Some basic topics in analysis related to shift operators

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Abstract

Some topics related to shift operators are discussed, in connection with absolute value functions on fields, and norms on vector spaces over such fields.

Contents

Part I

Basic notions

1 Vector spaces and linear mappings

Let k be a field, and let V, W be vector spaces over k. The space of linear mappings from V into W will be denoted $\mathcal{L}(V, W)$, and is a vector space over k with respect to pointwise addition and scalar multiplication. If $V = W$, then this space may be denoted $\mathcal{L}(V)$. This is an associative algebra over k, with composition of linear mappings as multiplication. The identity mapping from V into itself may be denoted I or I_V, which is the multiplicative identity element in $\mathcal{L}(V)$.

As usual, a *linear functional* on V is a linear mapping from V into k, where k is considered as a one-dimensional vector space over itself. The space of linear functionals on V is known as the *algebraic dual* of V, which may be denoted V^{alg}. This is the same as $\mathcal{L}(V, k)$, using the notation in the preceding paragraph, and in particular this is a vector space over k. If V has finite dimension, then it is well known that V^{alg} has the same dimension. A version of this will be discussed in the next section.

Let W be another vector space over k, and let T be a linear mapping from V into W. If λ is a linear functional on W, then the composition $\lambda \circ T$ of λ and T defines a linear functional on V. Put

$$T^{\mathrm{alg}}(\lambda) = \lambda \circ T, \tag{1.1}$$

which defines a linear mapping from W^{alg} into V^{alg}, dual to T. Note that

$$T \mapsto T^{\mathrm{alg}} \tag{1.2}$$

defines a linear mapping from $\mathcal{L}(V, W)$ into $\mathcal{L}(W^{\mathrm{alg}}, V^{\mathrm{alg}})$. If $V = W$, then we can apply this to $T = I_V$, and it is easy to see that

$$(I_V)^{\mathrm{alg}} = I_{V^{\mathrm{alg}}}, \tag{1.3}$$

the identity mapping on V^{alg}.

Let V, W, and Z be vector spaces over k, let T_1 be a linear mapping from V into W, and let T_2 be a linear mapping from W into Z, so that the composition $T_2 \circ T_1$ is a linear mapping from V into W. If λ is a linear functional on Z, then

$$(T_2 \circ T_1)^{\mathrm{alg}}(\lambda) = \lambda \circ (T_2 \circ T_1) = (\lambda \circ T_2) \circ T_1 = (T_2^{\mathrm{alg}}(\lambda)) \circ T_1 = T_1^{\mathrm{alg}}(T_2^{\mathrm{alg}}(\lambda)). \tag{1.4}$$

This implies that

$$(T_2 \circ T_1)^{\mathrm{alg}} = T_1^{\mathrm{alg}} \circ T_2^{\mathrm{alg}} \tag{1.5}$$

as linear mappings from V_3^{alg} into V_1^{alg}. Suppose now that T is a one-to-one linear mapping from V onto W, and let T^{-1} be the corresponding inverse mapping from W onto V. Under these conditions, T^{alg} is a one-to-one linear mapping from W^{alg} onto V^{alg}, with

$$(T^{\mathrm{alg}})^{-1} = (T^{-1})^{\mathrm{alg}} \tag{1.6}$$

as linear mappings from V^{alg} onto W^{alg}.

Let V be a vector space over k again, and put

$$L_v(\lambda) = L_{V,v}(\lambda) = \lambda(v) \tag{1.7}$$

for each $v \in V$ and $\lambda \in V^{\mathrm{alg}}$. This defines a linear functional on V^{alg} for each $v \in V$, which is to say an element of $(V^{\mathrm{alg}})^{\mathrm{alg}}$. Moreover,

$$v \mapsto L_v \tag{1.8}$$

defines a linear mapping from V into $(V^{\mathrm{alg}})^{\mathrm{alg}}$. Let W be another vector space over k, and let T be a linear mapping from V into W, so that T^{alg} is defined as a linear mapping from W^{alg} into V^{alg} as before. If $v \in V$, then $L_{V,v} \circ T^{\mathrm{alg}}$ is defined as a linear functional on W^{alg}, so that $(L_{V,v} \circ T^{\mathrm{alg}})(\mu)$ is defined as an element of k for each $\mu \in W^{\mathrm{alg}}$. Observe that

$$\begin{aligned} (L_{V,v} \circ T^{\mathrm{alg}})(\mu) = L_{V,v}(T^{\mathrm{alg}}(\mu)) &= L_{V,v}(\mu \circ T) \\ &= (\mu \circ T)(v) = \mu(T(v)) = L_{W,T(v)}(\mu) \end{aligned} \tag{1.9}$$

for every $v \in V$ and $\mu \in W^{\mathrm{alg}}$, where $L_{W,w}$ is defined as an element of $(W^{\mathrm{alg}})^{\mathrm{alg}}$ for every $w \in W$ as in (1.7). Thus

$$L_v \circ T^{\mathrm{alg}} = L_{W,T(v)} \tag{1.10}$$

as linear functionals on W^{alg} for every $v \in V$. Note that $(T^{\mathrm{alg}})^{\mathrm{alg}}$ can be defined as a linear mapping from $(V^{\mathrm{alg}})^{\mathrm{alg}}$ into $(W^{\mathrm{alg}})^{\mathrm{alg}}$ as before. Using (1.10), we get that

$$(T^{\mathrm{alg}})^{\mathrm{alg}}(L_{V,v}) = L_{V,v} \circ T^{\mathrm{alg}} = L_{W,T(v)} \tag{1.11}$$

for each $v \in V$.

2 k-Valued functions

Let k be a field, and let X be a nonempty set. The space of k-valued functions on X will be denoted $c(X,k)$, which is a vector space over k with respect to pointwise addition and scalar multiplication. The *support* of a k-valued function f on X is defined by

$$\operatorname{supp} f = \{x \in X : f(x) \neq 0\}. \tag{2.1}$$

The space of k-valued functions on X whose support have only finitely many elements is denoted $c_{00}(X, k)$, and is a linear subspace of $c(X, k)$. If $y \in X$, then $\delta_y = \delta_{X,y}$ is defined as a k-vaued function on X by

$$\begin{aligned} \delta_y(x) = \delta_{X,y}(x) &= 1 \quad \text{when } x = y \\ &= 0 \quad \text{when } x \neq y. \end{aligned} \tag{2.2}$$

Thus $\delta_y \in c_{00}(X, k)$ for each $y \in X$, and

$$\{\delta_y : y \in X\} \tag{2.3}$$

is a basis for $c_{00}(X, k)$ as a vector space over k. Of course, if X has only finitely many elements, then $c_{00}(X, k)$ is the same as $c(X, k)$.

If $f \in c_{00}(X, k)$, then

$$\sum_{x \in X} f(x) \tag{2.4}$$

reduces to a finite sum in k, and thus defines an element of k. The mapping from $f \in c_{00}(X, k)$ to the sum (2.4) defines a linear functional on $c_{00}(X, k)$.

Let $g \in c(X, k)$ be given. If $f \in c_{00}(X, k)$, then the product $f\,g$ of f and g has finite support in X as well. Thus

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x) \tag{2.5}$$

defines an element of k, as in the preceding paragraph. Hence (2.5) defines a linear functional on $c_{00}(X, k)$. Note that

$$\lambda_g(\delta_y) = g(y) \tag{2.6}$$

for every $y \in X$. If λ is any linear functional on $c_{00}(X, k)$, then we can put

$$g_\lambda(y) = \lambda(\delta_y) \tag{2.7}$$

for each $y \in X$. This defines a k-valued function on X, and one can check that

$$\lambda = \lambda_{g_\lambda} \tag{2.8}$$

as linear functionals on $c_{00}(X, k)$. It follows that $g \mapsto \lambda_g$ defines a one-to-one linear mapping from $c(X, k)$ onto $c_{00}(X, k)^{\mathrm{alg}}$, with inverse given by $\lambda \mapsto g_\lambda$.

Similarly, if $h \in c_{00}(X, k)$, then

$$\mu_h(g) = \sum_{x \in X} g(x)\, h(x) \tag{2.9}$$

defines a linear functional on $c(X, k)$, where the sum on the right side of (2.9) also reduces to a finite sum in k. The restriction of μ_h to $c_{00}(X, k)$ is the same as λ_h as defined in the preceding paragraph. In particular,

$$\mu_h(\delta_y) = h(y) \tag{2.10}$$

for every $y \in X$, so that $h \mapsto \mu_h$ defines a one-to-one linear mapping from $c_{00}(X, k)$ into $c(X, k)^{\mathrm{alg}}$. If $f \in c_{00}(X, k)$, then we can define L_f as a linear functional on $c_{00}(X, k)^{\mathrm{alg}}$ as in (1.7). This linear functional is given by

$$L_f(\lambda_g) = \lambda_g(f) = \mu_f(g) \tag{2.11}$$

for each $g \in c(X, k)$, where λ_g is as in (2.5).

3 Bilateral shift operators

Let k be a field, and let $c(\mathbf{Z}, k)$ be the space of k-valued mappings from the set $\mathbf{Z}$ of integers into k, as in the previous section. Consider the linear mapping T from $c(\mathbf{Z}, k)$ into itself defined by

$$(T(f))(j) = f(j - 1) \tag{3.1}$$

for every $f \in c(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$. This is the standard forward shift operator on $c(\mathbf{Z}, k)$, which is a one-to-one linear mapping from $c(\mathbf{Z}, k)$ onto itself. Note that T maps $c_{00}(\mathbf{Z}, k)$ onto itself. The inverse of T is given by

$$(T^{-1}(f))(j) = f(j + 1) \tag{3.2}$$

for each $f \in c(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$, which is the standard backward shift operator on $c(\mathbf{Z}, k)$.

Let T^l be the lth power of T as a linear mapping from $c(\mathbf{Z}, k)$ into itself with respect to composition for each positive integer l. If $l = 0$, then T^l is interpreted as being the identity mappping on $c(\mathbf{Z}, k)$. In this situation, we can also define T^l for negative integers l, by taking powers of T^{-1} on $c(\mathbf{Z}, k)$. Thus T^l is defined as a linear mapping on $c(\mathbf{Z}, k)$ for every integer l, and it is easy to see that

$$(T^l(f))(j) = f(j - l) \tag{3.3}$$

for every $f \in c(\mathbf{Z}, k)$ and $j, l \in \mathbf{Z}$.

Let $\delta_n = \delta_{\mathbf{Z},n}$ be the k-valued function defined on $\mathbf{Z}$ as in (2.2) for each $n \in \mathbf{Z}$. Observe that

$$(T^l(\delta_n))(j) = \delta_n(j - l) = \delta_{n+l}(j) \tag{3.4}$$

for every $j, l, n \in \mathbf{Z}$. Thus

$$T^l(\delta_n) = \delta_{n+l} \tag{3.5}$$

for every $l, n \in \mathbf{Z}$. If $A \subseteq \mathbf{Z}$ and $l \in \mathbf{Z}$, then let $A + l$ be the subset of $\mathbf{Z}$ defined by

$$A + l = \{a + l : a \in A\}. \tag{3.6}$$

Using (3.3), we get that

$$\operatorname{supp} T^l(f) = (\operatorname{supp} f) + l \tag{3.7}$$

for every $f \in c(\mathbf{Z}, k)$ and $l \in \mathbf{Z}$.

If $g \in c(\mathbf{Z}, k)$, then

$$\lambda_g(f) = \sum_{j=-\infty}^{\infty} f(j)\, g(j) \tag{3.8}$$

defines a linear functional on $c_{00}(\mathbf{Z}, k)$, as in (2.5). Observe that

$$\lambda_g(T(f)) = \sum_{j=-\infty}^{\infty} f(j-1)\, g(j) = \sum_{j=-\infty}^{\infty} f(j)\, g(j+1) = \lambda_{T^{-1}(g)}(f) \tag{3.9}$$

for every $f \in c_{00}(\mathbf{Z}, k)$ and $g \in c(\mathbf{Z}, k)$. This says that the algebraic dual of T as a linear mapping from $c_{00}(\mathbf{Z}, k)$ into itself corresponds to T^{-1} as a linear mapping from $c(\mathbf{Z}, k)$ into itself. This uses the identification of $c_{00}(\mathbf{Z}, k)^{\mathrm{alg}}$ with $c(\mathbf{Z}, k)$ given by (3.8), as in the previous section. Similarly, if $h \in c_{00}(\mathbf{Z}, k)$, then

$$\mu_h(g) = \sum_{j=-\infty}^{\infty} g(j)\, h(j) \tag{3.10}$$

defines a linear functional on $c(\mathbf{Z}, k)$, as in (2.9). As before, we have that

$$\mu_h(T(g)) = \mu_{T^{-1}(h)}(g) \tag{3.11}$$

for every $g \in c(\mathbf{Z}, k)$ and $h \in c_{00}(\mathbf{Z}, k)$. In this situation, we can identify $c_{00}(\mathbf{Z}, k)$ with a linear subspace of $c(\mathbf{Z}, k)^{\mathrm{alg}}$, using (3.10). With respect to this identification, the restriction of the algebraic dual of T to this subspace of $c(\mathbf{Z}, k)^{\mathrm{alg}}$ corresponds to the restriction of T^{-1} to $c_{00}(\mathbf{Z}, k)$, by (3.11).

4 Nonnegative sums

Let $f(x)$ be a nonnegative real-valued function on a nonempty set X. If X has only finitely many elements, or if f has finite support in X, then the sum

$$\sum_{x \in X} f(x) \tag{4.1}$$

can be defined as a nonnegative real number in the usual way. Otherwise, (4.1) can be defined as a nonnegative extended real number as the supremum of

$$\sum_{x \in E} f(x) \tag{4.2}$$

over all nonempty finite subsets E of X. If X is the set $\mathbf{Z}_+$ of positive integers, then (4.1) is the same as

$$\sum_{j=1}^{\infty} f(j) = \sup_{n \ge 1} \sum_{j=1}^{n} f(j), \tag{4.3}$$

where more precisely the supremum is taken over all positive integers n. Similarly, if $X = \mathbf{Z}$, then (4.1) is the same as

$$\sum_{j=-\infty}^{\infty} f(j) = \sup_{l,n \geq 1} \sum_{j=-l}^{n} f(j) = \sup_{n \geq 1} \sum_{j=-n}^{n} f(j). \tag{4.4}$$

Let X be any nonempty set again, and let f, g be nonnegative real-valued functions on X. If X has only finitely many elements, or if f and g have finite support in X, then

$$\sum_{x \in X} (f(x) + g(x)) = \Big(\sum_{x \in X} f(x)\Big) + \Big(\sum_{x \in X} g(x)\Big). \tag{4.5}$$

One can check that (4.5) holds without these additional conditions, by approximating these sums by finite subsums. This also uses standard conventions for sums of nonnegative extended real numbers. Similarly,

$$\sum_{x \in X} t\, f(x) = t \sum_{x \in X} f(x) \tag{4.6}$$

for every nonnegative real number t when X has finitely many elements, or f has finite support in X. This also holds without these additional conditions when $t > 0$, where the right side is interpreted as being $+\infty$ when (4.1) is infinite. If $t = 0$, then it is customary to interpret the right side as being 0 even when (4.1) is infinite.

If f is a nonnegative real-valued function on X and r is a positive real number, then we put

$$\|f\|_r = \Big(\sum_{x \in X} f(x)^r\Big)^{1/r}, \tag{4.7}$$

which is interpreted as being $+\infty$ when the sum is infinite. As usual, we can extend this to $r = \infty$ by putting

$$\|f\|_\infty = \sup_{x \in X} f(x), \tag{4.8}$$

which is defined as a nonnegative extended real number. Observe that

$$\|f\|_\infty \leq \|f\|_r \tag{4.9}$$

for every $r > 0$. If $0 < r_1 \leq r_2 < \infty$, then we have that

$$\|f\|_{r_2}^{r_2} = \sum_{x \in X} f(x)^{r_2} \leq \|f\|_\infty^{r_2 - r_1} \sum_{x \in X} f(x)^{r_1} = \|f\|_\infty^{r_2 - r_1}\, \|f\|_{r_1}^{r_1} \leq \|f\|_{r_1}^{r_2}, \tag{4.10}$$

using (4.9) in the last step. This implies that

$$\|f\|_{r_2} \leq \|f\|_{r_1} \tag{4.11}$$

when $r_1 \leq r_2$, which also works when $r_2 = \infty$, by (4.9).

If f, g are nonnegative real-valued functions on X and $1 < r < \infty$, then it is well known that

$$\|f + g\|_r \le \|f\|_r + \|g\|_r, \tag{4.12}$$

by Minkowski's inequality for sums.. Of course, equality holds trivially when $r = 1$, and it is easy to check directly that this inequality also holds when $r = \infty$. If $0 < r \le 1$, then

$$(a + b)^r \le a^r + b^r \tag{4.13}$$

for all nonnegative real numbers a, b. This follows from (4.11), with $r_1 = r$, $r_2 = 1$, and where X has exactly two elements. It follows that

$$\begin{aligned} \|f + g\|_r^r &= \sum_{x \in X} (f(x) + g(x))^r \\ &\le \sum_{x \in X} (f(x)^r + g(x)^r) \\ &= \sum_{x \in X} f(x)^r + \sum_{x \in X} g(x)^r = \|f\|_r^r + \|g\|_r^r \end{aligned} \tag{4.14}$$

for all nonnegative real-valued functions f, g on X when $0 < r \le 1$, using (4.13) in the second step, and (4.5) in the third step.

5 q-Semimetrics

Let X be a set, and let q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined on $X \times X$ is said to be a *q-semimetric* on X if it satisfies the following three conditions. First,

$$d(x, x) = 0 \quad \text{for every } x \in X. \tag{5.1}$$

Second,

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in X. \tag{5.2}$$

Third,

$$d(x, z)^q \le d(x, y)^q + d(y, z)^q \quad \text{for every } x, y, z \in X. \tag{5.3}$$

If we also have that

$$d(x, y) > 0 \tag{5.4}$$

for every $x, y \in X$ with $x \ne y$, then $d(\cdot, \cdot)$ is said to be a *q-metric* on X. A q-semimetric or q-metric with $q = 1$ is also known as a *semimetric* or *metric*, as appropriate. Note that (5.3) may be reformulated as saying that

$$d(x, z) \le (d(x, y)^q + d(y, z)^q)^{1/q} \tag{5.5}$$

for every $x, y, z \in X$. The right side of (5.5) decreases monotonically in q, by (4.11). This means that the property of being a q-semimetric or a q-metric becomes more restrictive as q increases.

A nonnegative real-valued function $d(x, y)$ defined on $X \times X$ is said to be a *semi-ultrametric* on X if it satisfies (5.1) and (5.2), and also

$$d(x, z) \le \max(d(x, y), d(y, z)) \tag{5.6}$$

for every $x, y, z \in X$. If (5.4) holds as well, then $d(\cdot, \cdot)$ is said to be an *ultrametric* on X. Observe that the right side of (5.6) is less than or equal to the right side of (5.5), so that (5.6) implies (5.5). This means that a semi-ultrametric is a q-semimetric for each $q > 0$, and similarly an ultrametric is a q-metric for every $q > 0$. Semi-ultrametrics and ultrametrics will be considered as q-semimetrics and q-metrics with $q = \infty$, respectively.

The *discrete metric* is defined on any set X by putting $d(x, y)$ equal to 1 when $x \ne y$, and equal to 0 when $x = y$. It is easy to see that this defines an ultrametric on X. Let $d(x, y)$ be any q-semimetric on a set X, $0 < q \le \infty$, and let a be a positive real number. Under these conditions, one can check that

$$d(x, , y)^a \tag{5.7}$$

defines a (q/a)-semimetric on X. Similarly, if $d(x, y)$ is a q-metric on X, then (5.7) is a (q/a)-metric on X.

Let $d(\cdot, \cdot)$ be a q-semimetric on a set X for some $q > 0$ again. The *open ball* in X centered at $x \in X$ with radius $r > 0$ with respect to $d(\cdot, \cdot)$ is defined as usual by

$$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}. \tag{5.8}$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \ge 0$ with respect to $d(\cdot, \cdot)$ is defined by

$$\overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in M : d(x, y) \le r\}. \tag{5.9}$$

Let a be a positive real number, so that (5.7) is a (q/a)-semimetric on X, as before. If $x \in X$, then

$$B_{d^a}(x, r^a) = B_d(x, r) \tag{5.10}$$

for every $r > 0$, and

$$\overline{B}_{d^a}(x, r) = \overline{B}_d(x, r) \tag{5.11}$$

for every $r \ge 0$.

A subset U of X is said to be an *open set* with respect to $d(\cdot, \cdot)$ if for every $x \in U$ there is an $r > 0$ such that

$$B(x, r) \subseteq U, \tag{5.12}$$

which defines a topology on X. If a is a positive real number, then (5.7) defines a (q/a)-metric on X, and the topology determined on X by (5.7) is the same as the topology determined by $d(\cdot, \cdot)$, because of (5.10). One can verify directly that the topology determined on X by $d(\cdot, \cdot)$ satisfies many of the same properties as for ordinary semimetrics, but one can also use the preceding remark to reduce to that case. In particular, open balls in X with respect to $d(\cdot, \cdot)$ are open sets, and closed balls are closed sets. If $d(\cdot,\cdot)$ is a q-metric on X, then it follows that X is Hausdorff with respect to this topology.

Suppose for the moment that $d(\cdot,\cdot)$ is a semi-ultrametric on X. If $x, y \in X$ satisfy $d(x,y) < r$ for some $r > 0$, then it is easy to see that

$$B(y,r) \subseteq B(x,r). \tag{5.13}$$

It follows that

$$B(x,r) = B(y,r) \tag{5.14}$$

under these conditions, since we can interchange the roles of x and y. Similarly, if $d(x,y) \le r$ for some $r \ge 0$, then

$$\overline{B}(y,r) \subseteq \overline{B}(x,r), \tag{5.15}$$

and hence

$$\overline{B}(x,r) = \overline{B}(y,r), \tag{5.16}$$

by interchanging the roles of x and y again. Using (5.15), we get that $\overline{B}(x,r)$ is an open set in X when $r > 0$, and one can check that open balls in X are closed sets in this situation too.

Let d_X be a q_X-metric on X for some $q_X > 0$. One can define the notions of Cauchy sequences in X and completeness of X with respect to d_X in the usual way. If a is a positive real number, then $d_X(\cdot,\cdot)^a$ determines the same collection of Cauchy sequences in X, and the same notion of completeness. Similarly, if Y is another set, and d_Y is a q_Y-metric on Y for some $q_Y > 0$, then one can define the notion of uniform continuity of mappings from X into Y with respect to d_X and d_Y in the usual way. If a, b are positive real numbers, then $d_X(\cdot,\cdot)^a$ and $d_Y(\cdot,\cdot)^b$ determine the same collection of uniformly continuous mappings from X into Y.

6 q-Absolute value functions

Let k be a field, and let q be a positive real number. A nonnegative real-valued function $|x|$ defined on k is said to be a *q-absolute value function* on k if it satisfies the following three conditions. First,

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{6.1}$$

More precisely, this means that $|x| = 0$ as a real number if and only if $x = 0$ as an element of k. Second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x, y \in k. \tag{6.2}$$

Third,

$$|x+y|^q \le |x|^q + |y|^q \quad \text{for every } x, y \in k. \tag{6.3}$$

If these conditions hold with $q = 1$, then $|\cdot|$ is said to be an *absolute value function* on k. It is well known that the standard absolute value functions on the fields $\mathbf{R}$ and $\mathbf{C}$ of real and complex numbers, respectively, are absolute value functions in this sense.

Using (6.1) and (6.2), one can check that $|1| = 1$, where the first 1 is the multiplicative identity element in k, and the second 1 is the multiplicative identity element in $\mathbf{R}$. If $x \in k$ satisfies $x^n = 1$ for some positive integer n, then it follows that $|x| = 1$, using (6.2) again. In particular, this holds when $x = -1$, the additive inverse of 1 in k. This implies in turn that $|-y| = |y|$ for every $y \in k$, using (6.2). If $|\cdot|$ is a q-absolute value function on k, then we get that

$$d(x, y) = |x - y| \tag{6.4}$$

defines a q-metric on k.

As in the previous section, (6.3) is equivalent to asking that

$$|x + y| \le (|x|^q + |y|^q)^{1/q} \tag{6.5}$$

for every $x, y \in k$. The right side of (6.5) decreases monotonically in q, as in (4.11), so that the property of being a q-absolute value function becomes more restrictive as q increases. A nonnegative real-valued function $|\cdot|$ on k is said to be an *ultrametric absolute value function* if it satisfies (6.1), (6.2), and

$$|x + y| \le \max(|x|, |y|) \tag{6.6}$$

for every $x, y \in k$. This implies that $|\cdot|$ is a q-absolute value function on k for every $q > 0$, because the right side of (6.6) is less than or equal to the right side of (6.5). If $|\cdot|$ is an ultrametric absolute value function on k, then (6.4) is an ultrametric on k. As before, an ultrametric absolute value function on k may be considered as a q-absolute value function with $q = \infty$. The *trivial absolute value function* is defined on any field k by putting $|x| = 1$ when $x \ne 0$ and $|0| = 0$. It is easy to see that this defines an ultrametric absolute value function on k, for which the corresponding ultrametric is the discrete metric.

If $|\cdot|$ is not the trivial absolute value function on k, then there is an $x \in k$ such that $x \ne 0$ and $|x| \ne 1$. This implies that there are $y, z \in k$ such that $0 < |y| < 1$ and $|z| > 1$, using x and $1/x$. It follows that $|y^j| = |y|^j \to 0$ and $|z^j| = |z|^j \to \infty$ as $j \to \infty$ under these conditions.

If $|\cdot|$ is a q-absolute value function on a field k for some $q > 0$ and a is a positive real number, then one can verify that

$$|x|^a \tag{6.7}$$

defines a (q/a)-absolute value function on k. Of course, the (q/a)-metric on k associated to (6.7) as in (6.4) is given by

$$|x - y|^a. \tag{6.8}$$

This corresponds to the q-metric (6.4) associated to $|\cdot|$ on k as in (5.7), so that many of the remarks in the previous section are applicable in this situation. Let

$|\cdot|_1$, $|\cdot|_2$ be q_1, q_2-absolute value functions on k, respectively, for some $q_1, q_2 > 0$. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{6.9}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* on k.

If p is a prime number, then the *p-adic absolute value* $|x|_p$ is defined on the field $\mathbf{Q}$ of rational numbers as follows. Of course, we put $|0|_p = 0$. Otherwise, if x is a nonzero rational number, then x can be expressed as $p^j\,(a/b)$, where $a, b, j \in \mathbf{Z}$, $a, b \neq 0$, and a, b are not integer multiples of p. In this case, we put

$$|x|_p = p^{-j}, \tag{6.10}$$

and one can check that this defines an ultrametric absolute value function on $\mathbf{Q}$. A famous theorem of Ostrowski implies that every q-absolute value function on $\mathbf{Q}$ is either trivial, or equivalent to the standard absolute value function on $\mathbf{R}$, or equivalent to the p-adic absolute value function for some prime p.

Let $|\cdot|$ be a q-absolute value function on a field k for some $q > 0$. If k is not already complete with respect to the associated q-metric (6.4), then one can pass to a completion. This leads to a field k_1 with a q-absolute value function $|\cdot|_1$, such that k_1 is complete with respect to the associated q-metric, and an isomorphism from k onto a subfield of k_1. This isomorphism should be isometric in the sense that $|\cdot|$ on k corresponds to $|\cdot|_1$ on the image, which implies that the associated q-metrics correspond in the same way. The image of k should also be dense in k_1, with respect to the q-metric associated to $|\cdot|_1$. This completion is unique up to isometric isomorphic equivalence. If p is a prime number, then the completion $\mathbf{Q}_p$ of $\mathbf{Q}$ with respect to the p-adic absolute value function is known as the field of *p-adic numbers*.

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$ again. If $x \in k$ and $n \in \mathbf{Z}_+$, then we let $n \cdot x$ be the sum of n x's in k. If there are positive integers n such that $|n \cdot 1|$ is arbitrarily large, then $|\cdot|$ is said to be *archimedian* on k. Otherwise, if there is a finite upper bound for $|n \cdot 1|$ with $n \in \mathbf{Z}_+$, then $|\cdot|$ is said to be *non-archimedian* on k. It is easy to see that $(n \cdot 1)^j = n^j \cdot 1$ in k for every $j, n \in \mathbf{Z}_+$, so that

$$|n^j \cdot 1| = |(n \cdot 1)^j| = |n \cdot 1|^j. \tag{6.11}$$

If $|n \cdot 1| > 1$, then it follows that

$$|n^j \cdot 1| = |n \cdot 1|^j \to \infty \quad \text{as } j \to \infty, \tag{6.12}$$

which means that $|\cdot|$ is archimedian on k. Equivalently, if $|\cdot|$ is non-archimedian on k, then it follows that

$$|n \cdot 1| \le 1 \tag{6.13}$$

for every $n \in \mathbf{Z}_+$. Ultrametric absolute value functions satisfy (6.13), and hence are non-archimedian. Conversely, it is well known that non-archimedian q-absolute value functions are ultrametric absolute value functions.

Let $|\cdot|$ be a q-absolute value function on a field k for some $q > 0$ again. Observe that

$$\{|x| : x \in k,\ x \neq 0\} \tag{6.14}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. If 1 is not a limit point of (6.14) with respect to the standard topology on $\mathbf{R}$, then $|\cdot|$ is said to be *discrete* on k. In this case, one can check that (6.14) has no limit points in $\mathbf{R}_+$ with respect to the standard topology on $\mathbf{R}$. However, 0 is a limit point of (6.14) in $\mathbf{R}$ when $|\cdot|$ is nontrivial on k.

In order to be more precise, put

$$\rho_1 = \sup\{|x| : x \in k,\ |x| < 1\}. \tag{6.15}$$

Thus $0 \le \rho_1 \le 1$ automatically. It is easy to see that $\rho_1 < 1$ if and only if $|\cdot|$ is discrete on k, and that $\rho_1 > 0$ if and only if $|\cdot|$ is nontrivial on k. If $|\cdot|$ is discrete and nontrivial on k, then the supremum in (6.15) is attained, and (6.14) consists of integer powers of ρ_1. If $|\cdot|$ is not discrete on k, then one can verify that (6.14) is dense in $\mathbf{R}_+$ with respect to the topology induced by the standard topology on $\mathbf{R}$.

One can also show that $|\cdot|$ is non-archimedian when $|\cdot|$ is discrete on k. This is trivial when k has positive characteristic, and so it suffices to consider the case where k has characteristic 0. In this case, there is a natural embedding of $\mathbf{Q}$ into k, so that $|\cdot|$ induces a discrete q-absolute value function on $\mathbf{Q}$. Using discreteness, it is easy to see that this induced absolute value function on $\mathbf{Q}$ cannot be equivalent to the standard absolute value function. Ostrowski's theorem implies that the induced absolute value function on $\mathbf{Q}$ is non-archimedian, and hence that $|\cdot|$ is non-archimedian on k.

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and suppose that $|\cdot|$ is archimedian on k, and that k is complete with respect to the corresponding q-metric (6.4). Under these conditions, another theorem of Ostrowski implies that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, where $|\cdot|$ corresponds to a q-absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard one.

7 q-Seminorms

Let k be a field with a q_k-absolute value function for some $q_k > 0$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *q-seminorm* on V for some positive real number q and with respect to $|\cdot|$ on k if it satisfies the following two conditions. First,

$$N(t\,v) = |t|\,N(v) \quad \text{for every } t \in k \text{ and } v \in V. \tag{7.1}$$

Second,

$$N(v+w)^q \le N(v)^q + N(w)^q \quad \text{for every } v, w \in V. \tag{7.2}$$

Note that the first condition implies that $N(0) = 0$, by taking $t = 0$. If we also have that

$$N(v) > 0 \tag{7.3}$$

for every $v \in V$ with $v \neq 0$, then N is said to be a *q-norm* on V. A q-seminorm or q-norm with $q = 1$ is also known as a *seminorm* or *norm*, respectively. As before, (7.2) is equivalent to asking that

$$N(v + w) \le (N(v)^q + N(w)^q)^{1/q} \tag{7.4}$$

for every $v, w \in V$. The right side of (7.4) decreases monotonically in q, by (4.11), so that this condition becomes more restrictive as q increases.

Similarly, a nonnegative real-valued function N on V is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k if it satisfies (7.1) and

$$N(v + w) \le \max(N(v), N(w)) \tag{7.5}$$

for every $v, w \in V$. If N also satisfies (7.3), then N is said to be an *ultranorm* on V with respect to $|\cdot|$ on k. The right side of (7.5) is less than or equal to the right side of (7.4), so that (7.5) implies (7.4). Thus semi-ultranorms and ultranorms are automatically q-seminorms and q-norms for every $q > 0$, respectively. As usual, semi-ultranorms and ultranorms will be considered as q-seminorms and q-norms with $q = \infty$, respectively. If N is a q-seminorm or q-norm on V for some $q > 0$, then it is easy to see that

$$d(v, w) = d_N(v, w) = N(v - w) \tag{7.6}$$

defines a q-semimetric or q-metric on V, as appropriate. Of course, $|\cdot|$ may be considered as a q_k-norm on k, where k is considered as a one-dimensional vector space over itself.

If N is a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, and if $N(v_0) > 0$ for some $v_0 \in V$, then one can check that $|\cdot|$ has to be a q-absolute value function on k, using (7.1). More precisely, this is trivial when $q \le q_k$, and otherwise one can get the stronger version of the triangle inequality for $|\cdot|$ from the one for N when $q > q_k$. If $|\cdot|$ is the trivial absolute value function on k, then the *trivial ultranorm* is defined on V by putting $N(0) = 0$ and $N(v) = 1$ for every $v \in V$ with $v \neq 0$. It is easy to see that this defines an ultranorm on V, for which the corresponding ultrametric as in (7.6) is the discrete metric.

Let $|\cdot|$ be any q_k-absolute value function on k for some $q_k > 0$ again. If a is a positive real number, then $|\cdot|^a$ defines a (q_k/a)-absolute value function on k, as in the previous section. Similarly, if N is a q-seminorm or q-norm on V with respect to $|\cdot|$ on k, then

$$N(v)^a \tag{7.7}$$

defines a (q/a)-seminorm or (q/a)-norm on V with respect to $|\cdot|^a$ on k, as appropriate. Of course, the (q/a)-semimetric or (q/a)-metric on V associated to (7.7) as before is the same as the ath power of the q-semimetric or q-metric on V associated to N, respectively.

