TALK NOTES: AN EXAMPLE OF A DERIVED ZETA FUNCTION

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1. Preliminaries

Definition 1.1. The *Grothendieck ring of varieties*, $K_0(\mathbf{Var}_k)$ is the free abelian group on varieties over k, modulo the relation that for any closed immersion $Y \hookrightarrow X$ we have

$$[X] = [Y] + [X \backslash Y].$$

The ring structure is given by $[X][Y] = [X \times Y]$ for any varieties X and Y.

This ring is very complicated; for example, it has zero divisors. One way to study its structures is through motivic measures.

Definition 1.2. A *motivic measure* is a (ring) homomorphism $K_0(\mathbf{Var}_k) \longrightarrow A$. In other words, a motivic measure is an invariant μ on the Grothendieck ring of varieties which satisfies the relation that $\mu(X) = \mu(Y) + \mu(X \backslash Y)$ for any closed immersion $Y \hookrightarrow X$. If this invariant is also multiplicative, in the sense that $\mu(X \times Y) = \mu(X)\mu(Y)$ then it is a ring homomorphism.

Example 1.3. When k is finite: point counting over k (to $\mathbb{Z}$).

Example 1.4. When $k = \mathbf{C}$: Let $MHS_{\mathbf{Q}}$ be the category of finite-dimensional $\mathbf{Q}$-vector spaces with mixed Hodge structures. We can define $K_0(MHS_{\mathbf{Q}})$ to be the free abelian group generated by finite-dimensional $\mathbf{Q}$-vector spaces with mixed Hodge structures, modulo the relation that when we have an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of vector spaces with mixed Hodge structures, then $[B] = [A] + [C]$. We can define a motivic measure by

$$\chi_{Hodge}(X) = \sum_{i \geq 0} (-1)^i [H_c^i(X(\mathbf{C}), \mathbf{Q})].$$

Example 1.5. When k is finite: Let $\mathrm{Rep}_{cts}(G_k; \mathbf{Q}_\ell)$ be the category of continuous representations of the Galois group of k. We can define $K_0(\mathrm{Rep}_{cts}(G_k; \mathbf{Q}_\ell)$ analogously to the above. We can define a motivic measure with values in $K_0(\mathrm{Rep}_{cts}(G_k; \mathbf{Q}_\ell)$ by

$$\chi_\ell(X) = \sum_{i \geq 0} (-1)^i [H_c^i(X, \mathbf{Q}_\ell)].$$

Here, H_c^i denotes the ℓ-adic cohomology of X.

Example 1.6. Suppose that $k = \mathbb{F}_q$. The local zeta function of X, $Z(X,t)$, is defined by

$$Z(X,t) = \exp \sum_{r \geq 1} \frac{1}{r} |X(F_{q^r})| t^r.$$

This gives a homomorphism $K_0(\mathbf{Var}_k) \longrightarrow (1 + \mathbb{Z}[[t]], \times)$.

Note that two of these measures look like Euler characteristics. This is a common feature of such examples, and, in fact, if we forget the Hodge structure in the second example we get the usual Euler characteristic with compact supports.

From the Lefschetz fixed point formula we can relate Examples 1.5 and 1.6 since

$$|X(k)| = \sum_i (-1)^i \operatorname{tr} F_q|_{H_c^i(X, \mathbf{Q}_\ell)}$$

(where F_q is the Frobenius) all of the data of the local zeta function is contained in the action of $G_{\mathbb{F}_q}$ on $H_c^*(X, \mathbf{Q}_\ell)$. The advantage of stopping here is that this is a category with a lot of structure, and which has a K-theory that we can try to analyze. It remembers that local zeta functions are rational, unline the formulation given in Example 1.6.

Thus we can now think of $Z(\cdot, t)$ as a function $K_0(\mathbf{Var}_k) \longrightarrow K_0(\mathrm{Rep}_{cts}(G_k; \mathbf{Q}_\ell))$. Once we think of it this way it is reasonable to ask whether there is a map of spaces[1] which produces $Z(\cdot, t)$ after applying π_0.

