Midterm Examination #2

1. The integrand has poles of order 1 whenever $z^4 - 1 = 0$, which occurs when $z$ is equal to $i$, $-i$, 1, or $-1$. Since $a \in \mathbb{R}$, $a > 1$, the only one of these poles that lies inside the circle $|z-a| = a$ is the pole at $z = 1$, as the following illustration makes apparent:

![Illustration of poles and contour]

The residue of the integrand at $z = 1$ is

$$
\text{Res}_{z=1} \frac{z}{z^4 - 1} = \lim_{z \to 1} \frac{z}{(z + i)(z - i)(z + 1)}
$$

$$
= \frac{1}{(1 + i)(1 - i)(1 + 1)}
$$

$$
= \frac{1}{4}.
$$

Therefore, by the Residue Theorem

$$
\int_{|z-a|=a} \frac{z}{z^4 - 1} \, dz = 2\pi i \cdot \frac{1}{4}
$$

$$
= \left[ \frac{\pi}{2} \right]
$$

2. Since $b$ lies inside the contour, by Cauchy’s Integral Formula for the derivatives,

$$
\int_{C} \frac{ze^z}{(z-b)^2} \, dz = \frac{2\pi i}{2!} \left[ \frac{d^2}{dz^2} [ze^z] \right]_{z=b}
$$

$$
= \pi i \left[ \frac{d}{dz} [ze^z + e^z] \right]_{z=b}
$$

$$
= \pi i [ze^z + 2e^z]_{z=b}
$$

$$
= [-\pi ie^b(b+2)]
$$
3. The integrand has a pole of order 1 at $z = 0$ and a pole of order 3 at $z = 1$. The residues of the integrand corresponding to each of these poles are

$$\text{Res}_{z=0} \frac{e^z}{z(1-z)^3} = \lim_{z \to 0} \frac{e^z}{z(1-z)^3} = \left[ \frac{e^z}{(1-z)^3} \right]_{z=0} = e^0 \frac{e}{(1-0)^3} = 1$$

and

$$\text{Res}_{z=1} \frac{e^z}{z(1-z)^3} = \frac{1}{2!} \lim_{z \to 1} \left[ \frac{d^2}{dz^2} \left( (z-1)^3 \frac{e^z}{z(1-z)^3} \right) \right] = \frac{1}{2} \lim_{z \to 1} \left[ \frac{d}{dz} \left[ -(z-1)e^z \right] \right] = \frac{1}{2} \lim_{z \to 1} \left[ -z^{-1}e^z + e^z z^{-2} + z^{-2}e^z - 2z^{-3}e^z \right] = \frac{1}{2} \left( -e + e + e - 2e \right) = -\frac{1}{2} e.$$

There are then four cases, depending on the status of the points 0 and 1 with respect to the contour $C$. (For simplicity, we assume that the winding number of $C$ about each pole is 1. For full generality, each residue will need to be multiplied by the winding number corresponding to its associated pole.)

**Case 1: Both 0 and 1 Inside $C$:** In this case,

$$\int_C \frac{e^z}{z(1-z)^3} \, dz = \frac{2\pi i}{2} \left( 1 - \frac{1}{2} e \right).$$

**Case 2: 0 Inside $C$, but 1 Outside $C$:** Now,

$$\int_C \frac{e^z}{z(1-z)^3} \, dz = \frac{2\pi i}{2}.$$

**Case 3: 1 Inside $C$, but 0 Outside $C$:** Here,

$$\int_C \frac{e^z}{z(1-z)^3} \, dz = -\pi i e.$$

**Case 4: Both 0 and 1 Outside $C$:** In this case, the integrand is actually analytic in the region bounded by $C$, and so
\[ \int_C \frac{e^z}{z(1-z)^3} \, dz = 0. \]

