Properties of BMO functions whose reciprocals are also BMO

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The main result says that a non-negative BMO-function $w$, whose reciprocal is also in BMO, belongs to $\bigcap_{p>1} A_p$, and that an arbitrary $u \in BMO$ can be written as $u = w - 1/w$, for $w$ as above. This leads then to some observations concerning the John-Nirenberg distribution inequality for $F \circ u, u \in BMO$ and $F \in \text{Lip } \alpha$.

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1. Introduction

We will consider the question of when a function $w$ and its reciprocal $1/w$ are in BMO. If we assume that $w : R^n \to R_+$ and consider this question for various spaces $X$, we obtain distinct results. The answer for $L^p(R^n)$ is that if $w, 1/w \in L^p(R^n)$, then $p = \infty$ while $w, 1/w \in L^\infty$ implies that $w \approx 1$ which is also equivalent to the fact that $w, 1/w \in A_1$ (for the precise definition of the $A_p$ classes see below). It is known that $BMO$ is the right space to consider in place of $L^p$ as $p \to \infty$ in a number of situations and we will give the answer to this question for $BMO$ in this paper.

The definition of $BMO$ is that $f \in BMO$ if

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx = ||f||_* < +\infty$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$, and $Q$ is a cube with sides parallel to the coordinate axes. It is important to know that the $L^1$ norm can be replaced by the $L^p$ norm for $0 < p < \infty$,

$$\sup_Q \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p} = ||f||_{*,p} \approx ||f||_*.$$

We need also to recall the John-Nirenberg lemma, the reason for the above result, for functions of bounded mean oscillation. If $f \in BMO$, there are constants $c_1, c_2 > 0$ independent of $f$ and $Q$ such that

$$|\{t \in Q : |f(t) - f_Q| > \lambda\}| \leq c_1 e^{-c_2 \lambda/||f||_*|Q|},$$

for all $\lambda > 0$. Of course, bounded functions are in $BMO$ and $\ln 1/|x|$ is an unbounded function in $BMO$. The precise space we will study is

$$BMO_* = \{w : R^n \to R_+ : w, 1/w \in BMO\}.$$
We need to recall the $A_p$ weights which are defined by the condition

$$A_p(w) = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < +\infty,$$

where $Q$ is again a cube. The $A_p$ weights solve the problem of characterizing when the Hardy-Littlewood maximal function maps $L^p_w$ into $L^p_w$, where $Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy$, and the result is

$$\int |Mf(x)|^p w(x) \, dx \leq C^p \int |f(x)|^p w(x) \, dx \quad \iff \quad w \in A_p.$$  

We will also need to consider $A_1 = \{ w | Mw(x) \leq C w(x) \}$, with the smallest such $C$ being denoted $A_1(w)$ and $A_\infty = \bigcup_{p>1} A_p$. Since the $A_p$ constants decrease by Hölder’s inequality, we can set $A_\infty(w) = \lim_{p \to \infty} A_p(w)$. We have the set inclusions

$$A_1 \subseteq A_p \subseteq A_q \subseteq A_\infty,$$

where $1 \leq p \leq q \leq \infty$. The $A_p$ weights also solve the corresponding problem for the Hilbert transform

$$Hf(x) = \lim_{\epsilon \to 0} \int_{\epsilon < |x-y| < 1/\epsilon} \frac{f(y)}{x-y} \, dy.$$  

It is known that if $w, 1/w \in A_p$, then $w \in A_2$, and we may limit our study to the case $1 \leq p \leq 2$ by the inclusion properties of $A_p$. It is also known that $[1, \text{p. 474}]$

$$w, 1/w \in \bigcap_{p>1} A_p \iff \ln w \in \text{clo} BMO L^\infty. \quad (1)$$

We say that $w \in RH_{p_0}$ (reverse Hölder) if

$$\left( \frac{1}{|Q|} \int_Q w^{p_0} \right)^{1/p_0} \leq \frac{C}{|Q|} \int_Q w,$$

and we abbreviate by $RH_{p_0}(w)$ the infimum of all such $C$. We will use the fact, due to Strömberg and Wheeden, that $w \in RH_{p_0}$ if and only if $w^{p_0} \in A_\infty$. An alternate proof of this fact can be found in [3, Lemma 3.1].