2. K-theory

Let's step back for a minute and think about K-theory. Algebraic K-theory takes a Grothendieck group (such as the ones mentioned above) and produces a space where the Grothendieck group is given by the connected components. This space is constructed in a manner so that the higher homotopy groups of the space give further interesting algebraic information.

Definition 2.1. A *Waldhausen category* $\mathcal{E}$ is a category with two subcategories $c\mathcal{E}$ and $w\mathcal{E}$, called the *cofibrations* (denoted $\hookrightarrow$) and *weak equivalences* (denoted $\xrightarrow{\sim}$). These satisfy the following axioms:

- $c\mathcal{E}$ and $w\mathcal{E}$ contain all isomorphisms.
- $\mathcal{E}$ has a zero object 0 and for all $A \in \mathcal{E}$, $0 \longrightarrow A$ is a cofibration.
- Pushouts along cofibrations exist and the pushout of a cofibration is a cofibration.
- For any diagram

$$\begin{tikzcd} C \arrow[d, "\wr"] & A \arrow[l] \arrow[r, hook] \arrow[d, "\wr"] & B \arrow[d, "\wr"] \\ C' & A' \arrow[l] \arrow[r, hook] & B' \end{tikzcd}$$

  the map $B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$ is a weak equivalence.

To a Waldhausen category $\mathcal{E}$ we can assign a space $K(\mathcal{E})$; then $K_i(\mathcal{E})$ is defined to be $\pi_i(\mathcal{E})$. These exist for all $i \geq 0$ and are always abelian groups.

Example 2.2. Let R be a commutative ring with unit. The category Mod_R^{fgp} is the category of finitely-generated projective R-modules. This is a Waldhausen category where the weak equivalences are the isomorphisms and the cofibrations are the injections with projective cokernel.

Example 2.3. Moreover, the category $\mathrm{Ch}^f(R)$ of bounded chain complexes of finitely-generated projective R-modules is a Waldhausen category, where the weak equivalences are the quasi-isomorphisms and the cofibrations are the injections with projective cokernel. There is an inclusion $Mod_R^{fgp} \longrightarrow \mathrm{Ch}^f(R)$ which puts an R-module at level 0; this induces an equivalence on K-theory.

Example 2.4. The category $\mathrm{Aut}(R)$ has objects pairs (P, f), where $P \in Mod_R^{fgp}$ and $f \in \mathrm{Aut}(P)$ and morphisms $h\colon (P, f) \longrightarrow (Q, g)$ are morphisms $h\colon P \longrightarrow Q$ such that $hf = gh$. This inherits a Waldhausen structure from Mod_R^{fgp}.

Example 2.5. $\mathrm{Rep}(G_k, \mathbf{Q}_\ell)$ and $MHS_{\mathbf{Q}}$ are also a Waldhausen categories.

Thus if we want to straightforwardly lift $Z(\cdot, t)$ to a K-theory map then we need to show how to obtain $K_0(\mathbf{Var}_k)$ as the K_0 of a Waldhausen category.

Lie 2.6. *There is a Waldhausen category* **Var** *where the objects are varieties, the cofibrations are closed immersions and the weak equivalences are isomorphisms.*

(To see the truth, see Jonathan Campbell's paper on the Grothendieck spectrum of varieties.)

This lie is morally true, however. And we then get a theorem:

Theorem 2.7 (Campbell–Wolfson–Z)**.** *The assignment $X \mapsto G_k \cdot H^*_{et,c}(X \times_k k^s; \mathbf{Q}_\ell)$ lifts to a map of K-theory spectra*

$$K(\mathbf{Var}_k) \longrightarrow K(\mathrm{Rep}_{cts}(G_k; \mathbf{Q}_\ell)).$$

[1] Actually, what we're interested is a map of *spectra*; for those not comfortable with spectra, thinking of this as a map of spaces will not lose anything in the context of this talk.

Specializing to Frobenius acting on the compactly-supported cohomology, we get a map $K(\mathbf{Var}_k) \longrightarrow K(\mathrm{Aut}(\mathbf{Q}_\ell))$ *which gives exactly the data necessary for the construction of the local zeta function.*

To recover the local zeta function from $K_0(\mathrm{Aut}(\mathbf{Q}_\ell))$ we take a pair (P, f) to $\det(1 - f^*t; P)$.

