1 Disclaimer

Use these notes at your own risk. This copy of my transcribed notes is here for your convenience, but neither the instructor nor I will be held responsible for any mistakes contained therein. In particular, this isn’t an “officially sanctioned” transcript of the class notes.

2 Cauchy’s Integral Formula

Given a holomorphic on the disk, how do we find \( f(0) \) knowing \( f(z), |z| = R \)?

We apply Cauchy’s Integral Formula.

\[
f(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(\zeta)}{\zeta} d\zeta.
\]

We write \( \zeta = Re^{i\theta} \). Then \( d\zeta = iRe^{i\theta} d\theta \). Thus \( d\zeta/\zeta = i d\theta \) on \( C \). This is clear also because

\[
\frac{d\zeta}{\zeta} = d\log \zeta = d\log R + id\theta.
\]

Thus

\[
f(0) = \frac{1}{2\pi} \oint_{|\zeta|=R} f(\zeta) d\theta.
\]

This is known as the Mean Value property for holomorphic functions.

The Mean Value property implies the general form of the Cauchy Integral Formula. Given a point \( z_0 \) in the unit disk, the transformation \( w = (z - z_0)/(1 - \bar{z}_0 z) \) preserves the unit circle and sends \( z_0 \) to the origin. The inverse map is \( z = (w + z_0)/(\bar{z}_0 w + 1) \). \( f(z(w)) \) is again holomorphic. Thus

\[
f(z_0) = \frac{1}{2\pi i} \oint_{|w|=R} \frac{f(w)}{w} dw.
\]

We have

\[
dw = \frac{1 - |z_0|^2}{(1 - \bar{z}_0 z)^2}.
\]

and so we can derive the general Cauchy formula (Exercise)

The Mean Value Property also implies the Maximum Principle. For if \( f(0) = \frac{1}{2\pi} \oint f(\zeta) d\theta \), and so \( |f(0)| \leq \frac{1}{2\pi} \oint |f(\zeta)| d\theta \), with equality possible iff \( f \) is a constant on the circle.

3 Schwarz’s Theorem

**Theorem 3.1.** A biholomorphic self-map of \( \mathbb{D} \) is a Mobius Transformation. If \( f(0) = 0 \), then \( |f(z)| \leq |z| \). Applying Schwarz’s Lemma to \( f^{-1} \), we also obtain \( |f(z)| \geq |z| \). By the second clause of Schwarz’s Lemma, \( f(z) \) is a rotation. If \( f(0) \) is not zero, then by an appropriate Mobius transformation we can send \( f(0) \) to 0.
4 Harmonic Functions

$u : \Omega \rightarrow \mathbb{R}$ is harmonic if $\delta u = 0$.

**Proposition 4.1.** A harmonic function $u$ always satisfies the Mean Value Property:

$$u(0) = \frac{1}{2\pi} \int_{|z|=R} u(z) d\theta.$$ 

Note that if $u$ is harmonic (and $u \in C^2$), then $f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is holomorphic. Warning: $\log r$ is a harmonic function which is not the real part of a holomorphic function. $\log r = \text{Re} \log(z)$, since $\log(z) = \log r + i\theta$, but $\theta$ is not a well-defined function! Thus harmonic functions are the real part of a holomorphic function locally, but not necessarily globally.

**Proof.** Let $f = u + iv$ and let us write Cauchy’s Integral Formula.

$$u(z) + iv(z) = \frac{1}{2i\pi} \int \frac{u(\zeta) + iv(\zeta)}{\zeta - z} d\zeta.$$ 

Exercise: Express $u$ in terms of $u$. 

\[\square\]