Some topics in analysis related to metrics, ultrametrics, and topological groups

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Preface

Some topics related to topological groups are discussed, in connection with translation-invariant metrics and ultrametrics in particular.

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Chapter 1

Topological groups and semimetrics

1.1 Metrics and semimetrics

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$. If $d(x, y)$ satisfies the following three conditions, then $d(x, y)$ is said to be a *semimetric* on X. First,

$$d(x, x) = 0 \quad \text{for every } x \in X. \tag{1.1.1}$$

Second,

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in X. \tag{1.1.2}$$

Third,

$$d(x, z) \le d(x, y) + d(y, z) \quad \text{for every } x, y, z \in X. \tag{1.1.3}$$

If we also have that

$$d(x, y) > 0 \quad \text{for every } x, y \in X \text{ with } x \ne y, \tag{1.1.4}$$

then $d(\cdot, \cdot)$ is said to be a *metric* on X. The *discrete metric* on X is defined as usual by putting $d(x, y)$ equal to 1 when $x \ne y$, and to 0 when $x = y$. It is easy to see that this defines a metric on X.

Let $d(x, y)$ be a semimetric on X. The *open ball* in X centered at $x \in X$ with radius $r > 0$ with respect to $d(\cdot, \cdot)$ is defined by

$$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}. \tag{1.1.5}$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \ge 0$ with respect to $d(\cdot, \cdot)$ is defined by

$$\overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \le r\}. \tag{1.1.6}$$

A subset U of X is said to be an *open set* with respect to $d(\cdot, \cdot)$ if for every $x \in U$ there is an $r > 0$ such that

$$B(x, r) \subseteq U. \tag{1.1.7}$$

This defines a topology on X, by standard arguments. One can check that open balls in X with respect to $d(\cdot, \cdot)$ are open sets with respect to the topology determined by $d(\cdot, \cdot)$, and that closed balls are closed sets. If $d(\cdot, \cdot)$ is a metric on X, then X is Hausdorff with respect to the topology determined by $d(\cdot, \cdot)$.

Let Y be a subset of X, and observe that the restriction of $d(x, y)$ to $x, y \in Y$ defines a semimetric on Y. Let $B_X(x, r)$ be the open ball in X centered at $x \in X$ with radius $r > 0$ with respect to $d(\cdot, \cdot)$, and let $B_Y(x, r)$ be the open ball in Y centered at $x \in Y$ with radius r with respect to the restriction of $d(\cdot, \cdot)$ to Y. Clearly

$$B_Y(x, r) = B_X(x, r) \cap Y \tag{1.1.8}$$

for every $x \in Y$ and $r > 0$. It is well known that the topology determined on Y by the restriction of $d(x, y)$ to $x, y \in Y$ is the same as the topology induced on Y by the topology determined on X by $d(\cdot, \cdot)$. More precisely, if U is an open subset of X with respect to the topology determined by $d(\cdot, \cdot)$, then it is easy to see that $U \cap Y$ is an open set in Y with respect to the topology determined by the restriction of $d(x, y)$ to $x, y \in Y$, using (1.1.8). This implies that every open set in Y with respect to the induced topology is an open set with respect to the topology determined on Y by the restriction of $d(\cdot, \cdot)$ to Y. In the other direction, (1.1.8) is an open set in Y with respect to the induced topology for every $x \in Y$ and $r > 0$, because $B_X(x, r)$ is an open set in X. If $E \subseteq Y$ is an open set with respect to the topology determined on Y by the restriction of $d(\cdot, \cdot)$, then E can be expressed as a union of open balls in Y. This means that E can be expressed as the union of open subsets of Y with respect to the induced topology, so that E is an open set with respect to the induced topology on Y, as desired. Of course, if $d(x, y)$ is a metric on X, then its restriction to $x, y \in Y$ is a metric on Y.

Let $x \in X$ be given, and consider

$$f_x(y) = d(x, y) \tag{1.1.9}$$

as a real-valued function of y on X. Note that

$$f_x(y) \leq f_x(y') + d(y, y') \tag{1.1.10}$$

for every $y, y' \in X$, by the triangle inequality. Hence

$$f_x(y) - f_x(y') \leq d(y, y') \tag{1.1.11}$$

for every $y, y' \in X$, and similarly with the roles of y and y' exchanged. This implies that

$$|f_x(y) - f_x(y')| \leq d(y, y') \tag{1.1.12}$$

for every $y, y' \in X$, where $|t|$ is the absolute value of a real number t. It follows that f_x is continuous on X with respect to the topology determined on X by $d(\cdot, \cdot)$, and the standard topology on the real line.

Let us say that $d(\cdot, \cdot)$ is compatible with a topology τ on X at a point $x \in X$ if for every $r > 0$ there is an open set $V \subseteq X$ with respect to τ such that $x \in V$ and

$$V \subseteq B(x, r). \tag{1.1.13}$$

This is the same as saying that (1.1.9) is continuous at x, as a real-valued function of y on X, and with respect to τ. If $d(\cdot, \cdot)$ is compatible with τ at every $x \in X$, then we may simply say that $d(\cdot, \cdot)$ is compatible with τ on X. In this case, τ is at least as strong as the topology determined on X by $d(\cdot, \cdot)$. More precisely, if $U \subseteq X$ is an open set with respect to the topology determined on X by $d(\cdot, \cdot)$, then U can be expressed as a union of open sets in X with respect to τ, so that U is an open set with respect to τ. Conversely, if every open subset of X with respect to $d(\cdot, \cdot)$ is an open set with respect to τ, then $d(\cdot, \cdot)$ is compatible with τ on X, because one can take $V = B(x, r)$ in (1.1.13). If $d(\cdot, \cdot)$ is compatible with τ on X, then (1.1.9) is continuous as a real-valued function of y on X with respect to τ for every $x \in X$.

1.2 Topological groups

Let G be a group, in which the group operations are expressed multiplicatively, and suppose that G is also equipped with a topology. If the group operations are continuous, then G is said to be a *topological group*. More precisely, continuity of multiplication on G means that multiplication is continuous as a mapping from $G \times G$ into G, using the product topology on $G \times G$ corresponding to the given topology on G. Similarly,

$$x \mapsto x^{-1} \tag{1.2.1}$$

should be continuous as a mapping from G into itself. This implies that (1.2.1) is a homeomorphism from G onto itself, because (1.2.1) is its own inverse mapping.

Let G be a topological group, and let $a, b \in G$ be given. It is easy to see that the left translation mapping

$$x \mapsto a\,x \tag{1.2.2}$$

is continuous as a mapping from G into itself, because of continuity of multiplication on G. In fact, (1.2.2) is a homeomorphism from G onto itself, because the inverse mapping corresponding to (1.2.2) is given by translation on the left by a^{-1}. Similarly, the right translation mapping

$$x \mapsto x\,b \tag{1.2.3}$$

is a homeomorphism from G onto itself.

Let e be the identity element in G. If $\{e\}$ is a closed set in G, then every subset of G with only one element is a closed set, by continuity of translations.

In this case, G satisfies the first separation condition as a topological space. We shall be primarily concerned with topological groups with this property, which is sometimes included in the definition of a topological group.

If G is any group, then G is a topological group with respect to the discrete topology. Similarly, G is a topological group with respect to the indiscrete topology, if $\{e\}$ is not required to be a closed set in G. The real line $\mathbf{R}$ is a commutative topological group with respect to addition and the standard topology. If G is a topological group and H is a subgroup, then H is a topological group with respect to the topology induced by the topology on G. Note that the closure of H in G is also a subgroup of G in this case.

Let G be a group, and let A, B be subsets of G. If $a, b \in G$, then put

$$A\,b = \{x\,b : x \in A\} \tag{1.2.4}$$

and

$$a\,B = \{a\,y : y \in B\}. \tag{1.2.5}$$

Also put

$$A\,B = \{x\,y : x \in A,\ y \in B\}, \tag{1.2.6}$$

so that

$$A\,B = \bigcup_{a \in A} a\,B = \bigcup_{b \in B} A\,b. \tag{1.2.7}$$

Similarly, put

$$A^{-1} = \{x^{-1} : x \in A\}, \tag{1.2.8}$$

and let us say that A is *symmetric* about e in G when

$$A^{-1} = A. \tag{1.2.9}$$

Note that

$$A \cap A^{-1} \tag{1.2.10}$$

is automatically symmetric about e in G.

Now let G be a topological group, and let A, B be subsets of G again. If $a, b \in G$ and A, B are open subsets of G, then $A\,b$ and $a\,B$ are open sets as well, by continuity of translations. If A or B is an open set in G, then it follows that $A\,B$ is an open set in G, because $A\,B$ is a union of open sets in G, as in (1.2.7). If A is an open set, then A^{-1} is an open set too, because (1.2.1) is a homeomorphism on G. This implies that (1.2.10) is an open set in G, so that every open set that contains e contains an open set that contains e and is symmetric about e.

If $W \subseteq G$ is an open set with $e \in W$, then there are open sets $U, V \subseteq G$ such that $e \in U, V$ and

$$U\,V \subseteq W. \tag{1.2.11}$$

This follows from the continuity of multiplication on G at (e, e). More precisely, this also uses the fact that if O is an open set in $G \times G$ with respect to the product topology and $(e, e) \in O$, then there are open sets $U, V \subseteq G$ such that

$e \in U, V$ and $U \times V \subseteq O$. Of course, we can take $U = V$ in (1.2.11), by replacing U and V with their intersection.

Let $x \in G$ and $E \subseteq G$ be given, and remember that x is an element of the closure $\overline{E}$ of E in G if and only if for every open set $U_1 \subseteq G$ with $x \in U_1$, we have that

$$U_1 \cap E \neq \emptyset. \tag{1.2.12}$$

This is equivalent to saying that for every open set $U_0 \subseteq G$ with $e \in U_0$, we have that

$$(U_0\, x) \cap E \neq \emptyset, \tag{1.2.13}$$

because of continuity of translations. Note that (1.2.13) holds exactly when

$$x \in U_0^{-1}\, E. \tag{1.2.14}$$

Because (1.2.1) is a homeomorphism on G, U_0^{-1} may be considered as an arbitrary open set in G that contains e, It follows that

$$\overline{E} = \bigcap \{U\, E : U \subseteq G \text{ is an open set, with } e \in U\}. \tag{1.2.15}$$

Similarly,

$$\overline{E} = \bigcap \{E\, V : V \subseteq G \text{ is an open set, with } e \in V\}. \tag{1.2.16}$$

This could also be obtained from (1.2.15), using the fact that (1.2.1) is a homeomorphism on G.

1.3 Regular topological spaces

A topological space X is said to be *regular in the strict sense* if for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $E \subseteq V$. This is equivalent to asking that for every $x \in X$ and open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$ and the closure $\overline{U}$ of U in X is contained in W, by a standard argument. Let us say that X is *regular in the strong sense* when X is regular in the strict sense and X satisfies the first separation condition, in which case we may also say that X satisfies the *third separation condition.* This implies that X is Hausdorff, and in fact it would suffice to ask that X satisfy the 0th separation condition instead of the first separation condition.

If the topology on X is determined by a semimetric $d(\cdot, \cdot)$, then it is easy to see that X is regular in the strict sense. If $d(\cdot, \cdot)$ is a metric on X, then X is regular in the strong sense.

Let G be a topological group, and let us check that G is regular as a topological space in the strict sense. Let W be an open subset of G that contains the identity element e, and let U, V be open subsets of G that contain e and satisfy (1.2.11). This implies that

$$\overline{U}, \overline{V} \subseteq W, \tag{1.3.1}$$

by (1.2.15) and (1.2.16). One can use continuity of translations to get an analogous statement for neighborhoods of any element of G, as desired.

If $\{e\}$ is a closed set in G, then it follows that G is regular as a topological space in the strong sense, and in particular that G is Hausdorff. One can also check more directly that G is Hausdorff in this situation, as follows. If $y \in G$, then $\{y\}$ is a closed set in G, so that $W = G \setminus \{y\}$ is an open set. If $y \neq e$, then $e \in W$, and one can use (1.2.11) to get disjoint open subsets of G that contain e and y. This implies that G is Hausdorff, using continuity of translations to deal with any two distinct elements of G.

Let X be a topological space that is regular in the strict sense. If K is a compact subset of X, $W \subseteq X$ is an open set, and $K \subseteq W$, then there is an open set $U \subseteq X$ such that $K \subseteq U$ and $\overline{U} \subseteq W$. Indeed, for each $x \in K$, there is an open set $U(x) \subseteq X$ such that $x \in U(x)$ and $\overline{U(x)} \subseteq W$, because X is regular in the strict sense. One can use compactness of K to cover K by finitely many of these open sets, and it is easy to see that the union U of these finitely many open sets has the desired properties.

Suppose that the topology on X is determined by a semimetric $d(\cdot, \cdot)$. If $A \subseteq X$ and r is a positive real number, then put

$$A_r = \bigcup_{x \in A} B(x, r), \tag{1.3.2}$$

where $B(x, r)$ is as in (1.1.5). Thus $A \subseteq A_r$, and A_r is an open set in X, because A_r is a union of open sets. It is easy to see that $\overline{A} \subseteq A_r$, and in fact

$$\overline{A} = \bigcap_{r > 0} A_r, \tag{1.3.3}$$

where more precisely the intersection is taken over all positive real numbers r.

Let $K \subseteq X$ be compact, let $W \subseteq X$ be an open set, and suppose that $K \subseteq W$, as before. If $x \in K$, then there is a positive real number $r(x)$ such that

$$B(x, r(x)) \subseteq W. \tag{1.3.4}$$

The collection of open balls $B(x, r(x)/2)$ with $x \in K$ forms an open covering of K, so that there are finitely many elements $x_1, \ldots, x_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} B(x_j, r(x_j)/2), \tag{1.3.5}$$

by compactness. If we put

$$r = \min_{1 \le j \le n} (r(x_j)/2) > 0, \tag{1.3.6}$$

then one can check that

$$K_r \subseteq W, \tag{1.3.7}$$

using the triangle inequality.

Let G be a topological group, let K be a compact subset of G, let W be an open subset of G, and suppose that $K \subseteq W$. If $x \in K$, then $x^{-1} W$ is an open subset of G that contains e, and hence there is an open subset $U(x)$ of G such that $e \in U(x)$ and

$$U(x)\,U(x) \subseteq x^{-1}\,W, \tag{1.3.8}$$

as in (1.2.11). The collection of open sets of the form $U(x)\,x$, $x \in K$, is an open covering of K, and so there are finitely many elements $x_1, \ldots, x_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} U(x_j)\,x_j, \tag{1.3.9}$$

by compactness. Put

$$U = \bigcap_{j=1}^{n} U(x_j), \tag{1.3.10}$$

which is an open subset of G that contains e. Observe that

$$U\,K \subseteq \bigcup_{j=1}^{n} U\,U(x_j)\,x_j \subseteq \bigcup_{j=1}^{n} U(x_j)\,U(x_j)\,x_j \subseteq W, \tag{1.3.11}$$

using (1.3.9) in the first step, (1.3.10) in the second step, and (1.3.8) in the third step. Similarly, there is an open subset V of G such that $e \in V$ and

$$K\,V \subseteq W. \tag{1.3.12}$$

This could also be obtained from (1.3.11), using the fact that (1.2.1) is a homeomorphism on G.

1.4 Translation-invariant semimetrics

Let G be a group, and let $d(x, y)$ be a semimetric on G. If

$$d(a\,x, a\,y) = d(x, y) \tag{1.4.1}$$

for every $a, x, y \in G$, then $d(\cdot, \cdot)$ is said to be *invariant under left translations* on G. Similarly, if

$$d(x\,a, y\,a) = d(x, y) \tag{1.4.2}$$

for every $a, x, y \in G$, then $d(\cdot, \cdot)$ is said to be *invariant under right translations* on G. Of course, if G is commutative, then (1.4.1) and (1.4.2) are equivalent, in which case we may simply say that $d(\cdot, \cdot)$ is *invariant under translations* on G.

If $d(\cdot, \cdot)$ is invariant under left or right translations on G, then we have that

$$d(x, e) = d(e, x^{-1}) \tag{1.4.3}$$

for every $x \in G$. This is the same as saying that

$$d(e,x) = d(e,x^{-1}) \tag{1.4.4}$$

for every $x \in G$, because of (1.1.2). This implies that open and closed balls in G centered at the identity element e with respect to d are symmetric about e.

If $d(\cdot,\cdot)$ is any semimetric on G again, then

$$\widetilde{d}(x,y) = d(x^{-1},y^{-1}) \tag{1.4.5}$$

defines a semimetric on G too. It is easy to see that $d(\cdot,\cdot)$ is invariant under left translations on G if and only if $\widetilde{d}(\cdot,\cdot)$ is invariant under right translations on G. Similarly, $d(\cdot,\cdot)$ is invariant under right translations on G if and only if $\widetilde{d}(\cdot,\cdot)$ is invariant under left translations on G. If $d(\cdot,\cdot)$ is invariant under both left and right translations on G, then one can verify that

$$\widetilde{d}(x,y) = d(x,y) \tag{1.4.6}$$

for every $x,y \in G$. More precisely, we have that

$$d(x,y) = d(x^{-1}\,x\,y^{-1}, x^{-1}\,y\,y^{-1}) = d(y^{-1},x^{-1}) \tag{1.4.7}$$

for every $x,y \in G$, using invariance under left translation by x^{-1} and right translation by y^{-1} in the first step.

Let $d(\cdot,\cdot)$ be a semimetric on G, and let $x,y \in G$ be given. If $d(\cdot,\cdot)$ is invariant under left translations on G, then

$$d(e,x\,y) \le d(e,x) + d(x,x\,y) = d(e,x) + d(e,y). \tag{1.4.8}$$

Similarly, if $d(\cdot,\cdot)$ is invariant under right translations on G, then

$$d(e,x\,y) \le d(e,y) + d(y,x\,y) = d(y,e) + d(x,e). \tag{1.4.9}$$

In both cases, we get that

$$B(e,r)\,B(e,t) \subseteq B(e,r+t) \tag{1.4.10}$$

for every $r,t > 0$, where these open balls are defined as in (1.1.5).

Let $d(x,y)$ be a semimetric on G, and let $r > 0$ be given. If $d(\cdot,\cdot)$ is invariant under left translations on G, then

$$B(x,r) = x\,B(e,r) \tag{1.4.11}$$

for every $x \in G$. If A is a subset of G and A_r is defined as in (1.3.2), then we get that

$$A_r = \bigcup_{x \in A} x\,B(e,r) = A\,B(e,r). \tag{1.4.12}$$

Similarly, if $d(\cdot,\cdot)$ is invariant under right translations on G, then

$$B(x,r) = B(e,r)\,x \tag{1.4.13}$$

for every $x \in G$. This implies that

$$A_r = \bigcup_{x \in A} B(e,r)\, x = B(e,r)\, A \tag{1.4.14}$$

for every subset A of G.

Suppose now that G is a topological group, and let $d(\cdot,\cdot)$ be a semimetric on G. Suppose that $d(\cdot,\cdot)$ is compatible with the given topology on G at e, as in Section 1.1. This means that for every $r > 0$ there is an open subset V_r of G, with respect to the topology given on G, such that $e \in V_r$ and

$$V_r \subseteq B(e,r). \tag{1.4.15}$$

If $d(\cdot,\cdot)$ is invariant under left translations on G, then it follows that

$$x\, V_r \subseteq x\, B(e,r) = B(x,r) \tag{1.4.16}$$

for every $x \in G$ and $r > 0$. Similarly, if $d(\cdot,\cdot)$ is invariant under right translations on G, then

$$V_r\, x \subseteq B(e,r)\, x = B(x,r) \tag{1.4.17}$$

for every $x \in G$ and $r > 0$. In both cases, we get that $d(\cdot,\cdot)$ is compatible with the given topology on G, as in Section 1.1. This implies that $d(e,x)$ is continuous as a real-valued function of x on G, with respect to the given topology on G, as in Section 1.1.

If there is a local base for the topology of G at e with only finitely or countably many elements, then a famous theorem states that there is a semimetric on G that is invariant under left translations on G, and which determines the same topology on G. If $\{e\}$ is a closed set in G, then this semimetric on G is a metric. Of course, there is an analogous statement with invariance under right translations instead of left translations. Note that the existence of a local base for the topology of G at e with only finitely or countably many elements is necessary to have a semimetric on G that determines the same topology on G. More precisely, if $d(\cdot,\cdot)$ is a semimetric on a set X and $x \in X$, then the collection of open balls of the form $B(x, 1/j)$ with j in the set $\mathbf{Z}_+$ of positive integers is a local base for the topology determined on X by $d(\cdot,\cdot)$ at x.

1.5 Collections of semimetrics

Let X be a set, and let $d_1, \ldots, d_n$ be finitely many semimetrics on X. It is easy to see that

$$d(x,y) = \max_{1 \le j \le n} d_j(x,y) \tag{1.5.1}$$

also defines a semimetric on X. If $x \in X$ and $r > 0$, then

$$B_d(x,r) = \bigcap_{j=1}^{n} B_{d_j}(x,r), \tag{1.5.2}$$

where these open balls are defined as in (1.1.5), as usual. Alternatively,

$$d'(x,y) = \sum_{j=1}^{n} d_j(x,y) \tag{1.5.3}$$

is a semimetric on X too, and

$$d(x,y) \le d'(x,y) \le n\, d(x,y) \tag{1.5.4}$$

for every $x, y \in X$. In particular, this implies that (1.5.1) and (1.5.3) determine the same topology on X.

Now let $\mathcal{M}$ be a nonempty collection of semimetrics on X. Let us say that $U \subseteq X$ is an *open set* with respect to $\mathcal{M}$ if for every $x \in U$ there are finitely many elements $d_1, \ldots, d_n$ of $\mathcal{M}$ and positive real numbers $r_1, \ldots, r_n$ such that

$$\bigcap_{j=1}^{n} B_{d_j}(x, r_j) \subseteq U. \tag{1.5.5}$$

One can also take the r_j's to be the same, by replacing them by their minimum. This defines a topology on X, which contains the topologies determined on X by each of the elements of $\mathcal{M}$. If $\mathcal{M}$ has only finitely many elements, then the topology determined on X by $\mathcal{M}$ is the same as the topology determined by the semimetric on X obtained by taking the maximum of the elements of $\mathcal{M}$.

One can check that the topology determined on X by $\mathcal{M}$ is regular in the strict sense, as in Section 1.3. Let us say that $\mathcal{M}$ is *nondegenerate* on X if for every $x, y \in X$ with $x \ne y$ there is a $d \in \mathcal{M}$ such that

$$d(x,y) > 0. \tag{1.5.6}$$

In this case, X is Hausdorff with respect to the topology determined by $\mathcal{M}$. If $\mathcal{M}$ is nondegenerate and $\mathcal{M}$ has only finitely many elements, then the sum and maximum of the elements of $\mathcal{M}$ are metrics on X.

Let Y be a subset of X, and for each $d \in \mathcal{M}$, let $d_Y(x,y)$ be the restriction of $d(x,y)$ to $x, y \in Y$. Thus

$$\mathcal{M}_Y = \{d_Y : d \in \mathcal{M}\} \tag{1.5.7}$$

is a nonempty collection of semimetrics on Y, which determines a topology on Y as before. One can check that the topology determined on Y by $\mathcal{M}_Y$ is the same as the topology induced on Y by the topology determined on X by $\mathcal{M}$. This is analogous to the case of a single semimetric, as in Section 1.1. More precisely, let $B_{X,d}(x,r)$ be the open ball in X centered at $x \in X$ with radius $r > 0$ with respect to $d \in \mathcal{M}$, and let $B_{Y,d_Y}(x,r)$ be the open ball in Y centered at $x \in Y$ with radius $r > 0$ with respect to d_Y. Note that

$$B_{Y,d_Y}(x,r) = B_{X,d}(x,r) \cap Y \tag{1.5.8}$$

for every $x \in Y$, $r > 0$, and $d \in \mathcal{M}$, as in (1.1.8). Using this, one can verify that every open subset of Y with respect to the induced topology is also an open set with respect to the topology determined by $\mathcal{M}_Y$. If $d \in \mathcal{M}$, $x \in X$, and $r > 0$, then $B_{X,d}(x, r)$ is an open set in X with respect to the topology determined by d, and hence with respect to the topology determined on X by $\mathcal{M}$. This implies that (1.5.8) is an open set in Y with respect to the topology induced on Y by the topology determined on X by $\mathcal{M}$. It follows that finite intersections of subsets of Y of this form are open sets with respect to the induced topology as well. If $E \subseteq Y$ is an open set with respect to $\mathcal{M}_Y$, then E can be expressed as a union of finite intersections of sets of this form. This implies that E is an open set with respect to the induced topology, as desired. If $\mathcal{M}$ is nondegenerate on X, then $\mathcal{M}_Y$ is clearly nondegenerate on Y.

If G is a topological group, then it is well known that there is a collection $\mathcal{M}$ of semimetrics on G such that every element of $\mathcal{M}$ is invariant under left translations on G, and the topology determined on G by $\mathcal{M}$ is the same as the given topology on G. If $\{e\}$ is a closed set in G, then $\mathcal{M}$ is nondegenerate on G. As before, there is an analogous statement with invariance under right translations instead of left translations. Note that if $d_1, \ldots, d_n$ are finitely many semimetrics on G that are invariant under left translations, then their sum and maximum are invariant under left translations on G, and similarly for invariance under right translations.

1.6 Sequences of semimetrics

Let X be a set, and let $d(x, y)$ be a semimetric on X. Also let t be a positive real number, and put

$$d_t(x, y) = \min(d(x, y), t) \tag{1.6.1}$$

for every $x, y \in X$. One can check that this defines a semimetric on X too, which is a metric when $d(x, y)$ is a metric. If $x \in X$ and r is another positive real number, then

$$\begin{aligned} B_{d_t}(x, r) &= B_d(x, r) \quad \text{when } r \le t \\ &= X \qquad\qquad \text{when } r > t, \end{aligned} \tag{1.6.2}$$

where these open balls are defined as in (1.1.5). This implies that the topologies determined on X by d and d_t are the same. Note that $d_t(x, y) = 0$ exactly when $d(x, y) = 0$. In particular, if $d(x, y)$ is a metric on X, then $d_t(x, y)$ is a metric on X.

Now let $d_1, d_2, d_3, \ldots$ be an infinite sequence of semimetrics on X, and put

$$d_j'(x, y) = \min(d_j(x, y), 1/j) \tag{1.6.3}$$

for every $x, y \in X$ and positive integer j. Thus d_j' is a semimetric on X that determines the same topology on X as d_j for each $j \in \mathbf{Z}_+$, as in the preceding paragraph. More precisely, for each $x \in X$, $r > 0$, and $j \in \mathbf{Z}_+$, we have that

$$\begin{aligned} B_{d_j'}(x, r) &= B_{d_j}(x, r) \quad \text{when } r \le 1/j \\ &= X \qquad\qquad \text{when } r > 1/j, \end{aligned} \tag{1.6.4}$$

as in (1.6.2). Put

$$d(x,y) = \max_{j \ge 1} d_j'(x,y) \tag{1.6.5}$$

for each $x, y \in X$, which is equal to 0 when $d_j'(x,y) = 0$ for every $j \in \mathbf{Z}_+$. If $d_l'(x,y) > 0$ for some $l \in \mathbf{Z}_+$, then $d_j'(x,y) \le 1/j \le d_l'(x,y)$ for all sufficiently large j, so that the right side of (1.6.5) reduces to the maximum of finitely many terms. Thus the right side of (1.6.5) is defined as a nonnegative real number for every $x, y \in X$, and one can check that (1.6.5) defines a semimetric on X. If the collection of semimetrics d_j, $j \in \mathbf{Z}_+$, is nondegenerate on X, as in the previous section, then the collection of semimetrics d_j', $j \in \mathbf{Z}_+$, is nondegenerate on X, and (1.6.5) is a metric on X.

If $x \in X$ and $r > 0$, then

$$B_d(x,r) = \bigcap_{j=1}^{\infty} B_{d_j'}(x,r), \tag{1.6.6}$$

by the definition (1.6.5) of d. Let $[1/r]$ be the largest nonnegative integer less than or equal to $1/r$, as usual. Combining (1.6.4) and (1.6.6), we get that

$$\begin{aligned} B_d(x,r) &= \bigcap_{j=1}^{[1/r]} B_{d_j}(x,r) \quad \text{when } r \le 1 \\ &= X \quad \text{when } r > 1. \end{aligned} \tag{1.6.7}$$

Using this, one can check that the topology determined on X by d is the same as the topology determined on X by the collection of semimetrics d_j, $j \in \mathbf{Z}_+$, as in the previous section. More precisely, $B_{d_j}(x,r)$ is an open set in X with respect to d_j for every $x \in X$, $r > 0$, and $j \in \mathbf{Z}_+$, as in Section 1.1. This implies that $B_{d_j}(x,r)$ is an open set in X with respect to the collection of d_l's, $l \in \mathbf{Z}_+$, for every $x \in X$, $r > 0$, and $j \in \mathbf{Z}_+$. It follows that $B_d(x,r)$ is an open set in X with respect to the collection of d_l's, $l \in \mathbf{Z}_+$, for every $x \in X$ and $r > 0$, because of (1.6.7). If $U \subseteq X$ is an open set with respect to d, then U can be expressed as a union of open balls in X with respect to d, which implies that U is an open set with respect to the collection of d_l's, $l \in \mathbf{Z}_+$. Conversely, if $U \subseteq X$ is an open set with respect to the collection of d_l's, $l \in \mathbf{Z}_+$, then one can verify that U is an open set with respect to d, using (1.6.7).

Let G be a group. If $d(x,y)$ is a semimetric on G that is invariant under left translations, then (1.6.1) is invariant under left translations for every $t > 0$, and similarly for right translations. Now let $d_1, d_2, d_3, \ldots$ be an infinite sequence of semimetrics on G, each of which is invariant under left translations. This implies that (1.6.3) is invariant under left translations for every $j \in \mathbf{Z}_+$, and hence that (1.6.5) is invariant under left translations. Similarly, if d_j is invariant under right translations for every $j \in \mathbf{Z}_+$, then (1.6.3) is invariant under right translations for every $j \in \mathbf{Z}_+$, so that (1.6.5) is invariant under right translations.

1.7 Continuity conditions on groups

Let G and H be topological groups, and let ϕ be a group homomorphism from G into H. If ϕ is continuous at the identity element e in G, then it is easy to see that ϕ is continuous at every point in G, using continuity of translations on G and H. More precisely, this works when G and H are equipped with topologies for which left translations are continuous. In this case, left translations on G and H are homeomorphisms, as before. Similarly, if right translations on G and H are continuous, and ϕ is continuous at the identity element in G, then ϕ is continuous at every point in G.

Let G be a group again, and suppose that G is equipped with a topology τ. In order to check that G is a topological group with respect to τ, it is often helpful to simplify the continuity conditions that need to be verified. Suppose for the moment that left and right translations are continuous on G with respect to τ, which implies that they are homeomorphisms on G with respect to τ, as before. Under suitable conditions, one would like to show that multiplication on G is continuous, as a mapping from $G \times G$ into G, and with respect to the corresponding product topology on G. If multiplication on G is continuous as a mapping from $G \times G$ into G at (e, e), then one can use continuity of left and right translations on G to get that multiplication on G is continuous at every point in $G \times G$.

Similarly, suppose that $x \mapsto x^{-1}$ is continuous at e as a mapping from G into itself. Using continuity of left and right translations on G, one can get that $x \mapsto x^{-1}$ is continuous at every point in G.

Let $x \in G$ be given, and put

$$C_x(y) = x\,y\,x^{-1} \tag{1.7.1}$$

for every $y \in G$. This is a group automorphism on G, which is the inner automorphism defined by conjugation by x. If left and right translations are continuous on G, then C_x is a continuous mapping from G into itself for every $x \in G$. Of course, if C_x is a continuous mapping from G into itself for every $x \in G$, then C_x is a homeomorphism on G for every $x \in G$, because $C_{x^{-1}}$ is the inverse mapping associated to C_x.

Suppose for the moment that left translations on G are continuous. If C_x is continuous on G for every $x \in G$, then it follows that right translations on G are continuous as well. In order to check that C_x is continuous on G for some $x \in G$, it suffices to verify that C_x is continuous at the identity element e in G, by the remarks at the beginning of the section. Similarly, if right translations on G are continuous, and if C_x is continuous at e for some $x \in G$, then C_x is continuous at every point in G. If this holds for every $x \in G$, then it follows that left translations are also continuous on G.

Let $d(x, y)$ be a semimetric on G. If $d(\cdot, \cdot)$ is invariant under left translations on G, then left translations on G are automatically homeomorphisms with respect to the topology determined on G by $d(\cdot, \cdot)$. Similarly, if $d(\cdot, \cdot)$ is invariant under right translations on G, then right translations on G are automatically homeomorphisms with respect to the topology determined on G by $d(\cdot,\cdot)$. If $d(\cdot,\cdot)$ is invariant under both left and right translations on G, then both left and right translations on G are automatically homeomorphisms on G with respect to the topology determined by $d(\cdot,\cdot)$.

Suppose that $d(\cdot,\cdot)$ is invariant under left translations on G, or under right translations on G, or both. In this case, one can use (1.4.4) to get that $x \mapsto x^{-1}$ is continuous at e as a mapping from G into itself, with respect to the topology determined on G by $d(\cdot,\cdot)$. Similarly, (1.4.10) implies that multiplication on G is continuous as a mapping from $G \times G$ into G at (e, e), with respect to the topology determined on G by $d(\cdot,\cdot)$, and the corresponding product topology on $G \times G$. If $d(\cdot,\cdot)$ is invariant under both left and right translations on G, then it follows that G is a topological group with respect to the topology determined by $d(\cdot,\cdot)$.

Now let $\mathcal{M}$ be a nonempty collection of semimetrics on G. If every element of $\mathcal{M}$ is invariant under left translations on G, then left translations on G are automatically homeomorphisms with respect to the topology determined on G by $\mathcal{M}$ as in Section 1.5. Similarly, if every element of $\mathcal{M}$ is invariant under right translations on G, then right translations on G are automatically homeomorphisms with respect to the topology determined on G by $\mathcal{M}$. In both cases, $x \mapsto x^{-1}$ is continuous at e as a mapping from G into itself, with respect to the topology determined on G by $\mathcal{M}$. We also have that multiplication on G is continuous as a mapping from $G \times G$ into G at (e, e) in both situations, with respect to the topology determined on G by $\mathcal{M}$, and the corresponding product topology on $G \times G$. If every element of $\mathcal{M}$ is invariant under both left and right translations, then it follows that G is a topological group with respect to the topology determined by $\mathcal{M}$. Otherwise, one should look at the continuity properties of the translations on G under which the elements of $\mathcal{M}$ are not necessarily invariant.

If $d(\cdot,\cdot)$ is a semimetric on G, then

$$\widetilde{d}(x, y) = d(x^{-1}, y^{-1}) \tag{1.7.2}$$

is a semimetric on G as well, as in (1.4.5). Let $\mathcal{M}$ be a nonempty collection of semimetrics on G again, so that

$$\widetilde{\mathcal{M}} = \{\widetilde{d} : d \in \mathcal{M}\} \tag{1.7.3}$$

is a nonempty collection of semimetrics on G too. If the elements of $\mathcal{M}$ are invariant under left translations on G, then the elements of $\widetilde{\mathcal{M}}$ are invariant under right translations on G, as in Section 1.4. Similarly, if the elements of $\mathcal{M}$ are invariant under right translations on G, then the elements of $\widetilde{\mathcal{M}}$ are invariant under left translations on G. Remember that $\widetilde{d} = d$ when a semimetric d on G is invariant under left and right translations, as in (1.4.6). In particular, if this holds for every $d \in \mathcal{M}$, then

$$\widetilde{\mathcal{M}} = \mathcal{M}. \tag{1.7.4}$$

Of course, this implies that the topologies determined on G by $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are the same.

If $\mathcal{M}$ is any nonempty collection of semimetrics on G, then $x \mapsto x^{-1}$ is automatically a homemorphism from G with the topology determined by $\mathcal{M}$ onto G with the topology determined by $\widetilde{\mathcal{M}}$. Thus $x \mapsto x^{-1}$ is a homeomorphism from G onto itself with respect to the topology determined by $\mathcal{M}$ on both the domain and range if and only if the topologies determined on G by $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are the same. This is a necessary condition for G to be a topological group with respect to the topology determined by $\mathcal{M}$. Suppose now that the elements of $\mathcal{M}$ are invariant under left translations on G, or that the elements of $\mathcal{M}$ are invariant under right translations. If the topologies determined on G by $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are the same, then it follows that both left and right translations on G are continuous with respect to this topology. In this case, G is a topological group with respect to this topology, by the earlier arguments. This condition holds automatically when the elements of $\mathcal{M}$ are invariant under both left and right translations on G, as before.

1.8 Cartesian products

Let I be a nonempty set, and let X_j be a set for each $j \in I$. Consider the *Cartesian product*

$$X = \prod_{j \in I} X_j \tag{1.8.1}$$

of the X_j's, $j \in I$. If $x \in X$ and $j \in I$, then x_j denotes the jth coordinate of x in X_j. Similarly, let p_j be the natural projection from X into X_j for each $j \in I$, so that

$$p_j(x) = x_j \tag{1.8.2}$$

for every $x \in X$ and $j \in I$.

Let E_l be a subset of X_l for some $l \in I$. Put $W_l = E_l$, and $W_j = X_j$ for every $j \in I$ with $j \neq l$. Observe that

$$p_l^{-1}(E_l) = \prod_{j \in I} W_j \tag{1.8.3}$$

as subsets of X.

Let d_l be a semimetric on X_l for some $l \in I$, and put

$$\widehat{d_l}(x, y) = d_l(x_l, y_l) \tag{1.8.4}$$

for each $x, y \in X$. It is easy to see that $\widehat{d_l}$ defines a semimetric on X. Let $B_{X_l,d_l}(x_l, r)$ be the open ball in X_l centered at $x_l \in X_l$ with radius $r > 0$ with respect to d_l, as in (1.1.5). Similarly, let $B_{X,\widehat{d_l}}(x, r)$ be the open ball in X centered at $x \in X$ with radius $r > 0$ with respect to $\widehat{d_l}$. If $x \in X$ and $x_l = p_l(x)$, then

$$B_{X,\widehat{d_l}}(x, r) = p_l^{-1}(B_{X_l,d_l}(x_l, r)) \tag{1.8.5}$$

for every $r > 0$.

Let $\mathcal{M}_l$ be a nonempty collection of semimetrics on X_l for each $l \in I$. Put

$$\widehat{\mathcal{M}_l} = \{\widehat{d_l} : d_l \in \mathcal{M}_l\} \tag{1.8.6}$$

for every $l \in I$, where $\widehat{d_l}$ is associated to d_l as in (1.8.4). Thus $\widehat{\mathcal{M}_l}$ is a nonempty collection of semimetrics on X for each $l \in I$, so that

$$\mathcal{M} = \bigcup_{l \in I} \widehat{\mathcal{M}_l} \tag{1.8.7}$$

is a nonempty collection of semimetrics on X too. Remember that $\mathcal{M}_l$ determines a topology on X_l for each $l \in I$, as in Section 1.5, and similarly $\mathcal{M}$ determines a topology on X. One can check that the topology determined on X by $\mathcal{M}$ is the same as the product topology corresponding to the topologies determined on the X_l's by the $\mathcal{M}_l$'s, $l \in I$.

Let I be a nonempty set again, and let G_j be a group for each $j \in I$. The *Cartesian product*

$$G = \prod_{j \in I} G_j \tag{1.8.8}$$

is also a group, where the group operations are defined coordinatewise. If G_j is a topological group for each $j \in I$, then one can verify that G is a topological group with respect to the corresponding product topology. Let d_l be a semimetric on G_l for some $l \in I$, and let $\widehat{d_l}$ be the corresponding semimetric on G, as in (1.8.4). If d_l is invariant under left or right translations on G_l, then $\widehat{d_l}$ has the analogous property on G.

1.9 Cauchy sequences

Let X be a set, and let $d(x, y)$ be a semimetric on X. A sequence $\{x_j\}_{j=1}^\infty$ of elements of X is said to be a *Cauchy sequence* with respect to $d(\cdot, \cdot)$ if for every $\epsilon > 0$ there is a positive integer L such that

$$d(x_j, x_l) < \epsilon \tag{1.9.1}$$

for every $j, l \geq L$. Note that $\{x_j\}_{j=1}^\infty$ converges to an element x of X with respect to the topology determined by $d(\cdot, \cdot)$ if and only if

$$\lim_{j \to \infty} d(x_j, x) = 0. \tag{1.9.2}$$

This implies that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $d(\cdot, \cdot)$, by a standard argument. If $d(\cdot, \cdot)$ is a metric on X, and if every Cauchy sequence of elements of X with respect to $d(\cdot, \cdot)$ converges to an element of X with respect to $d(\cdot, \cdot)$, then X is said to be *complete* with respect to $d(\cdot, \cdot)$.

Let $\mathcal{M}$ be a nonempty collection of semimetrics on X. A sequence $\{x_j\}_{j=1}^\infty$ of elements of X is said to be a *Cauchy sequence* with respect to $\mathcal{M}$ if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to every $d \in \mathcal{M}$. Observe that $\{x_j\}_{j=1}^\infty$ converges to an element x of X with respect to the topology determined by $\mathcal{M}$ if and only if $\{x_j\}_{j=1}^\infty$ converges to x with respect to every $d \in \mathcal{M}$. This implies that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $\mathcal{M}$, as before.

If $\mathcal{M}$ has only finitely many elements, then the sum and maximum of the elements of $\mathcal{M}$ are semimetrics on X that determine the same topology on X as $\mathcal{M}$, as in Section 1.5. In particular, the convergence of a sequence $\{x_j\}_{j=1}^\infty$ of elements of X to an element x of X with respect to the topology determined by $\mathcal{M}$ is equivalent to the convergence of $\{x_j\}_{j=1}^\infty$ to x with respect to the sum or maximum of the elements of $\mathcal{M}$. Similarly, the Cauchy condition for a sequence $\{x_j\}_{j=1}^\infty$ of elements of X with respect to $\mathcal{M}$ is equivalent to the corresponding Cauchy conditions for $\{x_j\}_{j=1}^\infty$ with respect to the sum and maximum of the elements of $\mathcal{M}$ in this case.

Suppose for the moment that $\mathcal{M}$ is nondegenerate on X. Let us say that X is *sequentially complete* with respect to $\mathcal{M}$ if every Cauchy sequence of elements of X converges to an element of X with respect to the topology determined by $\mathcal{M}$. If $\mathcal{M}$ has only finitely many elements, then this is equivalent to the completeness of X with respect to the sum or maximum of the elements of $\mathcal{M}$.

Let $d(x, y)$ be a semimetric on X again, and let t be a positive real number. Remember that

$$d_t(x, y) = \min(d(x, y), t) \tag{1.9.3}$$

is a semimetric on X, which determines the same topology on X as $d(x, y)$. It is easy to see that a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is a Cauchy sequence with respect to (1.9.3) if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is a metric on X, then (1.9.3) is a metric on X too. In this case, X is complete with respect to (1.9.3) if and only if X is complete with respect to $d(\cdot, \cdot)$.

Let $d_1, d_2, d_3, \ldots$ be a sequence of semimetrics on X, and put

$$d_j'(x, y) = \min(d_j(x, y), 1/j) \tag{1.9.4}$$

for every $x, y \in X$ and $j \geq 1$. We have seen that

$$d(x, y) = \max_{j \geq 1} d_j'(x, y) \tag{1.9.5}$$

defines a semimetric on X, and that the topology determined on X by (1.9.5) is the same as the topology determined by the collection of semimetrics d_j, $j \geq 1$. Let $r > 0$ be given, and let $[1/r]$ be the largest nonnegative integer less than or equal to $1/r$, as usual. Observe that $x, y \in X$ satisfy

$$d(x, y) < r \tag{1.9.6}$$

if and only if

$$d_j'(x, y) < r \tag{1.9.7}$$

for every $j \geq 1$. Of course, (1.9.7) holds automatically when $1/j < r$, which is to say that $j > 1/r$. In particular, (1.9.7) holds automatically for every $j \geq 1$

when $r > 1$. Otherwise, if $r \le 1$, and $1 \le j \le [1/r]$, then $1/j \le r$, and (1.9.7) holds if and only if

$$d_j(x, y) < r. \tag{1.9.8}$$

Let $\{x_k\}_{k=1}^\infty$ be a sequence of elements of X. If $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence with respect to (1.9.5), then $\{x_k\}_{k=1}^\infty$ is clearly a Cauchy sequence with respect to d'_j for every $j \ge 1$. Equivalently, this means that $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence with respect to d_j for every $j \ge 1$, as before. Conversely, if $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence in X with respect to d_j for every $j \ge 1$, then one can check that $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence with respect to (1.9.5), using the remarks in the preceding paragraph.

If the collection of semimetrics d_j, $j \ge 1$, is nondegenerate on X, then (1.9.5) is a metric on X. In this case, it follows that X is complete with respect to (1.9.5) if and only if X is sequentially complete with respect to the collection of semimetrics d_j, $j \ge 1$.

1.10 Cauchy sequences and Cartesian products

Let I be a nonempty set, let X_j be a set for each $j \in I$, and let $X = \prod_{j \in I} X_j$ be the corresponding Cartesian product. If $x \in X$ and $l \in I$, then $p_l(x) = x_l$ denotes the lth coordinate of x in X_l, as before. Let $\mathcal{M}_l$ be a nonempty collection of semimetrics on X_l for each $l \in I$, and put

$$\widehat{\mathcal{M}_l} = \{\widehat{d_l} : d_l \in \mathcal{M}_l\} \tag{1.10.1}$$

for every $l \in I$, where $\widehat{d_l}$ is the semimetric on X associated to a semimetric d_l on X_l as in (1.8.4). Thus $\mathcal{M} = \bigcup_{l \in I} \widehat{\mathcal{M}_l}$ is a nonempty collection of semimetrics on X, and we have seen that the topology determined on X by $\mathcal{M}$ is the same as the product topology, using the topology determined on X_j by $\mathcal{M}_j$ for each $j \in I$.

Let $\{x(k)\}_{k=1}^\infty$ be a sequence of elements of X, so that $p_l(x(k)) = x_l(k)$ is the lth coordinate of $x(k)$ in X_l for every $k \ge 1$ and $l \in I$. Observe that $\{x(k)\}_{k=1}^\infty$ is a Cauchy sequence in X with respect to $\mathcal{M}$ if and only if $\{x_l(k)\}_{k=1}^\infty$ is a Cauchy sequence in X_l with respect to $\mathcal{M}_l$ for every $l \in I$. Similarly, $\{x(k)\}_{k=1}^\infty$ converges to an element x of X with respect to the topology determined by $\mathcal{M}$ if and only if $\{x_l(k)\}_{k=1}^\infty$ converges to x_l with respect to the topology determined on X_l by $\mathcal{M}_l$ for every $l \in I$.

Suppose that $\mathcal{M}_l$ is nondegenerate on X_l for every $l \in I$, which implies that $\mathcal{M}$ is nondegenerate on X. If X_l is sequentially complete with respect to $\mathcal{M}_l$ for every $l \in I$, then it follows that X is sequentially complete with respect to $\mathcal{M}$.

Suppose for the rest of the section that $\mathcal{M}_l$ consists of a single metric d_l on X_l for each $l \in I$. In particular, this implies that $\mathcal{M}$ is nondegenerate on X. Suppose for the moment that I has only finitely many elements. In this case, the sum and maximum of the corresponding semimetrics $\widehat{d_l}$ on X, $l \in I$, define metrics on X, and the topologies determined on X by these metrics are the same as the topology determined by $\mathcal{M}$. As in the previous section, the sum and maximum of $\widehat{d}_l$, $l \in I$, also determine the same Cauchy sequences in X as $\mathcal{M}$. Thus the sequential completeness of X with respect to $\mathcal{M}$ is equivalent to the completeness of X with respect to the sum or maximum of $\widehat{d}_l$, $l \in I$, as before. If X_l is complete with respect to d_l for each $l \in I$, then it follows that X is complete with respect to the sum and maximum of d_l, $l \in I$.

Suppose now that $I = \mathbf{Z}_+$, so that $\mathcal{M}$ consists of the sequence of semimetrics $\widehat{d}_l$, $l \in \mathbf{Z}_+$. Put

$$\widehat{d}_l'(x, y) = \min(\widehat{d}_l(x, y), 1/l) \tag{1.10.2}$$

for every $x, y \in X$ and $l \geq 1$, as in (1.9.4). Similarly, put

$$d(x, y) = \max_{l \geq 1} \widehat{d}_l'(x, y) \tag{1.10.3}$$

for every $x, y \in X$, as in (1.9.5). Note that the collection of semimetrics (1.10.2) is nondegenerate on X, because $\mathcal{M}$ is nondegenerate on X, and hence (1.10.3) is a metric on X. Remember that the topology determined on X by (1.10.3) is the same as the topology determined by $\mathcal{M}$. A sequence of elements of X is a Cauchy sequence in X with respect to (1.10.3) if and only if it is a Cauchy sequence with respect to $\mathcal{M}$, as in the previous section. Thus X is complete with respect to (1.10.3) if and only if X is sequentially complete with respect to $\mathcal{M}$, as before. If X_l is complete with respect to d_l for each $l \in I$, then it follows that X is complete with respect to (1.10.3).

Chapter 2

Boundedness and supremum semimetrics

2.1 Bounded sets and mappings

Let Y be a set, and let d_Y be a semimetric on Y. A subset E of Y is said to be *bounded* with respect to d_Y if the set of nonnegative real numbers of the form $d_Y(y,z)$ with $y,z \in E$ has an upper bound in $\mathbf{R}$. If y_0 is any element of Y, then this implies that E is contained in a ball in Y centered at y_0 with some finite radius with respect to d_Y. Conversely, if E is contained in a ball in Y of finite radius with respect to d_Y, then it is easy to see that E is bounded, using the triangle inequality. If E is a compact subset of Y with respect to the topology determined by d_Y, and if y_0 is any element of Y, then E is contained in an open ball in Y centered at y_0 with respect to d_Y, so that E is bounded with respect to d_Y.

Let X be another set, and let A be a nonempty subset of X. A mapping f from X into Y is said to be *bounded* on A if $f(A)$ is a bounded subset of Y with respect to d_Y. Let $\mathcal{B}_A(X,Y)$ be the space of all mappings from X into Y that are bounded on A, and let $f,g \in \mathcal{B}_A(X,Y)$ be given. If y_0 is any element of Y, then

$$d_Y(f(x),g(x)) \le d_Y(f(x),y_0) + d_Y(y_0,g(x)) \tag{2.1.1}$$

for every $x \in A$, by the triangle inequality. Note that the right side of (2.1.1) has an upper bound in $\mathbf{R}$, by hypothesis. Put

$$\theta_A(f,g) = \sup_{x \in A} d_Y(f(x),g(x)), \tag{2.1.2}$$

which is defined as a nonnegative real number. This is equal to 0 when $f = g$, and otherwise (2.1.2) is symmetric in f and g, by the corresponding properties of d_Y on Y. If h is another element of $\mathcal{B}_A(X,Y)$, then

$$\begin{aligned} d_Y(f(x),h(x)) &\le d_Y(f(x),g(x)) + d_Y(g(x),h(x)) \\ &\le \theta_A(f,g) + \theta_A(g,h) \end{aligned} \tag{2.1.3}$$

for every $x \in A$, using the triangle inequality for d_Y in the first step. This implies that

$$\theta_A(f,h) \le \theta_A(f,g) + \theta_A(g,h), \tag{2.1.4}$$

by taking the supremum of the left side of (2.1.3) over $x \in A$. This shows that θ_A defines a semimetric on $\mathcal{B}_A(X,Y)$, which is the *supremum semimetric* associated to A.

