1 Disclaimer

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2 Zeroes of Analytic Functions

If \( f(z_0) = 0 \), with the image of \( f \) nonzero, then there exists smallest \( h \) such that \( f^{(h)}(z_0) \neq 0 \).

\[
f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k = (z - z_0)^k \left( \sum_{k=0}^{\infty} c_k z^k \right), \quad c_0 \neq 0
\]

\( f(z)/(z - z_0)^h \) is a holomorphic function. We call \( h \) the order of the zero.

**Theorem 2.1.** If \( f \) is holomorphic in \( \Omega \setminus z_0 \) and \( \lim_{z \to z_0} (z - z_0)f(z) = 0 \) (alternatively, this function \( f \) is bounded in the neighborhood of \( z_0 \)), then, in fact \( \exists G \) holomorphic in all of \( \Omega \) and such that \( f = g \).

**Example** Consider \( f(x, y) = xy/(x^2 + y^2) \). Approaching 0 from the \( x \) or \( y \) axis gives us 0, but approaching 0 from the \( x = y \) line gives us 1/2.

**Proof of Theorem.** Cauchy’s Theorem is applicable to \( f \) and yields

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta.
\]

But this function is meaningful at \( z_0 \), and thus the function is holomorphic in the whole domain.

Thus a bounded function on a neighborhood at \( z_0 \) cannot have a singularity at \( z_0 \). Now let \( f \) be holomorphic in \( \Omega \setminus z_0 \) and assume \( \lim_{z \to z_0} f(z) = \infty \). This means that \( \lim_{z \to z_0} \frac{1}{f(z)} = 0 \). But this means that \( 1/f(z) = g(z) \), where \( g \) is holomorphic and 0 around \( z_0 \), and \( g(z) = (z - z_0)^h \phi(z), \phi(z_0) \neq 0, f(z) = \frac{1}{g(z)} = \psi(z)/(z - z_0)^h, \psi(z_0) \neq 0 \). Such singularities are called poles. Representing \( \psi \) as a Taylor series around \( z_0 \), write \( \psi(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k, A_0 \neq 0 \), and so

\[
f(z) = \frac{A_0}{(z - z_0)^h} + \frac{A_1}{(z - z_0)^{h-1}} + \ldots + \frac{A_{k-1}}{z - z_0} + \tilde{f}(z).
\]

Where \( \tilde{f} \) is holomorphic.

**Definition 1.** A singularity which is not a pole is called an essential singularity.
Theorem 2.2. In any neighborhood of an essential singularity, the function $f$ takes values arbitrarily close to any given $A \in \mathbb{C}$.

Proof. Otherwise $g(z) = 1/(f(z) - A)$ is holomorphic at $z_0$, then $f(z) = A + 1/g(z)$. Thus it cannot be an essential singularity. \qed

Example Consider $\exp(1/z)$ in a neighborhood of 0.

Definition 2. Functions with no essential singularities are known as meromorphic functions.

We can see a pole as a removable singularity with value $\infty$. Thus meromorphic functions are holomorphic functions on the sphere, as opposed to the complex plane.

Let $f$ be holomorphic in $\{z : |z| = 1\} \setminus 0$ and have a pole at 0. As before, write

$$f(z) = A_{-h} \frac{z^{-h}}{z^{k-1}} + A_{1-h} \frac{z^{-1}}{z^{k-1}} + \ldots + A_{-1} \frac{1}{z} + A_0 + A_1 z + \ldots$$

Thus $\oint_{|z|=1} f(z)dz = 2\pi i \sum \text{Res} f(z)$.

Theorem 2.3 (Residue Theorem). Let $f$ be meromorphic in the unit disk $\{z : |z| = 1\}$, with finitely many poles inside the unit disk at $z_1, \ldots, z_k$. Then $\oint f(z)dz = 2\pi i \sum \text{Res} f(z)$.

Let us do $\oint_{|z|=2} \frac{z}{z^4 - 1} dz$. The poles are at 1, $-1, i, -i$. Only 1 is in the circle. We consider

$$\frac{z}{z^4 - 1} = \frac{z}{(z - 1)(z + 1)(z^2 + 1)}$$

The value of the residue is $1/4$. In general, if $f(z) = g(z)/(z - z_0)^k$, the value of the residue is $(1/(k - 1)! g^{(k-1)}(z_0)$.