+j_r(g)}|, \max_{0\le j\le j_r(f)-1}(|f_j|\,|g_{j_r(f)+j_r(g)-j}|), \max_{0\le l\le j_r(g)-1}(|f_{j_r(f)+j_r(g)-l}|\,|g_l|)\Big).$$

Note that

$$(23.12) \qquad \|f(T)\|_{\infty,r}\,\|g(T)\|_{\infty,r} = |f_{j_r(f)}|\,|g_{j_r(g)}|\,r^{j_r(f)+j_r(g)},$$

by (23.3). Multiplying both sides of (23.11) by $r^{j_r(f)+j_r(g)}$, we get that (23.12) is less than or equal to the maximum of

$$(23.13) \qquad |h_{j_r(f)+j_r(g)}|\,r^{j_r(f)+j_r(g)},$$

$$(23.14) \qquad \max_{0\le j\le j_r(f)-1}((|f_j|\,r^j)\,(|g_{j_r(f)+j_r(g)-j}|\,r^{j_r(f)+j_r(g)-j})), \text{ and}$$

$$(23.15) \qquad \max_{0\le l\le j_r(g)-1}((|f_{j_r(f)+j_r(g)-l}|\,r^{j_r(f)+j_r(g)-l})\,(|g_l|\,r^l)).$$

Each of the terms in (23.14) and (23.15) is strictly less than (23.12), because of (23.4), (23.5), and the definition of $\|f(T)\|_{\infty,r}$, $\|g(T)\|_{\infty,r}$. It follows that (23.6) holds, and hence (23.2), as desired.

Suppose now that $f(T), g(T) \in PS_r^\infty(k)$, and let a positive real number $r_1 < r$ be given. Thus $f(T), g(T) \in PS_{0,r_1}(k)$, as in (21.6). Using the previous argument applied to r_1, we get that

$$(23.16) \qquad \|f(T)\|_{\infty,r_1}\,\|g(T)\|_{\infty,r_1} \le \|h(T)\|_{\infty,r_1}.$$

It follows that

$$(23.17) \qquad \|f(T)\|_{\infty,r_1}\,\|g(T)\|_{\infty,r_1} \le \|h(T)\|_{\infty,r},$$

using the monotonicity property (19.6) on the right side of (23.16). This implies that

$$(23.18) \qquad (|f_j|\,r_1^j)\,(|g_{j'}|\,r_1^{j'}) \le \|h(T)\|_{\infty,r}$$

for all nonnegative integers j and j', by the definition (19.2) of $\|f(T)\|_{\infty,r_1}$, $\|g(T)\|_{\infty,r_1}$. Hence

$$(23.19) \qquad (|f_j|\,r^j)\,(|g_{j'}|\,r^{j'}) \le \|h(T)\|_{\infty,r}$$

for all nonnegative integers j, j', by approximating r by r_1 on the left side. This implies (23.2), by taking the supremum over $j, j' \ge 0$.

24 Rescaling

Let k be a field, let T be an indeterminate, and let $a \in k$ be given. If $f(T)$ is an element of $k[[T]]$, then put

$$R_a(f(T)) = \sum_{j=0}^{\infty} a^j\, f_j\, T^j. \tag{24.1}$$

As usual, a^j is interpreted as being the multiplicative identity element 1 in k when $j = 0$, even if $a = 0$. Thus $R_a(f(T))$ is also a formal power series in T, which might be considered informally as $f(a\,T)$. Clearly R_a defines a linear mapping from $k[[T]]$ into itself, which sends $k[T]$ into itself. More precisely, R_a defines an algebra homomorphism from $k[[T]]$ into itself. Indeed, if $f(T)$, $g(T)$ are elements of $k[[T]]$, then

$$\begin{aligned} R_a(f(T))\, R_a(g(T)) &= \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} (a^j\, f_j)\, (a^{n-j}\, g_j) \Big)\, T^n \\ &= \sum_{n=0}^{\infty} a^n \Big(\sum_{j=0}^{n} f_j\, g_{n-j} \Big)\, T^n = R_a(f(T)\, g(T)), \end{aligned} \tag{24.2}$$

as desired.

If $a, b \in k$ and $f(T) \in k[[T]]$, then

$$R_a(R_b(f(T))) = R_a\Big(\sum_{j=0}^{\infty} b^j\, f_j\, T^j \Big) = \sum_{j=0}^{\infty} a^j\, b^j\, f_j\, T^j = R_{a\,b}(f(T)), \tag{24.3}$$

so that

$$R_a \circ R_b = R_{a\,b}. \tag{24.4}$$

Note that R_1 is the identity mapping on $k[[T]]$, where 1 is the multiplicative identity element in k again. If $a \in k$ and $a \neq 0$, then R_a is invertible on $k[[T]]$, with inverse equal to $R_{1/a}$. Remember that $k[[T]]$ can be defined precisely as $c(\mathbf{Z}_+ \cup \{0\}, k)$, so that R_a corresponds to a linear mapping from $c(\mathbf{Z}_+ \cup \{0\})$ into itself. This linear mapping is the same as the multiplication operator associated to a^j as a k-valued function of j on $\mathbf{Z}_+ \cup \{0\}$, as in Section 12.

It is easy to see that

$$\|R_a(f(T))\|_{q,r} = \|f(T)\|_{q,|a|\,r} \tag{24.5}$$

for every $a \in k$, $f(T) \in k[[T]]$, $0 \leq r < \infty$, $0 < q \leq \infty$, directly from the definition of $\|f(T)\|_{q,r}$ in Section 19. This implies that R_a maps $PS^q_{|a|\,r}(k)$ into $PS^q_r(k)$. If $a \neq 0$, then R_a maps $PS^q_{|a|\,r}(k)$ onto $PS^q_r(k)$. Similarly, R_a maps $PS_{0,|a|\,r}(k)$ into $PS_{0,r}(k)$ for every $a \in k$ and $r \geq 0$, and this mapping is surjective when $a \neq 0$. One can also look at these properties of R_a in terms of multiplication operators, as in Section 12.

Of course,

$$R_0(f(T)) = f_0 \tag{24.6}$$

is a constant formal power series for every $f(T) \in k[[T]]$, which is to say that the coefficient of T^j in $R_0(f(T))$ is equal to 0 for every $j \in \mathbf{Z}_+$. In particular, the radius of convergence of $R_0(f(T))$ is equal to $+\infty$. If $a \neq 0$, then one can check that

$$\mathrm{rad}(R_a(f(T))) = \mathrm{rad}(f(T))/|a| \tag{24.7}$$

for every $f(T) \in k[[T]]$, using the remarks in the preceding paragraph. This implies that R_a maps PS_r into $PS_{r/|a|}$ for every nonnegative extended real number r and $a \in k$ with $a \neq 0$. One can verify that this mapping is surjective as well.

25 Related functions on algebras

Let k be a field, let T be an indeterminate, and let

$$f(T) = \sum_{j=0}^{n} f_j\, T^j \tag{25.1}$$

be an element of $k[T]$, so that $f_j \in k$ for each $j = 0, 1, \ldots, n$. Also let $\mathcal{A}$ be an algebra over k, with a multiplicative identity element e. If $x \in \mathcal{A}$, then put

$$f(x) = \sum_{j=0}^{n} f_j\, x^j, \tag{25.2}$$

where x^j is interpreted as being equal to e when $j = 0$. Thus $f(x) \in \mathcal{A}$ for every $x \in \mathcal{A}$, which defines a mapping from $\mathcal{A}$ into itself. Let $x \in \mathcal{A}$ be given, so that

$$f(T) \mapsto f(x) \tag{25.3}$$

defines a mapping from $k[T]$ into $\mathcal{A}$. It is easy to see that this mapping is an algebra homomorphism. This uses the remarks about Cauchy products for finite sums in Section 15.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. It is convenient to ask that $N(e) = 1$ too. In particular, this implies that $|\cdot|$ is a q-absolute value function on k, as in Section 5, so that we may as well suppose that $q \leq q_k$. Also let $x \in \mathcal{A}$ and a nonnegative real number r be given, with $N(x) \leq r$. If $q < \infty$, then

$$N(f(x))^q \leq \sum_{j=0}^{n} N(f_j\, x^j)^q \leq \sum_{j=0}^{n} |f_j|^q\, N(x)^{q\,j} \leq \sum_{j=0}^{n} |f_j|^q\, r^{q\,j}, \tag{25.4}$$

using the q-seminorm version of the triangle inequality in the first step. Here $N(x)^{q\,j}$ and $r^{q\,j}$ are interpreted as being equal to 1 when $j = 0$, as usual. Similarly, if $q = \infty$, then

$$N(f(x)) \le \max_{0 \le j \le n} N(f_j\, x^j) \le \max_{0 \le j \le n} (|f_j|\, N(x)^j) \le \max_{0 \le j \le n} (|f_j|\, r^j) \tag{25.5}$$

using the semi-ultranorm version of the triangle inequality in the first step. In both cases, we get that

$$N(f(x)) \le \|f(T)\|_{q,r}, \tag{25.6}$$

where $\|f(T)\|_{q,r}$ is as in Section 19.

Let us suppose in addition that N is a q-norm on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N, from now on in this section. Let $f(T) = \sum_{j=0}^\infty f_j\, T^j$ be an element of $k[[T]]$, so that we would like to put

$$f(x) = \sum_{j=0}^\infty f_j\, x^j. \tag{25.7}$$

Suppose for the moment that $q < \infty$ and $f(T) \in PS_r^q(k)$, where $PS_r^q(k)$ is as in (20.1). In this case,

$$\sum_{j=0}^\infty N(f_j\, x^j)^q \le \sum_{j=0}^\infty |f_j|^q\, N(x)^{q\,j} \le \sum_{j=0}^\infty |f_j|^q\, r^{q\,j} \le \|f(T)\|_{q,r}^q < \infty, \tag{25.8}$$

so that the sum on the right side of (25.7) converges q-absolutely with respect to N, and hence converges in $\mathcal{A}$ with respect to N, as in Section 9. Thus $f(x)$ may be defined as an element of $\mathcal{A}$ as in (25.7), and it satisfies (25.6), because of (9.6) and (25.8). Remember that $PS_r^q(k)$ is a subalgebra of $k[[T]]$, because $q \le q_k$, as in Section 22. One can check that (25.3) is an algebra homomorphism from $PS_r^q(k)$ into $\mathcal{A}$, using the remarks in Section 15.

Suppose now that $q = \infty$ and $f(T) \in PS_{0,r}$, where $PS_{0,r}(k)$ is as in (20.4). This implies that

$$N(f_j\, x^j) \le |f_j|\, N(x)^j \le |f_j|\, r^j \to 0 \quad \text{as } j \to \infty, \tag{25.9}$$

so that the series on the right side of (25.7) converges in $\mathcal{A}$ with respect to N, as in Section 9 again. If $f(x)$ is defined as an element of $\mathcal{A}$ as in (25.7), then it satisfies (25.6), because of (9.8) and (25.9). As in Section 22, $PS_{0,r}(k)$ is a subalgebra of $k[[T]]$. One can verify that (25.3) is an algebra homomorphism from $PS_{0,r}(k)$ into $\mathcal{A}$, using the remarks in Section 15 again.

As a variant of this case, suppose that $q = \infty$, $f(T) \in PS_r^\infty(k)$, and $\|x\| < r$. Let r_1 be a real number such that $\|x\| \le r_1 < r$, and remember that $PS_r^\infty(k)$ is contained in $PS_{0,r_1}(k)$, as in Section 21. This permits us to define $f(x)$ as an element of k, as in the preceding paragraph. We also get that (25.6) holds with r replaced by r_1, as before. Of course, this implies that (25.6) holds, by (19.6). As in Section 22, $PS_r^\infty(k)$ is a subalgebra of $k[[T]]$, and more precisely $PS_r^\infty(k)$ is a subalgebra of $PS_{0,r_1}(k)$ in this situation. It follows that (25.3) is an algebra homomorphism from $PS_r^\infty(k)$ into $\mathcal{A}$, because of the analogous statement for $PS_{0,r_1}(k)$, as in the previous paragraph.

