ON POSITIVE LYAPUNOV EXPONENT FOR THE SKEW-SHIFT POTENTIAL

HELGE KRÜGER

Abstract. I will prove that for most energies $E$ and some frequencies $\alpha$ the Lyapunov exponent of the Schrödinger operator with potential $V(n) = 2\lambda \cos(2\pi \alpha n^2)$ has the behavior $L(E) \gtrsim \lambda^2$ as $\lambda \to 0$. This improves upon earlier results by Bourgain.

1. Introduction

In this paper, I wish to analyze the skew-shift Schrödinger operator at small coupling. That is for $\lambda > 0$ small and $\alpha$ irrational the potential

$$V_{\alpha, \lambda}(n) = 2\lambda \cos(2\pi \alpha n^2)$$

and the associated Schrödinger operator $H_{\alpha, \lambda} = \Delta + V_{\alpha, \lambda}$. The Lyapunov exponent $L_{\alpha, \lambda}(E)$ describes the maximal exponential growth of solutions of $H_{\alpha, \lambda} u = Eu$. We will give a definition in (2.4). See also [7] or [15] for background.

It is known that for irrational $\alpha$ and $\lambda > 1$ large enough, we have $L_{\alpha, \lambda}(E) \gtrsim \log \lambda$. This can be shown by using Herman’s subharmonicity trick from [11], or under suitable additional assumptions on $\alpha$ or $E$ by the more constructive methods of Bourgain [3], Bourgain, Goldstein, and Schlag [6], or myself [12].

My main goal is to improve the result of Bourgain from [1] on positivity of the Lyapunov exponent for this model for small $\lambda > 0$, certain irrational $\alpha$, and most energies $E$. I will prove the following quantitative version of Bourgain’s result.

Theorem 1.1. Given $\delta > 0$ and $\tau > 0$. There are $\lambda_0 = \lambda_0(\delta, \tau) > 0$ and $\gamma_0 = \gamma_0(\delta, \tau) > 0$.

For each $0 < \lambda < \lambda_0$, there is $\alpha_0(\lambda) > 0$ such that for $0 < \alpha < \alpha_0$ irrational, there exists a set $E_b = E_b(\lambda, \alpha)$ of measure

$$|E_b| \leq \tau$$

such that for $E \in ((-2 + \delta, -\delta] \cup [\delta, 2 - \delta]) \setminus E_b$ 

$$L_{\alpha, \lambda}(E) \geq \gamma_0 \lambda^2.$$ 

The main improvement is the quantitative lower bound $L_{\alpha, \lambda}(E) \gtrsim \lambda^2$. This behavior is expected to hold for all irrational $\alpha$, $E \in (-2, 2) \setminus \{0\}$, and $0 < \lambda \leq 1$. However even positivity is unknown (see e.g. [4], [14]). In my earlier paper [12], I established a similar result for $E \in (-2, 2) \setminus \{0\}$. However, I had to replace the

Date: April 27, 2010.

2000 Mathematics Subject Classification. Primary 81Q10; Secondary 37D25.

Key words and phrases. Lyapunov Exponents, Schrödinger Operators.

H. K. was supported by NSF grant DMS–0800100 and a Nettie S. Autrey Fellowship.
power 2 in (1.1) by a higher one, so that \( V(n) = 2\lambda \cos(2\pi \alpha n^k) \) for an irrational \( \alpha \) and large enough \( k \) depending on \( \lambda > 0 \).

That \( |E_b| \) doesn’t converge to 0 as \( \lambda \to 0 \) comes from that I wish to ensure the behavior \( L_{\alpha,\lambda}(E) \gtrsim \lambda^2 \). One could also show for \( \varepsilon > 0 \) that \( L_{\alpha,\lambda}(E) \gtrsim \lambda^{2+\varepsilon} \) with \( |E_b| \lesssim \lambda^\varepsilon \) and uniform constants. For this and a discussion of the size of \( \alpha \), see the end of Section 3.

Combining the proof of the previous theorem with the results of Bourgain, Goldstein, and Schlag in [6], I can furthermore show

**Theorem 1.2.** Let \( \lambda \) and \( \alpha \) satisfy the same conditions as in Theorem 1.1. Assume furthermore that \( \alpha \) satisfies a Diophantine condition

\[
\text{dist}(n\alpha, \mathbb{Z}) \geq \frac{\kappa}{|n|^t}
\]

for \( n \in \mathbb{Z} \setminus \{0\} \) and some fixed \( \kappa, t > 0 \). Then the set \( E_b \) can be chosen to be a finite union of intervals.