## 2. Preliminary results

Our first result shows that Hölder continuous functions operate on $BMO$.

**Lemma 1:** If $F$ is Hölder continuous of order $\alpha$, where $0 < \alpha \leq 1$ and $f \in BMO$, then $F \circ f \in BMO$ and $\|F \circ f\|_\ast \leq 2 \|F\|_{\text{Lip } \alpha} \|f\|_\ast^\alpha$.

**Proof.** If there is a constant $c$ such that $\frac{1}{|Q|} \int_Q |f(x) - c| \, dx \leq A$, then it is well known that $\|f\|_\ast \leq 2A$. We compute

$$\left( \frac{1}{|Q|} \int_Q |F(f(x)) - F(f_Q)|^p \, dx \right)^{1/p} \leq \left( \frac{1}{|Q|} \|F\|_{\text{Lip } \alpha}^p \int_Q |f(x) - f_Q|^{\alpha p} \, dx \right)^{1/p}.$$  

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Thus we obtain with \( p = 1/\alpha \), \( \| F \circ f \|_* \leq 2 \| F \|_{\text{Lip} \alpha} \| f \|^* \).

This has been, at least partially, observed by many people. If \( f \in \text{BMO} \), then \( |f|^\alpha \in \text{BMO} \), for \( 0 < \alpha \leq 1 \) and \( \max\{f, g\} \) and \( \min\{f, g\} \) are in \( \text{BMO} \) if \( f, g \) are in \( \text{BMO} \).

We haven’t noticed the converse observed, but it is true. If \( \| F \circ f \|_* \leq A \| f \|_*^\alpha \), then \( F \in \text{Lip} \). The proof may be found in [2], but as this is not generally available, we give the proof here. Without loss of generality, we may assume \( F(0) = 0 \) and consider only cubes centered at the origin since \( \text{BMO} \) is translation invariant. Suppose that \( Q = [-\frac{d}{2}, \frac{d}{2}]^n \) and that

\[
f(x) = \begin{cases} x_1 & \text{on the double of } Q \\ 0 & \text{outside the double of } Q. \end{cases}
\]

One checks that

\[
F(f(x)) = \begin{cases} F(x_1) & \text{for } x \in 2Q, \\ 0 & \text{outside the double of } Q. \end{cases}
\]

and since \( \| f \|_* \leq \| f \|_\infty \leq \frac{d}{2} \), one finds

\[
\frac{1}{d} \int_{-\frac{d}{2}}^{\frac{d}{2}} |F(x_1) - F_Q| \, dx_1 \leq Ad^\alpha,
\]

where \( Q_1 \) is the one-dimensional cube \( [-\frac{d}{2}, \frac{d}{2}] \), and by the Campanato-Meyer theorem [4], this proves the result.

We can use the lemma to show that there is a close connection between \( \text{BMO} \) and \( \text{BMO}_* \).

**Theorem 1:** A real valued function \( u \) is in \( \text{BMO} \iff \) there exists a \( w \in \text{BMO}_* \) such that \( u = w - 1/w \) and \( \| w \|_* + \| 1/w \|_* \simeq \| u \|_* \).

**Proof.** If \( u \) admits the decomposition, it is clear that \( u \in \text{BMO} \). If we are given a \( u \in \text{BMO} \), it is easy to see that the equation for \( w \) leads to a quadratic equation with a solution of \( w = \frac{1}{2}(u + \sqrt{u^2 + 4}) \). The function \( F(x) = \frac{1}{2}(x + \sqrt{x^2 + 4}) \) is everywhere differentiable with derivative bounded by 1. By Lemma 1, \( w \in \text{BMO} \).

**Remark.** We note that the same proof proves the corresponding result for functions of vanishing mean oscillation, which are defined as is \( \text{BMO} \) but when the sup is taken over cubes of side \( r \), and the resulting sup goes to 0 as \( r \to 0+ \).

Another application of Lemma 1 is to the determination of conditions under which the square of a function belongs to \( \text{BMO} \). By Lemma 1 with \( F(x) = \sqrt{x} \), it follows that such a function belongs to \( \text{BMO} \). We show that more is true.