8 r-Summable functions

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, let X be a nonempty set, and let r be a positive real number. A k-valued function f on

X is said to be r-*summable* with respect to $|\cdot|$ on k if $|f|^r$ is summable as a nonnegative real-valued function on X, in the sense that

$$\sum_{x \in X} |f(x)|^r < \infty. \tag{8.1}$$

We may also simply say that f is summable on X when this holds with $r = 1$, especially for $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function. Let $\ell^r(X, k)$ be the space of r-summable k-valued functions on X. If f is any k-valued function on X, then

$$\|f\|_r = \|f\|_{\ell^r(X,k)} = \Big(\sum_{x \in X} |f(x)|^r\Big)^{1/r} \tag{8.2}$$

is defined as a nonnegative extended real number, which is finite exactly when f is r-summable on X. Similarly, let $\ell^\infty(X, k)$ be the space of k-valued functions f on X that are bounded, which means that $|f|$ is bounded as a nonnegative real-valued function on X. As before,

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,k)} = \sup_{x \in X} |f(x)| \tag{8.3}$$

is defined as a nonnegative extended real-number for any k-valued function f on X, and is finite exactly when f is bounded on X. If $0 < r_1 \leq r_2 \leq \infty$, then we have that

$$\|f\|_{r_2} \leq \|f\|_{r_1} \tag{8.4}$$

for every k-valued function f on X, as in (4.11). It follows that

$$\ell^{r_1}(X, k) \subseteq \ell^{r_2}(X, k) \tag{8.5}$$

under these conditions, by the definition of $\ell^r(X, k)$.

If f is a k-valued function on X, $t \in k$, and $t \neq 0$, then it is easy to see that

$$\|t\, f\|_r = |t|\, \|f\|_r \tag{8.6}$$

for every $r > 0$. This also holds trivially when $t = 0$ and $f \in \ell^r(X, k)$, so that the right side is defined. If g is another k-valued function on X, then we have that

$$\|f + g\|_r^r \leq \|f\|_r^r + \|g\|_r^r \tag{8.7}$$

when $r \leq q$ and $r < \infty$,

$$\|f + g\|_r^q \leq \|f\|_r^q + \|g\|_r^q \tag{8.8}$$

when $q \leq r$ and $q < \infty$, and

$$\|f + g\|_\infty \leq \max(\|f\|_\infty, \|g\|_\infty) \tag{8.9}$$

when $q = \infty$. More precisely, (8.7) and (8.8) are the same when $r = q < \infty$, in which case they can be verified directly from the definitions. Similarly, (8.9) corresponds to $r = q = \infty$, and it is easy to check directly. If $r \leq q$, then $|\cdot|$ is also an r-absolute value function on k, as in Section 6. This permits one to obtain (8.7) as in the $r = q$ case. One can get (8.8) using Minkowski's inequality for sums, applied to the exponent $q/r \geq 1$. It follows that $\ell^r(X,k)$ is a vector space over k with respect to pointwise addition and scalar multiplication for every $r > 0$, and more precisely $\ell^r(X,k)$ is a linear subspace of the space $c(X,k)$ of all k-valued functions on X. We also get that $\|f\|_r$ is an r-norm on $\ell^\infty(X,k)$ when $r \leq q$, and that $\|f\|_r$ is a q-norm on $\ell^r(X,k)$ when $r \geq q$. Of course, the space $c_{00}(X,k)$ of k-valued functions with finite support on X is contained in $\ell^r(X,k)$ for every $r > 0$. If $r < \infty$, then one can check that $c_{00}(X,k)$ is dense in $\ell^r(X,k)$ with respect to the r or q-metric associated to $\|f\|_r$, as appropriate. If $|\cdot|$ is the trivial absolute value function on k, then $\ell^r(X,k)$ is equal to $c_{00}(X,k)$ when $r < \infty$, and $\ell^\infty(X,k)$ is the same as $c(X,k)$.

A k-valued function f on X is said to *vanish at infinity* if for each $\epsilon > 0$ we have that

$$|f(x)| < \epsilon \tag{8.10}$$

for all but finitely many $x \in X$. The space of k-valued functions on X that vanish at infinity is denoted $c_0(X,k)$. This is a linear subspace of $\ell^\infty(X,k)$, which is a closed set with respect to the q-metric associated to $\|f\|_\infty$. It is easy to see that

$$\ell^r(X,k) \subseteq c_0(X,k) \tag{8.11}$$

when $0 < r < \infty$, and in particular $c_{00}(X,k)$ is contained in $c_0(X,k)$. We also have that $c_{00}(X,k)$ is dense in $c_0(X,k)$ with respect to the q-metric associated to $\|f\|_\infty$, and the two spaces are the same when $|\cdot|$ is the trivial absolute value function on k.

Suppose now that k is complete with respect to the q-metric associated to $|\cdot|$. Using standard arguments, one can check that $\ell^r(X,k)$ is complete for every $r > 0$, with respect to the r or q-metric associated to $\|f\|_r$, as appropriate. More precisely, if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\ell^r(X,k)$ for some $r > 0$, then $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in k for every $x \in X$. If k is complete, then it follows that $\{f_j(x)\}_{j=1}^\infty$ converges to an element $f(x)$ of k for each $x \in X$. One can verify that $f \in \ell^r(X,k)$ under these conditions, and that $\{f_j\}_{j=1}^\infty$ converges to f with respect to $\|\cdot\|_r$.

9 Bounded linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-norms on V, W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. A linear mapping T from V into W is said to be *bounded* with respect to N_V and N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \leq C\, N_V(v) \tag{9.1}$$

for every $v \in V$. This implies that

$$N_W(T(v) - T(v')) = N_W(T(v - v')) \le C\, N_V(v - v') \tag{9.2}$$

for every $v, v' \in V$. In particular, it follows that T is uniformly continuous with respect to the q_V, q_W-metrics on V, W associated to N_V, N_W, respectively.

Suppose for the moment that $|\cdot|$ is not the trivial absolute value function on k. If a linear mapping T from V into W is continuous at 0, then one can check that T is bounded in the sense of (9.1). More precisely, it suffices to ask that $N_W(T(v))$ be bounded on a ball in V with respect to N_V centered at 0 with positive radius. This uses the fact that there are $t \in k$ such that $|t|$ is within a bounded factor of any positive real number, and the homogeneity property (7.1) of q-norms.

The space of bounded linear mappings from V into W is denoted $\mathcal{BL}(V, W)$, or simply $\mathcal{BL}(V)$ when $V = W$ and $N_V = N_W$. If T is a bounded linear mapping from V into W, then we put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \ge 0 : (9.1) \text{ holds}\}, \tag{9.3}$$

where more precisely the infimum is taken over all nonnegative real numbers C for which (9.1) holds. It is easy to see that the infimum is always attained, so that (9.1) holds with $C = \|T\|_{op}$. If $t \in k$, then $t\,T$ is a bounded linear mapping from V into W too, and

$$\|t\,T\|_{op} = |t|\,\|T\|_{op}. \tag{9.4}$$

This uses the analogous homogeneity properties of N_V and N_W.

Suppose that T_1, T_2 are bounded linear mappings from V into W, and let C_1, C_2 be nonnegative real numbers such that

$$N_W(T_j(v)) \le C_j\, N_V(v) \tag{9.5}$$

for every $v \in V$ and $j = 1, 2$. This implies that

$$N_W(T_1(v) + T_2(v))^{q_W} \le (C_1^{q_W} + C_2^{q_W})\, N_V(v)^{q_W} \tag{9.6}$$

for every $v \in V$ when $q_W < \infty$, and that

$$N_W(T_1(v) + T_2(v)) \le \max(C_1, C_2)\, N_V(v) \tag{9.7}$$

for every $v \in V$ when $q_W = \infty$. In both cases, it follows that $T_1 + T_2$ is bounded as a linear mapping from V into W. Thus $\mathcal{BL}(V, W)$ is a vector space over k with respect to pointwise addition and scalar multiplication, which is a linear subspace of the space $\mathcal{L}(V, W)$ of all linear mappings from V into W. It also follows easily from this that (9.3) defines a q_W-norm on $\mathcal{BL}(V, W)$.

Let T be a linear mapping from V into W, and consider the following condition on T:

$$N_W(T(v)) \text{ is bounded for } v \in V \text{ with } N_V(v) \le 1. \tag{9.8}$$

This means that

$$\sup\{N_W(T(v)) : v \in V,\ N_V(v) \le 1\} \tag{9.9}$$

is defined as a nonnegative real number. If T is a bounded linear mapping from V into W, then (9.8) holds, and (9.9) is less than or equal to $\|T\|_{op}$. If $|\cdot|$ is nontrivial on k, then (9.8) implies that T is bounded as a linear mapping from V into W, as before. In this case, one can verify that (9.1) holds with C equal to a constant multiple of (9.9), where the extra constant factor depends on $|\cdot|$. It follows that $\|T\|_{op}$ is less than or equal to a constant multiple of (9.9) in this situation. If $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function, then this constant multiple is equal to 1, so that $\|T\|_{op}$ is equal to (9.9). This also works when $|\cdot|$ is not discrete on k, for any field k.

Let Z be another vector space over k, and let N_Z be a q_Z-norm on Z with respect to $|\cdot|$ on k, for some $q_Z > 0$. Suppose that T_1 is a bounded linear mapping from V into W, and that T_2 is a bounded linear mapping from W into Z. Thus there are nonnegative real numbers C_1, C_2 such that

$$N_W(T_1(v)) \le C_1\, N_V(v) \tag{9.10}$$

for every $v \in V$, and

$$N_Z(T_2(w)) \le C_2\, N_W(w) \tag{9.11}$$

for every $w \in W$. Combining these two statements, we get that

$$N_Z(T_2(T_1(v))) \le C_2\, N_W(T_1(v)) \le C_1\, C_2\, N_V(v) \tag{9.12}$$

for every $v \in V$. Hence the composition $T_2 \circ T_1$ of T_1 and T_2 is bounded as a linear mapping from V into W, with

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}. \tag{9.13}$$

In particular, $\mathcal{BL}(V)$ is closed under composition of mappings. Note that the identity mapping I_V on V is bounded as a linear mapping from V into itself, with $\|I_V\|_{op} = 1$ when $V \ne \{0\}$.

If T is a one-to-one linear mapping from V onto W, then the inverse T^{-1} of T is a linear mapping from W onto V. A one-to-one bounded linear mapping T from V onto W is considered to be *invertible* as a bounded linear mapping if T^{-1} is bounded as a linear mapping from W onto V. As before, the boundedness of the inverse means that

$$N_V(T^{-1}(w)) \le C'\, N_W(w) \tag{9.14}$$

for some $C' \ge 0$ and every $w \in W$. In this situation, this is the same as saying that

$$N_V(v) \le C'\, N_W(T(v)) \tag{9.15}$$

for every $v \in V$. Note that (9.15) implies that the kernel of T is trivial, so that T is injective.

A linear mapping T from V into W is said to be an *isometry* if

$$N_W(T(v)) = N_V(v) \tag{9.16}$$

for every $v \in V$. This is the same as saying that (9.1) and (9.15) hold with $C, C' = 1$. If T is an isometric linear mapping from V onto W, then the inverse mapping T^{-1} is an isometry from W onto V. Of course, the identity mapping I_V is an isometric linear mapping from V onto itself. The shift operator T discussed in Section 3 defines an isometry from $\ell^r(\mathbf{Z}, k)$ onto itself for every $r > 0$, and T maps $c_0(\mathbf{Z}, k)$ onto itself as well.

Remember that k may be considered as a one-dimensional vector space over itself, and that $|\cdot|$ may be considered as a q_k-norm on k as a vector space over itself. A *bounded linear functional* on V is a linear functional λ on V that is bounded as a linear mapping from V into k, with respect to $|\cdot|$ as a norm on k. The corresponding q_k-operator norm of λ is known as the *dual q_k-norm* of λ.

If W is complete with respect to the q_W-metric associated to N_W, then $\mathcal{BL}(V, W)$ is complete with respect to the q_W-metric associated to (9.3), by standard arguments. More precisely, if $\{T_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{BL}(V, W)$ with respect to this q_W-metric, then $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in W for every $v \in V$, with respect to the q_W-metric associated to N_W. Hence this sequence converges in W for every $v \in V$, because W is complete, by hypothesis. It is easy to see that the limit defines a linear mapping T from V into W. One can verify that T is bounded, and that $\{T_j\}_{j=1}^\infty$ converges to T with respect to the operator q_W-norm, as desired.

10 Summable functions

In this section, we take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. Let X be a nonempty set, and let f be a real or complex-valued summable function on X, so that $\sum_{x \in X} |f(x)| < \infty$. It is well known that one can define the sum

$$\sum_{x \in X} f(x) \tag{10.1}$$

as a real or complex number, as appropriate, in a natural way under these conditions. More precisely, if f is a nonnegative real-valued function on X, then this sum can be defined as in Section 4. Otherwise, if f is a real or complex-valued summable function on X, then f can be expressed as a linear combination of nonnegative real-valued summable functions on X, which permits one to reduce to the previous case. This is a bit simpler in the real case, where f can be expressed as a difference of nonnegative real-valued summable functions on X. The complex case can be reduced to the real case, because the real and imaginary parts of a complex-valued function on X are summable as well. Of course, one should also verify that the value of the sum (10.1) does not depend on the particular representation of f in terms of nonnegative real-valued summable functions on X. This uses the linearity properties of the sum for nonnegative real-valued functions on X mentioned in Section 4.

Remember that the spaces $\ell^1(X, \mathbf{R})$, $\ell^1(X, \mathbf{C})$ of real and complex-valued summable functions on X, respectively, are vector spaces over $\mathbf{R}$, $\mathbf{C}$ with respect to pointwise addition and scalar multiplication, as in Section 8. One can check that

$$f \mapsto \sum_{x \in X} f(x) \tag{10.2}$$

defines a linear functional on these spaces, using the linearity properties for sums of nonnegative real-valued functions as in Section 4 again, or the remarks in the preceding paragraph about how the sum (10.1) is well defined when f is summable on X. Moreover, we have that

$$\left| \sum_{x \in X} f(x) \right| \le \sum_{x \in X} |f(x)| \tag{10.3}$$

for every real or complex-valued summable function f on X. In the real case, this can be obtained by decomposing f into its positive and negative parts. In the complex case, one can get a slightly weaker estimate with an extra factor of 2 on the right side using the real case applied to the real and imaginary parts of a complex-valued summable function on X. Of course, if a complex-valued function f on X has finite support, then (10.3) reduces to the triangle inequality for the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$. One can use this and an approximation argument to get that (10.3) holds for every complex-valued summable function on X.

It is easy to see that (10.2) is a bounded linear functional on $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, with dual norm equal to 1, using (10.3). Remember that $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ are dense linear subspaces of $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, respectively, as in Section 8. It follows that (10.2) is uniquely determined as a bounded linear functional on $\ell^1(X, \mathbf{R})$ or $\ell^1(X, \mathbf{C})$ by its restriction to $c_{00}(X, \mathbf{R})$ or $c_{00}(X, \mathbf{C})$, respectively, because bounded linear mappings are continuous. Of course, if a real or complex-valued function f on X has finite support, then (10.1) reduces to a finite sum in $\mathbf{R}$ or $\mathbf{C}$, as appropriate.

If $X = \mathbf{Z}_+$, then the sum (10.1) may be treated as an infinite series. In this case, the hypothesis that f be summable on $\mathbf{Z}_+$ says exactly that this infinite series is absolutely convergent. Similarly, if $X = \mathbf{Z}$, then the sum may be treated as a doubly-infinite series, which reduces to a sum of two ordinary infinite series.

11 Hilbert spaces

In this section, we take $k = \mathbf{R}$ or $\mathbf{C}$ again, equipped with the standard absolute value function. Let V be a vector space over $\mathbf{R}$ or $\mathbf{C}$, and let $\langle v, w \rangle$ be an inner product on V, which is Hermitian in the complex case. Thus $\langle v, v \rangle$ is a nonnegative real number for every $v \in V$, and we put

$$\|v\| = \langle v, v \rangle^{1/2} \tag{11.1}$$

for each $v \in V$, using the nonnegative square root of $\langle v, v \rangle$ on the right side. It is well known that

$$|\langle v, w \rangle| \le \|v\| \, \|w\| \tag{11.2}$$

for every $v, w \in V$, which is the Cauchy–Schwarz inequality in this context, and that $\|v\|$ defines a norm on V. If V is complete with respect to the metric associated to this norm, then V is said to be a *Hilbert space*.

Let X be a nonempty set, and let $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ be as in Section 8. If f and g are elements of either of these spaces, then it is well known that $|f(x)| \, |g(x)|$ is summable on X. This can be obtained from the Cauchy–Schwarz inequality, or by comparing this function more directly with $|f(x)|^2 + |g(x)|^2$. The standard inner product on $\ell^2(X, \mathbf{R})$ is defined as usual by

$$\langle f, g \rangle = \sum_{x \in X} f(x) \, g(x), \tag{11.3}$$

and similarly the standard inner product on $\ell^2(X, \mathbf{C})$ is defined by

$$\langle f, g \rangle = \sum_{x \in X} f(x) \, \overline{g(x)}, \tag{11.4}$$

where $\overline{a}$ denotes the complex conjugate of a complex number a. The sums on the right sides of (11.3) and (11.4) are defined as in the previous section, and it is easy to see that the norms corresponding to these inner products are the same as the ℓ^2 norms defined in Section 8.

Now let $(X, \mathcal{A}, \mu)$ be a measure space, and let $L^2(X)$ be the corresponding space of real or complex-valued square-integrable functions on X. If f, g are square integrable functions on X, then it is well known that $|f(x)| \, |g(x)|$ is also integrable on X, as before. This permits us to define

$$\langle f, g \rangle = \int_X f(x) \, g(x) \, d\mu(x) \tag{11.5}$$

in the real case, and

$$\langle f, g \rangle = \int_X f(x) \, \overline{g(x)} \, d\mu(x) \tag{11.6}$$

in the complex case. These determine inner products on the real and complex versions of $L^2(X)$, for which the corresponding norm is the usual L^2 norm

$$\Big(\int_X |f(x)|^2 \, d\mu(x) \Big)^{1/2}. \tag{11.7}$$

Remember that square-integrable functions on X are considered the same in $L^2(X)$ when they are equal almost everywhere on X with respect to μ.

Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be inner product spaces, both real or both complex. A linear mapping T from V onto W is said to be *unitary* if

$$\langle T(v), T(v') \rangle_W = \langle v, v' \rangle_V \tag{11.8}$$

for every $v, v' \in V$. This implies that T is an isometry with respect to the corresponding norms, by taking $v = v'$. Conversely, if T is an isometric linear mapping from V into W, then it is well known that T satisfies (11.8), because of polarization identities. It is easy to see directly that the shift operator T in Section 3 defines a unitary transformation from $\ell^2(\mathbf{Z}, \mathbf{R})$ onto itself, and from $\ell^2(\mathbf{Z}, \mathbf{C})$ onto itself.

12 Infinite series

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q-norm N for some $q > 0$, with respect to $|\cdot|$ on k. As usual, an infinite series $\sum_{j=1}^\infty v_j$ with terms in V is said to *converge* in V if the corresponding sequence of partial sums $\sum_{j=1}^n v_j$ converges in V, in which case the value of the infinite series is defined to be the limit of the sequence of partial sums. Note that the sequence of partial sums is a Cauchy sequence in V if and only if for each $\epsilon > 0$ there is a positive integer L such that

$$N\Big(\sum_{j=l}^n v_j\Big) < \epsilon \tag{12.1}$$

for every $n \ge l \ge L$. In particular, this implies that $\{v_j\}_{j=1}^\infty$ converges to 0 in V, by taking $l = n$. Another necessary condition for the sequence of partial sums to be a Cauchy sequence in V is that the partial sums be bounded in V.

Suppose for the moment that $q < \infty$, so that

$$N\Big(\sum_{j=l}^n v_j\Big)^q \le \sum_{j=l}^n N(v_j)^q \tag{12.2}$$

for every $n \ge l \ge 1$. Let us say that $\sum_{j=1}^\infty v_j$ converges *q-absolutely* in V if $\sum_{j=1}^\infty N(v_j)^q$ converges as an infinite series of nonnegative real numbers. This implies that the sequence of partial sums $\sum_{j=1}^n v_j$ is a Cauchy sequence in V, using the reformulation of the Cauchy condition mentioned in the previous paragraph. If V is complete with respect to the q-metric associated to N, then it follows that $\sum_{j=1}^\infty v_j$ converges in V. Under these conditions, one can also check that

$$N\Big(\sum_{j=1}^\infty v_j\Big)^q \le \sum_{j=1}^\infty N(v_j)^q, \tag{12.3}$$

using (12.2) with $l = 1$.

Suppose now that $q = \infty$, so that

$$N\Big(\sum_{j=l}^n v_j\Big) \le \max_{l \le j \le n} N(v_j) \tag{12.4}$$

for every $n \ge l \ge 1$. If $\{v_j\}_{j=1}^\infty$ converges to 0 in V, then it is easy to see that the sequence of partial sums $\sum_{j=1}^n v_j$ forms a Cauchy sequence, using the earlier reformulation of the Cauchy condition again. If V is also complete with respect to the ultrametric associated to N, then $\sum_{j=1}^\infty v_j$ converges in V, and we get that

$$N\Big(\sum_{j=1}^\infty v_j\Big) \le \sup_{j \ge 1} N(v_j), \tag{12.5}$$

using (12.4) with $l = 1$. Note that the supremum on the right side of (12.5) is attained in this situation, because $N(v_j) \to 0$ as $j \to \infty$, by hypothesis.

Let us now take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function, and let $(V, \langle v, w\rangle)$ be a real or complex inner product space, with the corresponding norm $\|v\|$. Suppose that $\sum_{j=1}^\infty v_j$ is an infinite series of pairwise-orthogonal vectors in V, so that

$$\langle v_j, v_l\rangle = 0 \tag{12.6}$$

when $j \ne l$. This implies that

$$\Big\|\sum_{j=l}^n v_j\Big\|^2 = \sum_{j=l}^n \|v_j\|^2 \tag{12.7}$$

for each $n \ge l \ge 1$. If $\sum_{j=1}^\infty \|v_j\|^2$ converges as an infinite series of nonnegative real numbers, then it follows that the sequence of partial sums $\sum_{j=1}^n v_j$ is a Cauchy sequence in V. If V is a Hilbert space, then $\sum_{j=1}^\infty v_j$ converges in V, and

$$\Big\|\sum_{j=1}^\infty v_j\Big\|^2 = \sum_{j=1}^\infty \|v_j\|^2. \tag{12.8}$$

Of course, $\sum_{j=1}^\infty \|v_j\|^2$ converges if and only if the partial sums $\sum_{j=1}^n \|v_j\|^2$ are bounded. In this situation, this happens exactly when the partial sums $\sum_{j=1}^n v_j$ are bounded in V.

13 Completeness

Let (X, d_X) and (Y, d_Y) be q_X, q_Y-metric spaces for some $q_X, q_Y > 0$, and let E be a dense subset of X. Also let f be a uniformly continuous mapping from E into Y, with respect to the restriction of d_X to elements of E. If Y is complete with respect to d_Y, then there is a unique extension of f to a uniformly continuous mapping from X into Y. This is well known for ordinary metric spaces, and essentially the same argument works for q-metric spaces. One can also reduce to the case of ordinary metric spaces, by replacing d_X or d_Y with $d_X^{q_X}$ or $d_Y^{q_Y}$ when $q_X \le 1$ or $q_Y \le 1$, respectively. More precisely, uniform continuity of f is needed for the existence of such an extension, but ordinary continuity of the extension is sufficient for its uniqueness. Similarly, completeness of Y is only used for the existence of the extension, and not uniqueness.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k with q_V, q_W-norms N_V, N_W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. Also let V_0 be a dense linear subspace of V, and let T be a bounded linear mapping from V_0 into W, with respect to the restriction of N_V to V_0. Thus T is uniformly continuous with respect to the q_V, q_W-metrics associated to N_V, N_W, respectively, as in Section 9. If W is complete with respect to the q_W-metric associated to N_W, then T has a unique extension to a uniformly continuous mapping from V into W, as in the preceding paragraph. In this context, one can check that this extension is a bounded linear mapping from V into W, with the same operator norm as T has on V_0.

Let X be a nonempty set, and let us take $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function for the moment. If f is a real or complex-valued function on X with finite support, then one can define $\sum_{x \in X} f(x)$ as a real or complex number by reducing to a finite sum. It is easy to see that

$$\left|\sum_{x \in X} f(x)\right| \le \sum_{x \in X} |f(x)| \tag{13.1}$$

under these conditions, using the ordinary triangle inequality on $\mathbf{R}$ or $\mathbf{C}$. Of course,

$$f \mapsto \sum_{x \in X} f(x) \tag{13.2}$$

defines linear functionals on $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$. More precisely, these are bounded linear functionals on $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ with respect to the restrictions of the corresponding ℓ^1 norms to these spaces, with dual norms equal to 1, because of (13.1). Remember that $c_{00}(X, \mathbf{R})$ and $c_{00}(X, \mathbf{C})$ are dense in $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$, with respect to the metrics corresponding to the ℓ^1 norms, as in Section 8. It follows that there are unique extensions of (13.2) to bounded linear functionals on $\ell^1(X, \mathbf{R})$ and $\ell^1(X, \mathbf{C})$ with dual norms equal to 1, as in the preceding paragraph, and because $\mathbf{R}$ and $\mathbf{C}$ are complete with respect to their standard metrics. These extensions can also be obtained as in Section 10.

Now let k be any field with an ultrametric absolute value function $|\cdot|$. If f is a k-valued function on X with finite support, then $\sum_{x \in X} f(x)$ can be defined as an element of k, by reducing to a finite sum, as before. In this case, we have that

$$\left|\sum_{x \in X} f(x)\right| \le \max_{x \in X} |f(x)|, \tag{13.3}$$

by the ultrametric version of the triangle inequality. This implies that (13.2) defines a bounded linear functional on $c_{00}(X, k)$ with respect to the supremum ultranorm, with dual norm equal to 1. As in Section 8, $c_{00}(X, k)$ is dense in $c_0(X, k)$ with respect to the ultrametric associated to the supremum ultranorm. If k is complete with respect to the ultrametric associated to $|\cdot|$, then there is a unique extension of (13.2) to a bounded linear functional on $c_0(X, k)$ with respect to the supremum ultranorm, and with dual norm equal to 1. This extension may be used as the definition of $\sum_{x \in X} f(x)$ when f is a k-valued function on X that vanishes at infinity and k is complete. If $X = \mathbf{Z}_+$, then this sum can be treated as an infinite series, as in the previous section. Similarly, if $X = \mathbf{Z}$, then the sum can be reduced to a sum of two infinite series.

14 Orthonormal systems

In this section, we take $k = \mathbf{R}$ or $\mathbf{C}$ again, with the standard absolute value function. Let $(V, \langle v, w \rangle_V)$ be a real or complex inner product space, and let $\|v\|_V$ be the corresponding norm on V. Suppose for the moment that $v_1, \ldots, v_n$ are finitely many orthonormal vectors in V, so that $\|v_j\|_V = 1$ for each $j = 1, \ldots, n$, and

$$\langle v_j, v_l \rangle_V = 0 \tag{14.1}$$

when $j \neq l$. Let $v \in V$ be given, and put

$$w = \sum_{j=1}^{n} \langle v, v_j \rangle_V \, v_j. \tag{14.2}$$

By construction, $\langle v, v_j \rangle_V = \langle w, v_j \rangle_V$ for each $j = 1, \ldots, n$, so that $v - w$ is orthogonal to v_j. This implies that $v - w$ is orthogonal to w, and hence

$$\begin{aligned} \|v\|_V^2 = \|(v - w) + w\|_V^2 &= \|v - w\|_V^2 + \|w\|_V^2 \\ &= \|v - w\|_V^2 + \sum_{j=1}^{n} |\langle v, v_j \rangle_V|^2. \end{aligned} \tag{14.3}$$

In particular, we have that

$$\sum_{j=1}^{n} |\langle v, v_j \rangle_V|^2 \le \|v\|_V^2. \tag{14.4}$$

If v is in the linear span of $v_1, \ldots, v_n$ in V, then $w = v$, and equality holds in (14.4). If u is any element of the linear span of $v_1, \ldots, v_n$ in V, then $w - u$ is also in the linear span of $v_1, \ldots, v_n$, so that $v - w$ is orthogonal to $w - u$. This implies that

$$\begin{aligned} \|v - u\|_V^2 = \|(v - w) + (w - u)\|_V^2 &= \|v - w\|_V^2 + \|w - u\|_V^2 \\ &\ge \|v - w\|_V^2, \end{aligned} \tag{14.5}$$

and hence that w minimizes the distance to v in V among elements of the linear span of $v_1, \ldots, v_n$.

Now let A be a nonempty set, and let $\{v_a\}_{a \in A}$ be an orthonormal family of vectors in V indexed by A. As before, this means that $\|v_a\|_V = 1$ for every $a \in A$, and that $\langle v_a, v_b \rangle_V = 0$ when $a \neq b$. If $v \in V$, then

$$\sum_{a \in A} |\langle v, v_a \rangle_V|^2 \le \|v\|_V^2. \tag{14.6}$$

More precisely, the sum on the left is defined as in Section 4, as the supremum of the corresponding sums over all finite subsets of A. These finite subsums can be estimated as in (14.4), to get (14.6). If v is an element of the linear span of the v_a's, $a \in A$, in V, then equality holds in (14.6), as before. Similarly, if v is an element of the closure of the linear span of the v_a's, $a \in A$, in V, then one can check that equality holds in (14.6).

Let $c_{00}(A)$ and $\ell^2(A)$ denote $c_{00}(A, k)$ and $\ell^2(A, k)$ with $k = \mathbf{R}$ or $\mathbf{C}$, as appropriate. If $f \in c_{00}(A)$, then

$$T(f) = \sum_{a \in A} f(a)\, v_a \tag{14.7}$$

reduces to a finite sum in V, so that T defines a linear mapping from $c_{00}(A)$ into V. It is easy to see that

$$\langle T(f), T(g) \rangle_V = \langle f, g \rangle_{\ell^2(A)} \tag{14.8}$$

for every $f, g \in c_{00}(A)$, where the right side of (14.8) is given by (11.3) or (11.4), as appropriate, with $X = A$. In particular, we have that

$$\|T(f)\|_V = \|f\|_{\ell^2(A)} \tag{14.9}$$

for every $f \in c_{00}(A)$, by taking $f = g$ in (14.8). Note that T maps $c_{00}(A)$ onto the linear span of the v_a's, $a \in A$, in V.

If V is a Hilbert space, then T has a unique extension to a bounded linear mapping from $\ell^2(A)$ into V, as in the previous section. This extension satisfies (14.8) for every $f, g \in \ell^2(A)$, and hence (14.9) for every $f \in \ell^2(A)$. One can check that this extension maps $\ell^2(A)$ onto the closure of the linear span of the v_a's, $a \in A$, in V. If $A = \mathbf{Z}_+$, then the right side of (14.7) can be treated as an infinite series in V, as in Section 12. Similarly, if $A = \mathbf{Z}$, then (14.7) can be reduced to a sum of two infinite series in V.

15 Fourier series

In this section, we take $k = \mathbf{C}$, with the standard absolute value function. Let

$$\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\} \tag{15.1}$$

be the unit circle in $\mathbf{C}$, and let μ be a complex Borel measure on $\mathbf{T}$. If $j \in \mathbf{Z}$, then the *jth Fourier coefficient* of μ is defined by

$$\widehat{\mu}(j) = \int_{\mathbf{T}} \overline{z}^j \, d\mu(z). \tag{15.2}$$

Observe that

$$|\widehat{\mu}(j)| \le |\mu|(\mathbf{T}) \tag{15.3}$$

for each $j \in \mathbf{Z}$, where $|\mu|$ denotes the total variation measure on $\mathbf{T}$ associated to μ. It is well known that μ is uniquely determined by its Fourier coefficients, in the following way.

Let $C(\mathbf{T})$ be the space of complex-valued continuous functions on $\mathbf{T}$, which is a vector space over $\mathbf{C}$ with respect to pointwise addition and scalar multiplication. Observe that

$$\lambda_\mu(\phi) = \int_{\mathbf{T}} \phi \, d\mu \tag{15.4}$$

defines a linear functional on $C(\mathbf{T})$, and that

$$|\lambda_\mu(\phi)| \le \Big(\sup_{z \in \mathbf{T}} |\phi(z)|\Big) \, |\mu|(\mathbf{T}) \tag{15.5}$$

for every $\phi \in C(\mathbf{T})$. Thus λ_μ is a bounded linear functional on $C(\mathbf{T})$ with respect to the supremum norm on $C(\mathbf{T})$, and the corresponding dual norm of λ_μ is less than or equal to $|\mu|(\mathbf{T})$. It is well known that μ automatically satisfies certain regularity properties in this situation, which implies that μ is uniquely determined by λ_μ on $C(\mathbf{T})$. In fact, $|\mu|(\mathbf{T})$ is equal to the dual norm of λ_μ with respect to the supremum norm on $C(\mathbf{T})$. It is also well known that every bounded linear functional on $C(\mathbf{T})$ with respect to the supremum norm corresponds to a complex Borel measure on $\mathbf{T}$ in this way. In order to show that μ is uniquely determined by its Fourier coefficients, it suffices to show that λ_μ is uniquely determined as a linear functional on $C(\mathbf{T})$ by the Fourier coefficients of μ.

Note that (15.2) is the same as λ_μ applied to $\overline{z}^j$ as a complex-valued continuous function on $\mathbf{T}$ for each $j \in \mathbf{Z}$. If $\widehat{\mu}(j) = 0$ for each $j \in \mathbf{Z}$, then it follows that

$$\lambda_\mu(\phi) = 0 \tag{15.6}$$

when ϕ is of the form $\overline{z}^j$ for some $j \in \mathbf{Z}$. This implies that (15.6) also holds when ϕ is a linear combination of finitely many such functions, by linearity. It is well known that these linear combinations are dense in $C(\mathbf{T})$ with respect to the supremum norm. Under these conditions, we get that (15.6) holds for every $\phi \in C(\mathbf{T})$, because λ_μ is bounded on $C(\mathbf{T})$ with respect to the supremum norm, as desired.

Let ν be the complex Borel measure on $\mathbf{T}$ defined by putting

$$\nu(E) = \int_E z \, d\mu(z) \tag{15.7}$$

for every Borel set $E \subseteq \mathbf{T}$. It is well known that

$$\int_{\mathbf{T}} \phi(z) \, d\nu(z) = \int_{\mathbf{T}} \phi(z) \, z \, d\mu(z) \tag{15.8}$$

for every bounded complex-valued Borel measurable function ϕ on $\mathbf{T}$. In particular, for each $j \in \mathbf{Z}$, we have that

$$\widehat{\nu}(j) = \int_{\mathbf{T}} \overline{z}^j \, d\nu(z) = \int_{\mathbf{T}} \overline{z}^{j-1} \, d\mu(z) = \widehat{\mu}(j-1). \tag{15.9}$$

Note that $|\nu| = |\mu|$ as nonnegative Borel measures on $\mathbf{T}$.