4. Set \( f(z) = \frac{z^2 + z - 1}{z(z - 1)} \). This function has poles of order 1 at \( z = 0 \) and \( z = 1 \). The corresponding residues are:

\[
\text{Res}_{z=0} f(z) = \lim_{z \to 0} zf(z) = \left. \frac{z^2 + z - 1}{z - 1} \right|_{z=0} = \frac{0^2 + 0 - 1}{0 - 1} = 1.
\]

and

\[
\text{Res}_{z=1} f(z) = \lim_{z \to 1} (z - 1)f(z) = \left. \frac{z^2 + z - 1}{z} \right|_{z=1} = \frac{1^2 + 1 - 1}{1} = 1.
\]

5. \( \tan z = \frac{\sin z}{\cos z} \) has poles wherever \( \cos z = 0 \). From Problem #3 on Ahlfors, pg. 44 (worked in Homework #3), we know that \( \cos z = \cos \Re(z) \cosh \Im(z) - i \sin \Re(z) \sinh \Im(z) \). Thus, \( \cos z \) will be zero if and only if \( \cos \Re(z) \cosh \Im(z) \) and \( \sin \Re(z) \sinh \Im(z) \) are both simultaneously zero. \( \cosh \Im(z) \) is never zero, so we must have \( \cos \Re(z) = 0 \), but since sine and cosine have no common zeros, this implies that \( \sin \Re(z) \neq 0 \). Hence, \( \cos z = 0 \) if and only if \( \cos \Re(z) = 0 \) and \( \sinh \Im(z) = 0 \), but \( \sinh \Im(z) = 0 \) if and only if \( \Im(z) = 0 \). Hence, the zeroes of the complex cosine function are the same as those of the real cosine function, i.e., \( \cos z = 0 \) if and only if \( z = (2k + 1) \frac{\pi}{2} \) for some integer \( k \).

Thus, \( \tan z \) has poles at \( z = (2k + 1) \frac{\pi}{2} \), where \( k \) is any integer. The corresponding residues are

\[
\text{Res}_{z=(2k+1)\frac{\pi}{2}} \tan z = \lim_{z \to (2k+1)\frac{\pi}{2}} \left( z - (2k + 1) \frac{\pi}{2} \right) \tan z = \lim_{z \to (2k+1)\frac{\pi}{2}} \left( z - (2k + 1) \frac{\pi}{2} \right) \frac{\cos z}{\sin z} = \lim_{z \to (2k+1)\frac{\pi}{2}} \frac{(z - (2k + 1) \frac{\pi}{2}) \cos z + \sin z}{- \sin z} = \frac{\sin((2k + 1) \frac{\pi}{2})}{- \sin((2k + 1) \frac{\pi}{2})} = -1
\]

The expression \( \cos \left( \frac{1}{z-2} \right) \) has an essential isolated singularity at \( z = 2 \). We recall the Taylor expansion
\[
\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,
\]
and so for \( z \neq 2 \),

\[
\cos \left( \frac{1}{z - 2} \right) = 1 - \frac{1}{2!(z - 2)^2} + \frac{1}{4!(z - 2)^4} - \cdots.
\]

Viewing this as an expansion of the form

\[
\cos \left( \frac{1}{z - 2} \right) = \sum_{n=-\infty}^{\infty} A_n (z - 2)^n,
\]

i.e., a Laurent expansion about \( z = 2 \), we see that the coefficient \( A_{-1} = 0 \). Hence,

\[
\text{Res}_{z=2} \cos \left( \frac{1}{z - 2} \right) = 0.
\]

6. The expression \( z^3 \cos \left( \frac{1}{z - 2} \right) \) has an essential isolated singularity at \( z = 2 \). From the preceding problem, we have

\[
\cos \left( \frac{1}{z - 2} \right) = 1 - \frac{1}{2!(z - 2)^2} + \frac{1}{4!(z - 2)^4} - \cdots.
\]

for \( z \neq 2 \). Furthermore, we can “Taylor expand” the function \( f(z) = z^3 \) about \( z = 2 \) to obtain

\[
z^3 = [z^3]_{z=2} + \left[ \frac{d}{dz} z^3 \right]_{z=2} (z - 2) + \frac{1}{2!} \left[ \frac{d^2}{dz^2} z^3 \right]_{z=2} (z - 2)^2 + \frac{1}{3!} \left[ \frac{d^3}{dz^3} z^3 \right]_{z=2} (z - 2)^3
\]

\[
= 8 + (3 \cdot 2^2)(z - 2) + \frac{1}{2}(6 \cdot 2)(z - 2)^2 + \frac{1}{6}(z - 2)^3
\]

\[
= 8 + 12(z - 2) + 6(z - 2)^2 + (z - 2)^3.
\]

