1 Disclaimer

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2 Behavior of a Power Series at the Boundary of the Disk of Convergence

We recall that is the power series

\[ \sum_{n=0}^{\infty} a_n z^n, \]

converges for \( z = z_0 \), then it converges absolutely for all \( z : |z| < |z_0| \).

Consider the geometric power series \( \sum z^n \). It converges at no point on the boundary of convergence. Consider now the example \( \sum_{n=1}^{\infty} \frac{z^n}{n} \). This converges at every point at the boundary except \( z = 1 \). If we consider \( \sum \frac{z^n}{n^2} \), it yields a continuous function on the closed disc, and so converges everywhere.

Consider the series \(+1 - 1 + 1 - 1 + 1 - 1 + 1\). We consider partial sums, we know \( S_2 = 0 \), \( S_{2n+1} = 1 \). Cesàro convergence is that which \( (S_1 + \ldots + S_N)/N \) converges. So our series converges. Now consider \( 1 - t + t^2 - t^3 + \ldots = 1/(1 + t) \), we take \( t = 1 \). This is known as Abel’s convergence.

If \( a_n \rightarrow A \), then \( (a_1 + \ldots + a_n)/n \rightarrow A \). This is an exercise. If a series converges, does it follow that it converges in Abel’s sense?

One can think of Abel’s summation as discrete summation by parts.

Theorem 2.1 (Abel’s Theorem). Let \( a_n, b_n \) be bounded sequences, such that \( b_n \searrow 0 \) and \( |\sum_{n=1}^{N} a_n| < K \) for some \( M \). Then \( \sum_{n=1}^{\infty} a_n b_n \) converges.

Proof. We aim to show that \( \sum_{n=M}^{\infty} a_n b_n \) becomes small as \( M \rightarrow \infty \). \( A_k = a_M + a_{M+1} + \ldots + a_n \).

\[ \sum_{n=M}^{N} (A_{n+1} - A_n) b_n = \sum_{n=M}^{N} A_n (b_{n-1} - b_n) + A_{N+1} b_N - A_M b_{M-1} \]
\[ \leq \sum_{n=M}^{N} |A_n| (b_{n-1} - b_n) + |A_{N+1} b_N| + |A_M b_{M-1}| \]
\[ \leq K (b_{M-1} - b_N) + K b_N + K b_{M-1} \]

This gets arbitrarily small as \( M \) is large.

Corollary 2.2. For \( 0 < \theta < 2\pi \), the series \( \sum e^{i n \theta}/n \) converges.
proof of corollary.

\[ |\sum_{n=0}^{N} e^{in\theta}| \leq |(e^{i(N+1)\theta} - 1)/e^{i\theta} - 1| \leq 2/|e^{i\theta} - 1| \]

Exercise: derive \( \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} a_n t^n = S \)

We now show that for \( \sum_{n=0}^{\infty} a_n = S \), \( \lim_{t \to 1^-} a_n t^n = S \) (that is, the complex case). First assume \( S = 0 \). Without loss of generality. We aim to prove, as \( z \to 1 \), \( \sum_{n=0}^{\infty} a_n z^n \to 0 \). It suffices for any \( \epsilon > 0 \) to find \( M \) and a neighborhood \( U_\epsilon \) such that \( |\sum_{n=M}^{N} a_n z^n| < \epsilon \) for all \( z \in U_\epsilon \). Integrate by parts (use Abel’s summation) to write

\[ \sum_{n=M}^{N} a_n z^n = \sum_{n=M}^{N} (A_{n+1} - A_n) z^n \]

\[ = \sum_{n=M}^{N} A_n (z^{n-1} - z^n) + A_{N+1} z^N - A_M z^{M-1} \]

We estimate

\[ |\sum_{n=M}^{N} A_n (z^{n-1} - z^n)| = |(1 - z)(\sum_{n=M}^{N} A_n z^{n-1})| \]

\[ \leq |1 - z|\epsilon(|z|^{M-1} + |z|^M + \ldots + |z|^{N-1}) \]

If we have \( |1 - z| \) instead of \( (1 - |z|) \) we are done. Thus our full statement of the theorem is

**Theorem 2.3.** Choose arbitrary \( K > 0 \). Then \( \lim_{z \to 1} \sum_{n=0}^{\infty} a_n z^n = S \), provided \( |1 - z|/(1 - |z|) < K \).

We know have understanding of how a complex power series converges at the boundary.

3 The Joukowsky Transform

How do we draw the mapping \( w = (z + 1/z)/2 \)? We look at the polar coordinate grid in the \( z \) plane. The map is 2 to 1: as the points \( z_0, 1/z_0 \) get mapped into the same point. Other circles are mapped to ellipses. Consider now lines through the origin \( w = e^{i\theta} t \). We have \( w = e^{i\theta} t + e^{-i\theta}/2t = (1/2) \cos(\theta)(t + 1/t) + (1/2)i\sin(\theta)(t - 1/t) \).