1 Disclaimer

Use these notes at your own risk. This copy of my transcribed notes is here for your convenience, but neither the instructor nor I will be held responsible for any mistakes contained therein. In particular, this isn’t an “officially sanctioned” transcript of the class notes.

2 Index of a Point with respect to a Closed Curve.

We have a point $a$ and a point $\gamma$ that doesn’t intersect $a$. The index $n(\gamma, a) = (1/2\pi i) \oint_{\gamma} 1/(z - a) \, dz$. We know that the index is always an integer.

Claim 1. If $a_1, a_2$ lie in the same connected component of $\mathbb{C} \setminus \gamma$, then $n(\gamma, a_1) = n(\gamma, a_2)$.

Proof. Take a path between $a_1, a_2$. If the value of the winding number differs, it must differ by at least 1. But then there must be a discontinuous jump in the path, contradicting the continuity of $n$. \qed

Definition 1. A cycle is a formal finite sum of closed oriented curves $\sum_{i=1}^{r} \gamma_i$.

Definition 2. A cycle $\gamma$ in a domain $\Omega$ is said to be homologous to zero if for any $a$ not lying in $\Omega$ we have $n(\gamma, a) = 0$.

Note that $n(\gamma, a) = \sum_{i=1}^{r} n(\gamma_i, a)$.

Claim 2. If $a$ lies in the unbounded region of $\mathbb{C} \setminus \gamma$, then $n(\gamma, a) = 0$. More generally, if $\gamma$ lies outside of $\mathbb{C}$, then $n(\gamma, a) = 0$.

Theorem 2.1 (General Form of Cauchy’s Theorem). Let $f$ be holomorphic in $\Omega$, except at finitely many points $z_i$ at which $\lim_{z \to z_i} f(z) = 0$. Let $\gamma \sim 0$. Then $\int_{\gamma} f \, dz = 0$.

Definition 3. A domain $\Omega \subset \mathbb{C}$ is simply connected if its complement in $\mathbb{C}$ is connected.

Equivalently, every closed curve $i\gamma$ in $\Omega$ is homologous to 0.

Corollary 2.2. If $\Omega$ is simply connected and $f$ is holomorphic in $\Omega$, then $\int_{\gamma} f(z) \, dz = 0$ for any closed curve $\gamma$ in $\Omega$.

Corollary 2.3. Let $\gamma \sim 0$ in $\Omega$. Then $\int_{\gamma} f(z)/(z - a) \, dz = 2\pi i n(\gamma, a)f(a)$. In particular, if $\Omega$ is simply connected then this equation holds for any closed curve $\gamma$. 1
3 The Argument Principle

Let $\Omega$ be a domain, and let $\gamma$ be a closed curve in $\mathbb{D}$. Let $f$ be holomorphic, $f \neq 0$ on $\gamma$.

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} \, dz = n(\Gamma, 0) = \sum_{z_i \text{ zeroes of } f} \text{mult}(z_i)n(\gamma, z_i)
\]

Let $z_1, \ldots, z_k$ be the zeroes of $f$, each zero taken as many times as the multiplicity suggests. Then $(z - z_1)(z - z_2)\ldots(z - z_k)\phi(z)$, where $\phi(z) \neq 0$ in $\mathbb{D}$.

\[
\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \ldots + \frac{1}{z - z_k} + \frac{\phi'(z)}{\phi(z)}.
\]

So

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum n(\gamma, z_i).
\]

In other words, the numbers of times $f(\partial\mathbb{D})$ goes around 0 is the sum of multiplicities of 0’s inside $\mathbb{D}$.

**Exercise**  Formulate the Theorem in the most general form.