**Lemma 2:** If \( f = F(u), F \in \text{Lip} \alpha, u \in \text{BMO} \), then

\[
|\{ x \in Q : |f(x) - F(u_Q)| > \lambda \}| \leq c_1 e^{-c_2 \lambda^{1/\alpha} / \| F \|_{\text{Lip} \alpha}^{1/\alpha} \| u \|_*} |Q|.
\]

**Proof.** Because \( u \in \text{BMO} \), by the John-Nirenberg lemma, there are constants \( c_1 \) and \( c_2 \) such that

\[
|\{ t \in Q : |u(t) - u_Q| > \lambda \}| \leq c_1 e^{-c_2 \lambda / \| u \|_*} |Q|. \]

Hence, since

\[
\{ t \in Q : |f(t) - F(u_Q)| > \lambda \} \subseteq \left\{ t \in Q : |u(t) - u_Q| > \left( \frac{\lambda}{\| F \|_{\text{Lip} \alpha}} \right)^{1/\alpha} \right\},
\]

we have the inequality

\[
|\{ t \in Q : |f(t) - F(u_Q)| > \lambda \}| \leq c_1 e^{-c_2 (\| F \|_{\text{Lip} \alpha}^{\lambda / \| u \|_*})^{1/\alpha} |Q|},
\]

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which is the desired result.

**Corollary 1**: For any $\epsilon < c_2$, 
\[
\int_Q \left( e^{\frac{(c_2 - \epsilon) |f(x) - F(u_Q)|^{1/\alpha}}{||F||_{Lip, \alpha}||u||_*}} - 1 \right) \, dx \leq c_1 \left( \frac{c_2 - \epsilon}{\epsilon} \right) |Q|.
\]

**Proof.** Let $\phi(x) = e^{Ax^{1/\alpha}} - 1$, which is increasing with $\phi'(x) = \frac{A}{\alpha} x^{1/\alpha - 1} e^{Ax^{1/\alpha}}$. As long as $A$ is positive, 
\[
\int_Q e^{A |f(x) - F(u_Q)|^{1/\alpha}} \, dx - |Q| = \frac{A}{\alpha} \int_0^\infty \frac{|\{x \in Q : |f(x) - F(u_Q)| > \lambda\}| \lambda^{1/\alpha} - e^{\lambda^{1/\alpha}} d\lambda}{\lambda^{1/\alpha}} - \lambda^{1/\alpha-1} e^{\lambda^{1/\alpha}} d\lambda \leq \frac{A c_1}{\alpha} \int_0^\infty e^{-\left( \frac{c_2}{||F||_{Lip, \alpha}||u||_*} - A \right) \lambda^{1/\alpha}} \lambda^{1/\alpha-1} d\lambda |Q|.
\]
If we choose $A$ less than the fraction, we can use the fact that 
\[
\frac{1}{\alpha} \int_0^\infty e^{-\epsilon \lambda^{1/\alpha}} \epsilon \lambda^{1/\alpha-1} d\lambda = \int_0^\infty e^{-\epsilon u} du = 1
\]
to obtain the above estimate.

