MOTIVES OF DELIGNE-LUSZTIG VARIETIES

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Abstract. Deligne-Lusztig varieties are varieties over finite fields acted on by finite groups of Lie type. We will discuss their motives, and in particular their endomorphisms and their rationality properties.

1. Cohomology of Varieties over $\mathbf{F}_q$

Let X be a variety over $\mathbf{F}_q$. Then there is an action of the q^{th} power map (the Frobenius) F, on the variety $\overline{X} := X \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$. Fixing some $\ell \neq p$, F induces an action on $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell)$, which we also call F.

When X is smooth and projective, the Weil Conjectures say that if α is an eigenvalue of F on $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell)$, then for any $\psi\colon \mathbf{Q}(\alpha) \hookrightarrow \mathbf{C}$ one has $|\psi(\alpha)| = q^{i/2}$. We say that α is a q^i *Weil number.*

Conjecture. *F acts semisimply on $H^i_{\acute{e}t}(\overline{X}, \mathbf{Q}_\ell)$.*

Now we can pose Hodge conjecture-esque questions about ℓ-adic cohomology. Which elements of $H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell)$ come from cycles?

There is a cycle map over $\mathbf{F}_q$:

$$c_i\colon A^i(X) \to H^{2i}_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell)$$

and it is stable under the action of F.

Conjecture. *(Tate and Beilinson) c_i is an isomorphism.*

It would then follow of course that rational, numerical, and homological equivalence are the same.

This conjecture is known to be true for (N.B. often in the literature, the T+B conjecture refers to just surjectivity or just $i = 1$):

- Curves
- Products of two curves (just surjectivity)
- K3 surfaces (Nygaard-Ogus, Maulik, Madapusi-Pera), at least for p odd
- Some other surfaces
- Products of elliptic curves
- Certain Fermat hypersurfaces $x_0^m + \cdots + x_{n-1}^m \subseteq \mathbf{P}^{n-1}$ for $p \nmid m$:
  - for $n = 3$ (idea: relate to product of two curves)
  - for $m \mid p^v + 1$ for some $v > 0$ (Tate)

The last class of examples mentioned above when $m \mid p^v + 1$ are Deligne-Lusztig varieties for $\mathrm{U}_n(\mathbf{F}_q)$.

When X is an aribitrary variety over $\mathbf{F}_q$, i.e. not a smooth projective variety, we can consider the étale cohomology with compact support $H^*_{c,\text{ét}}(\overline{X}, \mathbf{Q}_\ell)$. The following conjecture is of course already known when X is smooth and projective:

Conjecture.

- *$\dim H^i_{c,\acute{e}t}(\overline{X}, \mathbf{Q}_\ell)$ is independent of ℓ.*
- *The set of F-eigenvalues is independent of ℓ, which follows from*
- *The characteristic polynomial of F is independent of ℓ.*

2. Deligne-Lusztig varieties

Deligne-Lusztig varieties are finite field analogues of Shimura varieties and Drinfeld shtukas. Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbf{F}_q$. Let F be some automorphism of $\mathbf{G}(\overline{\mathbf{F}}_q)$ (for example, F could be the Frobenius), and let $G := G(\overline{\mathbf{F}}_q)^F$. Some examples:
Example: $\mathbf{G} = \mathrm{GL}_n$, $F((a_{ij})) = (a_{ij}^q)$, $G = \mathrm{GL}_n(\mathbf{F}_q)$.
Example: $\mathbf{G} = GL_n$, $F((a_{ij})) = ((a_{ij}^q)^{-1})$, $G = \mathrm{U}_n(\mathbf{F}_q)$.
Let $\mathbf{P}, \mathbf{P}'$ be parabolic subgroups of $\mathbf{G}$. The diagonal action of $\mathbf{G}$ on the projective variety $\mathbf{G}/\mathbf{P} \times \mathbf{G}/\mathbf{P}'$ has finitely many orbits. Recall that every parabolic subgroup has a Levi decomposition $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$, where $\mathbf{U}$ is the unipotent radical and $\mathbf{L} \subseteq \mathbf{P}$ is a Levi subgroup. Fix a Levi decomposition of $\mathbf{P}$, and assume that $F(\mathbf{L}) = \mathbf{L}$ and $F(\mathbf{P}) = \mathbf{P}'$. Then the *Deligne-Lusztig variety* X_Ω associated to a G orbit Ω on $\mathbf{G}/\mathbf{P} \times \mathbf{G}/\mathbf{P}'$ is given by

$$X_\Omega = \Omega \cap \{(\ell, F(\ell)) \mid \ell \in \mathbf{G}/\mathbf{P}\} \hookrightarrow \mathbf{G}/\mathbf{P}$$

where the immersion above is given by the first projection.
Example: $\mathbf{G} = \mathrm{GL}_n$, with F as in the second example above. Let $\mathbf{P}$ be the parabolic subgroup of consisting block upper triangular matrices, a 1×1 followed by $(n-1) \times (n-1)$, i.e.

