HODGE THEORY OF CLASSIFYING STACKS

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Abstract. The goal of this talk is to create a correspondence between the representation theory of algebraic groups and the topology of Lie groups. The idea is to study the Hodge theory of the classifying stack of a reductive group over a field of characteristic p, the case of characteristic 0 being well known. The approach yields new calculations in representation theory, motivated by topology.

Let X be a smooth variety over a field k. Let Ω^1 be the cotangent bundle. Then $\Omega^i = \wedge^i(\Omega^1)$ is a vector bundle ($\implies$ coherent sheaf). We have

$$\mathrm{CH}^i X \to H^{2i}_{dR}(X/k) = H^{2i}(X, 0 \to \Omega^0_X \to \Omega^1_X \to \cdots)$$

where $\Omega^0_X = \mathcal{O}_X$, and

$$\mathrm{CH}^i X \to H^i(X, \Omega^i).$$

If $k = \mathbb{C}$ and X is smooth proper over $\mathbb{C}$, then Hodge theory says

$$H^j_{sing}(X, \mathbb{C}) \simeq H^j_H(X/\mathbb{C}) = \oplus_l H^l(X, \Omega^{j-l}).$$

Example 1. *For a field k,*

$$H^0_H(\mathbb{A}^1/k) = \mathcal{O}(\mathbb{A}^1_k) = k[x],$$

which is ∞-dimensional as a k-vector space. So Hodge cohomology is not $\mathbb{A}^1$ homotopy invariant.

Example 2.

$$\begin{aligned} H^0_{dR}(\mathbb{A}^1/k) &= \ker(d : \mathcal{O}(\mathbb{A}^1) \to \Omega^1(\mathbb{A}^1)) \\ &= \begin{cases} k & \text{if char}\, k = 0, \\ k[x^p] & \text{if char}\, k = p > 0. \end{cases} \end{aligned}$$

So H_{dR} is not $\mathbb{A}^1$-homotopy invariant in characteristic p.

Question 3. *What should $H^*_{G,Hodge}(X/k)$ be for a variety X with an action of an algebraic group G?*

We take a detour to look at classifying spaces in topology. Let G be a topological group.

Definition 4. *The classifying space of G is $BG := EG/G$ where EG is any free contractible G-space.*

The homotopy type of BG is well defined. Hence we get invariants of G such as $H^*(BG, \mathbb{Z})$ (If G is a discrete group then $BG = K(G, 1)$). Likewise, we define equivariant cohomology for a G-space X:

$$H^*_G(X) := H^*((X \times EG)/G).$$

Example 5.
1. $B\mathbb{Z}/2 = \mathbb{RP}^\infty$.
2. $BS^1 = \mathbb{CP}^\infty$.
3. $BU(n) = Gr(\mathbb{C}^n \subset \mathbb{C}^\infty) \implies H^*(BU(n), \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \dots, c_n]$ *where c_i are Chern classes which are invariants of $\mathbb{C}$-vector bundles.*

Remark 6. *Every compact Lie group G has a complexification $G_{\mathbb{C}}$ that is a $\mathbb{C}$-reductive group and $BG \to BG_{\mathbb{C}}$ is a homotopy equivalence, e.g.,*

$$(B\mathbb{G}_m)_{\mathbb{C}} \simeq BS^1,$$

$$BGL(n, \mathbb{C}) \simeq BU(n).$$

A reasonable definition for Hodge cohomology of BG (G an algebraic group over a number field k) was suggested by Simpson and Teleman. The idea was to use the stack BG. For a scheme X/k a morphism $X \to BG$ is a principal G-bundle over X:

Definition 7 (in topology)**.** *If G is a topological group and X a topological space, then a principal G-bundle E over X is a topological space with free G-action together with a homeomorphism $E/G \xrightarrow{\sim} X$.*

A quasi-coherent sheaf on the stack BG is a representatiion of G (as a group scheme over k). Let $[X/G]$ be the quotient stack for action of G on a scheme X. Quasi-coherent sheaves on $[X/G]$ are G-equivariant quasi-coherent sheaves on X.

Definition 8. *The big étale site of a stack Y is the category of schemes X over Y with morphisms*

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

It is a covering if f is étale and surjective.

Definition 9. *The sheaf Ω^j/k on the big étale site of a stack Y is the sheaf*

$$\left(\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd} \right) \longmapsto \Omega^j(X/k)$$

This is not a quasi-coherent sheaf on this site because for a morphism $X_1 \to X_2$ over Y, the natural map $f^*\Omega^j_{X_2} \to \Omega^j_{X_1}$ is not an isomorphism of sheaves in general. So we define Hodge cohomology of a stack Y as $H^i(Y, \Omega^j)$, the cohomology of the big sheaf Ω^j.

