Topology Qualifying Exam January 2022

Instructions. Please do each of the problems below, **justifying your work** to the best of your ability in the allotted time. When you have finished, please write and sign the honor code somewhere on your exam

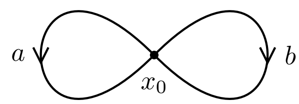
"On my honor, I have neither given nor received any unauthorized aid on this exam."

then email it to Chris at cjl12@rice.edu.

1. Let S be a compact, connected, orientable surface of genus 3.
   (a) Which compact, connected, orientable surfaces are covering spaces of S?
   (b) How many regular (i.e. normal), connected 7–sheeted covers of S are there, up to isomorphism?

2. Let G be a finitely generated abelian group. Construct a (path-connected) topological space X_G with $\pi_1(G) \cong G$. You must show all your work. You may use the classification of finitely generated abelian groups, which implies G has the form $\mathbb{Z}^k \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$, for some integers $k, p_1, \ldots, p_n$.

3. Consider X, the wedge of two circles, viewed as a 1-complex with oriented edges (1-cells) labeled a and b, as shown below.



   Let $f\colon X \to X$ be a map which sends x_0 to itself, sends the oriented edge a to the edge-path aba and the sends the edge b to the edge-path bab. Let $M_f = X \times [0,1]/(x,1) \sim (f(x),0)$ be the mapping torus of f; that is, the quotient space of $X \times [0,1]$ where every point $(x,1)$ is identified to the point $(f(x),0)$.
   (a) Compute $H_p(M_f)$ for all p.
   (b) Compute $H^p(M_f;\mathbb{Q})$ for all p, using the Universal Coefficient Theorem.

4. Let M be a compact, connected, orientable 4–dimensional manifold without boundary such that $\pi_1(M) \cong \mathbb{Z}_{15}$ and $H_2(M;\mathbb{Q}) \cong \mathbb{Q}^2$. Let N be a connected 3–fold covering space of M.
   (a) Calculate $\pi_1(N)$.
   (b) Calculate $H_p(M;\mathbb{Z})$ for each p
   (c) Calculate $\chi(M)$.
   (d) Calculate $H_p(N;\mathbb{Z})$ for each p
   (e) Prove that N admits no CW structure without 3–cells.

5. (a) Prove that every map $f : S^4 \to \mathbb{C}P^2$ has degree 0.
   (b) Suppose M is a closed, oriented manifold of dimension 4. Show there is a degree 1 map $f : M \to S^4$. You will need to explicitely describe this continuous map. (Note: Part (b) holds in all dimensions, not just dimension 4.)

6. Let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be the map given by $F(x,y,z) = z - x^2 - y^2$ and let $f\colon S^2 \to \mathbb{R}$ be the restriction to the sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$. (Alternatively, if $i\colon S^2 \to \mathbb{R}^3$ is inclusion, then $f = F \circ i$.) Prove that 0 is a regular value of f.