26 Another convergence condition

A sequence $\{a_j\}_{j=1}^{\infty}$ of nonnegative real numbers is said to be *submultiplicative* if

$$a_{j+l} \le a_j \, a_l \tag{26.1}$$

for every $j, l \ge 1$. In this case, if we put

$$\alpha = \inf_{j \ge 1} a_j^{1/j}, \tag{26.2}$$

then it is well known that

$$\lim_{j \to \infty} a_j^{1/j} = \alpha. \tag{26.3}$$

Of course,

$$\alpha \le \liminf_{j \to \infty} a_j^{1/j} \tag{26.4}$$

holds automatically, and so it suffices to check that

$$\limsup_{j \to \infty} a_j^{1/j} \le \alpha. \tag{26.5}$$

To do this, let $j, j_0 \in \mathbf{Z}_+$ be given, and let l_0, r_0 be nonnegative integers such that

$$j = j_0 \, l_0 + r_0 \tag{26.6}$$

and $r_0 < j_0$. Thus

$$a_j = a_{j_0 \, l_0 + r_0} \le a_{j_0}^{l_0} \, a_1^{r_0}, \tag{26.7}$$

by (26.1), so that

$$a_j^{1/j} \le (a_{j_0}^{1/j_0})^{j_0 \, l_0 / j} \, a_1^{r_0/j} = (a_{j_0}^{1/j_0})^{1-(r_0/j)} \, a_1^{r_0/j}. \tag{26.8}$$

This implies that

$$\limsup_{j \to \infty} a_j^{1/j} \le a_{j_0}^{1/j_0} \tag{26.9}$$

for every $j_0 \ge 1$, because $b^{1/j} \to 1$ as $j \to \infty$ for every $b > 0$. It follows that (26.5) holds, and hence (26.3), as desired.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-seminorm N with respect to $|\cdot|$ on k for some $q > 0$. If $x \in \mathcal{A}$, then it is easy to see that $a_j = N(x^j)$ defines a submultiplicative sequence of nonnegative real numbers. Put

$$N_\rho(x) = \inf_{j \ge 1} N(x^j)^{1/j}, \tag{26.10}$$

so that

$$\lim_{j \to \infty} N(x^j)^{1/j} = N_\rho(x), \tag{26.11}$$

as in (26.3). Of course,

$$N_\rho(x) \le N(x) \tag{26.12}$$

automatically. If $N_\rho(x) < r_1$ for some positive real number r_1, then

$$N(x^j)^{1/j} < r_1 \tag{26.13}$$

for all but finitely many $j \ge 1$, and hence

$$N(x^j) < r_1^j \tag{26.14}$$

for all but finitely many j.

Let us suppose from now on in this section that N is a q-norm on $\mathcal{A}$, that $\mathcal{A}$ is complete with respect to the q-metric associated to N, and that $\mathcal{A}$ has a nonzero multiplicative identity element e. This implies that $|\cdot|$ is a q-absolute value function on k, as in Section 5, and so we may as well suppose that $q \le q_k$. Let $x \in \mathcal{A}$ and a positive extended real number r be given, with

$$N_\rho(x) < r. \tag{26.15}$$

Also let T be an indeterminate, and let $f(T) \in PS_r(k)$ be given, where $PS_r(k)$ is as in Section 21. We would like to show that the right side of (25.7) converges in $\mathcal{A}$ with respect to N under these conditions.

Because of (26.15), we can choose a positive real number r_1 such that

$$N_\rho(x) < r_1 < r. \tag{26.16}$$

Suppose first that $q < \infty$, and remember that $f(T) \in PS^q_{r_1}(k)$, as in Section 21. In this case, it is easy to see that

$$\sum_{j=0}^{\infty} N(f_j\, x^j)^q = \sum_{j=0}^{\infty} |f_j|^q\, N(x^j)^q < \infty, \tag{26.17}$$

using (26.14). This means that the right side of (25.7) converges q-absolutely with respect to N, and hence converges in $\mathcal{A}$ with respect to N, as in Section 9. Similarly, if $q = \infty$, then we can use the fact that $f(T) \in PS_{0,r_1}(k)$, as in Section 21. This implies that

$$N(f_j\, x^j) = |f_j|\, N(x^j) \to 0 \quad \text{as } j \to \infty, \tag{26.18}$$

using (26.14). It follows that the right side of (25.7) converges in $\mathcal{A}$ with respect to N in this case as well, as in Section 9.

Remember that $PS_r(k)$ is a subalgebra of $k[[T]]$, as in Section 22. One can verify that (25.3) is an algebra homomorphism from $PS_r(k)$ into $\mathcal{A}$ in this situation, using the remarks in Section 15.

27 Trivial absolute values

Let k be a field, and let T be an indeterminate. Let us take k to be equipped with the trivial absolute value function, and consider the spaces $PS^q_r(k)$ defined in (20.1). Observe that

$$PS^q_r(k) = k[[T]] \tag{27.1}$$

for every $q > 0$ when $0 \leq r < 1$, and when $q = \infty$ and $r = 1$. Otherwise, we have that

$$PS_r^q(k) = k[T] \tag{27.2}$$

for every $q > 0$ when $r > 1$, and for $0 < q < \infty$ when $r = 1$. Similarly, if $PS_{0,r}(k)$ is as in (20.4), then

$$PS_{0,r}(k) = k[[T]] \tag{27.3}$$

when $0 \leq r < 1$, and

$$PS_{0,r}(k) = k[T] \tag{27.4}$$

when $r \geq 1$.

Let $\|f(T)\|_{q,r}$ be defined for $f(T) \in k[[T]]$ as in (19.1) and (19.2). It is easy to see that

$$\|f(T)\|_{\infty,1} \text{ is the trivial ultranorm on } k[[T]] \tag{27.5}$$

in this situation. Note that

$$\|f(T)\|_{\infty,1} \leq \|f(T)\|_{q,r} \tag{27.6}$$

for every $f(T) \in k[[T]]$, $q > 0$, and $r \geq 1$, as in (19.6) and (19.7). Of course, the trivial ultranorm on $k[[T]]$ corresponds to the discrete metric and topology on $k[[T]]$. If $r \geq 1$, then the topology determined on $k[T]$ by the q-metric associated to $\|f(T)\|_{q,r}$ is the same as the discrete topology, because of (27.6).

If $0 < r \leq 1$, then

$$\|f(T)\|_{\infty,r} = |f(T)|_r \tag{27.7}$$

for every $f(T) \in k[[T]]$, where $|f(T)|_r$ is as in Section 18. Remember that $k[[T]]$ can be defined precisely as $c(\mathbf{Z}_+ \cup \{0\}, k)$, which can be identified with the Cartesian product of copies of k indexed by nonnegative integers. If $0 < r < 1$, then the topology determined on $k[[T]]$ by the ultranorm associated to (27.7) is the same as the product topology corresponding to the discrete topology on k in each factor, as before.

Similarly, let $0 < r < 1$ and $0 < q < \infty$ be given, and let us check that the topology determined on $k[[T]]$ by the q-metric associated to $\|f(T)\|_{q,r}$ is the same as the topology corresponding to (27.7). Remember that (27.7) is less than or equal to $\|f(T)\|_{q,r}$ for every $f(T) \in k[[T]]$, as in (19.7). This implies that the topology determined on $k[[T]]$ by the q-metric associated to $\|f(T)\|_{q,r}$ is at least as strong as the topology determined by the ultrametric associated to (27.7). In the other direction, let $0 < r_0 < r$ be given, and remember that $\|f(T)\|_{q,r}$ is less than or equal to a constant times $\|f(T)\|_{\infty,r_0}$ for every $f(T)$ in $k[[T]]$, as in (21.8). This implies that the topology determined on $k[[T]]$ by the ultrametric associated to $\|f(T)\|_{\infty,r_0}$ is at least as strong as the topology determined by the q-metric associated to $\|f(T)\|_{q,r}$. The topologies determined on $k[[T]]$ by the ultrametrics associated to $\|f(T)\|_{\infty,r}$ and $\|f(T)\|_{\infty,r_0}$ are the same, as in the preceding paragraph. It follows that this is the same as the topology determined on $k[[T]]$ by the q-metric associated to $\|f(T)\|_{q,r}$, as desired.

28 Convergence in $PS_r^q(k)$

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let T be an indeterminate. Also let

$$a(T) = \sum_{j=0}^{\infty} a_j\, T^j \tag{28.1}$$

be a formal power series in T with coefficients in k. Of course, $a_j\, T^j$ may be considered as a formal polynomial in T with coefficients in k for each nonnegative integer j. Remember that the spaces $PS_r^q(k)$ are defined in (20.1) for $0 < q \le \infty$ and $0 \le r < \infty$. Let us look at the right side of (28.1) as an infinite series in these spaces.

Observe that

$$\|a_j\, T^j\|_{q,r} = |a_j|\, r^j \tag{28.2}$$

for every $j \ge 0$, $0 < q \le \infty$, and $0 \le r < \infty$, where $\|f(T)\|_{q,r}$ is as defined in (19.1) and (19.2). Of course, r^j is interpreted as being equal to 1 for every $r \ge 0$ when $j = 0$, as usual. In particular, $a_j\, T^j$ is an element of $PS_r^q(k)$ for every $j \ge 0$, $0 < q \le \infty$, and $0 \le r < \infty$. Similarly, if n is a nonnegative integer and r is a nonnegative real number, then

$$\biggl\|\sum_{j=0}^{n} a_j\, T^j\biggr\|_{q,r} = \Bigl(\sum_{j=0}^{n} |a_j|^q\, r^{q\,j}\Bigr)^{1/q} \tag{28.3}$$

when $0 < q < \infty$, and

$$\biggl\|\sum_{j=0}^{n} a_j\, T^j\biggr\|_{\infty,r} = \max_{0 \le j \le n} (|a_j|\, r^j). \tag{28.4}$$

The right sides of (28.3) and (28.4) increase monotonically in n, and

$$\|a(T)\|_{q,r} = \sup_{n \ge 0} \biggl\|\sum_{j=0}^{n} a_j\, T^j\biggr\|_{q,r} \tag{28.5}$$

as nonnegative extended real numbers for every $0 < q \le \infty$ and $0 \le r < \infty$.

Let n be a nonnegative integer again, and consider

$$a(T) - \sum_{j=0}^{n} a_j\, T^j = \sum_{j=n+1}^{\infty} a_j\, T^j. \tag{28.6}$$

If r is a nonnegative real number, then

$$\biggl\|a(T) - \sum_{j=0}^{n} a_j\, T^j\biggr\|_{q,r} = \Bigl(\sum_{j=n+1}^{\infty} |a_j|^q\, r^{q\,j}\Bigr)^{1/q} \tag{28.7}$$

when $0 < q < \infty$, and

$$\left\| a(T) - \sum_{j=0}^{n} a_j\, T^j \right\|_{\infty,r} = \sup_{j \ge n+1} (|a_j|\, r^j). \tag{28.8}$$

Let us take r to be a positive real number from now on in this section. Remember that $\|f(T)\|_{q,r}$ defines a q-norm on $PS_r^q(k)$ when $q \le q_k$, and a q_k-norm on $PS_r^q(k)$ when $q_k \le q$, as in Section 20. If $a(T) \in PS_r^q(k)$ and $0 < q < \infty$, then the right side of (28.1) converges to $a(T)$ as an infinite series in $PS_r^q(k)$, by (28.7). Conversely, the boundedness of the sequence of partial sums $\sum_{j=0}^{n} a_j\, T^j$ in $PS_r^q(k)$ implies that $a(T) \in PS_r^q(k)$, by (28.5). Of course, boundedness of a sequence is necessary for the sequence to converge, or even to be a Cauchy sequence.

Remember that $PS_{0,r}(k)$ is defined in (20.4). If $a(T) \in PS_{0,r}(k)$, then the right side of (28.1) converges with respect to the $\|f(T)\|_{\infty,r}$ q_k-norm, because of (28.8). Conversely, if the right side of (28.1) converges with respect to $\|f(T)\|_{\infty,r}$, then

$$\|a_j\, T^j\|_{\infty,r} \to \infty \quad \text{as } j \to \infty. \tag{28.9}$$

This means that $a(T) \in PS_{0,r}(k)$, because of (28.2). More precisely, (28.9) is necessary for the sequence of partial sums $\sum_{j=0}^{n} a_j\, T^j$ to be a Cauchy sequence with respect to $\|f(T)\|_{\infty,r}$.

Part III

Complex holomorphic functions

29 Holomorphic functions and power series

In this part, we take k to be the field $\mathbf{C}$ of complex numbers, equipped with the standard absolute value function. If U is a nonempty open set in $\mathbf{C}$, then we let $H(U)$ be the space of holomorphic complex-valued functions on U. This is a subalgebra of the algebra $C(U) = C(U, \mathbf{C})$ of continuous complex-valued functions on U. Let

$$U_r = \{z \in \mathbf{C} : |z| < r\} \tag{29.1}$$

be the open disk in $\mathbf{C}$ centered at 0 with radius r for each positive real number r. We can also allow $r = \infty$ here, in which case U_r is the complex plane $\mathbf{C}$.

