Theorem: The Euler-Lagrange equations for the energy functional are \( \tau(\phi) = 0 \), where \( \tau(\phi) = \tau(\phi)^i \partial_{x^i} \) is given by

Proof: In any local calculation, let’s adopt coordinates \( x^\alpha \) with Greek indices for the domain \( M \) and coordinates \( y^i \) with Latin indices for the target \( N \). Let \( \phi_t : M \times [0,1] \to N \) be a geodesic variation of \( \phi = \phi_0 \), e.g., \( \phi_t(x) = \exp_{\phi(x)}(t \psi) \) where \( \psi = \psi^i \partial_{y^i} \) is a vector field along \( \phi \). We compute:

\[
\frac{d}{dt} E(\phi_t) = \left. \frac{d}{dt} \right|_{t=0} \int_M \frac{1}{2} \| D\phi_t \|^2 \text{dvol}_M = \frac{d}{dt} \int_M \frac{1}{2} \langle D\phi_t, D\phi_t \rangle \text{dvol}_M
\]

\[
= \frac{1}{2} \int_M \left. \frac{d}{dt} \left( d\phi_t^i \otimes \partial_{y^i}, d\phi_t^j \otimes \partial_{y^j} \right) \right|_{t=0} \text{dvol}_M
\]

\[
= \int_M \left. \langle \nabla_{\partial_{y^i}} [d\phi_t^j \otimes \partial_{y^j}], d\phi_t^i \otimes \partial_{y^i} \rangle \right|_{t=0} \text{dvol}_M
\]

\[
= \int_M \left. \langle [\nabla_{\partial_{y^i}} d\phi_t^j], \partial_{y^i} + d\phi_t^j \otimes [\nabla_{\partial_{y^i}} \partial_{y^j}], \partial_{y^i} \rangle \right|_{t=0} \text{dvol}_M
\]