The problem with is that the methods from [6] are not directly applicable, since our initial condition only holds with low probability, and [6] needs probabilities extremely close to 1 (depending on the initial estimate on the Lyapunov exponent). We will demonstrate how this can be achieved using our methods at the end of the paper. Let me now discuss the proof of Theorem 1.1.

The proof of Theorem 1.1 essentially splits into two parts. First, I improve on my earlier results from [12]. The improvement is that the initial conditions are no longer required to hold with large probability, but can hold only with small probability.

Then the initial condition is verified by approximating the skew-shift model by the Almost–Mathieu operator with potential \( 2\lambda \cos(2\pi(\beta n + \omega)) \). It is essential here, that one notices that if \( \alpha \) is small one approximately has for \( m \) in a bounded range that \( \alpha(n_0 + m)^2 \approx \alpha n_0^2 + 2\alpha n_0 m \).

On a technical side except improving the lower bounds on the Lyapunov exponent, a second modification to the method of Bourgain in [1] is that we do not perturb the Green’s function directly, but work directly by perturbing the spectrum of the restricted operators. This enables one to prove the following extension

**Remark 1.3.** Theorem 1.1 remains valid, if one replaces \( V_{\alpha,\lambda}(n) \) from (1.1) by

\[
V_{\alpha,\lambda}(n) = 2\lambda \cos(2\pi \alpha n^2) + \lambda \kappa V_\omega(n)
\]

where \( V_\omega(n) \) is an ergodic potential and \( \kappa > 0 \) is small enough.

The organisation of the paper is as follows. In Section 2, we discuss general results that imply positive Lyapunov exponent for ergodic Schrödinger operators. In Section 3, we apply these to the skew-shift potential and prove Theorem 1.1. In Section 4, we prove the main result from Section 2. Finally, in Section 5, we discuss the modifications needed to proof Theorem 1.2.

## 2. Theorems that imply positive Lyapunov exponent

In this section, I will discuss results that imply positive Lyapunov exponent. They all have in common, that they require assumptions with low probability. This is an essential difference to my earlier work [12], where I assumed that the bad events have small probability. In Section 7 and 8 of [2], Bourgain has stated
similar results. However, he worked under Diophantine assumptions and used semi algebraic set techniques.

We will work in the generality of ergodic Schrödinger operators. So let $(\Omega, \mu)$ be a probability space, $T : \Omega \to \Omega$ an invertible ergodic transformation, and $f : \Omega \to \mathbb{R}$ a bounded measurable function. Given these, we introduce for $\omega \in \Omega$, the family of potentials

$$V_\omega(n) = f(T^n \omega)$$

for all integers $n$. We define the discrete Schrödinger operators $H_\omega$ by

$$H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$$

$$H_\omega u(n) = u(n + 1) + u(n - 1) + V_\omega(n)u(n).$$

These operators $H_\omega$ will be bounded and self-adjoint operators.

Define the $n$ step transfer matrix

$$A_\omega(E, n) = \prod_{j=0}^{n-1} \begin{pmatrix} E - V_\omega(n-j) & -1 \\ 1 & 0 \end{pmatrix}.$$ 

The Lyapunov exponent $L(E)$ is given by

$$L(E) = \lim_{n \to \infty} \frac{1}{n} \int_\Omega \log \|A_\omega(E, n)\| d\mu(\omega),$$

where the limit exists because of subadditivity.

For a subinterval $\Lambda \subseteq \mathbb{Z}$, we introduce $H_\omega, \Lambda$ as the restriction of $H_\omega$ to $\ell^2(\Lambda)$. Furthermore, we denote by $\sigma(H)$ the spectrum of the operator $H$.

The first version of our result is

**Theorem 2.1.** Given $\sigma > 0$, $N \geq 1$, $0 < \varepsilon < \frac{1}{1000}$, and $E_0 \in \mathbb{R}$. Assume that

$$\mu(\{\omega : \sigma(H_{\omega,[-N,N]}) \cap [E_0 - 2\varepsilon, E_0 + 2\varepsilon] = \emptyset\}) \geq \sigma.$$ 