Let T be an indeterminate, and let $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ be a formal power series in T with complex coefficients. Also let $0 < r \le \infty$ be given, and suppose that

$$\operatorname{rad}(f(T)) \ge r, \tag{29.2}$$

where $\operatorname{rad}(f(T))$ is the radius of convergence of $f(T)$, as in Section 21. This means that $f(T)$ is in the space $PS_r(\mathbf{C})$ defined in Section 21. If $0 \le r_1 < r$, then it follows that $f(T)$ is in the space $PS^1_{r_1}(\mathbf{C})$ defined in (20.1), as in Section 21. If $z \in U_r$, then we put

$$f(z) = \sum_{j=0}^{\infty} f_j \, z^j, \tag{29.3}$$

where the series on the right converges absolutely by the previous statement. It is well known that (29.3) defines a holomorphic function on U_r. Remember that $PS_r(\mathbf{C})$ is a subalgebra of $\mathbf{C}[[T]]$, as in Section 22. One can check that the mapping that sends $f(T)$ to (29.3) is an algebra homomorphism from $PS_r(\mathbf{C})$ into $H(U_r)$, using the remarks in Section 15, as usual.

Note that $f(T)$ is uniquely determined by the function (29.3), because the coefficients f_j can be obtained from the values of $f(z)$ and its derivatives at 0.

It is well known that every holomorphic function on U_r can be represented by a power series in this way. More precisely, let $0 < r \le \infty$, $f \in H(U_r)$, and $0 < r_1 < r$ be given. Put

$$f_j = \frac{1}{2\,\pi\, i} \int_{|w|=r_1} f(w)\, w^{-j-1}\, dw \tag{29.4}$$

for each nonnegative integer j, where more precisely the integral is an oriented contour integral over the circle of radius r_1 centered at 0. It is well known that f_j does not depend on r_1, because of Cauchy's theorem. Equivalently,

$$f_j = \frac{1}{2\,\pi\, r_1} \int_{|w|=r_1} f(w)\, w^{-j}\, |dw| \tag{29.5}$$

for each $j \ge 0$, where $|dw|$ refers to the element of arclength. This implies that

$$|f_j|\, r_1^j \le \frac{1}{2\,\pi\, r_1} \int_{|w|=r_1} |f(w)|\, |dw| \tag{29.6}$$

for each $j \ge 0$. If $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ is the corresponding formal power series, then (29.6) says that

$$\|f(T)\|_{\infty, r_1} = \sup_{j \ge 0} (|f_j|\, r_1^j) \le \frac{1}{2\,\pi\, r_1} \int_{|w|=r_1} |f(w)|\, |dw|, \tag{29.7}$$

where $\|f(T)\|_{\infty, r_1}$ is as in (19.2). Thus $f(T) \in PS^\infty_{r_1}(\mathbf{C})$, using the notation in (20.1). It follows that (29.2) holds, because the previous statement holds when $0 < r_1 < r$. It is well known that (29.3) holds for every $z \in U_r$ under these conditions, because of the Cauchy integral formula. More precisely, one can choose r_1 so that $|z| < r_1 < r$, and apply the Cauchy integral formula to the circle centered at 0 with radius r_1. The absolute convergence of the series on the right side of (29.3) follows from (29.6).

30 Bounded holomorphic functions

Let U be a nonempty open subset of $\mathbf{C}$, and let $H^\infty(U)$ be the space of bounded holomorphic functions on U. This is a subalgebra of the algebra $C_b(U) =$

$C_b(U, \mathbf{C})$ of bounded continuous complex-valued functions on U, and indeed

$$H^\infty(U) = H(U) \cap C_b(U). \tag{30.1}$$

It is well known that $H^\infty(U)$ is a closed set in $C_b(U)$, with respect to the supremum metric. More precisely, if f is a continuous complex-valued function on U that can be approximated by holomorphic functions on U uniformly on compact subsets of U, then f is holomorphic on U too. In particular, this holds when f can be approximated by holomorphic functions on U uniformly on U.

Let $0 < r \le \infty$ be given, and let U_r be as in (29.1). Also let $f \in C(U)$ and $0 \le r_1 < r$ be given, and consider

$$\sup\{|f(z)| : z \in \mathbf{C},\ |z| = r_1\}. \tag{30.2}$$

This is the same as the supremum seminorm of f associated to the circle in $\mathbf{C}$ centered at 0 with radius r_1, as in (7.3). Of course, the supremum norm of f on U_r,

$$\sup_{z \in U_r} |f(z)| \tag{30.3}$$

is the same as the supremum of (30.2) over $0 \le r_1 < r$. If $f \in H(U_r)$, then it is well known that (30.2) increases monotonically in r_1, by the maximum principle. This implies that (30.2) tends to (30.3) as $r_1 \to r-$ in this case. Remember that bounded holomorphic functions on $\mathbf{C}$ are constant, by Liouville's theorem.

Suppose now that $r < \infty$, and let $f \in H^\infty(U_r)$ be given. Also let T be an indeterminate, and let $f(T) \in \mathbf{C}[[T]]$ be the formal power series corresponding to f, as in the previous section. Using (29.6), we get that

$$|f_j|\, r_1^j \le \sup\{|f(w)| : w \in \mathbf{C},\ |w| = r_1\} \tag{30.4}$$

for every $0 < r_1 < r$ and nonnegative integer j. This also holds when $r_1 = 0$, because $f_0 = f(0)$. It follows that

$$|f_j|\, r_1^j \le \sup_{w \in U_r} |f(w)| \tag{30.5}$$

for every $j \ge 0$ and $0 \le r_1 < r$, since the right side of (30.4) is less than or equal to the right side of (30.5). Hence

$$|f_j|\, r^j \le \sup_{w \in U_r} |f(w)| \tag{30.6}$$

for every $j \ge 0$, by letting r_1 approach r in (30.5). This implies that

$$\|f(T)\|_{\infty, r} \le \sup_{w \in U_r} |f(w)|, \tag{30.7}$$

where $\|f(T)\|_{\infty, r}$ is as in (19.2). Thus $f(T)$ is an element of the space $PS_r^\infty(\mathbf{C})$ defined in (20.1).

If $f \in H^\infty(U_r)$, then it is well known that f has radial and in fact nontangential boundary values almost everywhere with respect to arclength measure. This defines an element of $L^\infty(\partial U_r)$ associated to f, whose L^∞ norm is equal to the supremum norm of f on U_r.

31 Continuity on the closure

Let U be a nonempty bounded open subset of $\mathbf{C}$, so that the closure $\overline{U}$ of U in $\mathbf{C}$ is compact. Consider the space $A(U)$ of continuous complex-valued functions on $\overline{U}$ that are holomorphic on U. This is a subalgebra of the algebra $C(\overline{U})$ of all continuous complex-valued functions on $\overline{U}$. Remember that elements of $C(\overline{U})$ are bounded on $\overline{U}$, because $\overline{U}$ is compact. It is easy to see that $A(U)$ is a closed set in $C(\overline{U})$ with respect to the supremum metric, for essentially the same reasons as in the previous section.

Let ∂U be the boundary of U in $\mathbf{C}$, as usual. If $f \in A(U)$, then

$$\sup_{z \in \partial U} |f(z)| = \sup_{z \in \overline{U}} |f(z)|, \tag{31.1}$$

by the maximum principle. In particular, this implies that f is uniquely determined on $\overline{U}$ by its restriction to ∂U.

Let r be a positive real number, and let U_r be the open unit disk in $\mathbf{C}$ centered at 0 with radius r, as in (29.1). Thus the closure

$$\overline{U_r} = \{z \in \mathbf{C} : |z| \le r\} \tag{31.2}$$

of U_r is the closed disk in $\mathbf{C}$ centered at 0 with radius r. Let T be an indeterminate, and let $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ be a formal power series in T with complex coefficients. Suppose that $f(T)$ is in the space $PS_r^1(\mathbf{C})$ defined in (20.1), so that

$$\|f(T)\|_{1,r} = \sum_{j=0}^{\infty} |f_j|\, r^j < \infty. \tag{31.3}$$

In this case, we can put

$$f(z) = \sum_{j=0}^{\infty} f_j\, z^j \tag{31.4}$$

for every $z \in \overline{U_r}$, where the series on the right converges absolutely, by the comparison test. It is well known that the partial sums of the series on the right side of (31.4) converge uniformly on $\overline{U_r}$ under these conditions, by a well-known criterion of Weierstrass. In particular, this implies that (31.4) defines a continuous function on $\overline{U_r}$. The restriction of (31.4) to U_r is holomorphic, as in Section 29, so that (31.4) defines an element of $A(U_r)$. Note that

$$|f(z)| \le \sum_{j=0}^{\infty} |f_j|\, |z|^j \le \sum_{j=0}^{\infty} |f_j|\, r^j \tag{31.5}$$

for every $z \in \overline{U_r}$, so that

$$\sup_{z \in \overline{U_r}} |f(z)| \le \|f(T)\|_{1,r}. \tag{31.6}$$

Remember that $PS_r^1(\mathbf{C})$ is a subalgebra of $\mathbf{C}[[T]]$, as in Section 22. As before, one can check that the mapping that sends $f(T)$ to (31.4) is an algebra homomorphism from $PS_r^1(\mathbf{C})$ into $A(U_r)$, using the remarks in Section 15.

Now let f be any element of $A(U_r)$, and observe that f is uniformly continuous on $\overline{U_r}$, because $\overline{U_r}$ is compact. If t is a positive real number, then it is easy to see that

$$f(t\,z) \tag{31.7}$$

defines an element of $A(U_{r/t})$ as a function of z. In particular, if $0 < t \le 1$, then the restriction of (31.7) to $z \in \overline{U_r}$ defines an element of $A(U_r)$. Because f is uniformly continuous on $\overline{U_r}$,

$$f(t\,z) \to f(z) \quad \text{as } t \to 1- \tag{31.8}$$

uniformly for $z \in \overline{U_r}$. Remember that $f(z)$ can be given by a convergent power series on U_r, as in Section 29. This leads to a convergent power series expansion for (31.7) on $U_{r/t}$ for each $t > 0$. If $0 < t < 1$, then we get an absolutely convergent power series expansion for (31.7) on $\overline{U_r}$, as in the previous paragraph. Using this expansion, we can approximate (31.7) uniformly by holomorphic polynomials on $\overline{U_r}$ when $0 < t < 1$. This permits us to approximate $f(z)$ uniformly by holomorphic polynomials on $\overline{U_r}$, because of (31.8).

32 Some related integrals

Let r_1 be a positive real number. If l is a nonzero integer, then it is well known that

$$\int_{|w|=r_1} w^{l-1}\,dw = 0, \tag{32.1}$$

where the integral is a contour integral over the circle in $\mathbf{C}$ centered at 0 with radius r_1. This is equivalent to the fact that

$$\int_{|w|=r_1} w^l\,|dw| = 0 \tag{32.2}$$

for every nonzero integer l, where $|dw|$ is the element of arclength, as before.

Let T be an indeterminate, let $f(T) = \sum_{j=0}^\infty f_j\,T^j$ be a formal power series in T with complex coefficients, and suppose that $f(T)$ has radius of convergence greater than or equal to some positive extended real number r. If U_r is as in (29.1), then the corresponding power series (29.3) converges absolutely for every $z \in U_r$, as before. In this situation, the integral expressions (29.4), (29.5) for f_j can be verified directly, using (32.1), (32.2). More precisely, the partial sums of the right side of (29.3) converge uniformly for $z \in \mathbf{C}$ with $|z| = r_1$ when $0 \le r_1 < r$, by the well-known criterion of Weierstrass. This permits one to interchange the order of summation and integration in the integrals in (29.4), (29.5).

If $0 < r_1 < r$, then one can also check that

$$(32.3) \qquad \sum_{j=0}^\infty |f_j|^2 \, r_1^{2\,j} = \frac{1}{2\,\pi\,r_1} \int_{|w|=r_1} |f(w)|^2 \, |dw|$$

under these conditions. Of course, $|f(w)|^2 = f(w)\,\overline{f(w)}$, where $\overline{f(w)}$ is the complex-conjugate of $f(w)$. One can express $f(w)$ as in (29.3), and interchange the order of summation and integration to get (32.3), using (32.2).

Note that

$$(32.4) \qquad \sum_{j=0}^\infty |f_j|^2 \, r_1^{2\,j}$$

increases monotonically in r_1, and in particular (32.4) is less than or equal to

$$(32.5) \qquad \sum_{j=0}^\infty |f_j|^2 \, r^{2\,j}$$

when $0 \le r_1 < r$. In fact, (32.4) tends to (32.5) as $r_1 \to r-$, which is the same as saying that (32.5) is equal to the supremum of (32.4) over $0 \le r_1 < r$. This is basically a version of the monotone convergence theorem for sums. If n is a nonnegative integer, then

$$(32.6) \qquad \sum_{j=0}^n |f_j|^2 \, r_1^{2\,j}$$

converges to

$$(32.7) \qquad \sum_{j=0}^n |f_j|^2 \, r^{2\,j}$$

as $r_1 \to r$. Of course, (32.6) is less than or equal to (32.4) for every $n \ge 0$. This implies that (32.7) is less than or equal to the supremum of (32.4) over $0 \le r_1 < r$ for every $n \ge 0$. It follows that (32.5) is less than or equal to the supremum of (32.4) over $0 \le r_1 < r$, as desired.

