# JACOBI FIELD COMPARISON

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## Abstract

Along a geodesic in a Riemannian manifold, the Jacobi fields encode how the pattern of near by geodesics differ from the corresponding pattern in Euclidean space. In particular, complete knowledge of a manifold's Jacobi fields completely determines its local geometry as well as much of its algebraic topology.

As Jacobi fields are the solutions of a $2^{nd}$ order linear ODE that involves curvature, it is not surprising that knowledge of curvature can be translated into knowledge of Jacobi fields. For manifolds with curvature bounded from below there is a result along these lines that is known as the Rauch Comparison Theorem. This classical theorem implies that in a manifold of nonnegative curvature, the Jacobi fields (and their derivatives) are smaller than the corresponding fields (and derivatives) in $\mathbb{R}^2$. The comparison only works for a field whose initial value or initial derivative is 0. This is a major drawback. In most cases, the inequality is strict. When it is strict at some time $t_1$, examples suggest that it typically is stricter at times $t > t_1$. Unfortunately, other examples show that this is not always the case.

Recently Guijarro and I remedied this situation by proving a comparison lemma. It shows that when the Rauch inequality is strict for some field at some time $t_1$, the expected "even stricter" estimate holds at future times for a field that is possibly different from the original one. The proof exploits a powerfull, but techncial tool—Wilking's Transverse Jacobi equation.

Our comparison lemma has several applications including the following optimal finiteness theorem.

**Theorem:** (Guijarro-W.) *Let $M$ be a compact Riemannian manifold. Given $D, r > 0$ the class $S$ of closed Riemannian manifolds that can be isometrically embedded into $M$ with focal radius $\geq r$ and intrinsic diameter $\leq D$ is precompact in the $C^{1,\alpha}$–topology. In particular, $S$ contains only finitely many diffeomorphism types.*

I will give a gentle survey of the metric aspects of Riemannian geometry, as they relate to Jacobi field comparison, with the goal of elucidating aspects of the statement and proof of this theorem.