If we modify the choice of $\phi$ slightly by putting $\psi(x) = e^{Ax^{1/\alpha}}$, we see that for $\alpha \leq 1$, $\psi$ is convex and we can apply Jensen’s formula to $Q, p = 1, f = |f(x) - F(u_Q)|$ and if we note that 
\[
|f_Q - F(u_Q)| = \left| \frac{1}{|Q|} \int_Q (f(x) - F(u_Q)) \right| \leq \frac{1}{|Q|} \int_Q |f(x) - F(u_Q)|,
\]
we can make the estimate 
\[
\psi(2|f_Q - F(u_Q)|) \leq \psi \left( \frac{1}{|Q|} \int_Q 2|f(x) - F(u_Q)| \right) \leq \frac{1}{|Q|} \int_Q e^{A_2^{1/\alpha} |f(x) - F(u_Q)|^{1/\alpha}}.
\]
We now combine this with Corollary 1 and obtain 
\[
\int_Q e^{A |f(x) - f_Q|^{1/\alpha}} = \int_Q e^{A |f(x) - F(u_Q) + F(u_Q) - f_Q|^{1/\alpha}} \leq \int_Q e^{A_2^{1/\alpha} |f(x) - F(u_Q)|^{1/\alpha} + A_2^{1/\alpha} |f_Q - F(u_Q)|^{1/\alpha}} = e^{A_2^{1/\alpha} |f_Q - F(u_Q)|^{1/\alpha}} \int_Q e^{A_2^{1/\alpha} |f(x) - F(u_Q)|^{1/\alpha}}.
\]
If we choose $A_2^{1/\alpha} = (c_2 - \epsilon)/(||F||_{Lip, \alpha}||u||_*),$ we can estimate this and 
\[
\int_Q e^{A |f(x) - f_Q|^{1/\alpha}} \leq \left( c_1 \left( \frac{c_2 - \epsilon}{\epsilon} \right) + 1 \right) |Q| \psi(2|f_Q - F(u_Q)|)
\]
by using Corollary 1 and now we apply Jensen’s inequality to get

\[
\int_Q e^{A|f(x) - f_Q|^{1/\alpha}} \leq \left( c_1 \left( \frac{c_2 - \epsilon}{\epsilon} \right) + 1 \right) |Q|^{1/|Q|} \int_Q e^{A^{21/\alpha} |f(x) - F(u_Q)|^{1/\alpha}} \\
\leq \left( c_1 \left( \frac{c_2 - \epsilon}{\epsilon} \right) + 1 \right)^2 |Q|.
\]

We can now state and prove the following.

**Theorem 2:** Consider the set of \( f = F(u), u \in BMO, 0 < \alpha \leq 1 \). The following two statements are equivalent.

(i) \( F \in \text{Lip} \alpha \)

(ii) there exists \( 0 < c_1, c_2 < \infty, 0 \leq A \leq \infty \), independent of \( Q, u \in BMO \) such that

\[
\{|x \in Q : |f(x) - f_Q| > \lambda\| \leq c_1 e^{-c_2 \lambda^{1/\alpha} |\mathcal{L}_Q|^\alpha} |Q|.
\]

and then \( A \simeq |F||\text{Lip} \alpha| \).

**Proof.** We will first prove that (i) implies (ii). By restricting the range of integration in the inequality derived after Corollary 1, we see that

\[
|E_\lambda| \equiv \{|x \in Q : |f(x) - f_Q| > \lambda\| \leq c_1 e^{-c_2 \lambda^{1/\alpha} |\mathcal{L}_Q|^\alpha} \int_Q e^{A|f(x) - f_Q|^{1/\alpha}} dx,
\]

since \( e^{-A^{1/\alpha} e^{A|f(x) - f_Q|^{1/\alpha}}} > 1 \) on \( E_\lambda \). This is the desired result if we choose \( \epsilon = \frac{c_2}{2} \) and \( A \) as above.

We next show that (ii) implies (i). We first observe that (ii) implies that for some constants \( 0 < c_3, c_4 < \infty \)

\[
\int_Q \exp \left( \frac{c_3 |f(x) - f_Q|^{1/\alpha}}{A|u|}\right) \leq c_4 |Q|.
\]

This implies that

\[
L_Q \equiv \frac{1}{|Q|} \int_Q \exp \left( \frac{c_3 |f(x) - f_Q|^{1/\alpha}}{A|u|}\right) \leq c_4.
\]

Hölder now gives us, since \( 1/\alpha \geq 1 \),

\[
L_Q \geq \left( \frac{1}{|Q|} \int_Q \frac{c_3^\alpha |f(x) - f_Q|}{A^\alpha|u|^\alpha} \right)^{1/\alpha}.
\]

Hence

\[
\frac{1}{|Q|} \int_Q |f(x) - f_Q| \leq C A^\alpha|u|^\alpha.
\]

The proof is now completed by an application of [2]; see the argument after Lemma 1.