16 Fourier series, continued

Let us continue to take $k = \mathbf{C}$ with the standard absolute value function. Of course, there is an analogue of Lebesgue measure on the unit circle $\mathbf{T}$, which corresponds to ordinary Lebesgue measure on $[0, 2\pi)$ with respect to an arclength parameterization of $\mathbf{T}$. If f is a complex-valued integrable function on $\mathbf{T}$ with respect to this measure, then the jth *Fourier coefficient* of f is defined for each $j \in \mathbf{Z}$ by

$$\widehat{f}(j) = \frac{1}{2\pi} \int_{\mathbf{T}} f(z)\, \overline{z}^j \, |dz|, \tag{16.1}$$

where $|dz|$ denotes the arclength measure on $\mathbf{T}$ just mentioned. In this case, we have that

$$|\widehat{f}(j)| \le \frac{1}{2\pi} \int_{\mathbf{T}} |f(z)| \, |dz| \tag{16.2}$$

for every $j \in \mathbf{Z}$, and

$$\lim_{|j| \to \infty} \widehat{f}(j) = 0, \tag{16.3}$$

by the Riemann–Lebesgue lemma. If we put

$$f_1(z) = z\, f(z) \tag{16.4}$$

for each $z \in \mathbf{T}$, then f_1 is integrable on $\mathbf{T}$ too, and

$$\widehat{f_1}(j) = \frac{1}{2\pi} \int_{\mathbf{T}} f(z)\, \overline{z}^{j-1} = \widehat{f}(j-1) \tag{16.5}$$

for every $j \in \mathbf{Z}$.

In this situation,

$$\mu_f(E) = \frac{1}{2\pi} \int_E f(z)\, |dz| \tag{16.6}$$

defines a complex Borel measure on $\mathbf{T}$, with

$$|\mu_f|(E) = \frac{1}{2\pi} \int_E |f(z)|\, |dz| \tag{16.7}$$

for every Borel set $E \subseteq \mathbf{T}$. If ϕ is a bounded complex-valued Borel measurable function on $\mathbf{T}$, then

$$\int_{\mathbf{T}} \phi \, d\mu_f = \int_{\mathbf{T}} \phi(z)\, f(z)\, |dz|. \tag{16.8}$$

In particular,

$$\widehat{\mu_f}(j) = \widehat{f}(j) \tag{16.9}$$

for each $j \in \mathbf{Z}$, where $\widehat{\mu_f}$ is as defined in (15.2). Similarly, (16.2) corresponds to (15.3) with $\mu = \mu_f$, μ_{f_1} corresponds to (15.7), and (16.5) corresponds to (15.9). Note that f is uniquely determined by its Fourier coefficients, as in the previous section.

Let $L^2(\mathbf{T})$ be the space of complex-valued square-integrable functions on $\mathbf{T}$ with respect to arclength measure. This is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \langle f, g \rangle_{L^2(\mathbf{T})} = \frac{1}{2\pi} \int_{\mathbf{T}} f(z) \, \overline{g(z)} \, |dz|, \tag{16.10}$$

and the corresponding norm

$$\|f\|_2 = \|f\|_{L^2(\mathbf{T})} = \Big(\frac{1}{2\pi} \int_{\mathbf{T}} |f(z)|^2 \, |dz|\Big)^{1/2}. \tag{16.11}$$

It is well known that

$$\int_{\mathbf{T}} z^j \, |dz| = 0 \tag{16.12}$$

for every $j \in \mathbf{Z}$ with $j \neq 0$, which implies that the functions z^j on $\mathbf{T}$ with $j \in \mathbf{Z}$ are orthonormal with respect to (16.10). If $f \in L^2(\mathbf{T})$, then $\widehat{f}(j)$ is the same as the inner product of f with z^j with respect to (16.10) for each $j \in \mathbf{Z}$. This implies that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2 \le \frac{1}{2\pi} \int_{\mathbf{T}} |f(z)|^2 \, |dz|, \tag{16.13}$$

as in (14.6). It is well known that the linear span of the functions z^j with $j \in \mathbf{Z}$ is dense in $L^2(\mathbf{T})$, since it is dense in $C(\mathbf{T})$ with respect to the supremum norm. Thus we actually have that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2 = \frac{1}{2\,\pi} \int_{\mathbf{T}} |f(z)|^2 \tag{16.14}$$

for every $f \in L^2(\mathbf{T})$, as in Section 14.

Let $\{a_j\}_{j=-\infty}^{\infty}$ be a doubly-infinite sequence of complex numbers that is square-summable, in the sense that

$$\sum_{j=-\infty}^{\infty} |a_j|^2 < \infty. \tag{16.15}$$

Thus $\{a_j\}_{j=-\infty}^{\infty}$ corresponds to an element of $\ell^2(\mathbf{Z}, \mathbf{C})$, and

$$f(z) = \sum_{j=-\infty}^{\infty} a_j \, z^j \tag{16.16}$$

determines an element of $L^2(\mathbf{T})$, because of the orthonormality of the functions z^j in $L^2(\mathbf{T})$, as in Sections 12 and 14. By construction,

$$\widehat{f}(j) = a_j \tag{16.17}$$

for each $j \in \mathbf{Z}$, and

$$\sum_{j=-\infty}^{\infty} a_{j-1} \, z^j = \sum_{j=-\infty}^{\infty} a_j \, z^{j+1} = z \, f(z). \tag{16.18}$$

If $\{a_j\}_{j=-\infty}^{\infty}$ is summable, in the sense that

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty, \tag{16.19}$$

then the right side of (16.16) can be treated as the sum of a summable complex-valued function on $\mathbf{Z}$ for each $z \in \mathbf{T}$, as in Section 10. In this case, the right side of (16.16) can also be considered as the sum of two absolute convergent series in $C(\mathbf{T})$ with respect to the supremum norm, as in Section 12.

17 Some bounded linear functionals

Let us begin with the case where $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. Let $(V, \langle v, w \rangle)$ be a real or complex inner product space. If $w \in V$, then

$$\lambda_w(v) = \langle v, w \rangle \tag{17.1}$$

defines a linear functional on V. This linear functional is bounded on V with respect to the norm $\|v\|$ corresponding to the inner product, because

$$|\lambda_w(v)| \le \|v\| \, \|w\| \tag{17.2}$$

for every $v \in V$, by the Cauchy–Schwarz inequality. More precisely, this implies that the corresponding dual norm of λ_w is less than or equal to $\|w\|$. It is easy to see that the dual norm of λ_w is equal to $\|w\|$, since

$$\lambda_w(w) = \langle w, w \rangle = \|w\|^2. \tag{17.3}$$

If V is a Hilbert space, then it is well known that every bounded linear functional on V is of this form. Note that this representation is unique, because $\lambda_w \ne 0$ when $w \ne 0$.

Let X be a nonempty set, and let us use the notation $c_{00}(X)$, $c_0(X)$, and $\ell^r(X)$ for the usual spaces of real or complex-valued functions on X, as appropriate. Suppose that $1 \le r, r' \le \infty$ are conjugate exponents, in the sense that

$$1/r + 1/r' = 1. \tag{17.4}$$

If $f \in \ell^r(X)$ and $g \in \ell^{r'}(X)$, then Hölder's inequality implies that their product $f(x)\, g(x)$ is summable on X, with

$$\sum_{x \in X} |f(x)| \, |g(x)| \le \|f\|_r \, \|g\|_{r'}. \tag{17.5}$$

Thus

$$\lambda_g(f) = \sum_{x \in X} f(x) \, g(x) \tag{17.6}$$

can be defined as a real or complex number, as appropriate, as in Section 10. We also have that

$$|\lambda_g(f)| \le \|f\|_r \, \|g\|_{r'}, \tag{17.7}$$

by (17.5). This implies that λ_g defines a bounded linear functional on $\ell^r(X)$ for each g in $\ell^{r'}(X)$, with dual norm less than or equal to $\|g\|_{r'}$. One can check that the dual norm of λ_g on $\ell^r(X)$ is equal to g, by considering suitable $f \in \ell^r(X)$.

If $1 \le r < \infty$, then it is well known that every bounded linear functional on $\ell^r(X)$ is of this form. Remember that any linear functional on $c_{00}(X)$ can be expressed as (17.6) for some real or complex-valued function g on X, as apprpriate, and as in Section 2. If this linear functional is bounded on $c_{00}(X)$ with respect to $\|f\|_r$, then one can verify that $g \in \ell^{r'}(X)$. More precisely, one can show that $\|g\|_{r'}$ is less than the corresponding dual norm of λ_g on $c_{00}(X)$ with respect to $\|f\|_r$ on $c_{00}(X)$, by considering suitable $f \in c_{00}(X)$, as in the preceding paragraph. It follows that λ_g defines a bounded linear functional on $\ell^r(X)$, as in the previous paragraph again. If $1 \le r < \infty$, then $c_{00}(X)$ is dense in $\ell^r(X)$, and the original bounded linear functional on $\ell^r(X)$ is equal to λ_g on all of $\ell^r(X)$. Essentially the same argument shows that a bounded linear functional on $c_0(X)$ with respect to the supremum norm can be expressed as (17.6) for some $g \in \ell^1(X)$. Note that the case where $r = r' = 2$ also corresponds to the earlier discussion of Hilbert spaces.

Suppose now that $g \in \ell^\infty(X)$, so that λ_g as in (17.6) defines a bounded linear functional on $\ell^1(X)$. In this case, (17.5) is very simple, and one can get that the dual norm of λ_g on $\ell^1(X)$ is equal to $\|g\|_\infty$ using (2.6). If $0 < r \le 1$, then $\ell^r(X)$ is contained in $\ell^1(X)$, and

$$\|f\|_1 \le \|f\|_r \tag{17.8}$$

for every $f \in \ell^r(X)$, as in (8.4) and (8.5). This implies that the restriction of λ_g to $\ell^r(X)$ defines a bounded linear functional on $\ell^r(X)$, with dual norm less than or equal to $\|g\|_\infty$. One can check that the dual norm of λ_g on $\ell^r(X)$ is equal to $\|g\|_\infty$ when $0 < r \le 1$, using (2.6) again.

It is well known that every bounded linear functional on $\ell^r(X)$ is of this form when $0 < r \le 1$. As before, the restriction of a bounded linear functional on $\ell^r(X)$ to $c_{00}(X)$ can be expressed as in (17.6) for some real or complex-valued function g on X. The boundedness of this linear functional on $c_{00}(X)$ with respect to $\|f\|_r$ implies that $g \in \ell^\infty(X)$, because of (2.6). This implies that λ_g extends to a bounded linear functional on $\ell^r(X)$ when $0 < r \le 1$, as in the preceding paragraph. The original bounded linear functional on $\ell^r(X)$ is equal to λ_g on $\ell^r(X)$, because $c_{00}(X)$ is dense in $\ell^r(X)$ when $r < \infty$.

Now let k be a field with an ultrametric absolute value function $|\cdot|$, and suppose that k is complete with respect to the associated ultrametric. If f, g are bounded k-valued functions on X and at least one of f, g vanishes at infinity on X, then it is easy to see that their product $f(x)\,g(x)$ vanishes at infinity on X too. This permits us to define $\lambda_g(f)$ as an element of k as in (17.6), as discussed in Section 13. We also get that

$$|\lambda_g(f)| \le \max_{x \in X}(|f(x)|\,|g(x)|) \le \|f\|_\infty \, \|g\|_\infty \tag{17.9}$$

where the first step is as in Section 13. This implies that λ_g defines a bounded linear functional on $c_0(X, k)$ with respect to the supremum ultranorm when g is in $\ell^\infty(X, k)$, and similarly that λ_g defines a bounded linear functional on $\ell^\infty(X, k)$ when $g \in c_0(X, k)$. In both cases, (17.9) implies that the corresponding dual norm of λ_g is less than or equal to $\|g\|_\infty$, and one can check that the dual norm of λ_g is equal to $\|g\|_\infty$, using (2.6). Of course, if X has only finitely many elements, then the completeness of k is not needed, and vanishing at infinity on X is trivial.

It is easy to see that every bounded linear functional on $c_0(X, k)$ is of this form, using the same type of argument as in the archimedean case. More precisely, every linear functional on $c_{00}(X, k)$ can be expressed as in (17.6) for some k-valued function g on X, as in Section 2. If this linear functional on $c_{00}(X, k)$ is bounded with respect to the supremum ultranorm, then one can verify that g is bounded on X, using (2.6) again. This implies that λ_g defines a bounded linear functional on $c_0(X, k)$ with respect to the supremum ultranorm, as in the preceding paragraph. The original bounded linear functional on $c_0(X, k)$ is equal to λ_g on all of $c_0(X, k)$, because $c_{00}(X, k)$ is dense in $c_0(X, k)$ with respect to the supremum norm.

Let g be any bounded k-valued function on X again, so that (17.6) defines a bounded linear functional λ_g on $c_0(X, k)$, as before. If r is a positive real number, then $\ell^r(X, k)$ is contained in $c_0(X, k)$, as in (8.11). We also have that

$$\|f\|_\infty \le \|f\|_r \tag{17.10}$$

for every $f \in \ell^r(X, k)$, as in (8.4). Hence the restriction of λ_g to $\ell^r(X, k)$ defines a bounded linear functional on $\ell^r(X, k)$, with dual norm less than or equal to $\|g\|_\infty$. As before, one can verify that the dual norm of λ_g on $\ell^r(X, k)$ is equal to $\|g\|_\infty$, using (2.6).

Every bounded linear functional on $\ell^r(X, k)$ is of this form when $0 < r < \infty$, by an argument like the one in the archimedean case. More precisely, any linear functional on $c_{00}(X, k)$ can be expressed as in (17.6) for some k-valued function g on X, as before. If this linear functional on $c_{00}(X, k)$ is bounded with respect to $\|f\|_r$, then g is bounded on X, because of (2.6). Thus λ_g extends to a bounded linear functional on $\ell^r(X, k)$ when $0 < r < \infty$, as in the preceding paragraph. It follows that the original linear functional on $\ell^r(X, k)$ is equal to λ_g on all of $\ell^r(X, k)$, because $c_{00}(X, k)$ is dense in $\ell^r(X, k)$ when $r < \infty$.

18 Some bounded linear mappings

Let k be a field, let X be a nonempty set, and let V be a vector space over k. Also let $a(x)$ be a V-valued function on X, and put

$$T_a(f) = \sum_{x \in X} f(x)\, a(x) \tag{18.1}$$

for every $f \in c_{00}(X, k)$. More precisely, the right side reduces to a finite sum in V, and so defines an element of V. Thus T_a defines a linear mapping from

$c_{00}(X, k)$ into V. Note that

$$T_a(\delta_y) = a(y) \tag{18.2}$$

for every $y \in X$, where $\delta_y \in c_{00}(X, k)$ is as in (2.2). It is easy to see that every linear mapping from $c_{00}(X, k)$ into V is of this form, since the δ_y's with $y \in X$ form a basis of $c_{00}(X, k)$ as a vector space over k. More precisely, a linear mapping T from $c_{00}(X, k)$ into V can be expressed as T_a, where $a(y) = T(\delta_y)$ for each $y \in X$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N_V be a q_V-norm on V with respect to $|\cdot|$ on k for some $q_V > 0$. Also let a be a V-valued function on X again, so that (18.1) defines a linear mapping from $c_{00}(X, k)$ into V, and let $0 < r \leq \infty$ be given. If T_a is bounded with respect to $\|f\|_r$ on $c_{00}(X, k)$ and N_V on V, then it is easy to see that $N_V(a(y))$ is less than or equal to the corresponding operator q-norm of V for every $y \in X$, because of (18.2). In the other direction, if $N_V(a(x))$ is bounded on X, and if $r \leq q_V$, then we have that

$$N_V(T_a(f)) \leq \Big(\sup_{x \in X} N_V(a(x))\Big) \|f\|_r \tag{18.3}$$

for every $f \in c_{00}(X, k)$. To see this, remember that N_V is an r-norm on V when $r \leq q_V$, as in Section 7. Using this, (18.3) follows easily from the definition of T_a and the r-norm version of the triangle inequality. Of course, (18.3) says that T_a is bounded with respect to $\|f\|_r$ on $c_{00}(X, k)$ and N_V on V, with operator norm less than or equal to $\sup_{x \in X} N_V(a(x))$. In this situation, the operator norm of T_a is equal to this supremum, since the reverse inequality holds for every $r > 0$, as before.

Suppose for the moment that V is complete with respect to the q_V-metric associated to N_V. Let us continue to ask that $N_V(a(x))$ be bounded on X, and that $0 < r \leq q_V$. If $r < \infty$, then $c_{00}(X, k)$ is dense in $\ell^r(X, k)$, and it follows that T_a extends to a bounded linear mapping from $\ell^r(X, k)$ into V, as in Section 13. Similarly, if $r = \infty$, then $c_{00}(X, k)$ is dense in $c_0(X, k)$ with respect to $\|f\|_\infty$, and T_a extends to a bounded linear mapping from $c_0(X, k)$ into V, using $\|f\|_\infty$ on $c_0(X, k)$. In these cases, the sum on the right side of (18.1) can be treated as an infinite series when $X = \mathbf{Z}_+$, or reduced to a sum of two infinite series when $X = \mathbf{Z}$.

If λ_x is a linear functional on V for each $x \in X$, then

$$v \mapsto \lambda_x(v) \tag{18.4}$$

defines a linear mapping from V into the space $c(X, k)$ of all k-valued functions on X. Clearly every linear mapping from V into $c(X, k)$ corresponds to a family of linear functionals on V indexed by X in this way. If λ_x is a bounded linear functional on V with respect to N_V for each $x \in X$, and if the corresponding dual norm of λ_x is uniformly bounded as a function of $x \in X$, then (18.4) defines a bounded linear mapping from V into $\ell^\infty(X, k)$. More precisely, the operator q_k-norm of this linear mapping from V into $\ell^\infty(X, k)$ is equal to the supremum of the dual norm of λ_x on V with respect to N_V over $x \in X$. As before, every bounded linear mapping from V into $\ell^\infty(X, k)$ corresponds to a family of bounded linear functionals on V indexed by X with uniformly bounded dual norms.

19 Multiplication operators

Let k be a field, and let X be a nonempty set. The space $c(X, k)$ of k-valued functions on X is a commutative algebra over k, with respect to pointwise multiplication of functions. The constant function $\mathbf{1}_X = \mathbf{1}_{X,k}$ on X equal to the multiplicative identity element $1 = 1_k$ in k at every point is the multiplicative identity element in $c(X, k)$. A k-valued function a has a multiplicative inverse in $c(X, k)$ if and only if $a(x) \neq 0$ for every $x \in X$, in which case the multiplicative inverse is given by $1/a(x)$. The space $c_{00}(X, k)$ of k-valued functions on X with finite support is an ideal in $c(X, k)$.

Put

$$M_a(f) = a\, f \tag{19.1}$$

for all k-valued functions a, f on X. This defines a linear mapping M_a from $c(X, k)$ into itself, which is the multiplication operator associated to a. Note that M_a sends $c_{00}(X, k)$ into itself. Clearly

$$M_a(\mathbf{1}_X) = a \tag{19.2}$$

for every $a \in c(X, k)$. If $y \in X$ and $\delta_y \in c_{00}(X, k)$ is as in (2.2), then

$$M_a(\delta_y) = a(y)\, \delta_y \tag{19.3}$$

for every $a \in c(X, k)$. The mapping

$$a \mapsto M_a \tag{19.4}$$

is an algebra homomorphism from $c(X, k)$ into the algebra of linear mappings from $c(X, k)$ into itself. In particular, this mapping sends $a = \mathbf{1}_X$ to the identity operator on $c(X, k)$.

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, so that $\ell^r(X, k)$ can be defined for each $r > 0$ as in Section 8. If a is a bounded k-valued function on X, then M_a maps $\ell^r(X, k)$ into itself for every $r > 0$. More precisely, the restriction of M_a to $\ell^r(X, k)$ is a bounded linear mapping with respect to the usual ℓ^r norm, and the corresponding operator norm of M_a is equal to the supremum norm $\|a\|_\infty$ of a. Similarly, M_a maps the space $c_0(X, k)$ of k-valued functions on X that vanish at infinity into itself when a is bounded on X. In this case, the operator norm of M_a with respect to the restriction of the supremum norm to $c_0(X, k)$ is equal to $\|a\|_\infty$ too.

Let us now take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. Let $(X, \mathcal{A}, \mu)$ be a measure space, so that X is a set, $\mathcal{A}$ is a σ-algebra of measurable subsets of X, and μ is a nonnegative measure defined on $\mathcal{A}$. The corresponding Lebesgue spaces $L^r(X)$ can be defined in the usual way for each $r > 0$. Remember that the L^r norm defines a norm on $L^r(X)$ in the usual sense when

$r \geq 1$, and it defines an r-norm on $L^r(X)$ when $0 < r \leq 1$. If a is an essentially bounded measurable real or complex-valued function on X, as appropriate, then the corresponding multiplication operator M_a defines a bounded linear mapping from $L^r(X)$ into itself for each $r > 0$. The operator norm of M_a on $L^r(X)$ is less than or equal to the essential supremum norm $\|a\|_\infty = \|a\|_{L^\infty(X)}$ of a on X for each $r > 0$. It is easy to see that the operator norm of M_a on $L^\infty(X)$ is equal to $\|a\|_\infty$. In order to get that the operator norm of M_a on $L^r(X)$ is equal to $\|a\|_\infty$ when $r < \infty$, an additional condition on X is needed. Namely, this works when every measurable set $A \subseteq X$ with $\mu(A) > 0$ has a measurable subset B such that $0 < \mu(B) < \infty$.

20 Dual linear mappings

Let k be a field with a q_k-absolute value function for some $q_k > 0$, let V be a vector space over k, and let N_V be a q_V-norm N_V on V with respect to $|\cdot|$ on k for some $q_V > 0$. The *dual space* V' associated to V is the space of bounded linear functionals on V, as in Section 9. This is the same as $\mathcal{BL}(V, k)$, where k is considered as a one-dimensional vector space over itself, and $|\cdot|$ is considered as a q_k-norm on k as a vector space. In particular, V' is a vector space over k, which may be considered as a linear subspace of the algebraic dual V^{alg}. The corresponding operator q_k-norm on V' is known as the dual q_k-norm, as before, and is denoted $N_{V'}$.

Let W be another vector space over k, and let N_W be a q_W-norm on W with respect to $|\cdot|$ on k for some $q_W > 0$. Also let T be a bounded linear mapping from V into W with respect to N_V and N_W, respectively. If λ is a bounded linear functional on W, then

$$T'(\lambda) = \lambda \circ T \tag{20.1}$$

defines a bounded linear functional on V. More precisely,

$$N_{V'}(T'(\lambda)) \leq \|T\|_{op,VW}\, N_{W'}(\lambda), \tag{20.2}$$

as in (9.13), where $\|T\|_{op,VW}$ is the corresponding operator q_W-norm of T, as in (9.3). It is easy to see that T' defines a linear mapping from W' into V', which is the same as the restriction of the algebraic dual linear mapping T^{alg} to V'. Using (20.2), we get that T' is bounded with respect to the corresponding dual norms, with

$$\|T'\|_{op,W'V'} \leq \|T\|_{op,VW}. \tag{20.3}$$

Observe that

$$T \mapsto T' \tag{20.4}$$

is a linear mapping from $\mathcal{BL}(V, W)$ into $\mathcal{BL}(W', V')$, as in (1.2). This linear mapping is bounded with respect to the corresponding operator norms, by (20.3).

Suppose for the moment that $V = W$, and that $N_V = N_W$. As in Section 9, the identity mapping I_V is a bounded linear mapping from V into itself. The corresponding dual mapping is the identity mapping $I_{V'}$ on V', so that

$$(I_V)' = I_{V'}, \tag{20.5}$$

as in (1.3).

Let Z be a third vector space over k, with a q_Z-norm N_Z with respect to $|\cdot|$ on k for some $q_Z > 0$. Suppose that T_1 and T_2 are bounded linear mappings from V into W and from W into Z, respectively, so that $T_2 \circ T_1$ is a bounded linear mapping from V into Z. Under these conditions, we have that

$$(T_2 \circ T_1)' = T_1' \circ T_2' \tag{20.6}$$

as linear mappings from Z' into V', as in (1.5). Suppose now that T is a one-to-one bounded linear mapping from V onto W whose inverse T^{-1} is bounded as a linear mapping from W onto V. In this case, one can check that T' is a one-to-one linear mapping from W' onto V'. The inverse of T' is given by

$$(T')^{-1} = (T^{-1})', \tag{20.7}$$

as linear mappings from V' into W', as in (1.3). In particular, the inverse of T' is bounded as a linear mapping from V' into W', with respect to the corresponding dual norms.

21 Hilbert space adjoints

In this section, $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be inner product spaces, both real or both complex, with corresponding norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. If $u \in W$, then

$$\|u\|_W = \sup\{|\langle u, w\rangle_W| : w \in W,\ \|w\|_W \le 1\}. \tag{21.1}$$

More precisely, the right side of (21.1) is less than or equal to $\|u\|_W$, by the Cauchy–Schwarz inequality. The opposite inequality is trivial when $u = 0$, and otherwise can be obtained by taking $w = u/\|u\|_W$.

Let T be a linear mapping from V into W, and consider the following condition on T:

(21.2) $|\langle T(v), w\rangle_W|$ is bounded for $v \in V$ and $w \in W$ with $\|v\|_V, \|w\|_W \le 1$.

Of course, this means that

$$\sup\{|\langle T(v), w\rangle_W| : v \in V,\ w \in W,\ \|v\|_V, \|w\|_W \le 1\} \tag{21.3}$$

is defined as a nonnegative real number. Observe that

$$\|T(v)\|_W = \sup\{|\langle T(v), w\rangle_W| : w \in W,\ \|w\|_W \le 1\} \tag{21.4}$$

for every $v \in V$, by applying (21.1) to $u = T(v)$. Thus (21.2) is equivalent to following condition on T:

$$\|T(v)\|_W \text{ is bounded for } v \in V \text{ with } \|v\|_V \le 1. \tag{21.5}$$

In this case, (21.3) is equal to

$$\sup\{\|T(v)\|_W : v \in V,\ \|v\|_V \le 1\}, \tag{21.6}$$

because of (21.4). Note that (21.5) corresponds to (9.8) in this situation, and that (21.6) corresponds to (9.9). Thus T is bounded as a linear mapping from V into W if and only if (21.5) holds, as in Section 9, in which case the operator norm $\|T\|_{op,VW}$ of T is equal to (21.6). It follows that (21.2) holds if and only if T is bounded as a linear mapping from V into W, in which case $\|T\|_{op,VW}$ is equal to (21.3).

Let T be a bounded linear mapping from V into W, and observe that

$$|\langle T(v), w\rangle_W| \le \|T(v)\|_W \,\|w\|_W \le \|T\|_{op,VW} \,\|v\|_V \,\|w\|_W \tag{21.7}$$

for every $v \in V$ and $w \in W$, using the Cauchy–Schwarz inequality in the first step. This implies that

$$v \mapsto \langle T(v), w\rangle_W \tag{21.8}$$

is a bounded linear functional on V for each $w \in W$, with dual norm less than or equal to $\|T\|_{op,VW} \,\|w\|_W$. If V is a Hilbert space, then for each $w \in W$ there is a unique element $T^*(w)$ of V such that

$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \tag{21.9}$$

for every $v \in V$, as in Section 17. One can check that T^* is a linear mapping from W into V, because $T^*(w)$ is uniquely determined by (21.9). We also have that

$$\|T^*(w)\|_V \le \|T\|_{op,VW} \,\|w\|_W \tag{21.10}$$

for every $w \in W$, because of the corresponding bound for the dual norm of (21.8). This implies that T^* is bounded as a linear mapping from W into V, with operator norm less than or equal to $\|T\|_{op,VW}$. In fact,

$$\|T^*\|_{op,WV} = \|T\|_{op,VW}, \tag{21.11}$$

because of (21.9) and the characterization of $\|T\|_{op,VW}$ as (21.3).

Let us continue to ask that V be a Hilbert space, so that

$$T \mapsto T^* \tag{21.12}$$

defines a mapping from $\mathcal{BL}(V, W)$ into $\mathcal{BL}(W, V)$. This mapping is linear in the real case, and conjugate-linear in the complex case. Of course, if W is a Hilbert space, then the adjoint of a bounded linear mapping from W into V is defined as a bounded linear mapping from V into W in the same way. If V and W are

Hilbert spaces, and T is a bounded linear mapping from V into W, then T^* is defined as a bounded linear mapping from W into V, and the adjoint $(T^*)^*$ of T^* is defined as a bounded linear mapping from V into W. It is easy to see that

$$(T^*)^* = T, \tag{21.13}$$

directly from the definitions.

Let us suppose that V and W are Hilbert spaces for the rest of the section. Remember that the identity mapping I_V on V is bounded as a linear mapping from V into itself. Observe that

$$I_V^* = I_V \tag{21.14}$$

as a bounded linear mapping from V into itself.

Let $(Z, \langle \cdot, \cdot \rangle_Z)$ be another Hilbert space, which is real or complex depending on whether V and W are real or complex. Also let T_1 be a bounded linear mapping from V into W, and let T_2 be a bounded linear mapping from W into Z. Thus the adjoint T_1^* of T_1 is defined as a bounded linear mapping from W into V, and the adjoint T_2^* of T_2 is defined as a bounded linear mapping from Z into W. If $v \in V$ and $z \in Z$, then we have that

$$\langle T_2(T_1(v)), z \rangle_Z = \langle T_1(v), T_2^*(z) \rangle_W = \langle v, T_1^*(T_2^*(z)) \rangle_V, \tag{21.15}$$

using the definition of T_2^* in the first step, and the definition of T_1^* in the second step. This implies that

$$(T_2 \circ T_1)^* = T_1^* \circ T_2^* \tag{21.16}$$

as bounded linear mappings from Z into V.

Suppose that T is a one-to-one bounded linear mapping from V onto W whose inverse is bounded as a linear mapping from W onto V. Thus T^* is defined as a bounded linear mapping from W into V, and $(T^{-1})^*$ is defined as a bounded linear mapping from V into W. Observe that

$$T^* \circ (T^{-1})^* = (T^{-1} \circ T)^* = I_V^* = I_V, \tag{21.17}$$

using (21.16) in the first step, and (21.14) in the third step. Similarly,

$$(T^{-1})^* \circ T^* = (T \circ T^{-1})^* = I_W^* = I_W. \tag{21.18}$$

This implies that T^* is invertible as a bounded linear mapping from W into V, with

$$(T^*)^{-1} = (T^{-1})^*. \tag{21.19}$$

Let T be any bounded linear mapping from V into W again. Remember that T is an isometry if and only if

$$\langle T(v), T(v') \rangle_W = \langle v, v' \rangle_V \tag{21.20}$$

for every $v, v' \in V$, as in (11.8). This condition is equivalent to asking that

$$\langle v, T^*(T(v')) \rangle_V = \langle v, v' \rangle \tag{21.21}$$

for every $v, v' \in V$, by definition of the adjoint. It is easy to see that (21.21) holds if and only if

$$T^* \circ T = I_V \tag{21.22}$$

as linear mappings from V into itself. Thus T is an isometry from V into W if and only if (21.22) holds.

Suppose now that T is a one-to-one bounded linear mapping from V onto W. In this case, (21.22) is the same as saying that

$$T^{-1} = T^*. \tag{21.23}$$

Thus T is a unitary mapping from V onto W if and only if (21.23) holds.

Part II
Invertibility and related topics

22 Invertibility of multiplication operators

Let k be a field, let X be a nonempty set, and let a be a k-valued function on X. Thus

$$M_a(f) = a\, f \tag{22.1}$$

defines a linear mapping from the space $c(X, k)$ of k-valued functions on X into itself, as in Section 19. If $a(x) \neq 0$ for every $x \in X$, then M_a is invertible on $c(X, k)$, with inverse equal to the multiplication operator $M_{1/a}$ associated to $1/a$. If a is any k-valued function on X, then the kernel of M_a on $c(X, k)$ consists of the k-valued functions f on X such that

$$\operatorname{supp} f \subseteq \{x \in X : a(x) = 0\}. \tag{22.2}$$

We also have that

$$\operatorname{supp} M_a(f) = \operatorname{supp}(a\, f) = (\operatorname{supp} a) \cap (\operatorname{supp} f) \subseteq \operatorname{supp} a \tag{22.3}$$

for every $f \in c(X, k)$. More precisely, M_a maps $c(X, k)$ onto the space of k-valued functions on X whose support is contained in the support of a. There are analogous statements for M_a as a linear mapping from $c_{00}(X, k)$ into itself.

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, and let a be a bounded k-valued function on X. Thus M_a defines a bounded linear mapping from $\ell^r(X, k)$ into itself for each $r > 0$, as in Section 19. As before, the kernel of M_a on $\ell^r(X, k)$ consists of the $f \in \ell^r(X, k)$ that satisfy (22.2). Using (22.3), we get that M_a maps $\ell^r(X, k)$ into the subspace of functions whose support is contained in the support of a. Remember that M_a maps $c_0(X, k)$ into itself as well under these conditions, for which there are analogous statements.

If $a(x) \neq 0$ for each $x \in X$, and if $1/a$ is bounded on X, then M_a has a bounded inverse on $\ell^r(X, k)$ for every $r > 0$, which is given by $M_{1/a}$. In this case, $M_{1/a}$ maps $c_0(X, k)$ into itself too, as before. In the other direction, suppose that for some $r > 0$ there is a positive real number c such that

$$\|M_a(f)\|_r \ge c\,\|f\|_r \tag{22.4}$$

for every $f \in \ell^r(X, k)$. This implies that

$$|a(y)| \ge c \tag{22.5}$$

for every $y \in X$, by taking f to be the function δ_y on X that is equal to 1 at y and to 0 on $X \setminus \{y\}$. In particular, this works as well when (22.4) holds for every $f \in c_0(X, k)$ and $r = \infty$.

Let us now take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function, and let $(X, \mathcal{A}, \mu)$ be a measure space. Also let a be an essentially bounded measurable real or complex-valued function on X, so that the corresponding multiplication operator M_a defines a bounded linear mapping from $L^r(X)$ into itself for every $r > 0$, as in Section 19. The kernel of M_a on $L^r(X)$ consists of the $f \in L^r(X)$ such that $f(x) = 0$ for μ-almost every $x \in X$ such that $a(x) \neq 0$. Similarly, M_a maps $L^r(X)$ into the subspace of functions that are equal to 0 for μ-almost every $x \in X$ such that $a(x) = 0$. If $a(x) \neq 0$ for μ-almost every $x \in X$, and if $1/a$ is essentially bounded on X, then M_a has a bounded inverse on $L^r(X)$, which is given by $M_{1/a}$. In the other direction, suppose that (22.4) holds for some $r > 0$, $c \in \mathbf{R}_+$, and every $f \in L^r(X)$. If $r = \infty$, then one can check that (22.5) holds for μ-almost every $y \in X$. If every measurable subset of X of positive measure contains a measurable subset of positive finite measure, then the analogous argument works when $r < \infty$ too.

