Math 542: Homework 6

1. Let $k = \mathbf{Q}(t)$ be a number field of degree n and let $t = t_1, t_2, \ldots, t_n$ be the Galois conjugates of t.
   Prove that the discriminant of the basis $\{1, t, t^2, \ldots, t^{n-1}\}$ is $\prod_{i<j}(t_i - t_j)$.
2. Let $k = \mathbf{Q}(\theta)$ where $\theta^4 + 5\theta^3 + 7\theta^2 + 3\theta + 1 = 0$.

(a) Prove that k has signature $(2, 1)$.

(b) Determine an integral basis compute the discriminant of k.

(c) Let $M = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold with invariant trace field k, and invariant quaternion algebra given by the Hilbert Symbol $\left(\frac{-1,\theta}{k}\right)$. Prove that M contains no totally geodesic surfaces.

3. Assume the following result:

Let k be a number field of degree $n = r_1 + 2r_2$ and discriminant d_k. Then every ideal class of k contains a non-zero integral ideal J whose norm satisfies:

$$N(J) \leq \frac{4^{r_2} n! \sqrt{|d_k|}}{\pi^{r_2} n^n}$$

   Prove that the field k in Ex 2 has class number 1.

4. Let Γ be a Kleinian group with trace-field k, all of whose traces are algebraic integers. Let

$$\mathcal{O}\Gamma = \{\Sigma a_i \gamma_i : a_i \in R_k, \gamma_i \in \Gamma\}$$

   (where all sums are finite).
   Prove that $\mathcal{O}\Gamma$ is an order of $A_0\Gamma$.

5. Let A be the quaternion algebra with Hilbert Symbol $\left(\frac{-3,5}{\mathbf{Q}}\right)$. Prove that $\mathcal{O} = \mathbf{Z}[1, i, \frac{1+j}{2}, \frac{i+ij}{2}]$ is an order of A.
6. Let k be a number field and I an integral ideal of k. Let $\mathcal{O}(I) \subset M_2(k)$ be the subset given by $\begin{pmatrix} R_k & I \\ I^{-1} & R_k \end{pmatrix}$. Prove that $O(I)$ is an order.
7. Let Γ be a (finitely generated) subgroup of $\mathrm{SL}(2, \mathbf{C})$ and Λ a finite index subgroup. Show that if Λ is discrete then Γ is discrete.
8. Let $\Gamma =< f, g >$ where f has order 6, g has order 2 and $z = \mathrm{tr([f,g])} - 2$ satisfies $z^3 + z^2 + 2z + 1 = 0$. Prove that Γ is discrete.

9. Let Γ be a finite co-volume Kleinian group that contains a copy of the alternating group A_4. Prove that $A\Gamma \cong \left(\frac{-1,-1}{k\Gamma}\right)$ (**Hint:** A_4 (or rather its extension in $\mathrm{SL}(2, \mathbf{C})$ is an irreducible subgroup).

10. Let B be a quaternion algebra over the number field k. By an *ideal* of B we mean a finitely generated R_k-module that contains a k-basis of B. For $I \subset B$ an ideal, define:

$$\mathcal{O}_l(I) = \{\alpha \in B : \alpha I \subset I\} \text{ and } \mathcal{O}_r(I) = \{\alpha \in B : I\alpha \subset I\}.$$

Prove that $\mathcal{O}_l(I)$ and $\mathcal{O}_r(I)$ are orders in B.