1 Disclaimer

Use these notes at your own risk. This copy of my transcribed notes is here for your convenience, but neither the instructor nor I will be held responsible for any mistakes contained therein. In particular, this isn’t an “officially sanctioned” transcript of the class notes.

2 The Joukowsky Mapping

We have \( z \rightarrow (z + 1/z)/2 \), and this is equal to

\[
\frac{1}{2} \cos(\theta)(t + 1/t) + \frac{1}{2}i \sin(\theta)(t - 1/t).
\]

We wish to see where straight lines in the \( z \)-plane map to. When \( \theta \) is constant, and neither \( \cos(\theta) \) nor \( \sin(\theta) = 0 \) we have

\[
\frac{u^2}{\cos(\theta)} - \frac{v^2}{\sin^2(\theta)} = 1.
\]

These are ellipses. Now we consider where circles in the real plane map to. When \( r \) constant we have

\[
\frac{u^2}{\frac{1}{2}(t + \frac{1}{t})^2} + \frac{v^2}{\frac{1}{2}(t - \frac{1}{t})^2} = 1.
\]

These are hyperbolas. Additionally, the real line and the imaginary line are preserved.

3 Complex Integration

If we have a parameterized curve \( \gamma(t) = (x(t), y(t)) \). Then the integrals are expressed as

\[
\int_\gamma f dx = \int_0^1 f(\gamma(t)) \cdot \dot{x}(t) dt.
\]

\[
\int_\gamma f dy = \int_0^1 f(\gamma(t)) \cdot \dot{y}(t) dt.
\]

Contrast with

\[
\int_\gamma f ds = \int_0^1 f(\gamma(t)) \cdot \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt.
\]

which is an entirely different type of integral, which we will not consider. We will consider instead integrals on differential forms, which have different behavior. We note also that if \( f \) is complex-valued, it is defined by

\[
\int f(z)dz = \int f(z)dx + i \int f(z)dy.
\]
We start with the connected domain $\Omega$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. We consider a point $A \in \Omega$, $B \in \Omega$. Let $\gamma$ join $A, B$. When does it hold that
\[ \int_{\gamma} p \, dx + q \, dy = U(B) - U(A) \] (1)

Where $U : \Omega \to \mathbb{C}$. The condition $\partial q / \partial x = \partial p / \partial y$ is necessary, by Green’s Theorem. Locally, this condition is sufficient as well. Globally, this is not true, as there may exist holes on $\Omega$.

We consider now the function $U$. If it exists, then $p = \partial U / \partial x, q = \partial U / \partial y$. This is a slightly stronger condition than our last one, and it is in fact necessary and sufficient.

When is it true that the integral $\int f(z) \, dz$ does not depend on the path? We have
\[ f(z) \, dz = f(z) \, dx + if(z) \, dy, \]
and so our condition must be
\[ f(z) = \frac{\partial U}{\partial x} \, if(z) = \frac{\partial U}{\partial y} \]
Note that $\partial U / \partial x = -i \partial U / \partial y$ (U complex-valued) is precisely the Cauchy-Riemann Equation. Thus $\int f(z) \, dz$ does not depend on the path precisely when $f$ is a complex derivative of a holomorphic function: $f = dU/dz$.

NB The validity of (1) automatically implies that $U$ is continuously differentiable.

**Theorem 3.1 (Cauchy Theorem).** Let $f$ be a holomorphic function in a rectangle $R$. Then for any piecewise smooth closed curve $\gamma \subset R$, we have
\[ \int_{\gamma} f(z) \, dz = 0. \] (2)

**Proof.** It suffices to prove this for when $\gamma$ is the oriented boundary of a rectangle. If $\oint_{\delta R} p \, dx + q \, dy = 0$ for $\gamma = \delta \tilde{R}$, $\forall \tilde{R}$, then $p = \partial U / \partial x, q = \partial U / \partial y$. But then for any closed $\gamma \oint_{\gamma} p \, dx + q \, dy = \oint_{\gamma} dU = U(P) - U(P) = 0$.

Under the additional assumption of $f$ being continuously differentiable, this follows easily from Green’s Theorem. Now if $\gamma = \delta \tilde{R}$, then
\[ \int_{\partial R} f \, dz = \int_{\partial \tilde{R}} f \, dx + if \, dy = \int_{\tilde{R}} \left( i \frac{\partial f}{\partial x} - \frac{df}{dy} \right) \, dx \, dy = 0. \] (3)
The last equality holds by the Cauchy-Riemann equations.
For any holomorphic function we see that the integral $\int f(z) \, dz$ does not depend on the path.

**Theorem 3.2.** Let $f : \mathbb{R} \to \mathbb{C}$ for rectangle $R$ be continuous and satisfy $\int_{\gamma} f(z) \, dz = 0$ for any piecewise-smooth closed $\gamma$. Then $f$ is holomorphic in $R$. 

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