COMPLEX ANALYSIS

THE SECOND MIDTERM

9 PROBLEMS

3 problems for a C; 5 problems for a B; 7 problems for an A.
This midterm is due on Friday 16 April at 11:59am in my office HB 444.
In all problems, please give detailed solutions.
Please clearly formulate all theorems that you are using.
You may use the textbook and your notes, but no other materials.
Consultation with other people is not allowed.
Please write the Honour Pledge in full.
No partial credit for wrong answers.

1 Zeros of analytic functions.

1. How many zeros do the following polynomials have inside the unit circle?

\[ z^9 - 2z^6 + z^2 - 8z - 2; \ z^7 - 5z^4 + z^2 - 2; \ 2z^5 - z^3 + 3z^2 - z + 8. \]

2. Find the number of the zeros in the right half-plane for the polynomial

\[ z^6 + z^5 + 6z^4 + 5z^3 + 8z^2 + 4z + 1 \]

3. How many zeros does the following polynomial have in each quadrant?

\[ 2z^4 - 3z^3 + 3z^2 - z + 1 \]

Remark. Zeros are always understood with multiplicities.

2 Integrals.

4. Let \( a, b \in \mathbb{R} \) and find

\[ \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 + b^2} dx. \]

5. For \( a \in \mathbb{R} \) find

\[ \int_{-\infty}^{\infty} \frac{\sin(ax)}{\sinhx} dx. \]

Reminder: \( \sinhx = (e^x - e^{-x})/2. \)
3 Series and products.

6. Take $a \in \mathbb{R}$ and find
\[
\sum_{n=0}^{\infty} \frac{1}{n^2 + a^2}.
\]

7. Take $a \in \mathbb{R}$, $a \notin \mathbb{Z}$ and find
\[
\sum_{n=-\infty}^{\infty} \frac{1}{(n + a)^2}.
\]

4 Conformal mappings.

8. Find a biholomorphism $\Psi$ of the semicircle $\{z : |z| < 1, \text{Im}(z) > 0\}$ onto
the unit disk such that $\Psi$ admits a continuous extension to the boundary
and $\Psi(1) = 1$, $\Psi(-1) = -1$, $\Psi(0) = -i$.

9. Find a biholomorphism from the domain $\{z : |z| > 1, \text{Im}(z) > 0\}$ onto the upper half-plane.