HODGE THEORY OF CLASSIFYING STACKS

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ABSTRACT. The goal of this talk is to create a correspondence between the representation theory of algebraic groups and the topology of Lie groups. The idea is to study the Hodge theory of the classifying stack of a reductive group over a field of characteristic p, the case of characteristic 0 being well known. The approach yields new calculations in representation theory, motivated by topology.

Let X be a smooth variety over a field k. Let Ω^1 be the cotangent bundle. Then $\Omega^i = \wedge^i(\Omega^1)$ is a vector bundle (\implies coherent sheaf). We have

$$\operatorname{CH}^{i} X \to H^{2i}_{dR}(X/k) = H^{2i}(X, 0 \to \Omega^{0}_{X} \to \Omega^{1}_{X} \to \cdots)$$

where $\Omega_X^0 = \mathcal{O}_X$, and

 $\operatorname{CH}^i X \to H^i(X, \Omega^i).$

If $k = \mathbb{C}$ and X is smooth proper over \mathbb{C} , then Hodge theory says

$$H^{j}_{sing}(X,\mathbb{C}) \simeq H^{j}_{H}(X/\mathbb{C}) = \bigoplus_{l} H^{l}(X,\Omega^{j-l}).$$

Example 1. For a field k,

$$H^0_H(\mathbb{A}^1/k) = \mathcal{O}(\mathbb{A}^1_k) = k[x],$$

which is ∞ -dimensional as a k-vector space. So Hodge cohomology is not \mathbb{A}^1 homotopy invariant.

Example 2.

$$H^0_{dR}(\mathbb{A}^1/k) = \ker(d: \mathcal{O}(\mathbb{A}^1) \to \Omega^1(\mathbb{A}^1))$$
$$= \begin{cases} k & \text{if char } k = 0, \\ k[x^p] & \text{if char } k = p > 0. \end{cases}$$

So H_{dR} is not \mathbb{A}^1 -homotopy invariant in characteristic p.

Question 3. What should $H^*_{G,Hodge}(X/k)$ be for a variety X with an action of an algebraic group G?

We take a detour to look at classifying spaces in topology. Let G be a topological group.

Definition 4. The classifying space of G is BG := EG/G where EG is any free contractible G-space.

The homotopy type of BG is well defined. Hence we get invariants of G such as $H^*(BG, \mathbb{Z})$ (If G is a discrete group then BG = K(G, 1)). Likewise, we define equivariant cohomology for a G-space X:

$$H^*_G(X) := H^*((X \times EG)/G).$$

Example 5.

- (1) $B\mathbb{Z}/2 = \mathbb{RP}^{\infty}$.
- (2) $BS^1 = \mathbb{CP}^\infty$.
- (3) $BU(n) = Gr(\mathbb{C}^n \subset \mathbb{C}^\infty) \implies H^*(BU(n),\mathbb{Z}) = \mathbb{Z}[c_1, c_2, \dots, c_n]$ where c_i are Chern classes which are invariants of \mathbb{C} -vector bundles.

Remark 6. Every compact Lie group G has a complexification $G_{\mathbb{C}}$ that is a \mathbb{C} -reductive group and $BG \to BG_{\mathbb{C}}$ is a homotopy equivalence, e.g.,

$$(B\mathbb{G}_m)_{\mathbb{C}} \simeq BS^1,$$

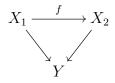
 $BGL(n,\mathbb{C}) \simeq BU(n).$

A reasonable definition for Hodge cohomology of BG (G an algebraic group over a number field k) was suggested by Simpson and Teleman. The idea was to use the stack BG. For a scheme X/k a morphism $X \to BG$ is a principal G-bundle over X:

Definition 7 (in topology). If G is a topological group and X a topological space, then a principal G-bundle E over X is a topological space with free G-action together with a homeomorphism $E/G \xrightarrow{\sim} X$.

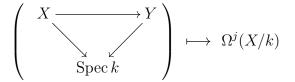
A quasi-coherent sheaf on the stack BG is a representation of G (as a group scheme over k). Let [X/G] be the quotient stack for action of G on a scheme X. Quasi-coherent sheaves on [X/G] are G-equivariant quasi-coherent sheaves on X.

Definition 8. The big étale site of a stack Y is the category of schemes X over Y with morphisms



It is a covering if f is étale and surjective.

Definition 9. The sheaf Ω^{j}/k on the big étale site of a stack Y is the sheaf



This is not a quasi-coherent sheaf on this site because for a morphism $X_1 \to X_2$ over Y, the natural map $f^*\Omega^j_{X_2} \to \Omega^j_{X_1}$ is not an isomorphism of sheaves in general. So we define Hodge cohomology of a stack Y as $H^i(Y, \Omega^j)$, the cohomology of the big sheaf Ω^j .

