TALK ON KODAIRA-SAITO VANISHING

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Rice, April, 2017

The following notation will be fixed from the beginning. k is a field, which is initially $\mathbb{C}$. X is a smooth projective variety over k with $d = \dim X$. Let L be an ample line bundle over k. $D \subset X$ is a divisor with simple normal crossings. $U = X - D$.

1. Background

We assume $k = \mathbb{C}$ throughout this section. The ancestor of all vanishing theorems, proved in the 1950's, is:

Theorem 1 (Kodaira-Akizuki-Nakano)**.**

$$H^i(X, \Omega_X^j \otimes L) = 0$$

when $i + j > d$.

This has many applications. For example, it implies the Lefschetz hyperplane theorem.

The original proofs were analytic in nature using harmonic forms. It was open problem to find purely algebraic proofs. Such proofs were eventually found by Faltings and then Deligne and Illusie in the mid 1980's. The second proof uses characteristic p methods, even though Kodaira can fail over such fields. They show that it works if $p > d$ and X lifts mod p^2 (i.e. X is the closed fibre of a scheme flat over $\operatorname{Spec} W(k)/(p^2)$, where $W(k)$ is the ring of Witt vectors).

M. Saito found a far reaching generalization of the Kodaira vanishing theorem in 1990 using his theory of mixed Hodge modules. Even the statement is rather complicated. So we will be content to explain a special case.

Suppose that $f : Y \to U$ is a family of smooth projective varieties. Associated to this is the following data:

(1) A local system $\mathcal{V} = R^i f_* \mathbb{C} =$ "$\bigcup_x H^i(Y_x, \mathbb{C})$". Recall that "local system" means that it comes with a monodromy representation $\pi_1(U, x) \to GL(H^i(Y_x))$. The Riemann-Hilbert correspondence says that local systems (of $\mathbb{C}$-vector spaces) are equivalent to flat vector bundles by which we mean vector bundles with integrable connections. In the case at hand, the vector bundle is $V' = \mathcal{V} \otimes \mathcal{O}_U$, and the connection is called the Gauss-Manin connection ∇.
(2) Hodge decompositions $H^i(Y_x, \mathbb{C}) = \bigoplus_{p+q=i} H^{pq}(Y_x)$. It is equivalent, and more convenient, to work with the Hodge filtration $F_x^p = \bigoplus_{p' \geq p} H^{p'q}(Y_x)$. These fit together as subbundles $F^p \subset V'$.
(3) Griffiths transversality, which says $\nabla(F^p) \subset \Omega_U^1 \otimes F^{p-1}$, holds.

(4) There exists appropriate bilinear forms on the fibres called polarizations. (This is technically important, but we won't say anything more it.)

Such a collection of data is called a *variation of Hodge structure* or simply a VHS. There are other examples which don't arise from geometry. A mixed Hodge module – which we won't define – is a generalization of a VHS allowing singularities. Let us fix a VHS $(\mathcal{V}, ...)$ on U, with the additional condition that the local monodromies about components of D are unipotent. In the geometric examples, this is a pretty mild assumption; it can always be achieved after pulling back to a branched cover. Under this assumption, we have a Deligne canonical extension V of V' to X, so that ∇ extends to log connection

$$\nabla : V \to \Omega^1_X(\log D) \otimes V$$

with nilpotent residues. Set

$$E = Gr_F(V) = \bigoplus F^p/F^{p+1}$$

This carries a twisted endomorphism

$$\theta = Gr_F(\nabla) : E \to \Omega^1_X(\log D) \otimes E$$

satisfying $\theta^2 := \theta \wedge \theta = 0$. A pair (E, θ) consisting of a vector bundle and a map θ satisfying $\theta^2 = 0$ is called a Higgs bundle[1]. We can form a "de Rham" complex

$$\mathrm{DR}(E, \theta) = E \xrightarrow{\theta} \Omega^1_X(\log D) \otimes E \xrightarrow{\theta} \ldots$$

Theorem 2 (Saito)**.**

$$H^i(X, \mathrm{DR}(E, \theta) \otimes L) = 0$$

for $i > d$*.*

When $(\mathcal{V}, V, \ldots) = (\mathbb{C}, \mathcal{O}_X, \ldots)$ is the trivial VHS, this recovers the Kodaira-Akizuki-Nakano theorem. It also implies Kollár's vanishing theorem. A number of applications of the full strength version of this theorem have been found recently.