**Corollary 2:** If \( b \in BMO \), then

\[
\{|x \in Q : |b(x) - b_Q| > \lambda\| \leq c_1 e^{-c_2 \lambda^{1/\alpha} |b||\text{Lip}\alpha|^\alpha} |Q|.
\]
Proof. Apply the above theorem with $u(x) = b^k, F(x) = x^{1/k}$ which is Lipschitz continuous of order $1/k$ with Lipschitz constant 1.

Remark. The argument actually shows that if

$$ |\{x \in Q : |u(x) - u_Q| > \lambda\} | \leq c_1 e^{-c_2 \lambda^k |Q|}, $$

then

$$ |\{x \in Q : |f(x) - f_Q| > \lambda\} | \leq (c_1 + 1)2e^{-c_2 \lambda^k \frac{1}{\|f\|_{Lip}^2}\lambda^{k/2}} |Q|. $$

Our main result connects the behavior of functions in $\text{BMO}_s$ with the $A_p$ classes.

**Theorem 3:** The set of nonnegative functions which are $\text{BMO}$ along with their reciprocals is contained in the intersection of all the $A_p$ classes for $p > 1$, i.e. $\text{BMO}_s \subseteq \bigcap_{p>1} A_p$.

Remarks. (1) Of course, if $b \in \text{BMO}_s$, then $1/b \in \text{BMO}_s \subseteq \bigcap_{p>1} A_p$ and (1) above implies $\ln b \in \text{clos}_{\text{BMO}} L^\infty$.

(2) The class $\text{BMO}_s$ is non-empty. For example, $b_1(x) = \max(\ln 1/|x|, e) \in \text{BMO}$ and $1/b_1 \in L^\infty \subseteq \text{BMO}$. Moreover, if we take

$$ b_2(x) = \max(\ln 1/|x|, 1/\ln(|x|^2)) $$

we get an example of a function which is unbounded and whose inverse is unbounded, yet both $b_2, 1/b_2 \in \text{BMO}$.

(3) The result is sharp in the sense that the function $b$ in the theorem cannot be in $A_1$ since if it were, $1/b$ would also be in $A_1$ and then by a result of Johnson and Neugebauer [3, Lemma 2.2], $b \approx 1$.

(4) The converse is, however, not true because with the same function $b_1$ as above, $b_1^2$ satisfies $1/b_1^2 \in L^\infty$ and $\ln b_1^2 = 2 \ln b_1 \in \text{clos}_{\text{BMO}} L^\infty$ and therefore $b_1^2 \in \bigcap_{p>1} A_p$ and $1/b_1 \in \bigcap_{p>1} A_p$, but $b_1^2 \notin \text{BMO}$.

We will prove Theorem 3 as a special case of a more general result, but let us indicate how it can be proved directly. The first step is a lemma.

**Lemma 3:** Let us denote by

$$ f_Q = \frac{1}{|Q|} \int_Q f(x)dx, $$

then we have

$$ (fg)_Q - f_Q g_Q = \frac{1}{|Q|} \int_Q (f(x) - f_Q)(g(x) - g_Q)dx. $$

Proof. Compute and use the fact that $g - g_Q$ has mean value zero.

We are ready for the first step in this version of the proof of Theorem 3.

**Theorem 4:** Suppose $b \in \text{BMO}_s$, then $b$ is in $A_2$. 

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Proof. Apply Lemma 3 to $b$ and $1/b$ which gives

$$1 - b_Q(1/b)_Q = \frac{1}{|Q|} \int_Q (b(x) - b_Q)(1/b(x) - (1/b)_Q)dx$$

and allows us to make the estimate $|1 - b_Q(1/b)_Q| \leq ||b||_s ||1/b||_s$. Hölder’s inequality shows that $1 \leq b_Q(1/b)_Q$ and the above becomes $1 \leq b_Q(1/b)_Q \leq 1 + ||b||_s ||1/b||_s$.

**Theorem 5**: If $b \in BMO_s$, then $b \in A_{3/2}$.

For the proof of this statement we have to estimate

$$\left(\frac{1}{|Q|} \int_Q b \right) \left(\frac{1}{|Q|} \int_Q \frac{1}{b} \right)^{1/2}.$$

First we require another lemma.