Therefore,

\[
z^3 \cos \left( \frac{1}{z - 2} \right) = z^3 - \frac{z^3}{2!(z - 2)^2} + \frac{z^3}{4!(z - 2)^4} - \cdots
\]

\[
= 8 + 12(z - 2) + 6(z - 2)^2 + (z - 2)^3 - \frac{8 + 12(z - 2) + 6(z - 2)^2 + (z - 2)^3}{2!(z - 2)^2}
\]

\[
+ \frac{8 + 12(z - 2) + 6(z - 2)^2 + (z - 2)^3}{4!(z - 2)^4} - \cdots.
\]

From this, we see that the term with a factor of \( (z - 2)^{-1} \) is
\[-\frac{12}{2!}(z-2)^{-1} + \frac{1}{4!}(z-2)^{-1} = -\frac{143}{24}(z-2)^{-1}.
\]

Thus,
\[
\text{Res}_{z=2} z^3 \cos \left( \frac{1}{z-2} \right) = \left[ -\frac{143}{24} \right].
\]

7. For the first integral, the integrand has an essential isolated singularity enclosed by the contour at \( z = 0 \) but is analytic everywhere else in the plane. We recall the Taylor expansion
\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots.
\]
Hence,
\[
\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{(3!)z^3} + \frac{1}{(5!)z^5} - \cdots.
\]
The coefficient of the \( z^{-1} \) term is 1. Hence,
\[
\text{Res}_{z=0} \sin \frac{1}{z} = 1.
\]
The Residue Theorem therefore implies
\[
\int_{C(r)} \sin \left( \frac{1}{z} \right) \, dz = 2\pi i \cdot 1
\]
\[
= \frac{2\pi i}{2\pi i}.
\]
For the second integral, observe that
\[
\cos 2z = 1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \cdots
\]
\[
= 1 - z^2 + \frac{1}{2} z^4 - \cdots.
\]
Hence,
\[
\sin^2 z = \frac{1 - \cos 2z}{2}
\]
\[
= \frac{1}{2} z^2 - \frac{1}{4} z^4 + \cdots,
\]
and so
\[
\sin^2 \frac{1}{z} = \frac{1}{z^2} - \frac{1}{z^4} + \cdots .
\]

As there is no \(z^{-1}\) term, we conclude that

\[
\text{Res}_{z=0} \sin^2 \frac{1}{z} = 0,
\]

whence

\[
\int_{C(r)} \sin^2 \left( \frac{1}{z} \right) \, dz = 0.
\]

8. Since

\[
x^2 - 2x + y^2 = 0 \\
(x - 1)^2 + y^2 = 0,
\]

the circle \(C\) may be described by the set of all points \(z\) satisfying \(|z - 1| = 1\). The integrand has poles wherever \(z^4 + 1 = 0\), or wherever \(z = (-1)^{\frac{1}{4}}\). Explicitly, the integrand has four poles of order 1:

\[
p_1 = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \\
p_2 = \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \\
p_3 = -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \\
p_4 = -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}.
\]

Of these, only \(p_1\) and \(p_2\) lie inside the circle \(|z - 1| = 1\). The corresponding residues are

\[
\text{Res}_{z=p_1} \frac{1}{z^4 + 1} = \frac{1}{(p_1 - p_2)(p_1 - p_3)(p_1 - p_4)} \\
= \frac{1}{i\sqrt{2}(\sqrt{2} + i\sqrt{2})\sqrt{2}} \\
= \frac{1}{i2\sqrt{2}(1 + i)}
\]

\[
\text{Res}_{z=p_2} \frac{1}{z^4 + 1} = \frac{1}{(p_2 - p_1)(p_2 - p_3)(p_2 - p_4)} \\
= \frac{1}{-i\sqrt{2}\sqrt{2}(\sqrt{2} - i\sqrt{2})} \\
= \frac{-1}{i2\sqrt{2}(1 - i)}
\]
Therefore,
\[
\int_C \frac{dz}{z^4 + 1} = 2\pi i \left( \frac{1}{i2\sqrt{2}(1 + i)} - \frac{1}{i2\sqrt{2}(1 - i)} \right)
\]
\[
= \frac{\pi}{\sqrt{2}} \left( \frac{(1 - i) - (1 + i)}{(1 + i)(1 - i)} \right)
\]
\[
= \frac{-i\pi}{\sqrt{2}}
\]