23 The usual series

Let k be a field, and let V be a vector space over k. Remember that the space $\mathcal{L}(V)$ of linear mappings from V into itself is an associative algebra with respect to composition of operators, which may also be expressed using the usual notation for multiplication. Let T be a linear mapping from V into itself, and let T^j be the jth power of T with respect to composition for each positive integer j. This is interpreted as being the identity operator $I = I_V$ on V when $j = 0$. It is well known and easy to see that

$$(I - T)\sum_{j=0}^{l} T^j = \Big(\sum_{j=0}^{l} T^j\Big)(I - T) = I - T^{l+1} \tag{23.1}$$

for each nonnegative integer l.

Suppose now that $|\cdot|$ is a q_k-absolute value function on k for some $q_k > 0$, and that N_V is a q_V-norm on V with respect to $|\cdot|$ on k for some $q_V > 0$. As in Section 9, $\mathcal{BL}(V)$ is the algebra of bounded linear mappings on V with respect to N_V, and $\|T\|_{op}$ denotes the corresponding operator q_V-norm. If T is a bounded linear mapping from V into itself, then

$$\|T^j\|_{op} \le \|T\|_{op}^j \tag{23.2}$$

for every $j \geq 1$, by (9.13). This also works when $j = 0$, with the right side interpreted as being equal to 1. Suppose from now on in this section that

$$\|T\|_{op} < 1, \tag{23.3}$$

so that

$$\lim_{j \to \infty} \|T^j\|_{op} = 0, \tag{23.4}$$

by (23.2). If q is any positive real number, then $\|T\|_{op}^q < 1$, and

$$\sum_{j=0}^{\infty} \|T^j\|_{op}^q \leq \sum_{j=0}^{\infty} \|T\|_{op}^{q\,j} = (1 - \|T\|_{op}^q)^{-1}, \tag{23.5}$$

by summing the geometric series in the second step. In particular, the sum on the left converges as an infinite series of nonnegative real numbers under these conditions.

Suppose for the rest of the section that V is complete with respect to the q_V-metric associated to N_V. This implies that $\mathcal{BL}(V)$ is complete with respect to the q_V-metric associated to $\|\cdot\|_{op}$, as in Section 9. Suppose for the moment that $q_V < \infty$, so that $\sum_{j=0}^{\infty} T^j$ converges q_V-absolutely with respect to $\|\cdot\|_{op}$, as in the previous paragraph. This implies that $\sum_{j=0}^{\infty} T^j$ converges in $\mathcal{BL}(V)$ with respect to $\|\cdot\|_{op}$, as in Section 12, because of completeness. We also get that

$$(I - T) \sum_{j=0}^{\infty} T^j = \Big(\sum_{j=0}^{\infty} T^j\Big) (I - T) = I, \tag{23.6}$$

by taking the limit as $l \to \infty$ in (23.1). It follows that $I - T$ is invertible on V, with

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j. \tag{23.7}$$

Note that

$$\|(I - T)^{-1}\|_{op}^{q_V} = \Big\|\sum_{j=0}^{\infty} T^j\Big\|_{op}^{q_V} \leq \sum_{j=0}^{\infty} \|T^j\|_{op}^{q_V} \leq (1 - \|T\|_{op}^{q_V})^{-1}, \tag{23.8}$$

using (12.3) in the second step, and (23.5) in the third step.

If $q_V = \infty$, then (23.4) implies that $\sum_{j=0}^{\infty} T^j$ converges in $\mathcal{BL}(V)$ with respect to $\|\cdot\|_{op}$, as in Section 12 again. It follows that $I - T$ is invertible on V, with inverse given as in (23.7), as before. In this case, we have that

$$\|(I - T)^{-1}\|_{op} = \Big\|\sum_{j=0}^{\infty} T^j\Big\|_{op} \leq \sup_{j \geq 0} \|T^j\|_{op} \leq 1, \tag{23.9}$$

using (12.5) in the second step. We also have that

$$\|I - T\|_{op} \leq \max(\|I\|_{op}, \|T\|_{op}) \leq 1, \tag{23.10}$$

because $\|\cdot\|_{op}$ is an ultranorm on $\mathcal{BL}(V)$. Thus $I - T$ is an isometry from V onto itself in this situation, as in Section 9.

24 Some consequences

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q_V-norm N_V with respect to $|\cdot|$ on k for some $q_V > 0$. Thus $\|\cdot\|_{op}$ defines a q_V-norm on $\mathcal{BL}(V)$, as in Section 9. Suppose that V is complete with respect to the q_V-metric associated to N_V, so that $\mathcal{BL}(V)$ is complete with respect to the q_V-metric associated to $\|\cdot\|_{op}$, as before.

Let T_1, T_2 be bounded linear mappings from V into itself, and suppose that T_1 has a bounded inverse on V. Observe that

$$T_1 - T_2 = T_1 \circ (I - T_1^{-1} \circ T_2). \tag{24.1}$$

If

$$\|T_1^{-1} \circ T_2\|_{op} < 1, \tag{24.2}$$

then $I - T_2 \circ T_1^{-1}$ has a bounded inverse on V, as in the previous section. This implies that $T_1 - T_2$ has a bounded inverse on V, with

$$(T_1 - T_2)^{-1} = (I - T_1^{-1} \circ T_2)^{-1} \circ T_1^{-1}, \tag{24.3}$$

by (24.1).

Under these conditions, we have that

$$\|(T_1 - T_2)^{-1}\|_{op} \le \|(I - T_1^{-1} \circ T_2)^{-1}\|_{op} \, \|T_1^{-1}\|_{op}, \tag{24.4}$$

as in (9.13). If $q_V < \infty$, then it follows that

$$\|(T_1 - T_2)^{-1}\|_{op} \le (1 - \|T_1^{-1} \circ T_2\|_{op}^{q_V})^{-1/q_V} \, \|T_1^{-1}\|_{op}, \tag{24.5}$$

because of (23.8). If $q_V = \infty$, then $I - T_1^{-1} \circ T_2$ is an isometry from V onto itself, as in the previous section. This means that $(I - T_1^{-1} \circ T_2)^{-1}$ is an isometry from V onto itself too. In particular, this implies that

$$\|(T_1 - T_2)^{-1}\|_{op} = \|T_1^{-1}\|_{op}, \tag{24.6}$$

using (24.3) again.

Note that

$$\|T_1^{-1} \circ T_2\|_{op} \le \|T_1^{-1}\|_{op} \, \|T_2\|_{op}, \tag{24.7}$$

as in (9.13). Thus (24.2) holds when the right side of (24.7) is strictly less than 1. In this case, (24.5) implies that

$$\|(T_1 - T_2)^{-1}\|_{op} \le (1 - \|T_1^{-1}\|_{op}^{q_V} \, \|T_2\|_{op}^{q_V})^{-1/q_V} \, \|T_1^{-1}\|_{op} \tag{24.8}$$

when $q_V < \infty$.

Suppose now that T_1 is an isometry from V onto itself, so that T_1^{-1} is an isometry on V as well. This implies that

$$\|T_1^{-1} \circ T_2\|_{op} = \|T_2\|_{op}. \tag{24.9}$$

If $\|T_2\|_{op} < 1$ and $q_V = \infty$, then

$$T_1 - T_2 \text{ is an isometry from } V \text{ onto itself,} \tag{24.10}$$

because of (24.1) and the fact that $I - T_1^{-1} \circ T_2$ is an isometry on V, as before.

Let T be a bounded linear mapping on V, and let $a \in k$ be given. Suppose for the moment that $a \neq 0$, so that

$$a\, I - T = a\, (I - a^{-1}\, T). \tag{24.11}$$

If

$$\|T\|_{op} < |a|, \tag{24.12}$$

then

$$\|a^{-1}\, T\|_{op} = |a|^{-1}\, \|T\|_{op} < 1, \tag{24.13}$$

and $I - a^{-1}\, T$ has a bounded inverse on V, as in the previous section. This implies that $a\, I - T$ has a bounded inverse on V, with

$$(a\, I - T)^{-1} = a^{-1}\, (I - a^{-1}\, T)^{-1}, \tag{24.14}$$

by (24.11). If $q_V < \infty$, then it follows that

$$\begin{aligned} \|(a\, I - T)^{-1}\|_{op} &= |a|^{-1}\, \|(I - a^{-1}\, T)^{-1}\|_{op} \\ &\leq |a|^{-1}\, (1 - |a|^{-q_V}\, \|T\|_{op}^{q_V})^{-1/q_V}, \end{aligned} \tag{24.15}$$

because of (23.8). If $q_V = \infty$, then $I - a^{-1}\, T$ is an isometry from V onto itself, as before. Of course, this implies that $(I - a^{-1}\, T)^{-1}$ is an isometry on V too.

Suppose that T has a bounded inverse on V, and let $a \in k$ be given again. Observe that

$$(a\, I - T) = -T \circ (I - a\, T^{-1}). \tag{24.16}$$

If

$$\|a\, T^{-1}\|_{op} = |a|\, \|T^{-1}\|_{op} < 1, \tag{24.17}$$

then $I - a\, T^{-1}$ has a bounded inverse on V, as in the previous section. This implies that $a\, I - T$ has a bounded inverse on V, with

$$(a\, I - T)^{-1} = -(I - a\, T^{-1})^{-1} \circ T^{-1}, \tag{24.18}$$

because of (24.16). If $q_V < \infty$, then

$$\begin{aligned} \|(a\, I - T)^{-1}\|_{op} &\leq \|(I - a\, T^{-1})^{-1}\|_{op}\, \|T^{-1}\|_{op} \\ &\leq (1 - |a|^{q_V}\, \|T^{-1}\|_{op}^{q_V})^{-1/q_V}\, \|T^{-1}\|_{op}, \end{aligned} \tag{24.19}$$

by (9.13) and (23.8). If $q_V = \infty$, then $I - a\, T^{-1}$ is an isometry on V, as before, so that $(I - a\, T^{-1})^{-1}$ is an isometry on V as well. This implies that

$$\|(a\, I - T)^{-1}\|_{op} = \|T^{-1}\|_{op}, \tag{24.20}$$

using (24.18).

Suppose now that T is an isometry from V onto itself, so that T^{-1} is an isometry on V too. In this case, the remarks in the previous two paragraphs imply that $a\, I - T$ has a bounded inverse on V when $|a| \neq 1$. If $|a| < 1$ and $q_V = \infty$, then $a\, I - T$ is an isometry on V, because of (24.16) and the fact that $I - a\, T^{-1}$ is an isometry on V, as before.

25 Eigenvalues and eigenvectors

Let k be a field, let V be a vector space over k, and let T be a linear mapping from V into itself. As usual, $a \in k$ is said to be an *eigenvalue* of T on V if there is a $v \in V$ such that $v \neq 0$ and

$$T(v) = a\,v. \tag{25.1}$$

In this case, v is said to be an *eigenvector* of T on V with respect to the eigenvalue a. Of course, $v = 0$ satisfies (25.1) trivially, and one may include $v = 0$ as an eigenvector associated to an eigenvalue a.

The collection of $v \in V$ that satisfy (25.1) for a given $a \in k$ is the same as the kernel of $a\,I - T$ on V. Thus $a \in k$ is an eigenvalue of T on V if and only if the kernel of $a\,I - T$ on V is not trivial, which means that $a\,I - V$ is not one-to-one on V. In particular, this implies that $a\,I - V$ is not invertible as a linear mapping on V. If V has finite dimension, then it is well known that a one-to-one linear mapping from V into itself is surjective, and hence invertible. In this case, $a \in k$ is an eigenvalue of T if and only if $a\,I - T$ is not invertible on V.

Let V^{alg} be the algebraic dual of V, and let T^{alg} be the dual linear mapping from V^{alg} into itself corresponding to T, as in Section 1. If $a \in k$ is an eigenvalue of T^{alg} on V^{alg}, then there is a nonzero linear functional λ on V such that

$$\lambda \circ T = T^{\mathrm{alg}}(\lambda) = a\,\lambda. \tag{25.2}$$

Equivalently, this means that

$$\lambda \circ (a\,I - T) = 0 \tag{25.3}$$

as a linear functional on V, so that $a\,I - T$ maps V into the kernel of λ. The condition that $\lambda \neq 0$ as a linear functional on V means that the kernel of λ is a proper linear subspace of V. It follows that $a\,I - T$ does not map V onto itself when a is an eigenvalue of T^{alg}.

Suppose for the moment that T is a one-to-one linear mapping from V onto itself, so that T is invertible as a linear mapping on V. In particular, this implies that 0 is not an eigenvalue of T on V. Let $a \in k$ with $a \neq 0$ be given, and observe that (25.1) holds for some $v \in V$ if and only if

$$T^{-1}(v) = (1/a)\,v. \tag{25.4}$$

Thus a is an eigenvalue for T on V if and only if $1/a$ is an eigenvalue for T^{-1} on V. In this case, the corresponding eigenspaces in V are the same.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N_V be a q_V-norm on V with respect to $|\cdot|$ on k for some $q_V > 0$. This leads to an operator q_V-norm $\|\cdot\|_{op}$ on $\mathcal{BL}(V)$, as in Section 9. Suppose now that T is a bounded linear mapping from V into itself with respect to N_V. If $a \in k$ and $v \in V$ satisfy (25.1), then we get that

$$|a|\,N_V(v) = N_V(a\,v) = N_V(T(v)) \leq \|T\|_{op}\,N_V(v). \tag{25.5}$$

This implies that

$$(25.6) \qquad |a| \le \|T\|_{op}$$

when $v \neq 0$.

Similarly, suppose that

$$(25.7) \qquad c\,N_V(v) \le N_V(T(v))$$

for some positive real number c and every $v \in V$. If $a \in k$ and $v \in V$ satisfy (25.1), then we have that

$$(25.8) \qquad c\,N_V(v) \le N_V(T(v)) = N_V(a\,v) = |a|\,N_V(v).$$

It follows that

$$(25.9) \qquad c \le |a|$$

when $v \neq 0$. In particular, if T has a bounded inverse on V, then (25.7) holds with $c = 1/\|T^{-1}\|_{op}$, as in Section 9.

Suppose now that T is an isometric linear mapping from V into itself. This implies that T is a bounded linear mapping on V, with $\|T\|_{op} = 1$ when $V \neq \{0\}$. In this case, (25.7) holds with $c = 1$. If $a \in k$ is an eigenvalue of T, then

$$(25.10) \qquad |a| = 1,$$

by (25.6) and (25.9).

Let V' be the dual space of bounded linear functionals on V, as in Section 20, with the corresponding dual q_k-norm $N_{V'}$. If T is any bounded linear mapping from V into itself, then the corresponding dual linear mapping T' is a bounded linear mapping from V' into itself, as before. Of course, V' is a linear subspace of V^{alg}, and T' is the same as the restriction of T^{alg} to V'. If $a \in k$ is an eigenvalue of T', then there is a nonzero bounded linear functional λ on V that satisfies (25.3). This is the same as saying that $a\,I - T$ maps V into the kernel of λ. Note that the kernel of λ is a closed linear subspace of V, because λ is continuous, and the kernel of λ is a proper linear subspace of V when $\lambda \neq 0$. Thus $a\,I - T$ maps V into a proper closed linear subspace of V when a is an eigenvalue of T'.

Now let $(V, \langle v, w\rangle_V)$ be a real or complex Hilbert space, and let T be a bounded linear mapping from V into itself. Thus the adjoint T^* of T is defined as a bounded linear mapping on V too, as in Section 21. Suppose that a is a real or complex number, as appropriate, which is an eigenvalue of T^*. Let $w \in V$ be an eigenvector associated to T^*, so that

$$(25.11) \qquad T^*(w) = a\,w.$$

If $v \in V$, then

$$(25.12) \qquad \langle T(v), w\rangle_V = \langle v, T^*(w)\rangle_V = \overline{a}\,\langle v, w\rangle_V,$$

using (21.9) in the first step. Of course, the complex-conjugate $\overline{a}$ of a is automatically equal to a in the real case. Equivalently,

$$(25.13) \qquad \langle(\overline{a}\,I - T)(v), w\rangle_V = \overline{a}\langle v, w\rangle_V - \langle T(v), w\rangle_V = 0$$

for every $v \in V$.

26 Eigenfunctions for bilateral shifts

Let k be a field, and let T be the forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3. Thus

$$(T(f))(j) = f(j-1) \tag{26.1}$$

for every k-valued function f on $\mathbf{Z}$ and $j \in \mathbf{Z}$, as before. Of course, T is one-to-one on $c(Z, k)$, so that 0 is not an eigenvalue of T. Let $a \in k$ with $a \neq 0$ be given, and put

$$e_a(j) = a^j \tag{26.2}$$

for each $j \in \mathbf{Z}$. Observe that

$$(T(e_a))(j) = e_a(j-1) = a^{j-1} = a^{-1}\, e_a(j) \tag{26.3}$$

for every $j \in \mathbf{Z}$, so that

$$T(e_a) = a^{-1}\, e_a \tag{26.4}$$

as elements of $c(\mathbf{Z}, k)$. This shows that e_a is an eigenvector of T on $c(\mathbf{Z}, k)$ with eigenvalue a^{-1}, and it is easy to see that every eigenvector of T on $c(\mathbf{Z}, k)$ with eigenvalue a^{-1} is a scalar multiple of e_a. Remember that T maps $c_{00}(\mathbf{Z}, k)$ into itself. As a mapping on $c_{00}(\mathbf{Z}, k)$, T has no eigenvalues.

Note that

$$T(f\, g) = T(f)\, T(g) \tag{26.5}$$

for every $f, g \in c(\mathbf{Z}, k)$. If $f \in c(\mathbf{Z}, k)$, $a \in k$, and $a \neq 0$, then it follows that

$$T(e_a\, f) = T(e_a)\, T(f) = a^{-1}\, e_a\, T(f), \tag{26.6}$$

by (26.4). This is the same as saying that

$$T \circ M_{e_a} = a^{-1}\, M_{e_a} \circ T, \tag{26.7}$$

where M_{e_a} is the multiplication operator on $c(\mathbf{Z}, k)$ corresponding to e_a, as before. Equivalently,

$$T \circ M_{e_a}^{-1} = a\, M_{e_a}^{-1} \circ T, \tag{26.8}$$

which can be obtained directly from (26.7), or using the facts that $1/e_a = e_{1/a}$ and $M_{e_a}^{-1} = M_{1/e_a} = M_{e_{1/a}}$. This implies that

$$M_{e_a} \circ T \circ M_{e_a}^{-1} = a\, T \tag{26.9}$$

as linear mappings from $c(\mathbf{Z}, k)$ into itself.

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. If $a \in k$ and $|a| = 1$, then

$$|e_a(j)| = |a^j| = |a|^j = 1 \tag{26.10}$$

for every $j \in \mathbf{Z}_+$, so that $e_a \in \ell^\infty(\mathbf{Z}, k)$. However, if $a \in k$, $a \neq 0$, and $|a| \neq 1$, then e_a is not bounded on $\mathbf{Z}$. It follows that e_a is not an element of $c_0(\mathbf{Z}, k)$ for any $a \in k \setminus \{0\}$, and in particular that e_a is not an element of $\ell^r(\mathbf{Z}, k)$ when $0 < r < \infty$. If $a \in k$ and $|a| = 1$, then (26.10) implies that M_{e_a} is an isometric linear mapping from $\ell^r(\mathbf{Z}, k)$ onto itself for every $r > 0$, and that M_{e_a} maps $c_0(\mathbf{Z}, k)$ onto itself.

Using (26.9), we get that

$$M_{e_a} \circ (I - T) \circ M_{e_a}^{-1} = I - M_{e_a} \circ T \circ M_{e_a}^{-1} = I - a\,T \tag{26.11}$$

for every $a \in k$ with $a \neq 0$, as linear mappings from $c(\mathbf{Z}, k)$ into itself. We may also consider T and M_{e_a} as linear mappings from $c_{00}(\mathbf{Z}, k)$ into itself. If $|a| = 1$, then we can consider T and M_{e_a} as linear mappings on $\ell^r(\mathbf{Z}, k)$ for any $r > 0$, or on $c_0(\mathbf{Z}, k)$, as in the previous paragraph. In these situations, (26.11) permits us to reduce questions about $I - a\,T$ to the case where $a = 1$. Thus we shall often focus on $I - T$ in the next sections.

27 Finite support in Z

Let k be a field. If $f \in c_{00}(\mathbf{Z}, k)$, then

$$\sum_{j=-\infty}^{\infty} f(j) \tag{27.1}$$

can be defined as an element of k, by reducing to a finite sum. The mapping from $f \in c_{00}(\mathbf{Z}, k)$ to the sum (27.1) defines a linear functional on $c_{00}(\mathbf{Z}, k)$. We also have that

$$\sum_{j=-\infty}^{\infty} (T(f))(j) = \sum_{j=-\infty}^{\infty} f(j-1) = \sum_{j=-\infty}^{\infty} f(j), \tag{27.2}$$

where T is as in (26.1). Thus

$$\sum_{j=-\infty}^{\infty} (f(j) - (T(f))(j)) = 0 \tag{27.3}$$

for every $f \in c_{00}(\mathbf{Z}, k)$, so that $I - T$ maps $c_{00}(\mathbf{Z}, k)$ into the kernel of the linear functional on $c_{00}(\mathbf{Z}, k)$ defined by (27.1).

If $f \in c_{00}(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$, then put

$$(R(f))(j) = \sum_{l=0}^{\infty} f(j-l), \tag{27.4}$$

where the sum on the right reduces to a finite sum in k, and hence defines an element of k. This defines $R(f)$ as a k-valued function on $\mathbf{Z}$, and R defines a linear mapping from $c_{00}(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$. Observe that

$$(R(f))(j) = \sum_{n=-\infty}^{\infty} f(n) \tag{27.5}$$

when $j \in \mathbf{Z}$ is sufficiently large, depending on $f \in c_{00}(\mathbf{Z}, k)$, because $f(n) = 0$ when $n \in \mathbf{Z}$ is sufficiently large. If

$$\sum_{n=-\infty}^{\infty} f(n) = 0, \tag{27.6}$$

then it follows that $R(f)(j) = 0$ when j is sufficiently large. This implies that

$$R(f) \in c_{00}(\mathbf{Z}, k) \tag{27.7}$$

when $f \in c_{00}(\mathbf{Z}, k)$ satisfies (27.6), because $f(n) = 0$ when $-n$ is sufficiently large, and hence $(R(f))(j) = 0$ when $-j$ is sufficiently large.

Let $f \in c_{00}(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$ be given, and observe that

$$(R(T(f)))(j) = \sum_{l=0}^{\infty} (T(f))(j-l) = \sum_{l=0}^{\infty} f(j-l-1) \tag{27.8}$$

and

$$(T(R(f)))(j) = (R(f))(j-1) = \sum_{l=0}^{\infty} f(j-1-l). \tag{27.9}$$

The sums on the right sides of (27.8) and (27.9) are both equal to

$$\sum_{l=1}^{\infty} f(j-l) = (R(f))(j) - f(j). \tag{27.10}$$

Thus

$$R(T(f)) = T(R(f)) = R(f) - f \tag{27.11}$$

for every $f \in c_{00}(\mathbf{Z}, k)$. This shows that

$$T \circ R = R \circ T = R - I \tag{27.12}$$

as linear mappings from $c_{00}(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$. More precisely, the first T in (27.12) is considered as a mapping from $c(\mathbf{Z}, k)$ into itself, the second T is considered as a mapping from $c_{00}(\mathbf{Z}, k)$ into itself, and I is the identity mapping on $c_{00}(\mathbf{Z}, k)$. It follows that

$$(I - T) \circ R = R \circ (I - T) = I \tag{27.13}$$

as mappings from $c_{00}(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$, where the first I and T are considered as mappings on $c(\mathbf{Z}, k)$, and the other I's and T are considered as mappings on $c_{00}(\mathbf{Z}, k)$. Alternatively,

$$(R(f))(j) = \sum_{l=0}^{\infty} (T^l(f))(j) \tag{27.14}$$

for every $f \in c_{00}(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$, because the lth power T^l of T with respect to composition is as in (3.3). Basically, R corresponds to $\sum_{l=0}^{\infty} T^l$ as a linear mapping from $c_{00}(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$, where this sum is interpreted as in (27.4) and (27.14).

One can check that $I - T$ is injective on $c_{00}(\mathbf{Z}, k)$. This is the same as saying that $1 \in k$ is not an eigenvalue of T on $c_{00}(\mathbf{Z}, k)$, and indeed T has no eigenvalues on $c_{00}(\mathbf{Z}, k)$, as mentioned in the previous section. The injectivity of $I - T$ on $c_{00}(\mathbf{Z}, k)$ also follows from the second equality in (27.13). If $f \in c_{00}(\mathbf{Z}, k)$, then

$$(I - T)(R(f)) = f, \tag{27.15}$$

because the left side of (27.13) is equal to the identity as a mapping from $c_{00}(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$. Remember that $R(f)$ has finite support in $\mathbf{Z}$ when f satisfies (27.6), as in (27.7). In this case, (27.15) implies that f is in the image of $I - T$ on $c_{00}(\mathbf{Z}, k)$. Thus the image of $I - T$ on $c_{00}(\mathbf{Z}, k)$ is the same as the kernel of (27.1) as a linear functional on $c_{00}(\mathbf{Z}, k)$, because the other inclusion follows from (27.3).

28 Arbitrary functions on Z

Let k be a field, and let f be a k-valued function on $\mathbf{Z}$. If $j \in \mathbf{Z}$, then put

$$\begin{aligned}(R_0(f))(j) &= \sum_{l=1}^{j} f(l) \qquad \text{when } j \geq 0 \\ &= -\sum_{l=j+1}^{0} f(l) \quad \text{when } j \leq 0,\end{aligned} \tag{28.1}$$

where both sums are interpreted as being 0 when $j = 0$. This defines a k-valued function on $\mathbf{Z}$, and R_0 defines a linear mapping from $c(\mathbf{Z}, k)$ into itself. If f has finite support in $\mathbf{Z}$, then (27.4) is the same as

$$(R(f))(j) = \sum_{l=-\infty}^{j} f(l). \tag{28.2}$$

In this case, we have that

$$(R_0(f))(j) = (R(f))(j) - \sum_{l=-\infty}^{0} f(l) \tag{28.3}$$

for every $j \in \mathbf{Z}$, where the sum on the right side of (28.3) reduces to a finite sum in k.

Let T be the forward shift operator on $c(\mathbf{Z}, k)$ again, as in (26.1). Let us check that

$$(R_0(T(f)))(j) = (R_0(f))(j) - f(j) + f(0) \tag{28.4}$$

for every $f \in c(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$. If $j = 0$, then both sides of (28.4) are equal to 0. If $j \geq 1$, then

$$\begin{aligned}(R_0(T(f))(j) = \sum_{l=1}^{j} (T(f))(l) &= \sum_{l=1}^{j} f(l-1) = \sum_{l=0}^{j-1} f(l) \\ &= (R_0(f))(j) - f(j) + f(0).\end{aligned} \tag{28.5}$$

If $j \leq -1$, then

$$\begin{aligned}(R_0(T(f))(j) = -\sum_{l=j+1}^{0} (T(f))(l)) &= -\sum_{l=j+1}^{0} f(l-1) = -\sum_{l=j}^{-1} f(l) \\ &= (R_0(f))(j) - f(j) + f(0).\end{aligned} \tag{28.6}$$

Similarly, let us check that

$$(T(R_0(f)))(j) = (R_0(f))(j-1) = R_0(f)(j) - f(j) \tag{28.7}$$

for every $f \in c(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$. Of course, the first step in (28.7) follows from the definition of T. If $j \geq 1$, then

$$(R_0(f))(j-1) = \sum_{l=1}^{j-1} f(l) = (R_0(f))(j) - f(j), \tag{28.8}$$

where the sum in the middle is interpreted as being equal to 0 when $j = 1$. If $j \leq 0$, then

$$(R_0(f))(j-1) = -\sum_{l=j}^{0} f(l) = (R_0(f))(j) - f(j). \tag{28.9}$$

Equivalently, (28.7) says that

$$T(R_0(f)) = R_0(f) - f \tag{28.10}$$

for every $f \in c(\mathbf{Z}, k)$. Thus

$$T \circ R_0 = R_0 - I \tag{28.11}$$

as linear mappings on $c(\mathbf{Z}, k)$. This implies that

$$(I - T) \circ R_0 = R_0 - T \circ R_0 = I \tag{28.12}$$

as linear mappings on $c(\mathbf{Z}, k)$. It follows that $I - T$ maps $c(\mathbf{Z}, k)$ onto itself.

Let λ_0 be the linear functional on $c(\mathbf{Z}, k)$ defined by

$$\lambda_0(f) = f(0) \tag{28.13}$$

for every $f \in c(\mathbf{Z}, k)$. As in Section 19, we let $\mathbf{1_Z} = \mathbf{1}_{\mathbf{Z},k}$ be the constant function equal to $1 = 1_k \in k$ on $\mathbf{Z}$. Using (28.4), we get that

$$R_0(T(f)) = R_0(f) - f + f(0)\,\mathbf{1_Z} \tag{28.14}$$

for every $f \in c(\mathbf{Z}, k)$. Hence

$$R_0 \circ T = R_0 - I + \lambda_0\,\mathbf{1_Z} \tag{28.15}$$

as linear mappings on $c(\mathbf{Z}, k)$. More precisely, $f \mapsto \lambda_0(f)\,\mathbf{1_Z}$ defines a linear mapping from $c(\mathbf{Z}, k)$ into itself, which sends $f \in c(\mathbf{Z}, k)$ to the constant function on $\mathbf{Z}$ equal to (28.13). It follows that

$$R_0 \circ (I - T) = R_0 - R_0 \circ T = I - \lambda_0\,\mathbf{1_Z} \tag{28.16}$$

as linear mappings on $c(\mathbf{Z}, k)$. Note that the kernel of $I - T$ on $c(\mathbf{Z}, k)$ consists of constant functions, which is the same as the kernel of the right side of (28.16).

29 Summable functions on Z

In this section, we take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. It will be convenient to let $\ell^r(\mathbf{Z})$ refer to either $\ell^r(\mathbf{Z}, \mathbf{R})$ or $\ell^r(\mathbf{Z}, \mathbf{C})$ for each $r > 0$, and similarly for $c_{00}(\mathbf{Z})$, $c_0(\mathbf{Z})$, and $c(\mathbf{Z})$.

If f is a summable real or complex-valued function on $\mathbf{Z}$, then

$$\sum_{j=-\infty}^{\infty} f(j) \tag{29.1}$$

can be defined as a real or complex number. This may be treated as a sum over $\mathbf{Z}$, as in Section 10, or as a sum of two absolutely convergent infinite series. Of course, the mapping from $f \in \ell^1(X)$ to the sum (29.1) defines a bounded linear functional on $\ell^1(X)$ with respect to the ℓ^1 norm. It is easy to see that (27.2) and (27.3) still hold in this situation, where T is as in (26.1) again. Thus $I - T$ maps $\ell^1(X)$ into the kernel of the linear functional defined by (29.1), as before.

Let f be a summable real or complex-valued function on $\mathbf{Z}$ again, and observe that

$$\sum_{l=0}^{\infty} f(j - l) \tag{29.2}$$

converges absolutely for each $j \in \mathbf{Z}$. This is the same as the right side of (27.4), so that $(R(f))(j)$ can be defined as before. Equivalently, this sum can be expressed as the right side of (28.2). This implies that

$$|(R(f))(j)| \le \sum_{l=-\infty}^{j} |f(l)| \le \|f\|_1 \tag{29.3}$$

for every $j \in \mathbf{Z}$. Thus $R(f)$ is a bounded real or complex-valued function on $\mathbf{Z}$, as appropriate, with

$$\|R(f)\|_\infty \le \|f\|_1. \tag{29.4}$$

Note that

$$\lim_{j \to -\infty} (R(f))(j) = 0 \tag{29.5}$$

for every $f \in \ell^1(\mathbf{Z})$, by the first inequality in (29.3). If (29.1) is equal to 0, then we have that

$$(R(f))(j) = - \sum_{l=j+1}^{\infty} f(l) \tag{29.6}$$

for each $j \in \mathbf{Z}$. This implies that

$$|(R(f))(j)| \le \sum_{l=j+1}^{\infty} |f(l)| \tag{29.7}$$

for each $j \in \mathbf{Z}$. It follows that

$$\lim_{j \to \infty} (R(f))(j) = 0 \tag{29.8}$$

under these conditions. Combining (29.5) and (29.8), we get that

$$R(f) \in c_0(\mathbf{Z}) \tag{29.9}$$

for every $f \in \ell^1(\mathbf{Z})$ such that (29.1) is equal to 0.

If $f \in \ell^1(\mathbf{Z})$, then (27.8) and (27.9) hold for every $j \in \mathbf{Z}$, for the same reasons as before. This implies that (27.11) holds for every $f \in \ell^1(\mathbf{Z})$, so that

$$T \circ R = R \circ T = R - I \tag{29.10}$$

as linear mappings from $\ell^1(\mathbf{Z})$ into $\ell^\infty(\mathbf{Z})$, as in (27.12). More precisely, the first T in (29.10) is considered as a mapping from $\ell^\infty(\mathbf{Z})$ into itself, the second T is considered as a mapping from $\ell^1(\mathbf{Z})$ into itself, I is the identity mapping on $\ell^1(\mathbf{Z})$, and R is considered as a mapping from $\ell^1(X)$ into $\ell^\infty(X)$. Hence

$$(I - T) \circ R = R \circ (I - T) = I \tag{29.11}$$

as linear mappings from $\ell^1(\mathbf{Z})$ into $\ell^\infty(\mathbf{Z})$, as in (27.13), where the first I and T in (29.11) are considered as mappings on $\ell^\infty(\mathbf{Z})$, and the other I's and T are considered as mappings on $\ell^1(\mathbf{Z})$.

Remember that T has no eigenvalues on $\ell^1(\mathbf{Z})$, as in Section 26. In particular, 1 is not an eigenvalue of T on $\ell^1(\mathbf{Z})$, which means that $I - T$ is injective on $\ell^1(\mathbf{Z})$. This can also be obtained from the second equality in (29.11), as in Section 27. More precisely, if $f \in \ell^1(\mathbf{Z})$, then

$$R(f - T(f)) = f, \tag{29.12}$$

by the second equality in (29.11). This implies that

$$\|f\|_\infty = \|R(f - T(f))\|_\infty \le \|f - T(f)\|_1, \tag{29.13}$$

using (29.4) in the second step.

Remember too that $I - T$ maps $\ell^1(\mathbf{Z})$ into

$$\Big\{g \in \ell^1(\mathbf{Z}) : \sum_{j=-\infty}^{\infty} g(j) = 0\Big\}, \tag{29.14}$$

as mentioned earlier. Note that (29.14) is a closed linear subspace of $\ell^1(X)$ with respect to the metric associated to the ℓ^1 norm, because it is the kernel of the bounded linear functional defined by (29.1). Of course,

$$\Big\{g \in c_{00}(\mathbf{Z}) : \sum_{j=-\infty}^{\infty} g(j) = 0\Big\} \tag{29.15}$$

is a linear subspace of (29.14). One can check that (29.15) is dense in (29.14) with respect to the metric associated to the ℓ^1 norm. More precisely, $c_{00}(\mathbf{Z})$ is dense in $\ell^1(\mathbf{Z})$, as in Section 8. An additional adjustment is needed to show that (29.15) is dense in (29.14), to get the condition on the sum. We have seen that $I - T$ maps $c_{00}(\mathbf{Z})$ onto (29.15), as in Section 27. It follows that $I - T$ maps $\ell^1(\mathbf{Z})$ onto a dense linear subspace of (29.14).

