RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2019

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Describe all connected 4-fold covers of $\mathbb{R}P^3 \# \mathbb{R}P^3$. For each cover, say whether it is regular or irregular and explain why. In addition, for each cover, say what the group of deck transformations is and explain why.

2. Let X be the space obtained from a solid octagon by identifying sides as shown Figure 1 below.

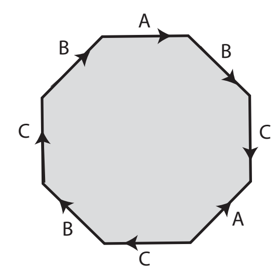
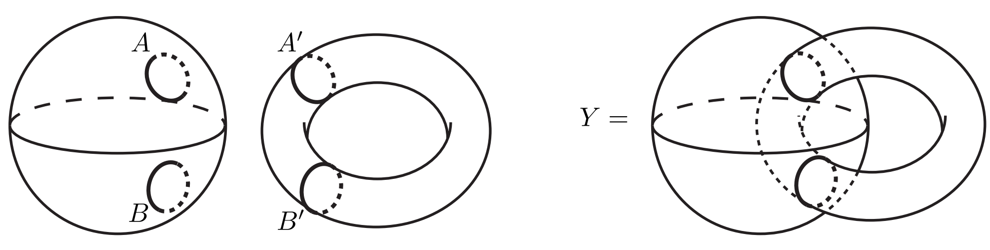


FIGURE 1

   (a) Give a CW-structure for X (be careful with the vertices) and describe the cellular chain complex.
   (b) Give a presentation for $\pi_1(X)$.
   (c) Calculate $H_n(X; \mathbb{Z}_3)$ and $H^n(X; \mathbb{Q})$ for all $n \geq 0$.
   (d) Prove or disprove: X has the homotopy type of a closed m-dimensional manifold for some $m \geq 0$ (not just $m = 2$).

3. Suppose Y is a topological space which is obtained from the union of a 2-sphere S^2 and a torus T by identifying the circle A to the circle A' and the circle B to the circle B' as shown below. Thus $S^2 \cap T \cong S^1 \sqcup S^1$.



   a) Use Mayer-Vietoris to calculate $H_i(Y; \mathbb{Z})$ for all i.
   b) Sketch or describe "geometric" representatives of the generators of $H_1(Y; \mathbb{Z})$ and $H_2(Y; \mathbb{Z})$.
   c) Calculate $\pi_1(X)$.
   d) Sketch or describe a connected 2-fold covering space of Y and the covering map.

4. Let X and Y be closed, connected, oriented 4-manifolds with $\pi_1(X) = \pi_1(Y) = 0$ and $H_2(X) \cong H_2(Y)$. Recall that closed means compact with no boundary.
   (a) Prove that $H_2(X) \cong \mathbb{Z}^g$ for some $g \geq 0$.
   (b) Show that $H_p(X) \cong H_p(Y)$ for all p.
   (c) Show that there are closed, connected, orientable 4-manifolds X and Y that have $\pi_1(X) = \pi_1(Y) = 0$ and $H_p(X) \cong H_p(Y)$ for all p but which are not homotopy equivalent (prove that they are not homotopy equivalent).
   (d) Prove that $\pi_2(X) \cong \pi_2(Y)$.

5. Give an example for each of the following or state that such an example does not exist. Give a brief justification in all cases. All spaces should be path connected CW-complexes.
   (a) Two spaces with isomorphic π_1 but non-isomorphic integral homology groups.
   (b) Two spaces with isomorphic π_1 and isomorphic integral homology groups that are NOT homotopy equivalent.
   (c) Two spaces with isomorphic H_1 and π_n for all $n \geq 2$ that are NOT homotopy equivalent.
   (d) A finite CW complex with H_n not finitely generated for some $n \geq 2$.
   (e) A finite CW complex with π_n not finitely generated for some $n \geq 2$.
   (f) A connected, closed, orientable 3-dimensional manifold M with $H_2(M) \cong \mathbb{Z}_3$.

6. Let $F : N \to M$ be a smooth map of smooth manifolds N and M. Suppose that F is transverse to an embedded submanifold $X \subset M$ and let $W = F^{-1}(X)$. For each $p \in W$, show that $T_pW = (dF_p)^{-1}(T_{F(p)}X)$. Conclude that if two embedded submanifolds $X, X' \subset M$ intersect transversely, then $T_p(X \cap X') = T_pX \cap T_pX'$ for every $p \in X \cap X'$.