A mapping f from X into Y is said to be *bounded* if it is bounded on X, and we let $\mathcal{B}(X,Y)$ be the space of all bounded mappings from X into Y. This corresponds to taking $A = X$ in the preceding paragraph, in which case we may also drop the subscript A from the left side of (2.1.2). If $X \ne \emptyset$ and d_Y is a metric on Y, then $\theta = \theta_X$ defines a metric on $\mathcal{B}(X,Y)$, which is the *supremum metric* associated to d_Y. Of course, every mapping from X into Y is bounded when Y is bounded with respect to d_Y.

Let X and Y be topological spaces, and let $C(X,Y)$ be the space of all continuous mappings from X into Y, as usual. Also let A be a nonempty compact subset of A. If $f \in C(X,Y)$, then it is well known that $f(A)$ is a compact subset of Y. Let d_Y be a semimetric on Y that is compatible with the given topology on Y, as in Section 1.1. If E is a compact subset of Y with respect to the given topology on Y, then it follows that E is compact with respect to the topology determined on Y by d_Y, and hence that E is bounded in Y with respect to d_Y. If $f \in C(X,Y)$, then it follows that $f(A)$ is bounded in Y with respect to d_Y, so that f is bounded on A. Thus

$$C(X,Y) \subseteq \mathcal{B}_A(X,Y) \tag{2.1.5}$$

under these conditions.

Let X be a topological space again, and let Y be a set with a semimetric d_Y, so that Y may be considered as a topological space with respect to the topology determined by d_Y. Consider the space

$$C_b(X,Y) = C(X,Y) \cap \mathcal{B}(X,Y) \tag{2.1.6}$$

of continuous mappings from X into Y that are also bounded on X with respect to d_Y on Y. Note that

$$C_b(X,Y) = C(X,Y) \tag{2.1.7}$$

when X is compact, and when Y is bounded with respect to d_Y, as in the previous paragraphs.

2.2 Uniform continuity

Let X, Y be sets with semimetrics d_X, d_Y, respectively, and let A be a subset of X. A mapping f from X into Y is said to be *uniformly continuous along A* if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d_Y(f(x), f(w)) < \epsilon \tag{2.2.1}$$

for every $x \in A$ and $w \in X$ with $d_X(x, w) < \delta$. If A consists of a single point, then this is the same as saying that f is continuous at that point. If A has only finitely many elements, then this condition holds exactly when f is continuous at every element of A. If $A = X$, then we simply say that f is uniformly continuous on X. If f is uniformly continuous along a subset A of X, then the restriction of f to A is uniformly continuous as a mapping from A into Y, with respect to the restriction of d_X to A. Of course, if f is uniformly continuous along $A \subseteq X$, then f is continuous at every element of A, as a mapping from X into Y.

If $f : X \to Y$ is continuous at every point in a compact subset A of X, then f is uniformly continuous along A, by standard arguments. To see this, let $\epsilon > 0$ be given. If $a \in A$, then there is a $\delta(a) > 0$ such that

$$d_Y(f(a), f(w)) < \epsilon/2 \tag{2.2.2}$$

for every $w \in X$ such that $d_X(a, w) < \delta(a)$, because f is continuous at a, by hypothesis. Let $B_X(a)$ be the open ball in X centered at $a \in A$ with radius $\delta(a)/2$ with respect to d_X. The collection of these open balls corresponding to elements a of A forms an open covering of A in X. Because A is compact in X, by hypothesis, there are finitely many elements $a_1, \ldots, a_n$ of A such that

$$A \subseteq \bigcup_{j=1}^{n} B_X(a_j). \tag{2.2.3}$$

Put

$$\delta = \min_{1 \le j \le n} (\delta(a_j)/2) > 0, \tag{2.2.4}$$

and let $x \in A$ and $w \in X$ be given, with $d_X(x, w) < \delta$. Because $x \in A$, there is a $j = 1, \ldots, n$ such that $x \in B_X(a_j)$, as in (2.2.3). Thus $d_X(a_j, x) < \delta(a_j)/2$, and hence

$$\begin{aligned} d_X(a_j, w) \le d_X(a_j, x) + d_X(x, w) &< \delta(a_j)/2 + \delta \\ &\le \delta(a_j)/2 + \delta(a_j)/2 = \delta(a_j). \end{aligned} \tag{2.2.5}$$

This permits us to apply the continuity condition (2.2.2) with $a = a_j$ to x and to w. It follows that

$$d_Y(f(x), f(w)) \le d_Y(f(x), f(a_j)) + d_Y(f(a_j), f(w)) < \epsilon/2 + \epsilon/2 = \epsilon, \tag{2.2.6}$$

as desired.

Let Z be another set with a semimetric d_Z. Suppose that a mapping f from X into Y is uniformly continuous along a subset A of X, and that a mapping g from Y into Z is uniformly continuous along a subset B of Y. If $f(A) \subseteq B$, then it is easy to see that the composition $g \circ f$ is uniformly continuous along A as a mapping from X into Z.

Let $UC(X, Y)$ be the space of all uniformly continuous mappings from X into Y, and let

$$UC_b(X, Y) = UC(X, Y) \cap \mathcal{B}(X, Y) \tag{2.2.7}$$

be the space of all bounded uniformly continuous mappings from X into Y. If X is compact, then every continuous mapping from X into Y is bounded and uniformly continuous.

2.3 Uniform homeomorphisms

Let X, Y be sets with semimetrics d_X, d_Y, respectively, again. Let us say that a one-to-one mapping f from X onto Y is a *uniform homeomorphism* if f is uniformly continuous as a mapping from X onto Y, and the inverse mapping f^{-1} is uniformly continuous as a mapping from Y onto X. Thus a uniform homeomorphism from X onto Y is automatically a homeomorphism from X onto Y as topological spaces, with respect to the topologies determined by d_X and d_Y, respectively. If X and Y are compact, and f is a homeomorphism from X onto Y, then it follows that f is a uniform homeomorphism.

Let us now take $X = Y$, and let d be a semimetric on X. Let $H(X)$ be the collection of all homeomorphisms from X onto itself, with respect to the topology determined on X by d. This is a group with respect to composition of mappings, and with the identity mapping on X as the identity element of $H(X)$. Similarly, let $UH(X)$ be the collection of all uniform homeomorphisms from X onto itself. This is a subgroup of $H(X)$, because the identity mapping on X is a uniform homeomorphism, and because compositions of uniformly continuous mappings are uniformly continuous, as mentioned in the previous section.

Let us suppose from now on in this section that X is nonempty and bounded with respect to d, and let

$$\theta(f, g) = \sup_{x \in X} d(f(x), g(x)) \tag{2.3.1}$$

be the corresponding supremum semimetric for mappings f, g of X into itself, as before. In particular, this defines a semimetric on $H(X)$. It is easy to see that θ is invariant under right translations on $H(X)$. More precisely, if $f, g, h \in H(X)$, then

$$\begin{aligned} \theta(f \circ h, g \circ h) &= \sup_{x \in X} d(f(h(x)), g(h(x))) \\ &= \sup_{y \in X} d(f(y), g(y)) = \theta(f, g), \end{aligned} \tag{2.3.2}$$

because h maps X onto itself. In fact, this works for all mappings f, g from X into itself, and all mappings h from X onto itself.

However, θ is not normally invariant under left translations on $H(X)$. As a partial substitute for this, suppose for the moment that h is a uniformly continuous mapping from X into itself. Let $\epsilon > 0$ be given, and let δ be a positive real number such that

$$d(h(x), h(y)) \le \epsilon \tag{2.3.3}$$

for every $x, y \in X$ with $d(x, y) \leq \delta$. Note that mappings f, g from X into itself satisfy

$$\theta(f, g) \leq \delta \tag{2.3.4}$$

exactly when

$$d(f(x), g(x)) \leq \delta \tag{2.3.5}$$

for every $x \in X$. In this case, we have that

$$d(h(f(x)), h(g(x))) \leq \epsilon \tag{2.3.6}$$

for every $x \in X$, so that

$$\theta(h \circ f, h \circ g) \leq \epsilon. \tag{2.3.7}$$

Let us now consider the restriction of the supremum semimetric θ to $UH(X)$. The remarks in the preceding paragraph imply that left translations on $UH(X)$ are uniformly continuous with respect to θ. Using this and the fact that θ is invariant under right translations on $UH(X)$, we get that $UH(X)$ is a topological group with respect to the topology determined by θ, as in Section 1.7.

2.4 Uniform convergence

Let X, Y be nonempty sets, and let d_Y be a semimetric on Y. Also let $\{f_j\}_{j=1}^\infty$ be a sequence of mappings from X into Y, and let f be another mapping from X into Y. As usual, $\{f_j\}_{j=1}^\infty$ is said to converge *pointwise* to f on X if for every $x \in X$, $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ as a sequence of elements of Y with respect to the topology determined on Y by d_Y. If for every $\epsilon > 0$ there is a positive integer L such that

$$d_Y(f_j(x), f(x)) < \epsilon \tag{2.4.1}$$

for every $x \in X$ and $j \geq L$, then $\{f_j\}_{j=1}^\infty$ is said to converge *uniformly* to f on X. Of course, uniform convergence implies pointwise convergence.

Suppose for the moment that $\{f_j\}_{j=1}^\infty$ is a sequence of bounded mappings from X into Y, and that f is a bounded mapping from X into Y. In this case, $\{f_j\}_{j=1}^\infty$ converges to f uniformly on X if and only if $\{f_j\}_{j=1}^\infty$ converges to f with respect to the supremum semimetric θ on the space $\mathcal{B}(X, Y)$ of bounded mappings from X into Y that corresponds to d_Y as in Section 2.1.

Let X be a topological space, and let x be an element of X. Also let $\{f_j\}_{j=1}^\infty$ be a sequence of mappings from X into Y that converges uniformly to a mapping f from X into Y. If f_j is continuous at x for each j, then it follows that f is continuous at x too, by a standard argument. If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of X that converges to x, then one can check that

$$\{f_j(x_j)\}_{j=1}^\infty \tag{2.4.2}$$

converges to $f(x)$ in Y in this situation. This uses the fact that $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y, because f is continuous at x.

Let $C_b(X, Y)$ be the space of bounded continuous mappings from X into Y, as in Section 2.1. This is a closed set in $\mathcal{B}(X, Y)$ with respect to the topology determined by the supremum semimetric θ. More precisely, if a sequence $\{f_j\}_{j=1}^\infty$ of elements of $C_b(X, Y)$ converges to an element f of $\mathcal{B}(X, Y)$ with respect to θ, then f is continuous, as in the preceding paragraph.

Suppose now that X is equipped with a semimetric d_X, and let A be a subset of X. Let $\{f_j\}_{j=1}^\infty$ be a sequence of mappings from X into Y that converges uniformly to a mapping f from X into Y again. If f_j is uniformly continuous along A for every j, then one can verify that f is uniformly continuous along A too. It follows that the space $UC_b(X, Y)$ of bounded uniformly continuous mappings from X into Y is a closed set in $\mathcal{B}(X, Y)$ with respect to the topology determined by the supremum semimetric θ.

Let $\{f_j\}_{j=1}^\infty$ be a sequence of homeomorphisms from X onto Y. Suppose that $\{f_j\}_{j=1}^\infty$ converges uniformly to a mapping f from X into Y, and that $\{f_j^{-1}\}_{j=1}^\infty$ converges uniformly to a mapping g from Y into X. Note that f and g are continuous, as before. Of course, we would like to have that

$$g(f(x)) = x \tag{2.4.3}$$

for every $x \in X$, so that $g \circ f$ is the identity mapping on X. Similarly, we would like to have that

$$f(g(y)) = y \tag{2.4.4}$$

for every $y \in Y$, so that $f \circ g$ is the identity mapping on Y.

To get (2.4.3), we would like to use the fact that $f_j^{-1}(f_j(x)) = x$ for every $x \in X$ and $j \in \mathbf{Z}_+$, and take the limit as $j \to \infty$. The earlier remarks about (2.4.2) imply that $\{f_j^{-1}(f_j(x))\}_{j=1}^\infty$ converges to $g(f(x))$ in X for every $x \in X$ under these conditions. If d_X is a metric on X, then it follows that (2.4.3) holds for every $x \in X$, because the limit of a convergent sequence in X is unique. Similarly, if d_Y is a metric on Y, then (2.4.4) holds for every $y \in Y$. This means that g is the inverse of f when d_X, d_Y are metrics on X, Y, respectively, so that f is a homeomorphism from X onto Y.

2.5 Completeness

Let (X, d) be a metric space, and let Z be a subset of X. Thus Z may be considered as a metric space too, using the restriction of $d(\cdot, \cdot)$ to Z. Note that a sequence of elements of Z is a Cauchy sequence in Z if and only if it is a Cauchy sequence in X. If X is complete with respect to d, then every Cauchy sequence in Z converges to an element of X. If Z is also a closed set in X with respect to d, then it follows that Z is complete with respect to the restriction of d to Z.

Now let X be a nonempty set, and let (Y, d_Y) be a metric space. Also let $\mathcal{B}(X, Y)$ be the space of bounded mappings from X into Y, as in Section 2.1, and let θ be the corresponding supremum metric on $\mathcal{B}(X, Y)$. If Y is complete with respect to d_Y, then it is well known that $\mathcal{B}(X, Y)$ is complete with respect to θ. To see this, let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence of bounded mappings from X into Y with respect to θ. This means that for each $\epsilon > 0$ there is a positive integer $L(\epsilon)$ such that

$$\theta(f_j, f_l) < \epsilon \tag{2.5.1}$$

for every $j, l \geq L(\epsilon)$. It follows that

$$d_Y(f_j(x), f_l(x)) < \epsilon \tag{2.5.2}$$

for every $x \in X$ and $j, l \geq L(\epsilon)$, so that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in Y for every $x \in X$. Because Y is complete, $\{f_j(x)\}_{j=1}^\infty$ converges to an element $f(x)$ of Y for every $x \in X$, which defines a mapping f from X into Y. Using (2.5.2), we get that

$$d_Y(f(x), f_l(x)) \leq \epsilon \tag{2.5.3}$$

for every $x \in X$ and $l \geq L(\epsilon)$, so that $\{f_l\}_{l=1}^\infty$ converges to f uniformly on X. In particular, this implies that f is a bounded mapping from X into Y, because f_l is bounded for each l, by hypothesis. Note that (2.5.3) is the same as saying that

$$\theta(f, f_l) \leq \epsilon \tag{2.5.4}$$

for every $l \geq L(\epsilon)$. Thus $f \in \mathcal{B}(X, Y)$, and $\{f_l\}_{l=1}^\infty$ converges to f with respect to the supremum metric θ, as desired.

If X is a topological space, then the space $C_b(X, Y)$ of bounded continuous mappings from X into Y is a closed set in $\mathcal{B}(X, Y)$ with respect to the supremum metric, as in the previous section. This implies that $C_b(X, Y)$ is complete with respect to the supremum metric when Y is complete with respect to d_Y. Similarly, if X is equipped with a semimetric, then the space $UC_b(X, Y)$ of bounded uniformly continuous mappings from X into Y is a closed set in $\mathcal{B}(X, Y)$ with respect to the supremum metric, as before. If Y is complete with respect to d_Y, then it follows that $UC_b(X, Y)$ is complete with respect to the supremum metric.

Let (X, d) be a nonempty metric space, and suppose that X is bounded with respect to d. Also let $H(X)$ be the group of homeomorphisms from X onto itself, as in Section 2.3, and let θ be the supremum metric on the space of mappings from X into itself corresponding to d, as in (2.3.1). Note that

$$\theta(f^{-1}, g^{-1}) \tag{2.5.5}$$

defines a metric on $H(X)$ too, as in (1.4.5). Thus

$$\max(\theta(f, g),\, \theta(f^{-1}, g^{-1})) \tag{2.5.6}$$

and

$$\theta(f, g) + \theta(f^{-1}, g^{-1}) \tag{2.5.7}$$

define metrics on $H(X)$ as well, as in Section 1.5. Of course, (2.5.6) and (2.5.7) determine the same topologies on $H(X)$, and the same collections of Cauchy sequences.

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of elements of $H(X)$ which is a Cauchy sequence with respect to (2.5.6) or (2.5.7). This is the same as saying that $\{f_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to θ and (2.5.5), which means that $\{f_j\}_{j=1}^{\infty}$ and $\{f_j^{-1}\}_{j=1}^{\infty}$ are Cauchy sequences with respect to θ. Suppose that X is complete with respect to d. In this case, it follows that $\{f_j\}_{j=1}^{\infty}$ converges uniformly to a continuous mapping f from X into itself, and that $\{f_j^{-1}\}_{j=1}^{\infty}$ converges uniformly to a continuous mapping g from X into itself. Under these conditions, g is the inverse of f, as in the previous section. In particular, $f \in H(X)$, and $\{f_j\}_{j=1}^{\infty}$ converges to f with respect to (2.5.6) and (2.5.7). Thus $H(X)$ is complete with respect to (2.5.6) and (2.5.7) when X is complete with respect to d.

Let $UH(X)$ be the subgroup of $H(X)$ consisting of uniform homeomorphisms from X onto itself, as in Section 2.3 again. It is easy to see that $UH(X)$ is a closed set in $H(X)$ with respect to (2.5.6) or (2.5.7), using analogous statements for uniform continuity and the supremum metric. Hence $UH(X)$ is also complete with respect to (2.5.6) or (2.5.7) when X is complete with respect to d.

The restriction of (2.5.5) to $UH(X)$ determines the same topology as the restriction of θ to $UH(X)$, because $UH(X)$ is a topological group with respect to the topology determined by the restriction of θ to $UH(X)$, as in Section 2.3. This implies that the restrictions of (2.5.6) and (2.5.7) to $UH(X)$ determine the same topology on $UH(X)$ as the restriction of θ to $UH(X)$.

2.6 Homeomorphisms on $[0, 1]$

Let $H([0, 1])$ be the group of homeomorphisms from the closed unit interval $[0, 1]$ in the real line $\mathbf{R}$ onto itself, as in Section 2.3. This example is mentioned in (c) on p212 of [14], in connection with completeness issues. Remember that 0, 1 are the only elements x of $[0, 1]$ such that $[0, 1] \setminus \{x\}$ is not connected. This implies that every homeomorphism from $[0, 1]$ onto itself maps $\{0, 1\}$ onto itself. The mapping from $f \in H([0, 1])$ to the restriction of f to $\{0, 1\}$ defines a group homomorphism from $H([0, 1])$ into the group of permutations on $\{0, 1\}$. It is easy to see that this homomorphism is surjective. Let $H_+([0, 1])$ be the collection of $f \in H([0, 1])$ such that $f(0) = 0$ and $f(1) = 1$, which is the kernel of the homomorphism just mentioned. It is well known that the elements of $H_+([0, 1])$ are strictly increasing on $[0, 1]$, because of the intermediate value theorem. In fact, $H_+([0, 1])$ is the same as the collection of continuous mappings from $[0, 1]$ onto itself that are strictly increasing. This uses the well-known fact that a one-to-one continuous mapping from a compact topological space onto a Hausdorff topological space is a homeomorphism.

Let $C([0, 1], \mathbf{R})$ be the space of all real-valued continuous functions on $[0, 1]$, as in Section 2.1. Of course, the elements of $C([0, 1], \mathbf{R})$ are bounded on $[0, 1]$, because $[0, 1]$ is compact. The supremum metric can be defined on $C([0, 1], \mathbf{R})$ using the standard metric on $\mathbf{R}$, as in Section 2.1 again. Note that $C([0, 1], \mathbf{R})$ is complete with respect to the supremum metric, as in Section 2.5, because $\mathbf{R}$ is complete with respect to the standard metric. However, it is easy to see that $H([0, 1])$ is not a closed set in $C([0, 1], \mathbf{R})$.

Let $C_+([0,1])$ be the space of $f \in C([0,1], \mathbf{R})$ such that $f(0) = 0$, $f(1) = 1$, and f increases monotonically on $[0,1]$. Thus f maps $[0,1]$ into itself, by monotonicity, and in fact f maps $[0,1]$ onto itself, by the intermediate value theorem. Observe that

$$H_+([0,1]) \subseteq C_+([0,1]), \tag{2.6.1}$$

and that $C_+([0,1])$ is a closed set in $C([0,1], \mathbf{R})$. Let us check that $C_+([0,1])$ is the closure of $H_+([0,1])$ in $C([0,1], \mathbf{R})$. Let $f \in C_+([0,1])$ and $\epsilon > 0$ be given, and put

$$f_\epsilon(x) = (1+\epsilon)^{-1} \, (f(x) + \epsilon \, x) \tag{2.6.2}$$

for every $x \in [0,1]$. Clearly f_ϵ is continuous on $[0,1]$, because f is continuous. One can check that f_ϵ is strictly increasing on $[0,1]$, because f increases monotonically on $[0,1]$. By construction, $f_\epsilon(0) = 0$ and $f_\epsilon(1) = 1$, so that $f \in C_+([0,1])$. This implies that f_ϵ maps $[0,1]$ onto itself, as before. It follows that $f_\epsilon \in H_+([0,1])$, because f is strictly increasing on $[0,1]$. It is easy to see that f_ϵ converges to f uniformly on $[0,1]$ as $\epsilon \to 0$, so that f is in the closure of $H([0,1])$ in $C([0,1], \mathbf{R})$, as desired.

It is well known that $C([0,1], \mathbf{R})$ is a vector space over the real numbers with respect to pointwise addition and scalar multiplication of functions. As usual, a subset E of $C([0,1], \mathbf{R})$ is said to be *convex* if for every $f, g \in E$ and $t \in [0,1]$ we have that

$$t \, f + (1-t) \, g \tag{2.6.3}$$

is an element of E. In particular, this implies that E is connected with respect to the supremum metric. It is easy to see that $C_+([0,1])$ is convex as a subset of $C([0,1], \mathbf{R})$. Similarly, $H_+([0,1])$ is a convex subset of $C([0,1], \mathbf{R})$, because $H_+([0,1])$ is the same as the set of $f \in C_+([0,1])$ such that f is strictly increasing on $[0,1]$. This uses the fact that the elements of $C_+([0,1])$ map $[0,1]$ onto itself, as in the previous paragraph. More precisely, if $f \in C_+([0,1])$, $g \in H_+([0,1])$, and $t \in [0,1)$, then (2.6.3) is in $H_+([0,1])$.

2.7 Isometric mappings

Let X be a set, and let d be a semimetric on X. A mapping f from X into itself is said to be an *isometry* with respect to d if

$$d(f(x), f(y)) = d(x,y) \tag{2.7.1}$$

for every $x, y \in X$. In particular, this implies that f is uniformly continuous with respect to d. Note that the composition of two isometries from X into itself is an isometry too. If d is a metric on X, then (2.7.1) implies that f is one-to-one on X.

Let $IH(X)$ be the set of one-to-one isometric mappings f from X onto itself. If $f \in IH(X)$, then $f^{-1} \in IH(X)$ too, so that f is a uniform homeomorphism from X onto itself. More precisely, $IH(X)$ is a subgroup of the group $UH(X)$ of uniform homeomorphisms from X onto itself, defined in Section 2.3.

Let A be a nonempty subset of X, and let $\mathcal{B}_A(X) = \mathcal{B}_A(X, X)$ be the space of mappings from X into itself that are bounded on A, as in Section 2.1. Also let θ_A be the corresponding supremum semimetric on $\mathcal{B}_A(X)$, as in (2.1.2). If $f, g \in \mathcal{B}_A(X)$ and h is an isometry from X into itself, then

$$\theta_A(h \circ f, h \circ g) = \sup_{x \in A} d(h(f(x)), h(g(x))) = \sup_{x \in A} d(f(x), g(x)) = \theta_A(f, g). \tag{2.7.2}$$

Let us suppose from now on in this section that X is nonempty and bounded with respect to d, and let $\theta(f, g)$ be the corresponding supremum semimetric for mappings f, g from X into itself, as in (2.3.1). If f, g are mappings from X into itself and h is an isometry from X into itself, then

$$\theta(h \circ f, h \circ g) = \theta(f, g), \tag{2.7.3}$$

by (2.7.2) with $A = X$. In particular, this means that the restriction of $\theta(f, g)$ to the group $H(X)$ of all homeomorphisms from X onto itself is invariant under left translations by elements of $IH(X)$. Remember that $\theta(f, g)$ is invariant under right translations on $H(X)$, as in (2.3.2), so that the restriction of $\theta(f, g)$ to $IH(X)$ is invariant under both left and right translations. It follows that

$$\theta(f^{-1}, g^{-1}) = \theta(f, g) \tag{2.7.4}$$

for all $f, g \in IH(X)$, as in Section 1.4.

Let us also suppose from now on in this section that d is a metric on X, and that X is complete with respect to d. We would like to check that $IH(X)$ is complete with respect to the supremum metric θ under these conditions. The space of all mappings from X into itself is complete with respect to the supremum metric in this situation, as in Section 2.5, and so it suffices to verify that $IH(X)$ is a closed set in this space. Equivalently, if $\{f_j\}_{j=1}^\infty$ is a sequence of elements of $IH(X)$ that converges to a mapping f from X into itself uniformly on X, then we would like to show that $f \in IH(X)$ too. It is easy to see that f satisfies the isometric property (2.7.1), because the f_j's are isometries that converge to f pointwise on X. In particular, this implies that f is one-to-one on X, because d is a metric on X, as before. It remains to show that f maps X onto itself. Observe first that $f(X)$ is dense in X, because $f_j(X) = X$ for each $j \geq 1$, and $\{f_j\}_{j=1}^\infty$ converges to f uniformly on X. We also have that $f(X)$ is complete as a metric space with respect to the restriction of d to $f(X)$, because X is complete, and f is an isometry. This implies that $f(X)$ is a closed set in X, by a standard argument. It follows that $f(X) = X$, as desired, because $f(X)$ is both closed and dense in X.

Alternatively, if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $IH(X)$ with respect to the supremum metric, then $\{f_j^{-1}\}_{j=1}^\infty$ is a Cauchy sequence with respect to the supremum metric too, by (2.7.4). Hence these sequences converge uniformly to isometric mappings f, g from X into itself, respectively, because of completeness. We also have that f and g are inverses of each other, as in Section 2.4. In particular, this implies that f maps X onto itself, so that f is an element of $IH(X)$, as desired. This is analogous to the completeness properties of $H(X)$ mentioned in Section 2.5.

If X is compact with respect to the topology determined by d, then $IH(X)$ is compact with respect to the topology determined by the supremum metric. This can be obtained from the usual Arzela–Ascoli type of arguments. Of course, X is complete when X is compact, so that $IH(X)$ is complete, as in the preceding paragraphs. Thus it is enough to show that $IH(X)$ is totally bounded with respect to the supremum metric. This can be verified using the fact that X is totally bounded with respect to d, because X is compact, and the equicontinuity of the elements of $IH(X)$.

2.8 Subadditive functions on $[0, \infty)$

A real-valued function α on $\mathbf{R}$ is an *additive function* if

$$\alpha(x + y) = \alpha(x) + \alpha(y) \tag{2.8.1}$$

for every $x, y \in \mathbf{R}$. This is the same as saying that α is a group homomorphism from $\mathbf{R}$ into itself, where $\mathbf{R}$ is considered as a commutative group with respect to addition. In this case, $\alpha(0) = 0$, and

$$\alpha(t\,x) = t\,\alpha(x) \tag{2.8.2}$$

for every $t \in \mathbf{Z}_+$ and $x \in \mathbf{R}$. More precisely, (2.8.2) also holds when $t = -1$, and hence when $t \in \mathbf{Z}$. Similarly, one can check that (2.8.2) holds when t is a rational number. If we consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of rational numbers, then α may be considered as a linear mapping from $\mathbf{R}$ into itself. If $a \in \mathbf{R}$, then

$$\alpha_a(x) = a\,x \tag{2.8.3}$$

defines an additive mapping from $\mathbf{R}$ into itself. If α is an additive mapping from $\mathbf{R}$ into itself that is also continuous with respect to the standard topology on $\mathbf{R}$, then α is of the form (2.8.3), with $a = \alpha(1)$.

Similarly, a real-valued function α on $\mathbf{R}$ is said to be *subadditive* if

$$\alpha(x + y) \le \alpha(x) + \alpha(y) \tag{2.8.4}$$

for every $x, y \in \mathbf{R}$. Note that α is additive on $\mathbf{R}$ if and only if α and $-\alpha$ are subadditive on $\mathbf{R}$. If $\alpha_1, \ldots, \alpha_n$ are finitely many subadditive functions on $\mathbf{R}$, then it is easy to see that

$$\alpha(x) = \max(\alpha_1(x), \ldots, \alpha_n(x)) \tag{2.8.5}$$

is subadditive on $\mathbf{R}$ as well. If α is any subadditive function on $\mathbf{R}$, then we can take $x = y = 0$ in (2.8.4), to get that

$$\alpha(0) \ge 0. \tag{2.8.6}$$

We also have that

$$\alpha(n\,x) \le n\,\alpha(x) \tag{2.8.7}$$

for every $n \in \mathbf{Z}_+$ and $x \in \mathbf{R}$.

Now let α be a real-valued function on the set $[0, \infty)$ of nonnegative real numbers. Let us say that α is *additive* if (2.8.1) holds for every $x, y \in [0, \infty)$. If α is an additive function on $\mathbf{R}$, then the restriction of α to $[0, \infty)$ is an additive function on $[0, \infty)$. Suppose that α is an additive function on $[0, \infty)$, and put

$$\alpha(x) = -\alpha(-x) \tag{2.8.8}$$

for every $x \in \mathbf{R}$ with $x < 0$. One can check that this extension of α to $\mathbf{R}$ is additive on $\mathbf{R}$.

A real-valued function α on $[0, \infty)$ is said to be *subadditive* if (2.8.4) holds for every $x, y \in [0, \infty)$. As before, α is additive on $[0, \infty)$ if and only if α and $-\alpha$ are subadditive on $[0, \infty)$. If α is a subadditive function on $\mathbf{R}$, then the restriction of α to $[0, \infty)$ is a subadditive function on $[0, \infty)$. If $\alpha_1, \ldots, \alpha_n$ are finitely many subadditive functions on $[0, \infty)$, then their maximum (2.8.5) is subadditive on $[0, \infty)$ too. If α is any subadditive function on $[0, \infty)$, then (2.8.6) holds, and (2.8.7) holds for every $n \in \mathbf{Z}_+$ and $x \ge 0$, as before.

Suppose that α is a monotonically increasing subadditive real-valued function on $[0, \infty)$ that satisfies $\alpha(0) = 0$. Note that $\alpha \ge 0$ on $[0, \infty)$, because of monotonicity. If $d(x, y)$ is a semimetric on a set X, then one can verify that

$$d_\alpha(x, y) = \alpha(d(x, y)) \tag{2.8.9}$$

defines a semimetric on X as well. If d is a metric on X, and $\alpha > 0$ on $(0, \infty)$, then (2.8.9) is a metric on X.

If t is a positive real number, then it is easy to see that

$$\alpha_t(r) = \min(r, t) \tag{2.8.10}$$

defines a subadditive function on $[0, \infty)$. Clearly $\alpha_t(0) = 0$, α_t is monotonically increasing on $[0, \infty)$, and $\alpha_t > 0$ on $(0, \infty)$. If we take $\alpha = \alpha_t$, then (2.8.9) is the same as (1.6.1).

Let α be a monotonically increasing subadditive real-valued function on $[0, \infty)$ again. Thus

$$0 \le \alpha(r + t) - \alpha(r) \le \alpha(t) \tag{2.8.11}$$

for every $r, t \ge 0$. If we also have that

$$\lim_{t \to 0+} \alpha(t) = 0, \tag{2.8.12}$$

then it follows that α is uniformly continuous on $[0, \infty)$, with respect to the restriction of the standard Euclidean metric on $\mathbf{R}$ to $[0, \infty)$.

2.9 Uniformly compatible semimetrics

Let X be a set, and let d_1, d_2 be semimetrics on X. Let us say that d_1 is *uniformly compatible* with d_2 on X if the identity mapping on X is uniformly continuous as a mapping from X equipped with d_2 into X equipped with d_1. This means that for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$ with

$$d_2(x, y) < \delta, \tag{2.9.1}$$

we have that

$$d_1(x, y) < \epsilon. \tag{2.9.2}$$

In particular, this implies that d_1 is compatible with the topology determined on X by d_2, as in Section 1.1.

Let us say that d_1, d_2 are *uniformly equivalent* on X if d_1 is uniformly compatible with d_2, and d_2 is uniformly compatible with d_1. This is the same as saying that the identity mapping on X is a uniform homeomorphism as a mapping from X equipped with d_1 onto X equipped with d_2. In this case, the topologies determined on X by d_1 and d_2 are the same.

Let d_3 be another semimetric on X. If d_1 is uniformly compatible with d_2 on X, and d_2 is uniformly compatible with d_3 on X, then d_1 is uniformly compatible with d_3 on X. Similarly, if d_1, d_2 are uniformly equivalent on X, and d_2, d_3 are uniformly equivalent on X, then d_1, d_3 are uniformly equivalent on X.

Let α be a monotonically increasing subadditive real-valued function on $[0, \infty)$ such that $\alpha(0) = 0$. Also let d be a semimetric on X, so that (2.8.9) defines a semimetric d_α on X, as before. If α satisfies (2.8.12) too, then it is easy to see that d_α is uniformly compatible with d on X.

Let d_1, d_2 be semimetrics on X again, and let α be a monotonically increasing nonnegative real-valued function on $[0, \infty)$ such that $\alpha(0) = 0$. Let us say that d_1 is *α-bounded* by d_2 if

$$\alpha(d_1(x, y)) \le d_2(x, y) \tag{2.9.3}$$

for every $x, y \in X$. In this case, if $x, y \in X$ satisfy

$$d_1(x, y) \ge \epsilon \tag{2.9.4}$$

for some $\epsilon > 0$, then we have that

$$d_2(x, y) \ge \alpha(\epsilon). \tag{2.9.5}$$

If we take $\delta = \alpha(\epsilon)$, then it follows that (2.9.1) implies (2.9.2). Hence d_1 is uniformly compatible with d_2 on X when $\alpha > 0$ on $(0, \infty)$.

Suppose that α is also subadditive on $[0, \infty)$, and let d be a semimetric on X. If d_α is as in (2.8.9), then d is automatically α-bounded by d_α. Thus d is uniformly compatible with d_α on X when $\alpha > 0$ on $(0, \infty)$.

Let d_1, d_2 be semimetrics on X such that d_1 is uniformly compatible with d_2. This means that for each $\epsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$

that satisfy (2.9.4), we have that

$$d_2(x, y) \geq \delta. \tag{2.9.6}$$

Put

$$\alpha(\epsilon) = \inf\{d_2(x, y) : x, y \in X,\ d_1(x, y) \geq \epsilon\} \tag{2.9.7}$$

for each $\epsilon > 0$. More precisely, if there are no $x, y \in X$ that satisfy (2.9.4), then (2.9.7) is interpreted as being $+\infty$. The hypothesis that d_1 be uniformly compatible with d_2 on X is the same as saying that

$$\alpha(\epsilon) > 0 \tag{2.9.8}$$

for every $\epsilon > 0$. Let us put $\alpha(0) = 0$, which is the same as (2.9.7) with $\epsilon = 0$ when $X \neq \emptyset$. It is easy to see that α increases monotonically on $[0, \infty)$, because the set whose infimum is taken on the right side of (2.9.7) gets smaller as ϵ increases.

In this situation, (2.9.3) holds by construction for every $x, y \in X$. Thus d_1 is α-bounded by d_2, but where α may take values in the nonnegative extended real numbers. This can be avoided by taking the minimum of α with any fixed positive real number.

2.10 Moduli of uniform continuity

Let X, Y be nonempty sets, and let d_X, d_Y be semimetrics on them, respectively. Also let f be a mapping from X into Y. If r is a positive real number, then we put

$$\beta_f^-(r) = \sup\{d_Y(f(x), f(x')) : x, x' \in X,\ d_X(x, x') < r\} \tag{2.10.1}$$

and

$$\beta_f(r) = \sup\{d_Y(f(x), f(x')) : x, x' \in X,\ d_X(x, x') \leq r\}, \tag{2.10.2}$$

where the suprema are defined as nonnegative extended real numbers. We can also define $\beta_f(r)$ when $r = 0$ as in (2.10.2). Note that the sets whose suprema are being taken are nonempty, because one can take $x' = x$. These sets get larger as r increases, which implies that (2.10.1) and (2.10.2) increase monotonically in r. If $r > 0$, then

$$\beta_f^-(r) \leq \beta_f(r), \tag{2.10.3}$$

and in fact

$$\beta_f^-(r) = \sup\{\beta_f(t) : 0 < t < r\}. \tag{2.10.4}$$

If $0 \leq r < t$, then

$$\beta_f(r) \leq \beta_f^-(t). \tag{2.10.5}$$

It is easy to see that f is bounded as a mapping from X into Y if and only if (2.10.1), (2.10.2) have finite upper bounds. If d_X is a metric on X, then $\beta_f(0) = 0$ automatically.

If f is uniformly continuous as a mapping from X into Y, then (2.10.2) is finite when r is sufficiently small, and

$$\lim_{r \to 0+} \beta_f(r) = 0. \tag{2.10.6}$$

In particular, this implies that $\beta_f(0) = 0$, because (2.10.2) increases monotonically in r. In the other direction, f is uniformly continuous when

$$\lim_{r \to 0+} \beta_f^-(r) = 0, \tag{2.10.7}$$

which implicitly includes the condition that (2.10.1) be finite when r is sufficiently small. Of course, the equivalence of (2.10.6) and (2.10.7) follows directly from (2.10.3) and (2.10.5).

Let β be a nonnegative extended real-valued function on $[0, \infty)$, and suppose that

$$d_Y(f(x), f(x')) \le \beta(d_X(x, x')) \tag{2.10.8}$$

for every $x, x' \in X$. If $\beta(0) = 0$ and

$$\lim_{r \to 0+} \beta(r) = 0, \tag{2.10.9}$$

then f is uniformly continuous as a mapping from X into Y. As usual, these conditions implicitly include the requirement that $\beta(r)$ be finite when r is sufficiently small. Of course,

$$d_Y(f(x), f(x')) \le \beta_f(d_X(x, x')) \tag{2.10.10}$$

for every $x, x' \in X$ automatically, by the definition (2.10.2) of β_f.

Now let β be any nonnegative extended real-valued function on $[0, \infty)$, and put

$$\widetilde{\beta}(r) = \sup\{\beta(t) : 0 \le t \le r\} \tag{2.10.11}$$

for each $r \ge 0$, where the supremum on the right is defined as a nonnegative extended real number. By construction, $\widetilde{\beta}(0) = \beta(0)$,

$$\beta(r) \le \widetilde{\beta}(r) \tag{2.10.12}$$

for every $r \ge 0$, and $\widetilde{\beta}$ increases monotonically on $[0, \infty)$. If β increases monotonically on $[0, \infty)$, then $\widetilde{\beta} = \beta$. If β satisfies $\beta(0) = 0$ and (2.10.9), then

$$\lim_{r \to 0+} \widetilde{\beta}(r) = 0, \tag{2.10.13}$$

which implicitly includes the finiteness of $\widetilde{\beta}(r)$ when r is sufficiently small. If (2.10.8) holds, then

$$\beta_f(r) \le \widetilde{\beta}(r) \tag{2.10.14}$$

for every $r \ge 0$.

2.11 Relating α's and β's

Let X, Y be nonempty sets again, with semimetrics d_X, d_Y, respectively. Also let f be a mapping from X into Y, and let α be a monotonically increasing nonnegative real-valued function on $[0, \infty)$ such that $\alpha(0) = 0$. Suppose that

$$\alpha(d_Y(f(x), f(x'))) \le d_X(x, x') \tag{2.11.1}$$

for every $x, x' \in X$. Thus for each $\epsilon > 0$,

$$d_Y(f(x), f(x')) \ge \epsilon \tag{2.11.2}$$

implies that

$$d_X(x, x') \ge \alpha(\epsilon). \tag{2.11.3}$$

Equivalently, this means that

$$d_Y(f(x), f(x')) < \epsilon \tag{2.11.4}$$

when

$$d_X(x, x') < \alpha(\epsilon). \tag{2.11.5}$$

If $\alpha > 0$ on $(0, \infty)$, then it follows that f is uniformly continuous on X. More precisely, for each $\epsilon > 0$ with $\alpha(\epsilon) > 0$, we get that

$$\beta_f^-(\alpha(\epsilon)) \le \epsilon, \tag{2.11.6}$$

where β_f^- is as in (2.10.1). This implies that (2.10.7) holds when $\alpha > 0$ on $(0, \infty)$.

Alternatively, put

$$\alpha_-(r) = \sup\{\alpha(t) : 0 \le t < r\} \tag{2.11.7}$$

for each positive real number r, which is the same as

$$\lim_{t \to r-} \alpha(t), \tag{2.11.8}$$

because α increases monotonically. It is convenient to put $\alpha_-(0) = 0$, and to let (2.11.7) be defined as a nonnegative extended real number when $r = +\infty$. This defines α_- as a nonnegative extended real-valued function on the set $[0, \infty]$ of nonnegative extended real numbers. Note that α_- increases monotonically on $[0, \infty]$, and that $\alpha_- \le \alpha$ on $[0, \infty)$. If $\alpha > 0$ on $(0, \infty)$, then $\alpha_- > 0$ on $(0, \infty]$.

Let $0 \le r < \infty$ be given. If $x, x' \in X$ and $d_X(x, x') \le r$, then

$$\alpha(d_Y(f(x), f(x'))) \le d_X(x, x') \le r, \tag{2.11.9}$$

by (2.11.1). Using this, one can check that

$$\alpha_-(\beta_f(r)) \le r, \tag{2.11.10}$$

where $\beta_f(r)$ is as in (2.10.2). If the supremum on the right side of (2.10.2) is attained, then

$$\alpha(\beta_f(r)) \le r. \tag{2.11.11}$$

Of course, if $0 < \beta_f(r) < \infty$ and α is left-continuous at $\beta_f(r)$, then (2.11.10) implies (2.11.11) automatically.

Now let β be any nonnegative extended real-valued function on $[0, \infty)$, and let α be a monotonically increasing nonnegative extended real-valued function on $[0, \infty]$. Suppose that

$$\alpha(\beta(r)) \le r \tag{2.11.12}$$

for every $r \ge 0$. This implies that for each $\epsilon > 0$ and $r \ge 0$ with

$$\beta(r) \ge \epsilon, \tag{2.11.13}$$

we have that

$$r \ge \alpha(\beta(r)) \ge \alpha(\epsilon). \tag{2.11.14}$$

Equivalently, for each $\epsilon > 0$ and $r \ge 0$ with

$$r < \alpha(\epsilon), \tag{2.11.15}$$

we have that

$$\beta(r) < \epsilon. \tag{2.11.16}$$

If $\alpha > 0$ on $(0, \infty)$, then it follows that

$$\beta(0) = 0 \tag{2.11.17}$$

and

$$\lim_{r \to 0+} \beta(r) = 0. \tag{2.11.18}$$

In particular, this implies that $\beta(r) < \infty$ when $r \ge 0$ is sufficiently small.

If $0 < t \le \infty$, then put

$$\widetilde{\beta}^{-}(t) = \sup\{\beta(r) : 0 \le r < t\}, \tag{2.11.19}$$

where the supremum on the right is defined as a nonnegative extended real number. If $\epsilon > 0$ and $\alpha(\epsilon) > 0$, then we get that

$$\widetilde{\beta}^{-}(\alpha(\epsilon)) \le \epsilon, \tag{2.11.20}$$

because (2.11.15) implies (2.11.16). Of course, this implies that (2.11.17) and (2.11.18) hold when $\alpha > 0$ on $(0, \infty)$.

Let β be a nonnegative extended real-valued function on $[0, \infty)$ again. If $0 \le t \le \infty$, then put

$$\alpha_\beta(t) = \inf\{u : 0 \le u < \infty,\ \beta(u) \ge t\}, \tag{2.11.21}$$

where the infimum is interpreted as being $+\infty$ when $\beta(u) < t$ for every u. This defines a nonnegative extended real-valued function on $[0, \infty]$, with $\alpha(0) = 0$.

Note that α increases monotonically on $[0, \infty]$, because the set whose infimum is being taken gets smaller as t increases. If $0 \le r < \infty$, then

$$\alpha_\beta(\beta(r)) \le r, \tag{2.11.22}$$

because $u = r$ is an element of the set whose infimum is taken in the right side of (2.11.21) when $t = \beta(r)$. If $0 \le t \le \infty$ and $0 \le u < \alpha_\beta(t)$, then

$$\beta(u) < t, \tag{2.11.23}$$

because u is not an element of the set whose infimum is taken in the right side of (2.11.21). This implies that

$$\widetilde{\beta}^-(\alpha_\beta(t)) \le t \tag{2.11.24}$$

when $\alpha_\beta(t) > 0$, where $\widetilde{\beta}^-$ is as in (2.11.19).

Suppose that β satisfies (2.11.17) and (2.11.18), and let $\epsilon > 0$ be given. By hypothesis, there is a $\delta > 0$ such that (2.11.16) holds when $0 \le r < \delta$. It follows that

$$\alpha_\beta(\epsilon) \ge \delta, \tag{2.11.25}$$

by the definition (2.11.21) of α_β.

2.12 Compatible supremum semimetrics

Let X, Y be nonempty sets, and let d_Y, d'_Y be semimetrics on Y. Suppose that d'_Y is uniformly compatible with d_Y on Y, as in Section 2.9. Thus for each $\epsilon > 0$ there is a $\delta > 0$ such that for every $y, z \in Y$ with

$$d_Y(y, z) < \delta, \tag{2.12.1}$$

we have that

$$d'_Y(y, z) < \epsilon. \tag{2.12.2}$$

Let $\{f_j\}_{j=1}^\infty$ be a sequence of mappings from X into Y, and let f be a mapping from X into Y. If $\{f_j\}_{j=1}^\infty$ converges to f uniformly on X with respect to d_Y on Y, then it is easy to see that $\{f_j\}_{j=1}^\infty$ converges to f uniformly on X with respect to d'_Y on Y.

Let $\mathcal{B}_{d_Y}(X, Y)$, $\mathcal{B}_{d'_Y}(X, Y)$ be the collections of bounded mappings from X into Y with respect to d_Y, d'_Y, respectively, as in Section 2.1. Consider the corresponding supremum semimetrics

$$\theta_{d_Y}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)), \tag{2.12.3}$$

$$\theta_{d'_Y}(f, g) = \sup_{x \in X} d'_Y(f(x), g(x)) \tag{2.12.4}$$

on $\mathcal{B}_{d_Y}(X, Y)$, $\mathcal{B}_{d'_Y}(X, Y)$, respectively. If d'_Y is uniformly compatible with d_Y on Y, then (2.12.4) is uniformly compatible with (2.12.3) on

$$\mathcal{B}_{d_Y}(X, Y) \cap \mathcal{B}_{d'_Y}(X, Y). \tag{2.12.5}$$

Indeed, let $\epsilon > 0$ be given, and let $\delta > 0$ be as in the preceding paragraph, so that (2.12.1) implies (2.12.2). If f, g are elements of (2.12.5) such that

$$\theta_{d_Y}(f, g) < \delta, \tag{2.12.6}$$

then

$$d_Y(f(x), g(x)) < \delta \tag{2.12.7}$$

for every $x \in X$. This implies that

$$d'_Y(f(x), g(x)) < \epsilon \tag{2.12.8}$$

for every $x \in X$, so that

$$\theta_{d'_Y}(f, g) \le \epsilon. \tag{2.12.9}$$

It follows in particular that (2.12.4) is compatible with the topology determined on (2.12.5) by (2.12.3).

Alternatively, let α be a monotonically increasing nonnegative real-valued function on $[0, \infty)$ such that $\alpha(0) = 0$ and

$$\alpha(d'_Y(y, z)) \le d_Y(y, z) \tag{2.12.10}$$

for every $y, z \in Y$. Let $\alpha_-(r)$ be defined for $r > 0$ as in (2.11.7), and put $\alpha_-(0) = 0$, as before. If f, g are elements of (2.12.5), then

$$\alpha_-(\theta_{d'_Y}(f, g)) \le \sup_{x \in X} \alpha(d'_Y(f(x), g(x))) \le \theta_{d_Y}(f, g). \tag{2.12.11}$$

If d'_Y is uniformly compatible with d_Y on Y, then we can choose α to be strictly positive on $(0, \infty)$, so that $\alpha_- > 0$ on $(0, \infty)$ too. In this case, it follows from (2.12.11) that (2.12.4) is uniformly compatible with (2.12.3) on (2.12.5).

As another version of this, let β be a monotonically increasing nonnegative extended real-valued function on $[0, \infty)$ such that

$$d'_Y(y, z) \le \beta(d_Y(y, z)) \tag{2.12.12}$$

for every $y, z \in Y$. If f, g are elements of (2.12.5), then

$$\theta_{d'_Y}(f, g) \le \sup_{x \in X} \beta(d_Y(f(x), g(x))) \le \beta(\theta_{d_Y}(f, g)). \tag{2.12.13}$$

If d'_Y is uniformly compatible with d_Y on Y, then we can choose β so that $\beta(0) = 0$ and $\beta(r) \to 0$ as $r \to 0+$. Using this condition and (2.12.13), we get that (2.12.4) is uniformly compatible with (2.12.3) on (2.12.5).

Chapter 3

Open subgroups and semi-ultrametrics

3.1 Semi-ultrametrics

A semimetric $d(x, y)$ on a set X is said to be a *semi-ultrametric* on X if

$$d(x, z) \le \max(d(x, y), d(y, z)) \quad \text{for every } x, y, z \in X. \tag{3.1.1}$$

Note that (3.1.1) automatically implies the ordinary triangle inequality (1.1.3). Similarly, a metric $d(x, y)$ on X is said to be an *ultrametric* on X if it satisfies (3.1.1). This discrete metric on X is an ultrametric, for instance. If $d(x, y)$ is a semi-ultrametric on X and Y is a subset of X, then the restriction of $d(x, y)$ to $x, y \in Y$ is a semi-ultrametric on Y.

Let $d(\cdot, \cdot)$ be a semi-ultrametric on X, and let r be a positive real number. Observe that

$$d(x, y) < r \tag{3.1.2}$$

defines an equivalence relation on X. The corresponding equivalence classes in X are the same as the open balls in X with radius r with respect to d. The complement of an open ball in X of radius r can be expressed as a union of other open balls of radius r, and in particular is an open set in X with respect to the topology determined by d. This implies that open balls in X are closed sets in X with respect to the topology determined by d.