**Lemma 4**: With the same notation as in Lemma 3, we have

$$\frac{1}{|Q|} \int_Q (f(t) - f_Q)(g(t) - g_Q)(h(t) - h_Q)(l(t) - l_Q)dt$$

$$= (fg hl) - f_Q(g hl)_Q - g_Q(f hl)_Q - h_Q(f gl)_Q - l_Q(f gh)_Q + f_Qg_Q(h l)_Q$$

$$+ f_Q h_Q (gl)_Q + f_Ql_Q (gh)_Q + g_Q h_Q (fl)_Q + g_Q l_Q (fh)_Q$$

$$+ h_Q l_Q (fg)_Q - 3f_Q g_Q h_Q l_Q.$$

**Proof.** We expand the integral and compute the resulting terms.

Take $f = h = b, g = l = \frac{1}{b}$. We obtain

$$1 - b_Q(\frac{1}{b})_Q - (\frac{1}{b})_Q b_Q - b_Q(\frac{1}{b})_Q - (\frac{1}{b})_Q b_Q$$

$$+ \left\{ b_Q(\frac{1}{b})_Q + (b_Q)^2(\frac{1}{b^2})_Q + b_Q(\frac{1}{b})_Q + (\frac{1}{b})_Q b_Q + ((\frac{1}{b})_Q)^2(\frac{1}{b^2})_Q + b_Q(\frac{1}{b})_Q \right\}$$

$$- 3(b_Q)^2((\frac{1}{b})_Q)^2$$

$$= \frac{1}{|Q|} \int_Q (b(t) - b_Q)^2 \left(\frac{1}{b}(t) - (\frac{1}{b})_Q \right)^2 dt.$$

This allows us to estimate

$$1 + (b_Q)^2(\frac{1}{b^2})_Q + \left(\frac{1}{b}_Q \right)^2(b^2)_Q - 3(b_Q)^2 \left(\frac{1}{b}_Q \right)^2 \leq ||b||^2_s ||1/b||^2_s$$

which means that

$$1 + (b_Q)^2(\frac{1}{b^2})_Q + \left(\frac{1}{b}_Q \right)^2(b^2)_Q \leq 3(b_Q)^2 \left(\frac{1}{b}_Q \right)^2 + ||b||^2_s ||1/b||^2_s.$$

In particular, $b_Q(\frac{1}{b^2})_Q^{1/2} \leq ||b||_s ||\frac{1}{b}||_s + \sqrt{A_2(b)}$, which proves that

$$A_{3/2}(b) \leq \sqrt{3} + (\sqrt{3} + 1)||b||_s ||\frac{1}{b}||_s.$$
3. $A_p$ weights whose reciprocals are $A_p$ weights

We will now obtain Theorem 3 as a special case of the next result.

**Theorem 6:** Suppose $1 < p_0 \leq 2$. Then the following are equivalent.

\[ w, 1/w \in A_{p_0} \]

\[ L_Q = \frac{1}{|Q|} \int_Q |w - w_Q|^{p_0 - 1} |\frac{1}{w} - (\frac{1}{w})_Q|^{p_0 - 1} \leq c < +\infty. \]

**Proof.** Suppose (2) holds. Let $r = p_0' - 1 \geq 1$. Note that

\[ L_Q \leq \frac{1}{|Q|} \int_Q \left| w^r - (w_Q)^r \right| \left| \frac{1}{w^r} - (\frac{1}{w})_Q^r \right| \]

\[ \leq 1 + (w^r)Q(\frac{1}{w})_Q^r + w_Q^r(\frac{1}{w})_Q^r + w^r_Q(\frac{1}{w})_Q^r \]

\[ \leq 1 + A_{p_0}(\frac{1}{w}) + A_{p_0}(w) + A_2(w)^r \leq c < +\infty, \]

because $w \in A_{p_0}$ implies $w \in A_2$.