Let $r \in \mathbf{R}_+$ and $f \in H^\infty(U)$ be given, and let $f(T) \in \mathbf{C}[[T]]$ be the corresponding formal power series, as in Section 29. Using (32.3), we get that

$$(32.8) \qquad \sum_{j=0}^\infty |f_j|^2 \, r_1^{2\,j} \le \Big(\sup\{|f(w)| : w \in \mathbf{C}, \, |w| = r_1\}\Big)^2$$

when $0 < r_1 < r$, which clearly holds when $r_1 = 0$ as well. Hence

$$(32.9) \qquad \sum_{j=0}^\infty |f_j|^2 \, r_1^{2\,j} \le \Big(\sup_{w \in U_r} |f(w)|\Big)^2$$

when $0 \le r_1 < r$. This implies that

$$(32.10) \qquad \sum_{j=0}^\infty |f_j|^2 \, r^{2\,j} \le \Big(\sup_{w \in U_r} |f(w)|\Big)^2,$$

by the remarks in the previous paragraph. In particular, it follows that $f(T)$ is in the space $PS_r^2(\mathbf{C})$ defined in (20.1) under these conditions.

33 Hardy spaces

Let f be a holomorphic function on the open unit disk

$$U = U_1 = \{z \in \mathbf{C} : |z| < 1\}. \tag{33.1}$$

Consider

$$\Big(\frac{1}{2\pi} \int_{|z|=1} |f(r\,z)|^p \, |dz|\Big)^{1/p} \tag{33.2}$$

for each positive real number p and $0 \le r < 1$, where more precisely the integral is taken over the unit circle in $\mathbf{C}$ with respect to arclength. This is the same as

$$\Big(\frac{1}{2\pi r} \int_{|w|=r} |f(w)|^p \, |dw|\Big)^{1/p} \tag{33.3}$$

when $0 < r < 1$, and otherwise (33.2) is equal to $|f(0)|$ when $r = 0$. Note that (33.2) increases monotonically in p, by a standard argument using Hölder's inequality or Jensen's inequality.

The analogue of (33.2) for $p = \infty$ is

$$\sup\{|f(r\,z)| : z \in \mathbf{C},\ |z| = 1\} = \sup\{|f(w)| : w \in \mathbf{C},\ |w| = r\}, \tag{33.4}$$

as usual. Of course, this also reduces to $|f(0)|$ when $r = 0$. It is easy to see that (33.2) is less than or equal to (33.4) for every $p > 0$ and $0 \le r < 1$.

As in Section 30, (33.4) increases monotonically in r, by the maximum principle. It is well known that (33.2) increases monotonically in r for each $p > 0$ as well. Let T be an indeterminate, and let $f(T) = \sum_{j=0}^\infty f_j\,T^j$ be the formal power series in T corresponding to f, as in Section 29. As in (32.3),

$$\sum_{j=0}^\infty |f_j|^2\,r^{2\,j} = \frac{1}{2\pi} \int_{|w|=1} |f(r\,w)|^2 \, |dw| \tag{33.5}$$

when $0 < r < 1$, and this clearly holds when $r = 0$ too. In particular, this implies that (33.2) increases monotonically in r when $p = 2$.

The *Hardy space* H^p is defined for each positive real number p to be the space of holomorphic functions f on U such that (33.2) is bounded, for $0 \le r < 1$. In this case, $\|f\|_{H^p}$ is defined to be the supremum of (33.2) over $0 \le r < 1$. This is the same as the limit of (33.2) as $r \to 1-$, because (33.2) increases monotonically in r, as before. Similarly, H^∞ is defined to be the space of bounded holomorphic functions on U, as in Section 30. If $f \in H^\infty$, then $\|f\|_{H^\infty}$ is defined to be the supremum norm of f on U, which is the same as the supremum of (33.4) over $0 \le r < 1$, or the limit of (33.4) as $r \to 1-$.

One can check that H^p is a vector space over $\mathbf{C}$ with respect to pointwise addition and scalar multiplication. If $p \ge 1$, then $\|f\|_{H^p}$ defines a norm on H^p, because of the integral version of Minkowski's inequality. If $0 < p \le 1$, then it is easy to see that $\|f\|_{H^p}$ defines a p-norm on H^p, using (1.9). If $0 < p_1 \le p_2 \le \infty$, then

$$H^{p_2} \subseteq H^{p_1}, \tag{33.6}$$

with

$$\|f\|_{H^{p_1}} \le \|f\|_{H^{p_2}} \tag{33.7}$$

for every $f \in H^{p_2}$. This follows from the monotonicity of (33.2) in p, mentioned earlier.

Let f be a holomorphic function on the open unit disk U again, and let $f(T) \in \mathbf{C}[[T]]$ be the corresponding formal power series. Observe that $f \in H^2$ if and only if $f(T)$ is in the space $PS_1^2(\mathbf{C})$ defined in (20.1), in which case

$$\|f(T)\|_{2,1}^2 = \sum_{j=0}^{\infty} |f_j|^2 = \|f\|_{H^2}^2. \tag{33.8}$$

Of course, the first step in (33.8) is basically the definition of $\|f(T)\|_{2,1}$, as in (19.1). Remember that the sum in the middle of (33.8) is the same as the supremum of the the sum on the left side of (33.5), which is the same as saying that the sum on the left side of (33.5) tends to the sum in the middle of (33.8) as $r \to 1-$, as in the previous section. Thus the second step in (33.8) follows from (33.5) by taking $r \to 1-$, or the supremum over $0 \le r < 1$.

Suppose that $0 < p_1, p_2, p_3 \le \infty$ satisfy

$$1/p_3 = 1/p_1 + 1/p_2, \tag{33.9}$$

and that $f \in H^{p_1}$, $g \in H^{p_2}$. Under these conditions, one can use Hölder's inequality to get that their product $f\,g$ is in H^{p_3}, with

$$\|f\,g\|_{H^{p_3}} \le \|f\|_{H^{p_1}}\,\|g\|_{H^{p_2}}. \tag{33.10}$$

Let $f \in H^1$ be given, and let $f(T)$ be the corresponding formal power series in T again. Note that

$$|f_j|\,r^j \le \frac{1}{2\,\pi}\int_{|z|=1} |f(r\,z)|\,|dz| \le \|f\|_{H^1} \tag{33.11}$$

for every $0 \le r < 1$ and nonnegative integer j. The first step follows from (29.6) when $0 < r < 1$, and is easy to see directly when $r = 0$, while the second step just uses the definition of $\|f\|_{H^1}$. This implies that

$$|f_j| \le \|f\|_{H^1} \tag{33.12}$$

for every $j \ge 0$, so that $f(T)$ is in the space $PS_1^\infty(\mathbf{C})$ defined in (20.1), with

$$\|f(T)\|_{\infty,1} = \sup_{j \ge 0} |f_j| \le \|f\|_{H^1}. \tag{33.13}$$

In fact, it is well known that

$$\lim_{j \to \infty} f_j = 0 \tag{33.14}$$

when $f \in H^1$, so that $f(T)$ is in the space $PS_{0,1}(\mathbf{C})$ defined in (20.4).

Part IV
Compositions

34 Composing formal polynomials

Let k be a field, and let T be an indeterminate. Also let

(34.1) $$f(T) = \sum_{j=0}^{n} f_j\, T^j$$

and $a(T)$ be a formal polynomials in T with coefficients in k. The *composition* $(f \circ a)(T)$ of $a(T)$ and $f(T)$ is defined as a formal polynomial in T by

(34.2) $$(f \circ a)(T) = \sum_{j=0}^{n} f_j\, a(T)^j.$$

Here $a(T)^j$ is interpreted as being the constant polynomial corresponding to the multiplicative identity element in k when $j = 0$, as usual.

Of course, the space $k[T]$ of formal polynomials in T with coefficients in k is an algebra over k with a multiplicative identity element, as in Section 16. Thus

(34.3) $$f(a(T)) = \sum_{j=0}^{n} f_j\, a(T)^j$$

is defined as an element of $k[T]$ as in (25.2). Clearly (34.2) is the same as (34.3). As in Section 25,

(34.4) $$f(T) \mapsto f(a(T))$$

defines an algebra homomorphism from $k[T]$ into itself for each $a(T) \in k[T]$.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e, and let $x \in \mathcal{A}$ be given. If $a(T) \in k[T]$, then $a(x)$ can be defined as an element of $\mathcal{A}$ as in Section 25. Similarly, if $f(T) \in k[T]$ is as in (34.1), then

(34.5) $$f(a(x)) = \sum_{j=0}^{n} f_j\, a(x)^j$$

is defined as an element of $\mathcal{A}$ as in (25.2). We also have that

(34.6) $$(f \circ a)(x)$$

is defined as an element of $\mathcal{A}$, as in Section 25, because $(f \circ a)(T) \in k[T]$. It is easy to see that (34.5) is the same as (34.6), using the fact that (25.3) is an algebra homomorphism.

Let $a(T), b(T), f(T) \in k[T]$ be given, so that their compositions are defined as elements of $k[T]$ too. One can check that

(34.7) $$((f \circ a) \circ b)(T) = (f \circ (a \circ b))(T)$$

as elements of $k[T]$. One way to do this is to use the remarks in the preceding paragraph, with $\mathcal{A} = k[T]$ and $x = b(T)$. This will be discussed more precisely in the next section, and one can also use the remarks in the next two paragraphs.

If $\mathcal{A}$ is any set, then the collection of all mappings from $\mathcal{A}$ into itself is a semigroup with respect to ordinary composition, and with the identity mapping on $\mathcal{A}$ as the multiplicative identity element. Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e again. If $f(T) \in k[T]$, then (25.2) defines a natural mapping $f_{\mathcal{A}}$ from $\mathcal{A}$ into itself. This defines a mapping

$$f(T) \mapsto f_{\mathcal{A}} \tag{34.8}$$

from $k[T]$ into the collection of all mappings from $\mathcal{A}$ into itself. This mapping (34.8) sends formal compositions of elements of $k[T]$ to ordinary compositions of mappings on $\mathcal{A}$. This corresponds to the fact that (34.5) is equal to (34.6) for every $f, a \in k[T]$ and $x \in \mathcal{A}$. Note that $f(T) = T$ corresponds to the identity mapping on $\mathcal{A}$ in this way.

In particular, we can take $\mathcal{A} = k[T]$, as before. In this case, we have that (34.8) is injective. This follows from the fact that $f(a(T)) = f(T)$ when $a(T) = T$.

Let $\alpha \in k$ be given, so that

$$a_\alpha(T) = \alpha\, T \tag{34.9}$$

may be considered as an element of $k[T]$. If $f(T) \in k[T]$ is as in (34.1), then

$$f(a_\alpha(T)) = f(\alpha\, T) = \sum_{j=0}^{n} f_j\, \alpha^j\, T^j. \tag{34.10}$$

This is the same as the rescaling $R_\alpha(f(T))$ of $f(T)$ by α, as in (24.1).

35 Polynomials and power series

Let k be a field, let T be an indeterminate, and let $f(T)$ be a formal polynomial in T with coefficients in k as in (34.1) again. If $a(T)$ is a formal power series in T with coefficients in k, then the *composition* $(f \circ a)(T)$ of $a(T)$ and $f(T)$ can be defined as a formal power series in T by (34.2), as before.

Remember that the space $k[[T]]$ of formal power series in T with coefficients in k is an algebra over k with a multiplicative identity element, as in Section 16. If $f(T) \in k[T]$ and $a(T) \in k[[T]]$, then $f(a(T))$ can be defined as an element of $k[[T]]$, as in (25.2) and (34.3). This is the same as $(f \circ a)(T)$ as defined in the preceding paragraph, for the same reasons as in the previous section. We also have that (34.4) defines an algebra homomorphism from $k[T]$ into $k[[T]]$ for each $a(T) \in k[[T]]$, as in Section 25.

Let $a(T), f(T) \in k[T]$ and $b(T) \in k[[T]]$ be given. Thus $(f \circ a)(T)$ is defined as an element of $k[T]$ as in the previous section, and

$$(a \circ b)(T) \quad = \quad a(b(T)), \tag{35.1}$$

$$((f \circ a) \circ b)(T) = (f \circ a)(b(T)), \text{ and} \tag{35.2}$$

$$(f \circ (a \circ b))(T) = f((a \circ b)(T)) = f(a(b(T))) \tag{35.3}$$

are defined as elements of $k[[T]]$. In this situation, we have that

$$f(a(b(T))) = (f \circ a)(b(T)) \tag{35.4}$$

as elements of $k[[T]]$. This follows from the equality between (34.5) and (34.6), with $\mathcal{A} = k[[T]]$ and $x = b(T)$. This shows that (34.7) holds as an equality between formal power series in T under these conditions.