30 Approximate eigenvalues

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k with q_V, q_W-norms N_V, N_W with respect to $|\cdot|$ on k for some $q_V, q_W > 0$, respectively. Also let T be a linear mapping from V into W, and consider the condition that there be a positive real number c such that

$$c\, N_V(v) \le N_W(T(v)) \tag{30.1}$$

for every $v \in V$. This implies that the kernel of T is trivial, so that T is injective. If T is a one-to-one linear mapping from V onto W, then this condition is equivalent to the boundedness of the inverse T^{-1} as a linear mapping from W onto V. In this case, (30.1) is the same as saying that

$$\|T^{-1}\|_{op,WV} \le 1/c, \tag{30.2}$$

where $\|T^{-1}\|_{op,WV}$ is the operator q_V-norm of T^{-1} associated to N_V and N_W, as in Section 9.

Let us now consider the condition that there is no $c > 0$ such that (30.1) holds for every $v \in V$. In particular, this implies that T does not have a bounded inverse mapping. This condition is the same as saying that for each $\epsilon > 0$ there is a $v_\epsilon \in V$ such that

$$N_W(T(v_\epsilon)) < \epsilon\, N_V(v_\epsilon). \tag{30.3}$$

Equivalently, this means that there is a sequence $\{v_j\}_{j=1}^\infty$ of nonzero vectors in V such that

$$\lim_{j\to\infty} N_W(T(v_j))\, N_V(v_j)^{-1} = 0. \tag{30.4}$$

If the kernel of T is nontrivial, then we can take $\{v_j\}_{j=1}^\infty$ to be a constant sequence in V.

Let us take $V = W$ for the rest of the section. Let T be a linear mapping from V into itself, and let $a \in k$ be given. Consider the condition that

$$c\, N_V(v) \le N_V((a\, I - T)(v)) = N_V(a\, v - T(v)) \tag{30.5}$$

for some $c > 0$ and every $v \in V$. This condition holds when $a\, I - T$ has a bounded inverse on V, as before. If this condition does not hold, then there is a sequence $\{v_j\}_{j=1}^\infty$ of nonzero vectors in V such that

$$\lim_{j\to\infty} N_V(a\, v_j - T(v_j))\, N_V(v_j)^{-1} = 0, \tag{30.6}$$

as in the preceding paragraph. In this case, one may say that a is an *approximate eigenvalue* of T on V. If a is an eigenvalue of T, then a is an approximate eigenvalue in this sense. If a is an approximate eigenvalue of T, then $a\, I - T$ does not have a bounded inverse on V.

Let $a \in k$ and a sequence $\{v_j\}_{j=1}^\infty$ of nonzero vectors in V be given. If $q_V < \infty$, then one can check that

$$|N_V(a\, v_j)^{q_V} - N_V(T(v_j))^{q_V}| \le N_V(a\, v_j - T(v_j))^{q_V} \tag{30.7}$$

for each j, using the q_V-norm version of the triangle inequality. Thus (30.6) implies that

$$\lim_{j\to\infty} |N_V(a\, v_j)^{q_V} - N_V(T(v_j))^{q_V}|\, N_V(v_j)^{-q_V} = 0. \tag{30.8}$$

Equivalently, this means that

$$\lim_{j\to\infty} \big||a|^{q_V}\, N_V(v_j)^{q_V} - N_V(T(v_j))^{q_V}\big|\, N_V(v_j)^{-q_V} = 0. \tag{30.9}$$

If $q_V = \infty$, then one can use the ultranorm version of the triangle inequality to get that

$$N_V(a\, v_j) = N_V(T(v_j)) \tag{30.10}$$

for every $j \ge 1$ such that

$$N_V(a\, v_j - T(v_j)) < \max(N_V(a\, v_j), N_V(T(v_j))). \tag{30.11}$$

This is the same as saying that

$$|a|\, N_V(v_j) = N_V(T(v_j)) \tag{30.12}$$

when

$$N_V(a\, v_j - T(v_j)) < \max(|a|\, N_V(v_j), N_V(T(v_j))). \tag{30.13}$$

If $a \neq 0$, then (30.6) implies that (30.13) holds for all sufficiently large j. Thus (30.12) holds for all sufficiently large j when $a \neq 0$ and (30.9) holds.

Suppose that T is a bounded linear mapping from V into itself, and let $\|T\|_{op}$ be the corresponding operator q_V-norm of T, as in Section 9. If $a \in k$ is an approximate eigenvalue of T on V, then

$$|a| \leq \|T\|_{op}. \tag{30.14}$$

To see this, let $\{v_j\}_{j=1}^{\infty}$ be a sequence of nonzero vectors in V that satisfies (30.6). Note that

$$N_V(T(v_j)) \leq \|T\|_{op}\, N_V(v_j) \tag{30.15}$$

for each j, by definition of $\|T\|_{op}$. If $|a| > \|T\|_{op}$ and $q_V < \infty$, then one can get a contradiction using (30.9) and (30.15). If $q_V = \infty$ and $a \neq 0$, then (30.12) holds for all sufficiently large j, as in the preceding paragraph. This implies that (30.14) holds when $q_V = \infty$ and $a \neq 0$, and of course (30.14) is trivial when $a = 0$.

Suppose now that

$$c\, N_V(v) \leq N_V(T(v)) \tag{30.16}$$

for some $c > 0$ and every $v \in V$. Let $a \in k$ be an approximate eigenvalue of T on V, and let us check that

$$c \leq |a|. \tag{30.17}$$

Let $\{v_j\}_{j=1}^{\infty}$ be a sequence of nonzero vectors in V that satisfies (30.6) again. It is easy to see directly that $a \neq 0$ in this case, which corresponds to the remarks at the beginning of the section. If $q_V < \infty$ and $|a| < c$, then one can get a contradiction using (30.9) and (30.16). If $q_V = \infty$ and $a \neq 0$, then (30.12) holds for all sufficiently large j, as before. Combining this with (30.16), we get (30.17), as desired.

If T is an isometric linear mapping from V into itself, then T is bounded in particular, with $\|T\|_{op} = 1$, unless $V = \{0\}$. In this case, (30.16) holds with $c = 1$ as well. If $a \in k$ is an approximate eigenvalue of T on V, then it follows that $|a| = 1$, by (30.14) and (30.17).

Let T be a one-to-one linear mapping from V onto itself, and suppose that T^{-1} is bounded as a linear mapping on V. This implies that (30.16) holds, with

$$c = 1/\|T^{-1}\|_{op}, \tag{30.18}$$

at least if $V \neq \{0\}$, so that $\|T^{-1}\|_{op} > 0$. Alternatively, if $a \in k$ is an approximate eigenvalue of T, then one can check that $a \neq 0$ and $1/a$ is an approximate eigenvalue of T^{-1}. More precisely, if $\{v_j\}_{j=1}^{\infty}$ is a sequence of nonzero vectors in V that satisfies (30.6), then $\{v_j\}_{j=1}$ satisfies the analogous condition for T^{-1} and $1/a$. This uses the fact that

$$\begin{aligned} N_V(a\, T^{-1}(v_j) - v_j) &= N_V(T^{-1}(a\, v_j - T(v_j))) \\ &\leq \|T^{-1}\|_{op}\, N_V(a\, v_j - T(v_j)) \end{aligned} \tag{30.19}$$

for each j. It follows that

$$1/|a| = |1/a| \le \|T^{-1}\|_{op}, \tag{30.20}$$

as in (30.14). This is the same as (30.17), with c as in (30.18).

31 Eigenvalues of multiplication operators

Let k be a field, and let X be a nonempty set. Also let b be a k-valued function on X, and let

$$M_b(f) = b\, f \tag{31.1}$$

be the corresponding multiplication operator on $c(X, k)$, as in Section 19. If $a \in k$ is an eigenvalue of M_b on $c(X, k)$, then there is a nonzero k-valued function f on X such that

$$b(x)\, f(x) = a\, f(x) \tag{31.2}$$

for every $x \in X$. This implies that

$$b(x) = a \tag{31.3}$$

for every $x \in X$ such that $f(x) \ne 0$. In particular, (31.3) holds for some $x \in X$, because f is not identically zero on X. In the other direction, (31.2) holds for every $f \in c(X, k)$ whose support is contained in the set of $x \in X$ such that (31.3) holds. If (31.3) holds for some $x \in X$, then a is an eigenvalue of M_b on $c(X, k)$. In this case, a is an eigenvalue for the restriction of M_a to $c_{00}(X, k)$ as well.

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$, and suppose that b is a bounded k-valued function on X. Thus M_b defines a bounded linear mapping from $\ell^r(X, k)$ into itself for every $r > 0$, and M_b maps $c_0(X, k)$ into itself, as in Section 19. As in the previous paragraph, $a \in k$ is an eigenvalue of M_b on any of these spaces if and only if (31.3) holds for some $x \in X$. In this case, the corresponding eigenfunctions are the functions in the appropriate space that are supported in the set of $x \in X$ such that (31.3) holds, as before.

Suppose that $a \in k$ is an element of the closure of $b(X)$ in k, with respect to the q-metric associated to $|\cdot|$. This means that there is a sequence $\{x_j\}_{j=1}^\infty$ of elements of X such that $\{b(x_j)\}_{j=1}^\infty$ converges to a in k. Let δ_{x_j} be the k-valued function on X equal to 1 at x_j and 0 everywhere else, as in (2.2). Thus

$$M_b(\delta_{x_j}) = b(x_j)\, \delta_{x_j} \tag{31.4}$$

for each j, as in (19.3), and

$$\|\delta_{x_j}\|_r = 1 \tag{31.5}$$

for every j and $r > 0$. This implies that a is an approximate eigenvalue of M_b on $\ell^r(X, k)$ for each $r > 0$, and also on $c_0(X, k)$, using the ℓ^∞ q-norm on $c_0(X, k)$.

Note that

$$M_{a\,\mathbf{1}_X - b} = a\,I - M_b \tag{31.6}$$

as linear mappings from $c(X,k)$ into itself. If $a \notin b(X)$, then $a\,\mathbf{1}_X - b$ is nonzero on X, so that

$$1/(a\,\mathbf{1}_X - b) \tag{31.7}$$

is defined as a k-valued function on X. The multiplication operator on $c(X,k)$ associated to (31.7) is the inverse of (31.6), as in Section 22. Suppose that a is not in the closure of $b(X)$ in k, so that there is a positive real number c such that

$$|a - b(x)| \ge c \tag{31.8}$$

for every $x \in X$. This implies that (31.7) is bounded on X, so that the multiplication operator associated to (31.7) is bounded on $\ell^r(X,k)$ for each $r > 0$. This means that (31.6) has a bounded inverse on $\ell^r(X,k)$ for each $r > 0$, and on $c_0(X,k)$. It follows that a is not an approximate eigenvalue of M_b on $\ell^r(X,k)$ for any $r > 0$. In particular, a is not an approximate eigenvalue of M_b on $c_0(X,k)$ with respect to the ℓ^∞ q-norm.

Let us take $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function for the rest of the section, and let $(X, \mathcal{A}, \mu)$ be a measure space. Let b be an essentially bounded measurable real or complex-valued function on X, so that the corresponding multiplication operator M_b defines a bounded linear mapping from $L^r(X)$ into itself for each $r > 0$. If a is a real or complex number, as appropriate, and $f \in L^r(X)$ for some $r > 0$, then

$$M_b(f) = a\,f \tag{31.9}$$

in $L^r(X)$ means that $b\,f = a\,f$ almost everywhere on X with respect to μ. If this holds for some f that is not equal to 0 almost everywhere on X, then

$$\mu(\{x \in X : b(x) = a\}) > 0. \tag{31.10}$$

If $E \subseteq X$ is a measurable set such that $b = a$ almost everywhere on E with respect to μ, then (31.9) holds with $f = \mathbf{1}_E$. Here $\mathbf{1}_E$ denotes the *indicator function* on X associated to E, which is equal to 1 on E and to 0 on $X \setminus E$. Of course, $\mathbf{1}_E$ is not equal to 0 almost everywhere on X with respect to μ when $\mu(E) > 0$. In this case, $\mathbf{1}_E$ is a nonzero eigenvector of M_b on $L^\infty(X)$ corresponding to the eigenvalue a. If $\mu(E) < \infty$ too, then $\mathbf{1}_E$ is a nonzero eigenvector of M_b on $L^r(X)$ for every $r > 0$, corresponding to the eigenvalue a.

A real or complex number a is said to be an element of the *essential range* of b if

$$\mu(\{x \in X : |b(x) - a| < \epsilon\}) > 0 \tag{31.11}$$

for every $\epsilon > 0$. Equivalently, this means that for each $\epsilon > 0$ there is a measurable set $E_\epsilon \subseteq X$ such that $\mu(E_\epsilon) > 0$ and

$$\mu(\{x \in E_\epsilon : |b(x) - a| \ge \epsilon\}) = 0. \tag{31.12}$$

Thus

$$|M_b(\mathbf{1}_{E_\epsilon}) - a\,\mathbf{1}_{E_\epsilon}| \le \epsilon\,\mathbf{1}_{E_\epsilon} \tag{31.13}$$

almost everywhere on X with respect to μ. This implies that a is an approximate eigenvalue of M_b on $L^\infty(X)$. If we can also choose E_ϵ so that $\mu(E_\epsilon) < \infty$ for each $\epsilon > 0$, then it is easy to see that a is an approximate eigenvalue of M_b on $L^r(X)$ for every $r > 0$.

If a real or complex number a, as appropriate, is not in the essential range of b on X, then $a\,\mathbf{1}_X - b$ is nonzero almost everywhere on X with respect to μ, and $1/(a\,\mathbf{1}_X - b)$ is essentially bounded on X. This implies that $a\,I - M_b$ has a bounded inverse on $L^r(X)$ for every $r > 0$. It follows that a is not an approximate eigenvalue of M_b on $L^r(X)$ for any $r > 0$.

32 Eigenvalues of bilateral shifts

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. Also let T be the forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3 and (26.1). The eigenvalues of T on various subspaces of $c(\mathbf{Z}, k)$ were discussed in Section 26, and we would like to consider approximate eigenvalues of T on some of these spaces in this section. If $a \in k$ and $|a| \ne 1$, then a cannot be an approximate eigenvalue of T on $\ell^r(\mathbf{Z}, k)$ for any $r > 0$. This follows from the remarks in Section 30, and the fact that T is an isometry on $\ell^r(\mathbf{Z}, k)$ for each $r > 0$. If $|a| = 1$, then we have seen that a is an eigenvalue of T on $\ell^\infty(\mathbf{Z}, k)$. We have also seen that a is not an eigenvalue of T on $\ell^r(\mathbf{Z}, k)$ when $0 < r < \infty$.

Let $a \in k$ with $|a| = 1$ and a positive real number r be given, and let j_1, j_2 be integers with $j_1 \le j_2$. Let f be the k-valued function defined on $\mathbf{Z}$ by

$$\begin{aligned} f(j) &= a^{-j} && \text{when } j_1 \le j \le j_2 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{32.1}$$

Thus

$$\begin{aligned} (T(f))(j) = f(j-1) &= a^{1-j} && \text{when } j_1 + 1 \le j \le j_2 + 1 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{32.2}$$

It follows that

$$\begin{aligned} a\,f(j) - (T(f))(j) &= a^{1-j_1} && \text{when } j = j_1 \\ &= -a^{1-j_2} && \text{when } j = j_2 \\ &= 0 && \text{otherwise.} \end{aligned} \tag{32.3}$$

This implies that

$$\|a\,f - T(f)\|_r = 2^{1/r}. \tag{32.4}$$

Similarly,

$$\|f\|_r = (j_2 - j_1 + 1)^{1/r}. \tag{32.5}$$

This shows that a is an approximate eigenvalue of T on $\ell^r(\mathbf{Z}, k)$, because we can choose j_1, j_2 so that (32.5) is arbitrarily large.

Let us now take $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function for the rest of the section. Let $a \in \mathbf{R}$ or $\mathbf{C}$ be given, as appropriate, with $|a| = 1$, and let a nonnegative integer n be given as well. Consider the real or complex-valued function f_n, as appropriate, defined on $\mathbf{Z}$ by

$$\begin{aligned} f_n(j) &= a^{-j}\,(n - |j|) \quad \text{when } |j| \le n \\ &= 0 \qquad\qquad\qquad \text{when } |j| > n. \end{aligned} \tag{32.6}$$

Note that $f_n(j) = 0$ when $|j| = n$. Using (32.6), we get that

$$\begin{aligned} (T(f_n))(j) = f_n(j-1) &= a^{1-j}\,(n - |j-1|) \quad \text{when } |j-1| \le n \\ &= 0 \qquad\qquad\qquad\qquad \text{when } |j-1| > n, \end{aligned} \tag{32.7}$$

which is equal to 0 when $|j - 1| = n$. If $-n + 1 \le j \le n$, so that $|j| \le n$ and $|j - 1| \le n$, then

$$\begin{aligned} a\, f_n(j) - (T(f_n))(j) &= a^{1-j}\,(n - |j|) - a^{j-1}\,(n - |j-1|) \\ &= a^{j-1}\,(|j-1| - |j|). \end{aligned} \tag{32.8}$$

This implies that

$$|a\, f_n(j) - (T(f_n))(j)| = 1 \tag{32.9}$$

when $-n + 1 \le j \le n$. Otherwise, if $j \ge n + 1$ or $j \le -n$, then $f_n(j) = f_n(j - 1) = 0$, which implies that

$$a\, f_n(j) - (T(f_n))(j) = 0. \tag{32.10}$$

It follows that

$$\|a\, f_n - T(f_n)\|_r = (2\, n)^{1/r} \tag{32.11}$$

for every $r > 0$, where the right side of (32.11) is interpreted as being equal to 1 when $r = \infty$, as usual. Observe that

$$\|f_n\|_\infty = |f_n(0)| = n, \tag{32.12}$$

and that

$$|f_n(j)| = n - |j| \ge n/2 \tag{32.13}$$

when $|j| \le n/2$. There are always at least n integers j with $|j| \le n/2$, so that

$$\|f_n\|_r \ge (n/2)\, n^{1/r} \tag{32.14}$$

when $0 < r < \infty$. Using (32.11) and (32.14), we get that a is an approximate eigenvalue of T on $\ell^r(\mathbf{Z}, \mathbf{R})$ and $\ell^r(\mathbf{Z}, \mathbf{C})$ when $0 < r < \infty$. Similarly, (32.11) and (32.12) imply that a is an approximate eigenvalue of T on $c_0(\mathbf{Z}, \mathbf{R})$ and $c_0(\mathbf{Z}, \mathbf{C})$ with respect to the supremum norm.

33 The ultrametric case

Let k be a field, and suppose that $|\cdot|$ is an ultrametric absolute value function on k. Also let f be a k-valued function on $\mathbf{Z}$, and let $a \in k$ be given, with $|a| = 1$. If j_1, j_2 are integers with $j_1 < j_2$, then

$$\begin{aligned} (33.1) \qquad a^{j_2}\, f(j_2) - a^{j_1}\, f(j_1) &= \sum_{j=j_1+1}^{j_2} (a^j\, f(j) - a^{j-1}\, f(j-1)) \\ &= \sum_{j=j_1+1}^{j_2} a^{j-1}\, (a\, f(j) - f(j-1)). \end{aligned}$$

This implies that

$$(33.2) \qquad |a^{j_2}\, f(j_2) - a^{j_1}\, f(j_1)| \le \max_{j_1+1 \le j \le j_2} |a\, f(j) - f(j-1)|,$$

by the ultrametric version of the triangle inequality. Equivalently,

$$(33.3) \qquad |a^{j_2}\, f(j_2) - a^{j_1}\, f(j_1)| \le \max_{j_1+1 \le j \le j_2} |a\, f(j) - (T(f))(j)|,$$

where T is the forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3 and (26.1).

If $a\, f - T(f)$ is bounded on $\mathbf{Z}$, then

$$(33.4) \qquad |a^{j_2}\, f(j_2) - a^{j_1}\, f(j_1)| \le \|a\, f - T(f)\|_\infty$$

for every $j_1, j_2 \in \mathbf{Z}$, by (33.3). Using the ultrametric version of the triangle inequality again, we get that f is bounded on $\mathbf{Z}$, with

$$(33.5) \qquad \|f\|_\infty \le \max\Big(\|a\, f - T(f)\|_\infty, \inf_{j \in \mathbf{Z}} |f(j)|\Big).$$

In particular, if f vanishes at infinity on $\mathbf{Z}$, then

$$(33.6) \qquad \|f\|_\infty \le \|a\, f - T(f)\|_\infty.$$

This shows that a is not an approximate eigenvalue of T on $c_0(\mathbf{Z}, k)$ with respect to the supremum ultranorm when $|a| = 1$. The analogous statement for $|a| \ne 1$ follows from the remarks in Section 30, as mentioned in the previous section.

More precisely, if $a\, f - T(f)$ is bounded on $\mathbf{Z}$, then

$$(33.7) \qquad \sup_{j_1, j_2 \in \mathbf{Z}} |a^{j_2}\, f(j_2) - a^{j_1}\, f(j_1)| = \|a\, f - T(f)\|_\infty.$$

This uses (33.4) to get that the left side of (33.7) is less than or equal to the right side. The opposite inequality follows directly from the definitions, with $j_2 = j_1 + 1$. Of course,

$$(33.8) \qquad \|a\, f - T(f)\|_\infty \le \max(\|a\, f\|_\infty, \|T(f)\|_\infty) = \|f\|_\infty$$

for every $f \in \ell^\infty(\mathbf{Z}, k)$, by the ultrametric version of the triangle inequality. Combining this with (33.6), we get that

$$\|a\, f - T(f)\|_\infty = \|f\|_\infty \tag{33.9}$$

when $f \in c_0(\mathbf{Z}, k)$.

Let $f \in \ell^\infty(\mathbf{Z}, k)$ be given, and let $R_0(f)$ be defined on $\mathbf{Z}$ as in (28.1). Using the ultrametric version of the triangle inequality, we get that $R_0(f)$ is bounded on $\mathbf{Z}$ too, with

$$\|R_0(f)\|_\infty \le \|f\|_\infty. \tag{33.10}$$

In particular, it follows that $I - T$ maps $\ell^\infty(\mathbf{Z}, k)$ onto itself, by (28.12). We also have that

$$\|f\|_\infty = \|(I - T)(R_0(f))\|_\infty \le \|R_0(f)\|_\infty, \tag{33.11}$$

using (28.12) in the first step, and (33.8) in the second step, with $a = 1$. Thus

$$\|R_0(f)\|_\infty = \|f\|_\infty, \tag{33.12}$$

by (33.10) and (33.12). Remember that

$$(R_0(f))(0) = 0 \tag{33.13}$$

automatically, by the definition (28.1) of $R_0(f)$. In fact, R_0 maps $\ell^\infty(\mathbf{Z}, k)$ onto the subspace of bounded k-valued functions on $\mathbf{Z}$ that are equal to 0 at 0, because of (28.16).

34 k Complete

Let k be a field with an ultrametric absolute value function $|\cdot|$ again, and suppose that k is complete with respect to the ultrametric associated to $|\cdot|$. If $f \in c_0(\mathbf{Z}, k)$, then the infinite series

$$\sum_{j=1}^\infty f(j), \quad \sum_{j=0}^\infty f(-j) \tag{34.1}$$

converge in k, as in Section 12. This permits us to define

$$\sum_{j=-\infty}^\infty f(j) \tag{34.2}$$

as an element of k, by combining the sums in (34.1). We also have that

$$\biggl|\sum_{j=-\infty}^\infty f(j)\biggr| \le \sup_{j \in \mathbf{Z}} |f(j)|, \tag{34.3}$$

because of the analogous statement for the sums in (34.1), as in (12.5). The sum (34.2) can be treated as a sum over $\mathbf{Z}$ as in Section 13 as well. Note that the mapping from $f \in c_0(\mathbf{Z}, k)$ to the sum (34.2) is linear in f. This mapping defines a bounded linear functional on $c_0(\mathbf{Z}, k)$ with respect to the supremum norm, with dual norm equal to 1.

If $f \in c_0(\mathbf{Z}, k)$, then one can check that

$$\sum_{j=-\infty}^{\infty} (T(f))(j) = \sum_{j=-\infty}^{\infty} f(j-1) = \sum_{j=-\infty}^{\infty} f(j), \tag{34.4}$$

as in (27.2), where T is the usual forward shift operator. This implies that

$$\sum_{j=-\infty}^{\infty} (f(j) - (T(f))(j)) = 0, \tag{34.5}$$

as in (27.3). It follows that $I - T$ maps $c_0(\mathbf{Z}, k)$ into

$$\Big\{g \in c_0(\mathbf{Z}, k) : \sum_{j=-\infty}^{\infty} g(j) = 0\Big\}. \tag{34.6}$$

Of course, this is the kernel of the linear functional defined on $c_0(\mathbf{Z}, k)$ by (34.2), as in the preceding paragraph. Thus (34.6) is a closed linear subspace of $c_0(\mathbf{Z}, k)$ with respect to the topology determined by the supremum metric, because (34.2) defines a bounded linear functional on $c_0(\mathbf{Z}, k)$ with respect to the supremum norm, as before.

If $f \in c_0(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$, then

$$\sum_{l=0}^{\infty} f(j-l) \tag{34.7}$$

converges in k, as in Section 12 again. Let $(R(f))(j)$ denote the value of this sum for each $j \in \mathbf{Z}$, as in (27.4). Equivalently, this sum can be expressed as

$$\sum_{l=-\infty}^{j} f(l) \tag{34.8}$$

for every $j \in \mathbf{Z}$, as in (28.2). It follows that

$$|(R(f))(j)| \le \sup_{l \le j} |f(l)| \le \|f\|_\infty \tag{34.9}$$

for every $j \in \mathbf{Z}$, using (12.5) in the first step. Thus $R(f)$ defines a bounded k-valued function on $\mathbf{Z}$, with

$$\|R(f)\|_\infty \le \|f\|_\infty. \tag{34.10}$$

The first inequality in (34.9) implies that

$$\lim_{j \to -\infty} |(R(f))(j)| = 0 \tag{34.11}$$

for every $f \in c_0(\mathbf{Z}, k)$. If (34.2) is equal to 0, then

$$(R(f))(j) = -\sum_{l=j+1}^{\infty} f(l) \tag{34.12}$$

for each $j \in \mathbf{Z}$. This implies that

$$|(R(f))(j)| \le \sup_{l \ge j+1} |f(l)| \tag{34.13}$$

for each $j \in \mathbf{Z}$, as in (12.5). Hence

$$\lim_{j \to \infty} |(R(f))(j)| = 0, \tag{34.14}$$

because f vanishes at infinity on $\mathbf{Z}$. This shows that

$$R(f) \in c_0(\mathbf{Z}, k) \tag{34.15}$$

when $f \in c_0(\mathbf{Z}, k)$ and (34.2) is equal to 0, by combining (34.11) and (34.14).

If $f \in c_0(\mathbf{Z}, k)$, then (27.8) and (27.9) hold for every $j \in \mathbf{Z}$, which implies that (27.11) holds in this situation as well. This means that

$$T \circ R = R \circ T = R - I \tag{34.16}$$

as linear mappings from $c_0(\mathbf{Z}, k)$ into $\ell^\infty(\mathbf{Z}, k)$, as in (27.12). More precisely, the first T in (34.16) is considered as a mapping from $\ell^\infty(\mathbf{Z}, k)$ into itself, the second T is considered as a mapping from $c_0(\mathbf{Z}, k)$ into itself, I is the identity mapping on $c_0(\mathbf{Z}, k)$, and R is considered as a mapping from $c_0(\mathbf{Z}, k)$ into $\ell^\infty(\mathbf{Z}, k)$. It follows that

$$(I - T) \circ R = R \circ (I - T) = I \tag{34.17}$$

as linear mappings from $c_0(\mathbf{Z}, k)$ into $\ell^\infty(\mathbf{Z}, k)$, as in (27.13), where the first I and T are considered as mappings on $\ell^\infty(\mathbf{Z}, k)$, and the other I's and T are considered as mappings on $c_0(\mathbf{Z}, k)$.

The second equality in (34.17) implies that

$$R(f - T(f)) = f \tag{34.18}$$

for every $f \in c_0(\mathbf{Z}, k)$. Thus

$$\|f\|_\infty = \|R(f - T(f))\|_\infty \le \|f - T(f)\|_\infty \tag{34.19}$$

for every $f \in c_0(\mathbf{Z}, k)$, using (34.10) in the second step. This gives another way to look at (33.6), with $a = 1$.

If $f \in c_0(\mathbf{Z}, k)$, then

$$(I - T)(R(f)) = f, \tag{34.20}$$

because the left side of (34.17) is equal to the identity as a mapping from $c_0(\mathbf{Z}, k)$ into $\ell^\infty(\mathbf{Z}, k)$. If we also have that the sum in (34.2) is equal to 0, then $R(f) \in c_0(\mathbf{Z}, k)$, as in (34.15). This shows that $f \in c_0(\mathbf{Z}, k)$ lies in the image of $I - T$ on $c_0(\mathbf{Z}, k)$ when the sum in (34.2) is equal to 0. We have already seen that $I - T$ maps $c_0(\mathbf{Z}, k)$ into (34.6), and the previous remarks imply that $I - T$ maps $c_0(\mathbf{Z}, k)$ onto (34.6).

35 The domain of R

Let k be a field, and let T be the usual forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3 and (26.1). If $f \in c(\mathbf{Z}, k)$, $j_1, j_2 \in \mathbf{Z}$, and $j_1 \le j_2$, then

$$\begin{aligned} \sum_{j=j_1}^{j_2} (f(j) - (T(f))(j)) &= \sum_{j=j_1}^{j_2} f(j) - \sum_{j=j_1}^{j_2} f(j-1) \\ &= \sum_{j=j_1}^{j_2} f(j) - \sum_{j=j_1-1}^{j_2-1} f(j) = f(j_2) - f(j_1 - 1). \end{aligned} \tag{35.1}$$

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. Consider the space $c_R(\mathbf{Z}, k)$ of k-valued functions f on $\mathbf{Z}$ such that

$$\sum_{j=0}^{\infty} f(-j) \tag{35.2}$$

converges in k. This is a linear subspace of $c(\mathbf{Z}, k)$, because linear combinations of convergent series converge as well. If $f \in c_R(\mathbf{Z}, k)$, then

$$\lim_{j \to -\infty} |f(j)| = 0, \tag{35.3}$$

as in Section 12. If $q = \infty$, and if k is complete with respect to the ultrametric associated to $|\cdot|$, then (35.3) implies the convergence of (35.2) in k, as in Section 12 again. Note that T maps $c_R(\mathbf{Z}, k)$ onto itself.

If $f \in c_R(\mathbf{Z}, k)$, then

$$\sum_{l=0}^{\infty} f(j - l) \tag{35.4}$$

converges in k for every $j \in \mathbf{Z}$, and we let $(R(f))(j)$ be the value of this sum, as in (27.4). This defines R as a linear mapping from $c_R(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$. Equivalently,

$$(R(f))(j) = \sum_{l=-\infty}^{j} f(l) \tag{35.5}$$

for every $f \in c_R(\mathbf{Z}, k)$ and $j \in \mathbf{Z}$, as in (28.2) and (34.8). It is easy to see that

$$\lim_{j \to -\infty} (R(f))(j) = 0 \tag{35.6}$$

in k for every $f \in c_R(\mathbf{Z}, k)$, because of the convergence of the series (35.2).

If $f \in c_R(\mathbf{Z}, k)$, then it is easy to see that (27.8) and (27.9) hold for every $j \in \mathbf{Z}$. This implies (27.11), as before, so that

$$T \circ R = R \circ T = R - I \tag{35.7}$$

as linear mappings from $c_R(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$, as in (27.12). As in previous situations, the first T in (35.7) is considered as a mapping from $c(\mathbf{Z}, k)$ into itself, the second T is considered as a mapping from $c_R(\mathbf{Z}, k)$ into itself, I is the identity mapping on $c_R(\mathbf{Z}, k)$, and R is considered as a mapping from $c_R(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$. Thus

$$(I - T) \circ R = R \circ (I - T) = I \tag{35.8}$$

as linear mappings from $c_R(\mathbf{Z}, k)$ into $c(\mathbf{Z}, k)$, as in (27.13), where the first I and T are considered as mappings on $c(\mathbf{Z}, k)$, and the other I's and T are considered as mappings on $c_R(\mathbf{Z}, k)$.

Let f be a k-valued function on $\mathbf{Z}$ that satisfies (35.3). Using (35.1), we get that

$$\sum_{l=j_1}^{j} (f(l) - (T(f))(l)) = f(j) - f(j_1 - 1) \to f(j) \quad \text{as } j_1 \to -\infty \tag{35.9}$$

for each $j \in \mathbf{Z}$. This implies that

$$f - T(f) \in c_R(\mathbf{Z}, k), \tag{35.10}$$

by taking $j = 0$ in (35.9). We also get that

$$(R(f - T(f)))(j) = \sum_{l=-\infty}^{j} (f(l) - (T(f))(l)) = f(j) \tag{35.11}$$

for each $j \in \mathbf{Z}$, by (35.5). This corresponds to the second equality in (35.8) when $f \in c_R(\mathbf{Z}, k)$.

36 Doubly-infinite sums

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$ again. If $f \in c(\mathbf{Z}, k)$ and

$$\sum_{j=1}^{\infty} f(j) \tag{36.1}$$

converges in k, then

$$\lim_{j \to \infty} |f(j)| = 0, \tag{36.2}$$

as in Section 12. As before, (36.2) implies that (36.1) converges in k when $q = \infty$ and k is complete with respect to the ultrametric associated to $|\cdot|$.

Let $c_S(\mathbf{Z}, k)$ be the space of k-valued functions f on $\mathbf{Z}$ such that (35.2) and (36.1) both converge in k. In this case,

$$\sum_{j=-\infty}^{\infty} f(j) \tag{36.3}$$

can be defined as the sum of (35.2) and (36.1). Of course, $c_S(\mathbf{Z}, k)$ is a linear subspace of the subspace $c_R(\mathbf{Z}, k)$ of $c(\mathbf{Z}, k)$ defined in the previous section. It is easy to see that the usual forward shift operator T maps $c_S(\mathbf{Z}, k)$ onto itself. If $f \in c_S(\mathbf{Z}, k)$, then one can check that

$$\sum_{j=-\infty}^{\infty} (T(f))(j) = \sum_{j=-\infty}^{\infty} f(j), \tag{36.4}$$

as in (27.2).

