THE KREIN-VON NEUMANN EXTENSION: SPECTRAL PROPERTIES, WEYL ASYMPTOTICS

FRITZ GESZTESY

Abstract. We discuss the unitary equivalence of the inverse of the Krein-von Neumann extension (on the orthogonal complement of its kernel) of a densely defined, closed, strictly positive operator $S$ in a Hilbert space to an abstract buckling problem operator. In the concrete case where $S$ represents the minimally defined Laplacian on a bounded Lipschitz domain, we will establish Weyl-type asymptotics of the eigenvalue counting function for strictly positive eigenvalues of the underlying Krein Laplacian.

1. Introduction

Consider a separable complex Hilbert space $\mathcal{H}$, and let $(\cdot,\cdot)$ denote its inner product. A linear operator $S$ from $\operatorname{dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is called non-negative if $(u,Su) \geq 0$ for all $u$ in $\operatorname{dom}(S)$. When $S$ is non-negative, we write $S \geq 0$.

We instead say $S$ is strictly positive if for some $\epsilon > 0$, $(u,Su) \geq \epsilon ||u||^2$ for $u \in \operatorname{dom}(S) \setminus \{0\}$. We write $S \geq \epsilon I_H$, where $I_H$ is the identity of $\mathcal{H}$. Similarly, one can define $S \geq T$ for an arbitrary domain $T$ (after addressing some technical considerations).

We say $A \subseteq B$ if $\operatorname{dom}(A) \subseteq \operatorname{dom}(B)$ and $A$ and $B$ agree on $\operatorname{dom}(A)$. We say that $B$ is an extension of $A$.

We say $T$ is a formal adjoint of $S$ if $(f,Sg) = (Tf,g)$ for all $f \in \operatorname{dom}(T), g \in \operatorname{dom}(S)$. We write $T = S^*$ ($T$ is the adjoint of $S$) if $T$ is the maximal formal adjoint of $S$. We say that $S$ is symmetric in $\mathcal{H}$ if $S \subseteq S^*$. We say that $S$ is self-adjoint if $S = S^*$.

It is then natural to ask about self-adjoint extensions of a symmetric operator $S$. In this talk, we restrict our attention to non-negative self-adjoint extensions of a non-negative symmetric operator $S$.

Given a densely defined, closed, symmetric, non-negative operator $S$, we define $S_K$ and $S_F$ respectively as the minimal and maximal non-negative self-adjoint extensions of $S$. We call $S_K$ the Krein-von Neumann extension and $S_F$ the Friedrichs extension.

2. A connection between the Krein Laplacian on bounded Lipschitz domains and the Buckling problem

Definition 1. Let $\Omega \subset \mathbb{R}^n$ open and nonempty and $R > 0$ fixed. Then $\Omega$ is called a bounded Lipschitz domain of there exists $r \in (0,R)$ such that for every $x_0 \in \partial \Omega$ one can find a rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$T(\Omega \cap B(x_0,R)) = T(B(x_0,R)) \cap \{(x',x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | x_n > \varphi(x')\}.$$ 

Let us define the minimal perturbed Laplacian in $L^2(\Omega)$,

$$H_{\Omega} u := (-\Delta + V) u, u \in \operatorname{dom}(H_{\Omega}) = \{u \in H^2(\Omega) | \gamma_D u = \gamma_n u = 0\},$$

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where $\gamma_D u = u|_{\partial \Omega}$, $\gamma_N u = \nu \cdot \nabla u|_{\partial \Omega}$ refer to Dirichlet and Neumann boundary conditions.

The Krein-von Neumann extension $H_{K,\Omega}$ of $H_{\Omega}$ is given by

$$H_{K,\Omega} u = (-\Delta + V) u,$$

with

$$\text{dom}(H_{K,\Omega}) = \text{dom}(H_{\Omega}) \oplus \{ v \in L^2(\Omega) | (-\Delta + V) v = 0 \text{ in } \Omega \},$$

where $\oplus$ denotes direct sum.

Let us now consider the generalized buckling problem:

$$(-\Delta + V)^2 u = \lambda (-\Delta + V) u,$$

in $\Omega$, $u \in \{ u \in H^2(\Omega) | \gamma_D u = \gamma_N u = 0 \}$. This is a generalized eigenvalue problem.

The main result of this talk is that the Krein Laplacian and the generalized buckling problem are closely related.

**Theorem 1.** If $u \in L^2(\Omega)$ is nonzero and solves the generalized buckling problem for some $\lambda \in \mathbb{C}$, then $\lambda$ is an eigenvalue of $H_{K,\Omega}$. Conversely, if $u \in L^2(\Omega)$ is nonzero and an eigenvalue of $H_{K,\Omega}$ then there exists a solution for the generalized buckling problem for that $\lambda$. 