Furthermore, assume $\varepsilon \log(\sigma^{-1}) \leq \frac{\sigma}{150}$ and

$$N \geq \frac{50}{\varepsilon^2}.$$ 

Then, there is a set $\mathcal{E}_b \subseteq [E_0 - \varepsilon, E_0 + \varepsilon]$ of measure $|\mathcal{E}_b| \leq 2\varepsilon e^{-\frac{\sigma}{150}}$ such that for $E \in [E_0 - \varepsilon, E_0 + \varepsilon] \setminus \mathcal{E}_b$

$$L(E) \geq \frac{1}{250}\sigma\varepsilon.$$ 

We will now reformulate this theorem in terms of the Green’s function, since this language is more convenient for the proof. Denote by $\{e_x\}_{x \in \mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$, that is

$$e_x(n) = \begin{cases} 1, & x = n; \\ 0, & \text{otherwise.} \end{cases}$$

Given an interval $\Lambda \subseteq \mathbb{Z}$, $E \in \mathbb{R}$, and $x, y \in \Lambda$, we introduce the Green’s function

$$G_{\omega,\Lambda}(E, x, y) = \langle e_x, (H_{\omega,\Lambda} - E)^{-1}e_y \rangle.$$
**Definition 2.2.** Given $\delta > 0$, $\mathcal{E} \subseteq \mathbb{R}$ an interval, and $N \geq 1$. We say that $H_\omega$ is $(\delta, N, \mathcal{E})$-good if there exists $k \in [a, b] \subseteq [-N, N]$ such that

$$\sup_{E \in \mathcal{E}} \sup_{x \in \{a, b\}} |G_{\omega, [a, b]}(E, k, x)| \leq e^{-2\delta}.$$ 

Now, we come to

**Theorem 2.3.** Let $\delta \geq 4$, $\mathcal{E} \subseteq \mathbb{R}$ an interval, and $\sigma > 0$. Assume that

$$\mu(\{\omega : H_\omega \text{ is } (\delta, N, \mathcal{E}) \text{-good}\}) \geq \sigma.$$ 

Furthermore assume the inequalities

$$\delta \geq 100 \log(|\mathcal{E}|^{-1})$$

(2.12)

$$e^{2/755} \geq \frac{2^{26} e^3}{\sigma}.$$ 

(2.13)

Then there is a set $\mathcal{E}_0 \subseteq \mathcal{E}$ such that

$$|\mathcal{E}_0| \geq (1 - e^{-\frac{199}{99}} \delta)|\mathcal{E}|$$

(2.14)

and for $E \in \mathcal{E}_0$

$$L(E) \geq e^{-\frac{199}{99} \sigma \delta}.$$ 

(2.15)

We now end this section, with that this theorem implies Theorem 2.1.

**Proof of Theorem 2.1.** This follows from the Combes–Thomas estimate (see e.g. Lemma 10.1 in [12]) with the choice $\delta = 2/\varepsilon$. \(\square\)

**Remark 2.4.** By changing the constants in Theorem 2.1 one could improve (2.6) to

$$N \geq C \frac{\log(\varepsilon^{-1})}{\varepsilon}$$

(2.16)

for a large enough constant $C > 0$. I have decided not to do so, to keep the formulas simple.

### 3. Applications to the skew-shift

We will begin by recasting the problem concerning the potential $V(n) = 2\lambda \cos(2\pi \alpha n^2)$ from (1.1) into the language of ergodic Schrödinger operators used in Section 2. Denote by $T = \mathbb{R}/\mathbb{Z}$ the unit circle. As probability space we will use the torus $\mathbb{T}^2$ equipped with the Lebesgue measure. For an irrational number $\alpha$ introduce the skew-shift $T_\alpha$ by

$$T_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$$

$$T_\alpha(x, y) = (x + \alpha, x + y) \pmod{1}.$$ 

(3.1)

It is known that $T_\alpha$ is an uniquely ergodic and minimal map. Iterating the map $T_\alpha$ on $(x, y)$, we find

$$T^n_\alpha(x, y) = (x + n\alpha, y + nx + \frac{n(n-1)}{2}\alpha) \pmod{1}.$$ 

(3.2)

Write $p_2$ for the second coordinate projection $p_2(x, y) = y$. We see that

$$p_2(T^n_\alpha(\frac{\alpha}{2}, 0)) = \frac{\alpha}{2} n^2.$$ 

(3.3)
Hence, it makes sense to introduce the skew-shift operator $H_{\alpha,x,y,\lambda}$ by
\begin{equation}
H_{\alpha,x,y,\lambda} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})
\end{equation}
\begin{equation*}
H_{\alpha,x,y,\lambda} u(n) = u(n+1) + u(n-1) + 2\lambda \cos \left(2\pi \cdot p_2(T_\alpha^n(x,y))\right) u(n).
\end{equation*}
We see that the operator $H_{\alpha,\lambda}$ from the introduction will just be $H_{2\alpha,\alpha,0,\lambda}$.

By (3.2), we approximately have for $n$ in a bounded range and $\alpha$ small enough that $p_2(T_\alpha^n(x,y)) \approx y + nx$. We will make this precise in Lemma 3.5. However, let us for now say, that we will try to approximate the skew-shift by rotations, and introduce these.