Suppose that $f \in c_S(\mathbf{Z}, k)$, so that $f \in c_R(\mathbf{Z}, k)$ in particular, and $R(f)$ can be defined as in the previous section. If (36.3) is equal to 0, then we have that

$$(R(f))(j) = -\sum_{l=j+1}^{\infty} f(l) \tag{36.5}$$

for each $j \in \mathbf{Z}$, because of (35.5). Note that the convergence of the sum on the right side of (36.5) follows from the convergence of (36.1). This implies that

$$\lim_{j \to \infty} (R(f))(j) = 0 \tag{36.6}$$

in k under these conditions. Combining (35.6) and (36.6), we get that

$$R(f) \in c_0(\mathbf{Z}, k). \tag{36.7}$$

Let f be a k-valued function on $\mathbf{Z}$ that satisfies (36.2). Observe that

$$\sum_{l=j}^{j_2} (f(j) - (T(f))(j)) = f(j_2) - f(j-1) \to -f(j-1) \quad \text{as } j_2 \to \infty \tag{36.8}$$

for each $j \in \mathbf{Z}$, using (35.1) in the first step. If we take $j = 1$, then we get that

$$\sum_{l=1}^{\infty} (f(l) - (T(f))(l)) = -f(0), \tag{36.9}$$

and in particular the series on the left converges in k. If $f \in c_0(\mathbf{Z}, k)$, then it follows that

$$f - T(f) \in c_S(\mathbf{Z}, k), \tag{36.10}$$

by combining the previous statement with (35.10). In this case, we also have that

$$\sum_{l=-\infty}^{0} (f(l) - (T(f))(l)) = f(0), \tag{36.11}$$

by (35.9). Combining (36.9) and (36.11), we obtain that

$$\sum_{l=-\infty}^{\infty} (f(l) - (T(f))(l)) = 0 \tag{36.12}$$

under these conditions. This could be derived from (36.4) as well when f is an element of $c_S(\mathbf{Z}, k)$.

37 Bounded partial sums

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$. If $f \in c(\mathbf{Z}, k)$, then

$$\|f\|_{BPS} = \|f\|_{BPS(\mathbf{Z},k)} = \sup\left\{\left|\sum_{j=j_1}^{j_2} f(j)\right| : j_1, j_2 \in \mathbf{Z},\ j_1 \le j_2\right\} \tag{37.1}$$

is defined as a nonnegative extended real number. Let us say that f has *bounded partial sums* on $\mathbf{Z}$ when (37.1) is finite. Observe that

$$\|f\|_\infty \le \|f\|_{BPS} \tag{37.2}$$

for every $f \in c(\mathbf{Z}, k)$, by taking $j_1 = j_2$ in (37.1). Let $BPS(\mathbf{Z}, k)$ be the space of $f \in c(\mathbf{Z}, k)$ with bounded partial sums, so that

$$BPS(\mathbf{Z}, k) \subseteq \ell^\infty(\mathbf{Z}, k), \tag{37.3}$$

by (37.2). One can check that $BPS(\mathbf{Z}, k)$ is a linear subspace of $\ell^\infty(\mathbf{Z}, k)$, and that (37.1) defines a q-norm on $BPS(\mathbf{Z}, k)$. In particular, (37.2) implies that $\|f\|_{BPS} > 0$ when $f(j) \neq 0$ for some $j \in \mathbf{Z}$.

Suppose for the moment that $q = \infty$. In this case, it is easy to see that

$$\|f\|_{BPS} \le \|f\|_\infty \tag{37.4}$$

for every $f \in c(\mathbf{Z}, k)$, by the ultrametric version of the triangle inequality. Of course, (37.2) and (37.4) imply that

$$\|f\|_{BPS} = \|f\|_\infty \tag{37.5}$$

for every $f \in c(\mathbf{Z}, k)$. Thus

$$BPS(\mathbf{Z}, k) = \ell^\infty(\mathbf{Z}, k) \tag{37.6}$$

in this situation.

Suppose now that $q < \infty$, and let $f \in c(\mathbf{Z}, k)$ be given. Observe that

$$\Big|\sum_{j=j_1}^{j_2} f(j)\Big| \le \Big(\sum_{j=j_1}^{j_2} |f(j)|^q\Big)^{1/q} \tag{37.7}$$

for every $j_1, j_2 \in \mathbf{Z}$ with $j_1 \le j_2$. This implies that

$$\|f\|_{BPS} \le \|f\|_q, \tag{37.8}$$

and hence

$$\ell^q(\mathbf{Z}, k) \subseteq BPS(\mathbf{Z}, k). \tag{37.9}$$

If $k = \mathbf{R}$ with the standard absolute value function, and f is a nonnegative real-valued function on $\mathbf{Z}$, then

$$\|f\|_{BPS} = \sum_{j=-\infty}^{\infty} f(j) = \|f\|_1. \tag{37.10}$$

In this case, f has bounded partial sums if and only if f is summable on $\mathbf{Z}$.

Let T be the usual forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3 and (26.1). If $f \in c(\mathbf{Z}, k)$, $j_1, j_2 \in \mathbf{Z}$, and $j_1 \le j_2$, then

$$\sum_{j=j_1}^{j_2} (T(f))(j) = \sum_{j=j_1}^{j_2} f(j-1) = \sum_{j=j_1-1}^{j_2-1} f(j). \tag{37.11}$$

It follows that

$$\|T(f)\|_{BPS} = \|f\|_{BPS}, \tag{37.12}$$

and that T maps $BPS(\mathbf{Z}, k)$ onto itself.

38 Normalized partial sums

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let $f \in c(\mathbf{Z}, k)$ be given. As before,

$$\|f\|_{BPS_+} = \sup\Big\{\Big|\sum_{j=1}^{j_2} f(j)\Big| : j_2 \in \mathbf{Z}_+\Big\} \tag{38.1}$$

and

$$\|f\|_{BPS_-} = \sup\Big\{\Big|\sum_{j=j_1}^{0} f(j)\Big| : j_1 \in \mathbf{Z},\ j_1 \le 0\Big\} \tag{38.2}$$

are defined as nonnegative extended real numbers. Put

$$\|f\|_{BPS_\pm} = \max(\|f\|_{BPS_+}, \|f\|_{BPS_-}), \tag{38.3}$$

which is also defined as a nonnegative extended real number. Clearly

$$\|f\|_{BPS_\pm} \le \|f\|_{BPS}, \tag{38.4}$$

because the partial sums in (38.1) and (38.2) are particular cases of the partial sums in (37.1). Similarly, we would like to estimate $\|f\|_{BPS}$ in terms of $\|f\|_{BPS_\pm}$.

Let $j_1, j_2 \in \mathbf{Z}$ be given, with $j_1 \le j_2$. If $j_1 \le 0$ and $1 \le j_2$, then

$$\sum_{j=j_1}^{j_2} f(j) = \sum_{j=j_1}^{0} f(j) + \sum_{j=1}^{j_2} f(j). \tag{38.5}$$

If $1 \le j_1$, then

$$\sum_{j=j_1}^{j_2} f(j) = \sum_{j=1}^{j_2} f(j) - \sum_{j=1}^{j_1-1} f(j), \tag{38.6}$$

where the second sum on the right side is interpreted as being equal to 0 when $j_1 = 1$. If $j_2 \le 0$, then

$$\sum_{j=j_1}^{j_2} f(j) = \sum_{j=j_1}^{0} f(j) - \sum_{j=j_2+1}^{0} f(j), \tag{38.7}$$

where the second sum on the right side is interpreted as being equal to 0 when $j_2 = 0$. In each of these three cases, one can check that

$$\left|\sum_{j=j_1}^{j_2} f(j)\right| \le 2^{1/q}\, \|f\|_{BPS_\pm}, \tag{38.8}$$

using the q-absolute value function version of the triangle inequality. Here $2^{1/q}$ is interpreted as being equal to 1 when $q = \infty$. This implies that

$$\|f\|_{BPS} \le 2^{1/q}\, \|f\|_{BPS_\pm}. \tag{38.9}$$

Thus f has bounded partial sums on $\mathbf{Z}$ if and only if $\|f\|_{BPS_\pm}$ is finite. It is easy to see that $\|f\|_{BPS_+}$, $\|f\|_{BPS_-}$ are q-seminorms on $BPS(\mathbf{Z}, k)$, and that $\|f\|_{BPS_\pm}$ is a q-norm on $BPS(\mathbf{Z}, k)$. If $q = \infty$, then

$$\|f\|_{BPS_\pm} = \|f\|_{BPS} = \|f\|_\infty \tag{38.10}$$

for every $f \in c(\mathbf{Z}, k)$, by (37.5), (38.4), and (38.9).

Let $f \in c(\mathbf{Z}, k)$ be given again, and let $R_0(f)$ be the k-valued function defined on $\mathbf{Z}$ as in (28.1). By construction,

$$\|R_0(f)\|_\infty = \|f\|_{BPS_\pm}. \tag{38.11}$$

In particular, $R_0(f)$ is bounded on $\mathbf{Z}$ if and only if f has bounded partial sums on $\mathbf{Z}$. If $q = \infty$, then (33.12) follows from (38.10) and (38.11).

Suppose that f is an element of the space $c_R(\mathbf{Z}, k)$ defined in Section 35. This means that the infinite series (35.2) converges in k, which implies that the corresponding sequence of partial sums is bounded. Thus

$$\|f\|_{BPS_-} < \infty. \tag{38.12}$$

If $R(f)$ is defined on $\mathbf{Z}$ as in Section 35, then

$$|(R(f))(j)| \le \|f\|_{BPS} \tag{38.13}$$

for every $j \in \mathbf{Z}$, by (35.5) and the definition (37.1) of $\|f\|_{BPS}$. It follows that

$$\|R(f)\|_\infty \le \|f\|_{BPS}. \tag{38.14}$$

If $j_1, j_2 \in \mathbf{Z}$ and $j_1 \le j_2$, then

$$\sum_{j=j_1}^{j_2} f(j) = (R(f))(j_2) - (R(f))(j_1 - 1), \tag{38.15}$$

by (35.5). This implies that

$$\biggl|\sum_{j=j_1}^{j_2} f(j)\biggr| \le 2^{1/q}\,\|R(f)\|_\infty, \tag{38.16}$$

and hence

$$\|f\|_{BPS} \le 2^{1/q}\,\|R(f)\|_\infty. \tag{38.17}$$

If $q = \infty$, then we get that

$$\|R(f)\|_\infty = \|f\|_{BPS} = \|f\|_\infty, \tag{38.18}$$

using (38.14) and (38.17) in the first step, and (37.5) in the second step.

If f is an element of the space $c_S(\mathbf{Z}, k)$ defined in Section 36, then f is an element of $c_R(\mathbf{Z}, k)$, and the infinite series (36.1) converges in k. It follows that the sequence of partial sums corresponding to (36.1) is bounded, so that

$$\|f\|_{BPS_+} < \infty. \tag{38.19}$$

Combining this with (38.12), we get that

$$c_S(\mathbf{Z}, k) \subseteq BPS(\mathbf{Z}, k), \tag{38.20}$$

using also (38.3) and (38.9).

39 Connections with $I - T$

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let T be the usual forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3 and (26.1). If $f \in c(\mathbf{Z}, k)$, $j_1, j_2 \in \mathbf{Z}$, and $j_1 \le j_2$, then

$$\Big|\sum_{j=j_1}^{j_2} (f(j) - (T(f))(j))\Big| \le |f(j_2) - f(j_1 - 1)|, \tag{39.1}$$

by (35.1). This implies that

$$\|f - T(f)\|_{BPS} = \sup_{l_1, l_2 \in \mathbf{Z}} |f(l_2) - f(l_1)|, \tag{39.2}$$

where $\|\cdot\|_{BPS}$ is as in (37.1). It follows that $f - T(f)$ has bounded partial sums on $\mathbf{Z}$ if and only if f is bounded on $\mathbf{Z}$. Note that

$$\sup_{l_1, l_2 \in \mathbf{Z}} |f(l_2) - f(l_1)| \le 2^{1/q} \, \|f\|_\infty, \tag{39.3}$$

and that

$$\|f\|_\infty \le \sup_{l_1, l_2 \in \mathbf{Z}} |f(l_2) - f(l_1)| \tag{39.4}$$

when $\inf_{l \in \mathbf{Z}} |f(l)| = 0$.

Suppose now that f has bounded partial sums on $\mathbf{Z}$. If $R_0(f)$ is defined on $\mathbf{Z}$ as in (28.1), then $R_0(f)$ is bounded on $\mathbf{Z}$, as in the previous section. We also have that

$$R_0(f) - T(R_0(f)) = f, \tag{39.5}$$

by (28.10). This implies that $I - T$ maps $\ell^\infty(\mathbf{Z}, k)$ onto $BPS(\mathbf{Z}, k)$. If $q = \infty$, then $BPS(\mathbf{Z}, k)$ is the same as $\ell^\infty(\mathbf{Z}, k)$, as in Section 37. In this case, the previous statement is the same as saying that $I - T$ maps $\ell^\infty(\mathbf{Z}, k)$ onto itself. This was mentioned more directly in Section 33.

Let us take $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function, for the rest of the section. Let f be a real or complex-valued function on $\mathbf{Z}$, and let $j_1, j_2 \in \mathbf{Z}$ be given, with $j_1 \le j_2$. Observe that

$$\frac{1}{j_2 - j_1 + 1} \Big|\sum_{j=j_1}^{j_2} f(j)\Big| \le \frac{1}{j_2 - j_1 + 1} \sum_{j=j_1}^{j_2} |f(j)| \le \|f\|_\infty. \tag{39.6}$$

Consider the collection of bounded functions f on $\mathbf{Z}$ such that

$$\frac{1}{j_2 - j_1 + 1} \sum_{j=j_1}^{j_2} f(j) \to 0 \tag{39.7}$$

uniformly as $j_2 - j_1 \to \infty$. More precisely, this means that for each $\epsilon > 0$ there should be a nonnegative integer L such that

$$\frac{1}{j_2 - j_1 + 1} \Big|\sum_{j=j_1}^{j_2} f(j)\Big| < \epsilon \tag{39.8}$$

for every $j_1, j_2 \in \mathbf{Z}$ such that $j_1 \le j_2$ and $j_2 - j_1 \ge L$. It is easy to see that this is a linear subspace of the space $\ell^\infty(\mathbf{Z})$ of bounded real or complex-valued functions on $\mathbf{Z}$, as appropriate. One can also check that this is a closed set in $\ell^\infty(\mathbf{Z})$ with respect to the supremum metric. Of course, nonzero constant functions on $\mathbf{Z}$ do not have this property.

Note that

$$\frac{1}{j_2 - j_1 + 1} \biggl| \sum_{j=j_1}^{j_2} f(j) \biggr| \le \frac{1}{j_2 - j_1 + 1} \, \|f\|_{BPS} \tag{39.9}$$

for every real or complex-valued function f on $\mathbf{Z}$ and $j_1, j_2 \in \mathbf{Z}$ with $j_1 \le j_2$. This implies that (39.7) holds when f has bounded partial sums on $\mathbf{Z}$. It follows that the space of real or complex-valued functions on $\mathbf{Z}$ with bounded partial sums is not dense in $\ell^\infty(\mathbf{Z})$ with respect to the supremum metric, by the remarks in the preceding paragraph.

40 Some density conditions

Let k be a field, and let T be the usual forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3 and (26.1). Remember that $I - T$ maps $c_{00}(\mathbf{Z}, k)$ onto

$$\biggl\{ f \in c_{00}(\mathbf{Z}, k) : \sum_{j=-\infty}^{\infty} f(j) = 0 \biggr\}, \tag{40.1}$$

as in Section 27. More precisely, $I - T$ maps $c_{00}(\mathbf{Z}, k)$ into (40.1), because of (27.3). The fact that $I - T$ maps $c_{00}(\mathbf{Z}, k)$ onto (40.1) was obtained from (27.15). In this section, we would like to use this to look at the image of $I - T$ on some other spaces of functions on $\mathbf{Z}$.

Let $| \cdot |$ be a q-absolute value function on k for some $q > 0$. Remember that $c_{00}(\mathbf{Z}, k)$ is dense in $\ell^r(\mathbf{Z}, k)$ when $0 < r < \infty$, as in Section 8. Similarly, $c_{00}(\mathbf{Z}, k)$ is dense in $c_0(\mathbf{Z}, k)$ with respect to the supremum q-metric.

Suppose first that $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. If $1 < r \le \infty$, then there are real or complex-valued functions f on $\mathbf{Z}$, as appropriate, such that f has finite support in $\mathbf{Z}$,

$$\sum_{j=-\infty}^{\infty} f(j) \tag{40.2}$$

is any given real or complex number, and $\|f\|_r$ is arbitrarily small. More precisely, if a is any real or complex number, $n \in \mathbf{Z}_+$, $f(j) = a/n$ for n integers j, and $f(j) = 0$ otherwise, then (40.2) is equal to a, and

$$\|f\|_r = |a| \, n^{(1/r)-1} \tag{40.3}$$

for every $r > 0$. If $r > 1$, then (40.3) tends to 0 as $n \to \infty$, as desired. Using this, one can check that (40.1) is dense in $c_{00}(\mathbf{Z}, k)$ with respect to the ℓ^r norm when $r > 1$. This implies that (40.1) is dense in $\ell^r(\mathbf{Z}, k)$ when $1 < r < \infty$, because $c_{00}(\mathbf{Z}, k)$ is dense in $\ell^r(\mathbf{Z}, k)$, as in the previous paragraph. It follows that $I - T$ maps $\ell^r(\mathbf{Z}, k)$ onto a dense linear subspace of itself when $1 < r < \infty$, because $I - T$ maps $c_{00}(\mathbf{Z}, k)$ onto (40.1). Similarly, (40.1) is dense in $c_0(\mathbf{Z}, k)$ with respect to the supremum metric, because $c_{00}(\mathbf{Z}, k)$ is dense in $c_0(\mathbf{Z}, k)$ with respect to the supremum metric, as before. Hence $I - T$ maps $c_0(\mathbf{Z}, k)$ onto a dense linear subspace of itself with respect to the supremum metric.

If $0 < r \le 1$, then r-summable functions on $\mathbf{Z}$ are summable on $\mathbf{Z}$, as in Section 8. If f is a summable real or complex-valued function on $\mathbf{Z}$, then the sum (40.2) can be defined as a real or complex-number, as appropriate. Thus the sum (40.2) can be defined as a real or complex number when f is a real or complex-valued r-summable function on $\mathbf{Z}$ and $0 < r \le 1$. In this case, we have that

$$\left| \sum_{j=-\infty}^{\infty} f(j) \right| \le \|f\|_1 \le \|f\|_r, \tag{40.4}$$

using (8.4) in the second step. This implies that (40.2) defines a bounded linear functional on $\ell^r(\mathbf{Z}, k)$ when $0 < r \le 1$, so that

$$\left\{ f \in \ell^r(\mathbf{Z}, k) : \sum_{j=-\infty}^{\infty} f(j) = 0 \right\} \tag{40.5}$$

is a closed linear subspace of $\ell^r(\mathbf{Z}, k)$ when $0 < r \le 1$.

Observe that $I - T$ maps $\ell^r(\mathbf{Z}, k)$ into (40.5) when $0 < r \le 1$. This was mentioned in Section 29 when $r = 1$, which implies the analogous statement for $0 < r \le 1$. In fact, $I - T$ maps $\ell^r(\mathbf{Z}, k)$ onto a dense linear subspace of (40.5) when $0 < r \le 1$. This was also mentioned in Section 29 when $r = 1$, and the analogous statement for $0 < r \le 1$ can be shown in essentially the same way. The main point is that (40.1) is dense in (40.5) with respect to the ℓ^r r-norm when $0 < r \le 1$, as before.

Now let k be a field with an ultrametric absolute value function $|\cdot|$, and suppose that k is complete with respect to the associated ultrametric. If f is an element of $c_0(\mathbf{Z}, k)$, then the sum (40.2) can be defined as an element of k, as in Section 34. More precisely, this sum defines a bounded linear functional on $c_0(\mathbf{Z}, k)$ with respect to the corresponding supremum norm, as before. Thus

$$\left\{ f \in c_0(\mathbf{Z}, k) : \sum_{j=-\infty}^{\infty} f(j) = 0 \right\} \tag{40.6}$$

is a closed linear subspace of $c_0(\mathbf{Z}, k)$, with respect to the supremum metric. Remember that $I - T$ maps $c_0(\mathbf{Z}, k)$ onto (40.6) under these conditions, as in Section 34. One can check directly that (40.1) is dense in (40.6) with respect to the supremum metric.

Let r be a positive real number, and remember that $\ell^r(\mathbf{Z}, k)$ is contained in $c_0(\mathbf{Z}, k)$, as in Section 8. If $f \in \ell^r(\mathbf{Z}, k)$, then it follows that the sum (40.2) can be defined as an element of k, as before. We also have that

$$\left|\sum_{j=-\infty}^{\infty} f(j)\right| \le \|f\|_\infty \le \|f\|_r, \tag{40.7}$$

using (34.3) in the first step, and (8.4) in the second step. Thus the sum (40.2) defines a bounded linear functional on $\ell^r(\mathbf{Z}, k)$ as well, which implies that (40.5) is a closed linear subspace of $\ell^r(\mathbf{Z}, k)$. Of course, $I - T$ maps $\ell^r(\mathbf{Z}, k)$ into (40.5), because $I - T$ maps $c_0(\mathbf{Z}, k)$ into (40.6), as before.

One can check directly that (40.1) is dense in (40.5) with respect to the ℓ^r r-norm in this situation. This is analogous to the corresponding statements for real and complex-valued functions on $\mathbf{Z}$, mentioned earlier. As before, this basically uses the density of $c_{00}(\mathbf{Z}, k)$ in $\ell^r(\mathbf{Z}, k)$, with an additional adjustment to deal with the condition on the sum. It follows that $I - T$ maps $\ell^r(\mathbf{Z}, k)$ onto a dense linear subspace of (40.5) with respect to the ℓ^r r-norm, because $I - T$ maps $c_{00}(\mathbf{Z}, k)$ onto (40.1).

41 Limits of partial sums

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let f be a k-valued function on $\mathbf{Z}$. To say that the limit

$$\lim_{\substack{j_1 \to -\infty \\ j_2 \to \infty}} \sum_{j=j_1}^{j_2} f(j) \tag{41.1}$$

exists and is equal to $a \in k$ means that for each $\epsilon > 0$ there is a nonnegative integer L such that

$$\left|\sum_{j=j_1}^{j_2} f(j) - a\right| < \epsilon \tag{41.2}$$

for every $j_1 \le -L$ and $j_2 \ge L$. It is easy to see that the limit a is unique when it exists, by standard arguments. In this case, (41.1) can be used as another way to define the sum of $f(j)$ over $j \in \mathbf{Z}$. If f is in the space $c_S(\mathbf{Z}, k)$ defined in Section 36, then this limit exists and is equal to the value of the sum defined there. Let $c_{S,2}(\mathbf{Z}, k)$ be the space of k-valued functions f on $\mathbf{Z}$ such that the limit (41.1) exists. This is a linear subspace of $c(\mathbf{Z}, k)$, and the value of the limit (41.1) defines a linear functional on $c_{S,2}(\mathbf{Z}, k)$.

Let $BPS_0(\mathbf{Z}, k)$ be the space of k-valued functions on $\mathbf{Z}$ with the following property: for each $\epsilon > 0$ there is a nonnegative integer L such that

$$\left|\sum_{j=j_1}^{j_2} f(j) - \sum_{j=j_1'}^{j_2'} f(j)\right| < \epsilon \tag{41.3}$$

for every $j_1, j_2, j_1', j_2' \in \mathbf{Z}$ such that $j_1, j_1' \le -L$ and $j_2, j_2' \ge L$. This may be considered as the Cauchy condition corresponding to the existence of the limit

(41.1). In particular, if the limit (41.1) exists, then this Cauchy condition holds, by standard arguments. If we restrict our attention to $j_1 = j_1'$ in this Cauchy condition, then we get that

$$\Big|\sum_{j=j_2+1}^{j_2'} f(j)\Big| < \epsilon \tag{41.4}$$

when $j_2' > j_2 \geq L$. Similarly, if we restrict our attention to $j_2 = j_2'$, then we get that

$$\Big|\sum_{j=j_1'}^{j_1-1} f(j)\Big| < \epsilon \tag{41.5}$$

when $j_1' < j_1 \leq -L$. Conversely, if these two conditions are satisfied, then one can check that the earlier Cauchy condition holds too. More precisely, one can use (41.4) and (41.5) to get a condition like (41.3), with an extra factor of $2^{1/q}$ on the right side.

The condition (41.4) is equivalent to saying that the sequence of partial sums corresponding to the infinite series $\sum_{j=1}^{\infty} f(j)$ is a Cauchy sequence in k. Similarly, the condition (41.5) is equivalent to saying that the sequence of partial sums corresponding to $\sum_{j=0}^{\infty} f(-j)$ is a Cauchy sequence. If k is complete with respect to the q-metric associated to $|\cdot|$, then these Cauchy conditions imply that the two series converge in k. This means that f is in the space $c_S(\mathbf{Z}, k)$ defined in Section 36. In particular, this implies that the limit (41.1) exists in k, as before.

If $f \in BPS_0(\mathbf{Z}, k)$, then one can check that f has bounded partial sums on $\mathbf{Z}$, as in Section 37. More precisely, $BPS_0(\mathbf{Z}, k)$ is a closed linear subspace of $BPS(\mathbf{Z}, k)$, with respect to the BPS q-norm. In fact, $BPS_0(\mathbf{Z}, k)$ is the closure of $c_{00}(\mathbf{Z}, k)$ in $BPS(\mathbf{Z}, k)$. In particular, $BPS_0(\mathbf{Z}, k)$ is contained in $c_0(\mathbf{Z}, k)$. If $q = \infty$, then $BPS_0(\mathbf{Z}, k)$ is the same as $c_0(\mathbf{Z}, k)$.

If $f \in c_{S,2}(\mathbf{Z}, k)$, then

$$\Big|\lim_{\substack{j_1 \to -\infty \\ j_2 \to \infty}} \sum_{j=j_1}^{j_2} f(j)\Big| \leq \|f\|_{BPS}, \tag{41.6}$$

where $\|f\|_{BPS}$ is the BPS q-norm of f, as in (37.1). Thus the mapping from f to the value of the limit (41.1) defines a bounded linear functional on $c_{S,2}(\mathbf{Z}, k)$ with respect to the BPS q-norm. The kernel of this linear functional is the collection of k-valued functions f on $\mathbf{Z}$ such that

$$\lim_{\substack{j_1 \to -\infty \\ j_2 \to \infty}} \sum_{j=j_1}^{j_2} f(j) = 0. \tag{41.7}$$

The boundedness of this linear functional implies that its kernel is relatively closed in $c_{S,2}(\mathbf{Z}, k)$, with respect to the topology determined by the BPS q-norm. In fact, one can check that the collection of k-valued functions f on $\mathbf{Z}$ for which (41.7) holds is a closed linear subspace of $BPS(\mathbf{Z}, k)$, with respect to the topology determined by the BPS q-norm.

42 The inverse of $I - a\,T$

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and suppose that k is complete with respect to the associated q-metric. Also let T be the usual forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3 and (26.1). Of course, T maps $\ell^r(\mathbf{Z}, k)$ isometrically onto itself for each $r > 0$. If $a \in k$ and $|a| < 1$, then $I - a\,T$ has a bounded inverse on $\ell^r(\mathbf{Z}, k)$ for every $r > 0$, as in Section 23. The inverse is given by

$$(I - a\,T)^{-1} = \sum_{l=0}^{\infty} a^l\, T^l, \tag{42.1}$$

where the series converges in $\mathcal{BL}(\ell^r(\mathbf{Z}, k))$, as before.

In particular, if $f \in \ell^r(\mathbf{Z}, k)$ for some $r > 0$, then

$$((I - a\,T)^{-1})(f) = \sum_{l=0}^{\infty} a^l\, T^l(f), \tag{42.2}$$

where the series converges in $\ell^r(\mathbf{Z}, k)$. It follows that

$$(((I - a\,T)^{-1})(f))(j) = \sum_{l=0}^{\infty} a^l\, (T^l(f))(j) \tag{42.3}$$

for each $j \in \mathbf{Z}$, where the series converges in k. Equivalently,

$$(((I - a\,T)^{-1})(f))(j) = \sum_{l=0}^{\infty} a^l\, f(j - l) \tag{42.4}$$

for each $j \in \mathbf{Z}$, using (3.3) on the right side. The convergence of the series on the right side of (42.4) in k for each $j \in \mathbf{Z}$ can be obtained directly from the remarks in Section 12 when $|a| < 1$ and f is bounded on $\mathbf{Z}$. Similarly, let us look more directly at some of the properties of the linear mapping defined by this series.

Suppose for the moment that $r \in \mathbf{R}_+$, $r \le q$, and $f \in \ell^r(\mathbf{Z}, k)$. Remember that $|\cdot|$ can also be considered as an r-absolute value function on k, because $r \le q$. Thus

$$\Big|\sum_{l=0}^{\infty} a^l\, f(j - l)\Big|^r \le \sum_{l=0}^{\infty} |a|^{l\,r}\, |f(j - l)|^r \tag{42.5}$$

for every $j \in \mathbf{Z}$, as in (12.3). This implies that

$$\sum_{j=-\infty}^{\infty} \Big|\sum_{l=0}^{\infty} a^l\, f(j - l)\Big|^r \le \sum_{j=-\infty}^{\infty} \sum_{l=0}^{\infty} |a|^{l\,r}\, |f(j - l)|^l. \tag{42.6}$$

Interchanging the order of summation, we get that this double sum is equal to

$$\sum_{l=0}^{\infty} \sum_{j=-\infty}^{\infty} |a|^{l\,r}\, |f(j - l)|^r = \sum_{l=0}^{\infty} \sum_{j=-\infty}^{\infty} |a|^{l\,r} |f(j)|^r = (1 - |a|^r)^{-1}\, \|f\|_r^r. \tag{42.7}$$

This corresponds to (23.8), with $V = \ell^r(\mathbf{Z}, k)$ and $q_V = r$. Note that the previous inequalities are equalities when $f(j) = 0$ for all but one $j \in \mathbf{Z}$. This implies that the operator r-norm of (42.1) on $\ell^r(\mathbf{Z}, k)$ is equal to

$$(1 - |a|^r)^{-1/r} \tag{42.8}$$

for every $a \in k$ with $|a| < 1$ when $r \le q$.

Suppose now that $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function, and that $1 \le r \le \infty$. In this case, the operator norm of (42.1) on $\ell^r(\mathbf{Z}, k)$ is equal to

$$(1 - |a|)^{-1} \tag{42.9}$$

when $|a| < 1$. The fact that the operator norm is less than or equal to (42.9) corresponds to (23.8) again, with $V = \ell^r(\mathbf{Z}, k)$ and $q_V = 1$. Of course, if $a = 0$, then (42.1) is the identity operator, which has operator norm 1. Otherwise, suppose that $a \ne 0$, and let b be the real or complex number, as appropriate, such that $a\,b = |a|$. Thus $|b| = 1$, so that b is an approximate eigenvalue of T on $\ell^r(\mathbf{Z}, k)$, as in Section 32. This implies that $a\,b$ is an approximate eigenvalue of $a\,T$ on $\ell^r(\mathbf{Z}, k)$, so that $1 - a\,b$ is an approximate eigenvalue of $I - a\,T$ on $\ell^r(\mathbf{Z}, k)$. It follows that

$$(1 - |a|)^{-1} = |1 - a\,b|^{-1} \tag{42.10}$$

is less than or equal to the operator norm of (42.1) on $\ell^r(\mathbf{Z}, k)$, as in (30.20). Thus the operator norm of (42.1) on $\ell^r(\mathbf{Z}, k)$ is equal to (42.9), as desired. Note that b is an eigenvalue of T on $\ell^\infty(\mathbf{Z}, k)$, as in Section 26, so that $1 - a\,b$ is an eigenvalue of $I - a\,T$ on $\ell^\infty(\mathbf{Z}, k)$. This does not work for the restriction of T to $c_0(\mathbf{Z}, k)$, but b is an approximate eigenvalue for the restriction of T to $c_0(\mathbf{Z}, k)$ with respect to the supremum norm, as in Section 32 again. Hence the operator norm of (42.1) on $c_0(\mathbf{Z}, k)$ with respect to the supremum norm is equal to (42.9) when $|a| < 1$, by the same type of argument as before.

Part III
Unilateral shift operators

43 Forward and backward shifts

Let k be a field, and let $\mathbf{Z}_{0+} = \mathbf{Z}_+ \cup \{0\}$ be the set of nonnegative integers. If f is a k-valued function on $\mathbf{Z}_{0+}$, then let $A(f)$ be the k-valued function defined on $\mathbf{Z}_{0+}$ by

$$\begin{aligned} (A(f))(j) &= f(j-1) \quad \text{when } j \ge 1 \\ &= 0 \qquad\qquad \text{when } j = 0. \end{aligned} \tag{43.1}$$

This defines a linear mapping A from $c(\mathbf{Z}_{0+}, k)$ into itself, which is the *forward shift operator* on $c(\mathbf{Z}_{0+}, k)$. Similarly, let $B(f)$ be the k-valued function defined on $\mathbf{Z}_{0+}$ by

(43.2) $$(B(f))(j) = f(j+1)$$

for every $j \geq 0$. This defines a linear mapping B from $c(\mathbf{Z}_{0+}, k)$ into itself, which is the *backward shift operator* on $c(\mathbf{Z}_{0+}, k)$.

More precisely, A is a one-to-one linear mapping from $c(\mathbf{Z}_{0+}, k)$ into itself, and B maps $c(\mathbf{Z}_{0+}, k)$ onto itself. If f is any k-valued function on $\mathbf{Z}_{0+}$, then

(43.3) $$(B(A(f)))(j) = f(j)$$

for every $j \in \mathbf{Z}_{0+}$. Thus

(43.4) $$B \circ A = I$$

as linear mappings from $c(\mathbf{Z}_{0+}, k)$ into itself. Similarly,

(43.5) $$\begin{aligned} (A(B(f)))(j) &= f(j) \quad \text{when } j \geq 1 \\ &= 0 \quad\;\;\;\, \text{when } j = 0 \end{aligned}$$

for every $f \in c(\mathbf{Z}_{0+}, k)$. Note that A maps $c(\mathbf{Z}_{0+}, k)$ onto the space of k-valued functions f on $\mathbf{Z}_{0+}$ such that $f(0) = 0$, and the kernel of B consists of the k-valued functions f on $\mathbf{Z}_{0+}$ whose support is contained in $\{0\}$.