If $f(T) \in k[T]$, then $f(x)$ can be defined as an element of k for every $x \in k$, as in Section 25. If $a(T) \in k[[T]]$, then $a(0)$ can be defined as an element of k, which is the same as the constant term a_0 in $a(T)$. It is easy to see that the constant term in $(f \circ a)(T)$ is equal to $f(a_0)$, which is the same as saying that $(f \circ a)(0)$ is equal to $f(a(0))$.

Let $0 < r < 1$ be given, and let $|\cdot|_r$ be defined on $k[[T]]$ as in Section 18. Let us take k to be equipped with the trivial absolute value function, so that $|\cdot|_r$ may be considered as an ultranorm on $k[[T]]$ as a vector space over k. More precisely, $k[[T]]$ is a commutative algebra over k, and we have seen that $|\cdot|_r$ is multiplicative as an ultranorm on $k[[T]]$. Remember that $k[[T]]$ may be identified with the space $c(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ as a vector space over k. This can also be identified with the Cartesian product of copies of k indexed by nonnegative integers. The topology determined on $k[[T]]$ by the ultrametric associated to $|\cdot|$ corresponds exactly to the product topology on this Cartesian product that uses the discrete topology on k in each factor. In particular, this topology does not depend on $r \in (0, 1)$. As before, $k[T]$ may be identified with the subspace $c_{00}(\mathbf{Z}_+ \cup \{0\}, k)$ of $c(\mathbf{Z}_+ \cup \{0\}, k)$ consisting of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ with finite support. It is easy to see that $k[T]$ is dense in $k[[T]]$ with respect to the topology just mentioned.

36 Composing formal power series

Let k be a field again, let T be an indeterminate, and let $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ be a formal power series in T with coefficients in k. Also let $a(T) \in k[[T]]$ be given, and suppose that the constant term in $a(T)$ is equal to 0. Equivalently, this means that

$$a(T) = c(T)\, T \tag{36.1}$$

for some $c(T) \in k[[T]]$, so that

$$a(T)^j = c(T)^j\, T^j \tag{36.2}$$

for every positive integer j. As usual, if $j = 0$, then we interpret both sides of (36.2) as being the power series whose constant term is the multiplicative identity element 1 in k, and for which the coefficient of T^l is equal to 0 when $l \geq 1$.

If n is a nonnegative integer, then

$$\sum_{j=0}^{n} f_j\, a(T)^j = \sum_{j=0}^{n} f_j\, c(T)^j\, T^j \tag{36.3}$$

is defined as a formal power series in T with coefficients in k. Note that the coefficient of T^l in (36.3) does not depend on n when $l \le n$. This permits us to define the *composition* $(f \circ a)(T)$ of $a(T)$ and $f(T)$ as a formal power series in T by

$$(f \circ a)(T) = \sum_{j=0}^{\infty} f_j\, a(T)^j = \sum_{j=0}^{\infty} f_j\, c(T)^j\, T^j. \tag{36.4}$$

More precisely, the coefficient of T^l in (36.4) is defined to be the same as the coefficient of T^l in (36.3) when $l \le n$. If $f(T) \in k[T]$, then this reduces to the definition of $(f \circ a)(T)$ in the previous section. The constant term in (36.4) is equal to the constant term f_0 in $f(T)$ for every $a(T) \in k[[T]]$. This is the same as saying that $(f \circ a)(0) = f(0)$ as elements of k, which is the same as $f(a(0))$, because $a(0) = 0$ by hypothesis.

Let us consider

$$f(T) \mapsto (f \circ a)(T) \tag{36.5}$$

as a mapping from $k[[T]]$ into itself, where $a(T) \in k[[T]]$ with $a(0) = 0$ is fixed for the moment. Clearly (36.5) is linear on $k[[T]]$ as a vector space over k. If $f(T), g(T) \in k[[T]]$, then one can check that

$$((f\, g) \circ a)(T) = (f \circ a)(T)\, (g \circ a)(T), \tag{36.6}$$

so that (36.5) is an algebra homomorphism from $k[[T]]$ into itself. Indeed, if $f(T), g(T) \in k[T]$, then (36.6) follows from the remarks in the previous paragraph. Otherwise, one can approximate $f(T), g(T) \in k[[T]]$ by elements of $k[T]$, using the remarks in the preceding paragraph.

Let $a(T), b(T) \in k[[T]]$ be given, with $a(0) = b(0) = 0$. Thus $(a \circ b)(T)$ can be defined as an element of $k[[T]]$ as before, with $(a \circ b)(0) = 0$. If $f(T) \in k[[T]]$, then it follows that $(f \circ a)(T)$, $((f \circ a) \circ b)(T)$, and $(f \circ (a \circ b))(T)$ can be defined as elements of $k[[T]]$ as well. One would like to verify that

$$((f \circ a) \circ b)(T) = (f \circ (a \circ b))(T) \tag{36.7}$$

for every $f(T) \in k[[T]]$, as usual. Observe that

$$f(T) \mapsto ((f \circ a) \circ b)(T), \quad f(T) \mapsto (f \circ (a \circ b))(T) \tag{36.8}$$

define algebra homomorphisms from $k[[T]]$ into itself, by the remarks in the previous paragraph. If $f(T)$ is a constant formal power series, so that $f_j = 0$ when $j \ge 1$, then both sides of (36.7) are equal to the same constant power series. Similarly, if $f(T) = T$, then $(f \circ a)(T) = a(T)$, and so on, so that (36.7) holds. This implies that (36.7) when $f(T) \in k[T]$, because the mappings in

(36.8) are algebra homomorphisms. If $f(T) \in k[[T]]$, then one can get the same conclusion by approximating $f(T)$ by elements of $k[T]$.

If $\alpha \in k$, then $a_\alpha(T) = \alpha\, T$ defines an element of $k[T]$ with $a_\alpha(0) = 0$. Thus $(f \circ a_\alpha)(T)$ is defined as an element of $k[[T]]$ when $f(T) \in k[[T]]$, in which case (36.4) reduces to

$$(f \circ a_\alpha)(T) = \sum_{j=0}^{\infty} f_j\, \alpha^j\, T^j. \tag{36.9}$$

This corresponds to (24.1), with slightly different notation, as in Section 34.

37 Polynomials and convergent series

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let T be an indeterminate. Also let $0 < q \le q_k$ and a positive real number r be given. Thus the space $PS_r^q(k)$ can be defined as in (20.1). Remember that $\|f(T)\|_{q,r}$ as defined in (19.1) and (19.2) is a q-norm on $PS_r^q(k)$, because $q \le q_k$, as in Section 20. We have seen that $PS_r^q(k)$ is a subalgebra of $k[[T]]$ when $q \le q_k$, and that $\|f(T)\|_{q,r}$ is submultiplicative on $PS_r^q(k)$, as in Section 22. The $\|\cdot\|_{q,r}$ q-norm of the multiplicative identity element in $k[[T]]$ is equal to 1, as before. If $q_k = \infty$, then the space $PS_{0,r}(k)$ defined in (20.4) is a subalgebra of $k[[T]]$ too, as in Section 22 again.

Let $f(T) = \sum_{j=0}^{n} f_j\, T^j$ be a formal polynomial with coefficients in k. If $a(T) \in k[[T]]$, then

$$f(a(T)) = \sum_{j=0}^{n} f_j\, a(T)^j \tag{37.1}$$

defines an element of $k[[T]]$, as in Section 35. This is the same as the formal composition $(f \circ a)(T)$, as before. If $a(T) \in PS_r^q(k)$, then (37.1) is an element of $PS_r^q(k)$ as well, because $PS_r^q(k)$ is a subalgebra of $k[[T]]$ that contains the multiplicative identity element. Similarly, if $a(T) \in PS_{0,r}(k)$, then (37.1) is an element of $PS_{0,r}(k)$.

Let r_1 be a nonnegative real number, and suppose for the moment that

$$\|a(T)\|_{q,r} \le r_1. \tag{37.2}$$

Under these conditions, we have that

$$\|f(a(T))\|_{q,r} \le \|f(T)\|_{q,r_1}, \tag{37.3}$$

as in (25.6). More precisely, the remarks in Section 25 are being used here with $\mathcal{A} = PS_r^q(k)$, $N = \|\cdot\|_{q,r}$, and $x = a(T)$. The r in the previous situation is taken to be r_1 here. Note that (37.3) also works when $r = 0$.

Now let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e, and let N be a submultiplicative q-norm on $\mathcal{A}$ with respect to $|\cdot|$ on k. Suppose that $\mathcal{A}$ is complete with respect to the q-metric associated to N, and let $x \in \mathcal{A}$ be given, with

$$N(x) \le r. \tag{37.4}$$

If $a(T) \in PS_r^q(k)$ and $q < \infty$, then $a(x)$ can be defined as an element of $\mathcal{A}$, as in Section 25. This permits us to define $f(a(x))$ as an element of $\mathcal{A}$, as in (34.5). Of course, $(f \circ a)(T) \in PS_r^q(k)$ too, as before, so that $(f \circ a)(x)$ can be defined as an element of $\mathcal{A}$ as in Section 25 again. One can check that

$$(f \circ a)(x) = f(a(x)) \tag{37.5}$$

under these conditions, which is corresponds to the equality of (34.5) and (34.6) in this situation. This uses the fact that $a(T) \mapsto a(x)$ is an algebra homomorphism from $PS_r^q(k)$ into $\mathcal{A}$, as in Section 25. There are analogous statements for $q = \infty$ and $a(T) \in PS_{0,r}(k)$, using the corresponding remarks in Section 25.

38 q-Summability

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let T be an indeterminate. Also let $0 < q \le q_k$ be given, with $q < \infty$, and let r be a positive real number. Thus $\|f(T)\|_{q,r}$ and $PS_r^q(k)$ are as defined in (19.1) and (20.1), respectively, $PS_r^q(k)$ is a subalgebra of $k[[T]]$, and $\|f(T)\|_{q,r}$ is a submultiplicative q-norm on $PS_r^q(k)$. Let us suppose throughout this section that k is complete with respect to the q_k-metric associated to $|\cdot|$. This implies that $PS_r^q(k)$ is complete with respect to the q-metric associated to $\|f(T)\|_{q,r}$. More precisely, let w_r be the positive real-valued function on $\mathbf{Z}_+ \cup \{0\}$ associated to r as in (19.4). Remember that $PS_r^q(k)$ corresponds exactly to the space $\ell^q_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ defined in Section 10, as in Section 20. Hence the completeness of $PS_r^q(k)$ is actually the same as the completeness of $\ell^q_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)$, which was mentioned in Section 10.

Let $a(T) \in PS_r^q(k)$ be given, with

$$\|a(T)\|_{q,r} \le r_1 \tag{38.1}$$

for some nonnegative real number r_1. Using r_1, the space $PS_{r_1}^q(k)$ can be defined as in (20.1) as well. Let $f(T) = \sum_{j=0}^\infty f_j\, T^j$ be an element of $PS_{r_1}^q(k)$. Under these conditions, we would like to define

$$f(a(T)) = \sum_{j=0}^\infty f_j\, a(T)^j \tag{38.2}$$

as a power series in T. More precisely, the right side of (38.2) converges q-absolutely as an infinite series with terms in $PS_r^q(k)$, as in Section 25. This uses the remarks in Section 25 with $\mathcal{A} = PS_r^q(k)$, $N = \|\cdot\|_{q,r}$, $x = a(T)$, and the r in the previous situation taken to be r_1 here. It follows that the right side of (38.2) converges in $PS_r^q(k)$, because $PS_r^q(k)$ is complete, as before. Thus (38.2) is defined as an element of $PS_r^q(k)$, and we have that

$$\|f(a(T))\|_{q,r} \le \|f(T)\|_{q,r_1}, \tag{38.3}$$

as in Section 25. We also have that

$$f(T) \mapsto f(a(T)) \tag{38.4}$$

defines an algebra homomorphism from $PS^q_{r_1}(k)$ into $PS^q_r(k)$, as in Section 25 again.

Let us take (38.2) to be the definition of the *composition* $(f \circ a)(T)$ of $a(T)$ and $f(T)$ as an element of $PS^q_r(k)$. Of course, this reduces to the discussion in the previous section when $f(T) \in k[T]$. In the present situation, if we also ask that $a(0) = 0$, then one can check that (38.2) determines the same formal power series in T as in Section 36.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e, and let N be a submultiplicative q-norm on $\mathcal{A}$ with respect to $|\cdot|$ on k. Suppose that $N(e) = 1$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Suppose also that $a(T) \in PS^q_r(k)$ satisfies (38.1) for some $r_1 \geq 0$ again, and that $x \in \mathcal{A}$ satisfies $N(x) \leq r$. Thus $a(x)$ can be defined as an element of $\mathcal{A}$, as in Section 25, with

$$N(a(x)) \leq \|a(T)\|_{q,r} \leq r_1. \tag{38.5}$$

If $f(T) \in PS^q_{r_1}(k)$, then we can also define $f(a(x))$ as an element of $\mathcal{A}$, as in Section 25. More precisely,

$$f(a(x)) = \sum_{j=0}^{\infty} f_j \, a(x)^j, \tag{38.6}$$

where the sum on the right converges q-absolutely with respect to N, and hence converges in $\mathcal{A}$ with respect to N. We can define $(f \circ a)(x)$ as an element of $\mathcal{A}$ too, as in Section 25, because $(f \circ a)(T) \in PS^q_r(k)$.