Similarly,

$$d(x, y) \le r \tag{3.1.3}$$

defines an equivalence relation on X for every nonnegative real number r. The corresponding equivalence classes in X are the same as the closed balls in X with radius r with respect to d. If $x, y \in X$ satisfy (3.1.3), then it follows that

$$\overline{B}(x, r) = \overline{B}(y, r), \tag{3.1.4}$$

where these closed balls are as defined in (1.1.6). This implies that closed balls in X of radius r are open sets in X with respect to the topology determined by d when $r > 0$. Note that (3.1.3) defines an equivalence relation on X for every semimetric $d(\cdot, \cdot)$ on X when $r = 0$.

Let $\mathcal{P}$ be a *partition* of X, which is to say a collection of pairwise-disjoint nonempty subsets of X whose union is equal to X. Put

$$d_{\mathcal{P}}(x, y) = 0 \tag{3.1.5}$$

when $x, y \in X$ are contained in the same element of $\mathcal{P}$, and

$$d_{\mathcal{P}}(x, y) = 1 \tag{3.1.6}$$

when $x, y \in X$ are contained in different elements of $\mathcal{P}$. One can check that $d_{\mathcal{P}}(x, y)$ defines a semi-ultrametric on X, which we shall call the *discrete semi-ultrametric* associated to $\mathcal{P}$. By construction, if $0 < r \le 1$, then the open balls in X with radius r with respect to $d_{\mathcal{P}}$ are the same as the elements of $\mathcal{P}$. If $0 \le r < 1$, then the closed balls in X with radius r with respect to $d_{\mathcal{P}}$ are the same as the elements of $\mathcal{P}$ too.

Let $d(x, y)$ be any semimetric on X such that for each $x, y \in X$, $d(x, y)$ is either 0 or 1. It is easy to see that $d(x, y)$ has to be a semi-ultrametric on X in this case, and we shall call $d(x, y)$ a *discrete semi-ultrametric* on X. As before,

$$d(x, y) = 0 \tag{3.1.7}$$

defines an equivalence relation on X, so that X is partitioned by the corresponding collection of equivalence classes. Observe that $d(\cdot, \cdot)$ is the same as the discrete semi-ultrametric on X associated to this partition, as in the preceding paragraph. Of course, the discrete semi-ultrametric associated to any partition $\mathcal{P}$ of X as in (3.1.5) and (3.1.6) is a discrete semi-ultrametric on X in this sense.

3.2 Translation-invariant semi-ultrametrics

Let G be a group, and let $d(\cdot, \cdot)$ be a semi-ultrametric on G. If $d(\cdot, \cdot)$ is invariant under left translations on G, then

$$d(e, x\,y) \le \max(d(e, x), d(x, x\,y)) = \max(d(e, x), d(e, y)) \tag{3.2.1}$$

for every $x, y \in G$. Similarly, if $d(\cdot, \cdot)$ is invariant under right translations on G, then

$$d(e, x\,y) \le \max(d(e, y), d(y, x\,y)) = \max(d(e, y), d(e, x)) \tag{3.2.2}$$

for every $x, y \in G$. It follows in both cases that open balls in G centered at the identity element e with respect to d are subgroups of G. This also uses the fact that open balls in G centered at e with respect to e are symmetric about e, as in Section 1.4. Similarly, closed balls in G centered at e with respect to d

are subgroups of G in both situations. If $d(\cdot, \cdot)$ is invariant under both left and right translations on G, then it is easy to see that open and closed balls in G centered at e with respect to d are normal subgroups of G.

Let H be a subgroup of G, and put

$$d_L(x, y) = 0 \quad \text{when } x\,H = y\,H \tag{3.2.3}$$
$$= 1 \quad \text{when } x\,H \neq y\,H,$$

where $x, y \in G$. This is the discrete semi-ultrametric on G associated to the partition of G consisting of the left cosets of H in G, as in the previous section. Observe that (3.2.3) is invariant under left translations on G, and right translations by elements of H. If $0 < r \leq 1$, then H is the same as the open ball in G centered at e with radius r with respect to (3.2.3). If $0 \leq r < 1$, then H is the same as the open ball in G centered at e with radius r with respect to (3.2.3).

Similarly, if $x, y \in G$, then we put

$$d_R(x, y) = 0 \quad \text{when } H\,x = H\,y \tag{3.2.4}$$
$$= 1 \quad \text{when } H\,x \neq H\,y.$$

This is the discrete semi-ultrametric on G associated to the partition of G consisting of right cosets of H in G. By construction, (3.2.4) is invariant under right translations on G, and left translations by elements of H. As before, H is the same as the open ball in G centered at e with respect to (3.2.4) when $0 < r \leq 1$. If $0 \leq r < 1$, then H is the same as the closed ball in G centered at e with radius r with respect to (3.2.4).

Note that $(x\,H)^{-1} = H^{-1}\,x^{-1} = H\,x^{-1}$ for every $x \in G$. If $x, y \in G$, then it follows that

$$x\,H = y\,H \quad \text{if and only if} \quad H\,x^{-1} = H\,y^{-1}. \tag{3.2.5}$$

This implies that

$$d_R(x^{-1}, y^{-1}) = d_L(x, y) \tag{3.2.6}$$

for every $x, y \in G$.

If H is a normal subgroup of G, then $x\,H = H\,x$ for every $x \in G$. This implies that

$$d_L(x, y) = d_R(x, y) \tag{3.2.7}$$

for every $x, y \in G$. In particular, (3.2.3) and (3.2.4) are invariant under both left and right translations on G in this case.

3.3 Open subgroups

Let G be a topological group, and suppose that U is a subgroup of G that is also an open set. This implies that the cosets of U in G are open sets too, by continuity of translations. It follows that U is a closed set, because the complement of U is a union of cosets, and hence an open set. In particular, if G is connected as a topological space, then G is its only open subgroup. Note that

$\{e\}$ is an open set in G if and only if G is equipped with the discrete topology, because of continuity of translations.

Of course, the set $\mathbf{Q}$ of rational numbers is a subgroup of $\mathbf{R}$, as a commutative group with respect to addition. We may also consider $\mathbf{Q}$ as a topological group with respect to addition, and the topology induced on $\mathbf{Q}$ by the standard topology on $\mathbf{R}$. One can check that $\mathbf{Q}$ is the only open subgroup of itself, even though $\mathbf{Q}$ is not connected as a topological space.

Let G be a topological group again, and let $d(\cdot, \cdot)$ be a semi-ultrametric on G that is invariant under left or right translations on G. Thus open and closed balls in G with respect to d centered at the identity element e are subgroups of G, as in the previous section. If $d(\cdot, \cdot)$ is compatible with the topology on G, as in Section 1.1, then it follows that open balls in G with respect to d centered at e are open subgroups of G with respect to the given topology on G. In this case, closed balls in G with respect to d centered at e with positive radius are open subgroups of G too. More precisely, closed balls in G with respect to d with positive radius are open sets with respect to the topology determined by d, as in Section 3.1, and hence with respect to the given topology on G, because d is supposed to be compatible with that topology.

Let U be an open subgroup in G again. Using the left and right cosets of U in G, we get discrete semi-ultrametrics d_L and d_R on G, as in (3.2.3) and (3.2.4). Remember that these semi-ultrametrics are invariant under left and right translations on G, respectively. It is easy to see that d_L and d_R are compatible with the given topology on G, because U is an open set. More precisely, it suffices to verify that d_L and d_R are compatible with the given topology on G at e, as in Section 1.4.

Let U be a subgroup of G. If e is an element of the interior of U in G, then U is an open subgroup of G. This uses the continuity of translations on G. Similarly, if the interior of U is nonempty, then U is an open subgroup of G.

Now let A be a subset of G that contains e, and suppose that A is symmetric about e. This can always be arranged by replacing A with $A \cap A^{-1}$, as in Section 1.2. Let us define A^j for each $j \in \mathbf{Z}_+$ by putting $A^1 = A$ and $A^{j+1} = A^j\, A$ for every $j \geq 1$. Equivalently, A^j consists of the elements of G that can be expressed as the product of exactly j elements of A. Thus

(3.3.1) $$A^j\, A^l = A^{j+l}$$

for every $j, l \in \mathbf{Z}_+$. Similarly, A^j consists of the elements of G that can be expressed as the product of at most j elements of A, because $e \in A$. We also have that

(3.3.2) $$(A^j)^{-1} = A^j$$

for every $j \in \mathbf{Z}_+$, because $A^{-1} = A$, by hypothesis. It follows that

(3.3.3) $$\bigcup_{j=1}^{\infty} A^j$$

is a subgroup of G.

If A is an open subset of G, then A^j is an open subset of G for every $j \in \mathbf{Z}_+$, as in Section 1.2. This implies that (3.3.3) is an open subset of G, and hence an open subgroup of G. Alternatively, if A has nonempty interior, then (3.3.3) has nonempty interior, which implies that (3.3.3) is an open set, as before.

3.4 U-Separated sets

Let G be a topological group, and let U be an open subset of G that contains the identity element e. Let us say that subsets A, B of G are *left-invariant U-separated* if

$$(A\,U) \cap B = \emptyset. \tag{3.4.1}$$

It is easy to see that this holds if and only if

$$A \cap (B\,U^{-1}) = \emptyset. \tag{3.4.2}$$

Thus A, B are left-invariant U-separated if and only if B, A are left-invariant U^{-1}-separated. If U is symmetric about e, so that $U^{-1} = U$, then A, B are left-invariant U-separated if and only if B, A are left-invariant U-separated.

Similarly, A, B are *right-invariant U-separated* if

$$(U\,A) \cap B = \emptyset. \tag{3.4.3}$$

This holds if and only if

$$(A^{-1}\,U^{-1}) \cap B^{-1} = \emptyset, \tag{3.4.4}$$

because $(U\,A)^{-1} = A^{-1}\,U^{-1}$. This means that A, B are right-invariant U-separated if and only if A^{-1}, B^{-1} are left-invariant U^{-1}-separated. We shall focus on left-invariant U-separated sets in this section, for simplicity.

Using the continuity of the group operations on G, we can find an open subset U_1 of G such that $e \in U_1$ and

$$U_1\,U_1^{-1} \subseteq U, \tag{3.4.5}$$

as in Section 1.2. If A, B are left-invariant U-separated subsets of G, then we get that

$$(A\,U_1\,U_1^{-1}) \cap B = \emptyset. \tag{3.4.6}$$

This is the same as saying that

$$(A\,U_1) \cap (B\,U_1) = \emptyset. \tag{3.4.7}$$

In particular, this implies that

$$\overline{A} \cap \overline{B} = \emptyset, \tag{3.4.8}$$

because of (1.2.16).

Suppose that A is compact, B is a closed set, and

$$A \cap B = \emptyset. \tag{3.4.9}$$

Under these conditions, there is an open subset U of G such that $e \in U$ and A, B are left-invariant U-separated. This is basically the same as (1.3.12).

If A is a compact open subset of G, then there is an open subset U of G such that $e \in U$ and

$$A\,U \subseteq A. \tag{3.4.10}$$

This follows from the remarks in the preceding paragraph, and can also be obtained from (1.3.12).

Let A, U be subsets of G such that $e \in U$, U is an open set, and (3.4.10) holds. As usual, we can take U to be symmetric about e, by replacing U with $U \cap U^{-1}$. Using (3.4.10), we get that

$$A\,U^j \subseteq A \tag{3.4.11}$$

for every $j \in \mathbf{Z}_+$, where U^j is as in the previous section. If we put

$$U_0 = \bigcup_{j=1}^{\infty} U^j, \tag{3.4.12}$$

then it follows that

$$A\,U_0 \subseteq A. \tag{3.4.13}$$

Of course, $A \subseteq A\,U \subseteq A\,U_0$, because $e \in U$, so that $A\,U_0 = A$.

Note that U_0 is an open subgroup of G under these conditions, as in the previous section. If $e \in A$, then we also have that

$$U_0 \subseteq A, \tag{3.4.14}$$

by (3.4.13).

3.5 Collections of subgroups

Let G be a group, and let $\mathcal{B}$ be a nonempty collection of subgroups of G. We would like to consider topologies on G such that the elements of $\mathcal{B}$ form a local sub-base at the identity element e, and with other appropriate properties. If $A \in \mathcal{B}$, then let $d_{A,L}$ and $d_{A,R}$ be the discrete semi-ultrametrics on G associated to the left and right cosets of A in G, as in (3.2.3) and (3.2.4), respectively. Thus

$$\mathcal{M}_L(\mathcal{B}) = \{d_{A,L} : A \in \mathcal{B}\} \tag{3.5.1}$$

and

$$\mathcal{M}_R(\mathcal{B}) = \{d_{A,R} : A \in \mathcal{B}\} \tag{3.5.2}$$

are nonempty collections of semi-ultrametrics on G that are invariant under left and right translations, respectively. Let $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$ be the topologies determined on G by $\mathcal{M}_L(\mathcal{B})$ and $\mathcal{M}_R(\mathcal{B})$ as in Section 1.5, respectively.

Equivalently, a subset U of G is an open set with respect to $\tau_L(\mathcal{B})$ if for every $x \in U$ there are finitely many elements $A_1, \ldots, A_n$ of $\mathcal{B}$ such that

$$\bigcap_{j=1}^{n} (x\, A_j) \subseteq U. \tag{3.5.3}$$

Similarly, U is an open set with respect to $\tau_R(\mathcal{B})$ if for every $x \in U$ there are finitely many elements $A_1, \ldots, A_n$ of $\mathcal{B}$ such that

$$\bigcap_{j=1}^{n} (A_j\, x) \subseteq U. \tag{3.5.4}$$

One can check directly that these define topologies on G, instead of using $\mathcal{M}_L(\mathcal{B})$ and $\mathcal{M}_R(\mathcal{B})$. It is easy to see that the elements of $\mathcal{B}$ are open sets with respect to $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$, because the elements of $\mathcal{B}$ are subgroups of G. Of course, $\mathcal{B}$ is a local sub-base for each of these topologies at e, by construction. Note that $\tau_L(\mathcal{B})$ is preserved by left translations on G, and that $\tau_R(\mathcal{B})$ is preserved by right translations on G. This follows from the fact that the elements of $\mathcal{M}_L(\mathcal{B})$ and $\mathcal{M}_R(\mathcal{B})$ are invariant under left and right translations on G, respectively, and it can also be obtained from the characterizations of $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$ just mentioned.

As in Section 1.7, $x \mapsto x^{-1}$ is continuous as a mapping from G into itself at e with respect $\tau_L(\mathcal{B})$, and with respect to $\tau_R(\mathcal{B})$. Multiplication on G is also continuous as a mapping from $G \times G$ into G at (e, e), using either $\tau_L(\mathcal{B})$ or $\tau_R(\mathcal{B})$, and the corresponding product topology. Both statements can be obtained from the characterizations of $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$ in the previous paragraph, as well as from earlier remarks about collections of translation-invariant semimetrics. Note that a subset U of G is an open set with respect to $\tau_L(\mathcal{B})$ if and only if U^{-1} is an open set with respect to $\tau_R(\mathcal{B})$, which is the same as saying that $x \mapsto x^{-1}$ is a homeomorphism as a mapping from G with $\tau_L(\mathcal{B})$ onto G with $\tau_R(\mathcal{B})$. This can be obtained from the characterizations of $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$ in the previous paragraph, or using (3.2.6).

The elements of $\mathcal{B}$ are automatically closed sets with respect to $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$. This can be obtained from either of the characterizations of $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$, as usual. Let us say that $\mathcal{B}$ is *nondegenerate* if

$$\bigcap_{A \in \mathcal{B}} A = \{e\}. \tag{3.5.5}$$

This implies that $\mathcal{M}_L(\mathcal{B})$ and $\mathcal{M}_R(\mathcal{B})$ are nondegenerate as collections of semimetrics. In this case, one can verify directly that G is Hausdorff with respect to $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$, using the characterizations of these topologies in terms of (3.5.3) and (3.5.4).

If

$$\tau_L(\mathcal{B}) = \tau_R(\mathcal{B}), \tag{3.5.6}$$

then left and right translations are continuous with respect to this common topology. This implies that G is a topological group with respect to this topology, as in Section 1.7, because of the properties of these topologies mentioned earlier. Conversely, if G is a topological group with respect to $\tau_L(\mathcal{B})$ or $\tau_R(\mathcal{B})$, then $x \mapsto x^{-1}$ is a homeomorphism with respect to that topology. This implies that (3.5.6) holds, because $x \mapsto x^{-1}$ automatically sends $\tau_L(\mathcal{B})$ onto $\tau_R(\mathcal{B})$, as before.

Let us say that $\mathcal{B}$ is *nice* if for every $x \in G$ and $A \in \mathcal{B}$ there are finitely many elements $A_1, \ldots, A_n$ of $\mathcal{B}$ such that

$$\bigcap_{j=1}^{n} A_j \subseteq x\, A\, x^{-1}. \tag{3.5.7}$$

Equivalently, (3.5.7) means that

$$\bigcap_{j=1}^{n} (x^{-1}\, A_j) \subseteq A\, x^{-1}, \tag{3.5.8}$$

and that

$$\bigcap_{j=1}^{n} (A_j\, x) \subseteq x\, A. \tag{3.5.9}$$

If $\mathcal{B}$ is nice, then one can check that (3.5.6) holds. This implies that G is a topological group with respect to (3.5.6), as before.

Conversely, suppose that G is a topological group with respect to $\tau_L(\mathcal{B})$ or $\tau_R(\mathcal{B})$. In particular, this implies that both left and right translations are continuous with respect to $\tau_L(\mathcal{B})$ or $\tau_R(\mathcal{B})$. If $x \in G$ and $A \in \mathcal{B}$, then it follows that $x\, A\, x^{-1}$ is an open set with respect to $\tau_L(\mathcal{B})$ or $\tau_R(\mathcal{B})$, because A is an open set with respect to both $\tau_L(\mathcal{B})$ and $\tau_R(\mathcal{B})$. This implies that there are finitely many elements $A_1, \ldots, A_n$ of $\mathcal{B}$ such that (3.5.7) holds, because $e \in x\, A\, x^{-1}$. This shows that $\mathcal{B}$ is nice under these conditions.

If every element of $\mathcal{B}$ is a normal subgroup of G, then $\mathcal{B}$ is automatically nice. In this case, the discrete semi-ultrametrics $d_{A,L}$ and $d_{A,R}$ mentioned at the beginning of the section are the same for every $A \in \mathcal{B}$, as in (3.2.7). In particular, this implies that (3.5.1) and (3.5.2) are the same. Remember that these semi-ultrametrics are invariant under both left and right translations in this situation, as in Section 3.2.

3.6 More on semi-ultrametrics

Let X be a set. If $d_1, \ldots, d_n$ are finitely many semimetrics on X, then we have seen that

$$\max_{1 \le j \le n} d_j(x, y) \tag{3.6.1}$$

and

$$\sum_{j=1}^{n} d_j(x,y) \tag{3.6.2}$$

define semimetrics on X too, as in Section 1.5. If $d_1, \ldots, d_n$ are semi-ultrametrics on X, then one can check that (3.6.1) is a semi-ultrametric on X as well. However, this does not normally work for the sum (3.6.2).

Similarly, if $d(x,y)$ is a semimetric on X and t is a positive real number, then we have seen that

$$\min(d(x,y), t) \tag{3.6.3}$$

defines a semimetric on X, as in Section 1.6. If $d(x,y)$ is a semi-ultrametric on X, then it is easy to see that (3.6.3) is also a semi-ultrametric on X.

Let $d_1, d_2, d_3, \ldots$ be an infinite sequence of semimetrics on X. As before,

$$d_j'(x,y) = \min(d_j(x,y), 1/j) \tag{3.6.4}$$

defines a semimetric on X for each positive integer j. We have seen that

$$\max_{j \geq 1} d_j'(x,y) \tag{3.6.5}$$

defines a semi-metric on X too, as in (1.6.5). If d_j is a semi-ultrametric on X for each $j \geq 1$, then (3.6.4) is also a semi-ultrametric on X for every $j \geq 1$, as in the preceding paragraph. In this case, one can check that that (3.6.5) is a semi-ultrametric on X as well.

Let Y be another set, and let d_Y be a semimetric on Y. Also let A be a nonempty subset of X, and remember that

$$\theta_A(f,g) = \sup_{x \in A} d_Y(f(x), g(x)) \tag{3.6.6}$$

defines a semimetric on the space $\mathcal{B}_A(X,Y)$ of mappings from X into Y that are bounded on A, as in Section 2.1. If d_Y is a semi-ultrametric on Y, then (3.6.6) is a semi-ultrametric on $\mathcal{B}_A(X,Y)$. More precisely, if $f, g, h \in \mathcal{B}_A(X,Y)$, then

$$\begin{aligned} d_Y(f(x), h(x)) &\leq \max\big(d_Y(f(x), g(x)), d_Y(g(x), h(x))\big) \\ &\leq \max(\theta_A(f,g), \theta_A(g,h)) \end{aligned} \tag{3.6.7}$$

for every $x \in A$ in this case. It follows that

$$\theta_A(f,h) \leq \max(\theta_A(f,g), \theta_A(g,h)), \tag{3.6.8}$$

as desired.

Let X, Y be sets again, and let d_Y be a semimetric on Y. If ϕ is a mapping from X into Y, then it is easy to see that

$$d_Y(\phi(x), \phi(x')) \tag{3.6.9}$$

defines a semimetric on X. Similarly, if d_Y is a semi-ultrametric on Y, then (3.6.9) is a semi-ultrametric on X. Note that this includes the situation considered in (1.8.4).

3.7 Some countable products

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of nonempty sets, and let

$$X = \prod_{j=1}^{\infty} X_j \tag{3.7.1}$$

be their Cartesian product. Thus the elements of X may be considered as sequences $x = \{x_j\}_{j=1}^{\infty}$, where $x_j \in X_j$ for each $j \in \mathbf{Z}_+$. To avoid degeneracies, one might as well ask that X_j have at least two elements for each $j \in \mathbf{Z}_+$. If $x, y \in X$ and $x \neq y$, then let $l(x, y)$ be the largest nonnegative integer such that

$$x_j = y_j \quad \text{when } j \le l(x, y). \tag{3.7.2}$$

Equivalently, $l(x, y) + 1$ is the smallest positive integer j such that $x_j \neq y_j$. In particular, if $x_1 \neq y_1$, then $l(x, y) = 0$. If $x = y$, then we take $l(x, y) = +\infty$, so that (3.7.2) holds for every $x, y \in X$. Note that $l(x, y)$ is symmetric in x, y. We also have that

$$l(x, z) \ge \min(l(x, y), l(y, z)) \tag{3.7.3}$$

for every $x, y, z \in X$. Indeed, if $j \in \mathbf{Z}_+$ satisfies

$$j \le \min(l(x, y), l(y, z)), \tag{3.7.4}$$

then $x_j = y_j$ and $y_j = z_j$. This implies that $x_j = z_j$ for these j, and hence that (3.7.3) holds.

Let $\{r_j\}_{j=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers that converges to 0, with respect to the standard metric on $\mathbf{R}$. If $x, y \in X$, then put

$$\begin{aligned} d(x, y) &= r_{l(x,y)} \quad \text{when } x \neq y \\ &= 0 \qquad\quad \text{when } x = y. \end{aligned} \tag{3.7.5}$$

It is convenient to put $r_\infty = 0$, so that the $x = y$ case may be given by the same expression as the $x \neq y$ case. Clearly $d(x, y)$ satisfies the positivity and symmetry requirements of a metric. If $x, y, z \in X$, then

$$d(x, z) = r_{l(x,z)} \le \max(r_{l(x,y)}, r_{l(y,z)}) = \max(d(x, y), d(y, z)). \tag{3.7.6}$$

This uses (3.7.3) and the hypothesis that the r_j's are decreasing in the second step. It follows that (3.7.5) defines an ultrametric on X.

If $x \in X$ and l is a nonnegative integer, then

$$\overline{B}(x, r_l) = \{y \in X : d(x, y) \le r_l\} \tag{3.7.7}$$

is the usual closed ball in X centered at x with radius r_l with respect to (3.7.5), as in (1.1.6). In this situation,

$$d(x, y) \le r_l \quad \text{if and only if} \quad l(x, y) \ge l, \tag{3.7.8}$$

because the r_j's are supposed to be strictly decreasing in j. Thus

$$\overline{B}(x, r_l) = \{y \in X : l(x, y) \geq l\}. \tag{3.7.9}$$

Equivalently,

$$\overline{B}(x, r_l) = \{y \in X : x_j = y_j \text{ when } j \leq l\}. \tag{3.7.10}$$

All other open and closed balls in X with respect to (3.7.5) can be characterized using this case.

The topology determined on X by (3.7.5) is the same as the product topology corresponding to the discrete topology on X_j for each $j \in \mathbf{Z}_+$. In particular, a sequence $x(1), x(2), x(3), \ldots$ of elements of X converges to an element x of X with respect to (3.7.5) if and only if for each $j \in \mathbf{Z}_+$, $x_j(l) = x_j$ for all sufficiently large l. Similarly, a sequence $x(1), x(2), x(3), \ldots$ of elements of X is a Cauchy sequence with respect to (3.7.5) if and only if for each $j \in \mathbf{Z}_+$, the sequence $\{x_j(l)\}_{l=1}^\infty$ of elements of X_j is eventually constant. It follows that Cauchy sequences in X are convergent, so that X is complete as a metric space with respect to (3.7.5). If X_j has only finitely many elements for each j, then X is compact.

3.8 Semi-ultrametrification

Let X be a set, and let $d(x, y)$ be a semimetric on X. If $x, y \in X$, then put

$$d_u(x, y) = \inf \Big\{ \max_{1 \leq j \leq n} d(w_j, w_{j-1}) : w_0, \ldots, w_n \in X, \; w_0 = x,\, w_n = y \Big\}. \tag{3.8.1}$$

More precisely, the infimum is taken over all finite sequences $w_0, \ldots, w_n$ of elements of X with $w_0 = x$ and $w_n = y$, where n is any positive integer. In particular, one can always take $n = 1$, $w_0 = x$, and $w_1 = y$, so that this set is nonempty. One can also allow $n = 0$ when $x = y$, with $w_0 = x = y$, and the maximum on the right side of (3.8.1) interpreted as being equal to 0. Observe that

$$0 \leq d_u(x, y) \leq d(x, y) \tag{3.8.2}$$

for every $x, y \in X$, using the $n = 1$ case just mentioned to get the second inequality. Thus

$$d_u(x, x) = 0 \tag{3.8.3}$$

for every $x \in X$, which follows from the $n = 0$ case just mentioned too. We also have that

$$d_u(x, y) = d_u(y, x) \tag{3.8.4}$$

for every $x, y \in X$, because finite sequences of elements of X going from x to y correspond to finite sequences of elements of X going from y to x, by reversing the order of the indices.

If $x, y, z \in X$, then

$$d_u(x, z) \le \max(d_u(x, y), d_u(y, z)). \tag{3.8.5}$$

To see this, let m and n be positive integers, let $v_0, \ldots, v_m$ be a finite sequence of elements of X such that $v_0 = x$ and $v_m = y$, and let $w_0, \ldots, w_n$ be a finite sequence of elements of X such that $w_0 = y$ and $w_n = z$. As before, we can allow $m = 0$ when $x = y$, and $n = 0$ when $y = z$. If we put

$$v_{m+j} = w_j \tag{3.8.6}$$

when $j = 1, \ldots, n$, then $v_0, \ldots, v_n$ is a finite sequence of elements of X going from x to z. Hence

$$d_u(x, z) \le \max_{1 \le j \le n} d(v_j, v_{j-1}). \tag{3.8.7}$$

In this situation,

$$\max_{1 \le j \le n} d(v_j, v_{j-1}) = \max\Big(\max_{1 \le j \le m} d(v_j, v_{j-1}),\ \max_{1 \le l \le n} d(w_l, w_{l-1})\Big), \tag{3.8.8}$$

by (3.8.6), which holds when $j = 0$ too. It follows that

$$d_u(x, z) \le \max\Big(\max_{1 \le j \le m} d(v_j, v_{j-1}),\ \max_{1 \le l \le n} d(w_l, w_{l-1})\Big). \tag{3.8.9}$$

It is easy to obtain (3.8.5) from (3.8.9), using (3.8.1). This shows that (3.8.1) defines a semi-ultrametric on X.

Let $\rho(x, y)$ be a semi-ultrametric on X, and suppose that

$$\rho(x, y) \le d(x, y) \tag{3.8.10}$$

for every $x, y \in X$. Let $x, y \in X$ be given, and let $w_0, \ldots, w_n$ be a finite sequence of elements of X going from x to y. Note that

$$\rho(x, y) \le \max_{1 \le j \le n} \rho(w_j, w_{j-1}), \tag{3.8.11}$$

because $\rho(\cdot, \cdot)$ is a semi-ultrametric on X. This implies that

$$\rho(x, y) \le \max_{1 \le j \le n} d(w_j, w_{j-1}), \tag{3.8.12}$$

because $\rho(w_j, w_{j-1}) \le d(w_j, w_{j-1})$ for each $j = 1, \ldots, n$, by hypothesis. It follows that

$$\rho(x, y) \le d_u(x, y), \tag{3.8.13}$$

by the definition (3.8.1) of $d_u(x, y)$. In particular, if $d(x, y)$ is a semi-ultrametric on X, then we can apply this to $\rho(x, y) = d(x, y)$. This implies that

$$d_u(x, y) = d(x, y) \tag{3.8.14}$$

for every $x, y \in X$ in this case, using also (3.8.2).

Of course, $d_u(x, y)$ is compatible with the topology determined on X by $d(x, y)$, as in Section 1.1, because of (3.8.2). Remember that open and closed balls in X with respect to d_u of positive radius are open and closed sets with respect to the topology determined on X by d_u, as in Section 3.1. This means that open and closed balls in X with respect to d_u of positive radius are open and closed sets with respect to the topology determined on X by d, because d_u is compatible with the topology determined on X by d. If X is connected with respect to the topology determined by d, then X is the only nonempty subset of itself that is both open and closed with respect to this topology. In particular, every open or closed ball in X with respect to d_u of positive radius is equal to X in this case. This implies that

$$d_u(x, y) = 0 \tag{3.8.15}$$

for every $x, y \in X$ when X is connected with respect to the topology determined by d. If $X = \mathbf{Q}$ and d is the restriction of the standard Euclidean metric on $\mathbf{R}$ to $\mathbf{Q}$, then it is easy to see that (3.8.15) holds, directly from the definitions.

3.9 Connection with uniform continuity

Let X, Y be sets, and let d_X, d_Y be semimetrics on them, respectively. Also let f be a uniformly continuous mapping from X into Y with respect to d_X, d_Y. Thus for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that

$$d_Y(f(x), f(x')) < \epsilon \tag{3.9.1}$$

for every $x, x' \in X$ with

$$d_X(x, x') < \delta(\epsilon). \tag{3.9.2}$$

Let $d_{X,u}$, $d_{Y,u}$ be the semi-ultrametrifications of d_X, d_Y on X, Y, respectively, as in (3.8.1). We would like to check that f is also uniformly continuous with respect to $d_{X,u}$, $d_{Y,u}$.

Let $\epsilon > 0$ be given, and suppose that $x, x' \in X$ satisfy

$$d_{X,u}(x, x') < \delta(\epsilon). \tag{3.9.3}$$

This means that there is a finite sequence $w_0, \ldots, w_n$ of elements of X such that $w_0 = x$, $w_n = x'$, and

$$\max_{1 \le j \le n} d_X(w_j, w_{j-1}) < \delta(\epsilon), \tag{3.9.4}$$

by the definition of $d_{X,u}(x, x')$. It follows that

$$\max_{1 \le j \le n} d_Y(f(w_j), f(w_{j-1})) < \epsilon, \tag{3.9.5}$$

because of the uniform continuity condition for f with respect to d_X, d_Y. Note that

$$d_{Y,u}(f(x), f(x')) \le \max_{1 \le j \le n} d_Y(f(w_j), f(w_{j-1})), \tag{3.9.6}$$

because $f(w_0), \ldots, f(w_n)$ is a finite sequence of elements of Y going from $f(x)$ to $f(x')$. Thus

$$d_{Y,u}(f(x), f(x')) < \epsilon, \tag{3.9.7}$$

as desired.

Suppose now that f is a uniform homeomorphism from X onto Y with respect to d_X, d_Y, respectively. Thus f is a one-to-one uniformly continuous mapping from X onto Y with respect to d_X, d_Y, whose inverse mapping f^{-1} is uniformly continuous as a mapping from Y onto X with respect to d_Y, d_X. The remarks in the previous paragraphs imply that f is also uniformly continuous as a mapping from X onto Y with respect to $d_{X,u}$, $d_{Y,u}$, respectively, and that f^{-1} is uniformly continuous as a mapping from Y onto X with respect to $d_{Y,u}$, $d_{X,u}$. This means that f is a uniform homeomorphism from X onto Y with respect to $d_{X,u}$, $d_{Y,u}$, respectively.

Let $UH_{d_X}(X)$ be the group of uniform homeomorphisms from X onto itself with respect to d_X, as in Section 2.3. Similarly, let $UH_{d_{X,u}}(X)$ be the group of uniform homeomorphisms from X onto itself with respect to $d_{X,u}$. The remarks in the previous paragraph imply that

$$UH_{d_X}(X) \subseteq UH_{d_{X,u}}(X). \tag{3.9.8}$$

The collection $IH_{d_X}(X)$ of one-to-one mappings from X onto itself that are isometries with respect to d_X is a subgroup of $UH_{d_X}(X)$, as in Section 2.7. Let $IH_{d_{X,u}}(X)$ be the collection of one-to-one mappings from X onto itself that are isometries with respect to $d_{X,u}$, which is a subgroup of $UH_{d_{X,u}}(X)$. If a one-to-one mapping f from X onto itself is an isometry with respect to d_X, then it is easy to see that f is also an isometry with respect to $d_{X,u}$, because of the way that $d_{X,u}$ is defined. This means that

$$IH_{d_X}(X) \subseteq IH_{d_{X,u}}(X). \tag{3.9.9}$$

3.10 Some related conditions

Let α be a motonically increasing real-valued function on $[0, \infty)$ with $\alpha(0) = 0$. Thus $\alpha \geq 0$ on $[0, \infty)$, by monotonicity. If $d(x, y)$ is a semi-ultrametric on a set X, then it is easy to see that

$$d_\alpha(x, y) = \alpha(d(x, y)) \tag{3.10.1}$$

is a semi-ultrametric on X too. This is a bit simpler than the analogous statement for semimetrics in Section 2.8, and in particular subadditivity of α is not needed here. If $d(x, y)$ is an ultrametric on X, and $\alpha > 0$ on $(0, \infty)$, then (3.10.1) is an ultrametric on X.

If

$$\lim_{t \to 0+} \alpha(t) = 0, \tag{3.10.2}$$

then (3.10.1) is uniformly compatible with $d(x, y)$ on X, as in Section 2.9. If $\alpha > 0$ on $(0, \infty)$, then we have also seen that $d(x, y)$ is uniformly compatible with (3.10.1) on X.

Let X, Y be sets, let d_X, d_Y be semimetrics on them, respectively, and let f be a mapping from X into Y. Suppose that

$$\alpha(d_Y(f(x), f(x'))) \le d_X(x, x') \tag{3.10.3}$$

for every $x, x' \in X$. Let $d_{X,u}$, $d_{Y,u}$ be the semi-ultrametrifications of d_X, d_Y on X, Y, respectively, as in (3.8.1) again. Observe that

$$\alpha(d_{Y,u}(y, y')) \le \alpha(d_Y(y, y')) \tag{3.10.4}$$

for every $y, y' \in Y$, because of (3.8.2) applied to d_Y, and the monotonicity of α. It follows that

$$\alpha(d_{Y,u}(f(x), f(x'))) \le d_X(x, x') \tag{3.10.5}$$

for every $x, x' \in X$, by taking $y = f(x)$, $y' = f(x')$, and using (3.10.3).

As in (3.10.1),

$$\alpha(d_{Y,u}(y, y')) \tag{3.10.6}$$

defines a semi-ultrametric on Y, because $d_{Y,u}$ is a semi-ultrametric on Y. This implies that

$$\alpha(d_{Y,u}(f(x), f(x'))) \tag{3.10.7}$$

is a semi-ultrametric on X, as in (3.6.9). Remember that $d_{X,u}$ is the largest semi-ultrametric on X that is less than or equal to d_X, as in (3.8.13). Thus

$$\alpha(d_{Y,u}(f(x), f(x'))) \le d_{X,u}(x, x') \tag{3.10.8}$$

for every $x, x' \in X$, because (3.10.7) is a semi-ultrametric on X that is less than or equal to $d_X(x, x')$, by (3.10.5).

Let r be a positive real number, so that

$$\beta_f^-(r) = \sup\{d_Y(f(x), f(x')) : x, x' \in X,\ d_X(x, x') < r\} \tag{3.10.9}$$

is defined as a nonnegative extended real number, as in (2.10.1). By construction, if $x, x' \in X$ satisfy

$$d_X(x, x') < r, \tag{3.10.10}$$

then

$$d_Y(f(x), f(x')) \le \beta_f^-(r). \tag{3.10.11}$$

Let $x, x' \in X$ be given, and suppose now that

$$d_{X,u}(x, x') < r. \tag{3.10.12}$$

This means that there is a finite sequence $w_0, \ldots, w_n$ of elements of X such that

$$\max_{1 \le j \le n} d_X(w_j, w_{j-1}) < r. \tag{3.10.13}$$

It follows that

$$d_Y(f(w_j), f(w_{j-1})) \le \beta_f^-(r) \tag{3.10.14}$$

for each $j = 1, \ldots, n$, by definition of $\beta_f^-(r)$. Of course,

$$d_{Y,u}(f(x), f(x')) \le \max_{1 \le j \le n} d_Y(f(w_j), f(w_{j-1})), \tag{3.10.15}$$

by definition of $d_{Y,u}$, and using $f(w_0), \ldots, f(w_n)$ as a finite sequence of elements of Y going from $f(x)$ to $f(x')$. This shows that

$$d_{Y,u}(f(x), f(x')) \le \beta_f^-(r) \tag{3.10.16}$$

when $x, x' \in X$ satisfy (3.10.12).

Alternatively, let β be a nonnegative extended real-valued function on $[0, \infty)$, and suppose that

$$d_Y(f(x), f(x')) \le \beta(d_X(x, x')) \tag{3.10.17}$$

for every $x, x' \in X$. If r is a nonnegative real number, then we put

$$\widetilde{\beta}(r) = \sup\{\beta(t) : 0 \le t \le r\}, \tag{3.10.18}$$

which is defined as a nonnegative extended real number, as in (2.10.11). Note that

$$\beta_f^-(r) \le \widetilde{\beta}(r) \tag{3.10.19}$$

for every $r > 0$, and indeed (2.10.14) holds. If $x, x' \in X$ satisfy (3.10.10), then

$$d_{Y,u}(f(x), f(x')) \le \widetilde{\beta}(r), \tag{3.10.20}$$

by (3.10.16).

If $0 \le t < \infty$, then put

$$\widetilde{\beta}^+(t) = \inf\{\widetilde{\beta}(r) : t < r < \infty\}, \tag{3.10.21}$$

which is defined as a nonnegative extended real number. This may be considered as the limit of $\widetilde{\beta}(r)$ as $r \to t+$, because $\widetilde{\beta}(r)$ increases monotonically, and with suitable interpretations for extended real numbers. Let $x, x' \in X$ be given, with

$$d_{X,u}(x, x') \le t. \tag{3.10.22}$$

If $t < r$, then (3.10.12) holds, and hence (3.10.20) holds, as before. It follows that

$$d_{Y,u}(f(x), f(x')) \le \widetilde{\beta}^+(t), \tag{3.10.23}$$

by taking the infimum over $r > t$. This implies that

$$d_{Y,u}(f(x), f(x')) \le \widetilde{\beta}^+(d_X(x, x')) \tag{3.10.24}$$

for every $x, x' \in X$, by taking $t = d_X(x, x')$. If $\beta(0) = 0$ and

$$\lim_{t \to 0+} \beta(t) = 0, \tag{3.10.25}$$

then $\widetilde{\beta}(0) = 0$ and

$$\lim_{r \to 0+} \widetilde{\beta}(r) = 0, \tag{3.10.26}$$

as in (2.10.13). In this case,

$$\widetilde{\beta}^+(0) = \lim_{t \to 0+} \widetilde{\beta}^+(t) = 0. \tag{3.10.27}$$

Chapter 4

Uniform continuity and total boundedness

4.1 Uniform continuity on topological groups

Let G be a topological group, let Y be a set, and let d_Y be a semimetric on Y. Also let f be a mapping from G into Y, and let A be a subset of G. Let us say that f is *left-invariant uniformly continuous along* A if for each $\epsilon > 0$ there is an open set U in G such that U contains the identity element e and

$$d_Y(f(a), f(a\,x)) < \epsilon \tag{4.1.1}$$

for every $a \in A$ and $x \in U$. Similarly, f is *right-invariant uniformly continuous along* A if for each $\epsilon > 0$ there is an open set U in G such that $e \in U$ and

$$d_Y(f(a), f(x\,a)) < \epsilon \tag{4.1.2}$$

for every $a \in A$ and $x \in U$. If A is the whole group G, then we simply say that f is left or right-invariant uniformly continuous on G, as appropriate.

If f is left or right-invariant uniformly continuous along a subset A of G, then f is continuous at every element of A, with respect to the topology determined on Y by d_Y. If A has only finitely many elements, and f is continuous at each element of A, then f is left and right-invariant uniformly continuous along A.

Put

$$\widetilde{f}(x) = f(x^{-1}) \tag{4.1.3}$$

for each $x \in G$. It is easy to see that f is left-invariant uniformly continuous along a subset A of G if and only if $\widetilde{f}$ is right-invariant uniformly continuous along A^{-1}. Similarly, f is right-invariant uniformly continuous along A if and only if $\widetilde{f}$ is left-invariant uniformly continuous along A^{-1}.

Suppose for the moment that the topology on G is determined by a semimetric d. If d is invariant under left translations on G, then f is left-invariant uniformly continuous along a subset A of G if and only if f is uniformly continuous along A with respect to d, as in Section 2.2. Similarly, if d is invariant under right translations on G, then f is right-invariant uniformly continuous along A if and only if f is uniformly continuous along A with respect to d.

Let us say that a semimetric d_0 on G is *left-invariant uniformly compatible* with the topology on G if the identity mapping on G is left-invariant uniformly continuous as a mapping into G with respect to d_0. Similarly, d_0 is *right-invariant uniformly compatible* with the topology on G if the identity mapping on G is right-invariant uniformly continuous as a mapping into G with respect to d_0. If d_0 is left or right-invariant uniformly compatible with the topology on G, then the identity mapping on G is continuous as a mapping into G with respect to the topology determined by d_0, as before. This means that d_0 is compatible with the topology on G, as in Section 1.1.

In the other direction, suppose for the moment that d_0 is compatible with the topology on G at e, as in Section 1.1 again. If d_0 is invariant under left translations, then it follows that d_0 is left-invariant uniformly compatible with the topology on G. This corresponds to (1.4.16). Similarly, if d_0 is invariant under right translations, then d_0 is right-invariant uniformly compatible with the topology on G, as in (1.4.17).

Let Z be another set with a semimetric d_Z, and let g be a mapping from Y into Z that is uniformly continuous along a subset B of Y, as in Section 2.2. Also let f be a mapping from G into Y and let A be a subset of G, as before, and suppose that $f(A) \subseteq B$. If f is left-invariant uniformly continuous along A as a mapping into Y, then one can check that $g \circ f$ is left-invariant uniformly continuous along A as a mapping into Z. Similarly, if f is right-invariant uniformly continuous along A as a mapping into Y, then $g \circ f$ is right-invariant uniformly continuous along A as a mapping into Z.

Suppose that A is a compact subset of G, and that f is a mapping from G into Y that is continuous at each point in A. Under these conditions, it is well known that f is left-invariant and right-invariant uniformly continuous along A. Let us briefly sketch the argument in the left-invariant case, which is analogous to the argument for mappings between semimetric spaces, as in Section 2.2. The right-invariant case is very similar, and the two cases are equivalent, by the earlier remarks about (4.1.3). Let $\epsilon > 0$ and $a \in A$ be given. Because f is continuous at a, there is an open subset $U(a)$ of G such that $e \in U(a)$ and

$$d_Y(f(a), f(a\,x)) < \epsilon/2 \tag{4.1.4}$$

for every $x \in U(a)$. Using continuity of multiplication on G at e, we can get an open subset $U_1(a)$ of G such that $e \in U_1(a)$ and

$$U_1(a)\,U_1(a) \subseteq U(a). \tag{4.1.5}$$

Note that $a \in a\,U_1(a)$ and

$$U_1(a) \subseteq U(a), \tag{4.1.6}$$

because $e \in U_1(a)$. In particular, A is covered by the open sets $a\,U_1(a)$, $a \in A$.

If A is compact, then there are finitely many elements $a_1, \ldots, a_n$ of A such that

$$A \subseteq \bigcup_{j=1}^{n} a_j \, U_1(a_j). \tag{4.1.7}$$

Put

$$U = \bigcap_{j=1}^{n} U_1(a_j), \tag{4.1.8}$$

so that U is an open set that contains e.

Let $a \in A$ and $x \in U$ be given, and let us check that (4.1.1) holds. Using (4.1.7), we get that $a \in a_j \, U_1(a_j)$ for some j, $1 \le j \le n$. Thus

$$a = a_j \, w \tag{4.1.9}$$

for some $w \in U_1(a_j)$. Observe that

$$w \, x \in U_1(a_j) \, U \subseteq U_1(a_j) \, U_1(a_j) \subseteq U(a_j), \tag{4.1.10}$$

using the definition (4.1.8) of U in the second step, and the analogue of (4.1.5) for a_j in the third step. It follows that

$$d_Y(f(a_j), f(a \, x)) = d_Y(f(a_j), f(a_j \, w \, x)) < \epsilon/2, \tag{4.1.11}$$

using (4.1.9) in the first step, and (4.1.10) and the analogue of (4.1.4) for a_j in the second step. Similarly,

$$d_Y(f(a_j), f(a)) = d_Y(f(a_j), f(a_j \, w)) < \epsilon/2, \tag{4.1.12}$$

because $w \in U_1(a_j) \subseteq U(a_j)$, by the analogue of (4.1.6) for a_j. Combining (4.1.11) and (4.1.12), we get that

$$\begin{aligned} d_Y(f(a), f(a \, x)) &\le d_Y(f(a), f(a_j)) + d_Y(f(a_j), f(a \, x)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned} \tag{4.1.13}$$

as desired.

4.2 Total boundedness and semimetrics

Let X be a set, and let $d(x, y)$ be a semimetric on X. A subset E of X is said to be *totally bounded* with respect to d if for each $\epsilon > 0$, E is contained in the union of finitely many open balls of radius ϵ in X. This is interpreted as holding when $E = \emptyset$, even if $X = \emptyset$. If $E \subseteq X$ is totally bounded with respect to d, then it is easy to see that E is bounded with respect to d, as in Section 2.1. If E is compact with respect to the topology determined on X by d, then E is totally bounded with respect to d.

If A is a nonempty subset of X, then the *diameter* of A with respect to d is the nonnegative extended real number defined by

$$\operatorname{diam} A = \operatorname{diam}_d A = \sup\{d(x,y) : x, y \in A\}. \tag{4.2.1}$$

This is finite exactly when A is bounded with respect to d. It is convenient to interpret the diameter of the empty set as being equal to 0. The diameter of a ball of radius r with respect to d is less than or equal to $2\,r$, by the triangle inequality. If d is a semi-ultrametric on X, then the diameter of a ball of radius r with respect to d is less than or equal to r. If A is a bounded subset of X and $a \in A$, then

$$A \subseteq \overline{B}_d(a, \operatorname{diam} A). \tag{4.2.2}$$

Of course, if $A_1 \subseteq A_2 \subseteq X$, then

$$\operatorname{diam} A_1 \le \operatorname{diam} A_2. \tag{4.2.3}$$

If $A \subseteq X$ and $\overline{A}$ is the closure of A in X with respect to the topology determined on X by d, then one can check that

$$\operatorname{diam} \overline{A} = \operatorname{diam} A. \tag{4.2.4}$$

It follows from some of the remarks in the preceding paragraph that $E \subseteq X$ is totally bounded with respect to d if and only if for every $r > 0$, E is contained in the union of finitely many subsets of X with diameter less than or equal to r. Let X_0 be a subset of X, so that the restriction of $d(x,y)$ to $x, y \in X_0$ defines a semimetric on X_0. If $E \subseteq X_0$, then one can use the characterization of total boundedness just mentioned to get that E is totally bounded as a subset of X if and only if E is totally bounded as a subset of X_0, with respect to the restriction of $d(x,y)$ to $x, y \in X_0$.

If $E \subseteq X$ is totally bounded with respect to d, then every subset of E is totally bounded with respect to d too. In this case, the closure $\overline{E}$ of E is totally bounded as well. If $E_1, E_2 \subseteq X$ are totally bounded, then their union $E_1 \cup E_2$ is totally bounded.

Let Y be another set with a semimetric d_Y, and let f be a uniformly continuous mapping from X into Y. If $E \subseteq X$ is totally bounded with respect to d, then $f(E)$ is totally bounded in Y with respect to d_Y.

Let $d_1, \ldots, d_n$ be finitely many semimetrics on X, and remember that

$$d'(x,y) = \max_{1 \le j \le n} d_j(x,y) \tag{4.2.5}$$

defines a semimetric on X, as in Section 1.5. If $A \subseteq X$, then

$$\operatorname{diam}_{d'} A \le \max_{1 \le j \le n} (\operatorname{diam}_{d_j} A). \tag{4.2.6}$$

Suppose that $E \subseteq X$ is totally bounded with respect to d_j for each $j = 1, \ldots, n$, and let us check that E is totally bounded with respect to d'.

Let $r > 0$ be given, and for each $j = 1, \ldots, n$, let $\mathcal{E}_j$ be a collection of finitely many subsets of X such that

$$\operatorname{diam}_{d_j} A_j \le r \tag{4.2.7}$$

for each $A_j \in \mathcal{E}_j$, and

$$E \subseteq \bigcup_{A_j \in \mathcal{E}_j} A_j. \tag{4.2.8}$$

Let $\mathcal{E}$ be the collection of subsets of X of the form

$$A = \bigcap_{j=1}^{n} A_j, \tag{4.2.9}$$

where $A_j \in \mathcal{E}_j$ for each $j = 1, \ldots, n$. If A is of this form, then

$$\operatorname{diam}_{d'} A \le \max_{1 \le j \le n} (\operatorname{diam}_{d_j} A) \le \max_{1 \le j \le n} (\operatorname{diam}_{d_j} A_j) \le r, \tag{4.2.10}$$

using (4.2.7) in the third step. We also have that

$$E \subseteq \bigcap_{j=1}^{n} \Big(\bigcup_{A_j \in \mathcal{E}_j} A_j \Big) = \bigcup_{A_1 \in \mathcal{E}_1} \cdots \bigcup_{A_n \in \mathcal{E}_n} \Big(\bigcap_{j=1}^{n} A_j \Big) = \bigcup_{A \in \mathcal{E}} A, \tag{4.2.11}$$

using (4.2.8) in the first step. Clearly $\mathcal{E}$ has only finitely many elements, because $\mathcal{E}_j$ has only finitely many elements for each j, as desired.