This is exactly the map that we wanted. So the further question now is: so what? can we get any useful information out of this?

3. Finding nontrivial elements in K_1

Let's step back for a minute and think of a simpler example. The category $\mathbf{FinSet}_*$ of pointed finite sets (with inclusions and isomorphisms) is a Waldhausen category. We have a functor $\mathbf{Var}_k \longrightarrow \mathbf{FinSet}_*$ given by $X \mapsto X(k)$. We can apply K to produce the K-theory of each side. The K-theory of $\mathbf{FinSet}_*$ is well-known: it is QS^0 and its homotopy groups are the stable homotopy groups of spheres. In particular, $K_0(\mathbf{FinSet}_*)$ is $\mathbb{Z}$, which corresponds to the size of a set, and the map $K_0(\mathbf{Var}_k) \longrightarrow K_0(\mathbf{FinSet}_*)$ is point counting. $K_1(\mathbf{FinSet}_*)$ is $\mathbb{Z}/2$ and encodes the sign of a permutation: given any automorphism of a set (permutation) it is taken to its sign. The functor $\mathbf{Var}_k \longrightarrow \mathbf{FinSet}_*$ has a section $\mathbf{FinSet}_* \longrightarrow \mathbf{Var}_k$, given by including finite sets as zero-dimensional varieties. Thus we definitely have non-trivial classes in $K_1(\mathbf{Var}_k)$ which are represented by permutations of zero-dimensional varieties. However, these are not very interesting, since they do not use any of the more interesting structure of varieties.

We can use the constructed zeta function to find a more interesting element in $K_1(\mathbf{Var}_k)$ whenever k is finite with $|k| \equiv 3 \pmod 4$. In general, for any object X in a Waldhausen category $\mathcal{E}$, there is a homomorphism $\mathrm{Aut}(X) \longrightarrow K_1(\mathcal{E})$. Thus to find an interesting element in $K_1(\mathbf{Var}_k)$ we want to find a homomorphism $K_1(\mathrm{Aut}(\mathbf{Q}_\ell)) \longrightarrow A$ and a variety X such that we can compute the composition

$$\mathrm{Aut}(X) \longrightarrow K_1(\mathbf{Var}) \longrightarrow K_1(\mathrm{Aut}(\mathbf{Q}_\ell)) \longrightarrow K_1(\mathrm{Aut}(\mathbb{Q}_\ell)) \longrightarrow A$$

on the image of $\pi_1(\mathbf{FinSet}_*)$ and an element in $\mathrm{Aut}(X)$ which does not map into that image.

An element in $K_1(\mathrm{Aut}(\mathbb{Q}_\ell))$ is represented by a pair of commuting automorphisms (f, g) of a finite dimensional $\mathbb{Q}_\ell$-vector space. Then there is a map (constructed by Milnor) which takes (f, g) to $f^{-1} \star g$, which is an element in $K_2^M(\mathbb{Q}_\ell)$. This in turn supports the 2-adic Hilbert symbol $(-.-)_2$ which is a map $K_2^M(\mathbb{Q}_\ell) \longrightarrow \mathbb{Z}/2$; this is defined on $\{\alpha, \beta\}$ to be 1 if $z^2 = \alpha x^2 + \beta y^2$ has a non-zero solution in $\mathbb{Q}_\ell^3$, and -1 if not.

Let X be a variety and φ an automorphism of X. Under the composition $K_1(\mathbf{Var}_k) \longrightarrow K_2(\mathbb{Q}_\ell)$ this maps to $Frob^{-1} \star \varphi$. If we trace through the definition of $\star$ (which is nontrivial) we see that this maps to $(-1, 1)_2 = 1$. On the other hand, if we let $X = \mathbb{P}^1 \amalg \mathbb{P}^1$ and let φ be the transposition of the two lines, this maps to $(-1, q)_2$, which when $q \equiv 3 \pmod 4$ is -1. Thus the element in $K_1(\mathbf{Var}_k)$ is nontrivial.