2. Main theorem

If (E, θ) is a Higgs bundle arising from a VHS with unipotent local monodromies, then it satisfies the following properties:

(1) The rational Chern classes of E are zero.
(2) (E, θ) is semistable in the sense that for any θ-invariant subsheaf $E' \subset E$,

$$\deg E' \leq 0$$

(3) θ is nilpotent, when viewed locally as a matrix of 1-forms.

Saito's theorem (as stated above) follows from a more general result.

Theorem 3 (A)**.** *Suppose that* (E, θ) *is a Higgs bundle satisfying (1), (2), and (3) above, and*

(a) $\mathrm{char}\, k = 0$ *or*
(b) $\mathrm{char}\, k = p \gg 0$ *and* (X, D) *is liftable mod* p^2*.*

Then

$$H^i(X, \mathrm{DR}(E, \theta) \otimes L) = 0$$

for $i > d$*.*

[1] Yes, *that* Higgs, but the connection to physics may a bit tenuous at this point

3. Background on Higgs bundles

Let us start with $k = \mathbb{C}$. We recall Simpson's results for motivation, although they won't be used.

Theorem 4 (Simpson). *Suppose $D = 0$.*

(1) *There exists a one to one correspondence between semistable Higgs bundles with vanishing Chern classes and semisimple flat bundles.*
(2) *If (E, θ) and (V, ∇) correspond,*

$$H^i(DR(E, \theta)) \cong H^i(DR(V, \nabla))$$

The proof of this is highly transcendental in nature. Nevertheless Ogus and Vologodsky found a nice analogue in positive characteristic. This was extended by Ogus's student Schepler to the log setting.

Theorem 5 (Ogus-Vologodsky, Schepler). *Suppose that* $\operatorname{char} p > d$, *and* (X, D) *lifts mod p^2.*

(1) *There is an equivalence the categories of*

$$\{\textit{certain Higgs bundles}\} \xrightarrow{C^{-1}} \{\textit{certain flat bundles}\}$$

where "certain" refers to appropriate nilpotence conditions on both sides.
(2) *If (E, θ) and (V, ∇) correspond, there is an isomorphism*

$$\mathrm{Fr}_* \mathrm{DR}(V, \nabla) \cong \mathrm{DR}(E, \theta)$$

in the derived category, where $\mathrm{Fr} : X \to X$ *is the Frobenius.*

In addition, we need some work of Langer.

Theorem 6 (Langer). (1) *Given a semistable flat bundle (V, ∇), there is a canonical filtration satisfying Griffiths transversality. Set*

$$\Lambda(V, \nabla) = (Gr_{F_{can}}(V), Gr_{F_{can}}(\nabla))$$
$$B(E, \theta) = \Lambda C^{-1}(E, \theta)$$

(2) *When $k = \overline{\mathbb{F}}_p$, if (E, θ) is semistable, the sequence*

$$(E, \theta), B(E, \theta), B^2(E, \theta), \ldots$$

is eventually periodic.

4. Proof of the main theorem

Standard arguments will show that case (a) follows case (b). And in fact, we can reduce to the case where $k = \overline{\mathbb{F}}_p$. The following is not difficult.

Lemma 4.1. $\dim H^i(\mathrm{DR}(E, \theta) \otimes L) \leq \dim H^i(\mathrm{DR}(B(E, \theta)) \otimes L^p)$

We are now ready to prove the main theorem. By Langer, the sequence

$$(E, \theta), B(E, \theta), B^2(E, \theta), \ldots$$

is eventually periodic. Say, for simplicity it's periodic of period n, then

$$\dim H^i(\mathrm{DR}(E, \theta) \otimes L) \leq \dim H^i(\mathrm{DR}((E, \theta)) \otimes L^{p^n})$$

By iteration, we can assume that the exponent on the right $\to \infty$. Now apply Serre vanishing to see that the cohomology must eventually be zero when $i > d$.

For more details, refinements and precise references, see my preprint Kodaira-Saito vanishing via Higgs bundles in positive characteristic on the ArXiv.