For an irrational number $\alpha$ denote by $R_\alpha : \mathbb{T} \to \mathbb{T}$ the rotation by
\begin{equation}
R_\alpha(x) = x + \alpha \pmod 1.
\end{equation}
Introduce for $\alpha$, $\omega$, and $\lambda > 0$ the operator $H_{R_\alpha,\omega,\lambda}$ by
\begin{equation}
H_{R_\alpha,\omega,\lambda} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})
\end{equation}
\begin{equation*}
H_{R_\alpha,\omega,\lambda} u(n) = u(n+1) + u(n-1) + 2\lambda \cos(2\pi(n\alpha + \omega)) u(n).
\end{equation*}
This family of operators is known as Almost–Mathieu Operator. We will need the following result

**Theorem 3.1.** Let $\delta > 0$ and $E$ satisfy
\begin{equation}
E \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta].
\end{equation}
There are constants $\kappa = \kappa(\delta) > 0$ and $\lambda_0 = \lambda_0(\delta) > 0$ such that for $0 < \lambda < \lambda_0$, there is a set $A \subseteq [0,1]$ satisfying:
\begin{enumerate}
\item $|A| \geq \kappa \lambda$
\item For $\alpha \in A$
\end{enumerate}
\begin{equation}
\sigma(H_\alpha) \cap [E - \lambda \kappa, E + \lambda \kappa] = \emptyset.
\end{equation}

**Proof.** This can be shown by a combination of Aubry Duality [9] as see in [1] and the results of [13] or [10] on the spectrum of $H_{R_\alpha,\omega,\lambda}$ for large $\lambda > 0$. \hfill $\square$

We will need the following lemma to pass from whole line operators to finite restrictions. Because of its abstract nature, we state it in the language of ergodic Schrödinger operators used in the previous section.

**Lemma 3.2.** Let $\varepsilon > 0$, $\kappa > 0$, $E_0 \in \mathbb{R}$, $Q \geq 1$, $\Lambda \subseteq \mathbb{Z}$ an interval, and $\Omega_0 \subseteq \Omega$ with $\mu(\Omega_0) > 0$. Assume for $\omega \in \Omega_0$ that
\begin{equation}
\sigma(H_\omega) \cap [E_0 - \varepsilon, E_0 + \varepsilon] = \emptyset.
\end{equation}
Introduce for $q = 0, \ldots, Q - 1$ the intervals
\begin{equation}
E_q = [E_0 - \varepsilon + 2\varepsilon \frac{q}{Q}, E_0 - \varepsilon + 2\varepsilon \frac{q+1}{Q}].
\end{equation}
There exists a set $\mathcal{Q} \subseteq [0, Q - 1]$ satisfying
\begin{equation}
\# \mathcal{Q} \leq \frac{8}{\kappa}
\end{equation}
such that for each $q \in \mathcal{Q}$, there is $\Omega_1 = \Omega_1(q) \subseteq \Omega_0$ such that
\begin{equation}
\mu(\Omega_1) \geq (1 - \kappa)\mu(\Omega_0)
\end{equation}
and for $\omega \in \Omega_1$

$$\sigma(H_{\omega,\Lambda}) \cap E_q = \emptyset. \quad (3.13)$$

**Proof.** Let $f(\omega, q)$ be given by

$$f(\omega, q) = \begin{cases} 
1 & \sigma(H_{\omega,\Lambda}) \cap E_q \neq \emptyset \\
0 & \text{otherwise.}
\end{cases}$$

As usual, we let $\Lambda^c = \mathbb{Z} \setminus \Lambda$. We have $H_{\omega} = H_{\omega,\Lambda} \oplus H_{\omega,\Lambda^c} + K$ with $K$ a rank 4 operator. Since $E_q \cap E_{\tilde{q}}$ consists of at most 1 point for $q \neq \tilde{q}$, we obtain that $\sum_{q=0}^{Q-1} f(\omega, q) \leq 8$. Hence, also

$$\sum_{q=0}^{Q-1} \frac{1}{\mu(\Omega_0)} \int_{\Omega_0} f(\omega, q) d\mu(\omega) \leq 8.$$

Define $Q = \{ q : \frac{1}{\mu(\Omega_0)} \int_{\Omega_0} f(\omega, q) d\mu(\omega) \geq \kappa \}$. By Markov’s inequality, we obtain (3.11). This finishes the proof. \[\square\]