9. We have
\[
\int_0^{2\pi} e^{i\varphi} \cos(n\varphi - \sin \varphi) d\varphi = \frac{1}{2} \int_0^{2\pi} e^{i\varphi} \left( e^{i(n\varphi - \sin \varphi)} + e^{-i(n\varphi - \sin \varphi)} \right) d\varphi.
\]
\[
= \frac{1}{2} \int_0^{2\pi} e^{i\varphi} e^{in\varphi} e^{-i\sin \varphi} d\varphi + \frac{1}{2} \int_0^{2\pi} e^{i\varphi} e^{-in\varphi} e^{i\sin \varphi} d\varphi
\]
\[
= \frac{1}{2} \int_0^{2\pi} e^{i\varphi - i\sin \varphi} e^{in\varphi} d\varphi + \frac{1}{2} \int_0^{2\pi} e^{i\varphi + i\sin \varphi} e^{-in\varphi} d\varphi
\]
\[
= \frac{1}{2} \int_0^{2\pi} e^{-i\varphi} (e^{i\varphi})^n d\varphi + \frac{1}{2} \int_0^{2\pi} e^{i\varphi} (e^{i\varphi})^{-n} d\varphi.
\]

Let \( z = e^{i\varphi} \). Then, \( dz = iz \, d\varphi \), and the above becomes
\[
\int_0^{2\pi} e^{i\varphi} \cos(n\varphi - \sin \varphi) d\varphi = \frac{-i}{2} \int_{|z|=1} e^{\frac{1}{i}z^{n-1}} dz + \frac{-i}{2} \int_{|z|=1} e^{-\frac{1}{i}z^{n-1}} dz.
\]

Recall that
\[
e^{z^2} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.
\]

Hence,
\[
e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{(2!)z^2} + \frac{1}{(3!)z^3} + \cdots,
\]

and so we have
\[
z^{-n-1} e^z = z^{-n-1} + z^{-n} + \frac{z^{-n+1}}{2!} + \frac{z^{-n+2}}{3!} + \cdots
\]
as well as
\[
z^{n-1} e^{\frac{1}{z}} = z^{n-1} + z^{n-2} + \frac{z^{n-3}}{2!} + \frac{z^{n-4}}{3!} + \cdots.
\]
There are now two cases, depending on the value of $n$.

**Case 1:** $n \geq 0$: In this case, both $z^{-n-1}e^z$ and $z^{n-1}e^{1/2}$ have an isolated singularity at $z = 0$. Using the series expansions from above, we find that

\[
\text{Res}_{z=0} z^{-n-1}e^z = \frac{1}{n!}
\]

and

\[
\text{Res}_{z=0} z^{n-1}e^{1/2} = \frac{1}{n!}.
\]

Hence,

\[
\int_0^{2\pi} e^{\cos \varphi} \cos(n \varphi - \sin \varphi) d\varphi = \frac{-i}{2} \frac{2\pi i}{n!} + \frac{-i}{2} \frac{2\pi i}{n!} = \frac{2\pi}{n!}
\]

**Case 2:** $n < 0$: Now, only $z^{n-1}e^{1/2}$ has an isolated singularity at $z = 0$, while $z^{-n-1}e^z$ is analytic on the whole plane. Looking at the series expansion for $z^{n-1}e^{1/2}$, we see that if $n < 0$, there is no $z^{-1}$ term, and so the residue at $z = 0$ is zero! Hence,

\[
\int_0^{2\pi} e^{\cos \varphi} \cos(n \varphi - \sin \varphi) d\varphi = 0
\]