Math 542: Homework 3

1. Let G be a group and $\Gamma_1, \Gamma_2 < G$. Say Γ_1 and Γ_2 are *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.

(a) Prove that commensurability is an equivalence relation.

(b) Let $\mathrm{Comm_G}(\Gamma) = \{\mathrm{g} \in \mathrm{G} | \mathrm{g}\Gamma\mathrm{g}^{-1} \text{ is commensurable with } \Gamma\}$. Prove that $\mathrm{Comm_G}(\Gamma)$ is a subgroup of G.

(c) Identify $\mathrm{Comm}_{\mathrm{PSL}(2,\mathbf{C})}(\mathrm{PSL}(2, \mathbf{Z}[\mathrm{i}]))$.

2. (a) Assume that Γ_1 is a Kleinian group, and Γ_2 is commensurable with Γ_1. Prove that Γ_2 is a Kleinian group.

(b) Assume that Γ_1 and Γ_2 are Kleinian groups. Prove that Γ_1 has (algebraic) integral traces if and only if Γ_2 has integral traces.

3. Prove that the set of parabolic fixed points is a commensurability invariant.

4. (a) Prove that a dodecahedron D with all dihedral angles $2\pi/5$ exists in $\mathbf{H}^3$ (**Hint:** Show the 2nd barycentric subdivision yields a particular tetrahedron in $\mathbf{H}^3$).

(b) Show that identifying opposite faces of D with a $3\pi/5$ twist produces a topological 3-manifold. (The Seifert-Weber Dodecahedral space).

(c) Compute the trace field of $\pi_1(D)$.

5. Below is the presentation of a co-compact Kleinian group Γ.

$$< a, b | a^4 = 1, waw^{-1}b^{-1} = 1, w = ab^{-1}a^{-1}b >$$

(a) Compute the trace-field of Γ.

(b) Find a subgroup of finite index in Γ whose trace-field is a proper subfield of the trace-field. (**Hint**: Index 2).

6. Prove that $\mathrm{tr}(\gamma^{\mathrm{N}})$ is a monic integer polynomial in $\mathrm{tr}(\gamma)$.

7. Prove that if $\mathrm{tr}(\gamma) = \lambda + 1/\lambda$, then $\mathrm{tr}(\gamma)$ is an algebraic integer if and only λ is a unit.

8. Compute the signatures of the following number fields.

(i) $\mathbf{Q}(t)$ where $t^3 + t^2 - 2t - 1 = 0$.

(ii) $\mathbf{Q}(t)$ where $t^4 - 2t^2 + 3t - 1 = 0$.

(iii) $\mathbf{Q}(i, \cos(\pi/12))$

9. Let k be a totally real number field and $t \in k$ negative. Suppose that all (non-identity) Galois conjugates of t are positive. What is the signature of the number field $\mathbf{Q}(\sqrt{t})$?

10. Let Γ be a non-elementary Kleinian group. The *limit set* $\Lambda(\Gamma)$ of Γ is the set of accumulation points on the sphere-at-infinity of Γ-orbits of points in $\mathbf{H}^3$.

(a) Show that $\Lambda(\Gamma)$ is the closure of the set of fixed points of hyperbolic elements in Γ.

(b) Show that $\Lambda(\Gamma)$ is the smallest non-empty, closed, Γ-invariant subset of the sphere-at-infinity.

(c) Let $\Omega(\Gamma)$ denote the complement of $\Lambda(\Gamma)$ in the sphere-at-infinity. Prove that Γ acts discontinuously on $\Omega(\Gamma)$.