In order to apply this lemma, we will need to subdivide $[-2 + \delta, -\delta] \cup [\delta, 2 - \delta]$ into small intervals. To simplify the notation, we will only work with $[\delta, 2 - \delta]$. First introduce for $\lambda > 0$ the subdivision

$$E_q^0(\lambda) = [\delta + q\kappa \lambda, \delta + (q + 1)\kappa \lambda],$$

where $q = 0, \ldots, Q^0(\lambda)$ with $Q^0(\lambda) = \lfloor \frac{2 - \delta}{\kappa \lambda} \rfloor$. Denote by $\tau > 0$ the number from Theorem 1.1. We will furthermore subdivide each of the $E_q^0(\lambda) = [E_0, E_1]$ intervals into $P = \lceil \frac{\tau}{2} \rceil$ pieces. These are given by

$$E_{q,p}^1(\lambda) = \left[ E_0 + \frac{p - 1}{P} (E_1 - E_0), E_1 + \frac{p + 2}{P} (E_1 - E_0) \right].$$

We note that the intervals $E_{q,p}^1(\lambda)$ overlap. However, if we only consider odd $p$ respectively even $p$, we get non overlapping sets (except for the boundaries). We furthermore introduce

$$E_{q,p}^2(\lambda) = \left[ E_0 + \frac{p}{P} (E_1 - E_0), E_1 + \frac{p + 1}{P} (E_1 - E_0) \right].$$

We have that

**Lemma 3.3.** We have that

$$|E_{q,p}^1(\lambda)| = \frac{3 \kappa \lambda}{P}, \quad |E_{q,p}^2(\lambda)| = \frac{\kappa \lambda}{P}$$

and $E_{q,p}^2(\lambda) \subseteq E_{q,p}^1(\lambda)$.

**Proof.** We observe that $|E_q^0(\lambda)| = \kappa \lambda$ by construction, and that we divide into pieces of length $\frac{3}{P} \cdot |E_q^0(\lambda)|$ and $\frac{1}{P} \cdot |E_q^0(\lambda)|$ respectively. \[\square\]

Now, we come to

**Corollary 3.4.** Let $N \geq 1$ and $0 \leq q \leq Q^0(\lambda)$. For $\lambda > 0$ small enough, there exists a set $P = P(q, \lambda) \subseteq [0, P - 1]$ satisfying $\#P \leq 64$. Let $p \notin P$. Then there is $\Omega_1(\lambda) \subseteq [0, 1]^2$ satisfying

$$|\Omega_1(\lambda)| \geq \frac{\kappa \lambda}{4} \quad (3.18)$$
and for \((\alpha, \omega) \in \Omega_1(\lambda)\), we have

\[(3.19) \quad \sigma(H^R_{\alpha,\omega,\lambda,[-N,N]}) \cap \mathcal{E}_{q,p}^1(\lambda) = \emptyset.\]

Proof. Let \(A\) be the set from Theorem 3.1 for \(E = \delta + (2q + 1)\kappa \lambda\). Introduce \(\Omega_0 = A \times [0, 1] \subseteq [0, 1]^2\). Then for \((\alpha, \omega) \in \Omega_0\), we have

\[\sigma(H^R_{\alpha,\omega,\lambda}) \cap \mathcal{E}_{q}^0(\lambda) = \emptyset\]

by Theorem 3.1. Now apply Lemma 3.2 with \(Q = P\), \(\kappa = \frac{1}{4}\), and \(\Lambda = [-N, N]\) to the set of \(\mathcal{E}_{q,p}^1\) with odd respectively even \(p\). The claim follows. \(\square\)

The next lemma follows from (3.2).

Lemma 3.5. Assume that \(|\alpha| \leq \frac{2\kappa}{N^2}\), then

\[(3.20) \quad |p_2(T^N_{\alpha}(x, y)) - R_y(x)| \leq \varepsilon\]

for \(|n| \leq N\).

We have the following lemma, which allows us to compare the operators \(H^R_{x,y,\lambda,[-N,N]}\) and \(H_{\alpha,x,y,\lambda,[-N,N]}\) for \(\alpha\) sufficiently small.

Lemma 3.6. Let \(\varepsilon > 0\) and \(N \geq 1\). Then for \(|\alpha| \leq \frac{2\kappa}{N^2}\), \(\delta > 0\), and \(E \in \mathbb{R}\) such that

\[(3.21) \quad \sigma(H^R_{x,y,\lambda,[-N,N]}) \cap [E - \delta, E + \delta] = \emptyset.\]

We also have that

\[(3.22) \quad \sigma(H_{\alpha,x,y,\lambda,[-N,N]}) \cap [E - (\delta - \lambda \varepsilon), E + (\delta - \lambda \varepsilon)] = \emptyset.\]

Proof. This follows from the last lemma, and that the spectrum we have for the spectrum

\[\text{dist}(\sigma(H), \sigma(H + V)) \leq ||V||,\]

where dist denotes the Hausdorff metric. \(\square\)

