1 Disclaimer

Use these notes at your own risk. This copy of my transcribed notes is here for your convenience, but neither the instructor nor I will be held responsible for any mistakes contained therein. In particular, this isn’t an “officially sanctioned” transcript of the class notes.

2 The Argument Principle

Proposition 2.1. If \( U \) is an open set and \( f \) is a nonconstant holomorphic mapping on \( U \), then \( f(U) \) is open.

Proof. Take any point \( z_0 \) on \( U \), and let \( w_0 \) be the image of \( z_0 \) via \( f \). Take \( \tilde{f} = f - w_0 \). Choose a circle \( C \) around \( z_0 \) so that no other zero of \( \tilde{f} \) is in the disk bounded by \( C \). Let \( C \) map to \( \gamma \) via \( \tilde{f} \). Consider another point \( z_1 \) in the interior of \( C \). It maps to \( w_1 \) - but then clearly \( \text{ind}(\gamma_1, w_1) = \text{ind}(\gamma_1, w_0) \). Since the index is locally constant, \( w_1 \) must lie in the same connected open region that contains \( w_0 \).

Corollary 2.2. The number of times the value \( w_0 \) is achieved in \( F \) is the index of \( \gamma \) with respect to \( w_0 \).

Corollary 2.3. If \( f \) is holomorphic at \( z_0 \) and \( f'(z_0) \neq 0 \), then \( f \) is locally invertible and \( f^{-1} \) is holomorphic.

3 The Maximum Principle

Theorem 3.1 (Maximum Principle). Let \( U \) be a domain with compact closure. Let \( f : U \to \mathbb{C} \) be holomorphic. Then \( \max_{U} |f| = \max_{\partial U} |f| \). Equivalently, if \( f \) is holomorphic and nonconstant, then \( |f| \) cannot have local maxima.

Proof. A neighborhood of \( z \) is mapped to an open neighborhood of \( w \). But then clearly in that neighborhood contains a point of higher absolute value.

Theorem 3.2 (Rouche’s Theorem). \( f, g : D \to \mathbb{C} \) holomorphic, and \( |g(z)| < |f(z)| \) on \( \partial D \). Then \( f, f + g \) have the same number of zeroes in \( D \).

Proof. Consider the integral \( (1/2\pi) \int_C (f'/tg')/(f+tg)dz \) as \( t \) ranges from 0 to 1. It is a continuous function that only takes constant values, and so it must be constant. But then the winding number of the image of \( f \) must be the same as the winding number of the images of \( f + g \), and so the number of zeroes with multiplicites is equal.

This gives a proof of the fundamental theorem of algebra. Consider a complex polynomial degree \( n \). Let \( f \) be the leading term, and \( g \) be the rest of the polynomial. We then apply Rouche’s Theorem to an infinitely large circle.