Remark 10.
1. *If Y is an algebraic space, then this is equivalent to the usual cohomology of Y (using the small étale site)*
2. *If Y is a scheme, then $H^i_{Zar}(X, \Omega^j) \simeq H^i_{et}(X, \Omega^j)$.*

General fact: Any algebraic stack Y has a smooth surjective morphism $U \to Y$ with U a scheme (e.g. G a smooth algebraic group, $X \to [X/G]$). Let E be a big sheaf then $H^*(Y, E)$ is computed by the Čech simplicial space associated to $U \to Y$:

$$\begin{tikzcd} U & U \times_Y U \arrow[l, shift left] \arrow[l, shift right] & U \times_Y U \times_Y U \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

Example 11. *Let $Y = BG$ and $U = \operatorname{Spec} k$. The Čech simplicial scheme BG is*

$$\begin{tikzcd} \operatorname{Spec} k & G \arrow[l, shift left] \arrow[l, shift right] & G^2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

So we get a spectral sequence

$$E^{ij} = H^j(U_Y^{i+1}, E) \implies H^{i+j}(Y, E).$$

Corollary 12. *Let G be a smooth affine group scheme over k. Then $H^*(BG, \Omega^\ell)$ is the cohomology of the complex of k-vector spaces*

$$0 \to \Omega^\ell_k(\operatorname{Spec} k) \to \Omega^\ell(G) \to \Omega^\ell(G^2) \to \cdots$$

Theorem 13 (Bott)**.** *If G is reductive,* $\operatorname{char} k = 0$*, then*

$$H^i(BG, \Omega^j) = \begin{cases} 0 & \text{if } i \neq j, \\ S^j(\mathfrak{g}^*)^G & \text{if } i = j. \end{cases}$$

where $\mathfrak{g}$ is the Lie algebra of G.

So $H^*_H(BG/k) \simeq \mathcal{O}(\mathfrak{g})^G$. This is not surprising: Chern-Weil theory says for G reductive over $\mathbb{C}$, we have $H^*(BG_{\mathbb{C}}, \mathbb{C}) \simeq \mathcal{O}(\mathfrak{g})^G$.

Theorem 14 (Totaro)**.** *If G is a smooth affine group over a field k, then*

$$H^i(BG, \Omega^i) \simeq H^{i-j}(G, S^j(\mathfrak{g}^*)).$$

Here, for a G-module M, $H^i(G, M) := \operatorname{Ext}^i_G(k, M)$.

This implies Bott's theorem: If G is reductive and $\operatorname{char} k = 0$, then a G-module is a semisimple category. So $H^{>0}(G, M) = 0$.

Example 15. *Let V be the standard representation of $GL(n)$ in characteristic 2. Then*

$$0 \to V^{(1)} \to S^2 V \to \wedge^2 V \to 0$$

is not split as representation of $GL(n)$.

Puzzle: How is

$$\begin{array}{ccc} H^*_{dR}(BG/\mathbb{F}_p) & \text{related to} & H^*(BG_{\mathbb{C}}, \mathbb{F}_p)? \\ G \text{ reductive } /\mathbb{F}_p & \longleftrightarrow & G \text{ reductive } /\mathbb{C} \end{array}$$

A guiding result from p-adic Hodge theory:

Theorem 16 (Bhatt, Morrow, Schultz)**.** *If X is a smooth projective scheme over $\mathbb{Z}_p$ then*

$$\dim_{\mathbb{F}_p} H^i_{dR}(X/\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H^i_{et}(X_{\overline{\mathbb{Q}_p}}, \mathbb{F}_p).$$

Although BG is not proper, it often acts as if it were proper.

Calculation: G reductive over k. To study BG, we use the fibration

$$G/B \to BB \to BG.$$

We have $\mathrm{CH}^*(G/B) \simeq H_H^*(G/B)$ and $H_H^*(BB) = H^*((\mathbb{CP}^\infty)^n) = k[x_1, \ldots, x_n]$ where $n = \operatorname{rk} G$.

Example 17. *Let $G = SO(n)$ over a field k of characteristic 2. Then*

$$H_H^*(BSO(n)/\mathbb{F}_2) = \mathbb{F}_2[w_2, w_3, \ldots, w_n], \qquad \begin{aligned} &w_i \in H^i \\ &w_{2a} \in H^a(BSO(n), \Omega^a) \\ &w_{2a+1} \in H^{a+1}(BSO(n), \Omega^a). \end{aligned}$$

Example 18. *One has*

$$\dim_{\mathbb{F}_2} H_{dR}^{32}(BSpin(11)/\mathbb{F}_2) > \dim_{\mathbb{F}_2} H_{et}^{32}(BSpin(11)_{\mathbb{C}}, \mathbb{F}_p).$$