4.3 Total boundedness in topological groups

Let G be a topological group, and let E be a subset of G. Let us say that E is *left-invariant totally bounded* in G if for every open subset U of G that contains the identity element e there are finitely many elements $a_1, \ldots, a_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} (a_j\, U). \tag{4.3.1}$$

Similarly, E is *right-invariant totally bounded* in G if for every open subset U of G with $e \in U$ there are finitely many elements $b_1, \ldots, b_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} (U\, b_j). \tag{4.3.2}$$

It is easy to see that E is left-invariant totally bounded in G if and only if E^{-1} is right-invariant totally bounded in G. Equivalently, E is right-invariant totally bounded in G if and only if E^{-1} is left-invariant totally bounded in G.

If E is a compact subset of G, then E is both left and right-invariant totally bounded in G. More precisely, if U is an open subset of G with $e \in U$, then E

is covered by the families of left and right translates of U. This implies that E can be covered by finitely many left and right translates of U, by compactness.

If E is either left or right-invariant totally bounded, then every subset of E has the same property. If E_1, E_2 are subsets of G that are both left-invariant totally bounded in G, or both right-invariant totally bounded, then their union $E_1 \cup E_2$ has the same property.

If E is either left or right-invariant totally bounded, then the closure $\overline{E}$ of E in G has the same property. To see this, let an open subset U of G with $e \in U$ be given. Remember that there is an open subset V of G such that $e \in V$ and $\overline{V} \subseteq U$, as in Section 1.3. By hypothesis, E can be covered by finitely many left or right translates of V, as appropriate. This implies that $\overline{E}$ is covered by the corresponding translates of $\overline{V}$, and hence by the corresponding translates of U, as desired.

Suppose for the moment that the topology on G is determined by a semimetric d. If d is invariant under left translations, then E is left-invariant totally bounded if and only if E is totally bounded with respect to d. Similarly, if d is invariant under right translations, then E is right-invariant totally bounded if and only if E is totally bounded with respect to d.

Let Y be a set with a semimetric d_Y, and let f be a mapping from G into Y. If E is left-invariant totally bounded, and f is left-invariant uniformly continuous, as in Section 4.1, then $f(E)$ it totally bounded in Y with respect to d_Y. Similarly, if E is right-invariant totally bounded, and f is right-invariant uniformly continuous, then $f(E)$ is totally bounded in Y.

Let d_0 be a semimetric on G. If E is left-invariant totally bounded, and d_0 is left-invariant uniformly compatible with the topology on G, as in Section 4.1, then E is totally bounded with respect to d_0. This follows from the corresponding statement about left-invariant uniformly continuous mappings in the preceding paragraph. Similarly, if E is right-invariant totally bounded, and d_0 is right-invariant uniformly compatible with the topology on G, then E is totally bounded with respect to d_0.

Let $\mathcal{M}$ be a nonempty collection of semimetrics on G, and suppose that the topology on G is determined by $\mathcal{M}$, as in Section 1.5. Suppose for the moment that the elements of $\mathcal{M}$ are invariant under left translations. This implies that each element of $\mathcal{M}$ is left-invariant uniformly compatible with the topology on G, as in Section 4.1. if E is left-invariant totally bounded, then it follows that E is totally bounded with respect to every element of $\mathcal{M}$. Conversely, suppose that E is totally bounded with respect to every element of $\mathcal{M}$. This implies that E is totally bounded with respect to the maximum of any finite number of elements of $\mathcal{M}$, as in the previous section. Using this, one can check that E is right-invariant totally bounded. Similarly, if the elements of $\mathcal{M}$ are invariant under right translations, then E is right-invariant totally bounded if and only if E is totally bounded with respect to each element of $\mathcal{M}$.

4.4 U-Small sets

Let G be a topological group, and let U be an open subset of G that contains the identity element e. Let us say that a subset A of G is *left-invariant U-small* if

$$A \subseteq a\,U \tag{4.4.1}$$

for every $a \in A$. Equivalently, this means that

$$a^{-1}\,A \subseteq U \tag{4.4.2}$$

for every $a \in A$, which is the same as saying that

$$A^{-1}\,A \subseteq U. \tag{4.4.3}$$

Similarly, A is *right-invariant U-small* if

$$A \subseteq U\,a \tag{4.4.4}$$

for every $a \in A$. This means that

$$A\,a^{-1} \subseteq U \tag{4.4.5}$$

for every $a \in A$, so that

$$A\,A^{-1} \subseteq U. \tag{4.4.6}$$

Observe that A is left-invariant U-small if and only if A^{-1} is right-invariant U-small. Thus A is right-invariant U-small if and only if A^{-1} is left-invariant U-small. Of course,

$$(A\,B)^{-1} = B^{-1}\,A^{-1} \tag{4.4.7}$$

for all subsets A, B of G, which implies that $A^{-1}\,A$ and $A\,A^{-1}$ are symmetric about e. It follows that A is left or right-invariant U-small if and only if A is left or right-invariant U^{-1}-small, respectively.

If A is left-invariant U-small, then every left translate of A is left-invariant U-small. Similarly, if A is right-invariant U-small, then every right translate of A is right-invariant U-small. If A is left or right-invariant U-small, then every subset of A has the same property.

Let V be an open subset of G such that $e \in V$ and

$$V^{-1}\,V \subseteq U, \tag{4.4.8}$$

so that V is left-invariant U-small. If E is a left-invariant totally bounded subset of G, then E is contained in the union of finitely many left translates of V. Each of these left translates of V is left-invariant U-small, so that E is contained in the union of finitely many left-invariant U-small sets. In the other direction, if E is contained in the union of finitely many left-invariant U-small sets, then E is contained in the union of finitely many left translates of U. This uses the fact that every left-invariant U-small set is contained in a left translate of U. It follows that E is left-invariant totally bounded if and only if for every open subset U of G with $e \in U$, E is contained in the union of finitely many left-invariant U-small sets.

Similarly, E is right-invariant totally bounded if and only if for every open subset U of G with $e \in U$, E is contained in the union of finitely many right-invariant U-small sets. This uses the fact that for each such U there is an open subset W of G such that $e \in W$ and

$$W\,W^{-1} \subseteq U, \tag{4.4.9}$$

so that W is right-invariant U-small. If E is right-invariant totally bounded, then E is contained in the union of finitely many right translates of W, each of which is right-invariant U-small.

Let H be a subgroup of G, equipped with the topology induced by the one on G. If U is an open subset of G that contains e, then $U \cap H$ is a relatively open set that contains e, and every relatively open subset of H that contains e is of this form. If A is a subset of H, then A is left-invariant U-small in G if and only if A is left-invariant $(U \cap H)$-small in H. Similarly, A is right-invariant U-small in G if and only if A is right-invariant $(U \cap H)$-small in H. It follows that a subset E of H is left or right-invariant totally bounded in H if and only if E is left or right-invariant totally bounded in G, respectively.

Let d_0 be a semimetric on G that is compatible with the topology on G. Also let r be a positive real number, so that $B_{d_0}(e, r)$ is an open subset of G that contains e. If d_0 is invariant under left translations, then a subset A of G is left-invariant $B_{d_0}(e, r)$ small if and only if

$$d_0(x, y) < r \tag{4.4.10}$$

for every $x, y \in A$. Similarly, if d_0 is invariant under right translations, then A is right-invariant $B_{d_0}(e, r)$-small if and only if (4.4.10) holds for every $x, y \in A$.

Let $U_1, \ldots, U_n$ be finitely many open subsets of G, each containing e. Thus $\bigcap_{j=1}^n U_j$ is an open set that contains e too. If a subset A of G is left-invariant U_j-small for each $j = 1, \ldots, n$, then A is left-invariant $\left(\bigcap_{j=1}^n U_j\right)$-small. Similarly, if A is right-invariant U_j-small for each $j = 1, \ldots, n$, then A is right-invariant $\left(\bigcap_{j=1}^n U_j\right)$-small.

4.5 Uniform continuity and open subgroups

Let G be a topological group, let Y be a set with a semimetric d_Y, and let f be a mapping from G into Y. Suppose for the moment that f is left-invariant uniformly continuous, as in Section 4.1, and that d_Y is a semi-ultrametric on Y. Let $\epsilon > 0$ be given, and let U be an open subset of G such that $e \in U$ and

$$d_Y(f(a), f(a\,x)) < \epsilon \tag{4.5.1}$$

for every $a \in G$ and $x \in U$. We may as well ask that U be symmetric about e, since otherwise we can replace U with $U \cap U^{-1}$. Let $a \in G$ and $x_1, \ldots, x_n \in U$

be given, and observe that

$$d_Y(f(a\,x_1\cdots x_{j-1}), f(a\,x_1\cdots x_{j-1}\,x_j)) < \epsilon \tag{4.5.2}$$

for each $j = 1, \ldots, n$, by (4.5.1). This implies that

$$d_Y(f(a), f(a\,x_1\cdots x_n)) < \epsilon, \tag{4.5.3}$$

by the ultrametric version of the triangle inequality. Thus, for each positive integer n, (4.5.1) holds for every $a \in G$ and $x \in U^n$. Here $U^n = U\cdots U$, with n U's on the right side, as in Section 3.3. Put

$$U_0 = \bigcup_{n=1}^{\infty} U^n, \tag{4.5.4}$$

which is an open subgroup of G, as in Section 3.3 again. It follows that (4.5.1) holds for every $a \in G$ and $x \in U_0$.

Let $\mathcal{B}_0$ be the collection of all open subgroups of G. If A is an open subgroup of G and x is an element of G, then $x\,A\,x^{-1}$ is an open subgroup of G as well. This implies that $\mathcal{B}_0$ is nice, in the sense of Section 3.5. Note that the intersection of finitely many elements of $\mathcal{B}_0$ is also an element of $\mathcal{B}_0$. Let τ_0 be the topology $\tau_L(\mathcal{B}_0) = \tau_R(\mathcal{B}_0)$ on G associated to $\mathcal{B}_0$ as in Section 3.5. Remember that the elements of $\mathcal{B}_0$ are open sets with respect to τ_0, and that $\mathcal{B}_0$ forms a local sub-base for τ_0 at e, by construction. In this situation, $\mathcal{B}_0$ is a local base for τ_0 at e, because $\mathcal{B}_0$ is closed under finite intersections. Of course, the given topology τ on B is automatically at least as strong as τ_0, because the elements of $\mathcal{B}_0$ are open sets with respect to τ. As before, G is a topological group with respect to τ_0, because $\mathcal{B}_0$ is nice.

Let f be a mapping from G into Y again, and suppose that d_Y is a semi-ultrametric on Y. If f is left-invariant uniformly continuous with respect to τ, then f is left-invariant uniformly continuous with respect to τ_0, by the earlier argument. Similarly, if f is right-invariant uniformly continuous with respect to τ, then f is right-invariant uniformly continuous with respect to τ_0. This can be shown in essentially the same way, or by reducing to the previous case applied to the mapping (4.1.3).

Now let d_Y be any semimetric on Y, and let $d_{Y,u}$ be the corresponding semi-ultrametrification of d_Y on Y, as in (3.8.1). Thus $d_{Y,u} \le d_Y$ on Y, as in (3.8.2). If a mapping f from G into Y is left or right-invariant uniformly continuous with respect to τ on G and d_Y on Y, then f is left or right-invariant uniformly continuous with respect to τ on G and $d_{Y,u}$ on Y, as appropriate. This implies that f is left or right-invariant uniformly continuous with respect to τ_0 on G and $d_{Y,u}$ on Y, as appropriate, by the remarks in the preceding paragraph.

4.6 Invariantization of semimetrics

Let G be a group, and let d be a semimetric on G. Put

$$d_L(x,y) = \sup_{a\in G} d(a\,x, a\,y) \tag{4.6.1}$$

for each $x, y \in G$, where the supremum on the right is defined as a nonnegative extended real number. Similarly, put

$$d_R(x, y) = \sup_{a \in G} d(x\,a, y\,a) \tag{4.6.2}$$

for every $x, y \in G$, where the supremum on the right is also defined as a nonnegative extended real number. It is easy to see that (4.6.1) and (4.6.2) satisfy the requirements of a semimetric on G, except that they may take values in $[0, \infty]$. Of course,

$$d(x, y) \le d_L(x, y),\ d_R(x, y) \tag{4.6.3}$$

for every $x, y \in G$. If

$$d(x, y) \le A \tag{4.6.4}$$

for some $A \ge 0$ and every $x, y \in G$, then

$$d_L(x, y),\ d_R(x, y) \le A \tag{4.6.5}$$

for every $x, y \in G$. By construction, (4.6.1) is invariant under left translations on G, and (4.6.2) is invariant under right translations on G. If d is a semi-ultrametric on G, then (4.6.1) and (4.6.2) satisfy the ultrametric version of the triangle inequality.

Suppose that G is a topological group with respect to a topology τ, and that d is left-invariant uniformly compatible with τ, as in Section 4.1. This means that for each $\epsilon > 0$ there is an open subset $U(\epsilon)$ of G such that $e \in U(\epsilon)$ and

$$d(x, y) < \epsilon \tag{4.6.6}$$

for every $x, y \in G$ such that $y \in x\,U(\epsilon)$. If $x, y \in G$ satisfy $y \in x\,U(\epsilon)$, then $a\,y \in a\,x\,U(\epsilon)$ for every $a \in G$, so that

$$d(a\,x, a\,y) < \epsilon. \tag{4.6.7}$$

It follows that

$$d_L(x, y) \le \epsilon \tag{4.6.8}$$

for every $x, y \in G$ such that $y \in x\,U(\epsilon)$. If (4.6.1) is finite for every $x, y \in G$, so that (4.6.1) defines a semimetric on G, then this semimetric is left-invariant uniformly compatible with τ on G. Conversely, if (4.6.1) is left-invariant uniformly compatible with τ on G, then d is left-invariant uniformly compatible with τ on G, because of (4.6.3). There are analogous statements for (4.6.2) and right-invariant uniform compatibility.

Suppose for the moment that d is a semi-ultrametric on G that is invariant under left or right translations. If d is compatible with τ on G, then open balls in G centered at e with respect to d are open subgroups with respect to τ, as in Section 3.3. This implies that d is compatible with the topology τ_0 defined on G as in the previous section, using open subgroups of G with respect to τ.

Now let d be any semi-ultrametric on G. If d is left-invariant uniformly compatible with τ, and (4.6.1) is finite, then (4.6.1) is a semi-ultrametric on

G that is invariant under left translations and compatible with τ. This implies that (4.6.1) is compatible with τ_0 on G, as in the preceding paragraph. More precisely, (4.6.1) is left-invariant uniformly compatible with τ_0, because of invariance under left translations. It follows that d is left-invariant uniformly compatible with τ_0, because of (4.6.3). Similarly, if d is right-invariant uniformly compatible with τ, and (4.6.2) is finite, then (4.6.2) is a semi-ultrametric on G that is invariant under right translations and compatible with τ. This implies that (4.6.2) is compatible with τ_0 on G, and hence that d is right-invariant uniformly compatible with τ_0 on G, as before.

4.7 Invariance under conjugations

Let G be a group. If U is a subset of G, then put

$$C(U) = \{a \in G : a\,U\,a^{-1} = U\} = \{a \in G : a\,U = U\,a\}. \tag{4.7.1}$$

It is easy to see that $C(U)$ is a subgroup of G. If $C(U) = G$, then U is said to be *invariant under conjugations* on G. Note that

$$C(U_1) \cap C(U_2) \subseteq C(U_1 \cap U_2) \tag{4.7.2}$$

for all subsets U_1, U_2 of G, and in particular that $U_1 \cap U_2$ is invariant under conjugations when U_1 and U_2 have this property.

Let G be a topological group, and let U be an open subset of G that contains the identity element e and is invariant under conjugations. If A is any subset of G, then A is left-invariant U-small, as in Section 4.4, if and only if A is right-invariant U-small. Suppose that there is a local base for the topology of G at e consisting of open sets that are invariant under conjugations. In this case, the left and right-invariant uniform continuity conditions discussed in Section 4.1 are equivalent. Similarly, the left and right total boundedness conditions discussed in Section 4.3 are equivalent in this situation.

Let d be a semimetric on G, and put

$$L(d) = \{a \in G : d(a\,x, a\,y) = d(x,y) \text{ for every } x, y \in G\}, \tag{4.7.3}$$

$$R(d) = \{a \in G : d(x\,a, y\,a) = d(x,y) \text{ for every } x, y \in G\}. \tag{4.7.4}$$

One can check that these are subgroups of G. Similarly,

$$C(d) = \{a \in G : d(a\,x\,a^{-1},\, a\,y\,a^{-1}) = d(x,y) \text{ for every } x, y \in G\} \tag{4.7.5}$$

is a subgroup of G. If $C(d) = G$, then d is said to be *invariant under conjugations* on G. Observe that

$$L(d) \cap R(d) \subseteq C(d), \tag{4.7.6}$$

$$L(d) \cap C(d) \subseteq R(d), \tag{4.7.7}$$

$$R(d) \cap C(d) \subseteq L(d). \tag{4.7.8}$$

Thus

$$L(d) = R(d) \quad \text{when } C(d) = G, \tag{4.7.9}$$

$$L(d) = C(d) \quad \text{when } R(d) = G, \tag{4.7.10}$$

$$R(d) = C(d) \quad \text{when } L(d) = G. \tag{4.7.11}$$

In particular, if d is invariant under both left and right translations, then d is invariant under conjugations. If d is invariant under conjugations, then open and closed balls centered at e with respect to d are invariant under conjugations.

Let $\mathcal{M}$ be a nonempty collection of semimetrics on G, which determines a topology on G as in Section 1.5. By construction, the open balls centered at e with respect to elements of $\mathcal{M}$ form a local sub-base for this topology at e. Of course, one can get a local base for the topology at e by taking finite intersections of these balls. If the elements of $\mathcal{M}$ are invariant under conjugations, then these balls centered at e are invariant under conjugations, as in the preceding paragraph. In this case, the finite intersections of these balls are invariant under conjugations too.

Let G be a topological group again, and suppose that there is a local base for the topology of G at e consisting of open sets that are invariant under conjugations. Under these conditions, it is well known that there is a collection $\mathcal{M}$ of semimetrics on G that determines the given topology on G, and for which each element of $\mathcal{M}$ is invariant under both left and right translations. If there is also a local base for the topology of G at e with only finitely or countably many elements, then the topology on G is determined by a single semimetric that is invariant under both left and right translations.

4.8 Equicontinuity of conjugations

Let G be a topological group, and put

$$C_x(y) = x\,y\,x^{-1} \tag{4.8.1}$$

for every $x, y \in G$, as before. This defines an inner automorphism on G for each $x \in G$, which is also a homeomorphism. Let E be a subset of G, and put

$$\mathcal{C}(E) = \{C_x : x \in E\}. \tag{4.8.2}$$

Let us say that $\mathcal{C}(E)$ is *equicontinuous* at the identity element e if for every open subset W of G with $e \in W$ there is an open subset V of G such that $e \in V$ and

$$C_x(V) = x\,V\,x^{-1} \subseteq W \tag{4.8.3}$$

for every $x \in E$. If E has only finitely many elements, then this condition can be obtained from the continuity of C_x for each $x \in E$. If W is invariant under conjugations, then $V = W$ satisfies (4.8.3) for every $x \in G$. If there is a local base for the topology of G at e consisting of open sets that are invariant under conjugations, then it follows that $\mathcal{C}(G)$ is equicontinuous at e.

Conversely, suppose that $\mathcal{C}(G)$ is equicontinuous at e, and let us show that there is a local base for the topology of G at e consisting of open sets that are invariant under conjugations. Let an open subset W of G with $e \in W$ be given, so that there is an open subset V of G such that $e \in V$ and (4.8.3) holds for every $x \in G$. Equivalently, (4.8.3) says that

$$V \subseteq x^{-1} W x \tag{4.8.4}$$

for every $x \in G$. Of course,

$$\bigcap_{x \in G} (x^{-1} W x) \tag{4.8.5}$$

is automatically invariant under conjugations. This set contains e, because $e \in W$, and is contained in W, because we can take $x = e$ in the intersection. We also have that V is contained in (4.8.5), by (4.8.4). Let W_0 be the interior of (4.8.5), so that $e \in W_0$ and $W_0 \subseteq W$. It is easy to see that W_0 is invariant under conjugations, as desired.

Let E be a right-invariant totally bounded subset of G, and let us check that $\mathcal{C}(E)$ is equicontinuous at e. Let an open subset W of G with $e \in W$ be given, and let U_1, U_2, U_3 be open subsets of G that contain e and satisfy

$$U_1 \, U_2 \, U_3 \subseteq W. \tag{4.8.6}$$

In particular, this means that

$$y \, U_2 \, y^{-1} \subseteq W \tag{4.8.7}$$

for every $y \in U_1 \cap U_3^{-1}$. Because E is right-invariant totally bounded, there are finitely many elements $b_1, \ldots, b_n$ of G such that

$$E \subseteq \bigcup_{j=1}^{n} ((U_1 \cap U_3^{-1}) \, b_j), \tag{4.8.8}$$

as in Section 4.3. Put

$$V = \bigcap_{j=1}^{n} (b_j^{-1} \, U_2 \, b_j), \tag{4.8.9}$$

which is an open set that contains e. Let $x \in E$ be given, so that x can be expressed as $y \, b_j$ for some $y \in U_1 \cap U_3^{-1}$ and $1 \le j \le n$. Observe that

$$x \, V \, x^{-1} = y \, b_j \, V \, b_j^{-1} \, y^{-1} \subseteq y \, U_2 \, y^{-1} \subseteq W, \tag{4.8.10}$$

as desired.

4.9 Local compactness and total boundedness

A topological space X is said to be *locally compact* if for every $x \in X$ there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U$ and $U \subseteq K$. If

X is Hausdorff, then it is well known that compact subsets of X are closed sets. In this case, if $U \subseteq K \subseteq X$ and K is compact, then it follows that the closure $\overline{U}$ of U in X is contained in K. This implies that $\overline{U}$ is compact, because closed sets contained in compact sets are compact as well. Thus local compactness of a Hausdorff topological space X is often formulated equivalently as saying that for every $x \in X$ there is an open set $U \subseteq X$ such that $x \in U$ and $\overline{U}$ is compact.

Now let X be a set with a semimetric $d(\cdot, \cdot)$. Let us say that X is *locally totally bounded* if for every $x \in X$ there is an $r > 0$ such that the open ball $B(x, r)$ in X centered at x with radius r with respect to $d(\cdot, \cdot)$ is totally bounded with respect to $d(\cdot, \cdot)$. If X is locally compact with respect to the topology determined by $d(\cdot, \cdot)$, then X is locally totally bounded with respect to $d(\cdot, \cdot)$, because compact subsets of X are totally bounded, and subsets of totally bounded sets are totally bounded.

Let G be a topological group. In order to check that G is locally compact, it suffices to find an open set $U \subseteq G$ and a compact set $K \subseteq G$ such that $e \in U$ and $U \subseteq K$, by continuity of translations. If $\{e\}$ is a closed set in G, so that G is Hausdorff as a topological space, then this is the same as saying that there is an open set $U \subseteq G$ such that $e \in U$ and the closure $\overline{U}$ of U in G is compact.

Let us say that G is *locally totally bounded* as a topological group if there is an open set $U \subseteq G$ such that $e \in U$ and U is either left or right-invariant totally bounded in G, as a topological group. Of course, if U is left or right-invariant totally bounded in G, then U^{-1} is right or left-invariant totally bounded in G, respectively. This implies that $U \cap U^{-1}$ is both left and right-invariant totally bounded in G. Thus G is locally totally bounded as a topological group if and only if there is an open set $U \subseteq G$ such that $e \in U$ and U is both left and right totally bounded in G. In this situation, we get that for each $x \in G$, $x\,U$ is left-invariant totally bounded in G, $U\,x$ is right-invariant totally bounded in G, and hence $(x\,U) \cap (U\,x)$ is both left and right-invariant totally bounded in G.

If G is locally compact, then G is locally totally bounded as a topological group, because compact subsets of G are left and right-invariant totally bounded in G. If the topology on G is determined by a semimetric $d(\cdot, \cdot)$ that is invariant under left or right translations, then G is locally totally bounded as a topological group if and only if G is locally totally boounded with respect to $d(\cdot, \cdot)$. More precisely, this holds if and only if there is a positive real number r such that the open ball in G centered at e with radius r with respect to $d(\cdot, \cdot)$ is totally bounded with respect to $d(\cdot, \cdot)$. This condition automatically implies that every open ball in G with radius r with respect to $d(\cdot, \cdot)$ is totally bounded with respect to $d(\cdot, \cdot)$, using invariance of $d(\cdot, \cdot)$ under left or right translations, as appropriate.

4.10 Cauchy sequences and topological groups

Let X be a set, and let d_X be a semimetric on X. If $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in X with respect to d_X, then the set of x_j's, $j \in \mathbf{Z}_+$, is totally bounded with respect to d_X. More precisely, for each $r > 0$, all but finitely many terms in the sequence are contained in a ball of radius r.

Let Y be another set with a semimetric d_Y, and let f be a uniformly continuous mapping from X into Y. If a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is a Cauchy sequence with respect to d_X, then it is easy to see that $\{f(x_j)\}_{j=1}^\infty$ is a Cauchy sequence in Y with respect to d_Y.

Let G be a topological group, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of G. Let us say that $\{x_j\}_{j=1}^\infty$ satisfies the *left-invariant Cauchy condition* if for every open set $U \subseteq G$ with $e \in U$ there is a positive integer L such that

$$x_j^{-1}\, x_l \in U \tag{4.10.1}$$

for every $j, l \ge L$. Equivalently, (4.10.1) means that

$$x_l \in x_j\, U \tag{4.10.2}$$

for every $j, l \ge L$. Similarly, $\{x_j\}_{j=1}^\infty$ satisfies the *right-invariant Cauchy condition* if for every open set $U \subseteq G$ with $e \in U$ there is an $L \in \mathbf{Z}_+$ such that

$$x_l\, x_j^{-1} \in U \tag{4.10.3}$$

for every $j, l \ge L$. This is the same as saying that

$$x_l \in U\, x_j \tag{4.10.4}$$

for every $j, l \ge L$, as before. If G is commutative, then the left and right-invarant Cauchy conditions are the same. One can check that $\{x_j\}_{j=1}^\infty$ satisfies the left-invariant Cauchy condition if and only if $\{x_j^{-1}\}_{j=1}^\infty$ satisfies the right-invariant Cauchy condition.

Suppose that $\{x_j\}_{j=1}^\infty$ satisfies the left-invariant Cauchy condition. This implies that the set of x_j's, $j \in \mathbf{Z}_+$, is left-invariant totally bounded in G. If d is a left-invariant semimetric on G that is compatible with the topology on G at e, then $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to d. If $\mathcal{M}$ is a collection of left-invariant semimetrics on G that determines the same topology on G, then $\{x_j\}_{j=1}^\infty$ satisfies the left-invariant Cauchy condition in G as a topological group if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $\mathcal{M}$. Let Y be a set with a semimetric d_Y, and let f be a mapping from G into Y that is left-invariant uniformly continuous. If $\{x_j\}_{j=1}^\infty$ satisfies the left-invariant Cauchy condition, then $\{f(x_j)\}_{j=1}^\infty$ is a Cauchy sequence in Y with respect to d_Y. Similarly, let H be another topological group, and let ϕ be a continuous homomorphism from G into H. If $\{x_j\}_{j=1}^\infty$ satisfies the left-invariant Cauchy condition in G, then $\{\phi(x_j)\}_{j=1}^\infty$ satisfies the left-invariant Cauchy condition in H. Of course, there are analogous statements for the right-invariant Cauchy condition.

Chapter 5

Equicontinuity and isometrization

5.1 Pointwise equicontinuity

Let X be a nonempty topological space, and let Y be a set with a semimetric d_Y. Also let $\mathcal{E}$ be a collection of mappings from X into Y, and let x be an element of X. As usual, $\mathcal{E}$ is said to be *equicontinuous* at x if for every $\epsilon > 0$ there is an open set $U \subseteq X$ such that $x \in U$ and

$$d_Y(f(x), f(w)) < \epsilon \tag{5.1.1}$$

for every $f \in \mathcal{E}$ and $w \in U$. This condition implies that each $f \in \mathcal{E}$ is continuous at x, with respect to the topology determined on Y by d_Y. If $\mathcal{E}$ has only finitely many elements, each of which is continuous at x, then $\mathcal{E}$ is equicontinuous at x.

Suppose for the moment that each element of $\mathcal{E}$ is bounded on X with respect to d_Y on Y, as in Section 2.1. Let $\theta(f, g)$ be the supremum semimetric on the space $\mathcal{B}(X, Y)$ of bounded mappings from X into Y corresponding to d_Y, as before. Suppose that $\mathcal{E}$ is totally bounded as a subset of $\mathcal{B}(X, Y)$ with respect to θ, as in Section 4.2. This means that for each $r > 0$, $\mathcal{E}$ is contained in the union of finitely many sets with diameter less than or equal to r with respect to θ, as before. If each element of $\mathcal{E}$ is continuous at x, then it follows that $\mathcal{E}$ is equicontinuous at x, by a standard argument.

Now let $\mathcal{E}$ be any collection of mappings from X into Y that is equicontinuous at every $x \in X$. Let $\epsilon > 0$ be given, so that for each $x \in X$ there is an open set $U(x, \epsilon) \subseteq X$ such that $x \in U(x, \epsilon)$ and (5.1.1) holds for every $f \in \mathcal{E}$ and $w \in U(x, \epsilon)$. Suppose that X is compact, so that there are finitely many elements $x_1, \ldots, x_n$ of X such that

$$X \subseteq \bigcup_{j=1}^{n} U(x_j, \epsilon). \tag{5.1.2}$$

If $f, g \in \mathcal{E}$, then

$$d_Y(f(w), g(w)) < 2\,\epsilon + \max_{1 \le j \le n} d_Y(f(x_j), g(x_j)) \tag{5.1.3}$$

for every $w \in X$. More precisely, if $w \in X$, then $w \in U(x_j, \epsilon)$ for some j, $1 \le j \le n$, by (5.1.2). In this case, we have that

$$\begin{aligned} d_Y(f(w), g(w)) &\le d_Y(f(w), f(x_j)) + d_Y(f(x_j), g(x_j)) + d_Y(g(x_j), g(w)) \\ &< 2\,\epsilon + d_Y(f(x_j), g(x_j)), \end{aligned} \tag{5.1.4}$$

using (5.1.1) for f and g in the second step, with $x = x_j$. This implies (5.1.3), as desired. Note that the elements of $\mathcal{E}$ are bounded as mappings from X to Y in this situation. Using (5.1.4), we get that

$$\theta(f, g) \le 2\,\epsilon + \max_{1 \le j \le n} d_Y(f(x_j), g(x_j)), \tag{5.1.5}$$

by taking the supremum of the left side of (5.1.3) over $w \in X$.

Put

$$\mathcal{E}(x) = \{f(x) : f \in \mathcal{E}\} \tag{5.1.6}$$

for each $x \in X$, which is a subset of Y. Suppose now that $\mathcal{E}(x)$ is totally bounded with respect to d_Y on Y for each $x \in X$, as in Section 4.2 again. In particular, this holds automatically when Y is totally bounded with respect to d_Y. This implies that for each $j = 1, \ldots, n$, $\mathcal{E}(x_j)$ can be expressed as the union of finitely many subsets with diameter less than or equal to ϵ. Equivalently, this means that for each $j = 1, \ldots, n$, $\mathcal{E}$ can be expressed as the union of finitely many subsets, where

$$d_Y(f(x_j), g(x_j)) \le \epsilon \tag{5.1.7}$$

for each f, g in the same subset. Using this, one can express $\mathcal{E}$ as the union of finitely many subsets, where

$$\max_{1 \le j \le n} d_Y(f(x_j), g(x_j)) \le \epsilon \tag{5.1.8}$$

for each f, g in the same subset. Combining this with (5.1.5), we get that

$$\theta(f, g) \le 3\,\epsilon \tag{5.1.9}$$

when f, g are in the same one of these finitely many subsets of $\mathcal{E}$. This means that $\mathcal{E}$ is totally bounded as a subset of $\mathcal{B}(X, Y)$ with respect to θ, as in the usual Arzela–Ascoli type of arguments.

5.2 Uniform equicontinuity

Let X, Y be nonempty sets with semimetrics d_X, d_Y, respectively, let A be a subset of X, and let $\mathcal{E}$ be a collection of mappings from X into Y. Let us say that $\mathcal{E}$ is *uniformly equicontinuous along* A if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d_Y(f(x), f(w)) < \epsilon \tag{5.2.1}$$

for every $f \in \mathcal{E}$, $x \in A$, and $w \in X$ with $d_X(x, w) < \delta$. This implies that each element of $\mathcal{E}$ is uniformly continuous along A as a mapping from X into Y, as in Section 2.2. If $\mathcal{E}$ has only finitely many elements, each of which is uniformly continuous as a mapping from X into Y along A, then $\mathcal{E}$ is uniformly equicontinuous along A.

If $A = X$, then we may simply say that $\mathcal{E}$ is uniformly equicontinuous on X. If $\mathcal{E}$ is uniformly equicontinuous along a subset A of X, then the restrictions of the elements of $\mathcal{E}$ to A are uniformly equicontinuous on A, with respect to the restriction of d_X to A.

If A consists of a single point, then uniform equicontinuity along A is the same as equicontinuity at that point, as in the previous section. If $\mathcal{E}$ is uniformly equicontinuous along any subset A of X, then $\mathcal{E}$ is equicontinuous at each point in A. If A has only finitely many elements, and $\mathcal{E}$ is equicontinuous at each point in A, then $\mathcal{E}$ is uniformly equicontinuous along A.

Suppose for the moment that each element of $\mathcal{E}$ is bounded on X with respect to d_Y. Let $\theta(f, g)$ be the supremum semimetric on the space $\mathcal{B}(X, Y)$ of bounded mappings from X into Y corresponding to d_Y again, as in Section 2.1. Suppose that $\mathcal{E}$ is totally bounded with respect to θ, as in Section 4.2. If each element of $\mathcal{E}$ is uniformly continuous along a subset A of X as a mapping from X into Y, then one can check that $\mathcal{E}$ is uniformly equicontinuous along A. This is analogous to the argument for pointwise equicontinuity mentioned in the previous section.

Let $\mathcal{E}$ be any uniformly equicontinuous collection of mappings from X into Y. Let $\epsilon > 0$ be given, so that there is a $\delta > 0$ such that (5.2.1) holds for every $f \in \mathcal{E}$ and $x, w \in X$ with $d_X(x, w) < \delta$. Suppose now that X is totally bounded with respect to d_X, so that there are finitely many elements $x_1, \ldots, x_n$ of X such that

$$X \subseteq \bigcup_{j=1}^{n} B_X(x_j, \delta). \tag{5.2.2}$$

Here $B_X(x, r)$ denotes the open ball in X centered at a point $x \in X$ with radius $r > 0$ with respect to d_X. Under these conditions, we have that

$$d_Y(f(w), g(w)) < 2\,\epsilon + \max_{1 \le j \le n} d_Y(f(x_j), g(x_j)) \tag{5.2.3}$$

for every $f, g \in \mathcal{E}$ and $w \in X$. Indeed, if $w \in X$, then $d_X(x_j, w) < \delta$ for some $j = 1, \ldots, n$, by (5.2.2). This implies that (5.1.4) holds in this situation, now using (5.2.1) for f and g in the second step, with $x = x_j$. It follows that (5.2.3) holds for every $f, g \in \mathcal{E}$ and $w \in X$, as desired. Note that the elements of $\mathcal{E}$ are bounded as mappings from X to Y in this case. As before, we can take the supremum of the left side of (5.2.3) over $w \in X$ to get that

$$\theta(f, g) \le 2\,\epsilon + \max_{1 \le j \le n} d_Y(f(x_j), g(x_j)) \tag{5.2.4}$$

for every $f, g \in \mathcal{E}$.

Let $\mathcal{E}(x) \subseteq Y$ be as in (5.1.6) for each $x \in X$ again. Suppose that $\mathcal{E}(x)$ is totally bounded in Y with respect to d_Y for every $x \in X$, as in Section 4.2, in addition to the hypotheses in the preceding paragraph. In particular, this means that $\mathcal{E}(x_j)$ is totally bounded in Y for each $j = 1, \ldots, n$. Using this and (5.2.4), we get that $\mathcal{E}$ can be expressed as a union of finitely many subsets, each of which has diameter less than or equal to $3\,\epsilon$ with respect to θ, as in the previous section. This implies that $\mathcal{E}$ is totally bounded as a subset of $\mathcal{B}(X, Y)$ with respect to θ, as before.

Now let $\mathcal{E}$ be any collection of mappings from X into Y, and suppose that $\mathcal{E}$ is equicontinuous at each point in a subset A of X, with respect to the topology determined on X by d_X. This means that for each $\epsilon > 0$ and $x \in X$ there is a $\delta > 0$ such that (5.2.1) holds for every $f \in \mathcal{E}$ and $w \in X$ with $d_X(x, w) < \delta$. If A is compact with respect to the topology determined by d_X, then it follows that $\mathcal{E}$ is uniformly equicontinuous along A, by a standard argument. This is very similar to the argument used to show that if a mapping f from X into Y is continuous at each point in A, then f is uniformly continuous along A, as in Section 2.2.

5.3 Some reformulations

Let X, Y be sets with semimetrics d_X, d_Y, respectively, and let $\mathcal{E}$ be a collection of mappings from X into Y. Also let α be a monotonically increasing nonnegative real-valued function on $[0, \infty)$ such that $\alpha(0) = 0$, and suppose that

$$\alpha(d_Y(f(x), f(x'))) \le d_X(x, x') \tag{5.3.1}$$

for every $x, x' \in X$. If

$$d_Y(f(x), f(x')) \ge \epsilon \tag{5.3.2}$$

for some $\epsilon > 0$, $f \in \mathcal{E}$, and $x, x' \in X$, then we have that

$$d_X(x, x') \ge \alpha(\epsilon), \tag{5.3.3}$$

by (5.3.1). Equivalently, if

$$d_X(x, x') < \alpha(\epsilon) \tag{5.3.4}$$

for some $\epsilon > 0$ and $x, x' \in X$, then

$$d_Y(f(x), f(x')) < \epsilon \tag{5.3.5}$$

for every $f \in \mathcal{E}$. If $\alpha > 0$ on $(0, \infty)$, then it follows that $\mathcal{E}$ is uniformly equicontinuous on X.

Put

$$\alpha_{\mathcal{E}}(\epsilon) = \inf\{d_X(x, x') : x, x' \in X,\ d_Y(f(x), f(x')) \ge \epsilon \text{ for some } f \in \mathcal{E}\} \tag{5.3.6}$$

for each $\epsilon > 0$. This is interpreted as being $+\infty$ when there are no $x, x' \in X$ such that (5.3.2) holds for some $f \in \mathcal{E}$. Let us put $\alpha_{\mathcal{E}}(0) = 0$, which is the same as (5.3.6) when X and $\mathcal{E}$ are nonempty. Observe that α increases monotonically on $[0, \infty)$, because the set whose infimum is taken on the right side of (5.3.6) gets smaller as ϵ increases. By construction,

$$\alpha_{\mathcal{E}}(d_Y(f(x), f(x'))) \le d_X(x, x') \tag{5.3.7}$$

for every $x, x' \in X$ and $f \in \mathcal{E}$. It is easy to see that $\mathcal{E}$ is uniformly equicontinuous on X if and only if

$$\alpha_{\mathcal{E}}(\epsilon) > 0 \tag{5.3.8}$$

for every $\epsilon > 0$. Of course, one can replace $\alpha_{\mathcal{E}}(\epsilon)$ with its minimum with any fixed positive real number to get a function that is finite on $[0, \infty)$.

Suppose that X, Y, and $\mathcal{E}$ are nonempty, and put

$$\beta^-_{\mathcal{E}}(r) = \sup\{d_Y(f(x), f(x')) : f \in \mathcal{E},\ x, x' \in X,\ d_X(x, x') < r\} \tag{5.3.9}$$

for every positive real number r, and

$$\beta_{\mathcal{E}}(r) = \sup\{d_Y(f(x), f(x')) : f \in \mathcal{E},\ x, x' \in X,\ d_X(x, x') \le r\} \tag{5.3.10}$$

for every nonnegative real number r. These suprema are defined as nonnegative extended real numbers, and the sets whose suprema are being taken are nonempty, because one can take $x = x'$. These sets get larger as r increases, so that (5.3.9) and (5.3.10) increase monotonically in r. If $0 \le r < t < \infty$, then

$$\beta_{\mathcal{E}}(r) \le \beta^-_{\mathcal{E}}(t) \le \beta_{\mathcal{E}}(t). \tag{5.3.11}$$

Note that $\beta_{\mathcal{E}}(0) = 0$ automatically when d_X is a metric on X. If $\mathcal{E}$ is uniformly equicontinuous on X, then

$$\beta_{\mathcal{E}}(r) < \infty \text{ when } r \ge 0 \text{ is sufficiently small} \tag{5.3.12}$$

and

$$\beta_{\mathcal{E}}(0) = \lim_{r \to 0+} \beta_{\mathcal{E}}(r) = 0. \tag{5.3.13}$$

In the other direction, if

$$\beta^-_{\mathcal{E}}(r) < \infty \text{ when } r > 0 \text{ is sufficiently small} \tag{5.3.14}$$

and

$$\lim_{r \to 0+} \beta^-_{\mathcal{E}}(r) = 0, \tag{5.3.15}$$

then $\mathcal{E}$ is uniformly equicontinuous on X. Of course, the equivalence of (5.3.12), (5.3.13) and (5.3.14), (5.3.15) follows directly from (5.3.11).

Let β be a nonnegative extended real-valued function on $[0, \infty)$, and suppose that

$$d_Y(f(x), f(x')) \le \beta(d_X(x, x')) \tag{5.3.16}$$

for every $f \in \mathcal{E}$ and $x, x' \in X$. If

$$\beta(r) < \infty \text{ when } r \ge 0 \text{ is sufficiently small} \tag{5.3.17}$$

and

$$\beta(0) = \lim_{r \to 0+} \beta(r) = 0, \tag{5.3.18}$$

then $\mathcal{E}$ is uniformly equicontinuous on X. Put

$$\widetilde{\beta}(r) = \sup\{\beta(t) : 0 \le t \le r\}, \tag{5.3.19}$$

for each nonnegative real number r, where the supremum is defined as a non-negative extended real number. Note that $\widetilde{\beta}$ increases monotonically on $[0, \infty)$, by construction, and that

$$\beta_{\mathcal{E}}(r) \le \widetilde{\beta}(r) \tag{5.3.20}$$

for every $r \ge 0$, because of (5.3.16). If (5.3.17) and (5.3.18) hold, then

$$\widetilde{\beta}(r) < \infty \text{ when } r \ge 0 \text{ is sufficiently small} \tag{5.3.21}$$

and

$$\widetilde{\beta}(0) = \lim_{r \to 0+} \widetilde{\beta}(r) = 0. \tag{5.3.22}$$

This implies that (5.3.12) and (5.3.13) hold, because of (5.3.20). Of course, (5.3.16) holds automatically when $\beta = \beta_{\mathcal{E}}$, in which case (5.3.17) and (5.3.18) correspond to (5.3.12) and (5.3.13).

5.4 Connection with semi-ultrametrification

Let X, Y be sets with semimetrics d_X, d_Y, respectively, and let $\mathcal{E}$ be a collection of mappings from X into Y again. Suppose that $\mathcal{E}$ is equicontinuous with respect to d_X, d_Y, so that for each $\epsilon > 0$ there is a $\delta(\mathcal{E}, \epsilon) > 0$ such that

$$d_Y(f(x), f(x')) < \epsilon \tag{5.4.1}$$

for every $f \in \mathcal{E}$ and $x, x' \in X$ with $d_X(x, x') < \delta(\mathcal{E}, \epsilon)$. Let $d_{X,u}$, $d_{Y,u}$ be the semi-ultrametrifications of d_X, d_Y on X, Y, respectively, as in Section 3.8. If $f \in \mathcal{E}$, then it follows that

$$d_{Y,u}(f(x), f(x')) < \epsilon \tag{5.4.2}$$

for every $x, x' \in X$ with $d_{X,u}(x, x') < \delta(\mathcal{E}, \epsilon)$, as in Section 3.9. Thus $\mathcal{E}$ is also uniformly equicontinuous with respect to $d_{X,u}$, $d_{Y,u}$ on X, Y, respectively.

Alternatively, let α be a monotonically increasing nonnegative real-valued function on $[0, \infty)$ such that $\alpha(0) = 0$ and

$$\alpha(d_Y(f(x), f(x'))) \le d_X(x, x') \tag{5.4.3}$$

for every $f \in \mathcal{E}$ and $x, x' \in X$. This implies that

$$\alpha(d_{Y,u}(f(x), f(x'))) \le d_{X,u}(x, x') \tag{5.4.4}$$

for every $f \in \mathcal{E}$ and $x, x' \in X$, as in Section 3.10. If $\mathcal{E}$ is uniformly equicontinuous with respect to d_X, d_Y on X, Y, respectively, then we can choose α so that $\alpha > 0$ on $(0, \infty)$, as in the previous section. In this case, (5.4.4) implies that $\mathcal{E}$ is uniformly equicontinuous with respect to $d_{X,u}$, $d_{Y,u}$ on X, Y, respectively, as before.

Let r be a positive real number, and let $\beta_{\mathcal{E}}^-(r)$ be as in (5.3.9). If $f \in \mathcal{E}$ and $x, x' \in X$ satisfy $d_{X,u}(x, x') < r$, then

$$d_{Y,u}(f(x), f(x')) \le \beta_{\mathcal{E}}^-(r), \tag{5.4.5}$$

as in Section 3.10. Put

$$\beta_{\mathcal{E},u}^-(r) = \sup\{d_{Y,u}(f(x), f(x')) : f \in \mathcal{E},\ x, x' \in X,\ d_{X,u}(x, x') < r\}, \tag{5.4.6}$$

where the supremum is defined as a nonnegative extended real number. This is the analogue of $\beta_{\mathcal{E}}^-(r)$ using $d_{X,u}$, $d_{Y,u}$ instead of d_X, d_Y, and we have that

$$\beta_{\mathcal{E},u}^-(r) \le \beta_{\mathcal{E}}^-(r), \tag{5.4.7}$$

by (5.4.5). If $\mathcal{E}$ is uniformly equicontinuous with respect to d_X, d_Y on X, Y, respectively, then (5.3.14) and (5.3.15) hold, as in the previous section. This implies that

$$\beta_{\mathcal{E},u}^-(r) < \infty \text{ when } r > 0 \text{ is sufficiently small} \tag{5.4.8}$$

and

$$\lim_{r \to 0+} \beta_{\mathcal{E},u}(r) = 0, \tag{5.4.9}$$

by (5.4.7). It follows that $\mathcal{E}$ is uniformly equicontinuous with respect to $d_{X,u}$, $d_{Y,u}$ on X, Y, respectively, as before.

Let β be a nonnegative extended real-valued function on $[0, \infty)$, and suppose that (5.3.16) holds for every $f \in \mathcal{E}$ and $x, x' \in X$. Also let $\widetilde{\beta}(r)$ be defined for $r \ge 0$ as in (5.3.19), and put

$$\widetilde{\beta}^+(t) = \inf\{\widetilde{\beta}(r) : t < r < \infty\} \tag{5.4.10}$$

for every nonnegative real number t, where the infimum is defined as a nonnegative extended real number. If $f \in \mathcal{E}$, then

$$d_{Y,u}(f(x), f(x')) \le \widetilde{\beta}^+(d_X(x, x')) \tag{5.4.11}$$

for every $x, x' \in X$, as in Section 3.10 again. If β satisfies (5.3.17) and (5.3.18), then $\widetilde{\beta}$ satisfies (5.3.21) and (5.3.22), as before. In this case,

$$\widetilde{\beta}^+(t) < \infty \text{ when } t \ge 0 \text{ is sufficiently small,} \tag{5.4.12}$$

and

$$\widetilde{\beta}^+(0) = \lim_{t \to 0+} \widetilde{\beta}^+(t) = 0. \tag{5.4.13}$$

If $\mathcal{E}$ is uniformly equicontinuous with respect to d_X, d_Y on X, Y, respectively, then one can find β satisfying (5.3.16), (5.3.17), and (5.3.18), as in the previous section. We have also seen that (5.4.11), (5.4.12), and (5.4.13) imply that $\mathcal{E}$ is uniformly equicontinuous with respect to $d_{X,u}$, $d_{Y,u}$ on X, Y, respectively.

5.5 Equicontinuity on topological groups

Let G be a topological group, and let Y be a set with a semimetric d_Y. Also let $\mathcal{E}$ be a collection of mappings from X into Y, and let A be a subset of G. Let us say that $\mathcal{E}$ is *left-invariant uniformly equicontinuous along* A if for every $\epsilon > 0$ there is an open subset U of G that contains the identity element e and has the property that

$$d_Y(f(a), f(a\,x)) < \epsilon \tag{5.5.1}$$

for every $f \in \mathcal{E}$, $a \in A$, and $x \in U$. Similarly, $\mathcal{E}$ is *right-invariant uniformly equicontinuous along* A if for every $\epsilon > 0$ there is an open subset U of G with $e \in U$ and

$$d_Y(f(a), f(x\,a)) < \epsilon \tag{5.5.2}$$

for every $f \in \mathcal{E}$, $a \in A$, and $x \in U$. We may simply say that $\mathcal{E}$ is left or right-invariant equicontinuous on G, as appropriate, when A is the whole group G.

If A has only one element, then left and right-invariant uniform equicontinuity along A are both the same as equicontinuity at that point, as in Section 5.1. If $\mathcal{E}$ is left or right-invariant uniformly equicontinuous along any subset A of G, then $\mathcal{E}$ is equicontinuous at each point in A. If A has only finitely many elements, and $\mathcal{E}$ is equicontinuous at each point in A, then $\mathcal{E}$ is both left and right-invariant uniformly equicontinuous along A.

If $\mathcal{E}$ is left or right-invariant uniformly equicontinuous along a subset A of G, then each element of $\mathcal{E}$ if left or right-invariant uniformly continuous along A, as appropriate, as in Section 4.1. If $\mathcal{E}$ has only finitely many elements, each of which is left-invariant uniformly continuous along A, then $\mathcal{E}$ is left-invariant uniformly equicontinuous along A, and similarly for the right-invariant case.

