1 Disclaimer

Use these notes at your own risk. This copy of my transcribed notes is here for your convenience, but neither the instructor nor I will be held responsible for any mistakes contained therein. In particular, this isn’t an “officially sanctioned” transcript of the class notes.

2 Geometric Meaning of the Derivative

We have $z \xrightarrow{f} w$ with derivative $f'$. $f'$ rotates by the angle of the argument and expands by the absolute value. Given a curve $\gamma$, the length of $f(\gamma)$ is

$$\int |d\gamma(t)|dt = \int |f'(\gamma(t))| \cdot |\gamma(t)|dt = \int |f'(s)|ds$$

Letting $s$ be the length parameter on $\gamma$.

Given a region $E$ that $f$ takes to region $E'$, $\text{Area}(E') = \int_E |f'(z)|^2 dxdy$.

Conformality of holomorphic mappings Let $f$ take the curves $\gamma_1, \gamma_2$ to curves $\tilde{\gamma}_1, \tilde{\gamma}_2$ respectively. If $f'(z) \neq 0$, this implies that the angle between two curves $\gamma_1, \gamma_2$ is $\arg(\dot{\gamma}_2/\dot{\gamma}_1)$. The angle between $\tilde{\gamma}_1, \tilde{\gamma}_2$ is then $\arg(\dot{\tilde{\gamma}}_2/\dot{\tilde{\gamma}}_1) = \arg(f'(z)\gamma_2/f'(z)\gamma_1)$, which in turn is equal to the angle between $\gamma_1, \gamma_2$.

Example Consider $f(z) = z^2 = w$, and we consider how it changes the $z$-plane. This map is clearly 2 to 1, and the upper $z$-plane should map to the entire $w$-plane. Consider the horizontal line $z = t + i\alpha$. We then have $u + iv = z^2 = t^2 - \alpha^2 + 2i\alpha t$. We have $u = t^2 - \alpha^2$, $v = 2\alpha t$. Thus it is a parabola symmetric to the $u$-axis, and they open to the right. Consider now the horizontal line $z = \alpha + it$. We then have $w = \alpha^2 - t^2 + 2i\alpha t$. We have parabolas, again symmetric to the $u$-axis, opening left.

Now we want to find what happens when we map the $w$-plane to the $z$-plane. In this case, it is easier to write equations rather than to parameterize. So we consider $v = c$ constant, the equation of a horizontal line. We have $u + iv(x + iy)^2 = x^2 = y^2 + 2ixy$. We then have $xy = c$, which is the equation of a hyperbola (whose asymptotes are horizontal and vertical lines). Similarly, if $u$ constant, then $x^2 - y^2$ is constant, which is again a hyperbola, with asymptotes the lines $y = \pm x$.

Excercise Visualize $w = (z + 1/z)/2$, the Zhukowski map.

3 Complex Power Series

Theorem 3.1. If a series $\sum a_n z^n$ converges (not necessarily absolutely) for $z = z_0$, then it converges absolutely for $|z| < |z_0|$.
Proof. Clearly \( \sup_n |a_n z^n_0| \leq C \) for some \( C \). But then \( |a_n z^n| = a_n z^n_0 (z/z_0)^n| \leq C|z/z_0|^n \), which is a convergent geometric series. \( \square \)

**Theorem 3.2.** An absolutely convergent series converges

Proof. A series \( \sum_{n=1}^\infty a_n \) converges iff \( \forall \epsilon > 0, \exists n_0 \) such that \( |a_n + \ldots + a_{n+k}| < \epsilon \) for all \( n > n_0 \), for all \( k \). Clearly however \( |a_n + \ldots + a_{n+k}| \leq |a_n| + \ldots + |a_{n+k}| \). \( \square \)

**Theorem 3.3** (Riemann Series Theorem for Complex Numbers). We can rearrange the terms of a conditionally convergent complex series so that it converges to any element of a linear space (either a line or a plane). For instance, if the series only contains real coefficients, we can make it converge to any real number (this is the Riemann Series Theorem for Real numbers).

**Next Time** Imagine that \( \sum_{n=1}^\infty a_n \) converges to \( S \). If we consider a coefficient with \( |z| < 1 \), certainly the series \( \sum_{n=1}^\infty a_n z^n \) converges. Does the convergent value approach \( S \) as \( z \) approaches 1?