Remark 10.

- (1) If Y is an algebraic space, then this is equivalent to the usual cohomology of Y (using the small étale site)
- (2) If Y is a scheme, then $H^i_{Zar}(X, \Omega^j) \simeq H^i_{et}(X, \Omega^j)$.

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General fact: Any algebraic stack Y has a smooth surjective morphism $U \to Y$ with U a scheme (e.g. G a smooth algebraic group, $X \to [X/G]$). Let E be a big sheaf then $H^*(Y, E)$ is computed by the Čech simplicial space associated to $U \to Y$:

$$U \underbrace{\bigcup}_{Y} U \underbrace{\bigvee}_{Y} U \underbrace{\bigvee}_{Y} U \times_{Y} U \cdots$$

Example 11. Let Y = BG and $U = \operatorname{Spec} k$. The Čech simplicial scheme BG is

$$\operatorname{Spec} k \biguplus G \biguplus G^2 \cdots$$

So we get a spectral sequence

$$E^{ij} = H^j(U_Y^{i+1}, E) \implies H^{i+j}(Y, E).$$

Corollary 12. Let G be a smooth affine group scheme over k. Then $H^*(BG, \Omega^{\ell})$ is the cohomology of the complex of k-vector spaces

$$0 \to \Omega^{\ell}_k(\operatorname{Spec} k) \to \Omega^{\ell}(G) \to \Omega^{\ell}(G^2) \to \cdots$$

Theorem 13 (Bott). If G is reductive, char k = 0, then

$$H^{i}(BG, \Omega^{j}) = \begin{cases} 0 & \text{if } i \neq j, \\ S^{j}(\mathfrak{g}^{*})^{G} & \text{if } i = j. \end{cases}$$

where \mathfrak{g} is the Lie algebra of G.

So $H^*_H(BG/k) \simeq \mathcal{O}(\mathfrak{g})^G$. This is not surprising: Chern-Weil theory says for G reductive over \mathbb{C} , we have $H^*(BG_{\mathbb{C}}, \mathbb{C}) \simeq \mathcal{O}(\mathfrak{g})^G$.

Theorem 14 (Totaro). If G is a smooth affine group over a field k, then

 $H^i(BG,\Omega^i) \simeq H^{i-j}(G,S^j(\mathfrak{g}^*)).$

Here, for a G-module M, $H^i(G, M) := \operatorname{Ext}^i_G(k, M)$.

This implies Bott's theorem: If G is reductive and char k = 0, then a G-module is a semisimple category. So $H^{>0}(G, M) = 0$.

Example 15. Let V be the standard representation of GL(n) in characteristic 2. Then

$$0 \to V^{(1)} \to S^2 V \to \wedge^2 V \to 0$$

is not split as representation of GL(n).

Puzzle: How is

$$\begin{array}{rcl} H^*_{dR}(BG/\mathbb{F}_p) & \text{related to} & H^*(BG_{\mathbb{C}},\mathbb{F}_p)?\\ G & \text{reductive } /\mathbb{F}_p & \longleftrightarrow & G & \text{reductive } /\mathbb{C} \end{array}$$

A guiding result from *p*-adic Hodge theory:

Theorem 16 (Bhatt, Morrow, Schultz). If X is a smooth projective scheme over \mathbb{Z}_p then

$$\dim_{\mathbb{F}_p} H^i_{dR}(X/\mathbb{F}_p) \ge \dim_{\mathbb{F}_p} H^i_{et}(X_{\overline{\mathbb{Q}_p}}, \mathbb{F}_p).$$

Although BG is not proper, it often acts as if it were proper. Calculation: G reductive over k. To study BG, we use the fibration

$$G/B \to BB \to BG.$$

We have $\operatorname{CH}^*(G/B) \simeq H^*_H(G/B)$ and $H^*_H(BB) = H^*((\mathbb{CP}^{\infty})^n) = k[x_1, \dots, x_n]$ where $n = \operatorname{rk} G$.

Example 17. Let G = SO(n) over a field k of characteristic 2. Then

$$H_H^*(BSO(n)/\mathbb{F}_2) = \mathbb{F}_2[w_2, w_3, \dots, w_n], \qquad w_i \in H^i$$
$$w_{2a} \in H^a(BSO(n), \Omega^a)$$
$$w_{2a+1} \in H^{a+1}(BSO(n), \Omega^a).$$

Example 18. One has

$$\dim_{\mathbb{F}_2} H^{32}_{dR}(BSpin(11)/\mathbb{F}_2) > \dim_{\mathbb{F}_2} H^{32}_{et}(BSpin(11)_{\mathbb{C}},\mathbb{F}_p).$$