Let l be a positive integer, and let A^l, B^l be the lth powers of A, B as linear mappings from $c(\mathbf{Z}_{0+}, k)$ into itself with respect to composition, as usual. If $f \in c(\mathbf{Z}_{0+}, k)$, then

(43.6) $$\begin{aligned} (A^l(f))(j) &= f(j-l) \quad \text{when } j \geq l \\ &= 0 \quad\quad\quad\;\; \text{when } 0 \leq j \leq l-1. \end{aligned}$$

Similarly,

(43.7) $$(B^l(f))(j) = f(j+l)$$

for every $j \geq 0$. In particular,

(43.8) $$B^l \circ A^l = I$$

as linear mappings from $c(\mathbf{Z}_{0+}, k)$ into itself, which can also be obtained from (43.4). If $f \in c(\mathbf{Z}_{0+}, k)$ again, then

(43.9) $$\begin{aligned} (A^l(B^l(f)))(j) &= f(j) \quad \text{when } j \geq l \\ &= 0 \quad\;\;\;\, \text{when } 0 \leq j \leq l-1. \end{aligned}$$

Let $n \in \mathbf{Z}_{0+}$ be given, and let $\delta_n(j) = \delta_{\mathbf{Z}_{0+},n}(j)$ be the k-valued function defined on $\mathbf{Z}_{0+}$ as in (2.2), which is equal to 1 when $j = n$ and to 0 otherwise. Also let $l \in \mathbf{Z}_+$ be given, and observe that

(43.10) $$\begin{aligned} (A^l(\delta_n))(j) &= \delta_n(j-l) = \delta_{n+l}(j) \quad \text{when } j \geq l \\ &= 0 = \delta_{n+l}(j) \quad\quad\quad\;\; \text{when } 0 \leq j \leq l-1. \end{aligned}$$

This implies that

(43.11) $$A^l(\delta_n) = \delta_{n+l}.$$

If $j \in \mathbf{Z}_{0+}$, then

$$(B^l(\delta_n))(j) = \delta_n(j+l) \quad = \quad \delta_{n-l}(j) \quad \text{when } l \le n \tag{43.12}$$
$$= \quad 0 \qquad \text{when } l > n.$$

It follows that

$$B^l(\delta_n) \quad = \quad \delta_{n-l} \quad \text{when } l \le n \tag{43.13}$$
$$= \quad 0 \qquad \text{when } l > n.$$

Let T be the usual forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3. Also let f be a k-valued function on $\mathbf{Z}$ such that

$$f(j) = 0 \tag{43.14}$$

when $j < 0$, and let f_0 be the restriction of f to $\mathbf{Z}_{0+}$. If $l \in \mathbf{Z}_+$, then

$$(T^l(f))(j) = (A^l(f_0))(j) \tag{43.15}$$

for every $j \ge 0$. Thus $A^l(f_0)$ is the same as the restriction of $T^l(f)$ to $\mathbf{Z}_{0+}$. Note that $(T^l(f))(j) = f(j-l) = 0$ when $j < 0$, $l \ge 1$, and f satisfies (43.14). Similarly,

$$(T^{-l}(f))(j) = f(j+l) = (B^l(f_0))(j) \tag{43.16}$$

for every $j \ge 0$ and $l \ge 1$, so that $B^l(f_0)$ is the same as the restriction of $T^{-l}(f)$ to $\mathbf{Z}_{0+}$. More precisely, (43.16) holds for every $f \in c(\mathbf{Z}, k)$, $j \ge 0$, and $l \ge 1$, without the additional condition (43.14).

44 Polynomials and power series

Let k be a field, and let X be an indeterminate. As in [4, 7], we use upper-case letters like X for indeterminates, and lower-case letters like x for elements of k. If $f \in c(\mathbf{Z}_{0+}, k)$, then

$$F(X) = \sum_{j=0}^{\infty} f(j)\, X^j \tag{44.1}$$

is a *formal power series* in X with coefficients in k. The space of formal power series in X with coefficients in k is typically denoted $k[[X]]$. Of course, a formal power series is characterized by its coefficients, so that the mapping from a k-valued function f on $\mathbf{Z}_{0+}$ to $F(X)$ is a one-to-one correspondence between $c(\mathbf{Z}_{0+}, k)$ and $k[[X]]$. Thus one may use $c(\mathbf{Z}_{0+}, k)$ as a precise definition of $k[[X]]$, and use (44.1) as notation for elements of $k[[X]]$. Note that $k[[X]]$ is a vector space over k with respect to termwise addition and scalar multiplication of formal power series, which correspond exactly to pointwise addition and scalar multiplication of k-valued functions on $\mathbf{Z}_{0+}$.

There is a natural way to multiply formal power series, where

$$X^j\, X^l = X^{j+l} \tag{44.2}$$

for all nonnegative integers j, l. It is well known and easy to see that $k[[X]]$ is a commutative algebra over k with respect to multiplication. Let $f \in c(\mathbf{Z}_{0+}, k)$ be given, and let $A(f)$ be as defined in (43.1). Thus f and $g = A(f)$ determine formal power series $F(X)$ and $G(X)$, as in (44.1). Observe that

$$
\begin{aligned}
(44.3) \qquad G(X) = \sum_{j=0}^{\infty} (A(f))(j)\, X^j &= \sum_{j=1}^{\infty} f(j-1)\, X^j \\
&= \sum_{j=0}^{\infty} f(j)\, X^{j+1} = F(X)\, X.
\end{aligned}
$$

A *formal polynomial* in X with coefficients in k may be considered as a formal power series in which all but finitely many coefficients are equal to 0. If $f \in c_{00}(\mathbf{Z}_{0+}, k)$, then the corresponding formal power series (44.1) is a formal polynomial. The space of formal polynomials in X with coefficients in k is typically denoted $k[X]$, and is a subalgebra of $k[[X]]$. As before, one can use $c_{00}(\mathbf{Z}_{0+}, k)$ as a precise definition of $k[X]$. Note that the shift operators A, B map $c_{00}(\mathbf{Z}_{0+}, k)$ into itself.

If $f \in c(\mathbf{Z}, k)$, then

$$
(44.4) \qquad F(X) = \sum_{j=-\infty}^{\infty} f(j)\, X^j
$$

may be considered as a *formal Laurent series* in X with coefficients in k. As usual, one can use $c(\mathbf{Z}, k)$ as a precise definition of the space of formal Laurent series in X with coefficients in k. Pointwise addition and scalar multiplication of k-valued functions on $\mathbf{Z}$ corresponds to termwise addition and scalar multiplication of formal Laurent series, by construction. Although the product of two formal Laurent series is not always defined, it can be defined in some situations. In particular, it is easy to multiply a formal Laurent series $F(X)$ with a monomial X^l for any $l \in \mathbf{Z}$. Let $f \in c(\mathbf{Z}, k)$ be given, and let $g = T(f)$ be as in (3.1). If $F(X)$ and $G(X)$ are the corresponding Laurent series, as in (44.4), then

$$
\begin{aligned}
(44.5) \qquad G(X) = \sum_{j=-\infty}^{\infty} (T(f))(j)\, X^j &= \sum_{j=-\infty}^{\infty} f(j-1)\, X^j \\
&= \sum_{j=-\infty}^{\infty} f(j)\, X^{j+1} = F(X)\, X.
\end{aligned}
$$

Of course, one can identify formal power series in X with formal Laurent series in X such that the coefficient of X^j is equal to 0 when $j < 0$.

45 Extension and restriction mappings

Let k be a field, let X be a nonempty set, and let Y be a nonempty subset of X. If f is a k-valued function on X, then let $R_Y(f)$ be the restriction of f to

Y. Thus R_Y defines a linear mapping from $c(X,k)$ onto $c(Y,k)$. Note that R_Y maps $c_{00}(X,k)$ onto $c_{00}(Y,k)$.

If f_0 is a k-valued function on Y, then let $E_Y(f_0)$ be the k-valued function on X defined by

$$(E_Y(f_0))(x) = f_0(x) \quad \text{when } x \in Y \tag{45.1}$$
$$= 0 \quad \text{when } x \in X \setminus Y.$$

This defines a one-to-one linear mapping from $c(Y,k)$ into $c(X,k)$. More precisely, E_Y maps $c(Y,k)$ onto the linear subspace

$$c_Y(X,k) = \{f \in c(X,k) : \operatorname{supp} f \subseteq Y\} \tag{45.2}$$

of $c(X,k)$, and

$$E_Y(c_{00}(Y,k)) = c_Y(X,k) \cap c_{00}(X,k). \tag{45.3}$$

Of course,

$$R_Y(E_Y(f_0)) = f_0 \tag{45.4}$$

for every $f_0 \in c(Y,k)$, so that $R_Y \circ E_Y$ is the identity mapping on $c(Y,k)$.

If $f \in c(X,k)$, then let $P_Y(f)$ be the k-valued function defined on X by

$$(P_Y(f))(x) = f(x) \quad \text{when } x \in Y \tag{45.5}$$
$$= 0 \quad \text{when } x \in X \setminus Y.$$

This defines a linear mapping from $c(X,k)$ onto $c_Y(X,k)$, which maps $c_{00}(X,k)$ onto (45.3). Observe that

$$P_Y \circ P_Y = P_Y, \tag{45.6}$$

so that P_Y defines a projection on $c(X,k)$. We also have that

$$E_Y(R_Y(f)) = P_Y(f) \tag{45.7}$$

for every $f \in c(X,k)$, so that $E_Y \circ R_Y = P_Y$ as linear mappings on $c(X,k)$. Note that P_Y is the same as the multiplication operator on $c(X,k)$ corresponding to the k-valued function on X that is equal to 1 on Y and to 0 on $X \setminus Y$, as in Section 19.

Let us now take $X = \mathbf{Z}$ and $Y = \mathbf{Z}_{0+}$, so that the restriction operator $R_{\mathbf{Z}_{0+}}$, the extension operator $E_{\mathbf{Z}_{0+}}$, and the projection $P_{\mathbf{Z}_{0+}}$ can be defined as before. Also let A, B be the forward and backward shift operators on $c(\mathbf{Z}_{0+},k)$, as in Section 43, and let T be the forward shift operator on $c(\mathbf{Z},k)$, as in Section 3. The condition (43.14) means that $f \in c_{\mathbf{Z}_{0+}}(\mathbf{Z},k)$, using the notation in (45.2). Thus (43.15) says that

$$R_{\mathbf{Z}_{0+}}(T^l(f)) = A^l(R_{\mathbf{Z}_{0+}}(f)) \tag{45.8}$$

for every $f \in c_{\mathbf{Z}_{0+}}(\mathbf{Z},k)$ and $l \geq 1$. Similarly,

$$R_{\mathbf{Z}_{0+}}(T^{-l}(f)) = B^l(R_{\mathbf{Z}_{0+}}(f)) \tag{45.9}$$

for every $f \in c(\mathbf{Z}, k)$ and $l \geq 1$, as in (43.16).

Observe that

$$T(c_{\mathbf{Z}_{0+}}(\mathbf{Z}, k)) \subseteq c_{\mathbf{Z}_{0+}}(\mathbf{Z}, k). \tag{45.10}$$

If $f_0 \in c(\mathbf{Z}_{0+}, k)$, then $f = E_{\mathbf{Z}_{0+}}(f_0) \in c_{\mathbf{Z}_{0+}}(\mathbf{Z}, k)$, and

$$T^l(E_{\mathbf{Z}_{0+}}(f_0)) = E_{\mathbf{Z}_{0+}}(A^l(f_0)) \tag{45.11}$$

for every $l \geq 1$. This is a more precise version of (43.15) and (45.8). If we identify $c(\mathbf{Z}_{0+}, k)$ with $c_{\mathbf{Z}_{0+}}(\mathbf{Z}, k)$ using $E_{\mathbf{Z}_{0+}}$, then A^l corresponds to the restriction of T^l to $c_{\mathbf{Z}_{0+}}(\mathbf{Z}, k)$ for each $l \geq 1$. We also have that

$$P_{\mathbf{Z}_{0+}}(T^{-l}(E_{\mathbf{Z}_{0+}}(f_0))) = E_{\mathbf{Z}_{0+}}(B^l(f_0)) \tag{45.12}$$

for every $f_0 \in c(\mathbf{Z}_{0+}, k)$ and $l \geq 1$, as in (43.16) and (45.9).

46 Dual mappings

Let k be a field. If $f \in c_{00}(\mathbf{Z}_{0+}, k)$ and $g \in c(\mathbf{Z}_{0+}, k)$, then

$$\lambda_g(f) = \sum_{j=0}^{\infty} f(j)\, g(j) \tag{46.1}$$

reduces to a finite sum in k. This defines a linear functional on $c_{00}(\mathbf{Z}_{0+}, k)$ for each $g \in c(\mathbf{Z}_{0+}, k)$, and every linear functional on $c_{00}(\mathbf{Z}_{0+}, k)$ is of this form, as in Section 2. Thus

$$g \mapsto \lambda_g \tag{46.2}$$

defines an isomorphism between $c(\mathbf{Z}_{0+}, k)$ and the algebraic dual $c_{00}(\mathbf{Z}_{0+}, k)^{\mathrm{alg}}$ of $c_{00}(\mathbf{Z}_{0+}, k)$ as vector spaces over k, as before.

Let A, B be the forward and backward shift operators on $c(\mathbf{Z}_{0+}, k)$, as in Section 43. Remember that A, B map $c_{00}(\mathbf{Z}_{0+}, k)$ into itself. If $f \in c_{00}(\mathbf{Z}_{0+}, k)$ and $g \in c(\mathbf{Z}_{0+}, k)$, then

$$\lambda_g(A(f)) = \sum_{j=1}^{\infty} f(j-1)\, g(j), \tag{46.3}$$

by the definition (43.1) of $A(f)$. It follows that

$$\lambda_g(A(f)) = \sum_{j=0}^{\infty} f(j)\, g(j+1) = \sum_{j=0}^{\infty} f(j)\, (B(g))(j) = \lambda_{B(g)}(f) \tag{46.4}$$

using the definition (43.2) of B in the second step. This shows that the algebraic dual A^{alg} of A on $c_{00}(\mathbf{Z}_{0+}, k)$ corresponds to B on $c(\mathbf{Z}_{0+}, k)$, with respect to the isomorphism between $c_{00}(\mathbf{Z}_{0+}, k)^{\mathrm{alg}}$ and $c(\mathbf{Z}_{0+}, k)$ mentioned in the previous paragraph. Similarly,

$$\lambda_g(B(f)) = \sum_{j=0}^{\infty} f(j+1)\, g(j) \tag{46.5}$$

for every $f \in c_{00}(\mathbf{Z}_{0+}, k)$ and $g \in c(\mathbf{Z}_{0+}, k)$, by the definition (43.2) of $B(f)$. Hence

$$\lambda_g(B(f)) = \sum_{j=1}^{\infty} f(j)\, g(j-1) = \sum_{j=0}^{\infty} f(j)\, (A(g))(j) = \lambda_{A(g)}(f), \tag{46.6}$$

using the definition (43.1) of A in the second step. Thus the algebraic dual B^{alg} of B on $c_{00}(\mathbf{Z}_{0+}, k)$ corresponds to A on $c(\mathbf{Z}_{0+}, k)$, in the same way as before.

Let X be a nonempty set, and let Y be a nonempty subset of X. The corresponding projection operator P_Y defined in (45.5) maps $c(X, k)$ into itself, and $c_{00}(X, k)$ into itself. If $f \in c_{00}(X, k)$ and $g \in c(X, k)$, then

$$\sum_{x \in X} (P_X(f))(x)\, g(x) = \sum_{x \in Y} f(x)\, g(x) = \sum_{x \in X} f(x)\, (P_Y(g))(x). \tag{46.7}$$

This implies that the algebraic dual $(P_Y)^{\mathrm{alg}}$ of P_Y on $c_{00}(X, k)$ corresponds to P_Y on $c(X, k)$, with respect to the usual identification between the algebraic dual $c_{00}(X, k)^{\mathrm{alg}}$ of $c_{00}(X, k)$ and $c(X, k)$, as in Section 2.

The restriction operator R_Y defined in the previous section maps $c(X, k)$ into $c(Y, k)$, and $c_{00}(X, k)$ into $c_{00}(Y, k)$. The extension operator E_Y defined in (45.1) maps $c(Y, k)$ into $c(X, k)$, and $c_{00}(Y, k)$ into $c_{00}(X, k)$. If $f \in c_{00}(X, k)$ and $g_0 \in c(Y, k)$, then

$$\sum_{x \in Y} (R_Y(f))(x)\, g_0(x) = \sum_{x \in Y} f(x)\, g_0(x) = \sum_{x \in X} f(x)\, (E_Y(g_0))(x). \tag{46.8}$$

This means that the algebraic dual $(R_Y)^{\mathrm{alg}}$ or R_Y as a linear mapping from $c_{00}(X, k)$ into $c_{00}(Y, k)$ corresponds to E_Y as a linear mapping from $c(Y, k)$ into $c(X, k)$. This uses the identification of the algebraic dual $c_{00}(X, k)^{\mathrm{alg}}$ of $c_{00}(X, k)$ with $c(X, k)$ discussed in Section 2, and the analogous identification of the algebraic dual $c_{00}(Y, k)^{\mathrm{alg}}$ or $c_{00}(Y, k)$ with $c(Y, k)$. If $f_0 \in c_{00}(Y, k)$ and $g \in c(X, k)$, then

$$\sum_{x \in X} (E_Y(f_0))(x)\, g(x) = \sum_{x \in Y} f_0(x)\, g(x) = \sum_{x \in Y} f_0(x)\, (R_Y(g))(x). \tag{46.9}$$

This implies that the algebraic dual $(E_Y)^{\mathrm{alg}}$ of E_Y as a linear mapping from $c_{00}(Y, k)$ into $c_{00}(X, k)$ corresponds to R_Y as a linear mapping from $c(X, k)$ into $c(Y, k)$, using the same identifications of the algebraic dual spaces as before.

47 Boundedness on ℓ^r spaces

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let $r > 0$ be given. If $f \in \ell^r(\mathbf{Z}_{0+}, k)$, then it is easy to see that

$$A(f),\ B(f) \in \ell^r(\mathbf{Z}_{0+}, k), \tag{47.1}$$

where $A(f)$ is as in (43.1), and $B(f)$ is as in (43.2). More precisely,

$$\|A(f)\|_r = \|f\|_r \tag{47.2}$$

for every $f \in \ell^r(\mathbf{Z}_{0+}, k)$, so that A defines an isometric linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself. We also have that

$$\|B(f)\|_r \le \|f\|_r \tag{47.3}$$

for every $f \in \ell^r(\mathbf{Z}_{0+}, k)$, so that B defines a bounded linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself, with operator q or r-norm less than or equal to 1. If $l \in \mathbf{Z}_+$ and $f \in \ell^r(\mathbf{Z}_{0+}, k)$ satisfies $f(j) = 0$ when $0 \le j < l$, then one can check that

$$\|B^l(f)\|_r = \|f\|_r, \tag{47.4}$$

using (43.7). In particular, this implies that the operator q or r-norm of B^l on $\ell^r(\mathbf{Z}_{0+}, k)$ is equal to 1 for every $l \ge 1$. Note that B maps $\ell^r(\mathbf{Z}_{0+}, k)$ onto itself.

Similarly, A maps $c_0(\mathbf{Z}_{0+}, k)$ into itself, and B maps $c_0(\mathbf{Z}_{0+}, k)$ onto itself. If $l \in \mathbf{Z}_+$, then the restriction of B^l to $c_0(\mathbf{Z}_{0+}, k)$ has operator q-norm equal to 1 with respect to the supremum q-norm on $c_0(\mathbf{Z}_{0+}, k)$. This uses (47.3) and (47.4), with $r = \infty$.

Let X be a nonempty set, let Y be a nonempty subset of X, and let P_Y be the projection operator defined in (45.5). If $f \in \ell^r(X, k)$ for some $r > 0$, then $P_Y(f) \in \ell^r(X, k)$, and

$$\|P_Y(f)\|_r \le \|f\|_r, \tag{47.5}$$

with equality when f is supported in Y. Thus P_Y defines a bounded linear mapping from $\ell^r(X, k)$ into itself, with operator q or r-norm equal to 1. Similarly, P_Y maps $c_0(X, k)$ into itself, with operator q-norm equal to 1 with respect to the supremum q-norm on $c_0(X, k)$. These statements about operator norms can also be obtained from the remarks in Section 19, because P_Y corresponds to multiplication by a k-valued function on X with supremum q-norm equal to 1, as in Section 45.

Let R_Y be the restriction mapping from $c(X, k)$ onto $c(Y, k)$, as in Section 45. If $f \in \ell^r(X, k)$ for some $r > 0$, then $R_Y(f) \in \ell^r(Y, k)$, and

$$\|R_Y(f)\|_{\ell^r(Y,k)} \le \|f\|_{\ell^r(X,k)}, \tag{47.6}$$

with equality when f is supported in Y. Hence R_Y defines a bounded linear mapping from $\ell^r(X, k)$ into $\ell^r(Y, k)$, with operator q or r-norm equal to 1. Similarly, R_Y maps $c_0(X, k)$ into $c_0(Y, k)$, with operator q-norm equal to 1 with respect to the corresponding supremum q-norms. Let E_Y be the extension mapping from $c(Y, k)$ into $c(X, k)$, as in (45.1). If $f_0 \in \ell^r(Y, k)$ for some $r > 0$, then $E_Y(f_0) \in \ell^r(X, k)$, and

$$\|E_Y(f_0)\|_{\ell^r(X,k)} = \|f_0\|_{\ell^r(Y,k)}, \tag{47.7}$$

so that E_Y defines an isometric linear mapping from $\ell^r(Y, k)$ into $\ell^r(X, k)$. Note that E_Y maps $c_0(Y, k)$ into $c_0(X, k)$ too. It is easy to see that R_Y maps $\ell^r(X, k)$ onto $\ell^r(Y, k)$ for every $r > 0$, and that R_Y maps $c_0(X, k)$ onto $c_0(Y, k)$, using (45.4).

48 Adjoint mappings

In this section, we take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. Remember that the standard inner products on $\ell^2(\mathbf{Z}_{0+}, \mathbf{R})$ and $\ell^2(\mathbf{Z}_{0+}, \mathbf{C})$ are given by

$$\langle f, g \rangle = \sum_{j=0}^{\infty} f(j)\, g(j) \tag{48.1}$$

and

$$\langle f, g \rangle = \sum_{j=0}^{\infty} f(j)\, \overline{g(j)}, \tag{48.2}$$

as in (11.3) and (11.4), respectively. The forward and backward shift operators A, B from Section 43 define bounded linear mappings from $\ell^2(\mathbf{Z}_{0+}, \mathbf{R})$ and $\ell^2(\mathbf{Z}_{0+}, \mathbf{C})$ into themselves, as in the previous section. If f, g are square-summable real or complex-valued functions on $\mathbf{Z}_{0+}$, then one can check that

$$\langle A(f), g \rangle = \langle f, B(g) \rangle. \tag{48.3}$$

This is analogous to (46.4). It follows that

$$A^* = B \tag{48.4}$$

as bounded linear mappings on $\ell^2(\mathbf{Z}_{0+}, \mathbf{R})$ or $\ell^2(\mathbf{Z}_{0+}, \mathbf{C})$, where A^* is the adjoint of A, as in Section 21. Of course,

$$B^* = A, \tag{48.5}$$

for essentially the same reasons, or by taking the adjoints of both sides of (48.4).

Let X be a nonempty set, and let Y be a nonempty subset of X. As before, the standard inner products on $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ are given by

$$\langle f, g \rangle_X = \sum_{x \in X} f(x)\, g(x) \tag{48.6}$$

and

$$\langle f, g \rangle_X = \sum_{x \in X} f(x)\, \overline{g(x)}, \tag{48.7}$$

respectively. The projection mapping P_Y in (45.5) determines a bounded linear mapping from each of $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ into itself, as in the previous section. If f, g are square-summable real or complex-valued functions on X, then

$$\langle P_Y(f), g \rangle_X = \langle f, P_Y(g) \rangle_X, \tag{48.8}$$

as in (46.7). This means that

$$P_Y^* = P_Y \tag{48.9}$$

as bounded linear mappings on $\ell^2(X, \mathbf{R})$ or $\ell^2(X, \mathbf{C})$.

The standard inner products on $\ell^2(Y, \mathbf{R})$ and $\ell^2(Y, \mathbf{C})$ are given by

$$\langle f_0, g_0 \rangle_Y = \sum_{x \in X} f_0(x)\, g_0(x) \tag{48.10}$$

and

$$\langle f_0, g_0 \rangle_Y = \sum_{x \in X} f_0(x)\, \overline{g_0(x)}, \tag{48.11}$$

respectively. As in the previous section, the restriction mapping R_Y from Section 45 defines a bounded linear mapping from $\ell^2(X, k)$ into $\ell^2(Y, k)$ for $k = \mathbf{R}$, $\mathbf{C}$. Similarly, the extension mapping E_Y in (45.1) defines a bounded linear mapping from $\ell^2(Y, k)$ into $\ell^2(X, k)$ for $k = \mathbf{R}$, $\mathbf{C}$. If f is a square-summable real or complex-valued function on X, and g_0 is a square-summable real or complex-valued function on Y, then

$$\langle R_Y(f), g_0 \rangle_Y = \langle f, E_Y(g_0) \rangle_X, \tag{48.12}$$

as in (46.8). This implies that

$$R_Y^* = E_Y \tag{48.13}$$

as bounded linear mappings from $\ell^2(Y, k)$ into $\ell^2(X, k)$, for $k = \mathbf{R}$, $\mathbf{C}$. We also have that

$$E_Y^* = R_Y \tag{48.14}$$

as bounded linear mappings from $\ell^2(X, k)$ into $\ell^2(Y, k)$, for $k = \mathbf{R}$, $\mathbf{C}$. This can be obtained in essentially the same way, or by taking adjoints of both sides of (48.13).

49 Eigenfunctions for unilateral shifts

Let k be a field, and let A and B be the forward and backward shift operators on $c(\mathbf{Z}_{0+}, k)$, as in Section 43. One can check that A has no nontrivial eigenvectors in $c(\mathbf{Z}_{0+}, k)$, so that A has no eigenvalues in k as a linear mapping from $c(\mathbf{Z}_{0+}, k)$ into itself. More precisely, 0 is not an eigenvalue of A, because A is injective on $c(\mathbf{Z}_{0+}, k)$. If $a \in k$, $a \neq 0$, $f \in c(\mathbf{Z}_{0+}, k)$, and $A(f) = a\, f$, then one can verify that $f \equiv 0$ on $\mathbf{Z}_{0+}$, using the definition of A. This corresponds to the fact that the bilateral shift operator T on $c(\mathbf{Z}, k)$ has no nontrivial eigenfunctions supported in $\mathbf{Z}_{0+}$, as in Section 26.

Let $a \in k$ be given, and let $e_{a,0}$ be the k-valued function defined on $\mathbf{Z}_{0+}$ by

$$e_{a,0}(j) = a^j \tag{49.1}$$

for every $j \geq 0$. This is interpreted as being equal to 1 when $j = 0$, as usual, even when $a = 0$. If $a \neq 0$, then $e_{a,0}$ is the same as the restriction to $\mathbf{Z}_{0+}$ of the function e_a defined on $\mathbf{Z}$ in (26.2). Observe that

$$(B(e_{a,0}))(j) = e_{a,0}(j+1) = a^{j+1} = a\, e_{a,0}(j) \tag{49.2}$$

for every $a \in k$ and $j \geq 0$, so that

$$B(e_{a,0}) = a\, e_{a,0} \tag{49.3}$$

as elements of $c(\mathbf{Z}_{0+}, k)$. Thus $e_{a,0}$ is an eigenvector for B on $c(\mathbf{Z}_{0+}, k)$ with eigenvalue a for each $a \in k$. It is easy to see that every eigenvector of B on $c(\mathbf{Z}_{0+})$ with eigenvalue a is a multiple of $e_{a,0}$. Note that $e_{a,0}$ has finite support in $\mathbf{Z}_{0+}$ only when $a = 0$, so that 0 is the only eigenvalue of B on $c_{00}(\mathbf{Z}_{0+}, k)$.

If $a \in k$, $a \neq 0$, and $f \in c(\mathbf{Z}_{0+}, k)$, then

$$(A(e_{a,0}\, f))(j) = a^{j-1}\, (A(f))(j) = a^{-1}\, e_{a,0}(j)\, (A(f))(j) \tag{49.4}$$

for every $j \geq 0$. Of course, each of these three expressions is equal to 0 when $j = 0$, by the definition (43.1) of A. Thus

$$A(e_{a,0}\, f) = a^{-1}\, e_{a,0}\, A(f) \tag{49.5}$$

as k-valued functions on $\mathbf{Z}_{0+}$. Let $M_{e_{a,0}}$ be the multiplication operator on $c(\mathbf{Z}_{0+}, k)$ corresponding to $e_{a,0}$, as in (19.1). Using this, (49.5) can be reformulated as saying that

$$A \circ M_{e_{a,0}} = a^{-1}\, M_{e_{a,0}} \circ A \tag{49.6}$$

as linear mappings from $c(\mathbf{Z}_{0+}, k)$ into itself. Note that (49.1) is nonzero for every $j \in \mathbf{Z}$ when $a \neq 0$, in which case $1/e_{a,0} = e_{1/a,0}$ and $M_{e_{a,0}}^{-1} = M_{1/e_{a,0}} = M_{e_{1/a,0}}$. It is easy to see that

$$A \circ M_{e_{a,0}}^{-1} = a\, M_{e_{a,0}}^{-1} \circ A \tag{49.7}$$

for every $a \in k \setminus \{0\}$, by rearranging the operators in (49.6), or by applying (49.6) to $1/a$. It follows that

$$M_{e_{a,0}} \circ A \circ M_{e_{a,0}}^{-1} = a\, A \tag{49.8}$$

for every $a \in k \setminus \{0\}$, as linear mappings from $c(\mathbf{Z}_{0+}, k)$ into itself.

If $f, g \in c(\mathbf{Z}_{0+}, k)$, then

$$B(f\, g) = B(f)\, B(g) \tag{49.9}$$

as k-valued functions on $\mathbf{Z}_{0+}$, by the definition (43.2) of B. In particular,

$$B(e_{a,0}\, f) = B(e_{a,0})\, B(f) = a\, e_{a,0}\, B(f) \tag{49.10}$$

for every $a \in k$ and $f \in c(\mathbf{Z}_{\mathbf{0}+}, k)$, using (49.3) in the second step. This implies that

$$B \circ M_{e_{a,0}} = a\, M_{e_{a,0}} \circ B \tag{49.11}$$

for every $a \in k$, as linear mappings from $c(\mathbf{Z}_{0+}, k)$ into itself. If $a \neq 0$, then we have that

$$B \circ M_{e_{a,0}}^{-1} = a^{-1}\, M_{e_{a,0}}^{-1} \circ B, \tag{49.12}$$

by rearranging the operators in (49.11), or applying the previous statement to $1/a$. Thus

$$M_{e_{a,0}} \circ B \circ M_{e_{a,0}}^{-1} = a^{-1}\, B \tag{49.13}$$

as linear mappings on $c(\mathbf{Z}_{0+}, k)$ when $a \neq 0$.

50 Eigenvalues of unilateral shifts

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. Also let $a \in k$ be given, and let $e_{a,0}$ be defined on $\mathbf{Z}_{0+}$ as in (49.1). Observe that

$$e_{a,0} \in \ell^\infty(\mathbf{Z}_{0+}, k) \text{ if and only if } |a| \le 1. \tag{50.1}$$

Similarly,

$$e_{a,0} \in c_0(\mathbf{Z}_{0+}, k) \text{ if and only if } |a| < 1. \tag{50.2}$$

If $0 < r < \infty$, then

$$e_{a,0} \in \ell^r(\mathbf{Z}_{0+}, k) \text{ if and only if } |a| < 1. \tag{50.3}$$

Let A, B be the usual forward and backward shift operators on $c(\mathbf{Z}_{0+}, k)$, as in Section 43. Remember that the restrictions of A and B to $\ell^r(\mathbf{Z}_{0+}, k)$ define bounded linear mappings from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself for every $r > 0$, as in Section 47. It follows from (50.1) and the remarks about eigenfunctions of B on $c(\mathbf{Z}_{0+}, k)$ in the previous section that $a \in k$ is an eigenvalue of B on $\ell^\infty(\mathbf{Z}_{0+}, k)$ if and only if $|a| \le 1$. Using (50.2), we get that $a \in k$ is an eigenvalue of B on $c_0(\mathbf{Z}_{0+}, k)$ if and only if $|a| < 1$. If $0 < r < \infty$, then $a \in k$ is an eigenvalue of B on $\ell^r(\mathbf{Z}_{0+}, k)$ if and only if $|a| < 1$, by (50.3).

Let us now consider approximate eigenvalues of A and B on $\ell^r(\mathbf{Z}_{0+}, k)$ and $c_0(\mathbf{Z}_{0+}, k)$. Of course, if $a \in k$ is an approximate eigenvalue of A or B on $c_0(\mathbf{Z}_{+}, k)$ with respect to the supremum norm, then a is an approximate eigenvalue of A or B, respectively, on $\ell^\infty(\mathbf{Z}_{0+}, k)$. Remember that A defines an isometric linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself for every $r > 0$, as in Section 47. If $a \in k$ is an approximate eigenvalue of A on $\ell^r(\mathbf{Z}_{0+}, k)$ for some $r > 0$, then it follows that $|a| = 1$, as in Section 30. Similarly, the restriction of B to $\ell^r(\mathbf{Z}_{0+}, k)$ has operator q or r-norm equal to 1, as appropriate, as in Section 47. If $a \in k$ is an approximate eigenvalue of B on $\ell^r(\mathbf{Z}_{0+}, k)$ for some $r > 0$, then $|a| \le 1$, as in (30.14). We shall restrict our attention to $|a| = 1$ for B as well, since the case where $|a| < 1$ is covered by the remarks in the preceding paragraph.

Let $a \in k$ with $|a| = 1$ be given, and let r be a positive real number. Also let T be the forward shift operator on $c(\mathbf{Z}, k)$, as in Section 3, and remember that A basically corresponds to the restriction of T to k-valued functions on $\mathbf{Z}$ that are supported in $\mathbf{Z}_{0+}$. We have seen that a is an approximate eigenvalue for T on $\ell^r(\mathbf{Z}, k)$, as in Section 32. The same type of construction can be used to show that a is an approximate eigenvalue for A on $\ell^r(\mathbf{Z}_{0+}, k)$. More precisely, one can use functions defined as in (32.1), with $j_1 \ge 0$, restricted to $\mathbf{Z}_{0+}$.

