1 Disclaimer

Use these notes at your own risk. This copy of my transcribed notes is here for your convenience, but neither the instructor nor I will be held responsible for any mistakes contained therein. In particular, this isn’t an “officially sanctioned” transcript of the class notes.

2 Analytic Functions

A holomorphic function can be represented as a power series in a neighborhood of any point.

Corollary 2.1 (Uniqueness Theorem). If two holomorphic functions \( f, g \) in a connected domain coincide on a set which has an accumulation point in the domain, then \( f = g \).

Proposition 2.2. Let \( f \) be \( C^\infty(\mathbb{R}) \). Its set can be any closed set.

Sketch of proof. Consider \( \exp(-1/d(x,K)) \), where \( K \) is your desired closed set and \( d(x,K) \) is the distance between \( x \) and \( K \). \( \square \)

An analytic function can be zero only on a discrete set. Consider how \( \sin(\pi z) \) has roots in the integer. In fact we can see a relationship between \( \sin(\pi z) \) and \( z(z - 1)(z + 1)(z - 2)(z + 2) \ldots \). To ensure the infinite product converges, we scale each term appropriately. We tack a constant \( \pi \) to make sure that the Taylor series match, and so now it makes sense heuristically to write

\[
\sin \pi z = \pi z (1 - z^2)(1 - z^2/4)(1 - z^2/9) \ldots.
\]

This is due to Euler.

Proof of the corollary. The Taylor series of \( f, g \) at the accumulation point must coincide. The functions must thus coincide at a neighborhood of the accumulation point. Consider the set \( \{ f = g \} \). This set is both open (consider the taylor series of any point in \( f = g \)- this is left as an exercise) and closed, and thus this set is the whole domain. \( \square \)

Consider the function \( 1/z \). This function is holomorphic everywhere but 0. The power series of \( 1/z \) at \( z_0 \neq 0 \) is represented as

\[
\frac{1}{z_0} \sum_{n=0}^{\infty} (-1)^n \left( \frac{z - z_0}{z_0} \right)^n.
\]

This series converges iff \( |z - z_0| < |z_0| \).