$$\mathbf{P} = \begin{pmatrix} GL_1 & * & \cdots & * & * \\ 0 & & & & \\ \vdots & & GL_{n-1} & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}.$$

Here, the Levi decomposition is $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$ and the unipotent radical is

$$\mathbf{U} = \begin{pmatrix} 1 & * & \cdots & * & * \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and there is an isomorphism with projective space $\mathbf{G}/\mathbf{P} \xrightarrow{\sim} \mathbf{P}^{n-1}$. Let Ω be an open orbit of $\mathbf{G}/\mathbf{P} \times \mathbf{G}/F(\mathbf{P})$. Then $X_\Omega \simeq \mathbf{P}^{n-1} \setminus Z$, where $Z = \{x_0^{q+1} + \cdots + x_{n-1}^{q+1} = 0\} \subseteq \mathbf{P}^{n-1}$. X_Ω has a $\mathrm{U}_n(\mathbf{F}_q) = G$-action.
A *unipotent representation* of G is an irreducible representation that occurs in $H^i_{c,\text{ét}}(\overline{X}_\Omega, \mathbf{Q}_\ell)$ for some $\mathbf{P}, \Omega, i$.

Theorem 1. *$H^i_{c,\acute{e}t}(\overline{X}_\Omega, \mathbf{Q}_\ell)$ is independent of ℓ.*

"Many" X_Ω satisfy the Tate-Beilinson conjecture (all X_Ω for $G = GL_n(\mathbf{F}_q)$, and Fermat hypersurfaces with $m = q+1$).

3. Motives over $\mathbf{F}_q$

The category of *Chow Motives*, denoted CM, is built as follows. Start with smooth projective varieties as the objects, with morphisms $X \to Y$ viewed as elements of the Chow ring $\mathrm{Chow}(X \times Y)$. Then add the images of the idempotent elements of the Chow ring. Finally, introduce a new element $\mathbf{L}$, defined by

$$\mathrm{Motive}(\mathbf{P}^1) := \mathrm{Motive}(\mathrm{pt}) \oplus \mathbf{L}$$

called the Lefschetz motive.

CM is not the nicest object in a lot of ways. It is not a nice abelian category. Thinking Hodge-theoretically, we want more than just smooth varieties in our categories (to do this, we want to use mixed structures). A better category is $\mathrm{DM}(\mathbf{F}_q)$, which is the triangulated category of motives of all varieties over $\mathbf{F}_q$, which is something like a derived version of CM.

There is an ℓ-adic cohomology functor $H^{\text{ét}}$

$$\mathrm{DM}(\mathbf{F}_q) \xrightarrow{H^{\text{ét}}} \mathbf{Q}_\ell\text{-graded vector spaces with } F\text{-action.}$$

The Tate-Beilinson conjecture is equivalent to $H^{\text{ét}}$ being fully faithful. Also, the semi-simplicity of F is equivalent to the semi-simplicity of the image of $H^{\text{ét}}$. Another way to say all this is that $\mathrm{DM}(\mathbf{F}_q)$ is semi-simple.

Let

$$\mathcal{C} = \{\text{simple objects of } \mathrm{DM}(\mathbf{F}_q)\}/\sim, \text{shift}$$
$$\mathcal{D} = \{q^n\text{-Weil numbers, } n \in \mathbf{Z}\}/\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}).$$

Then $C \simeq D$. Let $M \in \mathcal{C}$. Then $\mathrm{End}(M)$ is a central simple algebra over $\mathbf{Q}[F]$ which has the following properties:

- trivial at places above $\ell \neq p$.
- invariant $\frac{1}{2}$ at real places, if weight is odd
- there is some formula for the invariants at places above p.

The above leads one to conclude that $\mathrm{DM}(\mathbf{F}_q)$ is generated by abelian varieties, Artin motives, and $\mathrm{Spec}(\mathbf{F}_{q^n}/\mathbf{F}_q)$.

Theorem 2. $\mathrm{Motive}(X_\Omega \times X_{\Omega'})$ *is a direct sum of (shifted) Tate motives.*

Now if V is a unipotent representation of G, then one gets associated motive M_V, indemcomposable, of weight 1 or 0. If V is cuspidal, M_V depends only on the Harish-Chandra series. We have

$$\mathrm{End}_{\mathbf{Q}G}(V) \simeq \mathrm{End}(M_V).$$