There are a few minor differences between the analogous argument for B and the previous situation. Let j_0 be a nonnegative integer, and let f_0 be the k-valued function defined on $\mathbf{Z}_{0+}$ by

$$\begin{aligned} f_0(j) &= a^j \quad \text{when } 0 \le j \le j_0 \\ &= 0 \quad \text{when } j \ge j_0 + 1. \end{aligned} \tag{50.4}$$

This corresponds to (32.1), restricted to $\mathbf{Z}_{0+}$, with $j_1 = 0$, $j_2 = j_0$, and a replaced with $1/a$. Using the definition (43.2) of B, we get that

$$(B(f_0))(j) = f_0(j+1) = a^{j+1} \quad \text{when } j \le j_0 - 1 \tag{50.5}$$
$$= 0 \quad \text{when } j \ge j_0.$$

Thus

$$a\, f_0(j) - (B(f_0))(j) = a^{j_0+1} \quad \text{when } j = j_0 \tag{50.6}$$
$$= 0 \quad \text{otherwise},$$

so that

$$\|a\, f_0 - B(f_0)\|_r = 1. \tag{50.7}$$

We also have that

$$\|f_0\|_r = (j_0 + 1)^{1/r}, \tag{50.8}$$

which implies that a is an approximate eigenvalue for B on $\ell^r(\mathbf{Z}_{0+}, k)$. Of course, B basically corresponds to T^{-1} in the earlier discussion, and the restriction to $\mathbf{Z}_{0+}$ permits us to avoid an extra term in (50.6).

Suppose for the rest of the section that $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. Let a be a real or complex number with $|a| = 1$, and let f_n be the real or complex-valued function defined on $\mathbf{Z}$ for each nonnegative integer n as in (32.6). In order to get functions that are supported in $\mathbf{Z}_{0+}$, one can take

$$\widetilde{f}_n(j) = f_n(j - n), \tag{50.9}$$

for instance. Using the restrictions of these functions to $\mathbf{Z}_{0+}$, one can show that a is an approximate eigenvalue for A on $\ell^r(\mathbf{Z}_{0+}, k)$ when $0 < r < \infty$, and on $c_0(\mathbf{Z}_{0+}, k)$ with the supremum norm. This is basically the same as in Section 32, with additional translations as in (50.9), because A corresponds to the restriction of T to functions supported in $\mathbf{Z}_{0+}$.

As before, there are analogous arguments for B, with some minor differences. In order to use the same type of functions for B, one should replace a with $1/a$, because B corresponds to T^{-1}. It is not necessary to use translations as in (50.9), and instead one can simply restrict the functions to $\mathbf{Z}_{0+}$. However, instead of using the same type of functions as in (32.6), one can do the following. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real or complex numbers, as appropriate, that converges to a, and satisfies $|a_n| < 1$ for each n. If $e_{a_n,0}$ is defined on $\mathbf{Z}_{0+}$ as in (49.1), then $e_{a_n,0}$ is an eigenfunction for B with eigenvalue a_n for each n, as in (49.3). We also have that $e_{a_n,0} \in c_0(\mathbf{Z}_{0+}, k)$ for each n, by (50.2), and that $e_{a_n,0} \in \ell^r(\mathbf{Z}_{0+}, k)$ for every positive real number r and $n \ge 1$, by (50.3). Using this, one can check that a is an approximate eigenvalue of B on $\ell^r(\mathbf{Z}_{0+}, k)$ when $0 < r < \infty$, and on $c_0(\mathbf{Z}_{0+}, k)$ with respect to the supremum norm.

51 The case where $q = r = \infty$

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let $a \in k$ be given, with $|a| = 1$. If $f \in c(\mathbf{Z}_{0+}, k)$ and $A(f)$ is as in (43.1), then

$$(51.1) \qquad \begin{aligned} a\,f(j) - (A(f))(j) &= a\,f(j) - f(j-1) \quad \text{when } j \ge 1 \\ &= a\,f(0) \qquad\qquad\quad \text{when } j = 0. \end{aligned}$$

Let l be a nonnegative integer, and observe that

$$(51.2) \qquad a^l\,f(l) = \sum_{j=1}^{l} (a^j\,f(j) - a^{j-1}\,f(j-1)) + f(0),$$

where the summation on the right side is interpreted as being equal to 0 when $l = 0$. Equivalently,

$$(51.3) \qquad a^l\,f(l) = \sum_{j=0}^{l} a^{j-1}\,(a\,f(j) - (A(f))(j)),$$

where the $f(0)$ term on the right side of (51.2) corresponds to the $j = 0$ in the sum on the right side of (51.3). Thus

$$(51.4) \qquad |f(l)| \le \max_{0 \le j \le l} |a\,f(j) - (A(f))(j)|,$$

by the ultrametric version of the triangle inequality.

If $a\,f - A(f)$ is bounded on $\mathbf{Z}_{0+}$, then it follows that f is bounded on $\mathbf{Z}_{0+}$, with

$$(51.5) \qquad \|f\|_\infty \le \|a\,f - A(f)\|_\infty.$$

This could also be obtained from (33.5), applied to k-valued functions on $\mathbf{Z}$ supported in $\mathbf{Z}_{0+}$. It follows that a is not an approximate eigenvalue of A on $\ell^\infty(\mathbf{Z}_{0+}, k)$. Note that

$$(51.6) \qquad \|a\,f - A(f)\|_\infty \le \max(\|a\,f\|_\infty, \|A(f)\|_\infty) \le \|f\|_\infty$$

for every $f \in \ell^\infty(\mathbf{Z}_{0+}, k)$, by the ultrametric version of the triangle inequality. Hence

$$(51.7) \qquad \|a\,f - A(f)\|_\infty = \|f\|_\infty$$

for every $f \in \ell^\infty(\mathbf{Z}_{0+}, k)$, by (51.5) and (51.6).

Let $f \in c(\mathbf{Z}_{0+}, k)$ and nonnegative integers j_1, j_2 be given, with $j_1 < j_2$. Observe that

$$(51.8) \qquad \begin{aligned} a^{-j_1}\,f(j_1) - a^{-j_2}\,f(j_2) &= \sum_{j=j_1}^{j_2-1} (a^{-j}\,f(j) - a^{-j-1}\,f(j+1)) \\ &= \sum_{j=j_1}^{j_2-1} a^{-j-1}\,(a\,f(j) - (B(f))(j)), \end{aligned}$$

where $B(f)$ is as in (43.2). This implies that

$$|a^{-j_1} f(j_1) - a^{-j_2} f(j_2)| \le \max_{j_1 \le j \le j_2 - 1} |a\, f(j) - (B(f))(j)|, \tag{51.9}$$

by the ultrametric version of the triangle inequality. If $a\, f - B(f)$ is bounded on $\mathbf{Z}_{0+}$, then we have that

$$|a^{-j_1} f(j_1) - a^{-j_2} f(j_2)| \le \|a\, f - B(f)\|_\infty \tag{51.10}$$

for every $j_1, j_2 \ge 0$. It follows that f is bounded on $\mathbf{Z}_{0+}$, with

$$\|f\|_\infty \le \max\Big(\|a\, f - B(f)\|_\infty, \inf_{j \ge 0} |f(j)|\Big), \tag{51.11}$$

by the ultrametric version of the triangle inequality.

If f vanishes at infinity on $\mathbf{Z}_{0+}$, then

$$\|f\|_\infty \le \|a\, f - B(f)\|_\infty, \tag{51.12}$$

by (51.11). This means that a is not an approximate eigenvalue of B on $c_0(\mathbf{Z}_{0+}, k)$ with respect to the supremum ultranorm. As before,

$$\|a\, f - B(f)\|_\infty \le \max(\|a\, f\|_\infty, \|B(f)\|_\infty) \le \|f\|_\infty \tag{51.13}$$

for every $f \in \ell^\infty(\mathbf{Z}_{0+}, k)$, by the ultrametric version of the triangle inequality. Thus

$$\|a\, f - B(f)\|_\infty = \|f\|_\infty \tag{51.14}$$

for every $f \in c_0(\mathbf{Z}_{0+}, k)$. Note that

$$\sup_{j_1, j_2 \ge 0} |a^{-j_1} f(j_1) - a^{-j_2} f(j_2)| = \|a\, f - B(f)\|_\infty \tag{51.15}$$

for every $f \in \ell^\infty(\mathbf{Z}_{0+}, k)$, because of (51.10) and the definition of B.

52 Multiplicative inverses in $k[[X]]$

Let k be a field, and let X be an indeterminate. Of course, elements of k can be identified with formal polynomials in X of degree 0. More precisely, such a constant polynomial is a scalar multiple of X^0, which is typically omitted from the notation. This defines a natural embedding of k into $k[X]$. In particular, the multiplicative identity element 1 in k corresponds to a constant polynomial in X, which may be denoted 1 as well. This is the multiplicative identity element in $k[[X]]$. Thus $F(X) \in k[[X]]$ has a multiplicative inverse in $k[[X]]$ when there is a $G(X) \in k[[X]]$ such that

$$F(X)\, G(X) = 1. \tag{52.1}$$

Let $a \in k$ and a nonnegative integer n be given, so that $\sum_{j=0}^n a^j\, X^j$ defines an element of $k[X]$. As usual,

$$(1 - a\,X) \sum_{j=0}^{n} a^j\, X^j = 1 - a^{n+1}\, X^{n+1}. \tag{52.2}$$

Similarly, $\sum_{j=0}^\infty a^j\, X^j$ defines an element of $k[[X]]$, and

$$(1 - a\,X) \sum_{j=0}^{\infty} a^j\, X^j = 1. \tag{52.3}$$

Thus $1 - a\,X$ is invertible in $k[[X]]$, with

$$(1 - a\,X)^{-1} = \sum_{j=0}^{\infty} a^j\, X^j. \tag{52.4}$$

Now let $a(X) \in k[[X]]$ be given, so that $(a(X)\,X)^l = a(X)^l\, X^l$ is defined as a formal power series in X for every nonnegative integer l. If n is a nonnegative integer, then

$$(1 - a(X)\,X) \sum_{l=0}^{n} a(X)^l\, X^l = 1 - a(x)^{n+1}\, X^{n+1}, \tag{52.5}$$

as in (52.2). The sum

$$\sum_{l=0}^{\infty} a(X)^l\, X^l \tag{52.6}$$

can be defined as a formal power series in X as well, because the coefficient of X^j reduces to a finite sum for each $j \geq 0$. One can also check that

$$(1 - a(X)\,X) \sum_{l=0}^{\infty} a(X)^l\, X^l = 1, \tag{52.7}$$

as in (52.3). This implies that $1 - a(X)\,X$ is invertible in $k[[X]]$, with inverse equal to (52.6).

Let $F(X) \in k[[X]]$ be given. If $F(X)$ has a multiplicative inverse in $k[[X]]$, then it is easy to see that the constant term in $F(X)$ is not equal to 0. Conversely, if the constant term in $F(X)$ is not equal to 0, then $F(X)$ can be expressed as

$$F(X) = b\,(1 - a(X)\,X), \tag{52.8}$$

where $b \in k$, $b \neq 0$, and $a(X) \in k[[X]]$. This implies that $F(X)$ has a multiplicative inverse in $k[[X]]$, by the remarks in the preceding paragraph.

53 Inverting $I - a\,A$ on $c(\mathbf{Z}_{0+}, k)$

Let k be a field, and let $a \in k$ be given. If n is a nonnegative integer, then put

$$C_{a,n} = \sum_{l=0}^{n} a^l\, A^l, \tag{53.1}$$

where A is as in (43.1). This defines a linear mapping from $c(\mathbf{Z}_{0+}, k)$ into itself. Equivalently,

$$(C_{a,n}(f))(j) = \sum_{l=0}^{n} a^l\, (A^l(f))(j) = \sum_{l=0}^{\min(j,n)} a^l\, f(j-l) \tag{53.2}$$

for every $f \in c(\mathbf{Z}_{0+}, k)$ and $j \ge 0$, using (43.6) in the second step. Note that

$$(I - a\,A) \circ C_{a,n} = C_{a,n} \circ (I - a\,A) = I - a^{n+1}\, A^{n+1} \tag{53.3}$$

as linear mappings on $c(\mathbf{Z}_{0+}, k)$ for each $n \ge 0$, as in (23.1).

If $f \in c(\mathbf{Z}_{0+}, k)$ and $j \in \mathbf{Z}_{0+}$, then put

$$(C_a(f))(j) = \sum_{l=0}^{j} a^l f(j-l). \tag{53.4}$$

This defines a k-valued function on $\mathbf{Z}_{0+}$, and C_a defines a linear mapping from $c(\mathbf{Z}_{0+}, k)$ into itself. Comparing (53.4) with (53.2), we get that

$$(C_a(f))(j) = (C_{a,n}(f))(j) \tag{53.5}$$

when $j \le n$. Equivalently,

$$(C_a(f))(j) = \sum_{l=0}^{\infty} a^l\, (A^l(f))(j) \tag{53.6}$$

for every $j \ge 0$, where the right side of (53.6) reduces to the finite sum in (53.4), because of (43.6). Basically, C_a corresponds to $\sum_{l=0}^{\infty} a^l\, A^l$, which is made precise by (53.4) and (53.6).

Let $f \in c(\mathbf{Z}_{0+}, k)$ be given, and observe that

$$(C_a(a\,A(f)))(j) = \sum_{l=0}^{\infty} a^{l+1}\, (A^{l+1}(f))(j) \tag{53.7}$$

for each $j \ge 0$, by applying (53.6) to $A(f)$ in place of f. Similarly,

$$a\,(A(C_a(f)))(j) = \sum_{l=0}^{\infty} a^{l+1}\, (A^{l+1}(f))(j) \tag{53.8}$$

for each $j \geq 0$, by applying A to both sides of (53.6). We also have that

$$\sum_{l=0}^{\infty} a^{l+1}\,(A^{l+1}(f))(j) = \sum_{l=1}^{\infty} a^l\,(A^l(f))(j) = (C_a(f))(j) - f(j) \tag{53.9}$$

for each $j \geq 0$, using (53.6) in the second step. Combining this with (53.7) and (53.8), we get that

$$aA(C_a(f)) = C_a(aA(f)) = C_a(f) - f. \tag{53.10}$$

Thus

$$a\,A \circ C_a = a\,C_a \circ A = C_a - I \tag{53.11}$$

as linear mappings from $c(\mathbf{Z}_{0+}, k)$ into itself, so that

$$(I - a\,A) \circ C_a = C_a \circ (I - a\,A) = I. \tag{53.12}$$

Of course, this means that $I - a\,A$ is invertible as a linear mapping from $c(\mathbf{Z}_{0+}, k)$ into itself, with

$$(I - a\,A)^{-1} = C_a. \tag{53.13}$$

If $|\cdot|$ is an ultrametric absolute value function on k and $|a| \leq 1$, then it is easy to see that C_a maps $\ell^\infty(\mathbf{Z}_{0+}, k)$ into itself.

If $b \in k$ and $b \neq 0$, then it follows that $b\,I - A = b\,(I - b^{-1}\,A)$ is invertible as a linear mapping from $c(\mathbf{Z}_{0+}, k)$ into itself, with

$$(b\,I - A)^{-1} = b^{-1}\,(I - b^{-1}\,A)^{-1} = b^{-1}\,C_{1/b}. \tag{53.14}$$

We have already seen in Section 49 that A has no eigenvalues in k as a linear mapping from $c(\mathbf{Z}_{0+}, k)$ into itself, which is the same as saying that $b\,I - A$ is injective on $c(\mathbf{Z}_{0+}, k)$ for every $b \in k$. If $b = 0$, then $b\,I - A = -A$ does not map $c(\mathbf{Z}_{0+}, k)$ onto itself, and hence is not invertible on $c(\mathbf{Z}_{0+}, k)$.

54 ℓ^r Estimates, $r \leq q$, $r < \infty$

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and let r be a positive real number with $r \leq q$. Remember that the forward shift operator A in (43.1) defines an isometric linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself, as in Section 47. Let $a \in k$ and $n \in \mathbf{Z}_{0+}$ be given, and let $C_{a,n}$ be as in (53.1). If $f \in \ell^r(\mathbf{Z}_{0+}, k)$, then

$$\|C_{a,n}(f)\|_r^r = \Big\|\sum_{l=0}^{n} a^l\,A^l(f)\Big\|_r^r \leq \sum_{l=0}^{n} |a^l|^r\,\|A^l(f)\|_r^r = \Big(\sum_{l=0}^{n} |a|^{l\,r}\Big)\,\|f\|_r^r. \tag{54.1}$$

This uses the fact that $\|\cdot\|_r$ defines an r-norm on $\ell^r(\mathbf{Z}_{0+}, k)$ when $r \leq q$, as in Section 8, in the second step. Thus $C_{a,n}$ defines a bounded linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself, with operator r-norm less than or equal to

$$\Big(\sum_{l=0}^{n} |a|^{l\,r}\Big)^{1/r}. \tag{54.2}$$

If $f(j) = 0$ for every $j \geq 1$, then $(A^l(f))(j) = 0$ when $j \neq l$, as in (43.6). In this case, one can check that equality holds in the second step in (54.1). This implies that the operator r-norm of $C_{a,n}$ on $\ell^r(\mathbf{Z}_{0+}, k)$ is equal to (54.2).

Let C_a be as in (53.4), and let $f \in c(\mathbf{Z}_{0+}, k)$ be given. Observe that

$$\sum_{j=0}^{n} |(C_a(f))(j)|^r = \sum_{j=0}^{n} |(C_{a,n}(f))(j)|^r \tag{54.3}$$

for each $n \geq 0$, by (53.5). Of course,

$$\sum_{j=0}^{n} |(C_{a,n}(f))(j)|^r \leq \sum_{j=0}^{\infty} |(C_{a,n}(f))(j)|^r = \|C_{a,n}(f)\|_r^r \tag{54.4}$$

for every $n \geq 0$. If $f(j) = 0$ when $j \geq 1$, then $(C_{a,n}(f))(j) = 0$ when $j > n$, by (53.2). This implies that equality holds in the first step in (54.4) for every $n \geq 0$ in this situation.

Suppose that $|a| < 1$, so that (54.2) is less than or equal to

$$\Big(\sum_{l=0}^{\infty} |a|^{l\,r}\Big)^{1/r} = (1 - |a|^r)^{-1/r} \tag{54.5}$$

for each $n \geq 0$. If $f \in \ell^r(\mathbf{Z}_{0+}, k)$, then it follows that

$$\sum_{j=0}^{n} |(C_a(f))(j)|^r \leq (1 - |a|^r)^{-1}\, \|f\|_r^r \tag{54.6}$$

for every $n \geq 0$, by (54.1), (54.3), and (54.4). This implies that

$$\|C_a(f)\|_r^r = \sum_{j=0}^{\infty} |(C_a(f))(j)|^r \leq (1 - |a|^r)^{-1}\, \|f\|_r^r, \tag{54.7}$$

and in particular that $C_a(f) \in \ell^r(\mathbf{Z}_{0+}, k)$. Thus C_a defines a bounded linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself when $|a| < 1$, with operator r-norm less than or equal to (54.5). If $f(0) = 1$ and $f(j) = 0$ when $j \geq 1$, then

$$(C_a(f))(j) = a^j \tag{54.8}$$

for every $j \geq 0$, by (53.4). In this case, $\|C_a(f)\|_r$ is equal to (54.5), so that equality holds in the second step in (54.7). Hence the operator r-norm of C_a on $\ell^r(\mathbf{Z}_{0+}, k)$ is equal to (54.5).

By construction, $C_{a,n}$ is the same as the n partial sum of the infinite series

$$\sum_{l=0}^{\infty} a^l\, A^l \tag{54.9}$$

of linear mappings on $c(\mathbf{Z}_{0+}, k)$. Of course, the operator r-norm of $a^l\, A^l$ on $\ell^r(\mathbf{Z}_{0+}, k)$ is equal to $|a|^l$ for every $l \geq 0$, because A^l is an isometry on $\ell^r(\mathbf{Z}_{0+}, k)$.

If $|a| < 1$, then it follows that (54.9) converges r-absolutely as an infinite series of bounded linear mappings on $\ell^r(\mathbf{Z}_{0+}, k)$, with respect to the operator r-norm. One can check that $C_{a,n}$ converges to C_a as $n \to \infty$ as a sequence of bounded linear mappings on $\ell^r(\mathbf{Z}_{0+}, k)$ with respect to the operator r-norm when $|a| < 1$, using the same type of simple estimates as before. This means that (54.9) converges to C_a as an infinite series of bounded linear mappings on $\ell^r(\mathbf{Z}_{0+}, k)$. Note that k is not required to be complete with respect to the q-metric associated to $|\cdot|$ here. In this situation, we were able to define C_a directly, in terms of finite sums.

55 Estimates for $k = \mathbf{R}, \mathbf{C}$

Let us take $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function in this section, so that $q = 1$. We shall also restrict our attention to $r \geq 1$, since the case where $r \leq 1$ is covered by the discussion in the previous section. As before, the forward shift operator A in (43.1) defines an isometric linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself for each r. Let $a \in k$ and $n \in \mathbf{Z}_{0+}$ be given again, and let $C_{a,n}$ be as in (53.1). If $r \geq 1$ and $f \in \ell^r(\mathbf{Z}_{0+}, k)$, then

$$\|C_{a,n}(f)\|_r = \Big\|\sum_{l=0}^{n} a^l \, A^l(f)\Big\|_r \leq \sum_{l=0}^{n} |a|^l \, \|A^l(f)\|_r = \Big(\sum_{l=0}^{n} |a|^l\Big) \, \|f\|_r. \tag{55.1}$$

This uses the fact that $\|\cdot\|_r$ defines a norm on $\ell^r(\mathbf{Z}_{0+}, k)$ when $r \geq 1$, as in Section 8. Thus $C_{a,n}$ defines a bounded linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself when $r \geq 1$, with operator norm less than or equal to

$$\sum_{l=0}^{n} |a|^l. \tag{55.2}$$

Let C_a be as in (53.4) again, and let $f \in c(\mathbf{Z}_{0+}, k)$ be given. As in (54.3) and (54.4), we have that

$$\Big(\sum_{j=0}^{n} |(C_a(f))(j))|^r\Big)^{1/r} = \Big(\sum_{j=0}^{n} |(C_{a,n}(f))(j)|^r\Big)^{1/r} \leq \|C_{a,n}(f)\|_r \tag{55.3}$$

when $r < \infty$, and

$$\max_{0 \leq j \leq n} |(C_a(f))(j)| = \max_{0 \leq j \leq n} |(C_{a,n}(f))(j)| \leq \|C_{a,n}(f)\|_\infty, \tag{55.4}$$

using (53.5) in the first step of each. Suppose that $|a| < 1$, so that (55.2) is less than or equal to

$$\sum_{l=0}^{\infty} |a|^l = (1 - |a|)^{-1} \tag{55.5}$$

for each $n \geq 0$. If $r \geq 1$ and $f \in \ell^r(\mathbf{Z}_{0+}, k)$, then we get that

$$\|C_a(f)\|_r \leq (1 - |a|)^{-1} \, \|f\|_r, \tag{55.6}$$

using (55.1), (55.3), and (55.4). This shows that C_a defines a bounded linear mapping from $\ell^r(\mathbf{Z}_{0+}, k)$ into itself when $|a| < 1$ and $r \ge 1$, with operator norm less than or equal to (55.5).

As before, C_a basically corresponds to the infinite series

$$\sum_{l=0}^{\infty} a^l A^l. \tag{55.7}$$

If $|a| < 1$, then this series converges absolutely as an infinite series of bounded linear mappings on $\ell^r(\mathbf{Z}_{0+}, k)$ with respect to the corresponding operator norm, as in Section 23. One can also check that C_a maps $c_0(\mathbf{Z}_{0+}, k)$ into itself when $|a| < 1$. This can be verified directly from the original definition of C_a, or by considering (55.7) as an absolutely convergent series of bounded linear mappings on $c_0(\mathbf{Z}_{0+}, k)$ with respect to the supremum norm. This is basically the same as approximating C_a by $C_{a,n}$, and using the fact that $C_{a,n}$ maps $c_0(\mathbf{Z}_{0+}, k)$ into itself for every $n \ge 0$.

As in Section 42, one can show that the operator norm of C_a on $\ell^r(\mathbf{Z}_{0+}, k)$ is equal to (55.5) for every $r \ge 1$ when $|a| < 1$. Similarly, the operator norm of C_a on $c_0(\mathbf{Z}_{0+}, k)$ with respect to the supremum norm is equal to (55.5) when $|a| < 1$. Of course, C_a is the same as the inverse of $I - a\,A$ on these spaces when $|a| < 1$. If $b \in k$ and $|b| = 1$, then b is an approximate eigenvalue of A on these spaces, as in Section 50. This permits one to show that the operator norm of C_a on these spaces is greater than or equal to (55.5), in the same way as before.

56 Convergent power series

Let k be a field with a q-absolute value function $|\cdot|$ for some $q > 0$, and suppose that k is complete with respect to the associated q-metric. Let $f \in \ell^\infty(\mathbf{Z}_{0+}, k)$ and $x \in k$ be given, with $|x| < 1$. Under these conditions,

$$\sum_{j=0}^{\infty} f(j)\, x^j \tag{56.1}$$

converges in k, by the remarks in Section 12. More precisely, if $q < \infty$, then (56.1) converges q-absolutely, by comparison with the convergent geometric series $\sum_{j=0}^{\infty} |x|^{q\,j}$. If $q = \infty$, then it suffices to observe that the terms of the series converge to 0.

Let $F(x)$ denote the value of the sum (56.1). Put $g = A(f)$, where A is the forward shift operator defined in Section 43, as usual. Thus $g \in \ell^\infty(\mathbf{Z}_{0+}, k)$, as in Section 47, so that $G(x)$ can be defined as the sum of the corresponding series. In fact,

$$G(x) = \sum_{j=0}^{\infty} (A(f))(j)\, x^j = \sum_{j=1}^{\infty} f(j-1)\, x^j = \sum_{j=0}^{\infty} f(j)\, x^{j+1} = x\, F(x), \tag{56.2}$$

as in (44.3).

Put $\lambda_x(f) = F(x)$, considered now as a linear functional on $\ell^\infty(\mathbf{Z}_{0+}, k)$. Using (56.2), we get that

$$\lambda_x(A(f)) = x\,\lambda_x(f) \tag{56.3}$$

for every $f \in \ell^\infty(\mathbf{Z}_{0+}, k)$. It is easy to see that λ_x defines a bounded linear functional on $\ell^\infty(\mathbf{Z}_{0+}, k)$. This implies that the restriction of λ_x to $c_0(\mathbf{Z}_{0+}, k)$ is a bounded linear functional with respect to the supremum q-norm. Similarly, the restriction of λ_x to $\ell^r(\mathbf{Z}_{0+}, k)$ is a bounded linear functional for every positive real number r. Note that λ_x is not identically 0 on $\ell^r(\mathbf{Z}_{0+}, k)$ for any $r > 0$, or on $c_0(\mathbf{Z}_{0+}, k)$. It follows from (56.3) that $x\,I - A$ maps $\ell^r(\mathbf{Z}_{0+}, k)$ into the kernel of the restriction of λ_x to $\ell^r(\mathbf{Z}_{0+}, k)$ for each $r > 0$, and that $x\,I - A$ maps $c_0(\mathbf{Z}_{0+}, k)$ into the kernel of the restriction of λ_x to $c_0(\mathbf{Z}_{0+}, k)$. In particular, $x\,I - A$ is not surjective on these spaces, and hence not invertible.

Of course, if $f \in c_{00}(\mathbf{Z}_{0+}, k)$, then (56.1) reduces to a finite sum in k for every $x \in k$, and the value of the sum can be defined without asking k to be complete. If $g = A(f)$, then $g \in c_{00}(\mathbf{Z}_{0+}, k)$ too, and (56.2) holds for every $x \in k$. Thus $\lambda_x(f) = F(x)$ defines a linear functional on $c_{00}(\mathbf{Z}_{0+}, k)$ for every $x \in k$, and satisfies (56.3) for every $f \in c_{00}(\mathbf{Z}_{0+}, k)$ and $x \in k$. This means that $x\,I - A$ maps $c_{00}(\mathbf{Z}_{0+}, k)$ into the kernel of λ_x on $c_{00}(\mathbf{Z}_{0+}, k)$ for every $x \in k$. As before, λ_x is not identically 0 on $c_{00}(\mathbf{Z}_{0+}, k)$ for any $x \in k$, and so $x\,I - A$ is not surjective on $c_{00}(\mathbf{Z}_{0+}, k)$ for any $x \in k$.

57 Hardy spaces

In this section, we take $k = \mathbf{C}$, with the standard absolute value function. Let f be a complex-valued function on $\mathbf{Z}_{0+}$ such that

$$\sum_{j=0}^{\infty} |f(j)|\,\rho^j \tag{57.1}$$

converges for every nonnegative real number $\rho < 1$. The convergence of (57.1) implies that $|f(j)|\,\rho^j \to 0$ as $j \to \infty$, and hence that $|f(j)|\,\rho^j$ is bounded on $\mathbf{Z}_{0+}$. If

$$|f(j)|\,\rho_1^j \tag{57.2}$$

is bounded on $\mathbf{Z}_{0+}$ for some $\rho_1 > 0$, then (57.1) converges when $0 \le \rho < \rho_1$, by comparison with a convergent geometric series. Thus (57.1) converges for every $0 \le \rho < 1$ if and only if (57.2) is bounded on $\mathbf{Z}_{0+}$ for every $0 \le \rho_1 < 1$.

Under these conditions,

$$F(z) = \sum_{j=0}^{\infty} f(j)\,z^j \tag{57.3}$$

defines a holomorphic function on the open unit disk

$$U = \{z \in \mathbf{C} : |z| < 1\} \tag{57.4}$$

in the complex plane. It is well known that every holomorphic function on U can be expressed in this way. If $A(f)$ is the complex-valued function defined on $\mathbf{Z}_{0+}$ as in (43.1), then it is easy to see that $A(f)$ satisfies the same type of conditions as in the preceding paragraph. Thus $A(f)$ also determines a holomorphic function on U, which is the same as $z\,F(z)$, as in (56.2).

Let $0 \le \rho < 1$ be given, so that $F(\rho\, z)$ defines a continuous complex-valued function on the unit circle $\mathbf{T}$. If r is a positive real number, then

$$\Big(\frac{1}{2\pi}\int_{\mathbf{T}} |F(\rho\, z)|^r\,|dz|\Big)^{1/r} \tag{57.5}$$

can be defined using a Riemann integral with respect to the element $|dz|$ of arclength on $\mathbf{T}$. The analogue of this for $r = \infty$ is

$$\sup_{z \in \mathbf{T}} |F(\rho\, z)|, \tag{57.6}$$

where the supremum is attained because $\mathbf{T}$ is compact. This is the same as

$$\sup\{|F(w)| : w \in U,\ |w| \le \rho\}, \tag{57.7}$$

by the maximum principle. Clearly (57.7) increases monotonically in ρ, which means that (57.6) increases monotonically in ρ. It is well known that (57.5) aldo increases monotonically as a function of ρ for each $r \in \mathbf{R}_+$, because F is holomorphic on U. There are analogous statements for harmonic functions when $r \ge 1$.

If $0 < r < \infty$, then we put

$$\|F\|_{H^r} = \sup_{0 \le \rho < 1} \Big(\frac{1}{2\pi}\int_{\mathbf{T}} |F(\rho\, z)|^r\,|dz|\Big)^{1/r}, \tag{57.8}$$

where the supremum is defined as a nonnegative extended real number. The *Hardy space* H^r is defined to be the space of holomorphic functions F on U such that (57.8) is finite, which means that (57.5) is bounded. This is a linear subspace of the space of all holomorphic functions on U. If $r \ge 1$, then (57.8) defines a norm on H^r, and (57.8) defines an r-norm on H^r when $0 < r \le 1$. Equivalently,

$$\|F\|_{H^r} = \lim_{\rho \to 1-} \Big(\frac{1}{2\pi}\int_{\mathbf{T}} |F(\rho\, z)|^r\,|dz|\Big)^{1/r}, \tag{57.9}$$

because (57.5) increases monotonically in ρ.

Similarly, H^∞ is defined to be the space of bounded holomorphic functions on U. This is a subalgebra of the algebra of all holomorphic functions on U. The H^∞ norm

$$\|F\|_{H^\infty} = \sup_{z \in U} |F(z)| \tag{57.10}$$

is the same as the supremum norm on U, which can also be expressed as

$$\|F\|_{H^\infty} = \lim_{\rho \to 1-} \Big(\sup_{z \in \mathbf{T}} |F(\rho\, z)|\Big), \tag{57.11}$$

because of the monotonicity of (57.6).

It is well known that (57.5) increases monotonically as a function of r for each $\rho \in [0,1)$, by the inequalities of Jensen or Hölder. It is easy to see directly that (57.5) is less than or equal to (57.6) for every $r \in \mathbf{R}_+$ and $\rho \in [0,1)$. If $0 < r_1 \le r_2 \le \infty$, then it follows that

$$H^{r_2} \subseteq H^{r_1}, \tag{57.12}$$

with

$$\|F\|_{H^{r_1}} \le \|F\|_{H^{r_2}} \tag{57.13}$$

for every $F \in H^{r_2}$. If $F \in H^r$ for some $r > 0$, then one can check that $G(z) = z\,F(z)$ defines an element of H^r as well, with

$$\|G\|_{H^r} = \|F\|_{H^r}. \tag{57.14}$$

This uses (57.9) when $r < \infty$, and (57.11) when $r = \infty$.

If F is given as in (57.3), then

$$\frac{1}{2\pi} \int_{\mathbf{T}} |F(\rho\, z)|^2\, |dz| = \sum_{j=0}^{\infty} |f(j)|^2\, \rho^{2j} \tag{57.15}$$

for every $0 \le \rho < 1$. This follows from the orthonormality of the z^j's with respect to the usual integral inner product (16.10) for $L^2(\mathbf{T})$. Using this, we get that $F \in H^2$ if and only if $f \in \ell^2(\mathbf{Z}_{0+}, \mathbf{C})$, with

$$\|F\|_{H^2} = \Big(\sum_{j=0}^{\infty} |f(j)|^2\Big)^{1/2} = \|f\|_{\ell^2(\mathbf{Z}_{0+}, \mathbf{C})}. \tag{57.16}$$

Note that a complex-valued function f on $\mathbf{Z}_{0+}$ satisfies the conditions mentioned at the beginning of the section when f is bounded on $\mathbf{Z}_{0+}$, and in particular when $f \in \ell^2(\mathbf{Z}_{0+}, k)$.

If $F \in H^r$ for some $r > 0$, then it is well known that the limit of $F(\rho\, z)$ as $\rho \to 1-$ exists for almost every $z \in \mathbf{T}$ with respect to Lebesgue measure. More precisely, nontangential versions of this limit exist almost everywhere on $\mathbf{T}$. If $r < \infty$, then $F(\rho\, z)$ converges to the pointwise limit as $\rho \to 1-$ with respect to the L^r norm on $\mathbf{T}$. In particular, this implies that $\|F\|_{H^r}$ is equal to the L^r norm of the boundary value function with respect to normalized arclength measure on $\mathbf{T}$. If $r = \infty$, then $\|F\|_{H^\infty}$ is equal to the L^∞ norm of the boundary value function.

If $F \in H^r$ for some $r > 0$, then the product of F with a bounded holomorphic function on U is in H^r too. This defines another type of multiplication operator in this situation.

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