Combining this lemma with Corollary 3.4, we obtain

Corollary 3.7. Assume

\[(3.23) \quad |\alpha| \leq \frac{\kappa}{\mathcal{P} N^2}.\]

Given \(q\), denote by \(\mathcal{P}\) the set from Corollary 3.4. For \(p \notin \mathcal{P}\), there is a set \(\Omega_1\) of \((x, y)\) such that

\[(3.24) \quad |\Omega_1| \geq \frac{\kappa \lambda}{4}\]

and for \((x, y) \in \Omega_1\), we have that

\[(3.25) \quad \sigma(H_{\alpha,x,y,\lambda,[-N,N]}) \cap \mathcal{E}_{q,p}^2(\lambda) = \emptyset.\]

Proof. We first note that \(\mathcal{E}_{q,p}^1(\lambda)\) is the \(\frac{\kappa \lambda}{\mathcal{P}}\) neighborhood of \(\mathcal{E}_{q,p}^2(\lambda)\). Hence, we will apply Lemma 3.6 with \(\varepsilon = \frac{\kappa \lambda}{\mathcal{P}}\) to the sets from Corollary 3.4 to obtain the result. \(\square\)

We now come to
Proof of Theorem 1.1. We just prove the claim for \([\delta, 2-\delta]\), the interval \([-2+\delta, -\delta]\) can be dealt with similarly.

First observe by construction that
\[
|\delta, 2-\delta| \setminus \bigcup_{q,p \notin \mathcal{P}(q)} \mathcal{E}_{q,p}^2(\lambda) | \leq \frac{\tau}{4}.
\]
Furthermore by the previous corollary, we have that we may apply Theorem 2.1 with the choices
\[
\sigma = \kappa \lambda, \quad \varepsilon = \kappa \lambda P, \quad N = \left\lceil \frac{50}{\varepsilon^2} \right\rceil
\]
as long as \(\lambda\) is sufficiently small, and \(\alpha\) satisfies the conditions from the previous corollary. We see that by choosing again \(\lambda > 0\) sufficiently small, we can ensure that
\[
|\mathcal{E}_b| \geq \frac{\tau \lambda}{4 Q P},
\]
since \(QP \gtrsim \lambda^{-1} \). □

An inspection of the above shows, that we prove the bound
\[
L_{\alpha, \lambda}(E) \geq \frac{\kappa^2}{500} \frac{1}{\lambda^2}.
\]
Since \(P \leq \frac{300}{\tau^2}\), we conclude that
\[
L_{\alpha, \lambda}(E) \geq \frac{\kappa^2 \tau}{150000} \lambda^2,
\]
which justifies the discussion following the statement of Theorem 1.1.

The current proof shows that we have \(\alpha_0 \gtrsim \frac{1}{\lambda^7}\) in Theorem 1.1. Taking Remark 2.4 into account this can be improved to \(\alpha_0 \gtrsim \frac{\log(\lambda)}{\lambda^7}\).

4. Proof of the Theorem 2.3

We now proceed to prove Theorem 2.3. The proof will follow by an adaptation of the arguments from my earlier work [12]. The main difference is that we will have to deal with small probabilities.

We will need a variant of Theorem 3.3. from [12], which can be extracted from the proof.

**Theorem 4.1.** Let \(K \geq 1, \delta \geq 4\), and \(\mathcal{E} \subseteq \mathbb{R}\) an interval.

Assume that for a positive measure set of \(\omega\), we can find sequences \(N_t = N_t(\omega), L_t = L_t(\omega) \to \infty\) such that

\[
\lim_{t \to \infty} \frac{N_t}{L_t} \geq K
\]
and for \(t \geq 1\) there are integers \(k_l = k_l(t, \omega)\)

\[
0 \leq k_0 < k_1 < k_2 < \cdots < k_L < k_{L+1} \leq N_t
\]
and a set \(L_t = \mathcal{L}_t(\omega) \subseteq [1, L_t]\) such that for \(l \notin \mathcal{L}_t\), we have that

\[
|G_{\omega, [k_{l-1}, k_{l+1}]}(E, k_l, k_l \pm 1)| \leq \frac{1}{2} e^{-\delta}
\]
for \(E \in \mathcal{E}\). Furthermore assume the inequalities

\[
\delta \geq 100 \log(|\mathcal{E}|^{-1})
\]

\[
e^{-\frac{\delta}{\varepsilon}} \geq 2^{25} e^{3} K^3
\]

Then for a set \(\mathcal{E}_0\) satisfying

\[
|\mathcal{E}_0| \geq (1 - e^{-\frac{\delta}{\varepsilon}})|\mathcal{E}|
\]
we have for $E \in \mathcal{E}_0$

\begin{equation}
L(E) \geq e^{-\frac{199}{99} \delta} K.
\end{equation}

Proof. This result can be seen by verifying that our assumptions are the same as the conclusions of Lemma 9.1, after which one can proceed as in the proof of Theorem 3.3. in [12].

