1 Disclaimer

Use these notes at your own risk. This copy of my transcribed notes is here for your convenience, but neither the instructor nor I will be held responsible for any mistakes contained therein. In particular, this isn’t an “officially sanctioned” transcript of the class notes.

2 Evaluation of Definite Integrals

Consider the Dirichlet integral
\[ \int_0^\infty \frac{\sin x}{x} \, dx \]

This is equal to
\[ \lim_{R \to \infty} \int_0^R \frac{\sin x}{x} \, dx. \]

Note that \( \int_0^\infty |\sin(x)/x| \, dx = \infty \). Let’s agree that \( \int_{-R}^R dx/x = 0 \). Now
\[
\int_{-R}^R e^{ix}/x \, dx = \int_{-R}^R \frac{\cos x}{x} \, dx + i \int_{-R}^R \frac{\sin x}{x} \, dx
\]
\[= i \int_{-R}^R \frac{\sin x}{x} \, dx \]

Now let \( C \) be a semicircle whose diameter is the segment \([-R, R]\), with a little curve about the center so that the ‘semicircle’ does not intersect 0- so the diameter is composed of segments \((-R, \epsilon), (\epsilon, R)\). Let’s look at contributions of different arcs constituting the contour \( C \).

The horizontal arcs are \( 2i \int_{-R}^R \sin(x)/x \, dx \). The integral along the outer circular arc \( C_R \), \( \int_{C_R} e^{iz}/zdz \) goes to 0 as \( R \) goes to \( \infty \). As for the small arc, we have \( e^{iz}/z = 1/z + (e^{iz} - 1)/z \), and where the integral is over the arc of the small semicircle \( C_\epsilon \)

\[ \int e^{iz}/z = \int dz/z + \int \frac{e^{iz} - 1}{z} \, dz \]
\[= -\pi i \]

Thus \( \int_0^\infty \frac{\sin x}{x} \, dx = \pi/2 \). Thus the principal value

\[ \int_{-\infty}^{\infty} e^{ix}/x \, dx = \lim_{R \to \infty, \epsilon \to 0} \int_{-R}^{-\epsilon} e^{ix}/x \, dx + \int_{\epsilon}^R e^{ix}/x \, dx \]

For example, the Hilbert Transform

\[ Hf(x) = \text{vp} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-y)}{y} \, dy. \]
Consider now the Fresnel Integral $\int_0^\infty \cos(x^2)dx$. Let’s write $e^{iz^2} = \cos(z^2) + i\sin(z^2)$. Take our contour to be a horizontal line $L_1$, a line $x = y$ denoted $L_2$ and a curve connecting the two denoted $C_1$.

$$\int_{L_1} e^{iz^2} dz = \int_0^R (\cos x^2 + i\sin x^2)dx.$$

$$\int_{L_2} e^{iz^2} dz = -\int_{L_2} e^{-2x^2}(1 + i)dz.$$

As for $\int_{C_1} e^{iz^2} dz$, we may substitute $u = z^2$ to get $\int_{\tilde{C}_1} e^{iu}/\sqrt{u}du$. $\tilde{C}_1$ is a quarter-circle. This goes to 0 as $R$ goes to $\infty$.

We thus have $\int_0^\infty - (1 + i) \int_0^R e^{-2x^2} dx = \frac{\pi}{2\sqrt{2}}(1 + i)$ as $R$ goes to $\infty$. Thus $\int_0^\infty \cos x^2 dx = \int_0^{+\infty} \cos u/2\sqrt{u}du$.

3

Given an arbitrary $\gamma$ and a point $a$ around it. Then $\int_\gamma dz/(z - a) = 2\pi in(\gamma, a)$, where $n(\gamma, a)$ is the winding number of $\gamma$ with respect to $a$. We show that the winding number is an integer. Let $\gamma = \{z(t)\}_t$. Take $h(t) = \int_0^t dz(s)/(z(s) - a)$. $e^{-h(t)}(z(t) - a)$ is a constant. We check this by differentiation.