Geometry-Analysis Seminar: Localization in disordered quantum spin systems

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1 Introduction

Understanding many-body localization is a hard problem, and so we instead look at some toy models. Quantum spin systems fulfil this role. We consider particles, which are elements of a one-dimensional Hilbert space, and their associated spin, which are elements of a two-dimensional Hilbert space. Quantum spin systems can be used to model qubits, and thus have applications in the field of quantum information systems.

More precisely, a single quantum spin is a normalized vector in $\mathbb{C}^2$. That is, given

$$e_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

We express a single quantum spin as $v = \alpha e_\uparrow + \beta e_\downarrow$. A system of $n$ spins is written as

$$\mathcal{H}_n = \bigotimes_{j=1}^n \mathbb{C}^2,$$

with the $2^n$ basis vectors

$$e_{(j_1, \ldots, j_n)} = e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n},$$

with $e_{j_i} \in \{e_\uparrow, e_\downarrow\}$. Typically, we will consider $n$ large.

The interaction is now modelled by a Hamiltonian, that is, a self-adjoint operator on $\mathcal{H}_n$.

2 The XY spin chain in an exterior field

Let us recall the Pauli matrices:

$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define an XY spin chain in an exterior field as

$$H_n = \sum_{j=1}^{n-1} \mu_j [(1 + \gamma_j)\sigma_j^X \sigma_{j+1}^X + (1 - \gamma_j)\sigma_j^Y \sigma_{j+1}^Y] \quad \text{(Interaction Terms)}$$

$$+ \sum_{j=1}^n v_j \sigma_j^Z \quad \text{(Exterior Field)},$$
where \( \mu_j, \gamma_j, v_j \) are real and bounded uniformly in \( n \), with \( \mu_j \) in fact independent of \( n \). We assume that all three sequences are random, and ask if disorder in this model leads to localization.

**Theorem 1** (Lieb-Schultz-Mattis (1961) [1]). \( H_n \) is equivalent to a free Fermion system governed by an effective Hamiltonian \( M_n \), where

\[
M_n = \begin{pmatrix}
v_1 J & -\mu_1 S(\gamma_1) & \cdots \\
-\mu_1 S(\gamma_1)^T & \ddots & \cdots \\
\vdots & \ddots & -\mu_n S(\gamma_{n-1}) \\
-\mu_n S(\gamma_n) & \cdots & -\mu_n S(\gamma_n)^T & v_n J
\end{pmatrix},
\]

with

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}.
\]

We arrive at our main theorem,

**Theorem 2** (Hamza-Sims-Stolz (2012) [2]). Suppose \( M_n \) is dynamically localized, i.e.,

\[
E(\sup_{r \in \mathbb{R}} |\langle j| e^{-1+M_n}|k \rangle|) \leq Ce^{-\eta|j-k|},
\]

for \( C < \infty, \eta > 0 \) both independent of \( n \). Then \( H_n \) satisfies a zero-velocity Lieb-Robinson bound

\[
E(\sup_{t \in \mathbb{R}} ||[e^{itH_n} A e^{-itH_n}, B]||) \leq C' e^{-\eta'|j-k|},
\]

for all local observables \( A \in \mathcal{A}_j \) (that is, acting on the \( j \)th spin) and \( B \in \mathcal{A}_k \) (acting on the \( k \)th spin), and \( C' < \infty, \eta' > 0 \) both independent of \( n \).

In other words, there is no information transport through the chain, and we have many-body dynamical localization.

### 3 Examples

Throughout this section assume that \( \mu_j = \mu \neq 0 \).

#### 3.1 Isotropic XY-chain

Let \( \gamma_j = 0 \), and \( v_j \) iid, and random with a distribution \( \rho \) with \( L_\infty^\rho \) density. Then \( M_n = A_n \otimes (-A_n) \), where \( A_n \) is the \( \mathbb{D} \)-Anderson model. For this case (1) was proven in [3], through the Fractional Moment Method. (2) follows from our theorem.

#### 3.2 Anisotropic XY-chain with large disorder

Let \( \gamma_j = \gamma \in \mathbb{R} \) arbitrary, \( v_j = \lambda \tilde{v}_j \), with \( \tilde{v} \) iid with \( L_\infty^\rho \) density. We assume \( |\lambda| \) to be large. Then 1) was proven in [4], by adapting the Fractional Moment Method. (2) again follows from our theorem.
3.3 Anisotropic XY with any disorder

Let $\gamma_j = \gamma \in \mathbb{R}$, and $V_j$ iid, with nontrivial bounded support. Then we have the following theorem:

**Theorem 3** (Chapman-Stolz (2013) [5]). Let $J \in \mathbb{R} \setminus \{0\}$ be a compact interval and $0 < \rho < 1$. Then

$$E(\sup_t | \langle j | e^{-itM_n} \chi_j(M_n) | k \rangle |) \leq Ce^{-|j-k|\rho}.$$

**References**