Under these conditions, one can verify that

$$(f \circ a)(x) = f(a(x)). \tag{38.7}$$

This was already discussed in the previous section when $f \in k[T]$. If $r_1 = 0$, then (38.7) is trivial, because (38.1) implies that $a(T) = 0$, so that $a(x) = 0$, and $f(a(T)) = f_0$. If $r_1 > 0$, then one can approximate $f(T) \in PS^q_{r_1}(k)$ by elements of $k[T]$ with respect to the $\|\cdot\|_{q,r_1}$ q-norm. This leads to approximations of $(f \circ a)(T)$ in $PS^q_r(k)$ with respect to the $\|\cdot\|_{q,r}$ q-norm, because of (38.3). These approximations of $f(T)$ in $PS^q_{r_1}(k)$ lead to approximations of $f(a(x))$ in $\mathcal{A}$ with respect to N, because of (25.6). Similarly, these approximations of $(f \circ a)(T)$ in $PS^q_r(k)$ lead to approximations of $(f \circ a)(x)$ in $\mathcal{A}$ with respect to N.

39 Convergence when $q = \infty$

Let k be a field with an ultrametric absolute value function $|\cdot|$, let T be an indeterminate, and let r be a positive real number. Remember that $\|f(T)\|_{\infty,r}$ and $PS^\infty_r(k)$ are as defined in (19.2) and (20.1), respectively, $PS^\infty_r(k)$ is a subalgebra of $k[[T]]$, and $\|f(T)\|_{\infty,r}$ is a multiplicative ultranorm on $PS^\infty_r(k)$.

As in the previous section, we ask that k be complete with respect to the ultrametric associated to $|\cdot|$, which implies that $PS_r^\infty(k)$ is complete with respect to the ultrametric associated to $\|f(T)\|_{\infty,r}$. Remember also that $PS_{0,r}(k)$ is defined in (20.4), and that $PS_{0,r}(k)$ is a subalgebra of $PS_r^\infty(k)$. In addition, $PS_{0,r}(k)$ is a closed set in $PS_r^\infty(k)$ with respect to the topology determined by the ultrametric associated to $\|f(T)\|_{\infty,r}$, as in Section 20.

Let $a(T) \in PS_r^\infty(k)$ be given, with

$$\|a(T)\|_{\infty,r} \le r_1 \tag{39.1}$$

for some nonnegative real number r_1. If $f(T) = \sum_{j=0}^\infty f_j\, T^j \in PS_{0,r_1}(k)$, then we would like to define

$$f(a(T)) = \sum_{j=0}^\infty f_j\, a(T)^j \tag{39.2}$$

as a power series in T again. As before, the right side of (39.2) converges as an infinite series in $PS_r^\infty(k)$. This uses the remarks in Section 25, with $\mathcal{A} = PS_r^\infty(k)$, $N = \|\cdot\|_{\infty,r}$, $x = a(T)$, $q = \infty$, and the r in the previous situation taken to be r_1 here. We also have that

$$\|f(a(T))\|_{\infty,r} \le \|f(T)\|_{\infty,r_1} \tag{39.3}$$

under these conditions, and that

$$f(T) \mapsto f(a(T)) \tag{39.4}$$

is an algebra homomorphism from $PS_{0,r_1}(k)$ into $PS_r^\infty(k)$, as in Section 25.

As in the previous section, we take (39.2) to be the definition of the *composition* of $a(T)$ and $f(T)$ as an element of $PS_r^\infty(k)$. This reduces to the discussion in Section 37 when $f(T) \in k[T]$, as before. If we also ask that $a(0) = 0$ in the preceding paragraph, then one can verify that (39.2) determines the same formal power series as in Section 36. Note that

$$f(a(T)) \in PS_{0,r}(k) \tag{39.5}$$

when $a(T) \in PS_{0,r}(k)$ satisfies (39.1), because $PS_{0,r}(k)$ is a closed subalgebra of $PS_r^\infty(k)$.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e, and let N be a submultiplicative ultranorm on $\mathcal{A}$ with respect to $|\cdot|$ on k such that $N(e) = 1$. Suppose that $\mathcal{A}$ is complete with respect to the ultrametric associated to N, and that $a(T) \in PS_{0,r}(k)$ satisfies (39.1) for some $r_1 \ge 0$. If $x \in \mathcal{A}$ satisfies $N(x) \le r$, then $a(x)$ can be defined as an element of $\mathcal{A}$ as in Section 25, with

$$N(x) \le \|a(T)\|_{\infty,r} \le r_1. \tag{39.6}$$

Similarly, if $f(T) \in PS_{0,r_1}(k)$, then $f(a(x))$ can be defined as an element of $\mathcal{A}$ too, as in Section 25. We can define $(f \circ a)(x)$ as an element of $\mathcal{A}$ as well, as in Section 25, because $(f \circ a)(T) \in PS_{0,r}(k)$, as in (39.5).

As in the previous section, one can check that

$$(f \circ a)(x) = f(a(x)) \tag{39.7}$$

under these conditions. This follows from the remarks in Section 37 when $f(T)$ is a formal polynomial in T, as before. If $r_1 = 0$, then $a(T) = 0$, and (39.7) is trivial. If $r_1 > 0$, then one can approximate $f(T) \in PS_{0,r_1}(k)$ by formal polynomials with respect to the $\|\cdot\|_{\infty,r_1}$ ultranorm, which leads to approximations of $(f \circ a)(T)$ in $PS^{0,r}(k)$ with respect to the $\|\cdot\|_{\infty,r}$ ultranorm, by (39.3). These approximations lead to corresponding approximations of $f(a(x))$ and $(f \circ a)(x)$ in $\mathcal{A}$ with respect to N.

40 Associativity

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let T be an indeterminate. Suppose that k is complete with respect to the q_k-metric associated to $|\cdot|$, and let $0 < q \leq q_k$ and a positive real number r be given. Also let $a(T) \in PS^q_r(k)$ be given, with

$$\|a(T)\|_{q,r} \leq r_1 \tag{40.1}$$

for some nonnegative real number r_1. If $f(T) \in PS^q_{r_1}(k)$ and $q < \infty$, then $(f \circ a)(T)$ can be defined as an element of $PS^q_r(k)$, as in Section 38. Similarly, if $f(T) \in PS_{0,r_1}(k)$ and $q = \infty$, then $(f \circ a)(T)$ can be defined as an element of $PS^\infty_r(k)$, as in the previous section.

Let a positive real number r_0 be given, as well as $b(T) \in PS^q_{r_0}(k)$, with

$$\|b(T)\|_{q,r_0} \leq r. \tag{40.2}$$

If $q < \infty$, then $(a \circ b)(T)$ can be defined as an element of $PS^q_{r_0}(k)$, as in Section 38. If $q = \infty$ and $a(T) \in PS_{0,r}(k)$, then $(a \circ b)(T)$ can be defined as an element of $PS^\infty_{r_0}(k)$, as in the previous section. In both cases, we have that

$$\|(a \circ b)(T)\|_{q,r_0} \leq \|a(T)\|_{q,r} \leq r_1. \tag{40.3}$$

If $f(T) \in PS^q_{r_1}(k)$ and $q < \infty$, then $(f \circ a)(T) \in PS^q_r(k)$, and so

$$((f \circ a) \circ b)(T) \tag{40.4}$$

can be defined as an element of $PS^q_{r_0}(k)$, as in Section 38 . If $q = \infty$, $f(T)$ is an element of $PS_{0,r_1}(k)$, and $a(T) \in PS_{0,r}(k)$, then $(f \circ a)(T) \in PS_{0,r}(k)$ too, as in the previous section. This means that (40.4) can be defined as an element of $PS^\infty_{r_0}(k)$, as before. Note that

$$\|((f \circ a) \circ b)(T)\|_{q,r_0} \leq \|(f \circ a)(T)\|_{q,r} \leq \|f(T)\|_{q,r_1} \tag{40.5}$$

in both situations.

If $f(T) \in PS^q_{r_1}(k)$ and $q < \infty$, then

$$(f \circ (a \circ b))(T) \tag{40.6}$$

can be defined as an element of $PS^q_{r_0}(k)$, as in Section 38, and using (40.3). If $q = \infty$, $a(T) \in PS_{0,r}(k)$, and $f(T) \in PS_{0,r_1}(k)$, then (40.6) can be defined as an element of $PS^\infty_{r_0}(k)$, as in the previous section, and using (40.3) again. In both situations,

$$\|(f \circ (a \circ b))(T)\|_{q,r_0} \le \|f(T)\|_{q,r_1}, \tag{40.7}$$

as before.

If $q < \infty$, then the mappings from $f(T)$ to (40.4) and (40.6) define algebra homomorphisms from $PS^q_{r_1}(k)$ into $PS^q_{r_0}(k)$, because of the analogous statement for (38.4). Similarly, if $q = \infty$, and $a(T) \in PS_{0,r}(k)$, then the mappings from $f(T)$ to (40.4) and (40.6) define algebra homomorphisms from $PS_{0,r_1}(k)$ into $PS^\infty_{r_0}(k)$, because of the analogous statement for (39.4), and using (39.5).

Of course, we would like to say that (40.4) is equal to (40.6), under suitable conditions. Suppose first that $q < \infty$. If $f(T)$ is a constant power series, so that $f_j = 0$ when $j \ge 1$, or if $f(T) = T$, then the equality of (40.4) and (40.6) can be verified directly. This implies that (40.4) and (40.6) are the same when $f(T) \in k[T]$, because the mappings from $f(T)$ to (40.4) and (40.6) are algebra homomorphisms, as in the preceding paragraph. If $f(T)$ is any element of $PS^q_{r_1}(k)$, then one can check that (40.4) is equal to (40.6), by approximating $f(T)$ by formal polynomials.

Suppose now that $q = \infty$, and that $a(T) \in PS_{0,r}(k)$. If $f(T) \in k[T]$, then (40.4) is equal to (40.6), for the same reasons as before. If $f(T) \in PS_{0,r_1}(k)$, then one can get the same conclusion by approximating $f(T)$ by formal polynomials again.

41 Some variants for $q = \infty$

Let k be a field with an ultrametric absolute value function $|\cdot|$, let T be an indeterminate, and let r, r_2 be positive real numbers. As before, we ask that k be complete with respect to the ultrametric associated to $|\cdot|$, which implies that the space $PS^\infty_r(k)$ defined in (20.1) is complete with respect to the ultrametric associated to the ultranorm $\|\cdot\|_{\infty,r}$ defined in (19.2). Let $a(T) \in PS^\infty_r(k)$ and $f(T) = \sum_{j=0}^\infty f_j\, T^j \in PS^\infty_{r_2}(k)$ be given, and suppose for the moment that

$$\|a(T)\|_{\infty,r} < r_2. \tag{41.1}$$

As usual, we would like to define

$$f(a(T)) = \sum_{j=0}^\infty f_j\, a(T)^j \tag{41.2}$$

as a power series in T, where the right side of (41.2) converges as an infinite series in $PS^\infty_r(k)$. This corresponds to the second convergence condition described in

Section 25 with $q = \infty$, where $\mathcal{A} = PS_r^\infty(k)$, $N = \|\cdot\|_{\infty,r}$, $x = a(T)$, and the r in the previous situation is taken to be r_2 here. This amounts to applying the remarks in Section 39 to a nonnegative real number r_1 such that

$$\|a(T)\|_{\infty,r} \le r_1 < r_2. \tag{41.3}$$

Remember that $PS_{r_2}^\infty(k) \subseteq PS_{0,r_1}(k)$ when $r_1 < r_2$, as in Section 21, so that $f(T) \in PS_{0,r_1}(k)$. We also get that

$$\|f(a(T))\|_{\infty,r} \le \|f(T)\|_{\infty,r_1} \le \|f(T)\|_{\infty,r_2}, \tag{41.4}$$

using (39.3) in the first step, and (19.6) in the second step. As before,

$$f(T) \mapsto f(a(T)) \tag{41.5}$$

is an algebra homomorphism from $PS_{r_2}^\infty(k)$ into $PS_r^\infty(k)$ under these conditions, and $f(a(T)) \in PS_{0,r}(k)$ when $a(T) \in PS_{0,r}(k)$.