Conversely, if (3) holds, then we first note that $w \in A_2$. This follows from the next sequence of inequalities:

\[ c^{1/r} \geq L_Q^{1/r} \geq \frac{1}{|Q|} \int_Q |w - w_Q| \left| \frac{1}{w} - (\frac{1}{w})_Q \right| \]

\[ \geq \frac{1}{|Q|} \int_Q (w - w_Q)((\frac{1}{w})_Q - \frac{1}{w}) \]

\[ = w_Q(\frac{1}{w})_Q - 1 - w_Q(\frac{1}{w})_Q + w_Q(\frac{1}{w})_Q. \]

We use the fact that if $r \geq 1$, then $|a^r - b^r| \geq \frac{|a^r - b^r|}{2^r - 1}$. Write

\[ (w - w_Q)(\frac{1}{w} - (\frac{1}{w})_Q) = \left| wQ(\frac{1}{w})_Q + \frac{1}{w}w_Q - (w_Q(\frac{1}{w})_Q + 1) \right| \]

which allows us to estimate the integrand below by

\[ \left| (w - w_Q)(\frac{1}{w} - (\frac{1}{w})_Q) \right|^r \geq \frac{1}{2^{r-1}} \left\{ w(\frac{1}{w})_Q + \frac{1}{w}w_Q \right\}^r - \left( w_Q(\frac{1}{w})_Q + 1 \right)^r, \]

\[ \geq \frac{1}{2^{r-1}} \left\{ w^r(\frac{1}{w})_Q^r + \frac{1}{w}w_Q^r \right\} - \left( w_Q(\frac{1}{w})_Q + 1 \right)^r. \]

Now we take the average of this over $Q$ which gives

\[ \frac{1}{2^{r-1}} \left\{ (w^r)Q(\frac{1}{w})_Q^r + \frac{1}{w}w_Q^r \right\} \leq c + (A_2(w) + 1)^r, \]

and we conclude that $w, \frac{1}{w} \in A_{p_0}$.

Theorem 3 follows from this result; in fact, we obtain the estimate

\[ L_Q \leq \| w \|_{p_0', (p_0')^{-1}} \| \frac{1}{w} \|_{p_0', (p_0')^{-1}} \]

and as $BMO$ is characterized by $\| f \|_{p, p}$ for any $p > 0$, we can have any $p_0 > 1$ which proves the result.

Although we proved Theorem 6 for $A_p$, it immediately implies a result about $R^1_r$. 

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Theorem 7: The following statements are equivalent for $1 \leq r < \infty$:

$w, 1/w \in RH_r$  \hspace{1cm} (4)

$w, 1/w \in A_{1+1/r}$  \hspace{1cm} (5)

$w^r \in A_2.$  \hspace{1cm} (6)

Proof. (4) $\Rightarrow$ (5). Since $w^r, 1/w^r \in A_\infty$, we have that $w^r, 1/w^r \in A_2$, and hence $w, 1/w \in A_{1+1/r}$.

(5) $\Rightarrow$ (4). $w \in A_{1+1/r} \Rightarrow 1/w \in RH_r$. Similarly, $w \in RH_r$.

(4) $\Rightarrow$ (6). Since $w^r, 1/w^r \in A_\infty$, we have that $w^r \in A_2$ as above.

(6) $\Rightarrow$ (4). Since $w^r \in A_2$, $w \in A_{1+1/r} \Rightarrow 1/w \in RH_r$. From the fact that $w^r \in A_2$, it follows that $w^{-r} \in A_2$ and this implies that we can apply the above remark to $1/w$.

Theorem 8: Suppose $u \in BMO$ and $\alpha > 0$. Then $u^2 + \alpha \in \bigcap_{p>1} A_p$.

Proof. For any $\lambda > 0$, write $\lambda u = w_\lambda - 1/w_\lambda$, for some $w_\lambda \in BMO$. Then $\lambda^2 u^2 = w_\lambda^2 + \frac{1}{w_\lambda^2} - 2$.

By Theorem 7, $w_\lambda^2 \in A_\infty$ and since $w_\lambda \in \bigcap_{p>1} A_p$, by Lemma 2.1 in [3], $w_\lambda^2 \in \bigcap_{p>1} A_p$ and a similar result holds for $\frac{1}{w_\lambda^2}$. This shows that $\lambda^2 u^2 + 2 \in \bigcap_{p>1} A_p$ and hence, $u^2 + \frac{2}{\lambda^2} \in \bigcap_{p>1} A_p$.

Since $\lambda$ is an arbitrary positive number, the result follows.

References


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