Put

$$\widetilde{f}(x) = f(x^{-1}) \tag{5.5.3}$$

for every $f \in \mathcal{E}$ and $x \in G$, and

$$\widetilde{\mathcal{E}} = \{\widetilde{f} : f \in \mathcal{E}\}. \tag{5.5.4}$$

One can check that $\mathcal{E}$ is left-invariant uniformly equicontinuous along a subset A of G if and only if $\widetilde{\mathcal{E}}$ is right-invariant uniformly equicontinuous along A^{-1}. Similarly, $\mathcal{E}$ is right-invariant uniformly equicontinuous along A if and only if $\widetilde{\mathcal{E}}$ is left-invariant uniformly equicontinuous along A^{-1}.

Suppose for the moment that the topology on G is determined by a semimetric d, and let A be a subset of G again. If d is invariant under left translations on G, then $\mathcal{E}$ is left-invariant uniformly equicontinuous along A if and only if $\mathcal{E}$ is uniformly equicontinuous along A with respect to d, as in Section 5.2. Similarly, if d is invariant under right translations on G, then $\mathcal{E}$ is right-invariant uniformly equicontinuous along A if and only if $\mathcal{E}$ is uniformly equicontinuous along A with respect to d.

If A is a compact subset of G, and $\mathcal{E}$ is equicontinuous at each point in A, then $\mathcal{E}$ is both left and right-invariant uniformly equicontinuous along A. This is very similar to the case of a single mapping, as in Section 4.1.

Suppose for the moment that there is a local base for the topology of G at e consisting of open sets that are invariant under conjugation, as in Section 4.7. In this situation, left and right-invariant uniform equicontinuity conditions are equivalent.

Suppose for the moment again that every element of $\mathcal{E}$ is bounded on G with respect to d_Y, and let $\theta(f,g)$ be the supremum semimetric on the space $\mathcal{B}(G,Y)$ of bounded mappings from G into Y corresponding to d_Y, as in Section 2.1. Suppose also that $\mathcal{E}$ is totally bounded with respect to θ, as in Section 4.2. If every element of $\mathcal{E}$ is left-invariant uniformly continuous along a subset A of G, then one can verify that $\mathcal{E}$ is left-invariant uniformly equicontinuous along A. This is analogous to arguments mentioned in Sections 5.1 and 5.2. Similarly, if every element of $\mathcal{E}$ is right-invariant uniformly continuous along A, then $\mathcal{E}$ is right-invariant uniformly continuous along A in this case.

Suppose now that G is left-invariant totally bounded, as in Section 4.3. This is equivalent to right-invariant total boundedness, because G is automatically symmetric about e. It follows that there is a local base for the topology of G at e that is invariant under conjugations, as in Sections 4.7 and 4.8. Suppose that $\mathcal{E}$ is left-invariant uniformly equicontinuous on G, and let $\epsilon > 0$ be given. Thus there is an open subset U of G such that $e \in U$ and (5.5.1) holds for every $f \in \mathcal{E}$, $a \in G$, and $x \in U$. Left-invariant total boundedness implies that there are finitely many elements $a_1, \ldots, a_n$ of G such that

$$G = \bigcup_{j=1}^{n} (a_j\, U). \tag{5.5.5}$$

Let us check that

$$d_Y(f(w), g(w)) < 2\,\epsilon + \max_{1 \le j \le n} d_Y(f(a_j), g(a_j)) \tag{5.5.6}$$

for every $f, g \in \mathcal{E}$ and $w \in G$. If $w \in G$, then $w \in a_j\, U$ for some j, $1 \le j \le n$, and hence

$$\begin{aligned} d_Y(f(w), g(w)) &\le d_Y(f(w), f(a_j)) + d_Y(f(a_j), g(a_j)) + d_Y(g(a_j), g(w)) \\ &< 2\,\epsilon + d_Y(f(a_j), g(a_j)) \end{aligned} \tag{5.5.7}$$

for every $f, g \in \mathcal{E}$. This uses (5.5.1) for f and g in the second step, with $a = a_j$. This implies (5.5.6). Note that the elements of $\mathcal{E}$ are bounded as mappings from G into Y under these conditions. It follows that

$$\theta(f,g) \le 2\,\epsilon + \max_{1 \le j \le n} d_Y(f(a_j), g(a_j)) \tag{5.5.8}$$

for every $f, g \in \mathcal{E}$, by taking the supremum of the left side of (5.5.6) over w. If

$$\mathcal{E}(a) = \{f(a) : f \in \mathcal{E}\} \tag{5.5.9}$$

is totally bounded in Y with respect to d_Y for each $a \in G$, then one can use (5.5.8) to get that $\mathcal{E}$ is totally bounded in $\mathcal{B}(G,Y)$ with respect to θ, as in Sections 5.1 and 5.2.

Suppose that $\mathcal{E}$ is left-invariant uniformly equicontinuous on G again, and that d_Y is a semi-ultrametric on Y. Let $\epsilon > 0$ be given, so that there is an open subset U of G such that $e \in U$ and (5.5.1) holds for every $f \in \mathcal{E}$, $a \in G$, and $x \in U$. As usual, we may as well that U be symmetric about e too. It follows that for each positive integer n, (5.5.1) holds for every $f \in \mathcal{E}$, $a \in G$, and $x \in U^n$, as in Section 4.5. If

$$U_0 = \bigcup_{n=1}^{\infty} U^n, \tag{5.5.10}$$

then U_0 is an open subgroup of G, and (5.5.1) holds for every $f \in \mathcal{E}$, $a \in G$, and $x \in U_0$, as before.

Let $\mathcal{B}_0$ be the collection of open subgroups of G, and let τ_0 be the topology on G associated to $\mathcal{B}_0$ as in Section 4.5. Remember that the given topology τ on G is at least as strong as τ_0. If d_Y is a semi-ultrametric on Y, and $\mathcal{E}$ is left-invariant uniformly equicontinuous on G with respect to τ, then $\mathcal{E}$ is left-invariant uniformly equicontinuous on G with respect to τ_0, by the remarks in the preceding paragraph. Similarly, if d_Y is a semi-ultrametric on Y, and $\mathcal{E}$ is right-invariant uniformly equicontinuous on G with respect to τ, then $\mathcal{E}$ is right-invariant uniformly equicontinuous on G with respect to τ_0.

Let d_Y be any semimetric on Y, and let $d_{Y,u}$ be the semi-ultrametrification of d_Y on Y, as in Section 3.8. Thus $d_{Y,u} \le d_Y$ on Y, as before. If $\mathcal{E}$ is left or right-invariant uniformly equicontinuous with respect to τ on G and d_Y on Y, then $\mathcal{E}$ is left or right-invariant uniformly equicontinuous with respect to τ on G and $d_{Y,u}$ on Y, as appropriate. This implies that $\mathcal{E}$ is left or right-invariant uniformly equicontinuous with respect to τ_0 on G and $d_{Y,u}$, as appropriate, by the previous remarks.

5.6 Equicontinuity and pointwise convergence

Let X be a nonempty set, and let Y be a topological space. The space $M(X, Y)$ of all mappings from X into Y is the same as the Cartesian product of a family of copies of Y indexed by X. The *topology of pointwise convergence* on $M(X, Y)$ corresponds in this way to the product topology on the product of copies of Y indexed by X, using the given topology on Y. If $x_1, \ldots, x_n$ are finitely many elements of X, and $V_1, \ldots, V_n$ are open subsets of Y, then

$$\mathcal{U} = \{f \in M(X, Y) : f(x_j) \in V_j \text{ for each } j = 1, \ldots, n\} \tag{5.6.1}$$

is an open subset of $M(X, Y)$ with respect to the topology of pointwise convergence. The collection of open sets $\mathcal{U}$ of this form is a base for the topology of pointwise convergence on $M(X, Y)$.

Let d_Y be a semimetric on Y, and let us suppose from now on in this section that Y is equipped with the topology determined by d_Y. If $x \in X$, then

$$d_{x,Y}(f, g) = d_Y(f(x), g(x)) \tag{5.6.2}$$

defines a semimetric on $M(X, Y)$. In this situation, the topology of pointwise convergence on $M(X, Y)$ is the same as the topology determined on $M(X, Y)$ by the collection of semimetrics

$$\{d_{x,Y} : x \in X\}, \tag{5.6.3}$$

as in Section 1.5. Note that (5.6.3) is nondegenerate on $M(X, Y)$ when d_Y is a metric on Y. If d_Y is a semi-ultrametric on Y, then (5.6.2) is a semi-ultrametric on $M(X, Y)$ for every $x \in X$.

Suppose now that X is equipped with a topology, and let $\mathcal{E}$ be a collection of mappings from X into Y. Suppose also that $\mathcal{E}$ is equicontinuous at a point $x \in X$, as in Section 5.1. This is equivalent to saying that for each $\epsilon > 0$ there is an open set $U(\epsilon) \subseteq X$ such that $x \in U(\epsilon)$ and

$$d_Y(f(x), f(w)) \le \epsilon \tag{5.6.4}$$

for every $f \in \mathcal{E}$ and $w \in U(\epsilon)$. Let $\overline{\mathcal{E}}$ be the closure of $\mathcal{E}$ in $M(X, Y)$, with respect to the topology of pointwise convergence. Observe that (5.6.4) also holds for every $f \in \overline{\mathcal{E}}$ and $w \in U(\epsilon)$, because f can be approximated by elements of $\mathcal{E}$ on x, w. Thus $\overline{\mathcal{E}}$ is equicontinuous at x too. In particular, this implies that every $f \in \overline{\mathcal{E}}$ is continuous at x.

Suppose that $\mathcal{E}$ is equicontinuous at every $x \in X$, so that $\overline{\mathcal{E}}$ is equicontinuous at every $x \in X$ as well. Thus $\overline{\mathcal{E}}$ is contained in the space $C(X, Y)$ of continuous mappings from X into Y. The topology induced on $C(X, Y)$ by the topology of pointwise convergence on $M(X, Y)$ may be described as the topology of pointwise convergence on $C(X, Y)$. If $x \in X$, then the restriction of (5.6.2) to $f, g \in C(X, Y)$ defines a semimetric on $C(X, Y)$, as in Section 1.1. The topology determined on $C(X, Y)$ by the collection of the restrictions of these semimetrics to $C(X, Y)$ is the same as the topology induced on $C(X, Y)$ by the corresponding topology on $M(X, Y)$, as in Section 1.5. This is another description of the topology of pointwise convergence on $C(X, Y)$. Note that $\overline{\mathcal{E}}$ is the same as the closure of $\mathcal{E}$ in $C(X, Y)$ with respect to the topology of pointwise convergence in this case.

Similarly, let us refer to the topology induced on $\overline{\mathcal{E}}$ by the topology of pointwise convergence on $M(X, Y)$ or $C(X, Y)$ as the topology of pointwise convergence on $\overline{\mathcal{E}}$. If $x \in X$, then the restriction of (5.6.2) to $f, g \in \overline{\mathcal{E}}$ defines a semimetric on $\overline{\mathcal{E}}$, as before. The topology determined on $\overline{\mathcal{E}}$ by the collection of the restrictions of these semimetrics to $\overline{\mathcal{E}}$ is the same as the topology induced on $\overline{\mathcal{E}}$ by the corresponding topology on $M(X, Y)$ or $C(X, Y)$, as in Section 1.5 again. This is another description of the topology of pointwise convergence on $\overline{\mathcal{E}}$.

Let A be a nonempty subset of X, so that

$$\{d_{x,Y} : x \in A\} \tag{5.6.5}$$

is a nonempty collection of semimetrics on $M(X, Y)$. This collection determines a topology on $M(X, Y)$, as in Section 1.5. Let us refer to this as the *topology of* *pointwise convergence along* A on $M(X,Y)$. The topologies induced on $C(X,Y)$ and $\overline{\mathcal{E}}$ by this topology on $M(X,Y)$ may be described as the topologies of pointwise convergence along A on these spaces. These are the same as the topologies determined by the corresponding collections of restrictions of $d_{x,Y}$, $x \in X$, to these spaces, as before. Of course, the topology of pointwise convergence on $M(X,Y)$ is automatically at least as strong as the topology of pointwise convergence along A, and similarly for the corresponding topologies on $C(X,Y)$ and $\overline{\mathcal{E}}$. If A is dense in X and d_Y is a metric on Y, then the collection of restrictions of $d_{x,Y}$, $x \in A$, to $C(X,Y)$ is nondegenerate on $C(X,Y)$.

Suppose that A is dense in X, and let $x \in X$ and $\epsilon > 0$ be given. As before, there is an open set $U(\epsilon) \subseteq X$ such that $x \in U(\epsilon)$ and (5.6.4) holds for every $f \in \overline{\mathcal{E}}$ and $w \in U(\epsilon)$. It follows that

$$\begin{aligned} d_{x,Y}(f,g) &= d_Y(f(x),g(x)) \\ &\le d_Y(f(x),f(w)) + d_Y(f(w),g(w)) + d_Y(g(w),g(x)) \qquad (5.6.6) \\ &\le 2\,\epsilon + d_{w,Y}(f,g) \end{aligned}$$

for every $f, g \in \overline{\mathcal{E}}$ and $w \in U(\epsilon)$. In particular, we can take $w \in A \cap U(\epsilon)$, because A is dense in X. This implies that the topology of pointwise convergence on $\overline{\mathcal{E}}$ is the same as the topology of pointwise convergence along A on $\overline{\mathcal{E}}$.

Let $\mathcal{B}(X,Y)$ be the space of bounded mappings from X into Y with respect to d_Y on Y, as in Section 2.1, and let θ be the supremum semimetric on $\mathcal{B}(X,Y)$ corresponding to d_Y on Y. Thus

$$d_{x,Y}(f,g) = d_Y(f(x),g(x)) \le \theta(f,g) \qquad (5.6.7)$$

for every $f, g \in \mathcal{B}(X,Y)$ and $x \in X$. Let us refer to the topology induced on $\mathcal{B}(X,Y)$ by the topology of pointwise convergence on $M(X,Y)$ as the topology of pointwise convergence on $\mathcal{B}(X,Y)$. This is the same as the topology determined on $\mathcal{B}(X,Y)$ by the collection of restrictions of the semimetrics $d_{x,Y}$, $x \in X$, to $\mathcal{B}(X,Y)$, as usual. The topology determined on $\mathcal{B}(X,Y)$ by the supremum semimetric θ is automatically at least as strong as the topology of pointwise convergence, because of (5.6.7).

Suppose for the moment that X is compact, so that continuous mappings from X into Y are automatically bounded with respect to d_Y on Y. Under these conditions, one can check that the topology determined on $\overline{\mathcal{E}}$ by the restriction of $\theta(f,g)$ to $f, g \in \overline{\mathcal{E}}$ is the same as the topology of pointwise convergence on $\overline{\mathcal{E}}$. This uses some of the remarks in Section 5.1.

Let d_X be a semimetric on X, and suppose that X is equipped with the topology determined by d_X. Also let A be a subset of X, and suppose now that $\mathcal{E}$ is uniformly equicontinuous along A with respect to d_X on X, as in Section 5.2. This is equivalent to saying that for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that (5.6.4) holds for every $f \in \mathcal{E}$, $x \in A$, and $w \in X$ with $d_X(x,w) \le \delta(\epsilon)$. As before, the same condition holds for every element of the closure $\overline{\mathcal{E}}$ of $\mathcal{E}$ in $M(X,Y)$, with respect to the topology of pointwise convergence. Thus $\overline{\mathcal{E}}$ is uniformly equicontinuous along A as well.

Suppose that $\mathcal{E}$ is uniformly equicontinuous on X with respect to d_X, which corresponds to taking $A = X$ in the preceding paragraph. Thus $\overline{\mathcal{E}}$ is uniformly equicontinuous on X too, and in particular every element of $\overline{\mathcal{E}}$ is uniformly continuous on X. Suppose that X is also totally bounded with respect to X, as in Section 4.2. If $f \in \overline{\mathcal{E}}$, then it follows that $f(X)$ is totally bounded in Y, and hence that $f(X)$ is bounded in Y. One can check that the topology determined on $\overline{\mathcal{E}}$ by the restriction of the supremum semimetric $\theta(f, g)$ to $f, g \in \overline{\mathcal{E}}$ is the same as the topology of pointwise convergence on $\overline{\mathcal{E}}$ in this situation, using some of the remarks in Section 5.2.

Let G be a topological group, and let $\mathcal{E}$ be a collection of mappings from G into Y. Also let A be a subset of G, and suppose that $\mathcal{E}$ is left-invariant uniformly equicontinuous along A, as in the previous section. This is equivalent to saying that for each $\epsilon > 0$ there is an open subset $U(\epsilon)$ of G that contains the identity element e and has the property that

$$d_Y(f(a), f(a\,x)) \le \epsilon \tag{5.6.8}$$

for every $f \in \mathcal{E}$, $a \in A$, and $x \in U(\epsilon)$. Let $\overline{\mathcal{E}}$ be the closure of $\mathcal{E}$ in the space of all mappings from G into Y, with respect to the topology of pointwise convergence, as before. It is easy to see that (5.6.8) holds for every $f \in \overline{\mathcal{E}}$, $a \in A$, and $x \in U$, by approximating f by elements of $\mathcal{E}$ on a, $a\,x$. Hence $\overline{\mathcal{E}}$ is left-invariant uniformly equicontinuous along A too. Of course, there are analogous statements for right-invariant uniform equicontinuity along A.

Suppose that $\mathcal{E}$ is left-invariant uniformly equicontinuous on G, so that $\overline{\mathcal{E}}$ is left-invariant uniformly equicontinuous on G as well. Suppose also that G is left-invariant totally bounded, as in Section 4.3. In this case, every $f \in \overline{\mathcal{E}}$ is bounded on G, because f is left-invariant uniformly continuous on G. Let θ be the supremum semimetric on the space of all bounded mappings from G into Y corresponding to d_Y. One can verify that the topology determined on $\overline{\mathcal{E}}$ by the restriction of $\theta(f, g)$ to $f, g \in \overline{\mathcal{E}}$ is the same as the topology of pointwise convergence on $\overline{\mathcal{E}}$, using some of the remarks in the previous section.

5.7 Continuity of compositions

Let W, X be nonempty sets, and let Y be a nonempty topological space. Thus the topologies of pointwise convergence can be defined on the spaces $M(W, Y)$, $M(X, Y)$ of all mappings from W, X into Y, respectively, as in the previous section. If b is a mapping from W into X, and f is a mapping from X into Y, then the composition $f \circ b$ of b and f defines a mapping from W into Y, so that

$$R_b(f) = f \circ b \tag{5.7.1}$$

defines a mapping from $M(X, Y)$ into $M(W, Y)$. It is easy to see that R_b is continuous with respect to the topologies of pointwise convergence on $M(W, Y)$, $M(X, Y)$.

Let Z be another topological space, so that the topology of pointwise convergence can also be defined on the space $M(X, Z)$ of all mappings from X into

Z. If a is a mapping from Y into Z, and f is a mapping from X into Y, then the composition $a \circ f$ of a and f defines a mapping from X into Z, so that

$$L_a(f) = a \circ f \tag{5.7.2}$$

defines a mapping from $M(X,Y)$ into $M(X,Z)$. If a is a continuous mapping from Y into Z, then one can check that L_a is continuous with respect to the topologies of pointwise convergence on $M(X,Y)$, $M(X,Z)$.

Let d_Y be a semimetric on Y, and suppose for the moment that Y is equipped with the topology determined by d_Y. If $x \in X$, then

$$d_{x,X,Y}(f,g) = d_Y(f(x), g(x)) \tag{5.7.3}$$

defines a semimetric on $M(X,Y)$, as in (5.6.2). Similarly, if $w \in W$, then

$$d_{w,W,Y}(h,k) = d_Y(h(w), k(w)) \tag{5.7.4}$$

defines a semimetric on $M(W,Y)$. The topologies of pointwise convergence on $M(X,Y)$ and $M(X,Y)$ are the same as the topologies determined by the collections of semimetrics $d_{x,X,Y}$, $x \in X$, on $M(X,Y)$ and $d_{w,W,Y}$, $w \in W$, on $M(W,Y)$, respectively, as before. If b is a mapping from W into X and f, g are mappings from X into Y, then

$$\begin{aligned} d_{w,W,Y}(R_b(f), R_b(g)) &= d_{w,W,Y}(f \circ b, g \circ b) \\ &= d_Y(f(b(w)), g(b(w))) = d_{b(w),X,Y}(f,g) \end{aligned} \tag{5.7.5}$$

for every $w \in W$.

Let d_Z be a semimetric on Z, and suppose that Z is equipped with the topology determined by d_Z. If $x \in X$ and $y \in Y$, then

$$d_{x,X,Z}(f,g) = d_Z(f(x), g(x)) \tag{5.7.6}$$

defines a semimetric on $M(X,Z)$, and

$$d_{y,Y,Z}(h,k) = d_Z(h(y), k(y)) \tag{5.7.7}$$

defines a semimetric on $M(Y,Z)$. As usual, the topologies of pointwise convergence on $M(X,Z)$ and $M(Y,Z)$ are the same as the topologies determined by the collections of semimetrics $d_{x,X,Z}$, $x \in X$, on $M(X,Z)$, and $d_{y,Y,Z}$, $y \in Y$, on $M(Y,Z)$, respectively.

If f is a mapping from X into Y, and g is a mapping from Y into Z, then their composition $g \circ f$ defines a mapping from X into Z. Thus

$$(f,g) \mapsto g \circ f \tag{5.7.8}$$

defines a mapping from $M(X,Y) \times M(Y,Z)$ into $M(X,Z)$. Let f_0 be a mapping from X into Y, let g_0 be a mapping from Y into Z, and let x be an element of X. Observe that

$$\begin{aligned} d_{x,X,Z}(g \circ f, g_0 \circ f_0) &= d_Z(g(f(x)), g_0(f_0(x))) \\ &\le d_Z(g(f(x)), g(f_0(x))) + d_Z(g(f_0(x)), g_0(f_0(x))) \\ &= d_Z(g(f(x)), g(f_0(x))) + d_{f_0(x),Y,Z}(g, g_0) \end{aligned} \tag{5.7.9}$$

for all mappings f, g from X, Y into Y, Z, respectively.

Let $\mathcal{E}$ be a collection of mappings from Y into Z, and suppose that $\mathcal{E}$ is equicontinuous at $f_0(x)$, as in Section 5.1. Let $\epsilon > 0$ be given, so that there is an open set $V(\epsilon) \subseteq Y$ such that $f_0(x) \in V(\epsilon)$ and

$$d_Z(g(f_0(x)), g(y)) < \epsilon \tag{5.7.10}$$

for every $g \in \mathcal{E}$ and $y \in V(\epsilon)$. If f is a mapping from X into Y,

$$f(x) \in V(\epsilon), \tag{5.7.11}$$

and $g \in \mathcal{E}$, then

$$\begin{aligned} d_{x,X,Z}(g \circ f, g_0 \circ f_0) &= d_Z(g(f(x)), g_0(f_0(x))) \\ &< \epsilon + d_Z(g(f_0(x)), g_0(f_0(x))) \\ &= \epsilon + d_{f_0(x),Y,Z}(g, g_0), \end{aligned} \tag{5.7.12}$$

as in (5.7.9). If we also ask that

$$d_{f_0(x),Y,Z}(g, g_0) = d_Z(g(f_0(x)), g_0(f_0(x))) < \epsilon, \tag{5.7.13}$$

then we get that

$$d_{x,Y,Z}(g \circ f, g_0 \circ f_0) = d_Z(g(f(x)), g_0(f_0(x))) < 2\,\epsilon. \tag{5.7.14}$$

Suppose now that $\mathcal{E}$ is equicontinuous at every point $y \in Y$. As in the previous section, the topology of pointwise convergence on $\mathcal{E}$ is the topology induced on $\mathcal{E}$ by the topology of pointwise convergence on $M(Y, Z)$. Using the previous remarks, one can check that the restriction of (5.7.8) to $M(X, Y) \times \mathcal{E}$ is continuous as a mapping into $M(X, Z)$, using the topologies of pointwise convergence on $M(X, Y)$, $\mathcal{E}$, and $M(X, Z)$, and the corresponding product topology on $M(X, Y) \times \mathcal{E}$.

5.8 Continuity of inverses

Let X be a nonempty set with a semimetric d. If f, g are mappings from X into itself and $x \in X$, then put

$$d_x(f, g) = d(f(x), g(x)), \tag{5.8.1}$$

as in (5.6.2). This defines a semimetric on the space $M(X) = M(X, X)$ of all mappings from X into itself, as before. The topology of pointwise convergence on $M(X)$ is the same as the topology determined on $M(X)$ by the collection of these semimetrics d_x, $x \in X$, as in Section 1.5.

Let $\mathcal{E}_1$ be a collection of one-to-one mappings from X onto itself. Also let f_0, f be elements of $\mathcal{E}_1$, and let x be an element of X. Observe that

$$\begin{aligned} d_x(f^{-1}, f_0^{-1}) &= d(f^{-1}(x), f_0^{-1}(x)) \\ &= d(f^{-1}(f_0(f_0^{-1}(x))), f^{-1}(f(f_0^{-1}(x)))). \end{aligned} \tag{5.8.2}$$

Put

$$\mathcal{E}_2 = \{f^{-1} : f \in \mathcal{E}_1\}, \tag{5.8.3}$$

and suppose that $\mathcal{E}_2$ is equicontinuous at x, as in Section 5.1. Let $\epsilon > 0$ be given, so that there is a $\delta > 0$ such that

$$d(f^{-1}(x), f^{-1}(y)) < \epsilon \tag{5.8.4}$$

for every $f \in \mathcal{E}_1$ and $y \in X$ with $d(x, y) < \delta$. If $f \in \mathcal{E}_1$ satisfies

$$d(f_0(f_0^{-1}(x)), f(f_0^{-1}(x))) = d(x, f(f_0^{-1}(x))) < \delta, \tag{5.8.5}$$

then it follows that

$$\begin{aligned} & d(f^{-1}(f_0(f_0^{-1}(x))), f^{-1}(f(f_0^{-1}(x)))) \\ & \quad = \quad d(f^{-1}(x), f^{-1}(f(f_0^{-1}(x)))) < \epsilon, \end{aligned} \tag{5.8.6}$$

by taking $y = f(f_0^{-1}(x))$ in (5.8.4). Equivalently, this means that

$$d_x(f^{-1}, f_0^{-1}) = d(f^{-1}(x), f_0^{-1}(x)) < \epsilon, \tag{5.8.7}$$

by (5.8.2).

As before, the topology of pointwise convergence on $\mathcal{E}_1$ is the topology induced on $\mathcal{E}_1$ by the topology of pointwise convergence on $M(X)$. Suppose that $\mathcal{E}_2$ is equicontinuous at every point in X. One can check that $f \mapsto f^{-1}$ is continuous as a mapping from $\mathcal{E}_1$ into $M(X)$, with respect to the corresponding topologies of pointwise convergence, using the previous remarks.

Let $H(X)$ be the group of all homeomorphisms from X onto itself, as in Section 2.3. Let G be a subgroup of $H(X)$, and suppose that G is equicontinuous at every point in X. Under these conditions, the group operations on G are continuous, with respect to the topology of pointwise convergence. More precisely, the continuity of composition of mappings as multiplication on G follows from the remarks in the previous section. Similarly, the continuity of $f \mapsto f^{-1}$ on G follows from the remarks in the preceding paragraph.

Let $IH(X)$ be the subgroup of $H(X)$ consisting of one-to-one mappings from X onto itself that are isometries with respect to d, as in Section 2.7. Of course, $IH(X)$ is equicontinuous on X. Note that the semimetrics (5.8.1) are invariant under left translations on $IH(X)$.

5.9 Isometrization

Let X be a nonempty set with a semimetric d, and let G be a subgroup of the group $H(X)$ of homeomorphisms on X. Put

$$d_1(x, y) = \sup_{f \in G} d(f(x), f(y)) \tag{5.9.1}$$

for every $x, y \in X$, where the supremum is defined as a nonnegative extended real number. Clearly

$$d(x, y) \leq d_1(x, y) \tag{5.9.2}$$

for every $x, y \in X$, because the identity mapping on X is an element of G. If there is a nonnegative real number A such that

$$d(x, y) \leq A \tag{5.9.3}$$

for every $x, y \in X$, then we have that

$$d_1(x, y) \leq A \tag{5.9.4}$$

for every $x, y \in X$. One can check that (5.9.1) satisfies the requirements of a semimetric on X, except that it may take values in $[0, \infty]$. In particular, if (5.9.3) holds, then (5.9.1) is a semimetric on X. If d is a semi-ultrametric on X, then (5.9.1) satisfies the ultrametric version of the triangle inequality too.

Let $x, y \in X$ and $g \in G$ be given. Observe that

$$d_1(g(x), g(y)) = \sup_{f \in G} d(f(g(x)), f(g(y))) = \sup_{f \in G} d(f(x), f(y)) = d(x, y), \tag{5.9.5}$$

because G is a group with respect to composition, by hypothesis. Of course, if d is already invariant under G, then $d_1 = d$.

Suppose that G is equicontinuous at every point in X with respect to d, as in Section 5.1. This means that for each $x \in X$ and $\epsilon > 0$ there is a $\delta(x, \epsilon) > 0$ such that

$$d(f(x), f(y)) < \epsilon \tag{5.9.6}$$

for every $f \in G$ and $y \in X$ with $d(x, y) < \delta(x, \epsilon)$. It follows that

$$d_1(x, y) \leq \epsilon \tag{5.9.7}$$

for every $y \in X$ with $d(x, y) < \delta(x, \epsilon)$. Similarly, if G is uniformly equicontinuous on X with respect to d, as in Section 5.2, then for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that (5.9.6) holds for every $f \in G$ and $x, y \in X$ with $d(x, y) < \delta(\epsilon)$. This implies that (5.9.7) holds for every $x, y \in X$ with $d(x, y) < \delta(\epsilon)$.

Suppose that (5.9.1) is finite for every $x, y \in X$, so that it defines a semimetric on X. Of course, the topology determined on X by d_1 is automatically at least as strong as the topology determined on X by d, because of (5.9.2). If G is equicontinuous at every point in X with respect to d, then d_1 determines the same topology on X as d, by the remarks in the preceding paragraph. Similarly, if G is uniformly equicontinuous on X with respect to d, then d_1 is uniformly equivalent to d on X, as in Section 2.9.

Let A be a positive real number, and put

$$d'(x, y) = \min(d(x, y), A) \tag{5.9.8}$$

for every $x, y \in X$. We have seen that this defines a semimetric on X, which is uniformly equivalent to d on X. If G is equicontinuous at a point $x \in X$ with respect to d, then it is easy to see that G is also equicontinuous at x with respect to d'. Similarly, if G is uniformly equicontinuous on X with respect to d, then G is uniformly equicontinuous with respect to d'.

5.10 Invariantization and isometrization

Let X be a nonempty set with a semimetric d again, and put

$$d_x(f,g) = d(f(x), g(x)) \tag{5.10.1}$$

for all $x \in X$ and mappings f, g from X into itself, as in (5.8.1). Also let G be a subgroup of the group $H(X)$ of homeomorphisms from X onto itself, and let $d_1(x,y)$ be defined for $x, y \in X$ as in (5.9.1). Put

$$d_{1,x}(f,g) = d_1(f(x), g(x)) \tag{5.10.2}$$

for all $x \in X$ and mappings f, g from X into itself, which is the analogue of (5.10.1) for d_1 in place of d. Note that

$$d_x(f,g) \le d_{1,x}(f,g) \tag{5.10.3}$$

for all $x \in X$ and mappings f, g from X into itself, by (5.9.2). If $a \in G$, then

$$\begin{aligned} d_{1,x}(a \circ f, a \circ g) &= d_1(a(f(x)), a(g(x))) \\ &= d_1(f(x), g(x)) = d_{1,x}(f,g) \end{aligned} \tag{5.10.4}$$

for all $x \in X$ and mappings f, g from X into itself, using (5.9.5) in the second step.

Equivalently,

$$d_{1,x}(f,g) = \sup_{a \in G} d(a(f(x)), a(g(x))) = \sup_{a \in G} d_x(a \circ f, a \circ g) \tag{5.10.5}$$

for all $x \in X$ and mappings f, g from X into itself. This uses the definitions (5.10.2) of $d_{1,x}$ and (5.9.1) of d_1 in the first step, and the definition (5.10.1) of d_x in the second step. If $f, g \in G$, then the right side of (5.10.5) corresponds to the left-invariantization of d_x on G, as in (4.6.1).

Suppose that G is equicontinuous at every point in X with respect to d, as in Section 5.1, and let $\epsilon > 0$ be given. Thus for each $w \in X$ there is a $\delta(w, \epsilon) > 0$ such that

$$d_1(w,y) \le \epsilon \tag{5.10.6}$$

for every $y \in X$ with $d(w,y) < \delta(x, \epsilon)$, as in the previous section. It follows that

$$d_{1,x}(f,g) = d_1(f(x), g(x)) \le \epsilon \tag{5.10.7}$$

for all $x \in X$ and mappings f, g from X into itself such that

$$d_x(f,g) = d(f(x), g(x)) < \delta(f(x), \epsilon), \tag{5.10.8}$$

by taking $w = f(x)$ and $y = g(x)$ in the previous statement. Similarly, if G is uniformly equicontinuous on X with respect to d, as in Section 5.2, then there is a $\delta(\epsilon) > 0$ such that (5.10.6) holds for every $w, y \in X$ with $d(w,y) < \delta(\epsilon)$, as before. This implies that (5.10.7) holds for all $x \in X$ and mappings f, g from X into itself such that

$$d_x(f, g) = d(f(x), g(x)) < \delta(\epsilon). \tag{5.10.9}$$

Suppose that d_1 is finite on X, so that d_1 defines a semimetric on X, as in the previous section. This implies that (5.10.2) is finite for all $x \in X$ and mappings f, g from X into itself. In this case, (5.10.2) defines a semimetric on the space $M(X)$ of mappings from X into itself for every $x \in X$, as before. Of course, for each $x \in X$, the topology determined on $M(X)$ by $d_{1,x}$ is at least as strong as the topology determined by d_x, because of (5.10.3). If G is equicontinuous at every point in X with respect to d, then d_x and $d_{1,x}$ determine the same topology on $M(X)$ for every $x \in X$, by the remarks in the preceding paragraph. Similarly, if G is uniformly equicontinuous on X with respect to d, then d_x and $d_{1,x}$ are uniformly equivalent on $M(X)$ for every $x \in X$, as in Section 2.9, by (5.10.3) and the remarks in the preceding paragraph. These statements could also be obtained from the corresponding statements for d and d_1 on X in the previous section.

5.11 Some related continuity conditions

Let (X, τ) be a nonempty topological space, and let G be a subgroup of the group $H(X)$ of homeomorphisms from X onto itself. Also let x be an element of X, and let E_x be the evaluation map on G associated to x, defined by

$$E_x(f) = f(x) \tag{5.11.1}$$

for every $f \in G$. If U is a subset of X, then

$$E_x^{-1}(U) = \{f \in G : f(x) \in U\} \tag{5.11.2}$$

is a subset of G. Thus

$$\tau_x = \{E_x^{-1}(U) : U \in \tau\} \tag{5.11.3}$$

is a collection of subsets of G, which defines a topology on G. Of course, E_x is a continuous mapping from G into X with respect to this topology.

If $g \in G$ and $U \subseteq X$, then

$$g\, E_x^{-1}(U) = \{g \circ f : f \in E_x^{-1}(U)\} \tag{5.11.4}$$

is the left translate of $E_x^{-1}(U)$ by g, as usual. Note that $f \in E_x^{-1}(U)$ if and only if $g(f(x)) \in g(U)$, which is the same as saying that $g \circ f \in E_x^{-1}(g(U))$. Thus

$$g\, E_x^{-1}(U) = E_x^{-1}(g(U)). \tag{5.11.5}$$

It follows that τ_x is invariant under left translation by g, because g is a homeomorphism on X.

Similarly,

$$E_x^{-1}(U)\, g = \{f \circ g : f \in E_x^{-1}(U)\} \tag{5.11.6}$$

is the right translate of $E_x^{-1}(U)$ by g. Observe that

$$E_x^{-1}(U)\, g = E_{g^{-1}(x)}(U). \tag{5.11.7}$$

This means that right translation by g sends τ_x onto $\tau_{g^{-1}(x)}$.

Suppose now that τ is determined by a semimetric d on X. As before,

$$d_x(f, g) = d(f(x), g(x)) = d(E_x(f), E_x(g)) \tag{5.11.8}$$

defines a semimetric on G. It is easy to see that τ_x is the topology determined on G by d_x in this situation.

Suppose that G is equicontinuous at x, as in Section 5.1. Thus for each $\epsilon > 0$ there is a $\delta(x, \epsilon) > 0$ such that

$$d(a(x), a(y)) < \epsilon \tag{5.11.9}$$

for every $a \in G$ and $y \in X$ with $d(x, y) < \delta(x, \epsilon)$. It follows that

$$d(a(x), a(f(x))) < \epsilon \tag{5.11.10}$$

for every $a, f \in G$ with $d(x, f(x)) < \delta(x, \epsilon)$, by taking $y = f(x)$ in (5.11.9). This is the same as saying that

$$d(E_x(a), E_x(a \circ f)) < \epsilon \tag{5.11.11}$$

for every $a, f \in G$ with $d(x, f(x)) < \delta(x, \epsilon)$. Note that $d(x, f(x))$ is the same as the distance between f and the identity mapping on X with respect to d_x.

Under these conditions, we get that E_x satisfies the left-invariant version of uniform continuity on G with respect to τ_x, as in Section 4.1. Although this was discussed earlier in the context of topological groups, the same definition makes sense here.

If G is equicontinuous at every point in X, then G is a topological group with respect to the topology of pointwise convergence, as in Section 5.8. In this case, E_x is left-invariant uniformly continuous on G with respect to the topology of pointwise convergence for every $x \in X$.

5.12 Separability

Remember that a topological space X is said to be *separable* if there is a dense set $E \subseteq X$ such that E has only finitely or countably many elements. If there is a base for the topology of X with only finitely or countably many elements, then it is well known that X is separable. If the topology on X is determined by a semimetric $d(\cdot, \cdot)$, and if X is separable, then there is a base for the topology of X with only finitely or countably many elements. More precisely, if $E \subseteq X$ is a dense set, then the collection of open balls $B_d(x, 1/j)$ with $x \in E$ and $j \in \mathbf{Z}_+$ is a base for the topology of E. If E has only finitely or countably many elements, then there are only finitely or countably many of these balls.

Let X be a set with a semimetric $d(\cdot,\cdot)$ again. If $E \subseteq X$ is a dense set with respect to $d(\cdot,\cdot)$, then it is easy to see that

$$\bigcup_{x \in E} B_d(x,r) = X \tag{5.12.1}$$

for every $r > 0$. Suppose now that $E_1, E_2, E_3, \ldots$ is a sequence of subsets of X such that

$$\bigcup_{x \in E_j} B_d(x,1/j) = X \tag{5.12.2}$$

for every $j \geq 1$. Under these conditions, one can check that $\bigcup_{j=1}^{\infty} E_j$ is dense in X. If E_j also has only finitely or countably many elements for each $j \geq 1$, then $\bigcup_{j=1}^{\infty} E_j$ has only finitely or countably many elements, and hence X is separable with respect to the topology determined by $d(\cdot,\cdot)$. Using these remarks, one can verify that X is separable with respect to the topology determined by $d(\cdot,\cdot)$ if and only if for every $r > 0$, X can be covered by finitely or countably many sets with diameter less than or equal to r. In particular, if X is totally bounded with respect to d, then X is separable.

Let X, Y be nonempty sets with semimetrics d_X, d_Y, respectively, and let $\theta(f,g)$ be the supremum semimetric on the space $\mathcal{B}(X,Y)$ of bounded mappings from X into Y corresponding to d_Y, as in Section 2.1. Suppose that X is totally bounded with respect to d_X, so that every uniformly continuous mapping f from X into Y is bounded, because $f(X)$ is totally bounded in Y with respect to d_Y. If Y is separable with respect to d_Y, then the space $UC(X,Y)$ of uniformly continuous mappings from X into Y is separable with respect to θ. To see this, it suffices to show that $UC(X,Y)$ can be covered by finitely or countably many sets with arbitrarily small diameter with respect to θ, as before. Of course, if X is compact, then X is totally bounded, and every continuous mapping from X into Y is uniformly continuous.

Let $\epsilon > 0$ be given, and for each $\delta > 0$, let $\mathcal{E}(\epsilon,\delta)$ be the collection of uniformly continuous mappings f from X into Y such that

$$d_Y(f(x), f(w)) < \epsilon \tag{5.12.3}$$

for every $x, w \in X$ with $d_X(x,w) < \delta$. Let $\delta > 0$ be given, and let $E(\delta)$ be a finite subset of X such that

$$X \subseteq \bigcup_{x \in E(\delta)} B_X(x,\delta), \tag{5.12.4}$$

where $B_X(x,\delta)$ is the open ball in X centered at x with radius δ with respect to d_X. Of course, this uses the hypothesis that X be totally bounded with respect to d_X. If $f, g \in \mathcal{E}(\epsilon,\delta)$, then

$$\theta(f,g) \leq 2\,\epsilon + \max_{x \in E(\delta)} d_Y(f(x), g(x)), \tag{5.12.5}$$

as in (5.2.4). Thus

$$\theta(f,g) \le 3\,\epsilon \tag{5.12.6}$$

whenever $f, g \in \mathcal{E}(\epsilon, \delta)$ satisfy

$$\max_{x \in E(\delta)} d_Y(f(x), g(x)) \le \epsilon. \tag{5.12.7}$$

Because Y is separable, Y can be covered by finitely or countably many subsets with diameter less than or equal to ϵ with respect to d_Y. Using this, one can cover $\mathcal{E}(\epsilon, \delta)$ by finitely or countably many subsets, in such a way that (5.12.7) holds when f and g are in the same subset. This implies that $\mathcal{E}(\epsilon, \delta)$ can be covered by finitely or countably many subsets with diameter less than or equal to $3\,\epsilon$ with respect to θ, as in the preceding paragraph. Note that $UC(X, Y)$ is the union of $\mathcal{E}(\epsilon, 1/l)$ over $l \in \mathbf{Z}_+$, by the definition of uniform continuity. It follows that $UC(X, Y)$ can be covered by finitely or countably many subsets with diameter less than or equal to $3\,\epsilon$ with respect to θ, as desired.

Chapter 6

Absolute values, norms, and seminorms

6.1 Absolute value functions

A nonnegative real-valued function $|\cdot|$ defined on a field k is said to be an *absolute value function* on k if it satisfies the following three conditions. First,

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{6.1.1}$$

Second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x, y \in k. \tag{6.1.2}$$

Third,

$$|x + y| \le |x| + |y| \quad \text{for every } x, y \in k. \tag{6.1.3}$$

The standard absolute value functions on the fields $\mathbf{R}$ of real numbers and $\mathbf{C}$ of complex numbers are absolute value functions in this sense.

Let $|\cdot|$ be a nonnegative real-valued function on a field k that satisfies (6.1.1) and (6.1.2). It is easy to see that $|1| = 1$, where the first 1 is the multiplicative identity element in k, and the second 1 is the multiplicative identity element in $\mathbf{R}$. This uses the fact that $0 < |1| = |1^2| = |1|^2$. Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n, then $|x| = 1$, because $|x|^n = |x^n| = 1$. In particular, it follows that $|-1| = 1$, where -1 is the additive inverse of 1 in k, because $(-1)^2 = 1$.

If $|\cdot|$ is an absolute value function on k, then

$$d(x, y) = |x - y| \tag{6.1.4}$$

defines a metric on k. This uses the fact that $|-1| = 1$, to get that (6.1.4) is symmetric in x and y.

A nonnegative real-valued function $|\cdot|$ on k is said to be an *ultrametric absolute value function* on k if it satisfies (6.1.1), (6.1.2), and

$$|x + y| \le \max(|x|, |y|) \quad \text{for every } x, y \in k. \tag{6.1.5}$$

This implies that $|\cdot|$ is an absolute value function on k, because (6.1.5) implies (6.1.3). In this case, it is easy to see that (6.1.4) is an ultrametric on k.

The *trivial absolute value function* on k is defined by putting $|0| = 0$ and $|x| = 1$ when $x \neq 0$. This is an ultrametric absolute value function on k, for which the corresponding ultrametric (6.1.4) is the discrete metric on k.

If p is a prime number, then the *p-adic absolute value* of a rational number x is defined as follows. Of course, we put $|0|_p = 0$. Otherwise, if $x \neq 0$, then x can be expressed as $p^j\,(a/b)$ for some integers a, b, and j, where $a, b \neq 0$ and neither a nor b is an integer multiple of p, and we put

$$|x|_p = p^{-j}. \tag{6.1.6}$$

This defines an ultrametric absolute value function on the field $\mathbf{Q}$ of rational numbers, and the corresponding ultrametric

$$d_p(x, y) = |x - y|_p \tag{6.1.7}$$

is known as the *p-adic metric* on $\mathbf{Q}$.

Let k be a field, and let $|\cdot|$ be an absolute value function on k. The associated metric (6.1.4) determines a topology on k, as usual. One can check that addition and multiplication on k are continuous, in the sense that the corresponding mappings from $k \times k$ into k are continuous, using the product topology on $k \times k$. Similarly, $x \mapsto 1/x$ defines a continuous mapping from $k \setminus \{0\}$ into itself, with respect to the topology induced on $k \setminus \{0\}$ by the topology on k just mentioned. Thus k is a topological group with respect to addition and the topology determined by the metric associated to $|\cdot|$, and $k \setminus \{0\}$ is a topological group with respect to multiplication and the corresponding induced topology.

If k is not already complete with respect to (6.1.4), then one can pass to a completion, by standard arguments. More precisely, the completion of k is also a field with an absolute value function, which contains k as a dense subfield. The completion of k is unique up to a suitable isomorphic equivalence. If $|\cdot|$ is an ultrametric absolute value function on k, then the extension of $|\cdot|$ to the completion of k is an ultrametric absolute value function on the completion of k. If p is a prime number, then the field $\mathbf{Q}_p$ of *p-adic numbers* is obtained by completing $\mathbf{Q}$ with respect to the p-adic absolute value.

6.2 Equivalent absolute values

If a is a positive real number with $a \le 1$, then it is well known that

$$(r + t)^a \le r^a + t^a \tag{6.2.1}$$

for all nonnegative real numbers r and t. To see this, observe first that

$$\max(r, t) \le (r^a + t^a)^{1/a}. \tag{6.2.2}$$

Using this, we get that

$$r + t \le \max(r, t)^{1-a}\,(r^a + t^a) \le (r^a + t^a)^{(1-a)/a+1} = (r^a + t^a)^{1/a}, \tag{6.2.3}$$

which implies (6.2.1), as desired.

If $d(x, y)$ is a semimetric on a set X, then it is easy to see that

$$d(x, y)^a \tag{6.2.4}$$

also defines a semimetric on X when $0 < a \leq 1$, using (6.2.1). Similarly, if $d(x, y)$ is a metric on X, then (6.2.4) is a metric on X when $0 < a \leq 1$. If $d(x, y)$ is a semi-ultrametric on X, then one can check that (6.2.4) is a semi-ultrametric on X for every $a > 0$. If $d(x, y)$ is an ultrametric on X, then it follows that (6.2.4) is an ultrametric on X for every $a > 0$.

Let $d(x, y)$ be a semimetric on X, and suppose that (6.2.4) is also a semimetric on X for some $a > 0$. Observe that

$$B_{d^a}(x, r^a) = B_d(x, r) \tag{6.2.5}$$

for every $x \in X$ and $r > 0$, where these open balls in X corresponding to $d(\cdot, \cdot)$ and (6.2.4) are as defined in (1.1.5). Similarly,

$$\overline{B}_{d^a}(x, r^a) = \overline{B}_d(x, r) \tag{6.2.6}$$

for every $x \in X$ and $r \geq 0$, where these closed balls are defined as in (1.1.6). In particular, the topologies determined on X by $d(\cdot, \cdot)$ and (6.2.4) are the same, because of (6.2.5).

Let k be a field, and let $|\cdot|$ be an absolute value function on k. It is easy to see that

$$|x|^a \tag{6.2.7}$$

defines an absolute value function on k too when $0 < a \leq 1$, using (6.2.1). If $|\cdot|$ is an ultrametric absolute value function on k, then (6.2.7) is an ultrametric absolute value function for every $a > 0$. If (6.2.7) is an absolute value function on k for some $a > 0$, then the associated metric is the same as the ath power of the metric associated to $|\cdot|$.

Let $|\cdot|_1$ and $|\cdot|_2$ be absolute value functions on k. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{6.2.8}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* on k. This implies that the topologies determined on k by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same, as before. Conversely, if the topologies determined on k by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same, then it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k in this sense.

6.3 The archimedean property and discreteness

Let k be a field. If $x \in k$ and n is a positive integer, then let $n \cdot x$ be the sum of n x's in k. An absolute value function $|\cdot|$ on k is said to be *archimedean* if there are $n \in \mathbf{Z}_+$ such that $|n \cdot 1|$ can be arbitrarily large, where 1 refers to the multiplicative identity element in k. Thus $|\cdot|$ is *non-archimedean* on k if there is a finite upper bound for $|n \cdot 1|$, $n \in \mathbf{Z}_+$. If $|n \cdot 1| > 1$ for some $n \in \mathbf{Z}_+$, then $|\cdot|$ is archimedean on k, because

$$|n^j \cdot 1| = |(n \cdot 1)^j| = |n \cdot 1|^j \to \infty \tag{6.3.1}$$

as $j \to \infty$. This means that $|n \cdot 1| \le 1$ for every $n \in \mathbf{Z}_+$ when $|\cdot|$ is non-archimedean on k. If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that $|\cdot|$ is non-archimedean on k. Conversely, if $|\cdot|$ is non-archimedean on k, then it is well known that $|\cdot|$ is an ultrametric absolute value function on k.

A famous theorem of Ostrowski states that an absolute value function on the field $\mathbf{Q}$ of rational numbers is either equivalent to the standard absolute value function, or is the trivial absolute value function, or is equivalent to the p-adic absolute value function for some prime p. Another famous theorem of Ostrowski states that if k is a field with an archimedean absolute value function $|\cdot|$, and if k is complete with respect to the associated metric, then k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to an absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard absolute value function.

Let $|\cdot|$ be an absolute value function on a field k, and observe that

$$\{|x| : x \in k,\ x \ne 0\} \tag{6.3.2}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. If 1 is not a limit point of (6.3.2) with respect to the standard topology on $\mathbf{R}$, then $|\cdot|$ is said to be *discrete* on k. Put

$$\rho_1 = \sup\{|x| : x \in k,\ |x| < 1\}, \tag{6.3.3}$$

so that $0 \le \rho_1 \le 1$. One can check that $\rho_1 = 0$ if and only if $|\cdot|$ is the trivial absolute value function, and that $\rho_1 < 1$ if and only if $|\cdot|$ is discrete on k. If $|\cdot|$ is nontrivial and discrete on k, then it is not too difficult to show that the supremum in (6.3.3) is attained, and that (6.3.2) consists of the integer powers of ρ_1.

