The Hopf-Laplace Equation

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Abstract. The Hopf-Laplace equation arises in the study of the Dirichlet energy integral

$$\mathscr{E}[h] = \iint_{\mathbb{X}} |Dh|^2 = 2 \iint_{\mathbb{X}} \left(|h_z|^2 + |h_{\bar z}|^2 \right) \, \mathrm{d}z$$

for mappings $h \colon \mathbb{X} \to \mathbb{Y}$ between two designated domains $\mathbb{X}$ and $\mathbb{Y}$ in the complex plane. Here and throughout we take advantage of the complex partial derivatives, a notation indispensable for advancing this work. The inner variation of the energy leads to a second order nonlinear system of PDEs for a complex function h in the Sobolev space $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$,

$$(\star) \qquad \frac{\partial}{\partial \bar z} \left(h_z \overline{h_{\bar z}} \right) = 0, \quad \text{in the sense of distributions.}$$

This is what we call the *Hopf-Laplace* equation in its natural domain of definition. The classical solutions are the complex harmonic maps -the solutions of the linear Laplace system $h_{z\bar z} = 0$. It is true, though not obvious at all, that homeomorphic $\mathscr{W}^{1,2}$-solutions of $(\star)$ are indeed harmonic diffeomorphisms. Nevertheless, minimization of the Dirichlet energy among homeomorphisms often leads to noninjective extremal mappings, thus nonharmonic solutions of $(\star)$ (so-called squeezing phenomenon). We investigate the equation $(\star)$ for a certain class of topologically well behaved mappings which are almost homeomorphisms, called *Hopf deformations*. The associated Hopf quadratic differential $(h_z \overline{h_{\bar z}})\, \mathrm{d}z^2$ and its trajectories enter the stage. We have established Lipschitz continuity of Hopf deformations, the best possible regularity one can get, since in general Hopf deformations are not $\mathscr{C}^1$-smooth. Thus in particular, we show that the minimal-energy deformations are Lipschitz continuous, a result of considerable interest in the theory of minimal surfaces, calculus of variations and PDEs, with potential applications to elastic plates.

This talk is based on joint work with Jan Cristina, Leonid V. Kovalev and Jani Onninen.