We furthermore specialize to the case $\sigma \equiv \frac{1}{4}$. □

For the proof of Theorem 1.2, we will furthermore need Corollary 4.2. Denote by $\mathcal{E}_0$ the set from the previous theorem. Let $E \in \mathcal{E}_0$. Denote by $\Omega_0$ the set of $\omega$, where the assumptions of the previous theorem hold for fixed large enough $t$. Let $N = \lceil \frac{N_0}{2} \rceil$. For $\omega \in \Omega_0$, we have that

\begin{equation}
\frac{1}{N} \log \| A^t \omega (E, N) \| \geq e^{-\frac{199}{99} \delta} K
\end{equation}

(4.8)

\begin{equation}
\frac{1}{N} \log \| A^{Nt} \omega (E, N) \| \geq e^{-\frac{199}{99} \delta} K
\end{equation}

(4.9)

\begin{equation}
\frac{1}{2N} \log \| A^t \omega (E, 2N) \| \geq e^{-\frac{199}{99} \delta} K.
\end{equation}

(4.10)

Proof. This follows by the methods used to prove the previous theorem and the results in Section 5 of my earlier paper [12], in particular Lemma 5.2. □

We will furthermore need Lemma 4.1. from [12].

**Lemma 4.3.** Let $T : \Omega \to \Omega$ be an ergodic transformation, $\Omega_g \subseteq \Omega$, $K \geq 1$. Then there exists $\Omega_0 \subseteq \Omega$ such that for $\omega \in \Omega_0$ there is a sequence $L_t = L_t(\omega) \to \infty$ such that

\begin{equation}
\frac{1}{L_t} \# \{ 0 \leq l \leq L_t - 1 : T^{lK} \omega \in \Omega_g \} \geq \frac{\mu(\Omega_g)}{2}
\end{equation}

(4.11)

and $\mu(\Omega_0) > 0$.

We apply this lemma to set from (2.11). By this, we obtain

**Lemma 4.4.** Assume (2.11). For $\omega$ from a set of positive measure, we can find sequences $L_t, N_t \to \infty$ such that

(i) We have

\begin{equation}
\frac{L_t}{N_t} \geq \frac{\sigma}{2}.
\end{equation}

(4.12)

(ii) There are integers $a_l = a_{l}(t), b_l = b_{l}(t), k_l = k_{l}(t)$ such that

\begin{equation}
0 < a_1 < k_1 < b_1 < a_2 < k_2 < b_2 \cdots < k_{L_t} < b_{L_t} < N_t.
\end{equation}

(4.13)

Here the $k_l$ can depend on $t$.

(iii) For each $l$, we have that

\begin{equation}
\sup_{x \in [a_l, b_l]} |G_{\omega, [a_l, b_l]} (E, k_l, x) | \leq e^{-2\delta}.
\end{equation}

(4.14)

Proof. Apply the previous lemma to the set from (2.11) with $K = 2N + 1$. □

For the proof of Theorem 1.2 we remark
Remark 4.5. By replacing the use of Lemma 4.1. in [12] by Lemma 4.2. we can even ensure that the result of this lemma holds for a set of $\omega$, whose measure is arbitrarily close to 1.

Hence, by an inspection of the following, one can conclude that the conclusions of Corollary [4.2] hold for a set $\Omega_0$, whose measure is arbitrarily close to 1.

We now fix $\omega$ such that the conclusions of Lemma 4.4 hold. The main problem with (4.14) is that the intervals $[a_l, b_l]$ on which it holds are not overlapping. Hence, there is a priori no way one could iterate it to obtain an estimate for the Green’s function on all intervals. So, we will now proceed to make this estimate hold on overlapping intervals as required by the results from my earlier work.

The strategy of this will be similar to one in [12], we will obtain a priori bounds on the large intervals by being far from the spectrum, and then use this to improve our current bounds.

For this, we first need.