If k has positive characteristic, then there are only finitely many elements of k of the form $n \cdot 1$, and hence $|\cdot|$ is non-archimedean on k. Suppose that $|\cdot|$ is archimedean on k, so that k has characteristic 0. This implies that there is a natural embedding of $\mathbf{Q}$ into k, so that $|\cdot|$ induces an absolute value function on $\mathbf{Q}$. This induced absolute value function on $\mathbf{Q}$ is archimedean, because $|\cdot|$ is archimedean on k. The first theorem of Ostrowski mentioned earlier implies that the induced absolute value function on $\mathbf{Q}$ is equivalent to the standard absolute value function on $\mathbf{Q}$. It follows that $|\cdot|$ is not discrete on k, because the standard absolute value function on $\mathbf{Q}$ is not discrete. If $|\cdot|$ is a discrete absolute value function on k, then we get that $|\cdot|$ is non-archimedean on k.

Let $|\cdot|$ be an ultrametric absolute value function on a field k. If $x, y \in k$ satisfy

$$|x - y| < |x|, \tag{6.3.4}$$

then

$$|x| = |y|. \tag{6.3.5}$$

More precisely,

$$|y| \le \max(|x|, |x - y|) = |x| \tag{6.3.6}$$

when $|x-y| \le |x|$, by the ultrametric version of the triangle inequality. We also have that

$$|x| \le \max(|y|, |x - y|), \tag{6.3.7}$$

using the ultrametric version of the triangle inequality. If (6.3.4) holds, then (6.3.7) implies that $|x| \le |y|$, as desired.

6.4 p-Adic integers

Let k be a field, and let $|\cdot|$ be an absolute value function on k. If $x \in k$ and n is a nonnegative integer, then

$$(1 - x) \sum_{j=0}^{n} x^j = 1 - x^{n+1}, \tag{6.4.1}$$

where x^j is interpreted as being the multiplicative identity element 1 in k when $j = 0$, as usual. If $x \neq 1$, then it follows that

$$\sum_{j=0}^{n} x^j = \frac{1 - x^{n+1}}{1 - x} \tag{6.4.2}$$

for every $n \ge 0$. If $|x| < 1$, then we get that

$$\sum_{j=0}^{n} x^j \to \frac{1}{1 - x} \tag{6.4.3}$$

as $n \to \infty$ with respect to the metric on k associated to $|\cdot|$, because $|x^{n+1}| = |x|^{n+1} \to 0$ as $n \to \infty$.

Let p be a prime number, and let $|\cdot|_p$ be the p-adic absolute value function on the field $\mathbf{Q}_p$ of p-adic numbers, whose restriction to $\mathbf{Q}$ was defined in Section 6.1. If y is an integer, then $x = p\,y$ satisfies

$$|x|_p = p^{-1}\,|y|_p \le 1/p < 1, \tag{6.4.4}$$

so that the remarks in the preceding paragraph can be applied. It follows that $1/(1 - x)$ can be approximated by integers with respect to the p-adic metric, because $\sum_{j=0}^{n} x^j$ is an integer for every nonnegative integer n.

If $w \in \mathbf{Q}$ satisfies $|w|_p \le 1$, then w can be expressed as a/b, where a and b are integers, $b \neq 0$, and b is not a multiple of p. Note that there is an integer c such that $b\,c \equiv 1$ modulo p, because $\mathbf{Z}/p\,\mathbf{Z}$ is a field. Thus $b\,c$ can be expressed as $1 - p\,y$ for some integer y, and $w = a/b = (a\,c)/(b\,c) = a\,c\,(1 - p\,y)^{-1}$.

This implies that w can be approximated by integers with respect to the p-adic metric, because of the corresponding property of $(1 - p\,y)^{-1}$, mentioned in the previous paragraph.

The set of *p-adic integers* is defined by

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \le 1\}. \tag{6.4.5}$$

This is the same as the closed unit ball in $\mathbf{Q}_p$ with respect to the p-adic metric, and in particular this is a closed set with respect to the p-adic metrics. The set $\mathbf{Z}$ of all integers is contained in $\mathbf{Z}_p$, by the definition of $|\cdot|_p$ on $\mathbf{Q}$. Hence $\mathbf{Z}_p$ contains the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$ with respect to the p-adic metric. Let us check that $\mathbf{Z}_p$ is in fact the same as the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$.

By construction, $\mathbf{Q}$ is dense in $\mathbf{Q}_p$ with respect to the p-adic metric, because $\mathbf{Q}_p$ is supposed to be the completion of $\mathbf{Q}$. Let $z \in \mathbf{Z}_p$ be given. If $w \in \mathbf{Q}_p$ satisfies $|z - w|_p \le 1$, then $|w|_p \le 1$, by the ultrametric version of the triangle inequality. In particular, we can approximate z by $w \in \mathbf{Q}$, because $\mathbf{Q}$ is dense in $\mathbf{Q}_p$, as before. This shows that we can approximate z by $w \in \mathbf{Q}$ with $|w|_p \le 1$ with respect to the p-adic metric. We have also seen that $w \in \mathbf{Q}$ with $|w|_p \le 1$ can be approximated by integers with respect to the p-adic metric. It follows that z can be approximated by integers with respect to the p-adic metric as well. This implies that $\mathbf{Z}_p$ is the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$, as desired.

6.5 The residue field

Let k be a field, and let $|\cdot|$ be an ultrametric absolute value function on k. If r is a positive real number, then it is easy to see that the open ball

$$B(0, r) = \{x \in k : |x| < r\} \tag{6.5.1}$$

in k centered at 0 with radius r with respect to the ultrametric associated to $|\cdot|$ is a subgroup of k with respect to addition. Similarly, for each nonnegative real number r, the closed ball

$$\overline{B}(0, r) = \{x \in k : |x| \le r\} \tag{6.5.2}$$

in k centered at 0 with radius r is a subgroup of k with respect to addition. The closed unit ball $\overline{B}(0, 1)$ is a subring of k, that contains the multiplicative identity element 1 in k in particular. If $r \le 1$, then (6.5.1) and (6.5.2) are ideals in $\overline{B}(0, 1)$.

The quotient

$$\overline{B}(0, 1)/B(0, 1) \tag{6.5.3}$$

is in fact a field, which is the *residue field* associated to $|\cdot|$ on k. More precisely, every nonzero element of (6.5.3) comes from an element x of $\overline{B}(0, 1)$ not in $B(0, 1)$, which means that $|x| = 1$. This implies that $|1/x| = 1/|x| = 1$, so that $1/x$ is in $\overline{B}(0, 1)$. The image of $1/x$ in (6.5.3) is the multiplicative inverse of the image of x in (6.5.3), as desired. If $|\cdot|$ is the trivial absolute value function on k, then $\overline{B}(0, 1) = k$, $B(0, 1) = \{0\}$, and the residue field reduces to k itself.

Let p be a prime number, and let us consider the case where k is the field $\mathbf{Q}_p$ of p-adic numbers, with the p-adic absolute value function $|\cdot|_p$. In this situation, the closed unit ball is the same as the set $\mathbf{Z}_p$ of p-adic integers, discussed in the previous section. In particular, $\mathbf{Z}_p$ is a subring of $\mathbf{Q}_p$, as before. Put

$$p^j\,\mathbf{Z}_p = \{p^j\,x : x \in \mathbf{Z}_p\} \tag{6.5.4}$$

for each integer j. Equivalently,

$$p^j\,\mathbf{Z}_p = \{y \in \mathbf{Q}_p : |y|_p \le p^{-j}\} \tag{6.5.5}$$

is the closed ball in $\mathbf{Q}_p$ centered at 0 with radius p^{-j} with respect to the p-adic metric for every $j \in \mathbf{Z}$. If $x \in \mathbf{Q}_p$ and $x \neq 0$, then $|x|_p$ is an integer power of p, as one can show by approximating x by elements of $\mathbf{Q}$. This implies that all open and closed balls in $\mathbf{Q}_p$ centered at 0 with positive radius can be expressed as $p^j\,\mathbf{Z}_p$ for some $j \in \mathbf{Z}$.

Let $j \in \mathbf{Z}_+$ be given, and note that $p^j\,\mathbf{Z}_p$ is an ideal in $\mathbf{Z}_p$, so that the quotient

$$\mathbf{Z}_p/p^j\,\mathbf{Z}_p \tag{6.5.6}$$

may be considered as a commutative ring. There is a natural ring homomorphism from $\mathbf{Z}$ into (6.5.6), which is the composition of the natural inclusion of $\mathbf{Z}$ into $\mathbf{Z}_p$ with the quotient mapping from $\mathbf{Z}_p$ onto (6.5.6). One can check that this ring homomorphism from $\mathbf{Z}$ into (6.5.6) is surjective, because $\mathbf{Z}$ is dense in $\mathbf{Z}_p$ with respect to the p-adic metric. The kernel of this ring homomorphism from $\mathbf{Z}$ onto (6.5.6) is $\mathbf{Z} \cap (p^j\,\mathbf{Z}_p)$. This is the same as

$$p^j\,\mathbf{Z} = \{p^j\,x : x \in \mathbf{Z}\}, \tag{6.5.7}$$

by the definition of $|\cdot|_p$ on $\mathbf{Q}$. It follows that (6.5.6) is isomorphic as a ring to $\mathbf{Z}/p^j\,\mathbf{Z}$. The residue field corresponds to taking $j = 1$, because the open unit ball in $\mathbf{Q}_p$ is the same as $p\,\mathbf{Z}_p$.

6.6 Local total boundedness

Let k be a field, and let $|\cdot|$ be an absolute value function on k. Let us say that k is *locally totally bounded* with respect to $|\cdot|$ if there is a positive real number r such that $B(0,r)$ is totally bounded with respect to the metric associated to $|\cdot|$ on k. This is equivalent to saying that k is locally totally bounded with respect to the metric associated to $|\cdot|$, or that k is locally totally bounded as a topological group with respect to addition and the topology determined by the metric associated to $|\cdot|$, as in Section 4.9. In particular, if k is locally compact with respect to the topology determined by the metric associated to $|\cdot|$, then k is locally totally bounded with respect to $|\cdot|$, as before.

If k is complete with respect to the metric associated to $|\cdot|$, and k is locally totally bounded with respect to $|\cdot|$, then k is locally compact. More precisely, if k is locally totally bounded, then $\overline{B}(0,r_1)$ is totally bounded for some $r_1 > 0$.

If k is complete, then subsets of k that are both closed and totally bounded are compact, so that $\overline{B}(0, r_1)$ is compact.

If $t \in k$ and $E \subseteq k$, then put

$$t\,E = \{t\,x : x \in E\}. \tag{6.6.1}$$

Note that

$$t\,B(0, r) = B(0, |t|\,r) \tag{6.6.2}$$

for every $r > 0$ when $t \neq 0$, and that

$$t\,\overline{B}(0, r) = \overline{B}(0, |t|\,r) \tag{6.6.3}$$

for every $r \geq 0$. If $E \subseteq k$ is compact with respect to the topology determined by the metric associated to $|\cdot|$, then $t\,E$ is compact for every $t \in k$, because multiplication by t defines a continuous mapping from k into itself. If $E \subseteq k$ is totally bounded with respect to the metric associated to $|\cdot|$, then it is easy to see that $t\,E$ is totally bounded for every $t \in k$.

If $|\cdot|$ is the trivial absolute value function on k, then the associated metric is the discrete metric, and k is clearly locally compact. Let us suppose from now on in this section that $|\cdot|$ is nontrivial on k. This means that there is an $x \in k$ such that $x \neq 0$ and $|x| \neq 1$. It follows that there are $y, z \in k$ such that $0 < |y| < 1$ and $1 < |z|$, using x and $1/x$. Note that

$$|y^j| = |y|^j \to 0 \quad \text{and} \quad |z^j| = |z|^j \to \infty \tag{6.6.4}$$

as $j \to \infty$.

If k is locally compact with respect to the topology determined by the metric associated to $|\cdot|$, then $\overline{B}(0, r_0)$ is compact for some $r_0 > 0$. This implies that $\overline{B}(0, r)$ is compact for some arbitrarily large values of r, using (6.6.3) and the nontriviality of $|\cdot|$ on k. It follows that $\overline{B}(0, r)$ is compact for every $r \geq 0$, and in fact that all subsets of k that are both closed and bounded are compact.

Similarly, if k is locally totally bounded with respect to $|\cdot|$, then $B(0, r_0)$ is totally bounded with respect to the metric associated to $|\cdot|$ for some $r_0 > 0$. Hence $B(0, r)$ is totally bounded for some arbitrarily large values of r, using (6.6.2) and the nontriviality of $|\cdot|$ on k. This implies that $B(0, r)$ is totally bounded for every $r > 0$, and that all bounded subsets of k are totally bounded.

Let us also suppose from now on in this section that $|\cdot|$ is an ultrametric absolute value function on k. Suppose for the moment that k is locally totally bounded with respect to $|\cdot|$. This implies that $\overline{B}(0, 1)$ is totally bounded with respect to the metric associated to $|\cdot|$, as in the preceding paragraph. In particular, this implies that the residue field associated to $|\cdot|$ has only finitely many elements. One can also use the total boundedness of $B(0, 1)$ to get that $|\cdot|$ is discrete on k, by showing that (6.3.3) is strictly less than 1.

Conversely, if the residue field associated to $|\cdot|$ is finite, and if $|\cdot|$ is discrete on k, then k is locally totally bounded. To see this, let ρ_1 be as in (6.3.3) again, so that $0 < \rho_1 < 1$. In this case, the finiteness of the residue field implies that $\overline{B}(0, 1)$ is the union of finitely many closed balls of radius ρ_1. It follows that for each $j \in \mathbf{Z}$, $\overline{B}(0, \rho_1^j)$ is the union of finitely many closed balls of radius ρ_1^{j+1}. One can repeat the process to get that $\overline{B}(0, \rho_1^j)$ is the union of finitely many closed balls of radius ρ_1^{j+l} for any $l \in \mathbf{Z}_+$, as desired.

6.7 Norms and seminorms

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *seminorm* on V with respect to $|\cdot|$ or k if it satisfies the following two conditions. First,

$$N(t\,v) = |t|\,N(v) \quad \text{for every } v \in V \text{ and } t \in k. \tag{6.7.1}$$

Second,

$$N(v + w) \le N(v) + N(w) \quad \text{for every } v, w \in V. \tag{6.7.2}$$

Note that (6.7.1) implies that $N(0) = 0$. If we also have that

$$N(v) > 0 \quad \text{for every } v \in V \text{ with } v \neq 0, \tag{6.7.3}$$

then N is said to be a *norm* on V. Of course, k may be considered as a one-dimensional vector space over itself, and $|\cdot|$ may be considered as a norm on k with respect to itself.

A nonnegative real-valued function N on V is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k if it satisfies (6.7.1) and

$$N(v + w) \le \max(N(v), N(w)) \quad \text{for every } v, w \in V. \tag{6.7.4}$$

Clearly (6.7.4) implies (6.7.2), so that a semi-ultranorm is a seminorm in particular. If N satisfies (6.7.3) as well, then N is said to be an *ultranorm* on V. If N is a semi-ultranorm on V with respect to $|\cdot|$ on k, and if $N(v) > 0$ for some $v \in V$, then it is easy to see that $|\cdot|$ is an ultrametric absolute value function on k. If $|\cdot|$ is an ultrametric absolute value function on k, then $|\cdot|$ may be considered as an ultranorm on k with respect to itself, where k is considered as a vector space over itself.

If N is a seminorm on V with respect to $|\cdot|$ on k, then

$$d(v, w) = d_N(v, w) = N(v - w) \tag{6.7.5}$$

defines a semi-metric on V, which is a metric on V when N is a norm on V. If N is a semi-ultranorm on V, then (6.7.5) is a semi-ultrametric in V.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k. The *trivial ultranorm* is defined on V by putting $N(0) = 0$ and $N(v) = 1$ for every $v \in V$ with $v \neq 0$. It is easy to see that this is an ultranorm on V, for which the corresponding ultrametric (6.7.5) is the discrete metric on V.

Let n be a positive integer, and let k^n be the space of n-tuples of elements of k, as usual. This is a vector space over k, with respect to coordinatewise addition and scalar multiplication. Put

$$\|v\|_1 = \sum_{j=1}^{n} |v_j| \tag{6.7.6}$$

and

$$\|v\|_\infty = \max_{1 \le j \le n} |v_j| \tag{6.7.7}$$

for every $v \in k^n$. It is easy to see that (6.7.6) and (6.7.7) are norms on k^n with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then (6.7.7) is an ultrametric on k^n.

Clearly

$$\|v\|_\infty \le \|v\|_1 \le n\,\|v\|_\infty \tag{6.7.8}$$

for every $v \in k^n$. Let

$$d_1(v, w) = \|v - w\|_1 \tag{6.7.9}$$

and

$$d_\infty(v, w) = \|v - w\|_\infty \tag{6.7.10}$$

be the metrics on k^n associated to (6.7.6) and (6.7.7) as in (6.7.5). Thus

$$d_\infty(v, w) \le d_1(v, w) \le n\, d_\infty(v, w) \tag{6.7.11}$$

for every $v, w \in k^n$, by (6.7.8).

Let a be a positive real number, and suppose that $|\cdot|^a$ is an absolute value function on k too. This holds automatically when $a \le 1$, and when $|\cdot|$ is an ultrametric absolute value function on k, as in Section 6.2. Let V be a vector space over k again, and let N be a nonnegative real-valued function on V. If N satisfies (6.7.1), then N^a satisfies the analogous condition with respect to $|\cdot|^a$ on k. If N satisfies (6.7.2) and $a \le 1$, then

$$N(v + w)^a \le (N(v) + N(w))^a \le N(v)^a + N(w)^a \tag{6.7.12}$$

for every $v, w \in V$, using (6.2.1) in the second step. If N satisfies (6.7.4), then N^a satisfies the analogous condition for every $a > 0$. Of course, the analogue of (6.7.5) for N^a is the same as the ath power of (6.7.5).

6.8 Invertible matrices

Let R be a ring, and let n be a positive integer. The space $M_n(R)$ of $n \times n$ matrices with entries in R is a ring as well, with respect to entrywise addition of matrices, and the usual matrix multiplication. Suppose now that R has a multiplicative identity element e. The corresponding identity matrix $I \in M_n(R)$ is the matrix whose diagonal entries are equal to e and whose off-diagonal entries are the additive identity element 0 in R. This is the multiplicative identity element in $M_n(R)$. An element A of $M_n(R)$ is said to be *invertible* if A has a multiplicative inverse in $M_n(A)$. The multiplicative group of invertible elements of $M_n(R)$ is denoted $GL_n(R)$.

If R is commutative, then the determinant $\det A$ of $A \in M_n(R)$ can be defined as an element of R in the usual way. It is well known that A has a multiplicative inverse in $M_n(R)$ if and only if $\det A$ has a multiplicative inverse in R. In this case, A^{-1} can be given in terms of the entries of A using Cramer's rule.

Let k be a field, so that $A \in M_n(k)$ is invertible exactly when $\det A \neq 0$. Also let $|\cdot|$ be an absolute value function on k, which leads to a metric on k as before. This leads to a topology on $M_n(k)$, by identifying $M_n(k)$ with k^{n^2}, and using the product topology corresponding to the topology determined on k by the metric associated to $|\cdot|$. It is easy to see that matrix multiplication is continuous as a mapping from $M_n(k) \times M_n(k)$ into $M_n(k)$, using the corresponding product topology on the domain. This uses the analogous continuity properties of addition and multiplication on k. Similarly, the determinant defines a continuous mapping from $M_n(k)$ into k. In particular, $GL_n(k)$ is an open set in $M_n(k)$, because of the characterization of $GL_n(k)$ in terms of the determinant. One can check that $A \mapsto A^{-1}$ is continuous as a mapping from $GL_n(k)$ into itself, with respect to the topology induced on $GL_n(k)$ by the one already mentioned on $M_n(k)$, using Cramer's rule. Thus $GL_n(k)$ is a topological group with respect to this topology.

If $n = 1$, then $M_n(k)$ reduces to k, and $GL_n(k)$ reduces to $k \setminus \{0\}$. Of course,

$$\{x \in k : |x| = 1\} \tag{6.8.1}$$

is a subgroup of $k \setminus \{0\}$, and hence a topological group with respect to the topology induced on (6.8.1) by the usual ones on k or $k \setminus \{0\}$. Note that (6.8.1) is a closed set in k, and thus a relatively closed set in $k \setminus \{0\}$. If $x \in k$ and $x \neq 0$, then

$$|1 - 1/x| = |x - 1|/|x|. \tag{6.8.2}$$

In particular,

$$|1 - 1/x| = |x - 1| \tag{6.8.3}$$

when $|x| = 1$.

Suppose from now on in this section that $|\cdot|$ is an ultrametric absolute value function on k. If $x, y \in k$, $|x| = 1$, and $|x - y| < 1$, then $|y| = 1$, as in (6.3.5). Let $B(x, r)$ and $\overline{B}(x, r)$ be the usual open and closed balls in k centered at $x \in k$ with radii $r > 0$ with respect to the metric associated to $|\cdot|$. The previous statement implies that (6.8.1) is an open set in k, which could also be obtained from the facts that $\overline{B}(0, 1)$ is an open set in k, and $B(0, 1)$ is a closed set in k. More precisely, the previous statement also implies that $B(1, 1)$ is contained in (6.8.1). If $x, y \in k$, then

$$|x\,y - 1| = |x\,(y - 1) + x - 1| \leq \max(|x|\,|y - 1|, |x - 1|), \tag{6.8.4}$$

using the ultrametric version of the triangle inequality in the second step. In particular,

$$|x\,y - 1| \leq \max(|y - 1|, |x - 1|) \tag{6.8.5}$$

when $|x| = 1$. One can use (6.8.3), (6.8.5), and the fact that $B(1, 1)$ is contained in (6.8.1) to get that $B(1, r)$ is a subgroup of (6.8.1) when $0 < r \leq 1$. Similarly, $\overline{B}(1, r)$ is a subgroup of (6.8.1) when $0 < r < 1$.

Remember that $\overline{B}(0,1)$ is a subring of k. Let n be a positive integer again, so that $M_n(\overline{B}(0,1))$ is a subring of $M_n(k)$. Of course, $I \in M_n(\overline{B}(0,1))$, because $1 \in \overline{B}(0,1)$. Note that $M_n(\overline{B}(0,1))$ is both open and closed in $M_n(k)$, because $\overline{B}(0,1)$ is both open and closed in k. The group $GL_n(\overline{B}(0,1))$ of invertible elements of $M_n(\overline{B}(0,1))$ is a subgroup of $GL_n(k)$. As before, $GL_n(\overline{B}(0,1))$ consists of $A \in M_n(\overline{B}(0,1))$ such that $\det A$ has a multiplicative inverse in $\overline{B}(0,1)$. It is easy to see that $x \in \overline{B}(0,1)$ has a multiplicative inverse in $\overline{B}(0,1)$ if and only if $|x| = 1$, in which case one can use the multiplicative inverse $1/x$ of x in k. Thus

(6.8.6) $$GL_n(\overline{B}(0,1)) = \{A \in M_n(\overline{B}(0,1)) : |\det A| = 1\}.$$

This is both open and closed in $M_n(\overline{B}(0,1))$, because (6.8.1) is both open and closed in k, and because of the continuity of the determinant.

Consider

(6.8.7) $$\{A \in M_n(\overline{B}(0,1)) : A - I \text{ has entries in } B(0,r)\}$$

when $0 < r \leq 1$, and

(6.8.8) $$\{A \in M_n(\overline{B}(0,1)) : A - I \text{ has entries in } \overline{B}(0,r)\}$$

when $0 < r < 1$. Clearly I is an element of (6.8.7) and (6.8.8). One can check that the product of two elements of (6.8.7) is in (6.8.7) too, and similarly for (6.8.8). If A is in (6.8.7), then one can verify that

(6.8.9) $$\det A \in B(1,r).$$

Similarly, if A is in (6.8.8), then

(6.8.10) $$\det A \in \overline{B}(1,r).$$

In both cases, one can use Cramer's rule to get that A^{-1} is in (6.8.7) or (6.8.8), as appropriate. It follows that (6.8.7) and (6.8.8) are subgroups of $GL_n(\overline{B}(0,1))$, because $B(1,r)$ is contained in (6.8.1) when $0 < r \leq 1$, and $\overline{B}(0,r)$ is contained in (6.8.1) when $0 < r < 1$, as before. Note that (6.8.7) and (6.8.8) are each both open and closed in $M_n(\overline{B}(0,1))$.

6.9 Bounded linear mappings

Let k be a field with an absolute value function $|\cdot|$, and let V, W be vector spaces over k. Also let N_V, N_W be seminorms on V, W, respectively, and with respect to $|\cdot|$ on k. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

(6.9.1) $$N_W(T(v)) \leq C\, N_V(v)$$

for every $v \in V$. This implies that

(6.9.2) $$N_W(T(u) - T(v)) = N_W(T(u - v)) \leq C\, N_V(u - v)$$

for every $u, v \in V$. In particular, this means that T is uniformly continuous with respect to the semimetrics on V, W associated to N_V, N_W, respectively. If $|\cdot|$ is nontrivial on k, and if T is continuous at 0 with respect to these semimetrics, then one can check that T is bounded. More precisely, it suffices to know that $N_W(T(v))$ is bounded on a ball of positive radius in V in this case.

If T is a bounded linear mapping from V into W, then put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \geq 0 : (6.9.1) \text{ holds}\}, \tag{6.9.3}$$

where more precisely the infimum is taken over all nonnegative real numbers C for which (6.9.1) holds. This is the *operator seminorm* of T associated to N_V and N_W. It is easy to see that (6.9.1) holds with $C = \|T\|_{op}$, which is to say that the infimum on the right side of (6.9.3) is attained. Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W. One can check that this is a vector space over k with respect to pointwise addition and multiplication of mappings from V into W, and that (6.9.3) defines a seminorm on $\mathcal{BL}(V, W)$ with respect to $|\cdot|$ on k. If N_W is a norm on W, then (6.9.3) is a norm on $\mathcal{BL}(V, W)$. If N_W is a semi-ultranorm on W, then (6.9.3) is a semi-ultranorm on $\mathcal{BL}(V, W)$.

Let Z be another vector space over k, and let N_Z be a seminorm on Z with respect to $|\cdot|$ on k. If T_1 is a bounded linear mapping from V into W, and T_2 is a bounded linear mapping from W into Z, then it is easy to see that their composition $T_2 \circ T_1$ is bounded as a linear mapping from V into W, with

$$\|T_2 \circ T_1\|_{op,VZ} \leq \|T_1\|_{op,VW} \, \|T_2\|_{op,WZ}. \tag{6.9.4}$$

More precisely,

$$\begin{aligned} N_Z((T_2 \circ T_1)(v)) = N_Z(T_2(T_1(v))) &\leq \|T_2\|_{op,WZ} \, N_W(T_1(v)) \\ &\leq \|T_1\|_{op,VW} \, \|T_2\|_{op,WZ} \, N_V(v) \end{aligned} \tag{6.9.5}$$

for every $v \in V$.

Let n be a positive integer, and suppose that $V = k^n$, the space of n-tuples of elements of k. Let $e_1, \ldots, e_n$ be the standard basis vectors in k^n, so that the jth coordinate of e_l is equal to 1 when $j = l$, and to 0 when $j \neq l$. If T is a linear mapping from k^n into W, then

$$N_W(T(v)) = N_W\Big(\sum_{j=1}^n v_l \, T(e_l)\Big) \leq \sum_{l=1}^n |v_l| \, N_W(T(e_l)) \tag{6.9.6}$$

for every $v \in k^n$. If we take k^n to be equipped with the norm $\|v\|_1 = \sum_{j=1}^n |v_j|$, then it follows that T is a bounded linear mapping from k^n into W, with operator seminorm less than or equal to

$$\max_{1 \leq l \leq n} N_W(T(e_l)). \tag{6.9.7}$$

In fact, the operator seminorm of T is equal to (6.9.7), because $\|e_l\|_1 = 1$ for each $l = 1, \ldots, n$.

Let us now take k^n to be equipped with the norm $\|v\|_\infty = \max_{1 \le j \le n} |v_j|$. Let T be a linear mapping from k^n into W again, and observe that T is bounded with respect to $\|\cdot\|_\infty$ on k^n, with operator seminorm less than or equal to

$$\sum_{l=1}^{n} N_W(T(e_l)). \tag{6.9.8}$$

If N_W is a semi-ultranorm on W with respect to $|\cdot|$ on k, then

$$N_W(T(v)) = N_W\Big(\sum_{l=1}^{n} v_l\, T(e_l)\Big) \le \max_{1 \le l \le n} (|v_l|\, N_W(T(e_l))) \tag{6.9.9}$$

for every $v \in k^n$. This implies that the operator seminorm of T with respect to $\|\cdot\|_\infty$ on k^n is less than or equal to (6.9.7). As before, the operator seminorm of T is actually equal to (6.9.7) in this case, because $\|e_l\|_\infty = 1$ for every $l = 1, \ldots, n$.

6.10 Submultiplicative seminorms

Let k be a field, and let $\mathcal{A}$ be an (associative) *algebra* over k. This means that $\mathcal{A}$ is a vector space over k with a binary operation of multiplication, where multiplication is both bilinear over k and satisfies the associative law. If multiplication on $\mathcal{A}$ also satisfies the commutative law, then $\mathcal{A}$ is said to be a *commutative algebra* over k.

Let $|\cdot|$ be an absolute value function on k, and let N be a seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on $\mathcal{A}$. If

$$N(x\,y) \le N(x)\,N(y) \tag{6.10.1}$$

for every $x, y \in \mathcal{A}$, then N is said to be *submultiplicative* on $\mathcal{A}$.

Suppose that $\mathcal{A}$ has a multiplicative identity element e. If N is submultiplicative on $\mathcal{A}$, then

$$N(e) = N(e^2) \le N(e)^2. \tag{6.10.2}$$

This implies that $N(e) \ge 1$ when $N(e) > 0$. Otherwise, if $N(e) = 0$, then $N(x) = 0$ for every $x \in \mathcal{A}$, by (6.10.1).

Let V be a vector space over k, and let N_V be a seminorm on V with respect to $|\cdot|$ on k. Also let $\mathcal{BL}(V) = \mathcal{BL}(V, V)$ be the space of bounded linear mappings from V into itself, with respect to N_V on both the domain and the range. This is an algebra over k, with composition of linear mappings as multiplication. The corresponding operator seminorm $\|\cdot\|_{op}$ is submultiplicative on $\mathcal{BL}(V)$, as in the preceding section. Of course, the identity mapping $I = I_V$ on V is the multiplicative identity element in $\mathcal{BL}(V)$. It is easy to see that I is bounded with respect to N_V, with

$$\|I\|_{op} = 1 \tag{6.10.3}$$

as long as $N_V(v) > 0$ for some $v \in V$. Otherwise, if $N_V(v) = 0$ for every $v \in V$, then every linear mapping from V into itself is bounded, with operator seminorm equal to 0.

If n is a positive integer, then the space k^n of n-tuples of elements of k is a vector space over k, with respect to coordinatewise addition and scalar multiplication. Similarly, the space $M_n(k)$ of $n \times n$ matrices with entries in k is a vector space over k, with respect to entrywise addition and scalar multiplication. More precisely, $M_n(k)$ is an algebra over k, with respect to matrix multiplication. This can be identified with the algebra of linear mappings from k^n into itself in the usual way.

Let $\mathcal{A}$ be an algebra over k again, and let N be a submultiplicative seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k. One can check that multiplication on $\mathcal{A}$ is continuous as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, with respect to the topology determined on $\mathcal{A}$ by the semimetric associated to N, and the corresponding product topology on $\mathcal{A} \times \mathcal{A}$.

Suppose that $\mathcal{A}$ has a multiplicative identity element e. As usual, $a \in \mathcal{A}$ is said to be *invertible* if there is a $b \in \mathcal{A}$ such that

$$a\,b = b\,a = e. \tag{6.10.4}$$

In this case, b is unique, and denoted a^{-1}. The collection of invertible elements of $\mathcal{A}$ is a group with respect to multiplication.

If $x, y \in \mathcal{A}$ are invertible, then

$$x^{-1} - y^{-1} = x^{-1}\,y\,y^{-1} - x^{-1}\,x\,y^{-1} = x^{-1}\,(y - x)\,y^{-1}. \tag{6.10.5}$$

This implies that

$$N(x^{-1} - y^{-1}) \le N(x^{-1})\,N(y^{-1})\,N(x - y). \tag{6.10.6}$$

Suppose for the moment that

$$N(x^{-1})\,N(x - y) < 1. \tag{6.10.7}$$

Observe that

$$\begin{aligned} N(y^{-1}) &\le N(x^{-1}) + N(y^{-1} - x^{-1}) \\ &\le N(x^{-1}) + N(x^{-1})\,N(x - y)\,N(y^{-1}), \end{aligned} \tag{6.10.8}$$

and hence

$$(1 - N(x^{-1})\,N(x - y))\,N(y^{-1}) \le N(x^{-1}). \tag{6.10.9}$$

It follows that

$$N(y^{-1}) \le (1 - N(x^{-1})\,N(x - y))^{-1}\,N(x^{-1}) \tag{6.10.10}$$

when (6.10.7) holds. Combining this with (6.10.6), we get that

$$N(x^{-1} - y^{-1}) \le (1 - N(x^{-1})\,N(x - y))^{-1}\,N(x^{-1})^2\,N(x - y) \tag{6.10.11}$$

when (6.10.7) holds. Using this, one can check that $x \mapsto x^{-1}$ is a continuous mapping from the set of invertible elements in $\mathcal{A}$ into itself, with respect to the restriction of the semimetric associated to N to that set.

Suppose now that N is also a semi-ultranorm on $\mathcal{A}$. In this case, we have that

$$N(y^{-1}) \le \max(N(x^{-1}), N(y^{-1} - x^{-1})) \le \max(N(x^{-1}), N(x^{-1})\, N(x - y)\, N(y^{-1})), \tag{6.10.12}$$

using (6.10.6) in the second step. If (6.10.7) holds, then we get that

$$N(y^{-1}) \le N(x^{-1}). \tag{6.10.13}$$

This implies that

$$N(x^{-1} - y^{-1}) \le N(x^{-1})^2\, N(x - y) \tag{6.10.14}$$

when (6.10.7) holds, by (6.10.6).

6.11 Banach spaces and algebras

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N_V with respect to $|\cdot|$ on k. If V is complete with respect to the metric associated to N_V, then V is said to be a *Banach space* with respect to N_V. Otherwise, one can pass to a suitable completion of V, by standard arguments.

Let W be another vector space over k, and let N_W be a norm on W with respect to $|\cdot|$ on k. Also let $\|\cdot\|_{op}$ be the operator norm on the space $\mathcal{BL}(V, W)$ of bounded linear mappings from V into W corresponding to N_V and N_W, as in Section 6.9. If W is complete with respect to the metric associated to N_W, then it is well known that $\mathcal{BL}(V, W)$ is complete with respect to the metric associated to $\|\cdot\|_{op}$, by standard arguments.

Let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N. If $\mathcal{A}$ is complete with respect to the metric associated to N, then $\mathcal{A}$ is said to be a *Banach algebra* with respect to N. As usual, if $\mathcal{A}$ is not complete, then one can pass to a suitable completion of $\mathcal{A}$. The condition that $\mathcal{A}$ have a multiplicative identity element e with $N(e) = 1$ is sometimes included in the definition of a Banach algebra.

If V is a Banach space with respect to N_V, then the algebra $\mathcal{BL}(V)$ of bounded linear mappings from V into itself is a Banach algebra with respect to the corresponding operator norm $\|\cdot\|_{op}$. One may also ask that $V \ne \{0\}$, to get that the identity operator on V has operator norm equal to 1.

Let $\mathcal{A}$ be an algebra over k again, and suppose that $\mathcal{A}$ has a multiplicative identity element e. If $a \in \mathcal{A}$ and n is a nonnegative integer, then

$$(e - a) \sum_{j=0}^{n} a^j = \Big(\sum_{j=0}^{n} a^j\Big) (e - a) = e - a^{n+1}, \tag{6.11.1}$$

where a^j is interpreted as being equal to e when $j = 0$. Let N be a submultiplicative norm on $\mathcal{A}$, and suppose that

$$N(a) < 1. \tag{6.11.2}$$

If $j \in \mathbf{Z}_+$, then $N(a^j) \le N(a)^j$, by submultiplicativity, and hence $N(a^j) \to 0$ as $j \to \infty$. One can also check that the sequence of partial sums $\sum_{j=0}^n a^j$ is a Cauchy sequence in $\mathcal{A}$ with respect to the metric associated to N in this case. If $\mathcal{A}$ is complete with respect to the metric associated to N, then this sequence of partial sums converges in $\mathcal{A}$, and the limit is denoted $\sum_{j=0}^\infty a^j$, as usual. Under these conditions, we get that

$$(e - a) \sum_{j=0}^\infty a^j = \Big(\sum_{j=0}^\infty a^j\Big)(e - a) = e, \tag{6.11.3}$$

by taking the limit as $n \to \infty$ in (6.11.1). This means that $e - a$ is invertible in $\mathcal{A}$, with

$$(e - a)^{-1} = \sum_{j=0}^\infty a^j. \tag{6.11.4}$$

Suppose that $x \in \mathcal{A}$ is invertible, and that $y \in \mathcal{A}$ satisfies

$$N(x^{-1})\, N(x - y) < 1 \tag{6.11.5}$$

Observe that

$$y = x - (x - y) = x\,(e - x^{-1}\,(x - y)), \tag{6.11.6}$$

and that $N(x^{-1}\,(x - y)) < 1$, by (6.11.5). If $\mathcal{A}$ is complete with respect to the metric associated to N, then it follows that $e - x^{-1}\,(x - y)$ is invertible in $\mathcal{A}$, as before. This implies that y is invertible in $\mathcal{A}$ too, by (6.11.6). Thus the set of invertible elements of $\mathcal{A}$ is an open set with respect to the metric associated to N when $\mathcal{A}$ is complete.

Let $\{x_j\}_{j=1}^\infty$ be a sequence of invertible elements of $\mathcal{A}$ whose inverses are bounded with respect to N, so that

$$N(x_j^{-1}) \le C \tag{6.11.7}$$

for some nonnegative real number C and every $j \ge 1$. Suppose that $\{x_j\}_{j=1}^\infty$ also converges to some $x \in \mathcal{A}$ with respect to the metric associated to N. Note that

$$N(x_j^{-1} - x_l^{-1}) \le N(x_j^{-1})\, N(x_l^{-1})\, N(x_j - x_l) \le C^2\, N(x_j - x_l) \tag{6.11.8}$$

for every $j, l \ge 1$, using (6.10.6) in the first step, and (6.11.7) in the second step. This implies that $\{x_j^{-1}\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{A}$ with respect to the metric associated to N, because $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence, since it converges. If $\mathcal{A}$ is complete with respect to the metric associated to N, then it follows that $\{x_j^{-1}\}_{j=1}^\infty$ converges in $\mathcal{A}$. One can check that x in invertible in $\mathcal{A}$ under these conditions, with inverse equal to the limit of $\{x_j^{-1}\}_{j=1}^\infty$. This uses continuity of multiplication on $\mathcal{A}$, to get that the product of two convergent sequences in $\mathcal{A}$ converges to the product of the limits of the two sequences.

6.12 Some subgroups

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e and a submultiplicative seminorm N with respect to $|\cdot|$ on k. If x is an invertible element of $\mathcal{A}$, then

$$N(e) = N(x\,x^{-1}) \le N(x)\,N(x^{-1}). \tag{6.12.1}$$

Let us suppose from now on in this section that

$$N(e) = 1. \tag{6.12.2}$$

Consider the collection of invertible elements x of $\mathcal{A}$ such that

$$N(x),\ N(x^{-1}) \le 1. \tag{6.12.3}$$

In this case, we have that

$$N(x) = N(x^{-1}) = 1, \tag{6.12.4}$$

because of (6.12.1). Of course, e has this property, by (6.12.2). If $x \in \mathcal{A}$ is in this collection, then x^{-1} is in this collection too, because $(x^{-1})^{-1} = x$. If $y \in \mathcal{A}$ is another element of this collection, then it is easy to see that $x\,y$ is in this collection as well, using the submultiplicativity of N. Hence this collection is a subgroup of the group of invertible elements in $\mathcal{A}$.

Let us suppose from now on in this section that N is a semi-ultranorm on $\mathcal{A}$. In this case, the subgroup of the group of invertible elements of $\mathcal{A}$ described in the preceding paragraph is a relatively open set with respect to the metric associated to N. If $x \in \mathcal{A}$, then we let $B(x,r)$ be the open ball in $\mathcal{A}$ centered at x with radius $r > 0$ with respect to the semi-ultrametric associated to N, and $\overline{B}(x,r)$ be the closed ball in $\mathcal{A}$ centered at x with radius $r \ge 0$, as usual. Note that

$$\overline{B}(e,1) = \overline{B}(0,1), \tag{6.12.5}$$

because of (6.12.2) and the ultrametric version of the triangle inequality. If N is a norm on $\mathcal{A}$, and $\mathcal{A}$ is complete with respect to the metric associated to $\mathcal{A}$, then every element of $B(e,1)$ is invertible in $\mathcal{A}$, as in the previous section.

If $x, y \in \mathcal{A}$, then

$$\begin{aligned} N(x\,y - e) = N(x\,(y-e) + x - e) &\le \max(N(x\,(y-e)), N(x-e)) \\ &\le \max(N(x)\,N(y-e), N(x-e)), \end{aligned} \tag{6.12.6}$$

using the ultrametric version of the triangle inequality in the second step, and the submultiplicativity of N in the third step. Thus

$$N(x\,y - e) \le \max(N(x-e), N(y-e)) \tag{6.12.7}$$

when $N(x) \le 1$. If $0 < r \le 1$ and $x, y \in B(e,r)$, then it follows that

$$x\,y \in B(e,r). \tag{6.12.8}$$

This also uses the fact that $B(e,r) \subseteq \overline{B}(0,1)$ when $r \le 1$, by (6.12.5). Similarly, if $0 \le r \le 1$ and $x, y \in \overline{B}(e,r)$, then

$$x\,y \in \overline{B}(e,r), \tag{6.12.9}$$

by (6.12.7) and the fact that $\overline{B}(e,r) \subseteq \overline{B}(0,1)$.

If $x \in B(e,1)$ and x is invertible in $\mathcal{A}$, then

$$N(x^{-1}) \le 1, \tag{6.12.10}$$

by (6.10.13) applied to $y = x$, and using (6.12.2). We also have that

$$N(x^{-1} - e) \le N(x - e), \tag{6.12.11}$$

by (6.10.14) applied to $y = x$. If $0 < r \le 1$ and $x \in B(e,r)$, then we get that

$$x^{-1} \in B(e,r). \tag{6.12.12}$$

Similarly, if $0 \le r < 1$ and $x \in \overline{B}(e,r)$, then

$$x^{-1} \in \overline{B}(e,r). \tag{6.12.13}$$

6.13 Related conditions on linear mappings

Let k be a field with an absolute value function $|\cdot|$, and let V, W be vector spaces over k, with seminorms N_V, N_W, respectively, with respect to $|\cdot|$ on k. Let T be a linear mapping from V into W, and suppose that

$$c\,N_V(v) \le N_W(T(v)) \tag{6.13.1}$$

for some $c > 0$ and every $v \in V$. Let R be a bounded linear mapping from V into W with respect to N_V, N_W, with

$$\|R\|_{op,VW} < c. \tag{6.13.2}$$

If $v \in V$, then

$$\begin{aligned} N_W(T(v)) &\le N_W(T(v) + R(v)) + N_W(R(v)) \\ &\le N_W(T(v) + R(v)) + \|R\|_{op,VW}\,N_V(v). \end{aligned} \tag{6.13.3}$$

Combining this with (6.13.1), we get that

$$(c - \|R\|_{op,VW})\,N_V(v) \le N_W(T(v) + R(v)) \tag{6.13.4}$$

for every $v \in V$.

Suppose for the moment that N_W is a semi-ultranorm on W. In this case, we have that

$$\begin{aligned} N_W(T(v)) &\le \max(N_W(T(v) + R(v)), N_W(R(v))) \\ &\le \max(N_W(T(v) + R(v)), \|R\|_{op,VW}\,N_V(v)) \end{aligned} \tag{6.13.5}$$

for every $v \in V$. This implies that

$$c\, N_V(v) \le N_W(T(v) + R(v)) \tag{6.13.6}$$

for every $v \in V$. More precisely, (6.13.6) is trivial when $N_V(v) = 0$, and otherwise

$$\|R\|_{op,VW}\, N_V(v) < c\, N_V(v) \tag{6.13.7}$$

when $N_V(v) > 0$, by (6.13.2). This permits (6.13.6) to be obtained from (6.13.1) and (6.13.5), as desired.

If T is a one-to-one linear mapping from V onto W, then (6.13.1) is the same as saying that

$$N_V(T^{-1}(w)) \le (1/c)\, N_W(w) \tag{6.13.8}$$

for every $w \in W$. This means that the inverse mapping T^{-1} is bounded as a linear mapping from W into V, with

$$\|T^{-1}\|_{op,WV} \le 1/c. \tag{6.13.9}$$

If N_V is a norm on V, then (6.13.1) implies that the kernel of T is trivial, so that T is injective. If V and W are finite-dimensional vector spaces over k with the same dimension, then it is well known that any injective linear mapping from V into W is surjective as well.

A linear mapping T from V into W is said to be an *isometry* with respect to N_V and N_W if

$$N_W(T(v)) = N_V(v) \tag{6.13.10}$$

for every $v \in V$. This is the same as saying that T is a bounded linear mapping from V into W, with

$$\|T\|_{op,VW} \le 1, \tag{6.13.11}$$

and that (6.13.1) holds with $c = 1$. A one-to-one linear mapping from V onto W is an isometry if and only if T is a bounded linear mapping from V into W that satisfies (6.13.11), and T^{-1} is a bounded linear mapping from W into V with

$$\|T^{-1}\|_{op,WV} \le 1. \tag{6.13.12}$$

In this case, T^{-1} is an isometric linear mapping from W into V.

Of course, the identity mapping I on V is an isometric linear mapping from V into itself, using N_V on both the domain and the range. The collection of one-to-one isometric linear mappings from V onto itself is a group with respect to composition of mappings. If N_V is a norm on V, then isometric linear mappings from V into itself are automatically injective, as before. If V has finite dimension, then injective linear mappings from V into itself are automatically surjective.

6.14 Multiplicative total boundedness

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e and a submultiplicative seminorm N with respect to $|\cdot|$ on k. We have seen that the group of invertible elements of $\mathcal{A}$ is a topological group, with respect to the topology induced by the topology determined on $\mathcal{A}$ by the semimetric associated to N. More precisely, multiplication is continuous on $\mathcal{A}$, and in particular on the group of invertible elements in $\mathcal{A}$, with respect to the induced topology. We have also seen that $x \mapsto x^{-1}$ is continuous as a mapping from the group of invbertible elements into itself, with respect to the induced topology.

Let E be a set of invertible elements of $\mathcal{A}$. The notions of left and right-invariant total boundedness of E in the group of invertible elements of $\mathcal{A}$ can be defined as in Section 4.3. These notions can be reformulated equivalently in this situation as follows. If $x \in \mathcal{A}$ and $r > 0$, then the open ball in $\mathcal{A}$ centered at x with radius r with respect to the semimetric associated to N is denoted $B(x, r)$, as usual. The open balls centered at e form a local base for the topology determined on $\mathcal{A}$ by the semimetric associated to N at e, and so their intersections with the group of invertible elements forms a local base for the induced topology at e. It follows that E is left-invariant totally bounded in the group of invertible elements of $\mathcal{A}$ if and only if for every $r > 0$ there are finitely many invertible elements $x_1, \ldots, x_n$ of $\mathcal{A}$ such that

$$E \subseteq \bigcup_{j=1}^{n} x_j\, B(e, r). \tag{6.14.1}$$

Similarly, E is right-invariant totally bounded in the group of invertible elements of $\mathcal{A}$ if and only if for every $r > 0$ there are finitely many invertible elements $x_1, \ldots, x_n$ in $\mathcal{A}$ such that

$$E \subseteq \bigcup_{j=1}^{n} B(e, r)\, x_j. \tag{6.14.2}$$

Of course, these conditions become more restrictive as r decreases, and so it suffices to consider small r.

Let us suppose from now on in this section that $N(e) > 0$, which means that $N(e) \geq 1$, as before. Otherwise, if $N(e) = 0$, then $N(x) = 0$ for every $x \in \mathcal{A}$, and the total boundedness conditions mentioned in the preceding paragraph hold trivially. In particular, $N(x) > 0$ for every invertible element x of $\mathcal{A}$, because $N(e) > 0$. Let $x \in \mathcal{A}$ be given, with $N(x) > 0$, and observe that

$$x\, B(e, r),\ B(e, r)\, x \subseteq B(x, r\, N(x)) \tag{6.14.3}$$

for every $r > 0$. If $y \in B(x, r\, N(x))$, then

$$N(y) \leq N(x) + N(y - x) < (1 + r)\, N(x). \tag{6.14.4}$$

If $r < 1$, then

$$N(x) \leq N(y) + N(x - y) < N(y) + r\, N(x) \tag{6.14.5}$$

for every $y \in B(x, r\,N(x))$, and hence

$$(1 - r)\,N(x) < N(y). \tag{6.14.6}$$

If E is left or right-invariant totally bounded in the group of invertible elements in $\mathcal{A}$, then it is easy to see that E is bounded with respect to N, using (6.14.3) and (6.14.4).

In the definitions of left and right-invariant total boundedness, one can take the points $x_1, \ldots, x_n$ to be elements of E. This follows from the characterization of total boundedness in terms of coverings by small sets, as in Section 4.4. In the present situation, this implies that $N(x_1), \ldots, N(x_n)$ are bounded, as before. Alternatively, we may require that E intersects $x_j\,B(e, r)$ or $B(e, r)\,x_j$, as appropriate, for each $j = 1, \ldots, n$, since the x_j's for which this does not hold are not needed in (6.14.1) or (6.14.2), as appropriate. This leads to an upper bound for $N(x_j)$ when $r < 1$, using (6.14.6) and the boundedness of N on E.

If E is left or right-invariant totally bounded in the group of invertible elements of $\mathcal{A}$, then E is totally bounded with respect to the semimetric associated to N on $\mathcal{A}$. This uses (6.14.3) and the fact that we can take the x_j's in (6.14.1) or (6.14.2), as appropriate, so that $N(x_j)$ is bounded. One can also take $r \le 1/2$ here, for the second argument in the previous paragraph.

If y is an invertible element of $\mathcal{A}$, $0 < r < 1/N(e)$, and $y \in B(e, r)$, then

$$N(y^{-1}) \le (1 - N(e)\,r)^{-1}\,N(e), \tag{6.14.7}$$

by (6.10.10). If E is left or right-invariant totally bounded in the group of invertible elements of $\mathcal{A}$, then one can check that the inverses of the elements of E are bounded with respect to N, using (6.14.1) or (6.14.2), as appropriate, with $r = 1/2\,N(e)$.

If x is an invertible element of $\mathcal{A}$, then

$$x^{-1}\,B(x, t),\ B(x, t)\,x^{-1} \subseteq B(e, t\,N(x^{-1})) \tag{6.14.8}$$

for every $t > 0$. Equivalently, this means that

$$B(x, t) \subseteq x\,B(e, t\,N(x^{-1})),\ B(e, t\,N(x^{-1}))\,x \tag{6.14.9}$$

for every $t > 0$.