Let $a(T) = \sum_{j=0}^\infty a_j\, T^j \in PS_r^\infty(k)$ be given again, and observe that

$$\|a(T)\|_{\infty,r_0} = \max\Big(|a_0|, \sup_{j\ge 1}(|a_j|\, r_0^j)\Big) \tag{41.6}$$

for every nonnegative real number r_0, by the definition (19.2) of $\|a(T)\|_{\infty,r_0}$. If $0 \le r_0 \le r$, then

$$|a_j|\, r_0^j \le (r_0/r)\, |a_j|\, r^j \le (r_0/r)\, \|a(T)\|_{\infty,r} \tag{41.7}$$

for every $j \ge 1$. It follows that

$$\|a(T)\|_{\infty,r_0} \le \max(|a_0|, (r_0/r)\, \|a(T)\|_{\infty,r}) \tag{41.8}$$

when $0 \le r_0 \le r$.

Suppose now that

$$\|a(T)\|_{\infty,r} \le r_2, \tag{41.9}$$

instead of (41.1), and that

$$|a_0| < r_2. \tag{41.10}$$

If $0 \le r_0 < r$, then we get that

$$\|a(T)\|_{\infty,r_0} < r_2, \tag{41.11}$$

because of (41.8). Let $f(T) \in PS_{r_2}^\infty(k)$ be given again, so that $f(a(T))$ can be defined as an element of $PS_{r_0}^\infty(k)$ when $0 \le r_0 < r$, as before. Of course, the convergence of the series on the right side of (41.2) in $PS_{r_0}^\infty(k)$ is a stronger condition as r_0 increases. These convergence conditions determine the same formal power series in T when $r_0 > 0$. As in (41.4), we have that

$$\|f(a(T))\|_{\infty,r_0} \le \|f(T)\|_{\infty,r_2} \tag{41.12}$$

for every $0 \le r_0 < r$. Using this, one can check that $f(a(T)) \in PS_r^\infty(k)$, with

$$\|f(a(T))\|_{\infty,r} \le \|f(T)\|_{\infty,r_2}. \tag{41.13}$$

42 Some remarks about $A(U_r)$

In this section, we take $k = \mathbf{C}$, with the standard absolute value function. Let r_1 be a positive real number, and let

$$U_{r_1} = \{z \in \mathbf{C} : |z| < r_1\} \tag{42.1}$$

be the open disk in $\mathbf{C}$ centered at 0 with radius r_1, as before. Of course, the closure of U_{r_1} in $\mathbf{C}$ is the closed disk

$$\overline{U_{r_1}} = \{z \in \mathbf{C} : |z| \le r_1\} \tag{42.2}$$

centered at 0 with radius r_1. Remember that $A(U_{r_1})$ denotes the space of continuous complex-valued functions on $\overline{U_{r_1}}$ that are holomorphic on U_{r_1}, as in Section 31. The elements of $A(U_{r_1})$ are bounded on $\overline{U_{r_1}}$, because $\overline{U_{r_1}}$ is compact.

Let X be a nonempty topological space, and let $C_b(X) = C_b(X, \mathbf{C})$ be the algebra of bounded continuous complex-valued functions on X, as in Section 7. Also let $\mathcal{A}$ be a subalgebra of $C_b(X)$ that contains the constant functions on X, and suppose that $\mathcal{A}$ is a closed set in $C_b(X)$ with respect to the topology determined by the supremum metric. Let $a \in \mathcal{A}$ be given, and suppose that

$$|a(x)| \le r_1 \tag{42.3}$$

for every $x \in X$. Let $f \in A(U_{r_1})$ be given too, so that the composition $f \circ a$ is defined as a complex-valued function on X. Note that $f \circ a$ is continuous on X, because the composition of continuous functions is continuous. Similarly, $f \circ a$ is bounded on X, because f is bounded on $\overline{U_{r_1}}$, with

$$\sup_{x \in X} |(f \circ a)(x)| = \sup_{x \in X} |f(a(x))| \le \sup_{z \in \overline{U_{r_1}}} |f(z)|. \tag{42.4}$$

Let us check that

$$f \circ a \in \mathcal{A} \tag{42.5}$$

under these conditions. If $f(z)$ is the restriction to $\overline{U_{r_1}}$ of a polynomial in z, then (42.5) follows from the hypothesis that $\mathcal{A}$ be a subalgebra of $c_b(X)$ that contains the constant functions. If $f(z)$ is any element of $A(U_{r_1})$, then $f(z)$ can be approximated uniformly by polynomials in z on $\overline{U_{r_1}}$, as in Section 31. This implies that $f \circ a$ can be approximated by elements of $\mathcal{A}$ uniformly on X, by the previous case. It follows that (42.5) holds in this case as well, because $\mathcal{A}$ is a closed set in $C_b(X)$ with respect to the supremum metric.

Let U be a nonempty open subset of $\mathbf{C}$, equipped with the topology induced by the standard topology on $\mathbf{C}$. Remember that $H^\infty(U)$ denotes the space of bounded holomorphic complex-valued functions on U, as in Section 30. This is a subalgebra of the algebra $C_b(U)$ of bounded continuous complex-valued functions on U, and $H^\infty(U)$ contains the constant functions. It is well known that $H^\infty(U)$ is a closed set in $C_b(U)$ with respect to the topology determined by the supremum metric, as mentioned in Section 30. Thus we can take $X = U$

and $\mathcal{A} = H^\infty(U)$ in the preceding paragraph. Let $a \in H^\infty(U)$ be given, and suppose that

$$|a(w)| \le r_1 \tag{42.6}$$

for every $w \in U$. If $f \in A(U_{r_1})$, then

$$f \circ a \in H^\infty(U), \tag{42.7}$$

as in (42.5). Alternatively, it suffices to show that $f \circ a$ is holomorphic on U in this situation. If

$$|a(w)| < r_1 \tag{42.8}$$

for every $w \in U$, then the holomorphicity of $f \circ a$ on U follows from the holomorphicity of a on U, the holomorphicity of f on U_{r_1}, and the fact that compositions of holomorphic functions are holomorphic. Otherwise, suppose for the moment that U is connected. If a satisfies (42.6) and $|a(w)| = r_1$ for some $w \in U$, then it is well known that a is constant on U. This implies that $f \circ a$ is constant on U, so that $f \circ a$ is holomorphic on U in particular. If U is not connected, then one can verify that $f \circ a$ is holomorphic on each connected component of U, using the same type of argument.

Remember that $A(U_{r_1})$ is a subalgebra of the algebra $C(\overline{U_{r_1}})$ of continuous complex-valued functions on $\overline{U_{r_1}}$, that $A(\overline{U_{r_1}})$ contains the constant functions on $\overline{U_{r_1}}$, and that $A(\overline{U_{r_1}})$ is a closed set in $C(\overline{U_{r_1}})$ with respect to the topology determined by the supremum metric, as in Section 31. Thus $\mathcal{A} = A(\overline{U_{r_1}})$ satisfies the conditions mentioned earlier, with $X = \overline{U_{r_1}}$ equipped with the topology induced by the standard topology on $\mathbf{C}$. Put $a(z) = z$ for each $z \in \overline{U_{r_1}}$, so that $a \in A(\overline{U_{r_1}})$, and $|a(z)| \le r_1$ for every $z \in \overline{U_{r_1}}$. If $f \in C(\overline{U_{r_1}})$, then $f \circ a$ is defined as a continuous complex-valued function on $\overline{U_{r_1}}$, and is equal to f. In this case, (42.5) says that $f \in A(\overline{U_{r_1}})$, so that this condition is necessary for the earlier remarks to hold.

Part V
Invertibility

43 Invertible elements of algebras

Let k be a field, and let $\mathcal{A}$ be an algebra over k. Suppose that $\mathcal{A}$ has a nonzero multiplicative identity element e. As usual, an element a of $\mathcal{A}$ is said to be *invertible* if there is a $b \in \mathcal{A}$ such that

$$a\,b = b\,a = e. \tag{43.1}$$

It is well known and easy to see that b is unique when it exists, in which case it is denoted a^{-1}. If $a \in \mathcal{A}$ is invertible, then a^{-1} is invertible too, with $(a^{-1})^{-1} = a$. If $x, y \in \mathcal{A}$ are invertible, then $x\,y$ is invertible in $\mathcal{A}$ as well, with

$$(x\,y)^{-1} = y^{-1}\,x^{-1}. \tag{43.2}$$

Thus the invertbile elements in $\mathcal{A}$ form a group.

Let $b \in \mathcal{A}$ be given. If $a \in \mathcal{A}$ satisfies

$$a\,b = e, \tag{43.3}$$

then a is said to be a *left inverse* of b in $\mathcal{A}$. Similarly, if $c \in \mathcal{A}$ satisfies

$$b\,c = e, \tag{43.4}$$

then c is said to be a *right inverse* of b in $\mathcal{A}$. If b has a left inverse $a \in \mathcal{A}$ and a right inverse $c \in \mathcal{A}$, then $a = c$, and b is invertible in $\mathcal{A}$.

Let $x, y \in \mathcal{A}$ be given. If $x\,y$ is invertible in $\mathcal{A}$, then

$$x\,(y\,(x\,y)^{-1}) = (x\,y)\,(x\,y)^{-1} = e, \quad ((x\,y)^{-1}\,x)\,y = (x\,y)^{-1}\,(x\,y) = e. \tag{43.5}$$

In particular, this means that x has a right inverse in $\mathcal{A}$, and that y has a left inverse in $\mathcal{A}$. Similarly, if $y\,x$ is invertible in $\mathcal{A}$, then

$$y\,(x\,(y\,x)^{-1}) = (y\,x)\,(y\,x)^{-1} = e, \quad ((y\,x)^{-1}\,y)\,x = (y\,x)^{-1}\,(y\,x) = e, \tag{43.6}$$

which means that y has a right inverse in $\mathcal{A}$, and that x has a left inverse in $\mathcal{A}$. If $x\,y$ and $y\,x$ are both invertible in $\mathcal{A}$, then it follows that x and y are both invertible in $\mathcal{A}$.

Let $w, z \in \mathcal{A}$ be given, and suppose that w and z commute, so that $w\,z = z\,w$. If w is invertible in $\mathcal{A}$, then w^{-1} commutes with z too.

Let $x \in \mathcal{A}$ and a nonnegative integer be given. Using a standard argument, we get that

$$(e - x)\sum_{j=0}^{n} x^j = \Big(\sum_{j=0}^{n} x^j\Big)(e - x) = e - x^{n+1}, \tag{43.7}$$

where x^j is interpreted as being equal to e when $j = 0$, as usual. If $e - x^{n+1}$ is invertible in $\mathcal{A}$, then it follows that $e - x$ is invertible in $\mathcal{A}$, as before.

44 Invertibility and q-seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a nonzero multiplicative identity element e. Also let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. If $x \in \mathcal{A}$, then

$$N(x) = N(e\,x) \le N(e)\,N(x). \tag{44.1}$$

Thus $N(e) \ge 1$ when $N(x) > 0$ for some $x \in \mathcal{A}$. If $x \in \mathcal{A}$ is invertible in $\mathcal{A}$, then

$$N(e) = N(x\,x^{-1}) \le N(x)\,N(x^{-1}). \tag{44.2}$$

Suppose that $x, y \in \mathcal{A}$ are invertible in $\mathcal{A}$, and observe that

$$x^{-1} - y^{-1} = x^{-1}\,y\,y^{-1} - x^{-1}\,x\,y^{-1} = x^{-1}\,(y - x)\,y^{-1}. \tag{44.3}$$

It follows that

$$N(x^{-1} - y^{-1}) \le N(x^{-1})\,N(x-y)\,N(y^{-1}). \tag{44.4}$$

Using (44.3), we also get that

$$y^{-1} = x^{-1} + x^{-1}\,(y-x)\,y^{-1}. \tag{44.5}$$

Suppose for the moment that $q < \infty$, so that

$$N(y^{-1})^q \le N(x^{-1})^q + N(x^{-1}\,(x-y)\,y^{-1})^q, \tag{44.6}$$

by (44.5) and the q-seminorm version of the triangle inequality. This implies that

$$N(y^{-1})^q \le N(x^{-1})^q + N(x^{-1})^q\,N(x-y)^q\,N(y^{-1})^q, \tag{44.7}$$

and hence

$$\big(1 - N(x^{-1})^q\,N(x-y)^q\big)\,N(y^{-1})^q \le N(x^{-1})^q. \tag{44.8}$$

If

$$N(x^{-1})\,N(x-y) < 1, \tag{44.9}$$

then it follows that

$$N(y^{-1})^q \le N(x^{-1})^q\,\big(1 - N(x^{-1})^q\,N(x-y)^q\big)^{-1}. \tag{44.10}$$

Equivalently,

$$N(y^{-1}) \le N(x^{-1})\,\big(1 - N(x^{-1})^q\,N(x-y)^q\big)^{-1/q} \tag{44.11}$$

when (44.9) holds. Combining this with (44.4), we get that

$$N(x^{-1} - y^{-1}) \le N(x^{-1})^2\,N(x-y)\,\big(1 - N(x^{-1})^q\,N(x-y)^q\big)^{-1/q} \tag{44.12}$$

when (44.9) holds.

