SPECIAL CUBIC 4-FOLDS VS. APOLARITY

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1. Why cubic 4-folds

(1) Rationality: They are the simplest smooth hypersurfaces for which we don't understand rationality.
   - any smooth quadric with a rational point is rational
   - cubic surfaces are always rational
   - (quartic surface is always irrational for stupid reasons)
   - smooth cubic 3-folds are irrational (Clemens and Griffiths '72)
   - smooth quartic 3-folds are irrational (Isovskikh and Manin '71)
   - (quintic 3-fold irrational because it has kodaira dimension 0 (rather than $-\infty$))
   - very general smooth quartic 3-folds are not even stably rational (Colliot-Thélène and Pirutka '14)
   - very general smooth quartic 4-fold is irrational (Totaro '15)

(2) There are beautiful connections to K3 surfaces and hyperkähler varieties.
   - Deligne and Rapaport observed similarities in cohomology (H^2_{prim}(polarized K3) and H^4_{prim}(cubic 4-fold))

2. Associated K3 Surfaces

Hasset ('96) defined the following: A K3 surface, S, is associated to a cubic 4-fold, X, if there exists a primitive embedding of polarized Hodge structures

$$H^2_{\text{prim}}(S,\mathbb{Z}) \hookrightarrow H^4_{\text{prim}}(X,\mathbb{Z})(1).$$

- Cubics with an associated K3 surface of degree d form an irreducible divisor in moduli if $d = 14, 26, 38, 42, 62, 74, \ldots$. This contains all known rational cubic 4-folds. Many people believe that if a cubic 4-fold is rational, then it contains an associated K3 surface, and if you're optimistic, you might say "if and only if."
- You can upgrade this Hodge theoretic story to the derived category of coherent sheaves (Kuznetsov, '08). You'll say that a cubic 4-fold has an associated K3 if the derived category of the K3 surface appears in the derived category of the cubic 4-fold (see Addington-Thomas)
- In all known examples where X is rational, a K3 shows up geometrically (see Beauville-Donagi, Addington-Hassett-Tschinkel-Várilly-Alvarado)
- Let $F := \{\text{lines on } X\}$ (hyperkahler 4-fold). In 2014, the author showed that a cubic 4-fold has an associated K3 (in the Hodge theoretic sense) if and only if F is birational to a moduli space of sheaves on a K3.

- (Galkin and Shinder '14) If $\mathbb{A}^1$ is not a zero divisor in $K_0(\text{var})$, then X is rational implies that F is birational to $\text{Hilb}^2(\text{K3})$. Of course, now we know that this is not the case.

2.1. **Degree 14 (Beauville and Donagi '85).** Let $\omega_1, ..., \omega_6$ (generically chosen) be 2-forms on $\mathbb{C}^6$. Consider the K3

$$\{W \in \text{Gr}(2,6) : \omega_i|_W = 0 \forall i\},$$

and the associated cubic is

$$\{(a_1, ..., a_6) \in \mathbb{P}^5 : \Sigma(a_i\omega_i)^3 = 0\}.$$

Note that for generic ω_i, the K3 and the cubic above are smooth. There exists a correspondence on K3$\times X$ inducing the embeddings

$$H^2_{\text{prim}}(\text{K3}) \hookrightarrow H^4_{\text{prim}}(X)(1),$$
$$\mathcal{D}^b(\text{K3}) \hookrightarrow \mathcal{D}^b(X).$$

We also have a perfectly explicit description of the map

$$X \dashrightarrow \mathbb{P}^4$$

constructed using the kernels of the forms $\Sigma a_i\omega_i$.

2.2. **Degree 26.** There is an explicit description of these cubics due to Nuer, in the sense that the generic one contains a $\mathbb{P}^2$ blown up at 12 points embedded in some way. But there is no explicit description of the associated K3 in terms of equations, although there is a description due to Farkas and Verra as a component of the Hilbert scheme of certain scrolls on the cubic.

2.3. **Degree 38 (Mukai '89).** First description: Let $\omega_1, \omega_2, \omega_3$ be two forms on $\mathbb{C}^9$. There is a K3 defined by

$$\{W \in \text{Gr}(4,9) : \omega_i|_W = 0 \ \forall i\}.$$

Second description: Fix a plane sextic $g(x_0, x_1, x_2)$ of degree 6. Note that generic g can be written as $g = \ell_1^6 + \cdots + \ell_{10}^6$, and the variety of sums of powers is the set of all ways to do so. There is not a unique way to do this, there's a two parameter family of ways to do this, and it gives you a K3. In particular, we have a K3 defined by

$$\{(\ell_1, ..., \ell_{10}) \in \text{Hilb}^{10}(\mathbb{P}^2)^* : \text{ for some } a_i \in \mathbb{C},\ \Sigma a_i \ell_i^6 = g\}.$$

How do we obtain a cubic 4-fold from these two descriptions of degree 38 K3 surfaces?

First idea: We have a multiplication map

$$m\colon \text{Sym}^3 \underbrace{\text{Sym}^2 \mathbb{C}^3}_{\mathbb{C}^6} \to \text{Sym}^6 \mathbb{C}^3,$$

and the transpose

$$m^*\colon \text{Sym}^6 (\mathbb{C}^3)^* \to \text{Sym}^3 (\mathbb{C}^6)^*$$
$$g \mapsto m^* g =: f.$$

For generic g, the hypersurface $X \subset \mathbb{P}^5$ cut out by $f = m^*g$ is smooth, and we get an irreducible divisor, call it D_{V-ap}, in the moduli of cubic 4-folds.

If there's any justice, this K3 is the one associated (Hodge theoretically) to our cubic 4-fold. But there is no justice.

Cubics with associated K3 surfaces are irreducible components of the Noether–Lefschetz locus defined by

$$\{X \subset \mathbb{P}^5 : H^{2,2}_{\text{prim}}(X,\mathbb{Z}) := H^4_{\text{prim}}(X,\mathbb{Z}) \cap H^{2,2}(X) \neq 0\}.$$

Theorem 1. *(Ranestad and Voisin '13) D_{V-ap} is not a Noether–Lefschetz divisor.*

How could we prove this directly? Following van Luijk and Elsenhans-Jahnel, we use the following strategy: Write down some g with rational coefficients such that $f = m^*g$ has good reduction mod p for some p. Now we have that

$$\mathrm{rk}(H^{2,2}_{\text{prim}}(X,\mathbb{Z})) \leq \# \text{ of eigenvalues of Frob acting on } H^4_{\text{prim}}(X_{\overline{\mathbb{F}}_p},\mathbb{Q}_\ell) \text{ of the form } p^2\zeta,$$

where ζ is a root of unity (see Addington-Auel for references and details). Note that for this to work we use the fact that we know the Hodge conjecture for cubic 4-folds.

Let $\lambda_1, ..., \lambda_{22}$ be the eigenvalues. The Lefschetz fixed point theorem states

$$\#X(\mathbb{F}_{p^m}) = 1 + p^m + p^{2m} + p^{3m} + p^{4m} + \sum \lambda_i^m,$$

and thus, we count points up to $m = 22$. Since the eigenvalues come in conjugate pairs, with some care, it is usually sufficient to count up to $m = 11$.