Suppose that E is totally bounded with respect to the semimetric associated to N on $\mathcal{A}$, and that the inverses of the elements of E are bounded with respect to N. We would like to verify that E is both left and right-invariant totally bounded in the group of invertible elements of $\mathcal{A}$. If t is any positive real number, then there are finitely many elements $x_1, \ldots, x_n$ of $\mathcal{A}$ such that

$$E \subseteq \bigcup_{j=1}^{n} B(x_j, t), \tag{6.14.10}$$

by hypothesis. If $x_1, \ldots, x_n$ are invertible elements of $\mathcal{A}$, then it follows that

$$E \subseteq \bigcup_{j=1}^{n} x_j\,B(e, t\,N(x_j^{-1})),\ \bigcup_{j=1}^{n} B(e, t\,N(x_j^{-1}))\,x_j. \tag{6.14.11}$$

If $N(x_1^{-1}), \ldots, N(x_n^{-1})$ are also bounded, independently of t, then we can get the left and right-invariant total boundedness conditions for E from this.

In fact, we can choose $x_1, \ldots, x_n$ to be elements of E. This follows from the characterization of total boundedness with respect to semimetrics in terms of coverings by sets of small diameter, as in Section 4.2. If $x_1, \ldots, x_n \in E$, then the x_j's are invertible elements of $\mathcal{A}$ whose inverses are bounded with respect to N, by hypothesis.

Let $\{x_j\}_{j=1}^\infty$ be a sequence of invertible elements of $\mathcal{A}$. Observe that

$$N(x_j^{-1}\, x_l - e) = N(x_j^{-1}\,(x_l - x_j)) \le N(x_j^{-1})\, N(x_l - x_j) \tag{6.14.12}$$

for every $j, l \ge 1$. Similarly,

$$N(x_l\, x_j^{-1} - e) = N((x_l - x_j)\, x_j^{-1}) \le N(x_l - x_j)\, N(x_j^{-1}) \tag{6.14.13}$$

for every $j, l \ge 1$. If $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the semimetric on $\mathcal{A}$ associated to N, and if $\{x_j^{-1}\}_{j=1}^\infty$ is bounded with respect to N, then it follows that $\{x_j\}_{j=1}^\infty$ satisfies the left and right-invariant Cauchy conditions in the group of invertible elements of $\mathcal{A}$, as in Section 1.9.

Conversely, suppose that $\{x_j\}_{j=1}^\infty$ satisfies the left or right-invariant Cauchy condition in the group of invertible elements of $\mathcal{A}$. In particular, this implies that the set E of x_j's, $j \in \mathbf{Z}_+$, is left or right-invariant totally bounded in the group of invertible elements in $\mathcal{A}$, as in Section 1.9. It follows that $\{x_j\}_{j=1}^\infty$ and $\{x_j^{-1}\}_{j=1}^\infty$ are bounded with respect to N, as before. One can check that $\{x_j\}_{j=1}^\infty$ is also a Cauchy sequence with respect to the semimetric on $\mathcal{A}$ associated to N, using the boundedness of $\{x_j\}_{j=1}^\infty$ with respect to N.

Chapter 7

Haar measure and integration

7.1 Haar measure

Let G be a locally compact topological group such that $\{e\}$ is a closed set, so that G is Hausdorff. It is well known that there is a nonnegative Borel measure H_L on G, known as *left-invariant Haar measure* on G, with the following properties. First, $H_L(U) > 0$ for every nonempty open set $U \subseteq G$. Second, $H_L(K) < \infty$ when $K \subseteq G$ is compact, in which case K is a closed set, and hence a Borel set. Third, H_L is invariant under left translations, in the sense that

$$H_L(a\,E) = H_L(E) \tag{7.1.1}$$

for every Borel set $E \subseteq G$ and $a \in G$. Note that translates of Borel subsets of G are also Borel sets, because of continuity of translations. There are some additional regularity conditions that H_L should satisfy, that will be discussed in a moment. It is well known that H_L is unique, up to multiplication by a positive real number.

Similarly, there is a nonnegative Borel measure H_R on G, known as *right-invariant Haar measure* on G, with the following properties. As before, $H_R(U)$ is positive when U is a nonempty open subset of G, $H_R(K)$ is finite when $K \subseteq G$ is compact, and H_R should satisfy some additional regularity conditions. Invariance under right translations means that

$$H_R(E\,a) = H_R(E) \tag{7.1.2}$$

for every Borel set $E \subseteq G$ and $a \in G$. Any other nonnegative Borel measure on G with these properties is equal to H_R times a positive real number. Of course, invariance under left and right translations on G are the same when G is commutative.

If G is any group equipped with the discrete topology, then counting measure on G satisfies the requirements of left and right-invariant Haar measure. If

$G = \mathbf{R}^n$ for some positive integer n, as a commutative topological group with respect to addition and the standard topology, then n-dimensional Lebesgue measure satisfies the requirements of Haar measure. If G is a Lie group, then left and right-invariant Haar measures on G can be obtained using left and right-invariant volume forms on G, respectively. If G is a locally compact topological group such that $\{e\}$ is a closed set, and if H_L is a left-invariant Haar measure on G, then $H_L(E^{-1})$ satisfies the requirements of right-invariant Haar measure on G. Similarly, if H_R is a right-invariant Haar measure on G, then $H_R(E^{-1})$ satisfies the requirements of left-invariant Haar measure on G.

Let X be a locally compact Hausdorff topological space, and let μ be a nonnegative Borel measure on X. As usual, μ is said to be *outer regular* on X if

$$\mu(E) = \inf\{\mu(U) : U \subseteq X \text{ is an open set, and } E \subseteq U\} \tag{7.1.3}$$

for every Borel set $E \subseteq X$. A related *inner regularity* property for a Borel set $E \subseteq X$ is that

$$\mu(E) = \sup\{\mu(K) : K \subseteq X \text{ is compact, and } K \subseteq E\}. \tag{7.1.4}$$

The additional regularity properties for Haar measure mentioned earlier are outer regularity, and inner regularity for open sets, and for Borel sets of finite measure.

7.2 Haar integrals

Let X be a topological space, and let f be a real or complex-valued function on X. The *support* of f in X is defined to be the closure in X of the set of $x \in X$ such that $f(x) \neq 0$. Let $C_{com}(X, \mathbf{R})$ and $C_{com}(X, \mathbf{C})$ be the spaces of continuous real and complex-valued functions on X with compact support, respectively. These are vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, with respect to pointwise addition and scalar multiplication of functions. If X is a locally compact Hausdorff space, $K \subseteq X$ is compact, $U \subseteq X$ is an open set, and $K \subseteq U$, then a version of Urysohn's lemma implies that there is a nonnegative real-valued continuous function f on X with compact support contained in U such that $f(x) = 1$ for every $x \in K$, and $f(x) \leq 1$ for every $x \in X$.

Let G be a topological group, and let f be a real or complex-valued function on G. If $a \in G$, then let $L_a(f)$ be the real or complex-valued function defined on G by

$$(L_a(f))(x) = f(a\,x) \tag{7.2.1}$$

for every $x \in G$. Similarly, let $R_a(f)$ be the real or complex-valued function defined on G by

$$(R_a(f))(x) = f(x\,a) \tag{7.2.2}$$

for every $x \in G$. If f is continuous on G, then $L_a(f)$ and $R_a(f)$ are continuous on G for every $a \in G$, by continuity of translations. If f has compact support in G, then $L_a(f)$ and $R_a(f)$ have compact support for every $a \in G$ as well.

Suppose from now on in this section that G is locally compact, and that $\{e\}$ is a closed set in G. A *left-invariant Haar integral* on G is a linear functional I_L on the space of continuous real or complex-valued functions on G with compact support with the following properties. More precisely, I_L may be considered as a real-linear mapping from $C_{com}(G, \mathbf{R})$ into $\mathbf{R}$, or as a complex-linear mapping from $C_{com}(G, \mathbf{C})$ into $\mathbf{C}$. In the first case, I_L has a natural extension to a complex-linear functional on $C_{com}(G, \mathbf{C})$. In the second case, $I_L(f)$ should be a real number when f is a real-valued continuous function on G with compact support. In both cases, I_L should be invariant under left translations, in the sense that

$$I_L(L_a(f)) = I_L(f) \tag{7.2.3}$$

for every continuous real or complex-valued function f on G with compact support, and every $a \in G$. This linear functional should be nonnegative in the sense that

$$I_L(f) \geq 0 \tag{7.2.4}$$

when f is a nonnegative real-valued continuous function on G with compact support. If we also have that $f(x) > 0$ for some $x \in G$, then we ask that

$$I_L(f) > 0. \tag{7.2.5}$$

If H_L is a left-invariant Haar measure on G, then

$$I_L(f) = \int_G f\, dH_L \tag{7.2.6}$$

defines a left-invariant Haar integral on G. Conversely, if I_L is a left-invariant Haar integral on G, then one can get a corresponding left-invariant Haar measure using the Riesz representation theorem. It is well known that a left-invariant Haar integral is unique up to multiplication by a positive real number.

Similarly, a *right-invariant Haar integral* on G is a linear functional I_R on the space of continuous real or complex-valued functions on G with compact support with the following properties. As before, $I_R(f)$ should be a real number for every $f \in C_{com}(G, \mathbf{R})$, with $I_R(f) \geq 0$ when $f \geq 0$ on G, and $I_R(f) > 0$ when we also have that $f(x) > 0$ for some $x \in G$. Invariance under right translations means that

$$I_R(R_a(f)) = I_R(f) \tag{7.2.7}$$

for every continuous real or complex-valued function f on G with compact support, and every $a \in G$. If H_R is a right-invariant Haar measure on G, then

$$I_R(f) = \int_G f\, dH_R \tag{7.2.8}$$

is a right-invariant Haar integral on G, and conversely a right-invariant Haar measure on G can be obtained from a right-invariant Haar integral on G using the Riesz representation theorem. As usual, it is well known that a right-invariant Haar integral is unique up to multiplication by a positive real number.

If f is a continuous real or complex-valued function on G with compact support, then $f(x^{-1})$ is a continuous function on G with compact support as well, because $x \mapsto x^{-1}$ is a homeomorphism from G onto itself. This can be used to go from a left-invariant Haar integral to a right-invariant Haar integral, and vice-versa. If G is commutative, then invariance under left and right translations are the same.

7.3 Comparing left and right translations

Let G be a locally compact topological group such that $\{e\}$ is a closed set, and let H_L be a left-invariant Haar measure on G. If $a \in G$, then it is easy to see that $H_L(E\,a)$ also satisfies the requirements of a left-invariant Haar measure on G. The uniqueness of left-invariant Haar measure implies that there is a positive real number $\lambda(a)$ such that

$$H_L(E\,a) = \lambda(a)\,H_L(E) \tag{7.3.1}$$

for all Borel sets $E \subseteq G$. Remember that $H_L(U)$ is positive and finite when U is an open subset of G that contains e and is contained in a compact set. This implies that $\lambda(a)$ is uniquely determined by (7.3.1). One can check that λ defines a group homomorphism from G into the multiplicative group $\mathbf{R}_+$ of positive real numbers. By construction, $\lambda(a) = 1$ for every $a \in G$ exactly when H_L is invariant under right translations on G as well. In particular, this holds automatically when G is commutative. Note that λ does not depend on the choice of H_L.

If f is a nonnegative Borel measurable function on G and $a \in G$, then $f(x\,a)$ is also Borel measurable, by continuity of translations, and

$$\int_G f(x\,a)\,dH_L(x) = (1/\lambda(a)) \int_G f\,dH_L. \tag{7.3.2}$$

To see this, suppose first that f is the *indicator function* $\mathbf{1}_E(x)$ associated to a Borel set $E \subseteq G$, which is equal to 1 when $x \in E$ and to 0 when $x \notin E$. In this case,

$$\mathbf{1}_E(x\,a) = \mathbf{1}_{E\,a^{-1}}(x), \tag{7.3.3}$$

so that (7.3.2) follows from (7.3.1). Otherwise, one can approximate f by non-negative Borel measurable simple functions, as usual. If f is a real or complex-valued function on G that is integrable with respect to H_L, then it follows that $f(x\,a)$ is integrable with respect to H_L too, and that (7.3.2) holds.

Alternatively, let I_L be a left-invariant Haar integral on G. If $a \in G$, then $I_L(R_a(f))$ satisfies the requirements of a left-invariant Haar integral on G as well. The uniqueness of left-invariant Haar integrals implies that there is a positive real number $\lambda(a)$ such that

$$I_L(R_a(f)) = (1/\lambda(a))\,I_L(f) \tag{7.3.4}$$

for all continuous real or complex-valued functions f on G with compact support. There is a nonnegative real-valued continuous function f on G with compact support such that $f(e) > 0$, because of the version of Urysohn's lemma mentioned earlier. In this case, $I_L(f)$ is a positive real number, so that $\lambda(a)$ is uniquely determined by (7.3.4). As before, one can use (7.3.4) to verify that λ defines a group homomorphism from G into $\mathbf{R}_+$, and λ does not depend on the choice of I_L. This characterization of λ is equivalent to the previous one, because of the relationship between left-invariant Haar measures and integrals.

One can use (7.3.4) to show that λ is continuous as a mapping from G into $\mathbf{R}_+$, with respect to the topology induced on $\mathbf{R}_+$ by the standard topology on $\mathbf{R}$. It suffices to check that λ is continuous at e, because λ is a group homomorphism. Let f be a nonnegative real-valued continuous function on G with compact support such that $f(e) > 0$, so that $I_L(f) > 0$. It is enough to verify that $I_L(R_a(f))$ is continuous as a real-valued function of $a \in G$ at e, because of (7.3.4). Let U be an open subset of G that contains e and is contained in a compact set. There is a compact set $K \subseteq G$ that contains the support of $R_a(f)$ for every $a \in U$. This uses the fact that if $A, B \subseteq G$ are compact, then $A\,B$ is compact, because of the compactness of $A \times B$ and the continuity of multiplication. To get that $I_L(R_a(f))$ is close to $I_L(f)$ when a is sufficiently close to e, one can use uniform continuity along compact sets, as in Section 4.1.

If G is compact, then one can verify directly that $\lambda(a) = 1$ for every $a \in G$, so that a left-invariant Haar measure or integral on G is invariant under right translations too. If G_0 is a compact subgroup of G, then λ maps G_0 onto a compact subgroup of $\mathbf{R}_+$, because of the continuity of λ. However, the only compact subgroup of $\mathbf{R}_+$ is the trivial subgroup $\{1\}$. This means that $\lambda(a) = 1$ for every $a \in G_0$, so that a left-invariant Haar measure or integral on G is invariant under right translations by elements of G_0. Of course, we could have started with a right-invariant Haar measure or integral on G instead, and considered its behavior with respect to left translations.

7.4 Some additional comparisons

Let G be a locally compact topological group such that $\{e\}$ is a closed set again, let H_L be a left-invariant Haar measure on G, and let λ be as in the previous section. If f is a nonnegative Borel measurable function on G, and $a \in G$, then

$$\text{(7.4.1)} \quad \int_G f(x\,a)\,(1/\lambda(x))\,dH_L(x) = \int_G f(x\,a)\,(1/\lambda(x\,a))\,\lambda(a)\,dH_L(x) = \int_G f(x)\,(1/\lambda(x))\,dH_L(x),$$

using (7.3.2) in the second step. This implies that

$$\text{(7.4.2)} \qquad E \mapsto \int_E (1/\lambda(x))\,dH_L(x)$$

is invariant under right translations, and one can check that this satisfies the other requirements of a right-invariant Haar measure on G. Alternatively, if I_L is a left-invariant Haar integral on G, f is a continuous real or complex-valued function on G with compact support, and $a \in G$, then

$$I_L(R_a(f)/\lambda) = I_L(R_a(f/\lambda))\,\lambda(a) = I_L(f), \tag{7.4.3}$$

using (7.3.4) in the second step. This means that $I_L(f/\lambda)$ is invariant under right translations, and it is easy to see that $I_L(f/\lambda)$ satisfies the other requirements of a right-invariant Haar integral on G.

If H_L is a left-invariant Haar measure on G, then we have seen that $H_L(E^{-1})$ satisfies the requirements of a right-invariant Haar measure on G. It follows that there is a positive real number c such that

$$\int_E (1/\lambda(x))\,dH_L(x) = c\,H_L(E^{-1}) \tag{7.4.4}$$

for all Borel sets $E \subseteq G$, by the uniqueness of right-invariant Haar measure. It is not too difficult to show that

$$c = 1. \tag{7.4.5}$$

Let U be an open subset of G such that $e \in U$, U is contained in a compact set, and U is symmetric about e. The latter condition can always be arranged by replacing U with $U \cap U^{-1}$, as usual. In this case, $H_L(U)$ is positive and finite, and (7.4.4) implies that

$$\int_U (1/\lambda(x))\,dH_L(x) = c\,H_L(U). \tag{7.4.6}$$

Of course, $\lambda(e) = 1$, by definition of λ, and we have seen that λ is continuous on G. This means that we can choose U so that λ is as close to 1 as we want on U. We can use (7.4.6) to get that c has to be as close to 1 as we want, so that (7.4.5) holds.

If f is a real or complex-valued function on G, then put

$$\widetilde{f}(x) = f(x^{-1}) \tag{7.4.7}$$

for every $x \in G$. Note that $\widetilde{f}$ is continuous and has compact support when f is continuous and has compact support, by the continuity of $x \mapsto x^{-1}$. If I_L is a left-invariant Haar integral on G, then $I_L(\widetilde{f})$ satisfies the requirements of a right-invariant Haar integral on G. The uniqueness of right-invariant Haar integrals implies that

$$I_L(f/\lambda) = c\,I_L(\widetilde{f}) \tag{7.4.8}$$

for some $c > 0$ and all real or complex-valued continuous functions f on G with compact support, because $I_L(f/\lambda)$ is a right-invariant Haar integral on G too. One can check directly that $c = 1$ in this situation, using the same type of argument as before. More precisely, let us say that a real or complex-valued function f on G is *symmetric* about e if

$$\widetilde{f} = f. \tag{7.4.9}$$

In order to show that $c = 1$, one can use nonnegative real-valued continuous functions on G that are symmetric about e, positive at e, and have compact support contained in a small neighborhood of e.

In particular, if H_L is a left-invariant Haar measure on G that is also invariant under right translations, then

$$H_L(E) = H_L(E^{-1}) \tag{7.4.10}$$

for all Borel sets $E \subseteq G$. This can be seen more directly, using the fact that $H_L(E^{-1})$ is invariant under left translations. Uniqueness of left-invariant Haar measure implies that $H_L(E^{-1})$ is a constant multiple of $H(E)$, and one would like to show that this constant is equal to 1. To do this, one can take E to be an open set that contains e, is contained in a compact set, and is symmetric about e, so that $H_L(E)$ is positive and finite, and (7.4.10) holds. Similarly, if I_L is a left-invariant Haar integral on G that is invariant under right translations as well, then one can verify more directly that

$$I_L(f) = I_L(\widetilde{f}) \tag{7.4.11}$$

for every real or complex-valued continuous function f on G with compact support, using uniqueness of left-invariant Haar integrals, and nonnegative real-valued continuous functions on G with compact support that are symmetric about e and positive at e.

7.5 Automorphisms and Haar measure

If G is a topological group, then an *automorphism* of G as a topological group is a group automorphism of G that is also a homeomorphism. The collection of these automorphisms is a group with respect to composition of mappings. If $a \in G$, then

$$C_a(x) = a\,x\,a^{-1} \tag{7.5.1}$$

defines an automorphism on G as a topological group, which is the inner automorphism associated to a. Note that $a \mapsto C_a$ defines a group homomorphism from G into the group of automorphisms on G as a topological group.

Let G be a locally compact topological group such that $\{e\}$ is a closed set, and let H_L be a left-invariant Haar measure on G. If α is an automorphism of G as a topological group, then $H_L(\alpha(E))$ satisfies the requirements of left-invariant Haar measure on G. This implies that there is a positive real number $\Lambda(\alpha)$ such that

$$H_L(\alpha(E)) = \Lambda(\alpha)\, H_L(E) \tag{7.5.2}$$

for all Borel sets $E \subseteq G$, by the uniqueness of left-invariant Haar measure. Of course, $\Lambda(\alpha)$ is uniquely determined by (7.5.2), because there are Borel sets E such that $H_L(E)$ is positive and finite. It is easy to see that Λ defines a homomorphism from the group of automorphisms of G as a topological group into the multiplicative group $\mathbf{R}_+$ of positive real numbers, and that Λ does not depend on the choice of H_L. If $a \in G$, then

$$H_L(C_a(E)) = H_L(a\, E\, a^{-1}) = H_L(E\, a^{-1}) \tag{7.5.3}$$

for all Borel sets $E \subseteq G$, using invariance under left translations in the second step. This implies that

$$\Lambda(C_a) = \lambda(a^{-1}) = 1/\lambda(a), \tag{7.5.4}$$

where λ is as in Section 7.3.

Let α be an automorphism of G as a topological group again. If f is a nonnegative Borel measurable function on G, then $f \circ \alpha$ is Borel measurable too, and

$$\int_G f \circ \alpha \, dH_L = (1/\Lambda(\alpha)) \int_G f \, dH_L. \tag{7.5.5}$$

Indeed, if f is the indicator function $\mathbf{1}_E$ of a Borel set $E \subseteq G$, then (7.5.5) follows from (7.5.2) applied to α^{-1}, because

$$\mathbf{1}_E \circ \alpha = \mathbf{1}_{\alpha^{-1}(E)}. \tag{7.5.6}$$

Otherwise, one can reduce to this case, by approximating f by nonnegative Borel measurable simple functions. If f is a real or complex-valued function on G that is integrable with respect to H_L, then it follows that $f \circ \alpha$ is integrable with respect to H_L as well, and that (7.5.5) holds.

Alternatively, let I_L be a left-invariant Haar integral on G, and let α be an automorphism of G as a topological group. If f is a continuous real or complex-valued function on G with compact support, then $f \circ \alpha$ is a continuous function on G with compact support too. It is easy to see that $I_L(f \circ \alpha)$ also satisfies the requirements of a left-invariant Haar integral on G. The uniqueness of left-invariant Haar integrals implies that there is a positive real number $\Lambda(\alpha)$ such that

$$I_L(f \circ \alpha) = (1/\Lambda(\alpha))\, I_L(f) \tag{7.5.7}$$

for all continuous real or complex-valued functions f on G with compact support. As usual, $\Lambda(\alpha)$ is uniquely determined by (7.5.7), because there are such functions f such that $I_L(f) \neq 0$. One can use this to check that Λ is a homomorphism from the group of all automorphisms of G as a topological group into $\mathbf{R}_+$, and that Λ does not depend on the choice of I_L. This characterization of Λ is equivalent to the previous one, because of the relationship between left-invariant Haar measures and integrals.

If G is compact, then it is easy to see that $\Lambda(\alpha) = 1$ for every automorphism α of G as a topological group, using either of the previous characterizations of $\Lambda(\alpha)$.

7.6 More on regularity conditions

Let X be a locally compact Hausdorff topological space, and let μ be a nonnegative Borel measure on X such that $\mu(K) < \infty$ for every compact set $K \subseteq X$. Remember that compact subsets of X are closed sets, because X is Hausdorff. If every Borel set $E \subseteq X$ has the inner regularity property (7.1.4), then μ is said to be *inner regular* on X. If μ is both inner and outer regular on X, then μ is said to be *regular* on X.

A Borel set $E \subseteq X$ is said to be *σ-finite* with respect to μ if E can be expressed as the union of a sequence of Borel sets with finite measure with respect to μ. If the inner regularity condition (7.1.4) holds for all Borel subsets of X with finite measure with respect to μ, then one can check that (7.1.4) holds for all Borel subsets of X that are σ-finite with respect to μ. Of course, if X is σ-finite with respect to μ, then all Borel subsets of X are σ-finite with respect to μ. Thus if X is σ-finite with respect to μ, and if (7.1.4) holds for all Borel subsets of X with finite measure with respect to μ, then μ is inner regular on X.

A subset E of X is said to be *σ-compact* if E can be expressed as the union of a sequence of compact subsets of X. In particular, this implies that E is σ-finite with respect to μ, because of the hypothesis that compact subsets of X have finite measure with respect to μ. If E is σ-compact, then it is easy to see that E satisfies the inner regularity condition (7.1.4) with respect to μ. More precisely, if E is σ-compact, then E can be expressed as the union of an increasing sequence of compact sets, because the union of finitely many compact sets is compact too. In this case, the measures of these compact sets with respect to μ tend to $\mu(E)$.

A milder *inner regularity* property for a Borel set $E \subseteq X$ is that

$$\mu(E) = \sup\{\mu(A) : A \subseteq X \text{ is a closed set, and } A \subseteq E\}. \tag{7.6.1}$$

The inner regularity condition (7.1.4) automatically implies this one, because compact subsets of X are closed sets. If X is compact, then every closed set in X is compact, and (7.6.1) implies (7.1.4).

If $\mu(X) < \infty$, then one can check that (7.6.1) holds for every Borel set $E \subseteq X$ if and only if μ is outer regular on X, in the sense that (7.1.3) holds for every Borel set $E \subseteq X$. More precisely, (7.6.1) corresponds to the outer regularity condition (7.1.3) applied to $X \setminus E$ in this situation.

If X is σ-compact, then every closed set in X is σ-compact as well. This implies that closed subsets of X satisfy (7.1.4), as before. One can use this to check that (7.6.1) implies (7.1.4) for every Borel set $E \subseteq X$ in this case.

A subset of X is said to be an *F_σ set* if it can be expressed as the union of a sequence of closed sets. Of course, F_σ sets are Borel sets, and σ-compact subsets of X are F_σ sets, because X is Hausdorff. If $E \subseteq X$ is an F_σ-set, then E can be expressed as the union of an increasing sequence of closed sets, because the union of finitely many closed sets is closed as well. This implies that E satisfies (7.6.1), as before. If X is σ-compact, then F_σ sets in X are σ-compact.

A subset of X is said to be a G_δ *set* if it can be expressed as the intersection of a sequence of open sets. Thus G_δ sets are Borel sets, and a subset of X is an F_σ set if and only if its complement in X is a G_δ set. If the topology on X is determined by a metric, then it is well known that every closed set in X is a G_δ set. This can be obtained as in (1.3.3), but using the intersection over r of the form $1/j$, with $j \in \mathbf{Z}_+$. It follows that open subsets of X are F_σ sets in this case.

If every open subset of X is σ-compact, then μ is automatically regular on X, as in Theorem 2.18 on p50 in [21]. This condition on X is equivalent to asking that X be σ-compact, and that every open subset of X be an F_σ set, because σ-compact sets in X are F_σ sets, and because F_σ sets in X are σ-compact when X is σ-compact. If the topology on X is determined by a metric, then every open set in X is an F_σ set, and so it suffices to ask that X be σ-compact. It is well known that compact metric spaces are separable, which implies that σ-compact metric spaces are separable as well. Remember that separable metric spaces have bases for their topologies with only finitely or countably many elements.

Suppose now that X has a base for its topology with only finitely or countably many elements. This implies that X is σ-compact, because X is locally compact, and using Lindelöf's theorem. It is well known that X is regular as a topological space, because X is locally compact and Hausdorff. If $W \subseteq X$ is an open set and $x \in W$, then it follows that there is an open set $U \subseteq X$ such that $x \in U$, the closure $\overline{U}$ of U in X is contained in W, and $\overline{U}$ is compact. This implies that W is σ-compact, using Lindelöf's theorem again.

If a topological space is regular in the strong sense, and has a base for its topology with only finitely or countably many elements, then there is a metric that determines the topology, by famous theorems of Urysohn and Tychonoff. One can use Lindelöff's theorem to check directly that open sets are F_σ sets in this case.

7.7 Haar measure and products

Let $G_1, \ldots, G_n$ be finitely many locally compact topological groups, in which the set containing only the identity element is a closed set. Under these conditions,

$$G = \prod_{j=1}^{n} G_j \tag{7.7.1}$$

is a locally compact topological group with respect to the corresponding product topology, where the group operations are defined coordinatewise, and for which the set containing only the identity element is a closed set. Suppose for the moment that for each $j = 1, \ldots, n$ there is a base for the topology of G_j with only finitely or countably many elements. This implies that there is a base for the topology of G with only finitely or countably many elements, using products of the elements of the bases for the G_j's. If $H_{L,j}$ is a left-invariant Haar measure on G_j for each $j = 1, \ldots, n$, then one can get a left-invariant

Haar measure H_L on G using the standard product measure construction. Note that open subsets of G can be expressed as countable unions of products of open subsets of the G_j's, because of the hypothesis about countable bases. This implies that open subsets of G are measurable in the standard product measure construction, so that Borel sets in G are measurable in the standard product measure construction too.

If the G_j's are not asked to have countable bases for their topologies, then one should be more careful about the construction of a Borel product measure with suitable regularity properties. Alternatively, if $I_{L,j}$ is a left-invariant Haar integral on G_j for each $j = 1, \ldots, n$, then one can get a left-invariant Haar integral I_L on G. More precisely, if f is a continuous real or complex-valued function on G with compact support, then one can get $I_L(f)$ using $I_{L,j}$ in each variable separately. Of course, there are analogous statements for right-invariant Haar measures and integrals.

Now let I be a nonempty set, and let G_j be a compact topological group for each $j \in I$, in which the set containing the identity element is a closed set. In this case,

$$G = \prod_{j \in I} G_j \tag{7.7.2}$$

is a compact topological group with respect to the corresponding product topology, where the group operations are defined coordinatewise, and for which the set containing only the identity element is a closed set. Of course, this reduces to the previous situation when I has only finitely many elements. Suppose for the moment that I is countably infinite, and that for each $j \in I$ there is a base $\mathcal{B}_j$ for the topology of G_j with only finitely or countably many elements. Let $\mathcal{B}$ be the collection of subsets of G of the form $U = \prod_{j \in I} U_j$, where $U_j = G_j$ for all but finitely many $j \in I$, and $U_j \in \mathcal{B}_j$ otherwise. It is well known and not too difficult to show that $\mathcal{B}$ is a base for the product topology on G with only finitely or countably many elements. Let H_j be Haar measure on G_j for each $j \in I$, which is invariant under both left and right translations, because G_j is compact, and normalized so that

$$H_j(G_j) = 1. \tag{7.7.3}$$

One can get Haar measure H on G using a standard product construction for probability measures. Open subsets of G are measurable with respect to this type of product measure construction, because they can be expressed as unions of elements of $\mathcal{B}$. This implies that Borel subsets of G are measurable with respect to the product measure construction as well.

Otherwise, one should be more careful about the construction of a Borel product measure with suitable regularity properties again. As before, one could start with a Haar integral on G_j for each $j \in I$, and get a Haar integral on G. The Haar integral on G_j is invariant under both left and right translations, because G_j is compact, and should normalized to be equal to 1 for the constant function on G_j equal to 1. If f is a continuous real or complex-valued function on G, then one would like to define the Haar integral of f on G using the Haar integral on G_j in each variable separately. To do this, one can first use the Haar integral on G_j for finitely many $j \in I$, and then pass to a suitable limit.

7.8 Haar measure and open subgroups

Let X be a topological space, and let Y be a subset of X, so that Y may be considered as a topological space with respect to the induced topology. If E is a Borel subset of X, then $E \cap Y$ is a Borel set in Y, with respect to the induced topology on Y. This can be obtained from the continuity of the natural inclusion mapping from Y into X. One can argue more directly that the collection of subsets E of X such that $E \cap Y$ is a Borel set in Y is a σ-algebra of subsets of X. This σ-algebra contains the open subsets of X, and hence contains the Borel sets in X.

Suppose now that Y is a Borel set in X. The collection of subsets A of Y such that A is a Borel set in X is a σ-algebra of subsets of Y. If $A \subseteq Y$ is relatively open in Y, then $A = V \cap Y$ for some open set V in X, and hence A is a Borel set in X. If $A \subseteq Y$ is a Borel set in Y, then it follows that A is a Borel set in X too. Of course, if $E \subseteq X$ is a Borel set, then $E \cap Y$ is a Borel set in X.

Let G be a locally compact topological group such that $\{e\}$ is a closed set, and let U be an open subgroup of G. Thus U is also a topological group with respect to the induced topology, and U is a closed set in G, as in Section 3.3. It is easy to see that U is locally compact with respect to the induced topology as well. As in the previous paragraphs, a subset E of U is a Borel set in G if and only if E is a Borel set in U, where U is considered as a topological space with respect to the induced topology. The restriction of left or right-invariant Haar measure on G to Borel subsets of U satisfies the requirements of left or right-invariant Haar measure on U, as appropriate.

Let $H_{L,U}$ be a left-invariant Haar measure on U, and let us look at how this can be used to get a left-invariant Haar measure H_L on G. Let $E \subseteq G$ be a Borel set, and let $a \in G$ be given. Observe that $a^{-1}(E \cap a\,U)$ is a Borel set in G that is contained in U, and hence is a Borel set in U, as before. Thus

$$H_{L,U}(a^{-1}(E \cap a\,U)) \tag{7.8.1}$$

is defined. If $b \in G$ satisfies $a\,U = b\,U$, then (7.8.1) is equal to

$$H_{L,U}(b^{-1}(E \cap b\,U)) = H_{L,U}(b^{-1}(E \cap a\,U)), \tag{7.8.2}$$

because $H_{L,U}$ is invariant under left translations on U. This means that (7.8.1) only depends on the coset $a\,U$ of U in G, and not on the particular representation $a\,U$ of this coset. One might like to define $H_L(E)$ to be the sum of (7.8.1) over all left cosets $a\,U$ of U in G. This works when there are only finitely or countably many left cosets of U in G, or when $E \cap (a\,U) \neq \emptyset$ for only finitely or countably many left cosets $a\,U$ of U in G. Otherwise, one should put $H_L(E) = +\infty$, in order to get outer regularity. Of course, there are analogous statements for right-invariant Haar measure.

If f is a continuous real or complex-valued function on U with compact support, then f can be extended to a continuous function on G with compact support, by putting $f = 0$ on the complement of U. Given a left or right-invariant Haar integral on G, one can get a left or right-invariant Haar integral on U, as appropriate, by applying the Haar integral on G to this extension of f. This corresponds to restricting left or right-invariant Haar measure on G to U, as before.

Let f be a continuous real or complex-valued function on G with compact support. Note that the support of f can be covered by finitely many left cosets of U in G. The restriction of f to each left coset of U in G has compact support in that coset, because the cosets of U are closed sets in G. Let $I_{L,U}$ be a left-invariant Haar integral on U. If $a \in U$, then the restriction of $f(a\,x)$ to $x \in U$ is a continuous function on U with compact support, to which we can apply $I_{L,U}$. If $b \in G$ and $a\,U = b\,U$, then the restriction of $f(b\,x)$ to $x \in U$ corresponds to translating the restriction of $f(a\,x)$ to $x \in U$ on the left by an element of U. This means that $I_{L,U}$ applied to the restriction of $f(a\,x)$ to $x \in U$ is the same as $I_{L,U}$ applied to the restriction of $f(b\,x)$ to $x \in U$ when $a\,U = b\,U$. We can define $I_L(f)$ by applying $I_{L,U}$ to the restriction of $f(a\,x)$ to $x \in U$ for each left coset $a\,U$ of U in G, and then summing over the left cosets. This sum reduces to a finite sum, because the support of f is covered by finitely many left cosets of U in G. One can check that this satisfies the requirements of a left-invariant Haar integral on G, and there are analogous statements for right-invariant Haar integrals.

Let G be a locally compact topological group again, and let K be a compact subset of G such that e is an element of the interior of K. We may also ask that K be symmetric about e, by replacing K with $K \cup K^{-1}$, if necessary. As before, we define K^j for $j \in \mathbf{Z}_+$ by putting $K^1 = K$ and $K^{j+1} = K^j\,K$ for every $k \geq 1$. Under these conditions,

$$\bigcup_{j=1}^{\infty} K^j \tag{7.8.3}$$

is a subgroup of G, as in Section 3.3. In fact, (7.8.3) is an open subgroup of G, because e is an element of the interior of K. One can check that K^j is compact for every $j \in \mathbf{Z}_+$, using continuity of multiplication on G, and induction. Hence (7.8.3) is σ-compact.

7.9 Real and complex numbers

Of course, the real line $\mathbf{R}$ is a commutative topological group with respect to addition and the standard topology. Let $H_{\mathbf{R}}$ be one-dimensional Lebesgue measure on $\mathbf{R}$, which satisfies the requirements of Haar measure on $\mathbf{R}$, as mentioned in Section 7.1. If $t \in \mathbf{R} \setminus \{0\}$, then

$$\alpha_t(x) = t\,x \tag{7.9.1}$$

defines an automorphism of **R** as a topological group with respect to addition. It is well known that

$$H_{\mathbf{R}}(\alpha_t(E)) = |t| \, H_{\mathbf{R}}(E) \tag{7.9.2}$$

for all Borel sets $E \subseteq \mathbf{R}$, where $|\cdot|$ is the standard absolute value function on **R**.

We may also consider $\mathbf{R} \setminus \{0\}$ as a locally compact commutative topological group with respect to multiplication and the topology induced by the standard topology on **R**. It is easy to see that

$$E \mapsto \int_E |x|^{-1} \, dx \tag{7.9.3}$$

satisfies the requirements of Haar measure on $\mathbf{R} \setminus \{0\}$, where dx refers to one-dimensional Lebesgue measure on **R**. Note that $\mathbf{R} \setminus \{0\}$ is isomorphic as a topological group to $\mathbf{R}_+ \times \{\pm 1\}$, where $\mathbf{R}_+$ is the multiplicative group of positive real numbers, equipped with the topology induced by the standard topology on **R**, and the multiplicative group $\{\pm 1\}$ is equipped with the discrete topology. The exponential function is an isomorphism from **R** as a topological group with respect to addition onto $\mathbf{R}_+$ as a topological group with respect to multiplication.

The complex plane **C** is a commutative topological group with respect to addition and the standard topology as well. This can be identified with $\mathbf{R}^2$ as a commutative topological group with respect to addition and the standard topology. Let $H_{\mathbf{C}} = H_{\mathbf{R}^2}$ be 2-dimensional Lebesgue measure on $\mathbf{R}^2$, which satisfies the requirements of Haar measure on $\mathbf{C} = \mathbf{R}^2$, as in Section 7.1. If $t \in \mathbf{C} \setminus \{0\}$, then

$$\alpha_t(z) = t \, z \tag{7.9.4}$$

defines an automorphism on **C** as a topological group with respect to addition. In this case, we have that

$$H_{\mathbf{C}}(\alpha_t(E)) = |t|^2 \, H_{\mathbf{C}}(E) \tag{7.9.5}$$

for all Borel sets $E \subseteq \mathbf{C}$, where $|\cdot|$ is the standard absolute value function on **C**.

As before, $\mathbf{C} \setminus \{0\}$ is a locally compact commutative topological group with respect to multiplication and the topology induced by the standard topology on **C**. One can check that

$$E \mapsto \int_E |z|^{-2} \, dH_{\mathbf{C}}(z) \tag{7.9.6}$$

satisfies the requirements of Haar measure on $\mathbf{C} \setminus \{0\}$. Let

$$\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\} \tag{7.9.7}$$

be the unit circle in **C**, which is a compact subgroup of **C**. Thus **T** is a commutative topological group with respect to multiplication and the topology induced by the standard topology on **C**. Observe that $\mathbf{C} \setminus \{0\}$ is isomorphic as a topological group to $\mathbf{R}_+ \times \mathbf{T}$, equipped with the corresponding product topology.

The usual arclength measure on $\mathbf{T}$ satisfies the requirements of Haar measure. As in Section 7.7, one can get Haar measure on $\mathbf{R}_+ \times \mathbf{T}$ using Haar measures on $\mathbf{R}_+$ and $\mathbf{T}$ and the standard product measure construction. This corresponds to looking at Haar measure on $\mathbf{C} \setminus \{0\}$ in terms of polar coordinates.

7.10 Other fields

Let k be a field, and let $|\cdot|$ be an ultrametric absolute value function on k. If $|\cdot|$ is the trivial absolute value function on k, then the associated metric is the discrete metric, which determines the discrete topology on k. In this case, counting measure on k satisfies the requirements of Haar measure on k, as a commutative topological group with respect to addition. Similarly, counting measure on $k \setminus \{0\}$ satisfies the requirements of Haar measure on $k \setminus \{0\}$, as a commutative topological group with respect to multiplication. Let us suppose from now on in this section that $|\cdot|$ is nontrivial on k.

Let us also suppose from now on in this section that k is complete with respect to the ultrametric associated to $|\cdot|$, that $|\cdot|$ is discrete on k, and that the residue field associated to $|\cdot|$ has a finite number N of elements. As in Section 6.3, the discreteness and nontriviality of $|\cdot|$ on k means that if ρ_1 is the nonnegative real number defined in (6.3.3), then $0 < \rho_1 < 1$. We have also seen that the positive values of $|\cdot|$ are the same as the integer powers of ρ_1 in this case. In particular, the open unit ball $B(0,1)$ in k with respect to the ultrametric associated to $|\cdot|$ is the same as the closed ball $\overline{B}(0,\rho_1)$. The condition on the residue field means that the closed unit ball $\overline{B}(0,1)$ is the union of N pairwise-disjoint translates of $B(0,1)$. This is the same as saying that $\overline{B}(0,1)$ is the union of N pairwise-disjoint closed balls of radius ρ_1 in this situation. It follows that for every $x \in k$ and $j \in \mathbf{Z}_+$, $\overline{B}(x,\rho_1^j)$ can be expressed as N pairwise-disjoint closed balls of radius ρ_1^{j+1}.

Repeating the process, we get that for every $x \in k$, $j \in \mathbf{Z}$, and $l \in \mathbf{Z}_+$, $\overline{B}(x,\rho_1^j)$ can be expressed as the union of N^l pairwise-disjoint closed balls of radius ρ_1^{j+l}. In particular, closed balls in k are totally bounded with respect to the ultrametric associated to $|\cdot|$, as in Section 6.6. This implies that closed balls in k are compact, because k is complete with respect to the ultrametric associated to $|\cdot|$, as before. Thus k is locally compact with respect to the topology determined by the ultrametric associated to $|\cdot|$.

Let H_k be Haar measure on k, as a locally compact commutative topological group with respect to addition. We can normalize H_k so that

$$H_k(\overline{B}(0,1)) = 1, \tag{7.10.1}$$

by dividing by the Haar measure of $\overline{B}(0,1)$, which is positive and finite. Using this normalization, we get that

$$H_k(\overline{B}(x,\rho_1^j)) = N^{-j} \tag{7.10.2}$$

for every $x \in k$ and $j \in \mathbf{Z}$. More precisely, the left side of (7.10.2) does not depend on x, because of translation-invariance of Haar measure. Thus (7.10.2) follows from (7.10.1) when $j = 0$. If $j \geq 1$, then (7.10.2) follows from (7.10.1) and the fact that $\overline{B}(0, 1)$ can be expressed as the union of N^j pairwise-disjoint closed balls of radius ρ_1^j. Similarly, if $j \leq -1$, then (7.10.2) follows from (7.10.1) and the fact that $\overline{B}(x, \rho_1^j)$ can be expressed as the union of N^{-j} closed balls of radius 1.

Let $t \in k \setminus \{0\}$ be given, so that

$$|t| = \rho_1^{j_0} \tag{7.10.3}$$

for some $j_0 \in \mathbf{Z}$. Note that

$$\alpha_t(x) = t\,x \tag{7.10.4}$$

defines an automorphism on k as a topological group with respect to addition. If $E \subseteq k$ is a Borel set, then

$$H_k(\alpha_t(E)) = N^{-j_0}\, H_k(E). \tag{7.10.5}$$

More precisely, if E is a closed ball in k of radius ρ_1^j for some $j \in \mathbf{Z}$, then $\alpha_t(E)$ is a closed ball of radius $\rho_1^{j+j_0}$, and (7.10.5) reduces to (7.10.2). One can get the same conclusion for all Borel sets E using uniqueness of Haar measure. Remember that $N \geq 2$, so that there is a positive real number a such that

$$\rho_1 = N^{-a}. \tag{7.10.6}$$

Thus $|t| = N^{-j_0}$, so that (7.10.5) is the same as saying that

$$H_k(\alpha_t(E)) = |t|^a\, H_k(E) \tag{7.10.7}$$

for all Borel sets $E \subseteq k$.

In this situation,

$$\{x \in k : |x| = 1\} \tag{7.10.8}$$

is a compact subset of k. This is also an open set in k, because $|\cdot|$ is an ultrametric absolute value function on k. Of course, (7.10.8) is a commutative group with respect to multiplication, and in fact a topological group with respect to the topology determined by the restriction to (7.10.8) of the ultrametric on k associated to $|\cdot|$. One can check that the restriction of H_k to (7.10.8) satisfies the requirements of Haar measure on (7.10.8), as a compact topological group with respect to multiplication. More precisely, the invariance of H_k under multiplicative translations on (7.10.8) corresponds to (7.10.5) with $|t| = 1$, so that $j_0 = 0$.

We can also consider $k \setminus \{0\}$ as a commutative topological group with respect to multiplication, and the topology determined by the restriction to $k \setminus \{0\}$ of the ultrametric on k associated to $|\cdot|$. This is isomorphic as a topological group to the product of (7.10.8) and

$$\{\rho_1^j : j \in \mathbf{Z}\}, \tag{7.10.9}$$

where (7.10.9) is considered as a commutative topological group with respect to multiplication and the discrete topology. Of course, (7.10.9) is isomorphic as a topological group to $\mathbf{Z}$, considered as a commutative topological group with respect to addition and the discrete topology.

7.11 Estimating λ, Λ

Let G be a locally compact topological group such that $\{e\}$ is a closed set, and let H_L be a left-invariant Haar measure on G. Suppose that $E_1, E_2 \subseteq G$ are Borel sets such that $H_L(E_1) > 0$ and $H_L(E_2) < \infty$. If $a \in G$ satisfies

$$E_1\, a \subseteq E_2, \tag{7.11.1}$$

then

$$H_L(E_1\, a) \le H_L(E_2). \tag{7.11.2}$$

Let $\lambda(a) > 0$ be as in Section 7.3, so that

$$H_L(E_1\, a) = \lambda(a)\, H_L(E_1). \tag{7.11.3}$$

It follows that

$$\lambda(a) \le H_L(E_2)/H_L(E_1) \tag{7.11.4}$$

when (7.11.1) holds.

Suppose now that E_1 is compact, E_2 is an open set, and

$$E_1 \subseteq E_2. \tag{7.11.5}$$

Of course, we can get $H_L(E_1) > 0$ by taking E_1 to have nonempty interior, and in particular one might take E_1 to contain e in its interior. Similarly, we can get $H_L(E_2) < \infty$ by taking E_2 to be contained in a compact set, which can be obtained using local compactness. As in Section 1.3, there is an open set $V \subseteq G$ such that $e \in V$ and

$$E_1\, V \subseteq E_2. \tag{7.11.6}$$

This means that (7.11.4) holds for every $a \in V$. If $a \in V^{-1}$, then we can apply the previous statement to $a^{-1} \in V$, to get that

$$1/\lambda(a) = \lambda(a^{-1}) \le H_L(E_2)/H_L(E_1). \tag{7.11.7}$$

Thus

$$H_L(E_1)/H_L(E_2) \le \lambda(a) \le H_L(E_2)/H_L(E_1) \tag{7.11.8}$$

when $a \in V \cap V^{-1}$. This gives another way to look at the continuity of λ at e, by choosing E_1 and E_2 so that $H_L(E_1)$ and $H_L(E_2)$ are very close to each other. More precisely, one can start with a compact set E_1 such that $H_L(E_1) > 0$, and use outer regularity to get open sets E_2 containing E_1 such that $H_L(E_2)$ is as close as one wants to $H_L(E_1)$. Alternatively, one can start with a nonempty open set E_2 such that $H_L(E_2) < \infty$, and use inner regularity for open sets to get compact sets E_1 contained in E_2 such that $H_L(E_1)$ is as close as one wants to $H_L(E_2)$.

Let E_1 be a compact set with $H_L(E_1) > 0$ again, and let K be a nonempty compact subset of G. Suppose that

$$E_1\, K \subseteq E_2, \tag{7.11.9}$$

where $H_L(E_2) < \infty$. Note that $E_1\,K$ is compact, because of continuity of multiplication on G, and so we could simply take $E_2 = E_1\,K$. If $a \in K$, then (7.11.4) holds, as before. Similarly, (7.11.7) holds when $a \in K^{-1}$, and hence (7.11.8) holds when $a \in K \cap K^{-1}$. Of course, one could also use the continuity of λ to get that $\lambda(K)$ is a compact subset of $\mathbf{R}_+$. If K is a compact subgroup of G, then it follows that $\lambda(a) = 1$ for every $a \in K$, because $\lambda(K)$ is a bounded subgroup of $\mathbf{R}_+$, and the trivial subgroup $\{1\}$ is the only bounded subgroup of $\mathbf{R}_+$. This could be obtained as well from the fact that $E_1\,K$ is invariant under translations on the right by elements of K in this case.

Let E_1, E_2 be Borel sets with $H_L(E_1) > 0$ and $H_L(E_2) < \infty$ again. If α is an automorphism of G as a topological group such that

$$\alpha(E_1) \subseteq E_2, \tag{7.11.10}$$

then

$$H_L(\alpha(E_1)) \leq H_L(E_2). \tag{7.11.11}$$

Let $\Lambda(\alpha) > 0$ be as in Section 7.5, so that

$$H_L(\alpha(E_1)) = \Lambda(\alpha)\, H_L(E_1). \tag{7.11.12}$$

Thus

$$\Lambda(\alpha) \leq H_L(E_2)/H_L(E_1) \tag{7.11.13}$$

when (7.11.10) holds.

Let $\mathcal{A}$ be a collection of automorphisms of G as a topological group. Let us say that $\mathcal{A}$ is *equicontinuous* at e if for every open set $W \subseteq G$ with $e \in W$ there is an open set $U \subseteq G$ such that $e \in U$ and

$$\alpha(U) \subseteq W \tag{7.11.14}$$

for every $\alpha \in \mathcal{A}$. In particular, if W is contained in a compact set, then it follows that Λ is bounded on $\mathcal{A}$, as in the preceding paragraph. Of course, if $\mathcal{A}$ is a subgroup of the group of automorphisms of G as a topological group, then $\Lambda(\mathcal{A})$ is a subgroup of $\mathbf{R}_+$, because Λ is a group homomorphism. If Λ is also bounded on $\mathcal{A}$, then $\Lambda(\alpha) = 1$ for every $\alpha \in \mathcal{A}$, because $\{1\}$ is the only bounded subgroup of $\mathbf{R}_+$.

7.12 Haar measure on k^n

In this section, we let k be a field with an absolute value function $|\cdot|$, which is in one of the following four cases.

Case 1. $k = \mathbf{R}$ with the standard absolute value function.

Case 2. $k = \mathbf{C}$ with the standard absolute value function.