Lemma 4.6. There is $L_1 \subseteq [1, L_t]$ satisfying
\[
\frac{1}{L_t} \# L_1 \leq \frac{1}{8}
\]
such that for $l \notin L_1$
\[
k_{l+1} - k_{l-1} \leq \frac{32}{\sigma}.
\]

Proof. We have that
\[
\sum_{l=1}^{L_t} k_{l+1} - k_{l-1} \leq 2 \cdot N_t
\]
Hence, the claim follows by Markov’s inequality and (4.12). \qed

In order to satisfy (4.4), we will subdivide $E = [E_0, E_1]$ into intervals $E_q$. So define $Q = \lfloor \frac{1}{4}(E_1 - E_0)e^\delta \rfloor$ and
\[
E_q = [E_0 + q \frac{E_1 - E_0}{Q}, E_1 + (q + 1) \frac{E_1 - E_0}{Q}],
\]
for $q = 0, \ldots, Q - 1$. We then have that
\[
|E_q| \geq 4e^{-\delta}.
\]
We furthermore have that
\[
4e^{-\delta} \geq e^{-\delta/100}
\]
since by assumption $\delta \geq 1$. We will now need to eliminate eigenvalues. This will be accomplished by the following lemma.

Lemma 4.7. There exists a set $Q$ satisfying
\[
\# Q \leq \frac{800}{\sigma}
\]
and for $q \notin Q$, there exists $L_2 = L_2(q)$ such that
\[
\frac{1}{L_t} \# L_2 \leq \frac{1}{8}
\]
and for $l \notin L_1 \cup L_2$
\[
dist(E_q, \sigma(H_\omega, [k_{l-1+1}, k_{l+1-1}])) > 4e^{-\delta}.
\]
Proof. We follow the proof of Lemma 6.6. in [12]. For each \( l \)
introduce
\[
g(l) = \# \{ q : \text{dist}(E_q, \sigma(H_{\omega, [k_{l-1}+1, k_{l+1}-1]}) \leq 4e^{-\delta} \}.
\]
By the previous lemma, we have for \( l \not\in \mathcal{L}_1 \) that \( g(l) \leq 96/\sigma \leq 100/\sigma \). Hence,
\[
\sum_q 1_{\mathcal{L}_1} \# \{ l \not\in \mathcal{L}_1 : \text{dist}(E_q, \sigma(H_{\omega, [k_{l-1}+1, k_{l+1}-1]}) \leq 4e^{-\delta} \} \leq 100/\sigma.
\]
The claim now follows from Markov’s inequality. \( \square \)

We observe that the measure of the set of energies, we need to eliminate can be estimate by
\[
\frac{40000}{\sigma |\mathcal{E}|} e^{-\delta}.
\]

Lemma 4.8. If
\[
\text{dist}(E, \sigma(H_{\omega,[k_{l-1}+1, k_{l+1}-1]})) \geq 4e^{-\delta}.
\]
holds, then
\[
|G_{\omega,[k_{l-1}+1, k_{l+1}-1]}(E, k_l, k_{l\pm 1})| \leq \frac{1}{2} e^{-\delta}.
\]
Proof. By the resolvent equation, we have that
\[
G_{\omega,[k_{l-1}+1, k_{l+1}-1]}(E, k_l, k_{l\pm 1}) =
\]
\[
- G_{\omega,[k_{l-1}+1, k_{l+1}-1]}(E, a_l - 1, k_{l\pm 1}) G_{\omega,[a_l,b_l]}(E, k_l, a_l) \\
- G_{\omega,[k_{l-1}+1, k_{l+1}-1]}(E, b_l + 1, k_{l\pm 1}) G_{\omega,[a_l,b_l]}(E, k_l, b_l).
\]
Hence, we obtain that
\[
|G_{\omega,[k_{l-1}+1, k_{l+1}-1]}(E, k_l, k_{l\pm 1})| \leq 2 \cdot \left( \frac{1}{4} e^{\delta} \right) \cdot e^{-2\delta}.
\]
This finishes the proof. \( \square \)

Proof of Theorem 2.3. This follows from Theorem 1.1 since it is applicable by the previous lemmas. \( \square \)

5. PROOF OF THEOREM 1.2

Due to Corollary 4.2 and Remark 4.5, we are able to satisfy the conditions of Lemma 2.8. from Bourgain, Goldstein, and Schlag [6]. Hence, we might iterate it as described in Subsection 2.5. of [6]. This allows us to conclude Theorem 1.2.

It should be remarked here, that we are not able to improve Theorem 2.3, since the methods of [6] require the particular situation of the skew-shift with an analytic sampling function. Already the extension to higher skew-shifts as considered in Theorem 2.7. of [12] seems non trivial.

I believe that following the approach of [5] as explained in [6], one should also be able to conclude Anderson localization outside the set \( \mathcal{E}_b \). Unfortunately, we cannot ensure that there is spectrum there.
References


Department of Mathematics, Rice University, Houston, TX 77005, USA

E-mail address: helge.krueger@rice.edu

URL: http://math.rice.edu/~hk7/