If $q = \infty$, then

$$N(y^{-1}) \le \max\big(N(x^{-1}), N(x^{-1}\,(x-y)\,y^{-1})\big), \tag{44.13}$$

by (44.5) and the semi-ultranorm version of the triangle inequality. It follows that

$$N(y^{-1}) \le \max\big(N(x^{-1}), N(x^{-1})\,N(x-y)\,N(y^{-1})\big), \tag{44.14}$$

because N is submultiplicative on $\mathcal{A}$. If (44.9) holds, then (44.14) implies that

$$N(y^{-1}) \le N(x^{-1}). \tag{44.15}$$

More precisely, (44.15) is trivial when $N(y^{-1}) = 0$. Otherwise, if $N(y^{-1}) > 0$, then (44.9) implies that

$$N(x^{-1})\,N(x-y)\,N(y^{-1}) < N(y^{-1}). \tag{44.16}$$

This and (44.14) imply (44.15), as desired. Using (44.4) and (44.15), we get that

$$N(x^{-1} - y^{-1}) \le N(x^{-1})^2 \, N(x - y) \tag{44.17}$$

when $q = \infty$ and (44.9) holds.

Consider $y \mapsto y^{-1}$ as a mapping from the group of invertible elements of $\mathcal{A}$ into itself. This mapping is continuous with respect to the topology induced on the group of invertible elements of $\mathcal{A}$ by the topology determined on $\mathcal{A}$ by the q-semimetric associated to N. More precisely, the continuity of this mapping at a given invertible element x of $\mathcal{A}$ follows from (44.12) when $q < \infty$, and from (44.17) when $q = \infty$. In both cases, (44.9) holds when y is sufficiently close to x with respect to N. This permits us to use (44.12) and (44.17) to get that y^{-1} is close to x^{-1} with respect to N when y is close to x, as desired.

45 The inverse of $e - x$

Let k be a field, and let $\mathcal{A}$ be an algebra over k with a nonzero multiplicative identity element e. Suppose that $x \in \mathcal{A}$ has the property that $e - x$ is invertible in $\mathcal{A}$. Using (43.7), we get that

$$\sum_{j=0}^{n} x^j = (e - x)^{-1} \, (e - x^{n+1}) = (e - x^{n+1}) \, (e - x)^{-1} \tag{45.1}$$

for every nonnegative integer n. Equivalently,

$$(e - x)^{-1} - \sum_{j=0}^{n} x^j = (e - x)^{-1} \, x^{n+1} = x^{n+1} \, (e - x)^{-1} \tag{45.2}$$

for each $n \ge 0$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. Observe that

$$N(x^j) \le N(x)^j \tag{45.3}$$

for every $x \in \mathcal{A}$ and positive integer j. If $N(e) = 1$, then (45.3) also holds when $j = 0$, with x^j interpreted as being e and $N(x)^j$ interpreted as being 1, as usual.

If $x \in \mathcal{A}$ satisfies $N(x) < 1$, then (45.3) implies that

$$\lim_{j \to \infty} N(x^j) = 0. \tag{45.4}$$

If $e - x$ is invertible in $\mathcal{A}$, then (45.2) and (45.4) imply that

$$\lim_{n \to \infty} N\Big((e - x)^{-1} - \sum_{j=0}^{n} x^j\Big) = 0. \tag{45.5}$$

Of course, this is the same as saying that

$$\lim_{n \to \infty} N\Big((e - x)^{-1} - e - \sum_{j=1}^n x^j\Big) = 0, \tag{45.6}$$

because x^j is interpreted as being e when $j = 0$.

If $q < \infty$, then

$$N\Big(\sum_{j=1}^n x^j\Big)^q \le \sum_{j=1}^n N(x^j)^q \le \sum_{j=1}^n N(x)^{q\,j} \tag{45.7}$$

for every $x \in \mathcal{A}$ and positive integer n, using the q-seminorm version of the triangle inequality in the first step, and (45.3) in the second step. This implies that

$$N\Big(\sum_{j=1}^n x^j\Big)^q \le \sum_{j=1}^\infty N(x)^{q\,j} = N(x)^q\,(1 - N(x)^q)^{-1} \tag{45.8}$$

for every $n \ge 1$ when $N(x) < 1$, so that

$$N\Big(\sum_{j=1}^n x^j\Big) \le N(x)\,(1 - N(x)^q)^{-1/q} \tag{45.9}$$

for every $n \ge 1$. Similarly, if $q = \infty$, then

$$N\Big(\sum_{j=1}^n x^j\Big) \le \max_{1 \le j \le n} N(x^j) \le \max_{1 \le j \le n} N(x)^j \tag{45.10}$$

for every $x \in \mathcal{A}$ and positive integer n, using the semi-ultranorm version of the triangle inequality in the first step, and (45.3) in the second step. Hence

$$N\Big(\sum_{j=1}^n x^j\Big) \le N(x) \tag{45.11}$$

for every $n \ge 1$ when $N(x) \le 1$.

Suppose that $x \in \mathcal{A}$ satisfies $N(x) < 1$, and that $e - x$ is invertible in $\mathcal{A}$. If $q < \infty$, then we get that

$$N((e - x)^{-1} - e) \le N(x)\,(1 - N(x)^q)^{-1/q}, \tag{45.12}$$

using (45.6) and (45.9). Similarly, if $q = \infty$, then

$$N((e - x)^{-1} - e) \le N(x), \tag{45.13}$$

by (45.6) and (45.11).

46 Inverting $e - x$

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a nonzero multiplicative identity element e. Also let N be a submultiplicative q-norm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. Suppose that $x \in \mathcal{A}$ has the property that $\sum_{j=0}^{\infty} x^j$ converges as an infinite series in $\mathcal{A}$ with respect to N. In particular, this implies that

$$\lim_{j \to \infty} N(x^j) = 0, \tag{46.1}$$

as in (9.4). It follows that

$$(e - x) \sum_{j=0}^{\infty} x^j = \Big(\sum_{j=0}^{\infty} x^j\Big)(e - x) = e, \tag{46.2}$$

by taking the limit as $n \to \infty$ in (43.7). Thus $e - x$ has a multiplicative inverse in $\mathcal{A}$ under these conditions, with

$$(e - x)^{-1} = \sum_{j=0}^{\infty} x^j. \tag{46.3}$$

Note that (45.5) holds by construction in this situation.

Let us suppose from now on in this section that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Let $x \in \mathcal{A}$ be given, and suppose for the moment that $N(x) < 1$. If $q < \infty$ and $N(e) = 1$, then we can use (45.3) to get that

$$\sum_{j=0}^{\infty} N(x^j)^q \le \sum_{j=0}^{\infty} N(x)^{q\,j} = (1 - N(x)^q)^{-1}. \tag{46.4}$$

This means that $\sum_{j=0}^{\infty} x^j$ converges q-absolutely with respect to N, and hence $\sum_{j=0}^{\infty} x^j$ converges in $\mathcal{A}$ with respect to N, as in Section 9. Of course, the hypothesis that $N(e) = 1$ is only used to simplify (46.4), and is not needed to get that $\sum_{j=0}^{\infty} x^j$ converges q-absolutely. Similarly, (46.1) holds when $N(x) < 1$, because of (45.3). If $q = \infty$, then (46.1) implies that $\sum_{j=0}^{\infty} x^j$ converges in $\mathcal{A}$ with respect to N, as in Section 9 again.

Suppose now that $x \in \mathcal{A}$ satisfies

$$N(x^l) < 1 \tag{46.5}$$

for some positive integer l. This implies that $e - x^l$ has a multiplicative inverse in $\mathcal{A}$, as in the preceding paragraph. One can use this to get that $e - x$ has a multiplicative inverse in $\mathcal{A}$, as in Section 43.

Of course, (46.5) holds if and only if

$$N(x^l)^{1/l} < 1. \tag{46.6}$$

Thus there is an $l \in \mathbf{Z}_+$ such that (46.5) holds if and only if

$$\inf_{l \geq 1} N(x^l)^{1/l} < 1. \tag{46.7}$$

In this case, the convergence of $\sum_{j=1}^{\infty} x^j$ in $\mathcal{A}$ with respect to N can be obtained as in Section 26.

Let $y, z \in \mathcal{A}$ be given, and suppose that y is invertible in $\mathcal{A}$. Thus z can be expressed as

$$z = y - (y - z) = y\,\big(e - y^{-1}\,(y - z)\big). \tag{46.8}$$

If

$$N(y^{-1})\,N(y - z) < 1, \tag{46.9}$$

then $N(y^{-1}\,(y - z)) < 1$, so that $e - y^{-1}\,(y - z)$ has a multiplicative inverse in $\mathcal{A}$, as before. This implies that z has a multiplicative inverse in $\mathcal{A}$ too, with

$$z^{-1} = \big(e - y^{-1}\,(y - z)\big)^{-1}\,y^{-1}. \tag{46.10}$$

In particular, the group of invertible elements of $\mathcal{A}$ is an open set in $\mathcal{A}$ with respect to the q-metric associated to N under these conditions.

47 Some examples

Let k be a field, and let T be an indeterminate. Remember that the collection $k[[T]]$ of formal power series in T with coefficients in k is a commutative algebra over k, as in Section 16. As before, elements of k can be identified with formal power series whose 0th coefficient is the given element of k, and whose other coefficients are equal to 0. In particular, the multiplicative identity element 1 in k corresponds to the multiplicative identity element in $k[[T]]$ in this way. If $a \in k$, then $\sum_{j=0}^{\infty} a^j\,T^j$ defines an element of $k[[T]]$, where a^j is interpreted as being equal to 1 when $j = 0$, as usual. A standard argument shows that

$$(1 - a\,T)\sum_{j=0}^{\infty} a^j\,T^j = 1, \tag{47.1}$$

which may be considered as an instance of (16.7). Thus $1 - a\,T$ is invertible in $k[[T]]$, with

$$(1 - a\,T)^{-1} = \sum_{j=0}^{\infty} a^j\,T^j, \tag{47.2}$$

as before.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let r be a nonnegative real number. If $f(T) \in k[[T]]$, then $\|f(T)\|_{q,r}$ is defined as in (19.1) when q is a positive real number, and as in (19.2) when $q = \infty$. Let $a \in k$ be given, and observe that

$$\|a\,T\|_{q,r} = |a|\,r \tag{47.3}$$

for every $q > 0$. Suppose for the moment that $0 < q < \infty$, so that

$$\Big\|\sum_{j=0}^\infty a^j\,T^j\Big\|_{q,r} = \Big(\sum_{j=0}^\infty |a|^{q\,j}\,r^{q\,j}\Big)^{1/q}. \tag{47.4}$$

It follows that

$$\begin{aligned} \Big\|\sum_{j=0}^\infty a^j\,T^j\Big\|_{q,r} &= (1 - |a|^q\,r^q)^{1/q} \quad \text{when } |a|\,r < 1, \\ &= +\infty \qquad\qquad\quad \text{when } |a|\,r \ge 1. \end{aligned} \tag{47.5}$$

Similarly,

$$\begin{aligned} \Big\|\sum_{j=0}^\infty a^j\,T^j\Big\|_{\infty,r} = \sup_{j\ge 0}(|a|^{q\,j}\,r^{q\,r}) &= 1 \qquad \text{when } |a|\,r \le 1 \\ &= +\infty \quad \text{when } |a|\,r > 1. \end{aligned} \tag{47.6}$$

We also have that

$$|a^j|\,r^j = |a|^j\,r^j = (|a|\,r)^j \to 0 \quad \text{as } j \to \infty \tag{47.7}$$

exactly when $|a|\,r < 1$.

Remember that $PS_r^q(k)$ is defined in (20.1), and that $PS_{0,r}(k)$ is defined in (20.4). Using (47.5) and (47.6), we get that

$$\sum_{j=0}^\infty a^j\,T^j \in PS_r^q(k) \tag{47.8}$$

when $0 < q < \infty$ and $|a|\,r < 1$, and when $q = \infty$ and $|a|\,r \le 1$. Similarly,

$$\sum_{j=0}^\infty a^j\,T^j \in PS_{0,r}(k) \tag{47.9}$$

exactly when $|a|\,r < 1$, as in (47.7). If $|a|\,r < 1$, then $\sum_{j=0}^\infty a^j\,T^j$ converges as an infinite series in $PS_r^q(k)$ for every $q > 0$, as in Section 28. If $|a|\,r = 1$, then $\sum_{j=0}^\infty a^j\,T^j$ defines an element of $PS_r^\infty(k)$, but does not converge as an infinite series of elements of $PS_r^\infty(k)$ with respect to $\|\cdot\|_{\infty,r}$, as in Section 28 again.

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