Case 3. $|\cdot|$ is a nontrivial discrete ultrametric absolute value function on k, k is complete with respect to the ultrametric associated to $|\cdot|$, and the residue field associated to $|\cdot|$ on k is finite.

Case 4. $|\cdot|$ is the trivial absolute value function on k.

In each of these four cases, k may be considered as a locally compact commutative topological group with respect to addition, and using the topology determined by the metric associated to k. Let H_k be Haar measure on k, as in Sections 7.9 and 7.10.

Note that k is separable as a metric space in the first three cases, with respect to the metric associated to $|\cdot|$. This is well known in the first two cases. In the third case, this follows from the fact that closed balls in k centered at 0 are totally bounded, and that k is the union of a sequence of such balls. It follows that there is a countable base for the topology of k in the first three cases.

Let n be a positive integer, and let k^n be the space of n-tuples of elements of k, as usual. This may be considered as a locally compact commutative topological group with respect to addition, using the product topology corresponding to the topology determined on k by the metric associated to $|\cdot|$. Let H_{k^n} be Haar measure on k^n. In the first three cases, H_{k^n} can be obtained from H_k using the standard product measure construction. In the fourth case, the product topology on k^n is the discrete topology, and we can take H_{k^n} to be counting measure on k^n.

Let $a(k) > 0$ be as follows. In the first two cases, we put

$$a(\mathbf{R}) = 1, \quad a(\mathbf{C}) = 2. \tag{7.12.1}$$

This would have to be adjusted if we used a different power of the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$. In the third case, we take $a = a(k)$ to be as in (7.10.6). In the fourth case, we can take $a(k) = 1$.

Of course, k^n is also a vector space over k with respect to coordinatewise addition and scalar multiplication. Let T be a one-to-one linear mapping from k^n onto itself. This may also be considered as an automorphism on k^n as a topological group, because linear mappings from k^n into itself are continuous with respect to the product topology. Let us check that

$$H_{k^n}(T(E)) = |\det T|^{a(k)}\, H_{k^n}(E) \tag{7.12.2}$$

for every Borel set $E \subseteq k^n$. This is the same as saying that

$$|\Lambda(T)| = |\det T|^{a(k)}, \tag{7.12.3}$$

where $\Lambda(T) > 0$ is as in Section 7.5. This is trivial in the fourth case, and so we need only consider the first three cases. The argument is basically the same as in the classical case of real numbers.

It is well known that T can be expressed as the composition of finitely many "elementary" linear mappings, corresponding to elementary matrices. Thus it suffices to show that (7.12.2) holds when T is one of these elementary linear mappings, of which there are three types. The first type of elementary linear mapping corresponds to interchanging two coordinates of an element of k^n, which preserves Haar measure on k^n. The second type of elementary linear mapping corresponds to a diagonal matrix in which all but one diagonal entry is equal to 1, which can be handled using the remarks about the $n = 1$ case in Sections 7.9 and 7.10. The third type of elementary linear mapping corresponds to a matrix which is the same as the identity matrix except for one off-diagonal term, and this type of linear mapping also preserves Haar measure.

Let T be a linear mapping from $\mathbf{C}^n$ into itself, and let $\det_{\mathbf{C}} T$ be the usual complex determinant of T. If we identify $\mathbf{C}$ with $\mathbf{R}^2$ as a vector space over $\mathbf{R}$ in the usual way, then we can identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, and consider T as a real-linear mapping from $\mathbf{R}^{2n}$ into itself. It is well known that

$$\det_{\mathbf{R}} T = |\det_{\mathbf{C}} T|^2, \tag{7.12.4}$$

where $\det_{\mathbf{R}} T \in \mathbf{R}$ is the determinant of T as a real-linear mapping from $\mathbf{R}^{2n}$ into itself, and $|\cdot|$ is the standard absolute value function on $\mathbf{C}$. See Proposition 1.4.10 in [18].

7.13 Matrices and Haar measure

Let k be a field with an absolute value function $|\cdot|$, which is in one of the four cases mentioned at the beginning of the previous section. Also let n be a positive integer, and let $M_n(k)$ be the space of $n \times n$ matrices with entries in k, as before. This is a vector space over k with respect to coordinatewise addition and scalar multiplication, and in particular $M_n(k)$ is a commutative group with respect to addition. Using the topology determined on k by the metric associated to $|\cdot|$, we get a corresponding product topology on $M_n(k)$. In fact, $M_n(k)$ is a commutative topological group with respect to addition and this topology. Note that $M_n(k)$ is locally compact with respect to the product topology, because k is locally compact in this situation. Let $H_{M_n(k)}$ be Haar measure on $M_n(k)$. More precisely, $M_n(k)$ can be identified with k^{n^2} in a suitable way, so that $H_{M_n(k)}$ corresponds to $H_{k^{n^2}}$. Thus $H_{M_n(k)}$ can be obtained from H_k using a standard product measure construction in the first three cases, as in the previous section, and we can take $H_{M_n(k)}$ to be counting measure on $M_n(k)$ in the fourth case.

If $A, B, T \in M_n(k)$, then put

$$\alpha_A(T) = A\,T \tag{7.13.1}$$

and

$$\beta_B(T) = T\,B, \tag{7.13.2}$$

where $A\,T, T\,B \in M_n(k)$ are defined using matrix multiplication. Note that α_A and β_B define linear mappings from $M_n(k)$ into itself, as a vector space over k. Suppose that A, B are invertible with respect to matrix multiplication, so that α_A, β_B are invertible as linear mappings on $M_n(k)$. In particular, α_A and β_B may be considered as automorphisms of $M_n(k)$ as a commutative topological group with respect to addition. If $E \subseteq M_n(k)$ is a Borel set, then

$$H_{M_n(k)}(\alpha_A(E)) = |\det A|^{a(k)\,n}\, H_{M_n(k)}(E) \tag{7.13.3}$$

and

$$H_{M_n(k)}(\beta_B(E)) = |\det B|^{a(k)\,n}\, H_{M_n(k)}(E), \tag{7.13.4}$$

where $a(k) > 0$ is as in the previous section. This is the same as saying that

$$\Lambda(\alpha_A) = |\det A|^{a(k)\,n} \tag{7.13.5}$$

and

$$\Lambda(\beta_B) = |\det B|^{a(k)\,n}, \tag{7.13.6}$$

where Λ is as in Section 7.5. These statements may be considered as versions of the analogous statements in the previous section, using also the fact that the determinant of a matrix is equal to the determinant of its transpose.

If f is a nonnegative real-valued Borel measurable function on $M_n(k)$, then

$$\int_{M_n(k)} f \circ \alpha_A \, dH_{M_n(k)} = |\det A|^{-a(k)\,n} \int_{M_n(k)} f \, dH_{M_n(k)} \tag{7.13.7}$$

and

$$\int_{M_n(k)} f \circ \beta_B \, dH_{M_n(k)} = |\det B|^{-a(k)\,n} \int_{M_n(k)} f \, dH_{M_n(k)}, \tag{7.13.8}$$

as in Section 7.5. Similarly, if f is a real or complex-valued function on $M_n(k)$ that is integrable with respect to $H_{M_n(k)}$, then $f \circ \alpha_A$ and $f \circ \beta_B$ are integrable with respect to $H_n(k)$ too, and (7.13.7) and (7.13.8) hold, as before. Alternatively, if f is a continuous real or complex-valued function on $M_n(k)$ with compact support, then the corresponding Haar integral $I_{M_n(k)}(f)$ can be defined more directly. More precisely, in the first two cases for k in the previous section, $I_{M_n(k)}(f)$ can be obtained from an ordinary Riemann integral. One can basically obtain $I_{M_n(k)}(f)$ from a Riemann-type integral in the third case as well. In the fourth case, $M_n(k)$ is equipped with the discrete topology, and $I_{M_n(k)}(f)$ reduces to a finite sum when f has compact support. Of course, $f \circ \alpha_A$ and $f \circ \beta_B$ are continuous functions on $M_n(k)$ with compact support, and we have that

$$I_{M_n(k)}(f \circ \alpha_A) = |\det A|^{-a(k)\,n}\, I_{M_n(k)}(f) \tag{7.13.9}$$

and

$$I_{M_n(k)}(f \circ \beta_B) = |\det B|^{-a(k)\,n}\, I_{M_n(k)}(f). \tag{7.13.10}$$

Let $GL_n(k)$ be the group of invertible elements of $M_n(k)$ with respect to matrix multiplication, as in Section 6.8. Remember that $GL_n(k)$ is an open set in $M_n(k)$ with respect to the product topology mentioned earlier, and that $GL_n(k)$ is a topological group with respect to the topology induced by this topology on $M_n(k)$. In this situation, $GL_n(k)$ is locally compact, because k is locally compact. If $E \subseteq GL_n(k)$ is a Borel set, then put

$$H_{GL_n(k)}(E) = \int_E |\det T|^{-a(k)\,n}\, dH_{M_n(k)}(T). \tag{7.13.11}$$

One can check that this satisfies the requirements of left and right-invariant Haar measure on $GL_n(k)$.

The invariance of (7.13.11) under left and right translations on $GL_n(k)$ is basically the same as saying that if f is a nonnegative real-valued Borel measurable function on $GL_n(k)$ and $A, B \in GL_n(k)$, then

$$\int_{GL_n(k)} f(A\,T)\,|\det T|^{-a(k)\,n}\,dH_{M_n(k)} = \int_{GL_n(k)} f(T)\,|\det T|^{-a(k)\,n}\,dH_{M_n(k)} \tag{7.13.12}$$

and

$$\int_{GL_n(k)} f(T\,B)\,|\det T|^{-a(k)\,n}\,dH_{M_n(k)} = \int_{GL_n(k)} f(T)\,|\det T|^{-a(k)\,n}\,dH_{M_n(k)}. \tag{7.13.13}$$

It is easy to get (7.13.12) and (7.13.13) from (7.13.7) and (7.13.8), respectively. As usual, there are analogous statements for real and complex-valued functions on $GL_n(k)$ that are integrable with respect to $H_{GL_n(k)}$. One can also look at this in terms of Haar integrals. Let f be a continuous real or complex-valued function on $GL_n(k)$ with compact support. We can extend f to a continuous real or complex-valued function on $M_n(k)$ by putting $f(T) = 0$ when $T \in M_n(k)$ satisfies $\det T = 0$. The condition that f have compact support in $GL_n(k)$ implies that this extension has compact support in $M_n(k)$, and that the support of this extension does not contain any $T \in M_n(k)$ with $\det T = 0$. If we interpret

$$f(T)\,|\det T|^{-a(k)\,n} \tag{7.13.14}$$

as being equal to 0 when $\det T = 0$, then (7.13.14) defines a continuous function on $M_n(k)$ with compact support. The Haar integral $I_{GL_n(k)}(f)$ of f on $GL_n(k)$ can be defined by applying the Haar integral $I_{M_n(k)}$ on $M_n(k)$ to (7.13.14) as a continuous function on $M_n(k)$ with compact support. The invariance of $I_{GL_n(k)}$ under left and right translations on $GL_n(k)$ can be obtained from (7.13.9) and (7.13.10), as before.

Chapter 8

Spaces of continuous functions

8.1 Supremum semimetrics and compact sets

Let X be a nonempty topological space, and let Y be a nonempty set with a semimetric d_Y. Consider the space $\mathcal{B}^{com}(X,Y)$ of mappings f from X into Y that are bounded on compact subsets of X, so that $f(A)$ is a bounded subset of Y with respect to d_Y for every compact set $A \subseteq X$. Of course, if X is compact, then $\mathcal{B}^{com}(X,Y)$ is the same as the space $\mathcal{B}(X,Y)$ of all bounded mappings from X into Y. If A is a nonempty compact subset of X and $f, g \in \mathcal{B}^{com}(X,Y)$, then put

$$\theta_A(f,g) = \sup_{x \in A} d_Y(f(x), g(x)), \tag{8.1.1}$$

as in Section 2.1. This is the supremum semimetric on $\mathcal{B}^{com}(X,Y)$ associated to A and d_Y.

Consider the collection

$$\{\theta_A : A \subseteq X \text{ is nonempty and compact}\} \tag{8.1.2}$$

of all supremum semimetrics (8.1.1) associated to nonempty compact subsets of X, as a collection of semimetrics on $\mathcal{B}^{com}(X,Y)$. Note that (8.1.2) is nonempty, because finite subsets of X are compact. As in Section 1.5, (8.1.2) determines a topology on $\mathcal{B}^{com}(X,Y)$. If d_Y is a metric on Y, then (8.1.2) is nondegenerate on $\mathcal{B}^{com}(X,Y)$, because finite subsets of X are compact.

Let $A_1, \ldots, A_n$ be finitely many nonempty compact subsets of X, so that $A = \bigcup_{j=1}^{n} A_j$ is a nonempty compact subset of X as well. It is easy to see that

$$\theta_A(f,g) = \max_{1 \le j \le n} \theta_{A_j}(f,g) \tag{8.1.3}$$

for every $f, g \in \mathcal{B}^{com}(X,Y)$. Similarly, if A, B are nonempty compact subsets of X and $A \subseteq B$, then

$$\theta_A(f,g) \le \theta_B(f,g) \tag{8.1.4}$$

for every $f, g \in \mathcal{B}^{com}(X, Y)$.

Let $\mathcal{C}$ be a nonempty collection of nonempty compact subsets of X, so that

$$\{\theta_C : C \in \mathcal{C}\} \tag{8.1.5}$$

defines a nonempty collection of semimetrics on $\mathcal{B}^{com}(X, Y)$. Suppose that for every nonempty compact set $A \subseteq X$ there are finitely many elements $C_1, \ldots, C_n$ of $\mathcal{C}$ such that

$$A \subseteq \bigcup_{j=1}^{n} C_j. \tag{8.1.6}$$

This implies that

$$\theta_A(f, g) \le \max_{1 \le j \le n} \theta_{C_j}(f, g) \tag{8.1.7}$$

for every $f, g \in \mathcal{B}^{com}(X, Y)$, as in (8.1.3) and (8.1.4). It follows that the topology determined on $\mathcal{B}^{com}(X, Y)$ by (8.1.5) is the same as the topology determined by the collection of supremum semimetrics (8.1.2). If X is compact, then we can take $\mathcal{C} = \{X\}$.

If X is locally compact, then every compact set $K \subseteq X$ is contained in an open set $U \subseteq X$ such that U is contained in a compact subset of X. More precisely, for each $x \in K$ there is an open set $U(x) \subseteq X$ such that $x \in U(x)$ and $U(x)$ is contained in a compact set, because X is locally compact. The compactness of K implies that K is contained in the union U of finitely many such open sets, and U is contained in the union of finitely many compact sets, which is compact too.

Suppose that X is σ-compact, so that there is a sequence $K_1, K_2, K_3, \ldots$ of compact subsets of X such that

$$X = \bigcup_{j=1}^{\infty} K_j. \tag{8.1.8}$$

Of course, we may as well ask that $K_j \neq \emptyset$ for every $j \ge 1$. We may also ask that $K_j \subseteq K_{j+1}$ for every j, since otherwise we can replace K_j with $\bigcup_{l=1}^{j} K_j$.

If X is locally compact too, then we can replace the K_j's by somewhat larger compact sets, so that K_j is contained in the interior of K_{j+1} for every $j \ge 1$. In particular, the interior of K_j is contained in the interior of K_{j+1} for every $j \ge 1$, and the union of the interiors of the K_j's is equal to X.

If $A \subseteq X$ is compact, then it follows that A is contained in the interior of K_j for some j, and hence that $A \subseteq K_j$. This means that the collection of K_j's satisfies the condition mentioned earlier, so that the corresponding collection of supremum seminorms θ_{K_j} determines the same topology on $\mathcal{B}^{com}(X, Y)$ as (8.1.2).

8.2 Continuity on compact sets

Let X and Y be topological spaces, and let $C^{com}(X, Y)$ of mappings f from X into Y that are continuous on compact subsets of X. More precisely, this means that if K is a compact subset of X, then the restriction of f to K is continuous with respect to the topology induced on K by the given topology on X. Of course, if f is continuous as a mapping from X into Y, then the restriction of f to any subset E of X is continuous with respect to the induced topology on E. Thus

$$C(X,Y) \subseteq C^{com}(X,Y), \tag{8.2.1}$$

where $C(X,Y)$ is the space of all continuous mappings from X into Y, as before. If X is locally compact, then it is easy to see that

$$C^{com}(X,Y) = C(X,Y). \tag{8.2.2}$$

A mapping f from X into Y is said to be *sequentially continuous* at a point $x \in X$ if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of X that converges to x, $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y. If f is continuous at x, then it is well known and easy to verify that f is sequentially continuous at x. Suppose for the moment that there is a local base for the topology of X at x with only finitely or countably many elements. In this case, if f is sequentially continuous at x, then it is well known that f is continuous at x. More precisely, let $V \subseteq Y$ be an open set such that $f(x) \in V$. One would like to find an open set $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq V$. By hypothesis, there is a sequence $U_1, U_2, U_3, \ldots$ of open subsets of X containing x that form a local base for the topology of X at x. We may also ask that $U_{j+1} \subseteq U_j$ for each $j \ge 1$, by replacing U_j with $\bigcap_{l=1}^{j} U_l$. If $f(U_j) \not\subseteq V$ for any $j \ge 1$, then there is a sequence $\{x_j\}_{j=1}^\infty$ of elements of X such that $x_j \in U_j$ and $f(x_j) \notin V$ for each $j \ge 1$. This implies that $\{x_j\}_{j=1}^\infty$ converges to x in X, and that $\{f(x_j)\}_{j=1}^\infty$ does not converge to $f(x)$ in Y.

Let $C^{seq}(X,Y)$ be the space of mappings f from X into Y that are *sequentially continuous*, in the sense that f is sequentially continuous at every point in X. Of course,

$$C(X,Y) \subseteq C^{seq}(X,Y), \tag{8.2.3}$$

because continuous mappings are sequentially continuous, as in the preceding paragraph. If X satisfies the first countability condition, so that there is a local base for the topology of X at every $x \in X$ with only finitely or countably many elements, then

$$C^{seq}(X,Y) = C(X,Y), \tag{8.2.4}$$

as before.

Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of X that converges to an element x of X. It is easy to see that the set K consisting of the x_j's, $j \in \mathbf{Z}_+$, together with x, is compact in X. Let f be a mapping from X into Y whose restriction to k is continuous. This implies that the restriction of f to K is sequentially continuous, and hence that $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y. If the restriction of f to every compact subset of X is continuous, then it follows that f is sequentially continuous on X. This means that

$$C^{com}(X,Y) \subseteq C^{seq}(X,Y). \tag{8.2.5}$$

If X satisfies the first countability condition, then we get that (8.2.2) holds, because of (8.2.1) and (8.2.4).

8.3 Closure and completeness

Let X be a nonempty topological space, and let Y be a nonempty set with a semimetric d_Y. Thus Y is also a topological space with respect to the topology determined by d_Y. If a mapping f from X into Y is continuous on compact sets, then $f(A)$ is a compact subset of Y for every compact set $A \subseteq X$. This implies that f is bounded on compact subsets of X, because compact subsets of Y are bounded with respect to d_Y. It follows that

$$C^{com}(X,Y) \subseteq \mathcal{B}^{com}(X,Y), \tag{8.3.1}$$

where $\mathcal{B}^{com}(X,Y)$ is as in Section 8.1, and $C^{com}(X,Y)$ is as in the previous section.

If A is a nonempty compact subset of X, then we let θ_A denote the supremum semimetric on $\mathcal{B}^{com}(X,Y)$ associated to A and d_Y as in (8.1.1). The collection (8.1.2) of all such semimetrics determines a topology on $\mathcal{B}^{com}(X,Y)$ in the usual way. It is easy to see that $C^{com}(X,Y)$ is a closed set in $\mathcal{B}^{com}(X,Y)$ with respect to this topology. More precisely, if $f \in \mathcal{B}^{com}(X,Y)$ is in the closure of $C^{com}(X,Y)$ with respect to this topology, then the restriction of f to any nonempty compact set $A \subseteq X$ can be uniformly approximated by continuous mappings from A into Y, where A is equipped with the topology induced by the given topology on X. This implies that the restriction of f to A is continuous, as desired.

Let us suppose from now on in this section that d_Y is a metric on Y. If Y is complete with respect to d_Y, then $\mathcal{B}^{com}(X,Y)$ is sequentially complete with respect to the collection (8.1.2) of supremum semimetrics associated to nonempty compact subsets of X. To see this, let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence of elements of $\mathcal{B}^{com}(X,Y)$ with respect to (8.1.2). In particular, this implies that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in Y for every $x \in X$, because $\{x\}$ is a compact subset of X. It follows that $\{f_j\}_{j=1}^\infty$ converges pointwise to a mapping f from X into Y, because Y is complete. If $A \subseteq X$ is nonempty and compact, then one can check that $\{f_j\}_{j=1}^\infty$ converges uniformly to f on A, and that f is bounded on A, by standard arguments. This means that $f \in \mathcal{B}^{com}(X,Y)$, and that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the topology determined by (8.1.2), as desired.

Let $\mathcal{C}$ be a collection of nonempty compact subsets of X, and suppose that every compact subset of X is contained in the union of finitely many elements of $\mathcal{C}$. Remember that the collection (8.1.5) of supremum semimetrics associated to elements of $\mathcal{C}$ determines the same topology on $\mathcal{B}^{com}(X,Y)$ as the collection (8.1.2) of supremum semimetrics associated to all nonempty compact subsets of X, as in Section 8.1. Similarly, if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence of elements of $\mathcal{B}^{com}(X,Y)$ with respect to (8.1.5), then it is easy to see that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to (8.1.2). If Y is complete with respect to d_Y, then it follows that $\mathcal{B}^{com}(X,Y)$ is sequentially complete with respect to (8.1.5).

Note that $C^{com}(X,Y)$ is a closed set in $\mathcal{B}^{com}(X,Y)$ with respect to the topology determined by (8.1.5) under the conditions mentioned in the preceding paragraph. If Y is complete with respect to d_Y, then we get that $C^{com}(X,Y)$ is sequentially complete with respect to the restrictions of the elements of (8.1.5) to $C^{com}(X, Y)$.

8.4 Compatible semimetrics

Let Y be a nonempty set, and let d, d' be semimetrics on Y. Suppose that d' is compatible with the topology determined on Y by d, as in Section 1.1. This means that the topology determined on Y by d is at least as strong as the topology determined on Y by d', or equivalently that the identity mapping on Y is continuous as a mapping from Y equipped with d into Y equipped with d'. If K is a compact subset of Y with respect to the topology determined by d, then it follows that the identity mapping on Y is uniformly continuous along K as a mapping from Y equipped with d into Y equipped with d', as in Section 2.2. Thus for each $\epsilon > 0$ there is a $\delta(K, \epsilon) > 0$ such that

$$d'(y, z) < \epsilon \tag{8.4.1}$$

for every $y \in K$ and $z \in Y$ with $d(y, z) < \delta(K, \epsilon)$.

Let X be a nonempty topological space, and let

$$C^{com}(X, (Y, d)) \quad \text{and} \quad C^{com}(X, (Y, d')) \tag{8.4.2}$$

be the spaces of mappings from X into Y whose restrictions to compact subsets of X are continuous with respect to the topologies determined on Y by d and d', respectively, as in Section 8.2. Thus

$$C^{com}(X, (Y, d)) \subseteq C^{com}(X, (Y, d')), \tag{8.4.3}$$

because d' is supposed to be compatible with d on Y. Let A be a nonempty compact subset of X, so that

$$\theta_A(f, g) = \sup_{x \in A} d(f(x), g(x)) \tag{8.4.4}$$

defines a semimetric on $C^{com}(X, (Y, d))$, as before. Similarly, let

$$\theta'_A(f, g) = \sup_{x \in A} d'(f(x), g(x)) \tag{8.4.5}$$

be the supremum semimetric on $C^{com}(X, (Y, d'))$ associated to A and to d' on Y. Of course, the restriction of (8.4.5) to $f, g \in C^{com}(X, (Y, d))$ defines a semimetric on $C^{com}(X, (Y, d))$.

Let $f \in C^{com}(X, (Y, d))$ and $\epsilon > 0$ be given, and note that $f(A)$ is a compact subset of Y with respect to d. This implies that there is a $\delta(f(A), \epsilon) > 0$ such that

$$d'(f(x), z) < \epsilon \tag{8.4.6}$$

for every $x \in A$ and $z \in Y$ with $d(f(x), z) < \delta(f(A), \epsilon)$, as in (8.4.1). If $g \in C^{com}(X, (Y, d))$ satisfies

$$\theta_A(f, g) < \delta(f(A), \epsilon), \tag{8.4.7}$$

then we have that

$$d'(f(x), g(x)) < \epsilon \tag{8.4.8}$$

for every $x \in A$, by (8.4.6). This implies that

$$\theta'_A(f, g) < \epsilon. \tag{8.4.9}$$

More precisely, one can check that $d'(f(x), g(x))$ is continuous as a real-valued function of x on A, because f and g are continuous on A as mappings into Y equipped with d'. This implies that the supremum in (8.4.5) is attained, because A is compact. This permits us to get the strict inequality in (8.4.9) from the one in (8.4.8), although a non-strict inequality could also be used here. It follows that θ'_A is compatible with θ_A on $C^{com}(X, (Y, d))$.

Suppose now that d and d' determine the same topology on Y. In this case, $C^{com}(X, (Y, d))$ and $C^{com}(X, (Y, d'))$ are the same, and this space may be denoted simply $C^{com}(X, Y)$, as before. If A is a nonempty compact subset of X, then we get that θ_A and θ'_A determine the same topologies on $C^{com}(X, Y)$.

8.5 Total boundedness conditions

Let X be a set, and let $\mathcal{M}$ be a nonempty collection of semimetrics on X. Let us say that a subset E of X is *totally bounded with respect to* $\mathcal{M}$ if E is totally bounded with respect to every $d \in \mathcal{M}$, as in Section 4.2. In this case, if $d_1, \ldots, d_n$ are finitely many elements of $\mathcal{M}$, then E is totally bounded with respect to their maximum, as before. In particular, if $\mathcal{M}$ has only finitely many elements, then E is totally bounded with respect to $\mathcal{M}$ if and only if E is totally bounded with respect to the maximum of the elements of $\mathcal{M}$. More precisely, the "if" part of this statement can be verified directly, and the "only if" part follows from the previous statement.

Let $d(x, y)$ be a semimetric on X, and let t be a positive real number. Remember that

$$d_t(x, y) = \min(d(x, y), t) \tag{8.5.1}$$

defines a semimetric on X too. One can check that $E \subseteq X$ is totally bounded with respect to $d(x, y)$ if and only if E is totally bounded with respect to $d_t(x, y)$.

Let $d_1, d_2, d_3, \ldots$ be an infinite sequence of semimetrics on X, and put

$$d'_j(x, y) = \min(d_j(x, y), 1/j) \tag{8.5.2}$$

for every $x, y \in X$ and $j \geq 1$. Remember that

$$d(x, y) = \max_{j \geq 1} d'_j(x, y) \tag{8.5.3}$$

defines a semimetric on X, for which the corresponding topology on X is the same as the topology determined by the collection of semimetrics d_j, $j \geq 1$. If $E \subseteq X$ is totally bounded with respect to (8.5.3), then it is easy to see that E is totally bounded with respect to (8.5.2) for every $j \geq 1$. This implies that

E is totally bounded with respect to d_j for every $j \ge 1$, as in the preceding paragraph.

In order to deal with the converse, let $r > 0$ be given, and consider the set of $x, y \in X$ such that

$$d(x, y) < r. \tag{8.5.4}$$

This is the same as the set of $x, y \in X$ such that

$$d'_j(x, y) < r \tag{8.5.5}$$

for every $j \ge 1$, by the definition (8.5.3) of $d(x, y)$. Note that (8.5.5) holds for every $x, y \in X$ when $r > 1/j$. In particular, (8.5.4) holds for every $x, y \in X$ when $r > 1$. Suppose that $r \le 1$, and let $[1/r]$ be the largest integer less than or equal to $1/r$, as usual. If $1 \le j \le [1/r]$, then $r \le 1/j$, and (8.5.5) holds exactly when

$$d_j(x, y) < r, \tag{8.5.6}$$

as in Section 1.9. In this case, we get that (8.5.4) holds exactly when

$$\max_{1 \le j \le [1/r]} d_j(x, y) < r. \tag{8.5.7}$$

Suppose that $E \subseteq X$ is totally bounded with respect to d_j for every positive integer j. This implies that E is totally bounded with respect to the maximum of $d_1, \ldots, d_n$ for any positive integer n, as before. One can use this and the remarks in the previous paragraph to get that E is totally bounded with respect to (8.5.3).

8.6 Separate continuity of composition

Let X be a nonempty set with a semimetric $d(x, y)$, which determines a topology on X, as usual. If A is a nonempty compact subset of X, then

$$\theta_A(f, g) = \sup_{x \in A} d(f(x), g(x)) \tag{8.6.1}$$

defines a semimetric on the space $C(X, X)$ of continuous mappings from X into itself, as before. The collection

$$\{\theta_A : A \subseteq X \text{ is nonempty and compact}\} \tag{8.6.2}$$

determines a topology on $C(X, X)$ in the usual way, as in Section 8. Let an element h of $C(X, X)$ be given, and let us consider the continuity properties of

$$f \mapsto f \circ h \tag{8.6.3}$$

and

$$f \mapsto h \circ f, \tag{8.6.4}$$

as mappings from $C(X, X)$ into itself.

Let A be a nonempty compact subset of X, so that $h(A)$ is a nonempty compact subset of X as well. If $f, g \in C(X, X)$, then

$$\theta_A(f \circ h, g \circ h) = \sup_{x \in A} d(f(h(x)), g(h(x))) = \sup_{y \in h(A)} d(f(y), g(y)) = \theta_{h(A)}(f, g). \tag{8.6.5}$$

In particular, this implies that (8.6.3) is continuous as a mapping from $C(X, X)$ into itself, with respect to the topology determined by (8.6.2).

Let $f_0 \in C(X, X)$ be given, and let us consider the continuity of (8.6.4) at f_0. Let A be a nonempty compact subset of X again, which implies that $f_0(A)$ is a nonempty compact subset of X too. Remember that h is uniformly continuous along $f_0(A)$, as in Section 2.2, because h is continuous at every point in $f_0(A)$, and $f_0(A)$ is compact. Let $\epsilon > 0$ be given, so that there is a $\delta > 0$ such that

$$d(h(y), h(w)) < \epsilon \tag{8.6.6}$$

for every $y \in f_0(A)$ and $w \in X$ such that $d(y, w) < \delta$. Equivalently, this means that

$$d(h(f_0(x)), h(w)) < \epsilon \tag{8.6.7}$$

for every $x \in A$ and $w \in X$ such that $d(f_0(x), w) < \delta$.

Let $f \in C(X, X)$ be given, and suppose that

$$\theta_A(f_0, f) < \delta, \tag{8.6.8}$$

so that $d(f_0(x), f(x)) < \delta$ for every $x \in A$. This implies that

$$d(h(f_0(x)), h(f(x))) < \epsilon \tag{8.6.9}$$

for every $x \in A$, by taking $w = f(x)$ in (8.6.7). Hence

$$\theta_A(h \circ f_0, h \circ f) < \epsilon. \tag{8.6.10}$$

More precisely, this uses the fact that the supremum is attained in (8.6.1) for continuous mappings, because A is compact. However, the non-strict version of (8.6.10) would also suffice here.

Using the remarks in the previous paragraphs, one can check that (8.6.4) is continuous as a mapping from $C(X, X)$ into itself. Note that the open balls in $C(X, X)$ with respect to the supremum semimetrics (8.6.1) form a base for the topology determined by (8.6.2), because finitely many of these semimetrics can be combined into a single semimetric, as in (8.1.3). This makes it a bit easier to verify that (8.6.4) is continuous on $C(X, X)$, because continuity at $f_0 \in C(X, X)$ follows more directly from the condition in the preceding paragraph.

8.7 Joint continuity properties

Let X be a nonempty set with a semimetric $d(x, y)$ again, and consider the topology determined on the space $C(X, X)$ of continuous mappings from X into itself by the collection of supremum semimetrics (8.6.1) associated to nonempty compact subsets A of X. We would like to consider continuity properties of

$$(f, g) \mapsto f \circ g, \tag{8.7.1}$$

as a mapping from $C(X, X) \times C(X, X)$ into $C(X, X)$. More precisely, this uses the product topology on $C(X, X) \times C(X, X)$, corresponding to the topology on $C(X, X)$ just mentioned. Let $f_0, g_0 \in C(X, X)$ be given, and let us consider continuity properties of (8.7.1) at (f_0, g_0). Thus, if $f, g \in C(X, X)$ are close to f_0, g_0, respectively, then we would like to be able to show that $f \circ g$ is close to $f_0 \circ g_0$, under suitable conditions.

Let A be a nonempty compact subset of X, and let $f, g \in C(X, X)$ be given. Observe that

$$\theta_A(f_0 \circ g_0, f \circ g) \le \theta_A(f_0 \circ g_0, f_0 \circ g) + \theta_A(f_0 \circ g, f \circ g), \tag{8.7.2}$$

by the triangle inequality. This implies that

$$\theta_A(f_0 \circ g_0, f \circ g) \le \theta_A(f_0 \circ g_0, f_0 \circ g) + \theta_{g(A)}(f_0, f), \tag{8.7.3}$$

where the second term on the right side of (8.7.2) has been reexpressed as in (8.6.5). Let B be another nonempty compact subset of X, and suppose that

$$g(A) \subseteq B. \tag{8.7.4}$$

In this case, we get that

$$\theta_A(f_0 \circ g_0, f \circ g) \le \theta_A(f_0 \circ g_0, f_0 \circ g) + \theta_B(f_0, f), \tag{8.7.5}$$

where the second term on the right side of (8.7.3) is estimated as in (8.1.4).

Let us suppose from now on in this section that X is locally compact with respect to the topology determined by $d(\cdot, \cdot)$. Note that $g_0(A)$ is compact in X, because A is compact and g_0 is continuous. Using local compactness, we can get an open set $U \subseteq X$ and a compact set $B \subseteq X$ such that

$$g_0(A) \subseteq U \subseteq B. \tag{8.7.6}$$

More precisely, local compactness implies that every element of $g_0(A)$ is contained in an open set that is contained in a compact set. To get U and B, one can use compactness of $g_0(A)$ to cover $g_0(A)$ by finitely many open sets that are contained in compact sets.

If $E \subseteq X$ and $r > 0$, then put

$$E_r = \bigcup_{x \in E} B(x, r), \tag{8.7.7}$$

which is an open set in X that contains E. Remember that there is an $r_0 > 0$ such that

$$(g_0(A))_{r_0} \subseteq U, \tag{8.7.8}$$

as in Section 1.3, because $g_0(A)$ is compact, U is an open set, and $g_0(A) \subseteq U$. If $g \in C(X,X)$ satisfies

$$\theta_A(g_0, g) < r_0, \tag{8.7.9}$$

then it is easy to see that

$$g(A) \subseteq (g_0(A))_{r_0}. \tag{8.7.10}$$

Thus (8.7.9) implies (8.7.4), because of (8.7.8) and (8.7.10). Hence (8.7.5) holds when g satisfies (8.7.9), where B is as in the preceding paragraph.

Of course, we can make the second term on the right side of (8.7.5) as small as we like, by taking f close to f_0 in $C(X,X)$. We can also make the first term on the right side of (8.7.5) as small as we like, by taking $\theta_A(g_0, g)$ to be sufficiently small. This uses the fact that f_0 is uniformly continuous along $g_0(A)$, as in the previous section, and indeed this corresponds to the continuity of (8.6.4) on $C(X,X)$, as before. It follows that the left side of (8.7.5) is as small as we like when f and g are sufficiently close to f_0 and g_0, respectively, in $C(X,X)$, which includes the condition (8.7.9). Using this, one can check that (8.7.1) is continuous as a mapping from $C(X,X) \times C(X,X)$ into $C(X,X)$ when X is locally compact.

8.8 Inverse mappings and homeomorphisms

Let X be a nonempty set with a semimetric $d(x,y)$, and let $H(X)$ be the group of homeomorphisms from X onto itself with respect to the topology determined by $d(\cdot,\cdot)$, as before. If $A \subseteq X$ is nonempty and compact, then let θ_A be the supremum semimetric on the space $C(X,X)$ of continuous mappings from X into itself associated to A and $d(\cdot,\cdot)$, as in (8.6.1). Thus the restriction of $\theta_A(f,g)$ to $f,g \in H(X)$ defines a semimetric on $H(X)$. We would like to consider continuity properties of

$$f \mapsto f^{-1} \tag{8.8.1}$$

as a mapping from $H(X)$ to itself, related to these supremum semimetrics.

Let us consider

$$\{\theta_A : A \subseteq X \text{ is nonempty and compact}\} \tag{8.8.2}$$

as a collection of semimetrics on $H(X)$, using the restriction of $\theta_A(f,g)$ to $f,g \in H(X)$, as before. If $A \subseteq X$ is nonempty and compact, then

$$\widetilde{\theta}_A(f,g) = \theta_A(f^{-1}, g^{-1}) \tag{8.8.3}$$

defines a semimetric on $H(X)$ as well. Thus

$$\{\widetilde{\theta}_A : A \subseteq X \text{ is nonempty and compact}\} \tag{8.8.4}$$

is another collection of semimetrics on $H(X)$. Both (8.8.2) and (8.8.4) determine topologies on $H(X)$, as in Section 1.5. Similarly, the union of (8.8.2) and (8.8.4) determines a topology on $H(X)$.

Of course, (8.8.1) defines a homeomorphism from $H(X)$ with the topology determined by (8.8.2) onto $H(X)$ with the topology determined by (8.8.4). We can also consider (8.8.1) as a homeomorphism from $H(X)$ onto itself, using the topology determined on $H(X)$ by the union of (8.8.2) and (8.8.4) on both the domain and the range.

In the previous two sections, we considered continuity properties of composition on $C(X, X)$, with respect to the topology determined on $C(X, X)$ by the collection of supremum semimetrics associated to nonempty compact subsets of X. In particular, these continuity properties can be used for compositions on $H(X)$, with respect to the topology determined by (8.8.2).

If $f, g \in H(X)$, then

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}, \tag{8.8.5}$$

as usual. One can use this to get continuity properties for composition on $H(X)$ with respect to the topology determined by (8.8.4), from the continuity properties for composition on $H(X)$ with respect to (8.8.2).

More precisely, left and right translations on $H(X)$ are continuous with respect to the topology determined by (8.8.2), as in Section 8.6. Using this, one can get that left and right translations on $H(X)$ are continuous with respect to the topology determined by (8.8.4), as in the preceding paragraph. One can also check that left and right translations on $H(X)$ are continuous with respect to the topology determined by the union of (8.8.2) and (8.8.4).

Suppose for the moment that X is locally compact. In this case, composition of homeomorphisms defines a continuous mapping from $H(X) \times H(X)$ into $H(X)$, with respect to the topology determined on $H(X)$ by (8.8.2) and the corresponding product topology on $H(X) \times H(X)$, as in the previous section. One can use this and (8.8.5) to get that composition of homeomorphisms defines a continuous mapping from $H(X) \times H(X)$ into $H(X)$, with respect to the topology determined on $H(X)$ by (8.8.4) and the corresponding product topology on $H(X) \times H(X)$. Using the previous two statements, one can verify that composition of homeomorphisms defines a continuous mapping from $H(X) \times H(X)$ into $H(X)$, with respect to the topology determined on $H(X)$ by the union of (8.8.2) and (8.8.4), and the corresponding product topology on $H(X) \times H(X)$. This implies that $H(X)$ is a topological group with respect to the topology determined by the union of (8.8.2) and (8.8.4), because (8.8.1) is continuous with respect to this topology on $H(X)$.

If $d(\cdot, \cdot)$ is a metric on X, then (8.8.2) is nondegenerate on $H(X)$, because finite subsets of X are compact. Similarly, (8.8.4) is nondegenerate on $H(X)$ in this situation. In particular, this implies that the union of (8.8.2) and (8.8.4) is nondegenerate on $H(X)$.

Let $\mathcal{C}$ be a collection of nonempty compact subsets of X, and suppose that every nonempty compact subset of X is contained in the union of finitely many elements of $\mathcal{C}$. This implies that

$$\{\theta_C : C \in \mathcal{C}\} \tag{8.8.6}$$

determines the same topology on $H(X)$ as (8.8.2), as before. Similarly,

$$\{\widetilde{\theta}_C : C \in \mathcal{C}\} \tag{8.8.7}$$

determines the same topology on $H(X)$ as (8.8.4). In the same way, the union of (8.8.6) and (8.8.7) determines the same topology on $H(X)$ as the union of (8.8.2) and (8.8.4).

8.9 One-point compactification

Let (Y, d) be a nonempty compact metric space. If f and g are mappings from Y into itself, then put

$$\theta_Y(f, g) = \sup_{y \in Y} d(f(y), g(y)), \tag{8.9.1}$$

as usual. Remember that continuous mappings from Y into itself are uniformly continuous with respect to d, because Y is compact. Thus the group $H(Y)$ of homeomorphisms from Y onto itself is the same as the group $UH(Y)$ of uniform homeomorphisms from Y onto itself. It follows that $H(Y)$ is a topological group with respect to the topology determined by the restriction of (8.9.1) to $H(Y)$, as in Section 2.3.

Let p be an element of Y, and suppose that p is a limit point of Y. Thus $X = Y \setminus \{p\}$ is a dense open set in Y, and X is not compact. Note that X is locally compact with respect to the induced topology, which is the same as the topology determined by the restriction of d to X. In this situation, Y corresponds to the usual one-point compactification of X.

If f is a homeomorphism from X onto itself, then let $\widehat{f}$ be the extension of f to a mapping from Y into itself defined by putting

$$\widehat{f}(p) = p. \tag{8.9.2}$$

One can check that $\widehat{f}$ defines a homeomorphism from Y onto itself. The mapping

$$f \mapsto \widehat{f} \tag{8.9.3}$$

defines an injective group homomorphism from the group $H(X)$ of all homeomorphisms from X onto itself into $H(Y)$. This mapping sends $H(X)$ onto the subgroup of $H(Y)$ consisting of homeomorphisms on Y that send p to itself.

If f and g are mappings from X into itself, then put

$$\theta_X(f, g) = \sup_{x \in X} d(f(x), g(x)). \tag{8.9.4}$$

If $f, g \in H(X)$ and $\widehat{f}, \widehat{g} \in H(Y)$ are as in (8.9.2), then it is easy to see that

$$\theta_X(f, g) = \theta_Y(\widehat{f}, \widehat{g}). \tag{8.9.5}$$

Note that $f \in H(X)$ is uniformly continuous with respect to the restriction of d to X, because $\widehat{f}$ is uniformly continuous on Y. Similarly, f^{-1} is uniformly continuous on X when $f \in H(X)$, so that $H(X) = UH(X)$. It follows that $H(X)$ is a topological group with respect to the restriction of (8.9.4) to $H(X)$, which corresponds to the subgroup of $H(Y)$ of homeomorphisms on Y that send p to itself as a topological group with respect to the restriction of (8.9.1) to this subgroup.

Let A be a nonempty subset of X, and put

$$\theta_A(f,g) = \sup_{x \in A} d(f(x), g(x)) \tag{8.9.6}$$

for all mappings f, g from X into itself, as before. Clearly

$$\theta_A(f,g) \le \theta_X(f,g) \tag{8.9.7}$$

for all f, g. If $f, g \in H(X)$, then put

$$\widetilde{\theta}_A(f,g) = \theta_A(f^{-1}, g^{-1}), \tag{8.9.8}$$

as in the previous section. Note that

$$\widetilde{\theta}_A(f,g) \le \theta_X(f^{-1}, g^{-1}) = \widetilde{\theta}_X(f,g) \tag{8.9.9}$$

for every $f, g \in H(X)$, by (8.9.7).

The topology on $H(X)$ determined by θ_X is at least as strong as the topology determined on $H(X)$ by the collection (8.8.2) of θ_A, with $A \subseteq X$ nonempty and compact, because of (8.9.7). Similarly, the topology determined on $H(X)$ by $\widetilde{\theta}_X$ is at least as strong as the topology determined on $H(X)$ by the collection (8.8.4) of $\widetilde{\theta}_A$, with $A \subseteq X$ nonempty and compact, by (8.9.9).

Remember that $f \mapsto f^{-1}$ is a continuous mapping from $H(X)$ into itself with respect to the topology determined on $H(X)$ by θ_X, because $H(X)$ is a topological group with respect to this topology. More precisely, $f \mapsto f^{-1}$ is a homeomorphism from $H(X)$ onto itself with respect to the topology determined by θ_X, because this mapping is its own inverse. This means that the topology determined on $H(X)$ by $\widetilde{\theta}_X$ is the same as the topology determined on $H(X)$ by θ_X.

It follows from the remarks in the previous two paragraphs that the topology determined on $H(X)$ by θ_X is at least as strong as the topology determined on $H(X)$ by the union of the collections (8.8.2) and (8.8.4). In the next section, we shall show that these two topologies on $H(X)$ are the same.

8.10 The other direction

Let us continue with the same notation and hypotheses as in the previous section. If A is a nonempty proper subset of X, then

$$\theta_X(f,g) = \max(\theta_A(f,g), \theta_{X \setminus A}(f,g)) \tag{8.10.1}$$

for all mappings f, g from X into itself. In particular, if A is compact, then A is a proper subset of X, because X is not compact. Of course,

$$d(f(x), g(x)) \leq d(f(x), p) + d(p, g(x)) \tag{8.10.2}$$

for all mappings f, g from X into itself and $p \in X$, by the triangle inequality. This implies that

$$\theta_{X \setminus A}(f, g) \leq \sup_{x \in X \setminus A} d(f(x), p) + \sup_{x \in X \setminus A} d(p, g(x)) \tag{8.10.3}$$

for all mappings f, g from X into itself.

Let $B_Y(p, r)$ be the open ball in Y centered at p with radius r for each $r > 0$. This is an open set in Y, so that $Y \setminus B_Y(p, r)$ is a closed set, and hence compact, because Y is compact. We also have that $Y \setminus B_Y(p, r) \subseteq X$, because $p \in B_Y(p, r)$. Let $f \in H(X)$ and $r_1, r_2 > 0$ be given. Observe that

$$f(X \cap B_Y(p, r_1)) \subseteq X \cap B_Y(p, r_2) \tag{8.10.4}$$

exactly when

$$X \setminus B_Y(p, r_2) \subseteq f(X \setminus B_Y(p, r_1)), \tag{8.10.5}$$

because f is a one-to-one mapping from X onto itself. Of course, (8.10.5) is the same as saying that

$$f^{-1}(X \setminus B_Y(p, r_2)) \subseteq X \setminus B_Y(p, r_1). \tag{8.10.6}$$

Let $f_0 \in H(X)$ and $r_2 > 0$ be given. Because $\widehat{f_0}$ is continuous on Y at p, there is an $r_1 > 0$ such that

$$f_0(X \cap B_Y(p, r_1)) \subseteq X \cap B_Y(p, r_2). \tag{8.10.7}$$

This implies that

$$f_0^{-1}(X \setminus B_Y(p, r_2)) \subseteq X \setminus B_Y(p, r_1), \tag{8.10.8}$$

as in the preceding paragraph. Put

$$A(r) = X \setminus B_Y(p, r) = Y \setminus B_Y(p, r) \tag{8.10.9}$$

for every $r > 0$, which is a compact subset of X, as before. Suppose for the moment that $A(r_2) \neq \emptyset$, and that $f \in H(X)$ satisfies

$$\theta_{A(r_2)}(f^{-1}, f_0^{-1}) = \widetilde{\theta}_{A(r_2)}(f, f_0) \leq r_1/2. \tag{8.10.10}$$

If $x \in A(r_2)$, then

$$d(f_0^{-1}(x), p) \geq r_1, \tag{8.10.11}$$

by (8.10.8), and

$$d(f^{-1}(x), f_0^{-1}(x)) \leq r_1/2, \tag{8.10.12}$$

by (8.10.10). This implies that

$$r_1 \le d(f_0^{-1}(x), p) \le d(f_0^{-1}(x), f^{-1}(x)) + d(f^{-1}(x), p) \le r_1/2 + d(f^{-1}(x), p), \tag{8.10.13}$$

and hence

$$r_1/2 \le d(f^{-1}(x), p). \tag{8.10.14}$$

It follows that

$$f^{-1}(X \setminus B_Y(p, r_2)) = f^{-1}(A(r_2)) \subseteq X \setminus B_Y(p, r_1/2). \tag{8.10.15}$$

This means that

$$f(X \cap B_Y(p, r_1/2)) \subseteq X \cap B_Y(p, r_2), \tag{8.10.16}$$

as before. Note that (8.10.16) holds automatically when $A(r_2) = \emptyset$, because $Y = B_Y(p, r_2)$.

Suppose for the moment that $A(r_1/2) \ne \emptyset$, and note that

$$X \setminus A(r_1/2) = X \cap B_Y(p, r_1/2), \tag{8.10.17}$$

by the definition (8.10.9) of $A(r_1)$. Thus

$$\theta_X(f_0, f) = \max(\theta_{A(r_1/2)}(f_0, f), \theta_{X \cap B_Y(p, r_2)}(f_0, f)), \tag{8.10.18}$$

by (8.10.1) with $A = A(r_1)$. If $x \in X \cap B_Y(p, r_1/2)$, then

$$d(f_0(x), f(x)) \le d(f_0(x), p) + d(p, f(x)) \le r_2 + r_2 = 2\, r_2, \tag{8.10.19}$$

using (8.10.7) and (8.10.16) in the second step. This implies that

$$\theta_{X \cap B_Y(p, r_1/2)}(f_0, f) \le 2\, r_2, \tag{8.10.20}$$

so that

$$\theta_X(f_0, f) \le \max(\theta_{A(r_1/2)}(f_0, f), 2\, r_2), \tag{8.10.21}$$

because of (8.10.18). If $A(r_1/2) = \emptyset$, then $X = X \cap B_Y(p, r_1/2)$, and we simply get that

$$\theta_X(f_0, f) \le 2\, r_2, \tag{8.10.22}$$

by (8.10.20).

Remember that $f_0 \in H(X)$ is fixed, and that the previous statements hold for all $f \in H(X)$ such that (8.10.10) holds when $A(r_2) \ne \emptyset$. If $A(r_2) = \emptyset$, then (8.10.16) holds automatically, and (8.10.21) or (8.10.22) holds for all $f \in H(X)$, as appropriate.

Using (8.10.21) or (8.10.22), as appropriate, we can get that f is as close to f_0 as we want with respect to θ_X, by taking f sufficiently close to f_0 with respect to the topology determined on $H(X)$ by the union of the collections (8.8.2) and (8.8.4) of semimetrics. This means that the topology determined on $H(X)$ by the union of the collections (8.8.2) and (8.8.4) is at least as strong as the topology determined on $H(X)$ by θ_X. It follows that the topology determined on $H(X)$ by θ_X is the same as the topology determined on $H(X)$ by the union of the collections (8.8.2) and (8.8.4), by the remarks in the previous section.

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